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PROPER BOUNDARY POINTS OF THE SPECTRUM

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1. Introduction. Let A be a bounded operator on an infinite dimensional Hilbert space H. A point λ on the boundary $\partial\sigma(A)$ of the spectrum $\sigma(A)$ of A will be called *proper* if there is a bounded sequence $\{\lambda_k\}$ of points in the resolvent set $\rho(A)$ of A such that

$$(1) \qquad \|(\lambda_k-\lambda)(\lambda_k-A)^{-1}\| \to 1 .$$

Examples of proper boundary points of the spectrum are easily given.

1. If $|\lambda|=\|A\|$, then λ is a proper boundary point.
2. If λ in $\sigma(A)$ is a boundary point of the numerical range of A, then it is a proper boundary point.
3. If λ in $\sigma(A)$ is a boundary point of a spectral set X for A and there is a sequence $\{\lambda_k\}$ in the complement of X such that

$$(2) \qquad |\lambda_k-\lambda|/d(\lambda_k, X) \to 1 ,$$

then λ is a proper boundary point (here $d(z, X)$ denotes the distance from z to X).

4. If A is seminormal, λ is a boundary point of $\sigma(A)$ and there is a sequence of points in $\rho(A)$ satisfying (2) with $X=\sigma(A)$, then λ is a proper boundary point.

We shall verify these statements later. We shall also prove the following theorem and give several applications.

Theorem 1. *If λ is a proper boundary point of $\sigma(A)$, then for each bounded sequence $\{x_n\}$ in H we have*

$$(3) \qquad (A-\lambda)x_n \to 0 \quad \textit{iff} \quad (A^*-\bar{\lambda})x_n \to 0 .$$

The theorem generalizes results of Putnam [1], Saito [2], Sz-Nagy, Foias [3], Schreiber [4] and others. The proof will be given in the next section. Now we shall give some consequences and applications. The essential spectrum of A is defined as

$$\sigma_e(A) = \bigcap_{K \text{ compact}} \sigma(A+K) .$$

Theorem 2. *If λ is a proper boundary point of $\sigma(A)$ and it is also in the essential spectrum of A, then there is an orthonormal sequence $\{\varphi_n\}$ in H such that*

$$(A-\lambda)\varphi_n \to 0 \quad \text{and} \quad (A^*-\bar{\lambda})\varphi_n \to 0\,. \tag{4}$$

Proof. If λ is in the essential spectrum of A, then either $A-\lambda$ is not a Fredholm operator or its index is not 0 ([5, p. 180]). The latter case cannot occur, since λ is a boundary point of the spectrum. Hence there is an orthonormal sequence satisfying at least one of the statements in (4). By Theorem 1, it must satisfy both.

Corollary 3. *If λ is a proper boundary point of the spectrum of A and it is not isolated, then there is an orthonormal sequence satisfying* (4).

Proof. By [6, Theorem 2.12] or by [7, Theorem 1], a nonisolated boundary point of the spectrum must be in the essential spectrum. Apply Theorem 2.

We now give some applications suggested by the work of Putman [7].

Corollary 4. *Let A be a bounded operator in H having at least one proper boundary point of $\sigma(A)$ in the essential spectrum. Then the operator*

$$C = A^*A-AA^* \tag{5}$$

has 0 *in its essential spectrum.*

Proof. Let λ be one such point. By Theorem 2 there is an orthonormal sequence such that (4) holds. Since

$$C = (A^*-\bar{\lambda})(A-\lambda)-(A-\lambda)(A^*-\bar{\lambda})\,,$$

we see that $C\varphi_n \to 0$. Thus 0 is in the essential spectrum of C.

Corollary 5. *If A has at least one non-isolated proper boundary point of its spectrum, then* 0 *is in the essential spectrum of the operator C given by* (5).

Proof. Use Corollary 3.

Corollary 6. *If every boundary point of the spectrum of A is proper, then C has* 0 *in its essential spectrum.*

Proof. If not all of the boundary points of $\sigma(A)$ are isolated, the result follows from Corollary 5. If they are, then the spectrum of A consists of only a finite number of points. At least one of these points must be in the essential spectrum of A (if H is infinite dimensional). Now apply Corollary 4.

