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Introduction. One of interesting problems on integral group rings of finite groups is whether non-isomorphic groups can have isomorphic group rings. The character theory of finite groups gives a useful tool to this problem (G. Higman [4], J.A. Cohn and D. Livingstone [3], and D.S. Passman [9]).

On the other hand, in our previous paper ([8]), we investigated with the problem by a homological method.

The aim of this paper is to develop the study of the problem by fitting the both methods. Our motivation is the fact that the cohomology group $H^2(\Pi, A)$ of a group $\Pi$ can be regarded as the cohomology group $H^2(Z\Pi, A)$ of the group ring $Z\Pi$ of $\Pi$, so that the extension theory of groups can be reduced to that of group rings. For an example, any algebra automorphism of $Z\Pi$ which is commutative with the operation on $A$ induces an automorphism of $H^2(\Pi, A)$. Our problem is closely related to the question whether any automorphism of the cohomology group $H^2(\Pi, A)$ which is induced from an automorphism of $Z\Pi$ can be also induced from an automorphism of $\Pi$.

Owing to Cohn and Livingstone, any algebra automorphism of $Z\Pi$ of a finite group $\Pi$ gives an automorphism of the center of $\Pi$, so that these automorphisms induce the same automorphism of $H^2(\Pi, A)$ restricted to the center. Then we can show that if $G$ is a finite group with an abelian normal subgroup $A$, then the normal subgroup $H$ such that $H/A$ is equal to the center of the quotient $G/A$ is determined by the group ring $ZG$. In particular, we can obtain, as immediate corollaries, the Jackson’s result ([5]) that any metabelian group of finite order is determined by its group ring, and the Passman’s result ([9]) that the second center of a finite group is determined by its group ring.

The group ring of a non-abelian group can admit automorphisms which are not necessarily induced from group automorphisms. Indeed, we shall give an example of such an automorphism of the group ring $ZD_4$ of the dihedral group $D_4$ of order 8. Nevertheless, if $A$ has exponent 2, then we can show that any algebra automorphism of $ZD_4$ always coincides on $H^2(D_4, A)$ with some group automorphism. This implies that any 2-group with an elementary abelian
group as a normal subgroup and with the dihedral group as the quotient is determined by its group ring.

Finally, we apply our arguments to the Whitehead group \( \text{Wh}(G) \) of a finite group \( G \). We shall show that the reduced norm of the Whitehead group \( \text{Wh}(D_n) \) of the dihedral group \( D_n \) is equal to \((1, 1, 1, 1, 1)\), so that \( \text{Wh}(D_n) \) is isomorphic to the special Whitehead group \( \text{SK}^1(ZD_n) \).

The author would like to express his gratitude to Professor A. Hattori for many helpful suggestions.

1. Let \( \Pi \) be a finite group and \( Z\Pi \) be the group ring of \( \Pi \) over the ring \( Z \) of integers. Then \( Z\Pi \) is a supplemented algebra by the augmentation \( \varepsilon; Z\Pi \to Z, \varepsilon(\sum r_\sigma \sigma) = \sum r_\sigma \). The augmentation ideal \( I(\Pi) \) is a two-sided ideal of \( Z\Pi \) which is a free abelian group with the elements \( \sigma - 1 \) as basis \( (\sigma \in \Pi) \). If \( A \) is a \( \Pi \)-module, then by the augmentation \( \varepsilon \), \( A \) is regarded as a two-sided module over \( Z\Pi \) and the cohomology group \( H^2(\Pi, A) \) of \( \Pi \) (in the sense of Eilenberg-MacLane) coincides with the cohomology group \( H^2(Z\Pi, A) \) of the supplemented algebra \( Z\Pi \). Hence, there is a 1–1 correspondence between the equivalence classes of extensions \( E_G \) over \( \Pi \) with \( A \) as kernel and those of extensions \( E_A \) over \( Z\Pi \) with \( A \) as kernel. This correspondence is concretely given in Cartan-Eilenberg ([2]).

For convenience, we shall recall the constructions of \( E_G \) from \( E_A \) and of the converse. The exact sequence

\[
E_A : 0 \to A \xrightarrow{i^*} \Lambda \xrightarrow{f^*} Z\Pi \to 0
\]

is called an extension of the supplemented algebra if \( \Lambda \) is a \( Z \)-algebra, \( f^* \) is a \( Z \)-algebra homomorphism, \( i^* \) is a homomorphism of \( Z \)-modules, and for any \( a \in A \), \( \lambda \in \Lambda \)

\[
i^*\left(f^*(\lambda) \cdot a\right) = \lambda \cdot i^*(a), \quad i^*(\varepsilon(f^*(\lambda)) \cdot a) = i^*(a) \cdot \lambda.
\]

Given an extension

\[
E_G : 0 \to A \xrightarrow{i} G \xrightarrow{f} \Pi \to 1
\]

over \( \Pi \). If we identify \( A \) with the image \( i(A) \), a normal subgroup of \( G \), we then have an exact sequence

\[
0 \to I(A)ZG \xrightarrow{i} ZG \xrightarrow{f} Z\Pi \to 0 \quad \tag{1.1}
\]

of algebras. Since the two-sided ideal \( I(A)I(G) = I(A)ZG \cdot I(G) \) of \( ZG \) is contained in \( I(A)ZG \), the sequence (1.1) implies the exact sequence

\[
0 \to I(A)ZG/I(A)I(G) \xrightarrow{i^*} ZG/I(A)I(G) \xrightarrow{f^*} Z\Pi \to 0 \quad \tag{1.2}
\]
Lemma 1. The additive group $I(A)ZG/I(A)I(G)$ is isomorphic to $A$ and if we set $\Lambda = ZG/I(A)I(G)$, then the sequence (1.2) gives an extension of the supplemented algebra.

