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INTEGRAL GROUP RINGS OF FINITE GROUPS

Dedicated to Professor Keizō Asano on his 60th birthday

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Introduction. One of interesting problems on integral group rings of finite groups is whether non-isomorphic groups can have isomorphic group rings. The character theory of finite groups gives a useful tool to this problem (G. Higman [4], J.A. Cohn and D. Livingstone [3], and D.S. Passman [9]).

On the other hand, in our previous paper ([8]), we investigated with the problem by a homological method.

The aim of this paper is to develop the study of the problem by fitting the both methods. Our motivation is the fact that the cohomology group $H^2(\Pi, A)$ of a group Π can be regarded as the cohomology group $H^2(Z\Pi, A)$ of the group ring $Z\Pi$ of Π, so that the extension theory of groups can be reduced to that of group rings. For an example, any algebra automorphism of $Z\Pi$ which is commutative with the operation on A induces an automorphism of $H^2(\Pi, A)$. Our problem is closely related to the question whether any automorphism of the cohomology group $H^2(\Pi, A)$ which is induced from an automorphism of $Z\Pi$ can be also induced from an automorphism of Π.

Owing to Cohn and Livingstone, any algebra automorphism of $Z\Pi$ of a finite group Π gives an automorphism of the center of Π, so that these automorphisms induce the same automorphism of $H^2(\Pi, A)$ restricted to the center. Then we can show that if G is a finite group with an abelian normal subgroup A, then the normal subgroup H such that H/A is equal to the center of the quotient G/A is determined by the group ring ZG. In particular, we can obtain, as immediate corollaries, the Jackson's result ([5]) that any metabelian group of finite order is determined by its group ring, and the Passman's result ([9]) that the second center of a finite group is determined by its group ring.

The group ring of a non-abelian group can admit automorphisms which are not necessarily induced from group automorphisms. Indeed, we shall give an example of such an automorphism of the group ring ZD_4 of the dihedral group D_4 of order 8. Nevertheless, if A has exponent 2, then we can show that any algebra automorphism of ZD_4 always coincides on $H^2(D_4, A)$ with some group automorphism. This implies that any 2–group with an elementary abelian group as a normal subgroup and with the dihedral group as the quotient is determined by its group ring.

Finally, we apply our arguments to the Whitehead group Wh(G) of a finite group G. We shall show that the reduced norm of the Whitehead group Wh(D_4) of the dihedral group D_4 is equal to (1, 1, 1, 1, 1), so that Wh(D_4) is isomorphic to the special Whitehead group $SK^1(ZD_4)$.

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1. Let Π be a finite group and $Z\Pi$ be the group ring of Π over the ring Z of integers. Then $Z\Pi$ is a supplemented algebra by the augmentation ε; $Z\Pi \to Z$, $\varepsilon(\sum r_\sigma \cdot \sigma) = \sum r_\sigma$. The augmentation ideal $I(\Pi)$ is a two-sided ideal of $Z\Pi$ which is a free abelian group with the elements $\sigma - 1$ as basis ($\sigma \in \Pi$). If A is a Π-module, then by the augmentation ε, A is regarded as a two-sided module over $Z\Pi$ and the cohomology group $H^2(\Pi, A)$ of Π (in the sense of Eilenberg-MacLane) coincides with the cohomology group $H^2(Z\Pi, A)$ of the supplemented algebra $Z\Pi$. Hence, there is a 1–1 correspondence between the equivalence classes of extensions E_G over Π with A as kernel and those of extensions E_Λ over $Z\Pi$ with A as kernel. This correspondence is concretely given in Cartan-Eilenberg ([2]).

For convenience, we shall recall the constructions of E_G from E_Λ and of the converse. The exact sequence

$$E_\Lambda:\ 0 \to A \xrightarrow{i^*} \Lambda \xrightarrow{f^*} Z\Pi \to 0$$

is called an extension of the supplemented algebra if Λ is a Z-algebra, f^* is a Z-algebra homomorphism, i^* is a homomorphism of Z-modules, and for any $a \in A$, $\lambda \in \Lambda$

$$i^*(f^*(\lambda)\cdot a) = \lambda \cdot i^*(a)\,, \quad i^*(\varepsilon(f^*(\lambda))\cdot a) = i^*(a)\cdot \lambda\,.$$

Given an extension

$$E_G:\ 0 \to A \xrightarrow{i} G \xrightarrow{f} \Pi \to 1$$

over Π. If we identify A with the image $i(A)$, a normal subgroup of G, we then have an exact sequence

$$0 \to I(A)ZG \xrightarrow{i} ZG \xrightarrow{f} Z\Pi \to 0 \tag{1.1}$$

of algebras. Since the two-sided ideal $I(A)I(G) = I(A)ZG \cdot I(G)$ of ZG is contained in $I(A)ZG$, the sequence (1.1) implies the exact sequence

$$0 \to I(A)ZG/I(A)I(G) \xrightarrow{i^*} ZG/I(A)I(G) \xrightarrow{f^*} Z\Pi \to 0\,. \tag{1.2}$$

Lemma 1. *The additive group $I(A)ZG/I(A)I(G)$ is isomorphic to A and if we set $\Lambda=ZG/I(A)I(G)$, then the sequence* (1.2) *gives an extension of the supplemented algebra.*

Proof. Consider the commutative diagram of left A-modules:

$$\begin{array}{ccccccccc} 0 & \to & I(A) & \to & ZA & \to & Z & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 0 & \to & I(G) & \to & ZG & \to & Z & \to & 0 . \end{array}$$

