PROPER BOUNDARY POINTS OF THE SPECTRUM

MARTIN SCHECHTER

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1. Introduction. Let A be a bounded operator on an infinite dimensional Hilbert space H. A point λ on the boundary $\partial\sigma(A)$ of the spectrum $\sigma(A)$ of A will be called *proper* if there is a bounded sequence $\{\lambda_k\}$ of points in the resolvent set $\rho(A)$ of A such that

$$(1) \qquad \|(\lambda_k-\lambda)(\lambda_k-A)^{-1}\| \to 1 .$$

Examples of proper boundary points of the spectrum are easily given.

1. If $|\lambda|=\|A\|$, then λ is a proper boundary point.

2. If λ in $\sigma(A)$ is a boundary point of the numerical range of A, then it is a proper boundary point.

3. If λ in $\sigma(A)$ is a boundary point of a spectral set X for A and there is a sequence $\{\lambda_k\}$ in the complement of X such that

$$(2) \qquad |\lambda_k-\lambda|/d(\lambda_k, X) \to 1 ,$$

then λ is a proper boundary point (here $d(z, X)$ denotes the distance from z to X).

4. If A is seminormal, λ is a boundary point of $\sigma(A)$ and there is a sequence of points in $\rho(A)$ satisfying (2) with $X=\sigma(A)$, then λ is a proper boundary point.

We shall verify these statements later. We shall also prove the following theorem and give several applications.

Theorem 1. *If λ is a proper boundary point of $\sigma(A)$, then for each bounded sequence $\{x_n\}$ in H we have*

$$(3) \qquad (A-\lambda)x_n \to 0 \quad \text{iff} \quad (A^*-\bar\lambda)x_n \to 0 .$$

The theorem generalizes results of Putnam [1], Saito [2], Sz-Nagy, Foias [3], Schreiber [4] and others. The proof will be given in the next section. Now we shall give some consequences and applications. The essential spectrum of A is defined as

$$\sigma_e(A) = \bigcap_{K \text{ compact}} \sigma(A+K) .$$

Theorem 2. *If* λ *is a proper boundary point of* $\sigma(A)$ *and it is also in the essential spectrum of* A, *then there is an orthonormal sequence* $\{\varphi_n\}$ *in* H *such that*

(4) $(A-\lambda)\varphi_n \to 0$ and $(A^*-\bar{\lambda})\varphi_n \to 0$.

Proof.   If λ is in the essential spectrum of A, then either $A-\lambda$ is not a Fredholm operator or its index is not 0 ([5, p. 180]).   The latter case cannot occur, since λ is a boundary point of the spectrum.   Hence there is an orthonormal sequence satisfying at least one of the statements in (4).   By Theorem 1, it must satisfy both.

Corollary 3. *If* λ *is a proper boundary point of the spectrum of* A *and it is not isolated, then there is an orthonormal sequence satisfying* (4).

Proof.   By [6, Theorem 2.12] or by [7, Theorem 1], a nonisolated boundary point of the spectrum must be in the essential spectrum.   Apply Theorem 2.
We now give some applications suggested by the work of Putman [7].

Corollary 4. *Let* A *be a bounded operator in* H *having at least one proper boundary point of* $\sigma(A)$ *in the essential spectrum.   Then the operator*

(5) $C = A^*A - AA^*$

has 0 *in its essential spectrum.*

Proof.   Let λ be one such point.   By Theorem 2 there is an orthonormal sequence such that (4) holds.   Since

$$C = (A^*-\bar{\lambda})(A-\lambda) - (A-\lambda)(A^*-\bar{\lambda}),$$

we see that $C\varphi_n \to 0$.   Thus 0 is in the essential spectrum of C.

Corollary 5. *If* A *has at least one non-isolated proper boundary point of its spectrum, then* 0 *is in the essential spectrum of the operator* C *given by* (5).

Proof.   Use Corollary 3.

Corollary 6. *If every boundary point of the spectrum of* A *is proper, then* C *has* 0 *in its essential spectrum.*

Proof.   If not all of the boundary points of $\sigma(A)$ are isolated, the result follows from Corollary 5.   If they are, then the spectrum of A consists of only a finite number of points.   At least one of these points must be in the essential spectrum of A (if H is infinite dimensional).   Now apply Corollary 4.