Proof. Consider the commutative diagram of left $A$-modules:

$$
0 \rightarrow I(A) \rightarrow ZA \rightarrow Z \rightarrow 0
$$

Taking homology, we get a commutative diagram

$$
0 \rightarrow H_1(A, Z) \rightarrow I(A)/I(A)^2 \rightarrow Z \rightarrow 0
$$

where $H_1(A, Z) = A$, and the isomorphism $H_1(A, Z) \cong I(A)/I(A)^2$ is given by the mapping $a \mapsto a - 1 \mod I(A)^2$ ($a \in A$). Hence, the mapping $a \mapsto a - 1 \mod I(A)I(G)$ gives rise to a monomorphism of the multiplicative structure of $A$ into the additive structure of $I(G)/I(A)I(G)$. But the equality in $ZG$

$$(a-1)g = (a-1)+(a-1)(g-1), \quad a \in A, \quad g \in G \quad (1.3)$$

shows that the image of $A$ is precisely the subgroup $I(A)ZG/I(A)I(G)$. Thus, $I(A)ZG/I(A)I(G)$ is isomorphic to the additive group $A$.

Furthermore, the equality in $ZG$

$$g(a-1) = (gag^{-1}-1)g$$

and the equality (1.3) show that the sequence (1.2) is an extension of the supplemented algebra. This proves the lemma.

Conversely, given an extension $E_\Lambda$ over $Z\Pi$, let $G$ be the set of elements $\lambda$ of $\Lambda$ such that $f^*(\lambda) \in \Pi$. Then $G$ is a group under the multiplication of the ring $\Lambda$, and the epimorphism $f; G \rightarrow \Pi$ induced from $f^*$ and the monomorphism $i; A \rightarrow \Lambda, i(a) = i^*(a)+1$ give an extension $E_G$ over $\Pi$.

Lemma 2 ([2]). If $E_\Lambda$ (resp. $E_G$) is the extension constructed from $E_G$ (resp. $E_\Lambda$) as above, then $E_\Lambda$ and $E_G$ have the same characteristic class. Hence, the above constructions establish a $1-1$ correspondence between the equivalence classes of extensions $E_G$ over $\Pi$ and those of extensions $E_\Lambda$ over $Z\Pi$.

Remark. The cohomology group $H^2(\Pi, A)$ is also expressed as $\text{Ext}_{Z\Pi}^2(Z, A)$. Then each extension $E_G$ over $\Pi$ may be related to a $2$-fold extension of $Z\Pi$-modules. The homology sequence
0 \rightarrow H_*(A, Z) \rightarrow I(G)/I(A)I(G) \rightarrow Z\Pi \rightarrow Z \rightarrow 0

in the proof of Lemma 1 is the corresponding 2-fold extension of $Z\Pi$-modules.

In the next section, we also use the following lemma. Given a diagram of extensions of groups:

$$0 \rightarrow A' \rightarrow G' \rightarrow \Pi' \rightarrow 1$$

then $\psi$ induces an isomorphism $H(\psi); H^2(\Pi, A) \cong H^2(\Pi', A)$ where $A$ is regarded as $\Pi'$-module by $\psi$. In addition, if $\phi^*$ is a $\Pi'$-isomorphism, then $\phi^*$ induces an isomorphism $H(\phi^*); H^2(\Pi', A') \cong H^2(\Pi', A)$.

**Lemma 3** (S. Lang [7]). There exists an isomorphism $\psi; G' \cong G$ which makes the above diagram commutable, if and only if $\phi^*$ is a $\Pi'$-isomorphism and $H(\phi^*)(\alpha')=H(\psi)(\alpha)$ for the characteristic classes $\alpha$ and $\alpha'$ of the extensions $G$ and $G'$, respectively.

We shall remark that an analogous lemma also holds for extensions of supplemented algebras.

2. In this section, we consider finite groups $G'$ and $G$ with an isomorphism $\phi; ZG' \cong ZG$ as algebras. With a generality, we can assume that $\phi$ is commutative with the augmentations (see [8]). If $A'$ is a normal subgroup of $G'$ and $f'$ is the natural epimorphism of $ZG'$ onto $Z(G'/A')$, then a normal subgroup $\Phi(A')$ of $G$ is defined by setting

$$\Phi(A') = \{g \in G; f' \circ \phi^{-1}(g) = 1\} .$$

**Lemma 4** ([3], [9], and [8]). (a) $\Phi$ is an isomorphism of the lattice of normal subgroups of $G'$ onto that of $G$, (b) $\phi$ induces an algebra isomorphism $\bar{\phi}$ of $Z(G'/A')$ onto $Z(G/\Phi(A'))$ such that the induced diagram

$$
\begin{array}{ccc}
ZG' & \overset{f'}{\longrightarrow} & Z(G'/A') \\
\phi \mid \cong & & \bar{\phi} \mid \cong \\
ZG & \overset{f}{\longrightarrow} & Z(G/\Phi(A'))
\end{array}
$$

is commutative, and

(c) if $A'$ is abelian, $\Phi(A')$ is isomorphic to $A'$, and if $A'$ is central, then $\phi$ restricted to $A'$ gives itself an isomorphism of $A'$ onto $\Phi(A')$.