Taking homology, we get a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \to & \mathrm{H}_1(A, Z) & \to & I(A)/I(A)^2 & \longrightarrow & Z & \to & Z & \to & 0 \\ & & \downarrow = & & \downarrow & & \downarrow & & \downarrow = & & \\ 0 & \to & \mathrm{H}_1(A, Z) & \to & I(G)/I(A)I(G) & \to & Z\Pi & \to & Z & \to & 0 , \end{array}$$

where $\mathrm{H}_1(A, Z)=A$, and the isomorphism $\mathrm{H}_1(A, Z)\cong I(A)/I(A)^2$ is given by the mapping $a\to a-1 \bmod I(A)^2$ ($a\in A$). Hence, the mapping $a\to a-1 \bmod I(A)I(G)$ gives rise to a monomorphism of the multiplicative structure of A into the additive structure of $I(G)/I(A)I(G)$. But the equality in ZG

$$(a-1)g = (a-1)+(a-1)(g-1)\,, \quad a\in A\,, \quad g\in G \tag{1.3}$$

shows that the image of A is precisely the subgroup $I(A)ZG/I(A)I(G)$. Thus, $I(A)ZG/I(A)I(G)$ is isomorphic to the additive group A.

Furthermore, the equality in ZG

$$\begin{aligned} g(a-1) &= (gag^{-1}-1)g \\ &= (f(g)a-1)+(f(g)a-1)(g-1)\,, \quad a\in A\,, \quad g\in G \end{aligned}$$

and the equality (1.3) show that the sequence (1.2) is an extension of the supplemented algebra. This proves the lemma.

Conversely, given an extension E_Λ over $Z\Pi$, let G be the set of elements λ of Λ such that $f^*(\lambda)\in\Pi$. Then G is a group under the multiplication of the ring Λ, and the epimorphism $f;\ G\to\Pi$ induced from f^* and the monomorphism $i;\ A\to\Lambda$, $i(a)=i^*(a)+1$ give an extension E_G over Π.

Lemma 2 ([2]). *If E_Λ (resp. E_G) is the extension constructed from E_G (resp. E_Λ) as above, then E_Λ and E_G have the same characteristic class. Hence, the above constructions establish a $1-1$ correspondence between the equivalence classes of extensions E_G over Π and those of extensions E_Λ over $Z\Pi$.*

REMARK. The cohomology group $\mathrm{H}^2(\Pi, A)$ is also expressed as $\mathrm{Ext}^2_{Z\Pi}(Z, A)$. Then each extension E_G over Π may be related to a 2-fold extension of $Z\Pi$-modules. The homology sequence

$$0 \to H_1(A, Z) \to I(G)/I(A)I(G) \to Z\Pi \to Z \to 0$$

in the proof of Lemma 1 is the corresponding 2-fold extension of $Z\Pi$-modules.

In the next section, we also use the following lemma. Given a diagram of extensions of groups:

$$\begin{array}{ccccccccc} 0 & \to & A' & \to & G' & \to & \Pi' & \to & 1 \\ & & \phi^* \downarrow \cong & & & & \psi \downarrow \cong & & \\ 0 & \to & A & \to & G & \to & \Pi & \to & 1 , \end{array}$$

then ψ induces an isomorphism $H(\psi)$; $H^2(\Pi, A) \cong H^2(\Pi', A)$ where A is regarded as Π'-module by ψ. In addition, if ϕ^* is a Π'-isomorphism, then ϕ^* induces an isomorphism $H(\phi^*)$; $H^2(\Pi', A') \cong H^2(\Pi', A)$.

Lemma 3 (S. Lang [7]). *There exists an isomorphism ψ; $G' \cong G$ which makes the above diagram commutable, if and only if ϕ^* is a Π'-isomorphism and $H(\phi^*)(\alpha') = H(\psi)(\alpha)$ for the characteristic classes α and α' of the extensions G and G', respectively.*

We shall remark that an analogous lemma also holds for extensions of supplemented algebras.

2. In this section, we consider finite groups G' and G with an isomorphism ϕ; $ZG' \cong ZG$ as algebras. With a generality, we can assume that ϕ is commutative with the augmentations (see [8]). If A' is a normal subgroup of G' and f' is the natural epimorphism of ZG' onto $Z(G'/A')$, then a normal subgroup $\Phi(A')$ of G is defined by setting

$$\Phi(A') = \{g \in G : f' \circ \phi^{-1}(g) = 1\} .$$

Lemma 4 ([3], [9], and [8]). *(a) Φ is an isomorphism of the lattice of normal subgroups of G' onto that of G,*

(b) ϕ induces an algebra isomorphism $\bar{\phi}$ of $Z(G'/A')$ onto $Z(G/\Phi(A'))$ such that the induced diagram

$$\begin{array}{ccc} ZG' & \xrightarrow{f'} & Z(G'/A') \\ \phi \downarrow \cong & & \bar{\phi} \downarrow \cong \\ ZG & \xrightarrow{f} & Z(G/\Phi(A')) \end{array}$$

is commutative, and

(c) if A' is abelian, $\Phi(A')$ is isomorphic to A', and if A' is central, then ϕ restricted to A' gives itself an isomorphism of A' onto $\Phi(A')$.

For the proof of (a), (b), and the latter half of (c), see [3], [9], and of the first part of (c), see [8].

