

Title	Proper boundary points of the spectrum
Author(s)	Schechter, Martin
Citation	Osaka Journal of Mathematics. 1975, 12(1), p. 41-44
Version Type	VoR
URL	<a href="https://doi.org/10.18910/5114">https://doi.org/10.18910/5114</a>
rights	
Note	

*Osaka University Knowledge Archive : OUKA*

<https://ir.library.osaka-u.ac.jp/>

Osaka University

## PROPER BOUNDARY POINTS OF THE SPECTRUM

MARTIN SCHECHTER

(Received December 20, 1973)

**1. Introduction.** Let  $A$  be a bounded operator on an infinite dimensional Hilbert space  $H$ . A point  $\lambda$  on the boundary  $\partial\sigma(A)$  of the spectrum  $\sigma(A)$  of  $A$  will be called *proper* if there is a bounded sequence  $\{\lambda_k\}$  of points in the resolvent set  $\rho(A)$  of  $A$  such that

$$(1) \quad \|(\lambda_k - \lambda)(\lambda_k - A)^{-1}\| \rightarrow 1.$$

Examples of proper boundary points of the spectrum are easily given.

1. If  $|\lambda| = \|A\|$ , then  $\lambda$  is a proper boundary point.
2. If  $\lambda$  in  $\sigma(A)$  is a boundary point of the numerical range of  $A$ , then it is a proper boundary point.
3. If  $\lambda$  in  $\sigma(A)$  is a boundary point of a spectral set  $X$  for  $A$  and there is a sequence  $\{\lambda_k\}$  in the complement of  $X$  such that

$$(2) \quad |\lambda_k - \lambda| / d(\lambda_k, X) \rightarrow 1,$$

then  $\lambda$  is a proper boundary point (here  $d(z, X)$  denotes the distance from  $z$  to  $X$ ).

4. If  $A$  is seminormal,  $\lambda$  is a boundary point of  $\sigma(A)$  and there is a sequence of points in  $\rho(A)$  satisfying (2) with  $X = \sigma(A)$ , then  $\lambda$  is a proper boundary point.

We shall verify these statements later. We shall also prove the following theorem and give several applications.

**Theorem 1.** *If  $\lambda$  is a proper boundary point of  $\sigma(A)$ , then for each bounded sequence  $\{x_n\}$  in  $H$  we have*

$$(3) \quad (A - \lambda)x_n \rightarrow 0 \quad \text{iff} \quad (A^* - \bar{\lambda})x_n \rightarrow 0.$$

The theorem generalizes results of Putnam [1], Saito [2], Sz-Nagy, Foias [3], Schreiber [4] and others. The proof will be given in the next section. Now we shall give some consequences and applications. The essential spectrum of  $A$  is defined as

$$\sigma_e(A) = \bigcap_{K \text{ compact}} \sigma(A + K).$$

**Theorem 2.** *If  $\lambda$  is a proper boundary point of  $\sigma(A)$  and it is also in the essential spectrum of  $A$ , then there is an orthonormal sequence  $\{\varphi_n\}$  in  $H$  such that*

$$(4) \quad (A-\lambda)\varphi_n \rightarrow 0 \quad \text{and} \quad (A^*-\bar{\lambda})\varphi_n \rightarrow 0.$$

*Proof.* If  $\lambda$  is in the essential spectrum of  $A$ , then either  $A-\lambda$  is not a Fredholm operator or its index is not 0 ([5, p. 180]). The latter case cannot occur, since  $\lambda$  is a boundary point of the spectrum. Hence there is an orthonormal sequence satisfying at least one of the statements in (4). By Theorem 1, it must satisfy both.

**Corollary 3.** *If  $\lambda$  is a proper boundary point of the spectrum of  $A$  and it is not isolated, then there is an orthonormal sequence satisfying (4).*

*Proof.* By [6, Theorem 2.12] or by [7, Theorem 1], a nonisolated boundary point of the spectrum must be in the essential spectrum. Apply Theorem 2.

We now give some applications suggested by the work of Putman [7].

**Corollary 4.** *Let  $A$  be a bounded operator in  $H$  having at least one proper boundary point of  $\sigma(A)$  in the essential spectrum. Then the operator*

$$(5) \quad C = A^*A - AA^*$$

*has 0 in its essential spectrum.*

*Proof.* Let  $\lambda$  be one such point. By Theorem 2 there is an orthonormal sequence such that (4) holds. Since

$$C = (A^* - \bar{\lambda})(A - \lambda) - (A - \lambda)(A^* - \bar{\lambda}),$$

we see that  $C\varphi_n \rightarrow 0$ . Thus 0 is in the essential spectrum of  $C$ .

**Corollary 5.** *If  $A$  has at least one non-isolated proper boundary point of its spectrum, then 0 is in the essential spectrum of the operator  $C$  given by (5).*

*Proof.* Use Corollary 3.

**Corollary 6.** *If every boundary point of the spectrum of  $A$  is proper, then  $C$  has 0 in its essential spectrum.*

*Proof.* If not all of the boundary points of  $\sigma(A)$  are isolated, the result follows from Corollary 5. If they are, then the spectrum of  $A$  consists of only a finite number of points. At least one of these points must be in the essential spectrum of  $A$  (if  $H$  is infinite dimensional). Now apply Corollary 4.