For the proof of (a), (b), and the latter half of (c), see [3], [9], and of the first part of (c), see [8].
From now on, let $A'$ be an abelian normal subgroup of $G'$ and $\Pi'$ be the quotient group $G'/A'$. Then, by the above lemma (c) there exists an abelian normal subgroup $A$ of $G$ which is isomorphic to $A'$. Set $\Pi=G/A$ the quotient. From the lemma (b), we have an isomorphism $\overline{\phi}; Z\Pi' \cong Z\Pi$ and a commutative diagram

$$
\begin{array}{cccccc}
0 & \to & I(A')ZG' & \overset{i'}{\to} & ZG' & \overset{f'}{\to} & Z\Pi' & \to & 0 \\
\phi \downarrow \cong & & \phi \downarrow \cong & & \phi \downarrow \cong & & \phi \downarrow \cong \\
0 & \to & I(A)ZG & \overset{i}{\to} & ZG & \overset{f}{\to} & Z\Pi & \to & 0
\end{array}
$$

of algebras. Since $\phi$ is commutative with the augmentations, we get $\phi(I(G')) = I(G)$, so that $\phi(I(A')I(G')) = \phi(I(A')ZG'I(G')) = \phi(I(A')ZG'\cdot I(G')) = I(A)ZG\cdot I(G) = I(A)I(G)$. Therefore, we can obtain an isomorphism of extensions of the supplemented algebras:

$$
\begin{array}{cccccc}
0 & \to & I(A')ZG'/I(A')I(G') & \overset{j_{I(A')I(G')}}{\to} & ZG'/I(A')I(G') & \overset{j_{I(A)ZG'}}{\to} & Z\Pi' & \to & 0 \\
\phi \downarrow \cong & & \phi \downarrow \cong & & \phi \downarrow \cong & & \phi \downarrow \cong \end{array}
$$

We notice that if we identify $A$ with $I(A)ZG/I(A)I(G)$, then the isomorphism $\phi^*; A' \cong A$ is nothing but the isomorphism $\Phi(A') = A$ stated in the lemma (c) (see [8]).

If we regard $A$ as a $Z\Pi'$-module (hence, as a $\Pi'$-module) by the algebra isomorphism $\overline{\phi}$, then $\overline{\phi}$ induces an isomorphism $H(\overline{\phi}); H^p(\Pi, A) = H^p(Z\Pi, A) \cong H^p(\Pi', A)$.

**Lemma 5.** $\phi^*; A' \cong A$ is a $\Pi'$-isomorphism and $H(\phi^*)(\alpha') = H(\overline{\phi})(\alpha)$ for the characteristic classes $\alpha$ and $\alpha'$ of the extensions $G$ and $G'$, respectively.

**Proof.** By Lemma 2, $\alpha$ and $\alpha'$ are also the characteristic classes of the corresponding algebra extensions $ZG/I(A)I(G)$ and $ZG'/I(A')I(G')$, respectively. Hence, the lemma follows immediately from the commutativity of the diagram (2.1) and the remark after Lemma 3.

**Theorem 1.** Let $G'$ and $G$ be finite groups with an isomorphism $\phi; ZG' \cong ZG$. If $A'$ is an abelian normal subgroup of $G'$, then there exists an abelian normal subgroup $A$ of $G$ which is isomorphic to $A'$, and if $H'/A'$ and $H/A$ are the centers of the quotients $G'/A'$ and $G/A$, respectively, then $H'$ is isomorphic to $H$.

**Proof.** The first assertion has already been seen. Set $\Pi=G/A$ (resp. $\Pi'=G'/A'$) and $\Pi_0=H/A$ (resp. $\Pi_0'=H'/A'$). Then we have an isomorphism $\overline{\phi}; Z\Pi' \cong Z\Pi$ and the isomorphism (2.1) of extensions. By Lemma 4 (c), $\overline{\phi}$
restricted to the center $\Pi'_0$ gives rise to a group isomorphism $\bar{\phi}_0$; $\Pi'_0 \cong \Pi_0$. Then we get a diagram of extensions:

$$
\begin{array}{c}
0 \to A' \to H' \to \Pi'_0 \to 1 \\
\phi^* \downarrow \cong \downarrow \bar{\phi}_0 \\
0 \to A \to H \to \Pi_0 \to 1.
\end{array}
$$

Moreover, the operation of $\Pi'_0$ through $\bar{\phi}$ on $A$ coincides with that through $\bar{\phi}_0$, so that, under the latter operation, $\phi^*$ is a $\Pi'_0$-isomorphism (Lemma 5).

Let $\text{Res}_* H^i(\Pi, A) \to H^i(\Pi'_0, A)$ be the restriction map of cohomology groups. Then, we see easily that $\text{Res} \circ H(\phi^*) = H(\bar{\phi}_0^*) \circ \text{Res}$, and $\text{Res} \circ H(\bar{\phi}) = H(\bar{\phi}_0) \circ \text{Res} (\bar{\phi}_0$ is the $\bar{\phi}$ restricted to $\Pi'_0$). Let $\alpha$ and $\alpha'$ be the characteristic classes of the extensions $G$ and $G'$, respectively. Then, in Lemma 5, we have seen that $H(\phi^*)(\alpha') = H(\bar{\phi}_0)(\alpha)$. This implies that $H(\phi^*)(\text{Res}(\alpha')) = H(\bar{\phi}_0)(\text{Res}(\alpha))$. Therefore, by Lemma 3, we get an isomorphism $H' \cong H$, since $\text{Res}(\alpha')$ and $\text{Res}(\alpha)$ are the characteristic classes of the extensions $H'$ and $H$, respectively. This proves the theorem.