From now on, let A' be an abelian normal subgroup of G' and Π' be the quotient group G'/A'. Then, by the above lemma (c) there exists an abelian normal subgroup A of G which is isomorphic to A'. Set $\Pi=G/A$ the quotient. From the lemma (b), we have an isomorphism $\bar{\phi}$; $Z\Pi' \cong Z\Pi$ and a commutative diagram

$$\begin{array}{ccccccc} 0 \to I(A')ZG' & \xrightarrow{i'} & ZG' & \xrightarrow{f'} & Z\Pi' & \to 0 \\ \phi\downarrow\cong & & \phi\downarrow\cong & & \bar{\phi}\downarrow\cong & \\ 0 \to I(A)ZG & \xrightarrow{i} & ZG & \xrightarrow{f} & Z\Pi & \to 0 \end{array}$$

of algebras. Since ϕ is commutative with the augmentations, we get $\phi(I(G'))=I(G)$, so that $\phi(I(A')I(G'))=\phi(I(A')ZG'\cdot I(G'))=\phi(I(A')ZG')\cdot\phi(I(G'))=I(A)ZG\cdot I(G)=I(A)I(G)$. Therefore, we can obtain an isomorphism of extensions of the supplemented algebras:

$$\begin{array}{ccccccc} 0 \to I(A')ZG'/I(A')I(G') & \xrightarrow{i'^*} & ZG'/I(A')I(G') & \xrightarrow{f'^*} & Z\Pi' & \to 0 \\ \phi^*\downarrow\cong & & \phi^*\downarrow\cong & & \bar{\phi}\downarrow\cong & \\ 0 \to I(A)ZG/I(A)I(G) & \xrightarrow{i^*} & ZG/I(A)I(G) & \xrightarrow{f^*} & Z\Pi & \to 0\,. \end{array} \tag{2. 1}$$

We notice that if we identify A with $I(A)ZG/I(A)I(G)$, then the isomorphism ϕ^*; $A' \cong A$ is nothing but the isomorphism $A' \cong \Phi(A')=A$ stated in the lemma (c) (see [8]).

If we regard A as a $Z\Pi'$-module (hence, as a Π'-module) by the algebra isomorphism $\bar{\phi}$, then $\bar{\phi}$ induces an isomorphism $\mathrm{H}(\bar{\phi})$; $\mathrm{H}^2(\Pi, A)=\mathrm{H}^2(Z\Pi, A) \cong \mathrm{H}^2(Z\Pi', A)=\mathrm{H}^2(\Pi', A)$.

Lemma 5. *ϕ^*; $A' \cong A$ is a Π'-isomorphism and $\mathrm{H}(\phi^*)(\alpha')=\mathrm{H}(\bar{\phi})(\alpha)$ for the characteristic classes α and α' of the extensions G and G', respectively.*

Proof. By Lemma 2, α and α' are also the characteristic classes of the corresponding algebra extensions $ZG/I(A)I(G)$ and $ZG'/I(A')I(G')$, respectively. Hence, the lemma follows immediately from the commutativity of the diagram (2.1) and the remark after Lemma 3.

Theorem 1. *Let G' and G be finite groups with an isomorphism ϕ; $ZG' \cong ZG$. If A' is an abelian normal subgroup of G', then there exists an abelian normal subgroup A of G which is isomorphic to A', and if H'/A' and H/A are the centers of the quotients G'/A' and G/A, respectively, then H' is isomorphic to H.*

Proof. The first assertion has already been seen. Set $\Pi=G/A$ (resp. $\Pi'=G'/A'$) and $\Pi_0=H/A$ (resp. $\Pi'_0=H'/A'$). Then we have an isomorphism $\bar{\phi}$; $Z\Pi' \cong Z\Pi$ and the isomorphism (2.1) of extensions. By Lemma 4 (c), $\bar{\phi}$

restricted to the center Π_0' gives rise to a group isomorphism $\bar{\phi}_0$; $\Pi_0' \tilde{\rightarrow} \Pi_0$. Then we get a diagram of extensions:

$$\begin{array}{ccccccccc} 0 & \rightarrow & A' & \rightarrow & H' & \rightarrow & \Pi_0' & \rightarrow & 1 \\ & & \phi^* \downarrow \cong & & & & \bar{\phi}_0 \downarrow \cong & & \\ 0 & \rightarrow & A & \rightarrow & H & \rightarrow & \Pi_0 & \rightarrow & 1 . \end{array}$$

Moreover, the operation of Π_0' through $\bar{\phi}$ on A coincides with that through $\bar{\phi}_0$, so that, under the latter operation, ϕ^* is a Π_0'-isomorphism (Lemma 5).

Let Res; $H^2(\Pi, A) \rightarrow H^2(\Pi_0, A)$ be the restriction map of cohomology groups. Then, we see easily that $\mathrm{Res} \circ H(\phi^*) = H(\phi^*) \circ \mathrm{Res}$, and $\mathrm{Res} \circ H(\bar{\phi}) = H(\bar{\phi}_0) \circ \mathrm{Res}$ ($\bar{\phi}_0$ is the $\bar{\phi}$ restricted to Π_0'). Let α and α' be the characteristic classes of the extensions G and G', respectively. Then, in Lemma 5, we have seen that $H(\phi^*)(\alpha') = H(\bar{\phi})(\alpha)$. This implies that $H(\phi^*)(\mathrm{Res}(\alpha')) = H(\bar{\phi}_0)(\mathrm{Res}(\alpha))$. Therefore, by Lemma 3, we get an isomorphism $H' \tilde{\rightarrow} H$, since $\mathrm{Res}(\alpha')$ and $\mathrm{Res}(\alpha)$ are the characteristic classes of the extensions H' and H, respectively. This proves the theorem.

