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Osaka J. Math.
12 (1975), 41-44

PROPER BOUNDARY POINTS OF THE SPECTRUM

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(Received December 20, 1973)

1. Introduction. Let A be a bounded operator on an infinite dimensional Hilbert space H. A point λ on the boundary $\partial\sigma(A)$ of the spectrum $\sigma(A)$ of A will be called *proper* if there is a bounded sequence $\{\lambda_k\}$ of points in the resolvent set $\rho(A)$ of A such that

$$\|(\lambda_k-\lambda)(\lambda_k-A)^{-1}\|\rightarrow 1\,. \tag{1}$$

Examples of proper boundary points of the spectrum are easily given.

1. If $|\lambda|=\|A\|$, then λ is a proper boundary point.
2. If λ in $\sigma(A)$ is a boundary point of the numerical range of A, then it is a proper boundary point.
3. If λ in $\sigma(A)$ is a boundary point of a spectral set X for A and there is a sequence $\{\lambda_k\}$ in the complement of X such that

$$|\lambda_k-\lambda|/d(\lambda_k, X)\rightarrow 1\,, \tag{2}$$

then λ is a proper boundary point (here $d(z, X)$ denotes the distance from z to X).

4. If A is seminormal, λ is a boundary point of $\sigma(A)$ and there is a sequence of points in $\rho(A)$ satisfying (2) with $X=\sigma(A)$, then λ is a proper boundary point.

We shall verify these statements later. We shall also prove the following theorem and give several applications.

Theorem 1. *If λ is a proper boundary point of $\sigma(A)$, then for each bounded sequence $\{x_n\}$ in H we have*

$$(A-\lambda)x_n\rightarrow 0 \quad \textit{iff} \;\; (A^*-\bar{\lambda})x_n\rightarrow 0\,. \tag{3}$$

The theorem generalizes results of Putnam [1], Saito [2], Sz-Nagy, Foias [3], Schreiber [4] and others. The proof will be given in the next section. Now we shall give some consequences and applications. The essential spectrum of A is defined as

$$\sigma_e(A)=\bigcap_{K\text{ compact}}\sigma(A+K)\,.$$

Theorem 2. *If λ is a proper boundary point of $\sigma(A)$ and it is also in the essential spectrum of A, then there is an orthonormal sequence $\{\varphi_n\}$ in H such that*

$$(4) \qquad (A-\lambda)\varphi_n \to 0 \quad \text{and} \quad (A^*-\bar{\lambda})\varphi_n \to 0 .$$

Proof. If λ is in the essential spectrum of A, then either $A-\lambda$ is not a Fredholm operator or its index is not 0 ([5, p. 180]). The latter case cannot occur, since λ is a boundary point of the spectrum. Hence there is an orthonormal sequence satisfying at least one of the statements in (4). By Theorem 1, it must satisfy both.

Corollary 3. *If λ is a proper boundary point of the spectrum of A and it is not isolated, then there is an orthonormal sequence satisfying* (4).

Proof. By [6, Theorem 2.12] or by [7, Theorem 1], a nonisolated boundary point of the spectrum must be in the essential spectrum. Apply Theorem 2.

We now give some applications suggested by the work of Putman [7].

Corollary 4. *Let A be a bounded operator in H having at least one proper boundary point of $\sigma(A)$ in the essential spectrum. Then the operator*

$$(5) \qquad C = A^*A - AA^*$$

has 0 *in its essential spectrum.*

Proof. Let λ be one such point. By Theorem 2 there is an orthonormal sequence such that (4) holds. Since

$$C = (A^*-\bar{\lambda})(A-\lambda)-(A-\lambda)(A^*-\bar{\lambda}) ,$$

we see that $C\varphi_n \to 0$. Thus 0 is in the essential spectrum of C.

Corollary 5. *If A has at least one non-isolated proper boundary point of its spectrum, then* 0 *is in the essential spectrum of the operator C given by* (5).

Proof. Use Corollary 3.

Corollary 6. *If every boundary point of the spectrum of A is proper, then C has* 0 *in its essential spectrum.*

Proof. If not all of the boundary points of $\sigma(A)$ are isolated, the result follows from Corollary 5. If they are, then the spectrum of A consists of only a finite number of points. At least one of these points must be in the essential spectrum of A (if H is infinite dimensional). Now apply Corollary 4.