In particular, if $A'$ is the center of $G'$, then $H'$ is nothing but the second center of $G'$. On the other hand, if $G'$ is metabelian and $A'$ is the commutator of $G'$, then $A'$ and the quotient $G'/A'$ are both abelian. Hence, we obtain

**Corollary 1** ([9]). *If $ZG' \cong ZG$, then the second centers of $G'$ and $G$ are isomorphic.*

**Corollary 2** ([5]). *If $ZG' \cong ZG$ and $G'$ is metabelian, then $G' \cong G$.*

**Remark.** In [9], Passman shows the more general result that if $x$ and $y$ are any elements of a finite group $G$ which satisfy the commutator conditions: $[x, G], y] = [y, G], x] = 1$ and $[x, G] \cap [y, G]$ is contained in the hyper center of $G$, then the commutator $[x, y]$ is determined by the group ring $ZG$. His proof is based on the following property of augmentation ideals: if $A, B, C$ are three normal subgroups of $G$ such that $A \subseteq B \subseteq C$ and $B$ is contained in the hyper center of $G$, then $I(A)ZG \cap I(C) = I(A)ZG \subseteq I(A)$ and $I(C) \subseteq ZG$. But this is not necessarily true. In fact, if we set, especially, $B = C = G$, then this inclusion of augmentation ideals implies that the natural map $I(A)ZG/I(A)I(G) \to I(G)/I(G)^2$ is monomorphic, which means that the natural map $A/[A, A] \to G/[G, G]$ is also monomorphic (apply our arguments in the proof of Lemma 1 to the case where $A$ is not abelian). But this is not necessarily true even if $G$ is nilpotent.

Again, we consider finite groups $G'$ and $G$ with an isomorphism $ZG' \cong ZG$ and an abelian normal subgroup $A'$ of $G'$. Then there is an abelian normal subgroup $A$ of $G$ and we have the commutative diagram (2.1). The following proposition is an immediate consequence from Lemma 3 and Lemma 5.
Proposition 6. \( G' \) is isomorphic to \( G \), if there exists an isomorphism \( \psi \); \( \Pi' \cong \Pi \) such that the operation of \( \Pi' \) through \( \psi \) on \( A \) coincides with the operation through \( \phi \) and \( H(\psi)(\alpha)=H(\phi)(\alpha) \) for the characteristic class \( \alpha \) of \( G \).

Owing to G. Higman ([4]), it is known that if \( \Pi \) is the direct product of the quaternion group of order 8 and an elementary abelian 2-group, then any unit of \( \mathbb{Z}\Pi \) is a trivial unit \( \pm \sigma(\sigma G \Pi) \). Thus, if \( \Pi \) is such a group, any isomorphism \( \phi \); \( \Pi' \cong \Pi \) gives an group isomorphism \( \psi; \Pi' \cong \Pi \), that is, \( \psi=\phi \) restricted to \( \Pi' \). Therefore we get

Theorem 2. If \( G' \) is a finite group with an abelian normal subgroup such that the quotient is isomorphic to the direct product of the quaternion group of order 8 and an elementary abelian 2-group, then \( ZG'\cong ZG \) implies \( G' \cong G \).

3. Let \( D_4 \) be the dihedral group of order 8. In this section, we shall determine the automorphisms of \( \mathbb{Z}D_4 \). Any automorphism of \( \mathbb{Z}D_4 \) is given as the composition of a group automorphism and an algebra automorphism defined by a solution of certain simultaneous equations. By using a property of the solutions, we shall show

Theorem 3. If \( G' \) is a 2-group with an elementary abelian group as a normal subgroup and with the dihedral group of order 8 as the quotient, then \( ZG' \cong ZG \) implies \( G' \cong G \).

Let \( a \) and \( b \) be generators of \( D_4 \) with relations: \( a^4=b^2=1, \ ab=ba^3 \), and let \( A \) be the center of \( D_4 \). Then \( A \) is a cyclic group of order 2 and is generated by the element \( a^2 \).

Now, we consider two elements \( \tilde{a} \) and \( \tilde{b} \) of \( \mathbb{Z}D_4 \) of the forms:

\[
\begin{align*}
\tilde{a} &= a + r_a(1-a^2)a + r_b(1-a^2)b + r_{ab}(1-a^2)ab \\
\tilde{b} &= b + s_a(1-a^2)a + s_b(1-a^2)b + s_{ab}(1-a^2)ab
\end{align*}
\]

respectively, where \( r: r_a, r_b, r_{ab}, s: s_a, s_b, s_{ab} \) are all integers. By simple calculations, we get the equalities

\[
\begin{align*}
\tilde{a}^2 &= a^2 + 2(r_a^2 + r_{ab}^2 - r_a^2 - r_a)(1-a^2) \\
\tilde{b}^2 &= 1 + 2(s_a^2 + s_b^2 - s_a^2 + s_{ab}^2)(1-a^2) \quad \text{and} \quad (3.1) \\
\tilde{a}\tilde{b} - \tilde{b}\tilde{a} &= 2(2r_as_a - 2r_bs_b - 2r_{ab}s_{ab} - r_b + s_a)(1-a^2) \quad \text{and} \quad (3.2)
\end{align*}
\]

If \( \{r, s\} \) is a solution of the simultaneous equations:

\[
\begin{align*}
r_a(r_a+1) &= r_a^2 + r_{ab}^2 \quad \text{.........}(1) \\
s_b(s_b+1) &= s_a^2 - s_{ab}^2 \quad \text{.........}(2) \\
2(r_as_a - r_bs_b - r_{ab}s_{ab}) &= r_b - s_a \quad \text{.........}(3)
\end{align*}
\]

then \( \tilde{a} \) and \( \tilde{b} \) satisfy the relations:
These mean that \( \tilde{a} \) and \( \tilde{b} \) are units of \( ZD_4 \), and generate a group \( \tilde{D} \) isomorphic to \( D_4 \). Since the submodule \( Z\tilde{D} \) generated by \( \tilde{D} \) over \( Z \) coincides with the group ring \( ZD_4 \) (see [3], Theorem 3.2), then the map: \( a \mapsto \tilde{a}, \ b \mapsto \tilde{b} \) can be extended to an automorphism \( \varphi \) of \( ZD_4 \). In particular, if \( \{r, s\} \) is a solution consisting of even integers, this automorphism \( \varphi \) verifies the congruence:

\[
\varphi(x) \equiv x \operatorname{mod} I(A)I(D_4), \quad \text{for any } x \in D_4, \tag{3.3}
\]

since \( I(A) \) is generated by \( 1-a^2 \), so that \( 2I(A)ZD_4 = I(A)^2 \).)