In particular, if A' is the center of G', then H' is nothing but the second center of G'. On the other hand, if G' is metabelian and A' is the commutator of G', then A' and the quotient G'/A' are both abelian. Hence, we obtain

Corollary 1 ([9]). *If $ZG' \tilde{\rightarrow} ZG$, then the second centers of G' and G are isomorphic.*

Corollary 2 ([5]). *If $ZG' \tilde{\rightarrow} ZG$ and G' is metabelian, then $G' \tilde{\rightarrow} G$.*

REMARK. In [9], Passman shows the more general result that if x and y are any elements of a finite group G which satisfy the commutator conditions: $[[x, G], y] = [[y, G], x] = 1$ and $[x, G] \cap [y, G]$ is contained in the hyper center of G, then the commutator $[x, y]$ is determined by the group ring ZG. His proof is based on the following property of augmentation ideals: if A, B, C are three normal subgroups of G such that $A \subseteq B \subseteq C$ and B is contained in the hyper center of G, then $I(A)ZG \cap I(B)I(C)ZG \subseteq I(A)I(C)ZG$. But this is not necessarily true. In fact, if we set, especially, $B = C = G$, then this inclusion of augmentation ideals implies that the natural map $I(A)ZG/I(A)I(G) \rightarrow I(G)/I(G)^2$ is monomorphic, which means that the natural map $A/[A, A] \rightarrow G/[G, G]$ is also monomorphic (apply our arguments in the proof of Lemma 1 to the case where A is not abelian). But this is not necessarily true even if G is nilpotent.

Again, we consider finite groups G' and G with an isomorphism $ZG' \tilde{\rightarrow} ZG$ and an abelian normal subgroup A' of G'. Then there is an abelian normal subgroup A of G and we have the commutative diagram (2.1). The following proposition is an immediate consequence from Lemma 3 and Lemma 5.

Proposition 6. *G' is isomorphic to G, if there exists an isomorphism ψ; $\Pi' \cong \Pi$ such that the operation of Π' through ψ on A coincides with the operation through $\bar{\phi}$ and $\mathrm{H}(\psi)(\alpha)=\mathrm{H}(\bar{\phi})(\alpha)$ for the characteristic class α of G.*

Owing to G. Higman ([4]), it is known that if Π is the direct product of the quaternion group of order 8 and an elementary abelian 2-group, then any unit of $Z\Pi$ is a trivial unit $\pm\sigma$ $(\sigma\in\Pi)$. Thus, if Π is such a group, any isomorphism $\bar{\phi}$; $Z\Pi' \cong Z\Pi$ gives an group isomorphism ψ; $\Pi' \cong \Pi$, that is, $\psi=\bar{\phi}$ restricted to Π'. Therefore we get

Theorem 2. *If G' is a finite group with an abelian normal subgroup such that the quotient is isomorphic to the direct product of the quaternion group of order 8 and an elementary abelian 2-group, then $ZG' \cong ZG$ implies $G' \cong G$.*

3. Let D_4 be the dihedral group of order 8. In this section, we shall determine the automorphisms of ZD_4. Any automorphism of ZD_4 is given as the composition of a group automorphism and an algebra automorphism defined by a solution of certain simultaneous equations. By using a property of the solutions, we shall show

Theorem 3. *If G' is a 2-group with an elementary abelian group as a normal subgroup and with the dihedral group of order 8 as the quotient, then $ZG' \cong ZG$ implies $G' \cong G$.*

Let a and b be generators of D_4 with relations: $a^4=b^2=1$, $ab=ba^3$, and let A be the center of D_4. Then A is a cyclic group of order 2 and is generated by the element a^2.

Now, we consider two elements $\tilde{a}$ and $\tilde{b}$ of ZD_4 of the forms:

$$\tilde{a}=a+r_a(1-a^2)a+r_b(1-a^2)b+r_{ab}(1-a^2)ab \quad \text{and}$$
$$\tilde{b}=b+s_a(1-a^2)a+s_b(1-a^2)b+s_{ab}(1-a^2)ab\,,$$

respectively, where r: r_a, r_b, r_{ab}, s: s_a, s_b, s_{ab} are all integers. By simple calculations, we get the equalities

$$\begin{aligned}\tilde{a}^2 &= a^2+2(r_b^2+r_{ab}^2-r_a^2-r_a)(1-a^2)\,,\\ \tilde{b}^2 &= 1+2(s_b^2+s_b-s_a^2+s_{ab}^2)(1-a^2) \quad \text{and}\\ \tilde{a}\tilde{b}-\tilde{b}\tilde{a}\cdot a^2 &= 2(2r_a s_a-2r_b s_b-2r_{ab}s_{ab}-r_b+s_a)(1-a^2)\,.\end{aligned} \tag{3.1}$$

If $\{r, s\}$ is a solution of the simultaneous equations:

$$\begin{aligned} r_a(r_a+1) &= r_b^2+r_{ab}^2 \quad \cdots\cdots\cdots\cdots(1)\\ s_b(s_b+1) &= s_a^2-s_{ab}^2 \quad \cdots\cdots\cdots\cdots(2)\\ 2(r_a s_a-r_b s_b-r_{ab}s_{ab}) &= r_b-s_a \quad \cdots\cdots\cdots\cdots(3)\,,\end{aligned} \tag{3.2}$$

then $\tilde{a}$ and $\tilde{b}$ satisfy the relations:

$$\tilde{a}^2 = a^2\,,\quad \tilde{b}^2 = 1 \quad \text{and} \quad \tilde{a}\tilde{b} = \tilde{b}\tilde{a}\cdot a^2 = \tilde{b}\tilde{a}^3\,.$$