Corollary 7. *Suppose every boundary point of the spectrum of* A *is proper, and put* $A = L+iM$, *where* L *and* M *are self adjoint.   Then* $\sigma(L)$ *[resp.* $\sigma(M)$*] contains the projection of* $\sigma(A)$ *on the* x *[resp, y] axis.*

Proof.   Suppose λ is a boundary point of $\sigma(A)$.   Then there is a sequence of unit vectors satisfying one of the statements in (3).   By Theorem 1 it satisfies both.   Since $L=\frac{1}{2}(A+A^*)$, we see that $(L-Re\ \lambda)x_n \to 0$.   Thus $Re\ \lambda$ is in the spectrum of L.   If λ is an interior point, then there is a boundary point λ_1 such that $Re(\lambda-\lambda_1)=0$.   We use the point λ_1 in place of λ.   A similar proof works for M.

2. The proofs.   Let us first verify that the points described in section 1 are proper.   Consider the first statement.   By rotating we may assume $\lambda=||A||$. For $t>\lambda$, we have $||(t-A)u|| \geq (t-\lambda)||u||$.   Hence $||(t-A)^{-1}|| \leq 1/(t-\lambda)$.

Similar reasoning gives the second case.   Since the numerical range $W(A)$ of A is convex, it is contained on one side of line L going through λ.   Let z be any point on the other side of L such that $z-\lambda$ is orthogonal to it.   Thus $|z-\lambda|$ is the distance $d(z, W(A))$ from z to $W(A)$.   We know in general that

$$||(z-A)^{-1}|| \leq 1/d(z, W(A))$$

holds for any z not in the closure of $W(A)$.   This shows that

$$||(z-A)^{-1}|| \leq 1/|z-\lambda|$$

holds for z on the other side of L.   This proves the assertion for the second case.

If X, is a spectral set for A, then

$$(7) \qquad\qquad ||(z-A)^{-1}|| \leq 1/d(z, X)$$

holds for all z not in X.   If there is a sequence in the complement of X such that (2) holds, then we see that (1) holds for the same sequence.

If A is seminormal, then (7) holds for $X=\sigma(A)$.   Apply the same reasoning.

Proof of Theorem 1.   Assume $\lambda=0$, and set $W_k=\lambda_k(\lambda_k-A)^{-1}$. Then (1) says

$$(8) \qquad\qquad ||W_k|| \to 1 \ .$$

Now

$$I-W_k = (\lambda_k-A)(\lambda_k-A)^{-1} - \lambda_k(\lambda_k-A)^{-1} = -A(\lambda_k-A)^{-1} \ .$$

Thus

$$\begin{aligned}
||(I-W_k)x||^2 &= ||x||^2 - 2\,Re\,(x, W_k x) + ||W_k x||^2 \\
&= 2\,Re\,(x, [I-W_k]x) + ||W_k x||^2 - ||x||^2 \\
&= -2\,Re\,(A^*x, (\lambda_k-A)^{-1}x) + ||W_k x||^2 - ||x||^2 \\
&\leq ||2A^*x||\ ||(\lambda_k-A)^{-1}||\ ||x|| + (||W_k||^2-1)||x||^2 \ .
\end{aligned}$$

Now assume that $||x_n|| \leq K$ and that $|\lambda_k| \leq M$.   Let $\varepsilon > 0$ be given.   Take k so large that

$$||W_k||^2 - 1 < \varepsilon/2R^2K^2,$$

where $R = M + ||A||$.   Then fix k.   If $A^*x_n \to 0$, we can find an N so large that

$$||A^*x_n|| < \varepsilon/4KR^2||(\lambda_k - A)^{-1}||, \qquad n > N.$$

These last inequalities imply

$$||(I - W_k)x_n||^2 < \varepsilon/R^2, \qquad n > N.$$

Now

$$||Ax_n||^2 = ||(\lambda_k - A)(I - W_k)x_n||^2 \leq \varepsilon, \qquad n > N.$$

This shows that $Ax_n \to 0$.   A symmetrical argument shows the converse, that $Ax_n \to 0$ implies $A^*x_n \to 0$.   This comples the proof.

Yeshiva University

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