**Lemma 7.** Any algebra automorphism \( \varphi \) of \( ZD_4 \) which satisfies the congruence (3.3) is given by \( \varphi_{r,s} \) for a solution \( \{r, s\} \) consisting of even integers of the equations (3.2), where \( \varphi_{r,s} \) denotes the automorphism defined as above by the solution \( \{r, s\} \).

**Proof.** Let \( \varphi \) be any automorphism satisfying the congruence (3.3). Then \( \varphi(a) \) is written as \( \varphi(a) = a + r_a(1-a^2)a + r_b(1-a^2)b + r_{ab}(1-a^2)ab \) (\( r_a, r_b, r_{ab} \in Z \)), because \( \varphi(a) - a \in I(A)I(D_4) \subseteq I(A)ZD_4 \). However, we see that \( r_i = 0 \). Otherwise, the coefficient of the identity in \( \varphi(a) \) is not zero. But \( \varphi(a) \) is a unit of finite order, then \( \varphi(a) \) must be equal to 1 (see [3], Theorem 3.1), which is a contradiction. Therefore, \( \varphi(a) \) is written as \( \varphi(a) = a + r_a(1-a^2)a + r_b(1-a^2)b + r_{ab}(1-a^2)ab \), and similarly \( \varphi(b) = b + s_a(1-a^2)a + s_b(1-a^2)b + s_{ab}(1-a^2)ab \). On the other hand, by Lemma 4 (c), \( \varphi \) restricted to the center \( A \) gives rise to an automorphism of \( A \), which is the identity since \( A \) is of order 2. Thus, \( \varphi(a^2) = a^2 \), so that we have the equalities: \( \varphi(a)^2 = a^2 \), \( \varphi(b)^2 = 1 \) and \( \varphi(a) \cdot \varphi(b) = \varphi(b) \cdot \varphi(a) = \varphi(b) \cdot \varphi(a) \cdot a^2 \). Let \( \varphi(a) = \tilde{a} \) and \( \varphi(b) = \tilde{b} \). Then, from the equalities (3.1), the integers \( r_1 r_a, r_b, r_{ab}, s_1 s_a, s_b, s_{ab} \) must satisfy the equations (3.2), and the automorphism \( \varphi_{r,s} \) defined by this solution \( \{r, s\} \) clearly coincides with the given automorphism \( \varphi \). Therefore, the proof is finished once we show that the solution \( \{r, s\} \) consists of even integers. Since \( \varphi(a) - a \in I(A)I(D_4) \), then \( (r_a + r_b + r_{ab})(1-a^2) \in I(A)I(D_4) \), which shows that \( (r_a + r_b + r_{ab})I(1-a^2) = 2I(A) \) (recall the isomorphism \( I(A)/I(A)^2 \cong I(A)ZD_4/I(A)I(D_4) \) in Lemma 1). Then \( r_a + r_b + r_{ab} \) is even, and similarly \( s_a + s_b + s_{ab} \) is also even. On the other hand, in the equations (3.2), \( r_b \) and \( r_{ab} \) must be both even or odd, because the left hand side of the equality (1) is always even. Then \( r_a \) must be even. Now we shall assume that \( r_b \) and \( r_{ab} \) are both odd. Then \( r_a \) is not divisible by 4, so that \( r_a \) is written as \( 4n-2 \). Therefore, we have

\[
(4n-2)(4n-1) = r_a^2 + r_{ab}^2.
\]

Since the right hand side of this equality is a norm of an integer in the quadratic
field \(Q(\sqrt{-1})\), no primes which are congruent to 3 mod 4 divide \((4n-2)(4n-1)\) with square free. But \(4n-2\) and \(4n-1\) have no prime divisors in common, then any prime congruent to 3 mod 4 can not also occur in \(4n-1\) with square free. Therefore we have the congruence \(3 \equiv 3^{2k} \mod 4\). This is a contradiction.

Then, \(r_a, r_b\) and \(r_{ab}\) are all even, so that by the equalities (3) and (2) in the equations (3.2) \(s_a\) and \(s_{ab}\) are both even. Thus, \(s_b\) is also even.

For an example, \(\phi(a) = a+4(1-a^2)a+2(1-a^2)b+4(1-a^2)ab\), \(\phi(b) = b+2(1-a^2)a+2(1-a^2)ab\) define an automorphism of \(ZD_4\), but we can show that this automorphism is not an inner automorphism.

To prove Theorem 3, we need one more lemma, which restates Corollary 2 of Theorem 1, slight accurately.

**Lemma 8.** Let \(G'\) be a metabelian group of finite order, and let \(A'\) be an abelian normal subgroup of \(G'\) such that the quotient \(G'/A'\) is abelian. If \(\phi; ZG' \cong ZG\) is an isomorphism, then there exists an isomorphism \(\phi^*\) of \(A'\) onto a some abelian normal subgroup \(A\) of \(G\). Furthermore, this isomorphism can be extended to an isomorphism \(\Psi\) of \(G'\) onto \(G\) such that \(\phi(g') \equiv \Psi(g') \mod I(A)I(G)\) for any \(g' \in G'\).