These mean that $\tilde{a}$ and $\tilde{b}$ are units of ZD_4, and generate a group $\tilde{D}$ isomorphic to D_4. Since the submodule $Z\tilde{D}$ generated by $\tilde{D}$ over Z coincides with the group ring ZD_4 (see [3], Theorem 3.2), then the map: $a\rightarrow\tilde{a}$, $b\rightarrow\tilde{b}$ can be extended to an automorphism φ of ZD_4. In particular, if $\{r, s\}$ is a solution consisting of even integers, this automorphism φ verifies the congruence:

$$\varphi(x) \equiv x \bmod I(A)I(D_4), \qquad \text{for any} \quad x\in D_4\,, \tag{3. 3}$$

since $I(A)$ is generated by $1-a^2$, so that $2I(A)ZD_4=I(A)^2ZD_4\subseteq I(A)I(D_4)$.

Lemma 7. *Any algebra automorphism φ of ZD_4 which satisfies the congruence* (3.3) *is given by $\varphi_{r,s}$ for a solution $\{r, s\}$ consisting of even integers of the equations* (3.2), *where $\varphi_{r,s}$ denotes the automorphism defined as above by the solution $\{r, s\}$.*

Proof. Let φ be any automorphism satisfying the congruence (3.3). Then $\varphi(a)$ is written as $\varphi(a)=a+r_1(1-a^2)+r_a(1-a^2)a+r_b(1-a^2)b+r_{ab}(1-a^2)ab$ (r_1, r_a, r_b, $r_{ab}\in Z$), because $\varphi(a)-a\in I(A)I(D_4)\subseteq I(A)ZD_4$. However, we see that $r_1=0$. Otherwise, the coefficient of the identity in $\varphi(a)$ is not zero. But $\varphi(a)$ is a unit of finite order, then $\varphi(a)$ must be equal to 1 (see [3], Theorem 3.1), which is a contradiction. Therefore, $\varphi(a)$ is written as $\varphi(a)=a+r_a(1-a^2)a+r_b(1-a^2)b+r_{ab}(1-a^2)ab$, and similarly $\varphi(b)=b+s_a(1-a^2)a+s_b(1-a^2)b+s_{ab}(1-a^2)ab$. On the other hand, by Lemma 4 (c), φ restricted to the center A gives rise to an automorphism of A, which is the identity since A is of order 2. Thus, $\varphi(a^2)=a^2$, so that we have the equalities: $\varphi(a)^2=a^2$, $\varphi(b)^2=1$ and $\varphi(a)\cdot\varphi(b)=\varphi(b)\cdot\varphi(a)^3=\varphi(b)\cdot\varphi(a)\cdot a^2$. Let $\varphi(a)=\tilde{a}$ and $\varphi(b)=\tilde{b}$. Then, from the equalities (3.1), the integers $r: r_a$, r_b, r_{ab}, $s: s_a$, s_b, s_{ab} must satisfy the equations (3.2), and the automorphism $\varphi_{r,s}$ defined by this solution $\{r, s\}$ clearly coincides with the given automorphism φ. Therefore, the proof is finished once we show that the solution $\{r, s\}$ consists of even integers. Since $\varphi(a)-a\in I(A)I(D_4)$, then $(r_a+r_b+r_{ab})(1-a^2)\in I(A)I(D_4)$, which shows that $(r_a+r_b+r_{ab})(1-a^2)\in I(A)^2=2I(A)$ (recall the isomorphism $I(A)/I(A)^2\cong I(A)ZD_4/I(A)I(D_4)$ in Lemma 1). Then $r_a+r_b+r_{ab}$ is even, and similarly $s_a+s_b+s_{ab}$ is also even. On the other hand, in the equations (3.2), r_b and r_{ab} must be both even or odd, because the left hand side of the equality (1) is always even. Then r_a must be even. Now we shall assume that r_b and r_{ab} are both odd. Then r_a is not divisible by 4, so that r_a is written as $4n-2$. Therefore, we have

$$(4n-2)(4n-1) = r_b^2+r_{ab}^2\,.$$

Since the right hand side of this equality is a norm of an integer in the quadratic field $Q(\sqrt{-1})$, no primes which are congruent to 3 mod 4 divide $(4n-2)(4n-1)$ with square free. But $4n-2$ and $4n-1$ have no prime divisors in common, then any prime congruent to 3 mod 4 can not also occur in $4n-1$ with square free. Therefore we have the congruence $3\equiv 3^{2k} \bmod 4$. This is a contradiction. Then, r_a, r_b and r_{ab} are all even, so that by the equalities (3) and (2) in the equations (3.2) s_a and s_{ab} are both even. Thus, s_b is also even.

For an example, $\varphi(a)=a+4(1-a^2)a+2(1-a^2)b+4(1-a^2)ab$, $\varphi(b)=b+2(1-a^2)a+2(1-a^2)ab$ define an automorphism of ZD_4, but we can show that this automorphism is not an inner automorphism.

To prove Theorem 3, we need one more lemma, which restates Corollary 2 of Theorem 1, slight accurately.