Corollary 7. *Suppose every boundary point of the spectrum of A is proper, and put $A=L+iM$, where L and M are self adjoint. Then $\sigma(L)$ [resp. $\sigma(M)$] contains the projection of $\sigma(A)$ on the x [resp, y] axis.*

Proof. Suppose λ is a boundary point of $\sigma(A)$. Then there is a sequence of unit vectors satisfying one of the statements in (3). By Theorem 1 it satisfies both. Since $L=\frac{1}{2}(A+A^*)$, we see that $(L-Re\,\lambda)x_n\to 0$. Thus $Re\,\lambda$ is in the spectrum of L. If λ is an interior point, then there is a boundary point λ_1 such that $Re(\lambda-\lambda_1)=0$. We use the point λ_1 in place of λ. A similar proof works for M.

2. The proofs. Let us first verify that the points described in section 1 are proper. Consider the first statement. By rotating we may assume $\lambda=\|A\|$. For $t>\lambda$, we have $\|(t-A)u\|\geq(t-\lambda)\|u\|$. Hence $\|(t-A)^{-1}\|\leq 1/(t-\lambda)$.

Similar reasoning gives the second case. Since the numerical range $W(A)$ of A is convex, it is contained on one side of line L going through λ. Let z be any point on the other side of L such that $z-\lambda$ is orthogonal to it. Thus $|z-\lambda|$ is the distance $d(z, W(A))$ from z to $W(A)$. We know in general that

$$\|(z-A)^{-1}\|\leq 1/d(z, W(A))$$

holds for any z not in the closure of $W(A)$. This shows that

$$\|(z-A)^{-1}\|\leq 1/|z-\lambda|$$

holds for z on the other side of L. This proves the assertion for the second case.

If X, is a spectral set for A, then

$$\|(z-A)^{-1}\|\leq 1/d(z, X) \tag{7}$$

holds for all z not in X. If there is a sequence in the complement of X such that (2) holds, then we see that (1) holds for the same sequence.

If A is seminormal, then (7) holds for $X=\sigma(A)$. Apply the same reasoning.

Proof of Theorem 1. Assume $\lambda=0$, and set $W_k=\lambda_k(\lambda_k-A)^{-1}$. Then (1) says

$$\|W_k\|\to 1\,. \tag{8}$$

Now

$$I-W_k=(\lambda_k-A)(\lambda_k-A)^{-1}-\lambda_k(\lambda_k-A)^{-1}=-A(\lambda_k-A)^{-1}\,.$$

Thus

$$\begin{aligned}\|(I-W_k)x\|^2&=\|x\|^2-2\,Re(x, W_kx)+\|W_kx\|^2\\ &=2\,Re(x, [I-W_k]x)+\|W_kx\|^2-\|x\|^2\\ &=-2\,Re(A^*x, (\lambda_k-A)^{-1}x)+\|W_kx\|^2-\|x\|^2\\ &\leq\|2A^*x\|\,\|(\lambda_k-A)^{-1}\|\,\|x\|+(\|W_k\|^2-1)\|x\|^2\,.\end{aligned}$$

Now assume that $||x_n|| \leq K$ and that $|\lambda_k| \leq M$. Let $\varepsilon > 0$ be given. Take k so large that

$$||W_k||^2 - 1 < \varepsilon/2R^2K^2,$$

where $R = M + ||A||$. Then fix k. If $A^* x_n \to 0$, we can find an N so large that

$$||A^* x_n|| < \varepsilon/4KR^2||(\lambda_k - A)^{-1}||, \qquad n > N.$$

These last inequalities imply

$$||(I - W_k)x_n||^2 < \varepsilon/R^2, \qquad n > N.$$

Now

$$||Ax_n||^2 = ||(\lambda_k - A)(I - W_k)x_n||^2 \leq \varepsilon, \qquad n > N.$$

This shows that $Ax_n \to 0$. A symmetrical argument shows the converse, that $Ax_n \to 0$ implies $A^* x_n \to 0$. This comples the proof.

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Bibliography

[1] C.R. Putnam: *Eigenvalues and boundary spectra*, Illinois J. Math. **12** (1968), 278–283.

[2] T. Saito: *A theorem on boundary spectra*, Acta Math. Szeged **33** (1972), 101–104.

[3] B. Sz.-Nagy and C. Foiaz: *Une relation parmi les vecteurs propres d'un operateur de l'espace de Hilbert et de l'operateur adjoint*, ibid. **20** (1959), 91–96.

[4] M. Schreiber: *On the spectrum of a contraction*, Proc. Amer. Math. Soc. **12** (1961), 709–713.

[5] M. Schechter: Principles of Functional Analysis, Academic Press, New York, 1971.

[6] M. Schechter: *Operators obeying Weyl's theorem*, Scripta Math. **29** (1973), 67–75.

[7] C.R. Putnam: *The spectra of operators having resolvents of first order growth*, Trans. Amer. Math. Soc. **133** (1968), 505–510.

[8] A.E. Taylor: Introduction to Functional Analysis, Wiley, New York, 1958.