Proof. As in the proof of Theorem 1, let \(A\) be the normal subgroup of \(G\) which makes the diagram (2.1) commutable. Then the isomorphism \(\phi^*; A' \cong A\) (in the diagram) can be extended to an isomorphism \(\Psi; G' \cong G\). Therefore, it suffices to show that the isomorphism \(\Psi\) verifies the congruence required in the lemma. To see that, we shall concretely describe the isomorphism \(\Psi\). Let

\[
G' = \bigsqcup_{\sigma' \in \Pi'} A' \cdot g_{\sigma'} (g_{1} = 1), \quad G = \bigsqcup_{\sigma \in \Pi} A \cdot g_{\sigma} (g_{1} = 1)
\]

be the coset decompositions of \(G'\) and \(G\), respectively, and set

\[
\alpha' (\sigma', \tau') = g_{\sigma'} g_{\tau'}^{-1}, \quad \sigma', \tau' \in \Pi', \quad \alpha (\sigma, \tau) = g_{\sigma} g_{\tau}^{-1}, \quad \sigma, \tau \in \Pi.
\]

Then, \(\alpha' (\sigma', \tau')\) (resp. \(\alpha (\sigma, \tau)\)) is a normalized 2-cocycle representing the characteristic class \(\alpha'\) (resp. \(\alpha\)) of the extension \(G'\) (resp. \(G\)). Since \(\Pi'\) is abelian, \(\phi\) restricted to \(\Pi'\) gives an isomorphism \(\phi_0; \Pi' \cong \Pi\). Hence, by the commutativity of the diagram (2.1) we see that \(f^*((\phi(g_{\sigma'} g_{\bar{\phi}_0(\sigma')}) \mod I(A)I(G)) = 0\) for any \(\sigma' \in \Pi'.\) Therefore, for each \(\sigma'\) there exists uniquely an element \(a(\sigma')\) of \(A\) such that

\[
\phi(g_{\sigma'}) \equiv a(\sigma') g_{\bar{\phi}_0(\sigma')} \mod I(A)I(G), \quad (3.4)
\]

and we see easily that

\[
\phi^*(\alpha' (\sigma', \tau')) = a(\sigma') a(\tau')^{-1} \alpha(\bar{\phi}_0(\sigma'), \bar{\phi}_0(\tau'))
\]

(this equality means \(H(\phi^*) (\alpha') = H(\bar{\phi}_0 (\alpha))\). Then the mapping

\[
\Psi; a' g_{\sigma'} \to \phi^*(a') a(\sigma') g_{\bar{\phi}_0(\sigma')}, \quad (3.5)
\]
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G' \simeq G', which is clearly an extension of \( \phi^* \).
Furthermore, by the congruence (3.4) and the definition (3.5) of \( \Psi \), we verify the congruence 
\[ \phi(g') \equiv \Psi(g') \mod I(A)I(G) \]
for any \( g' \in G' \). This proves the lemma.

Proof of Theorem 3. Let \( G' \) be the group stated in the theorem and let \( A' \) be the abelian normal subgroup of exponent 2 such that the quotient \( \Pi' = G'/A' \) is the dihedral group of order 8. If \( \phi \) is an isomorphism of \( ZG' \) onto \( ZG \), then there is a normal subgroup \( A \) of \( G \) which is isomorphic to \( A' \) and \( \phi \) induces an isomorphism \( \phi; ZI \Pi' \simeq ZI \Pi \) of group rings of the quotients.

Let \( \Pi' \) be the center of \( \Pi' \), and apply Lemma 8 to the isomorphism 
\[ \bar{\phi}: ZI \Pi' \simeq ZI \Pi (\Pi' \text{ is metabelian}). \]
Then there exists an isomorphism \( \Psi; \Pi' \simeq \Pi \) such that 
\[ \bar{\phi}(\sigma') \equiv \Psi(\sigma') \mod I(\Pi')I(\Pi), \quad \sigma' \in \Pi', \]
so that we get an automorphism \( \Psi^{-1}\bar{\phi} \) such that 
\[ \Psi^{-1}\bar{\phi}(\sigma') \equiv \sigma' \mod I(\Pi)I(\Pi'). \]
Thus, by Lemma 7 each \( \bar{\phi}(\sigma') \) is written as \( \Psi(\sigma') + 2S \) by some \( S \) of \( I(\Pi)Z \Pi \). This means that the operation of \( \Pi' \) by \( \Psi \) on \( A \) coincides with that by \( \bar{\phi} \), and \( \Psi \) also coincides with \( \bar{\phi} \) on the cohomology group \( H^2(\Pi', A) \), because \( A \) has exponent 2, so that \( H^2(\Pi', A) \) has also exponent 2. Consequently, by Proposition 6 we obtain an isomorphism of \( G' \) onto \( G \), which proves the theorem.

**Theorem 4.** Every automorphism of the group ring \( ZD_4 \) is given as the composition \( \varphi \circ \Psi \) of an automorphism \( \Psi \) of \( D_4 \) and the automorphism \( \varphi \) of \( ZD_4 \) defined by a solution \( \{r, s\} \) consisting of even integers of the simultaneous equations (3.2).

Proof. Let \( A \) be the center of \( D_4 \) and apply Lemma 8 to any automorphism \( \phi; ZD_4 \simeq ZD_4 \). Then there exists an automorphism \( \Psi \) of \( D_4 \) such that 
\[ \phi(x) \equiv \Psi(x) \mod I(A)I(D_4), \quad x \in D_4. \]
Then the theorem is immediate from Lemma 7.