Lemma 8. *Let G' be a metabelian group of finite order, and let A' be an abelian normal subgroup of G' such that the quotient G'/A' is abelian. If ϕ; $ZG' \xrightarrow{\sim} ZG$ is an isomorphism, then there exists an isomorphism ϕ^* of A' onto a some abelian normal subgroup A of G. Furthermore, this isomorphism can be extended to an isomorphism Ψ of G' onto G such that $\phi(g')\equiv\Psi(g') \bmod I(A)I(G)$ for any $g'\in G'$.*

Proof. As in the proof of Theorem 1, let A be the normal subgroup of G which makes the diagram (2.1) commutable. Then the isomorphism ϕ^*; $A'\xrightarrow{\sim} A$ (in the diagram) can be extended to an isomorphism Ψ; $G'\xrightarrow{\sim} G$. Therefore, it suffices to show that the isomorphism Ψ verifies the congruence required in the lemma. To see that, we shall concretely describe the isomorphism Ψ. Let

$$G' = \bigcup_{\sigma'\in\Pi'} A'\cdot g_{\sigma'} \quad (g_1=1), \qquad G = \bigcup_{\sigma\in\Pi} A\cdot g_{\sigma} \quad (g_1=1)$$

be the coset decompositions of G' and G, respectively, and set

$$\alpha'(\sigma', \tau') = g_{\sigma'}g_{\tau'}g_{\sigma'\tau'}^{-1}, \quad \sigma', \tau'\in\Pi', \qquad \alpha(\sigma, \tau) = g_{\sigma}g_{\tau}g_{\sigma\tau}^{-1}, \quad \sigma, \tau\in\Pi.$$

Then, $\alpha'(\ ,\ )$ (resp. $\alpha(\ ,\ )$) is a normalized 2–cocycle representing the characteristic class α' (resp. α) of the extension G' (resp. G). Since Π' is abelian, $\bar{\phi}$ restricted to Π' gives an isomorphism $\bar{\phi}_0$; $\Pi'\xrightarrow{\sim}\Pi$. Hence, by the commutativity of the diagram (2.1) we see that $f^*((\phi(g_{\sigma'})-g_{\bar{\phi}_0(\sigma')}) \bmod I(A)I(G))=0$ for any $\sigma'\in\Pi'$. Therefore, for each σ' there exists uniquely an element $a(\sigma')$ of A such that

$$\phi(g_{\sigma'})\equiv a(\sigma')g_{\bar{\phi}_0(\sigma')} \bmod I(A)I(G)\,, \tag{3.4}$$

and we see easily that

$$\phi^*(\alpha'(\sigma', \tau')) = a(\sigma')a(\tau')^{\bar{\phi}_0(\sigma')}a(\sigma'\tau')^{-1}\alpha(\bar{\phi}_0(\sigma'), \bar{\phi}_0(\tau'))$$

(this equality means $\mathrm{H}(\phi^*)(\alpha')=\mathrm{H}(\bar{\phi}_0)(\alpha)$). Then the mapping

$$\Psi;\ a'g_{\sigma'} \to \phi^*(a')a(\sigma')g_{\bar{\phi}_0(\sigma')} \tag{3.5}$$

gives rise to an isomorphism Ψ; $G' \cong G$, which is clearly an extension of ϕ^*. Furthermore, by the congruence (3.4) and the definition (3.5) of Ψ, we verify the congruence $\phi(g') \equiv \Psi(g') \bmod I(A)I(G)$ for any $g' \in G'$. This proves the lemma.

Proof of Theorem 3. Let G' be the group stated in the theorem and let A' be the abelian normal subgroup of exponent 2 such that the quotient $\Pi' = G'/A'$ is the dihedral group of order 8. If ϕ is an isomorphism of ZG' onto ZG, then there is a normal subgroup A of G which is isomorphic to A' and ϕ induces an isomorphism $\bar{\phi}$; $Z\Pi' \cong Z\Pi$ of group rings of the quotients.

Let Π_0' be the center of Π', and apply Lemma 8 to the isomorphism $\bar{\phi}$: $Z\Pi' \cong Z\Pi$ (Π' is metabelian). Then there exists an isomorphism Ψ; $\Pi' \cong \Pi$ such that $\bar{\phi}(\sigma') \equiv \Psi(\sigma') \bmod I(\Pi_0)I(\Pi)$, $\sigma' \in \Pi'$, so that we get an automorphism $\Psi^{-1}\bar{\phi}$ such that $\Psi^{-1}\bar{\phi}(\sigma') \equiv \sigma' \bmod I(\Pi_0')I(\Pi')$. Thus, by Lemma 7 each $\bar{\phi}(\sigma')$ is written as $\Psi(\sigma') + 2S$ by some S of $I(\Pi_0)Z\Pi$. This means that the operation of Π' by Ψ on A coincides with that by $\bar{\phi}$, and Ψ also coincides with $\bar{\phi}$ on the cohomology group $H^2(\Pi', A)$, because A has exponent 2, so that $H^2(\Pi', A)$ has also exponent 2. Cosequently, by Proposition 6 we obtain an isomorphism of G' onto G, which proves the theorem.

Theorem 4. *Every automorphism of the group ring ZD_4 is given as the composition $\varphi_{r,s} \circ \Psi$ of an automorphism Ψ of D_4 and the automorphism $\varphi_{r,s}$ of ZD_4 defined by a solution $\{r, s\}$ consisting of even integers of the simultaneous equations* (3.2).

Proof. Let A be the center of D_4, and apply Lemma 8 to any automorphism ϕ; $ZD_4 \cong ZD_4$. Then there exists an automorphism Ψ of D_4 such that $\phi(x) \equiv \Psi(x) \bmod I(A)I(D_4)$, $x \in D_4$. Then the theorem is immediate from Lemma 7.