Corollary 7. *Suppose every boundary point of the spectrum of A is proper, and put $A=L+iM$, where L and M are self adjoint. Then $\sigma(L)$ [resp. $\sigma(M)$] contains the projection of $\sigma(A)$ on the x [resp, y] axis.*

Proof. Suppose λ is a boundary point of $\sigma(A)$. Then there is a sequence of unit vectors satisfying one of the statements in (3). By Theorem 1 it satisfies both. Since $L=\frac{1}{2}(A+A^*)$, we see that $(L-Re\,\lambda)x_n\to 0$. Thus $Re\,\lambda$ is in the spectrum of L. If λ is an interior point, then there is a boundary point λ_1 such that $Re(\lambda-\lambda_1)=0$. We use the point λ_1 in place of λ. A similar proof works for M.

2. The proofs. Let us first verify that the points described in section 1 are proper. Consider the first statement. By rotating we may assume $\lambda=||A||$. For $t>\lambda$, we have $||(t-A)u||\geq(t-\lambda)||u||$. Hence $||(t-A)^{-1}||\leq 1/(t-\lambda)$.

Similar reasoning gives the second case. Since the numerical range $W(A)$ of A is convex, it is contained on one side of line L going through λ. Let z be any point on the other side of L such that $z-\lambda$ is orthogonal to it. Thus $|z-\lambda|$ is the distance $d(z, W(A))$ from z to $W(A)$. We know in general that

$$||(z-A)^{-1}||\leq 1/d(z, W(A))$$

holds for any z not in the closure of $W(A)$. This shows that

$$||(z-A)^{-1}||\leq 1/|z-\lambda|$$

holds for z on the other side of L. This proves the assertion for the second case.

If X, is a spectral set for A, then

$$||(z-A)^{-1}||\leq 1/d(z, X) \tag{7}$$

holds for all z not in X. If there is a sequence in the complement of X such that (2) holds, then we see that (1) holds for the same sequence.

If A is seminormal, then (7) holds for $X=\sigma(A)$. Apply the same reasoning.

Proof of Theorem 1. Assume $\lambda=0$, and set $W_k=\lambda_k(\lambda_k-A)^{-1}$. Then (1) says

$$||W_k||\to 1\,. \tag{8}$$

Now

$$I-W_k=(\lambda_k-A)(\lambda_k-A)^{-1}-\lambda_k(\lambda_k-A)^{-1}=-A(\lambda_k-A)^{-1}\,.$$

Thus

$$\begin{aligned}||(I-W_k)x||^2&=||x||^2-2\,Re\,(x, W_kx)+||W_kx||^2\\&=2\,Re\,(x, [I-W_k]x)+||W_kx||^2-||x||^2\\&=-2\,Re\,(A^*x, (\lambda_k-A)^{-1}x)+||W_kx||^2-||x||^2\\&\leq ||2A^*x||\,||(\lambda_k-A)^{-1}||\,||x||+(||W_k||^2-1)||x||^2\,.\end{aligned}$$

Now assume that $||x_n|| \leq K$ and that $|\lambda_k| \leq M$. Let $\varepsilon > 0$ be given. Take k so large that

$$||W_k||^2 - 1 < \varepsilon / 2R^2K^2 ,$$

where $R = M + ||A||$. Then fix k. If $A^*x_n \to 0$, we can find an N so large that

$$||A^* x_n|| < \varepsilon / 4KR^2 ||(\lambda_k - A)^{-1}||, \qquad n > N .$$

These last inequalities imply

$$||(I - W_k)x_n||^2 < \varepsilon / R^2 , \qquad n > N .$$

Now

$$||Ax_n||^2 = ||(\lambda_k - A)(I - W_k)x_n||^2 \leq \varepsilon , \qquad n > N .$$

This shows that $Ax_n \to 0$. A symmetrical argument shows the converse, that $Ax_n \to 0$ implies $A^*x_n \to 0$. This comples the proof.

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