**Remark.** In this proof of the theorem, automorphisms are assumed implicitly to be commutative with the augmentation (recall our assumption at the beginning of section 2). If \( \phi \) is not commutative with the augmentation \( \varepsilon \); \( ZD_4 \to Z \), then we get a non-trivial map \( \phi_{\varepsilon}; x \mapsto \varepsilon(\phi(x))x(x \in D_4) \), which is clearly extended to an automorphism of \( ZD_4 \), and \( \phi \circ \phi_{-1} \) is commutative with the augmentation \( \varepsilon \). Therefore it suffices to determine the automorphisms \( \phi_{\varepsilon} \). But this is easy, because each \( \varepsilon(\phi(x)) \) is a unit of \( Z \), so that \( \varepsilon(\phi(x)) = \pm 1 \). Indeed, such automorphisms are given by the mappings: 
\[ a \to \pm a, \quad b \to \pm b, \]
where \( a \) and \( b \) denote generators of \( D_4 \) with \( a^4 = b^2 = 1, \quad ab = ba^3 \).

4. T. Y. Lam ([6]) showed that the Whitehead group \( \text{Wh}(S_3) \) of the symmetric group \( S_3 \) is trivial. His proof consists of following two parts: the reduced norm of \( \text{Wh}(S_3) \) is equal to \( (1, 1, 1) \), and \( \text{SK}^1(ZS_3) \) is trivial. But his
computation of the reduced norm is complicated, so that it seems impossible to apply his method to other cases. In this section, we shall give a simpler technique to compute the reduced norm of $\text{Wh}(S_3)$, and apply the technique to the $\text{Wh}(D_4)$ of the dihedral group $D_4$ of order 8. For notations used here, see ([6]) or (H. Bass [1]).

Let $G$ be a finite group with a normal subgroup $A$ and let the Whitehead group $\text{Wh}(G/A)$ of the quotient $G/A$ be trivial. Since $ZG$ has 'stable range 2', we may regard $K^1(ZG)$ as to be generated by 2 by 2 invertible matrices over $ZG$. Let

$$K^1(ZG) \to K^1(Z(G/A))$$

be the homomorphism of $K^1$-groups which is induced from the natural epimorphism $ZG \to ZG/I(A)ZG \cong Z(G/A)$. Then, by the assumption that

$$\text{Wh}(G/A) = K^1(Z(G/A))/\pm(G/A)$$

is trivial $K^1(ZG)$ is generated by $\pm G$ and invertible matrices $X$ such that

$$X = \begin{pmatrix} 1 + \alpha & \beta \\ \gamma & 1 + \delta \end{pmatrix}, \quad \alpha, \beta, \gamma, \delta \in I(A)ZG.$$ 

Consider elementary matrices

$$\begin{pmatrix} 1 & 0 \\ (-1)^{t-1} \gamma & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & (-1)^{t-1} \alpha^{t-2} \beta \gamma \alpha^{t-2} \beta \\ 1 & 1 \end{pmatrix},$$

then we see easily that

$$\begin{pmatrix} 1 & 0 \\ (-1)^{n-1} \gamma \alpha^{n-2} \\ 1 & 1 \end{pmatrix} X \begin{pmatrix} 1 & 0 \\ (-1)^{t-1} \gamma \alpha^{t-2} \beta \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 + \alpha & \beta \\ \gamma \alpha^{n-1} & 1 + \delta + \sum_{i=1}^{n} \gamma \alpha^{n-2}(\alpha - 1) \beta \end{pmatrix}.$$ 

Consequently, for any positive integer $n$, any element of $K^1(ZG)$ may be regarded as to be represented by an element of $\pm G$ and an invertible matrix

$$\begin{pmatrix} 1 + \alpha & \beta_n \\ \gamma_n & 1 + \delta \end{pmatrix}$$

such that $\alpha, \delta \in I(A)ZG$, $\beta_n, \gamma_n \in I(A)^nZG$.

Next, we consider the natural epimorphism $f; ZG \to ZG/I(A)I(G)$ and the induced homomorphism $f^*; K^1(ZG) \to K^1(ZG/I(A)I(G))$. Since $\beta_n, \gamma_n \in I(A)I(G)$ for any positive integer $n \geq 2$, we get

$$f^* \begin{pmatrix} 1 + \alpha & \beta_n \\ \gamma_n & 1 + \delta \end{pmatrix} = \begin{pmatrix} 1 + f(\alpha) & 0 \\ 0 & 1 + f(\delta) \end{pmatrix}.$$ 

On the other hand, the map: $a \mod [A, A] \to a - 1 \mod I(A)I(G)$ gives rise to an isomorphism $A/[A, A] \cong I(A)ZG/I(A)I(G)$ (in the case where $A$ is abelian, this isomorphism has been seen in Lemma 1). Thus, there
exist elements \( a \) and \( d \) of \( A \) such that \( f(\alpha) = a - 1 \mod I(A)I(G) \) and \( f(\delta) = d - 1 \mod I(A)I(G) \), respectively, so that \( f^*(\begin{pmatrix} 1 + \alpha & \beta_n \\ \gamma_n & 1 + \delta \end{pmatrix}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

Therefore, we have obtained the following proposition

**Proposition 9.** If \( G \) is a finite group with a normal subgroup \( A \) and if the Whitehead group \( \text{Wh}(G/A) \) is trivial, then, for any positive integer \( n \geq 2 \), any element of \( \text{Wh}(G) \) is represented by an invertible matrix \( X \) such that

\[
X = \begin{pmatrix} 1 + \alpha & \beta_n \\ \gamma_n & 1 + \delta \end{pmatrix}, \quad \text{where} \quad \alpha, \delta \in I(A)I(G), \quad \beta_n, \gamma_n \in I(A)^*ZG.
\]