Remark. In this proof of the theorem, automorphisms are assumed implicitly to be commutative with the augmentation (recall our assumption at the beginning of section 2). If ϕ is not commutative with the augmentation ε; $ZD_4 \to Z$, then we get a non-trivial map ϕ_ε; $x \to \varepsilon(\phi(x)) \cdot x \, (x \in D_4)$, which is clearly extended to an automorphism of ZD_4, and $\phi \circ \phi_\varepsilon^{-1}$ is commutative with the augmentation ε. Therefore it suffices to determine the automorphisms ϕ_ε. But this is easy, because each $\varepsilon(\phi(x))$ is a unit of Z, so that $\varepsilon(\phi(x)) = \pm 1$. Indeed, such automorphisms are given by the mappings: $a \to \pm a$, $b \to \pm b$, where a and b denote generators of D_4 with $a^4 = b^2 = 1$, $ab = ba^3$.

4. T. Y. Lam ([6]) showed that the Whitehead group Wh (S_3) of the symmetric group S_3 is trivial. His proof consists of following two parts: the reduced norm of Wh (S_3) is equal to (1, 1, 1), and SK^1 (ZS_3) is trivial. But his computation of the reduced norm is complicated, so that it seems impossible to apply his method to other cases. In this section, we shall give a simpler technique to compute the reduced norm of Wh (S_3), and apply the technique to the Wh (D_4) of the dihedral group D_4 of order 8. For notations used here, see ([6]) or (H. Bass [1]).

Let G be a finite group with a normal subgroup A and let the Whitehead group Wh (G/A) of the quotient G/A be trivial. Since ZG has 'stable range 2', we may regard $K^1(ZG)$ as to be generated by 2 by 2 invertible matrices over ZG. Let

$$K^1(ZG) \to K^1(Z(G/A))$$

be the homomorphism of K^1-groups which is induced from the natural epimorphism $ZG \to ZG/I(A)ZG \cong Z(G/A)$. Then, by the assumption that Wh $(G/A) = K^1(Z(G/A))/\pm(G/A)$ is trivial $K^1(ZG)$ is generated by $\pm G$ and invertible matrices X such that

$$X = \begin{pmatrix} 1+\alpha & \beta \\ \gamma & 1+\delta \end{pmatrix}, \qquad \alpha, \beta, \gamma, \delta \in I(A)ZG.$$

Consider elementary matrices $\begin{pmatrix} 1 & 0 \\ (-1)^{i-1}\gamma\alpha^{i-2} & 1 \end{pmatrix}, \begin{pmatrix} 1 & (-1)^{i-1}\alpha^{i-2}\beta \\ 0 & 1 \end{pmatrix}$, then we see easily that

$$\begin{pmatrix} 1 & 0 \\ (-1)^{n-1}\gamma\alpha^{n-2} & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix} X \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & (-1)^{n-1}\alpha^{n-2}\beta \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1+\alpha & (-\alpha)^{n-1}\beta \\ \gamma(-\alpha)^{n-1} & 1+\delta+\sum_{i=2}^{n} \gamma\alpha^{2(i-2)}(\alpha-1)\beta \end{pmatrix}.$$

Consequently, for any positive integer n, any element of $K^1(ZG)$ may be regarded as to be represented by an element of $\pm G$ and an invertible matrix $\begin{pmatrix} 1+\alpha & \beta_n \\ \gamma_n & 1+\delta \end{pmatrix}$ such that $\alpha, \delta \in I(A)ZG$, $\beta_n, \gamma_n \in I(A)^n ZG$.

Next, we consider the natural epimorphism f; $ZG \to ZG/I(A)I(G)$ and the induced homomorphism f^*; $K^1(ZG) \to K^1(ZG/I(A)I(G))$. Since $\beta_n, \gamma_n \in I(A)I(G)$ for any positive integer $n \geqq 2$, we get

$$f^*\begin{pmatrix} 1+\alpha & \beta_n \\ \gamma_n & 1+\delta \end{pmatrix} = \begin{pmatrix} 1+f(\alpha) & 0 \\ 0 & 1+f(\delta) \end{pmatrix}.$$

On the other hand, the map: $a \bmod [A, A] \to a-1 \bmod I(A)I(G)$ gives rise to an isomorphism $A/[A, A] \xrightarrow{\sim} I(A)ZG/I(A)I(G)$ (in the case where A is abelian, this isomorphism has been seen in Lemma 1). Thus, there exist elements a and d of A such that $f(\alpha)=a-1 \bmod I(A)I(G)$ and $f(\delta)=d-1 \bmod I(A)I(G)$, respectively, so that $f^*\left(\begin{pmatrix}1+\alpha & \beta_n\\ \gamma_n & 1+\delta\end{pmatrix}\begin{pmatrix}a^{-1} & 0\\ 0 & d^{-1}\end{pmatrix}\right)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Therefore, we have obtained the following proposition

Proposition 9. *If G is a finite group with a normal subgroup A and if the Whitehead group* Wh(G/A) *is trivial, then, for any positive integer $n\geqq 2$, any element of* Wh(G) *is represented by an invertible matrix X such that*

$$X=\begin{pmatrix}1+\alpha & \beta_n\\ \gamma_n & 1+\delta\end{pmatrix}, \quad \textit{where} \quad \alpha,\ \delta\in I(A)I(G), \quad \beta_n,\ \gamma_n\in I(A)^n ZG.$$

Using this proposition, we shall compute the reduced norm of Wh(S_3). Let G be the symmetric group S_3 and set $a=(1\ 2\ 3)$ and $b=(1\ 2)$. Then, G is generated by a and b. If A is the subgroup generated by a, then the quotient G/A is of order 2, so that Wh(G/A) is trivial ([1], [4]). Hence, we can apply the proposition to this case, and to determine the reduced norm of Wh(G), it suffices to compute the reduced norms of invertible matrices $X=\begin{pmatrix}1+\alpha & \beta_2\\ \gamma_2 & 1+\delta\end{pmatrix}$ such that $\alpha,\ \delta\in I(A)I(G)$, $\beta_2,\ \gamma_2\in I(A)^2ZG$.