**Corollary 7.** *Suppose every boundary point of the spectrum of  $A$  is proper, and put  $A=L+iM$ , where  $L$  and  $M$  are self adjoint. Then  $\sigma(L)$  [resp.  $\sigma(M)$ ] contains the projection of  $\sigma(A)$  on the  $x$  [resp.  $y$ ] axis.*

**Proof.** Suppose  $\lambda$  is a boundary point of  $\sigma(A)$ . Then there is a sequence of unit vectors satisfying one of the statements in (3). By Theorem 1 it satisfies both. Since  $L = \frac{1}{2}(A + A^*)$ , we see that  $(L - \operatorname{Re} \lambda)x_n \rightarrow 0$ . Thus  $\operatorname{Re} \lambda$  is in the spectrum of  $L$ . If  $\lambda$  is an interior point, then there is a boundary point  $\lambda_1$  such that  $\operatorname{Re}(\lambda - \lambda_1) = 0$ . We use the point  $\lambda_1$  in place of  $\lambda$ . A similar proof works for  $M$ .

**2. The proofs.** Let us first verify that the points described in section 1 are proper. Consider the first statement. By rotating we may assume  $\lambda = \|A\|$ . For  $t > \lambda$ , we have  $\|(t - A)u\| \geq (t - \lambda)\|u\|$ . Hence  $\|(t - A)^{-1}\| \leq 1/(t - \lambda)$ .

Similar reasoning gives the second case. Since the numerical range  $W(A)$  of  $A$  is convex, it is contained on one side of line  $L$  going through  $\lambda$ . Let  $z$  be any point on the other side of  $L$  such that  $z - \lambda$  is orthogonal to it. Thus  $|z - \lambda|$  is the distance  $d(z, W(A))$  from  $z$  to  $W(A)$ . We know in general that

$$\|(z - A)^{-1}\| \leq 1/d(z, W(A))$$

holds for any  $z$  not in the closure of  $W(A)$ . This shows that

$$\|(z - A)^{-1}\| \leq 1/|z - \lambda|$$

holds for  $z$  on the other side of  $L$ . This proves the assertion for the second case.

If  $X$  is a spectral set for  $A$ , then

$$(7) \quad \|(z - A)^{-1}\| \leq 1/d(z, X)$$

holds for all  $z$  not in  $X$ . If there is a sequence in the complement of  $X$  such that (2) holds, then we see that (1) holds for the same sequence.

If  $A$  is seminormal, then (7) holds for  $X = \sigma(A)$ . Apply the same reasoning.

**Proof of Theorem 1.** Assume  $\lambda = 0$ , and set  $W_k = \lambda_k(\lambda_k - A)^{-1}$ . Then (1) says

$$(8) \quad \|W_k\| \rightarrow 1.$$

Now

$$I - W_k = (\lambda_k - A)(\lambda_k - A)^{-1} - \lambda_k(\lambda_k - A)^{-1} = -A(\lambda_k - A)^{-1}.$$

Thus

$$\begin{aligned} \|(I - W_k)x\|^2 &= \|x\|^2 - 2 \operatorname{Re}(x, W_k x) + \|W_k x\|^2 \\ &= 2 \operatorname{Re}(x, [I - W_k]x) + \|W_k x\|^2 - \|x\|^2 \\ &= -2 \operatorname{Re}(A^* x, (\lambda_k - A)^{-1} x) + \|W_k x\|^2 - \|x\|^2 \\ &\leq \|2A^* x\| \|(\lambda_k - A)^{-1}\| \|x\| + (\|W_k\|^2 - 1)\|x\|^2. \end{aligned}$$

Now assume that  $\|x_n\| \leq K$  and that  $|\lambda_k| \leq M$ . Let  $\varepsilon > 0$  be given. Take  $k$  so large that

$$\|W_k\|^2 - 1 < \varepsilon/2R^2K^2,$$

where  $R = M + \|A\|$ . Then fix  $k$ . If  $A^*x_n \rightarrow 0$ , we can find an  $N$  so large that

$$\|A^*x_n\| < \varepsilon/4KR^2\|(\lambda_k - A)^{-1}\|, \quad n > N.$$

These last inequalities imply

$$\|(I - W_k)x_n\|^2 < \varepsilon/R^2, \quad n > N.$$

Now

$$\|Ax_n\|^2 = \|(\lambda_k - A)(I - W_k)x_n\|^2 \leq \varepsilon, \quad n > N.$$

This shows that  $Ax_n \rightarrow 0$ . A symmetrical argument shows the converse, that  $Ax_n \rightarrow 0$  implies  $A^*x_n \rightarrow 0$ . This completes the proof.

YESHIVA UNIVERSITY

---

### Bibliography

- [1] C.R. Putnam: *Eigenvalues and boundary spectra*, Illinois J. Math. **12** (1968), 278–283.
- [2] T. Saito: *A theorem on boundary spectra*, Acta Math. Szeged **33** (1972), 101–104.
- [3] B. Sz.-Nagy and C. Foias: *Une relation parmi les vecteurs propres d'un operateur de l'espace de Hilbert et de l'operateur adjoint*, ibid. **20** (1959), 91–96.
- [4] M. Schreiber: *On the spectrum of a contraction*, Proc. Amer. Math. Soc. **12** (1961), 709–713.
- [5] M. Schechter: *Principles of Functional Analysis*, Academic Press, New York, 1971.
- [6] M. Schechter: *Operators obeying Weyl's theorem*, Scripta Math. **29** (1973), 67–75.
- [7] C.R. Putnam: *The spectra of operators having resolvents of first order growth*, Trans. Amer. Math. Soc. **133** (1968), 505–510.
- [8] A.E. Taylor: *Introduction to Functional Analysis*, Wiley, New York, 1958.