Using this proposition, we shall compute the reduced norm of \( \text{Wh}(S_3) \). Let \( G \) be the symmetric group \( S_3 \) and set \( a = (1 \ 2 \ 3) \) and \( b = (1 \ 2) \). Then, \( G \) is generated by \( a \) and \( b \). If \( A \) is the subgroup generated by \( a \), then the quotient \( G/A \) is of order 2, so that \( \text{Wh}(G/A) \) is trivial ([1], [4]). Hence, we can apply the proposition to this case, and to determine the reduced norm of \( \text{Wh}(G) \), it suffices to compute the reduced norms of invertible matrices \( X = \begin{pmatrix} 1 + \alpha & \beta_2 \\ \gamma_2 & 1 + \delta \end{pmatrix} \)

such that \( \alpha, \delta \in I(A)I(G), \quad \beta_2, \gamma_2 \in I(A)^*ZG. \)

Since \( A \) is the commutator of \( G \), any element of \( I(A)ZG \) is represented to 0 by any representation of degree 1, hence each component of degree 1 of the reduced norm of \( X \) is equal to 1. The irreducible representation of \( G \) of degree 2 is given by

\[
\rho(a) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}.
\]

It is known that the reduced norm of \( \text{Wh}(S_3) \) of the symmetric group \( S_3 \) is of the form \( (\pm 1, \pm 1, \pm 1) \) ([1]), therefore it is harmless to carry out mod 3 the computation of the component \( \det(\rho(X)) \) of degree 2 of the reduced norm of \( X \). Since \( A \) is a cyclic group generated by the element \( a \), we see easily that \( I(A)^*ZG = (a - 1)^*ZG \). But \( \rho \) represents the element \((a - 1)^2\) to the matrix \( \begin{pmatrix} 0 & 3 \\ -3 & 3 \end{pmatrix} \), then \( \det(\rho(X)) \equiv \det\left(\begin{pmatrix} 1 + \rho(\alpha) & 0 \\ 0 & 1 + \rho(\delta) \end{pmatrix}\right) \mod 3. \)

On the other hand, we can easily see that any element of \( I(A)I(G)/I(A)^*ZG \) is written as \((x - 1)(b - 1) \mod I(A)^*ZG \) for some element \( x \) of \( A \). Therefore, it suffices to compute

\[
\det\left(\begin{pmatrix} 1 + \rho((x - 1)(b - 1)) & 0 \\ 0 & 1 + \rho((x' - 1)(b - 1)) \end{pmatrix}\right).
\]

Indeed,

\[
\det(1 + \rho((a - 1)(b - 1))) = \det\begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} \equiv 1 \mod 3,
\]

and
\[ \det(1 + \rho((a^2 - 1)(b - 1))) = \det \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \equiv 1 \mod 3. \]

Consequently, \( \det(\rho(X)) = 1 \), and the reduced norm of \( \text{Wh}(S_n) \) is equal to \((1, 1, 1, 1, 1)\).

Finally, we compute the reduced norm of \( \text{Wh}(D_4) \). Let \( G \) be the dihedral group \( D_4 \) of order 8, and let \( a \) and \( b \) be generators of \( G \) with relations: \( a^4 = b^2 = 1 \), \( ab = ba^3 \). Set \( A = \{1, a^2\} \), then the quotient \( G/A \) is an abelian group of type \((2, 2)\), so that \( \text{Wh}(G/A) \) is trivial ([1], [4]). Hence, we can also apply Proposition 9, and it suffices to compute the reduced norms of invertible matrices \( X = \begin{pmatrix} 1 + \alpha & \beta_s \\ \gamma_s & 1 + \delta \end{pmatrix} \) such that \( \alpha, \delta \in I(A), I(G) \), \( \beta_s, \gamma_s \in I(A)^* ZG \).

Since \( A \) is the commutator of \( G \), each component of degree 1 of the reduced norm of \( X \) is equal to 1. The irreducible representation of degree 2 is given by

\[ \rho(a) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

To begin with, we try to compute the component of degree 2 mod 4, that is, \( \det(\rho(X)) \equiv 4 \mod 4 \). \( A \) is generated by the element \( a^2 \), then we see that \( I(A)^* ZG = 4I(A)ZG \). Therefore, it suffices to compute

\[ \det \begin{pmatrix} 1 + \rho(\alpha) & 0 \\ 0 & 1 + \rho(\delta) \end{pmatrix} \mod 4. \]

Set \( \alpha = r_1(1 - a^2) + r_a(1 - a^2)a + r_b(1 - a^2)b + r_{ab}(1 - a^2)ab \). Then, we see easily that

\[ \det(1 + \rho(\alpha)) = \det \begin{pmatrix} 1 + 2(r_1 - r_{ab}) & 2(-r_a + r_b) \\ 2(r_a + r_b) & 1 + 2(r_1 + r_{ab}) \end{pmatrix}, \]

and this is congruent to 1 mod 4. Thus, \( \det(\rho(X)) \) is also congruent to 1 mod 4, so that \( \det(\rho(X)) \) can not be equal to \(-1 \). But it is known that the reduced norm of \( \text{Wh}(D_4) \) is of the form \((\pm 1, \pm 1, \pm 1, \pm 1, \pm 1)\) ([1]), hence \( \det(\rho(X)) = 1 \). Consequently, we have shown

**Theorem 5.** The reduced norm of the Whitehead group \( \text{Wh}(D_4) \) of the dihedral group \( D_4 \) of order 8 is equal to \((1, 1, 1, 1, 1)\), so that \( \text{Wh}(D_4) \) is isomorphic to the special Whitehead group \( \text{SK}'(ZD_4) \).

**Remark.** Apply the Witt-Berman and Swan-Lam's induction theorem ([1], [6]) to the Whitehead group \( \text{Wh}(S_4) \) of the symmetric group \( S_4 \), then, from Lam's result on \( S_4 \) and the above theorem, we can see that \( \text{Wh}(S_4) \) is isomorphic to the special Whitehead group \( \text{SK}'(ZS_4) \).

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References