Since A is the commutator of G, any element of $I(A)ZG$ is represented to 0 by any representation of degree 1, hence each component of degree 1 of the reduced norm of X is equal to 1. The irreducible representation of G of degree 2 is given by

$$\rho(a)=\begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}, \quad \rho(b)=\begin{pmatrix}1 & -1\\ 0 & -1\end{pmatrix}.$$

It is known that the reduced norm of Wh(S_3) of the symmetric group S_3 is of the form $(\pm 1, \pm 1, \pm 1)$ ([1]), therefore it is harmless to carry out mod 3 the computation of the component $\det(\rho(X))$ of degree 2 of the reduced norm of X. Since A is a cyclic group generated by the element a, we see easily that $I(A)^2ZG=(a-1)^2ZG$. But ρ represents the element $(a-1)^2$ to the matrix $\begin{pmatrix}0 & 3\\ -3 & 3\end{pmatrix}$, then $\det(\rho(X))\equiv\det\begin{pmatrix}1+\rho(\alpha) & 0\\ 0 & 1+\rho(\delta)\end{pmatrix} \bmod 3$.

On the other hand, we can easily see that any element of $I(A)I(G)/I(A)^2ZG$ is written as $(x-1)(b-1) \bmod I(A)^2ZG$ for some element x of A. Therefore, it suffices to compute

$$\det\begin{pmatrix}1+\rho((x-1)(b-1)) & 0\\ 0 & 1+\rho((x'-1)(b-1))\end{pmatrix}.$$

Indeed,

$$\det(1+\rho((a-1)(b-1)))=\det\begin{pmatrix}1 & 3\\ 0 & 4\end{pmatrix}\equiv 1 \bmod 3, \quad \text{and}$$

$$\det(1+\rho((a^2-1)(b-1)))=\det\begin{pmatrix}1 & 0\\ 0 & 4\end{pmatrix}\equiv 1 \bmod 3\,.$$

Consequently, $\det(\rho(X))=1$, and the reduced norm of $\mathrm{Wh}(S_3)$ is equal to $(1, 1, 1)$.

Finally, we compute the reduced norm of $\mathrm{Wh}(D_4)$. Let G be the dihedral group D_4 of order 8, and let a and b be generators of G with relations: $a^4=b^2=1$, $ab=ba^3$. Set $A=\{1, a^2\}$, then the quotient G/A is an abelian group of type $(2, 2)$, so that $\mathrm{Wh}(G/A)$ is trivial ([1], [4]). Hence, we can also apply Proposition 9, and it suffices to compute the reduced norms of invertible matrices $X=\begin{pmatrix}1+\alpha & \beta_3\\ \gamma_3 & 1+\delta\end{pmatrix}$ such that $\alpha, \delta\in I(A)I(G)$, $\beta_3, \gamma_3\in I(A)^3ZG$.

Since A is the commutator of G, each component of degree 1 of the reduced norm of X is equal to 1. The irreducible representation of degree 2 is given by

$$\rho(a)=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\quad \rho(b)=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

To begin with, we try to compute the component of degree 2 mod 4, that is, $\det(\rho(X))$ mod 4. A is generated by the element a^2, then we see that $I(A)^3ZG=4I(A)ZG$. Therefore, it suffices to compute

$$\det\begin{pmatrix}1+\rho(\alpha) & 0\\ 0 & 1+\rho(\delta)\end{pmatrix} \bmod 4\,.$$

Set $\alpha=r_1(1-a^2)+r_a(1-a^2)a+r_b(1-a^2)b+r_{ab}(1-a^2)ab$. Then, we see easily that

$$\det(1+\rho(\alpha))=\det\begin{pmatrix}1+2(r_1-r_{ab}) & 2(-r_a+r_b)\\ 2(r_a+r_b) & 1+2(r_1+r_{ab})\end{pmatrix},$$

and this is congruent to 1 mod 4. Thus, $\det(\rho(X))$ is also congruent to 1 mod 4, so that $\det(\rho(X))$ can not be equal to -1. But it is known that the reduced norm of $\mathrm{Wh}(D_4)$ is of the form $(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$ ([1]), hence $\det(\rho(X))=1$. Consequently, we have shown

Theorem 5. *The reduced norm of the Whitehead group* $\mathrm{Wh}(D_4)$ *of the dihedral group* D_4 *of order* 8 *is equal to* $(1, 1, 1, 1, 1)$, *so that* $\mathrm{Wh}(D_4)$ *is isomorphic to the special Whitehead group* $\mathrm{SK}^1(ZD_4)$.

REMARK. Apply the Witt-Berman and Swan-Lam's induction theorem ([1], [6]) to the Whitehead group $\mathrm{Wh}(S_4)$ of the symmetric group S_4, then, from Lam's result on S_3 and the above theorem, we can see that $\mathrm{Wh}(S_4)$ is isomorphic to the special Whitehead group $\mathrm{SK}^1(ZS_4)$.

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