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THE GENERALIZED SCHUBERT CYCLES AND THE POINCARÉ DUALITY

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0. Introduction

It is known that a simply connected compact kählerian homogeneous space admits the standard analytic cell decomposition, in which each cell is called a generalized Schubert cell and determines a generalized Schubert cycle (Borel-Kostant; cf. [4]). The set of these cycles forms a homology basis of the space and corresponds bijectively to a canonically-defined subset of the Weyl group of the transformation group of the space. The purpose of this paper is to express the Poincaré duality of the space by means of elements of the Weyl group.

In more detail, let X be a simply connected compact kählerian homogeneous space. According to Wang, we call X a *kählerian C-space* and we may identify X with the coset space G/U where G is a complex semi-simple Lie group and U is a parabolic subgroup of G. Take a Borel subgroup B in U and let M be the commutator subgroup of B, so that M is a maximal unipotent subgroup of G. Let W be the Weyl group of G, W_1 be the Weyl group of the reductive part of U, and consider W_1 as a subgroup of W. Then the M-orbit decomposition of X gives an analytic cell decomposition, which is parametrized by the right coset space $W_1\backslash W$. Namely we have

$$X=\bigcup_{(s)\in W_1\backslash W} V_{(s)} \qquad \text{(disjoint sum)}$$

where (s) denotes the class to which the element $s\in W$ belongs, and $V_{(s)}$ is an M-orbit (a generalized Schubert cell) marked with (s). On the other hand, there exists an involutive element κ of the Weyl group W, which is uniquely determined in a canonical way from the fixed Borel subgroup B in U (cf. §1). Then we see $\dim_c V_{(s)}+\dim_c V_{(s\kappa)}=\dim_c X$.

Theorem I. *The notation being as above, the correspondence of the generalized Schubert cycles* $\bar{V}_{(s)} \rightsquigarrow \bar{V}_{(s\kappa)}$ *gives the Poincaré duality of a kählerian C-space X. Precisely speaking, we have the intersection numbers;* $\bar{V}_{(s)}\cdot\bar{V}_{(s\kappa)}=1$ *and* $\bar{V}_{(s)}\cdot\bar{V}_{(t)}=0$ *for any cycles* $\bar{V}_{(t)}\neq\bar{V}_{(s\kappa)}$ *such that* $\dim_c\bar{V}_{(s)}+\dim_c\bar{V}_{(t)}=$

$\dim_c X$. *Here* $\bar{V}_{(s)}$ *denotes the cycle carried by the analytic subvariety of the closure of the cell* $V_{(s)}$.

The above result was conjectured by Takeuchi [5], who gave a proof in the case that the kählerian C-space X is a hermitian symmetric space. Further we refer to Ehresmann's thesis [1], where the fact was found out for classical hermitian symmetric spaces; i.e. complex Grassmann manifolds, complex quadrics and other two types. Hence Theorem I generalizes the results of Ehresmann-Takeuchi's. Our proof is based on the idea "dual" cells for Schubert cells introduced by Takeuchi [5]. However our method does not depend on the Borel-Weil imbedding nor the theory of reflection groups as used in [5]. We shall also prove an analogue to the Theorem I for some kind of real algebraic homogeneous spaces called Tits' real R-spaces (Theorem II), and this result also generalizes Ehresmann [2] and Takeuchi [5].

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1. Preliminaries

Let X be a kählerian C-space. Put $X=G/U$ as a homogenous space where G is a connected complex semi-simple Lie group and U is a parabolic subgroup in G. Henceforth, when a Roman capital letter denotes a Lie group, the corresponding German small letter shall denote the corresponding Lie algebra. Let B be a Borel subgroup in U. We can choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{b}$. Let Δ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and let

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$$

be the root space decomposition of $\mathfrak{g}$, $\mathfrak{g}_\alpha$ being the eigenspace belonging to a root $\alpha\in\Delta$. There exists then a lexicographical order in Δ such that

$$\mathfrak{b} = \mathfrak{h} + \sum_{\alpha\in\Delta_+} \mathfrak{g}_\alpha$$

where Δ_+ (resp. Δ_-) denotes the set of all positive (resp. negative) roots. As for $\mathfrak{u}\supset\mathfrak{b}$, there exists a subset Δ_+^1 in Δ_+ such that

$$\mathfrak{u} = \mathfrak{b} + \sum_{\beta\in\Delta_+^1} \mathfrak{g}_{-\beta}\,.$$

Now put

$$\mathfrak{g}_1 = \mathfrak{h} + \sum_{\alpha\in\Delta_+^1} \mathfrak{g}_\alpha + \sum_{\alpha\in\Delta_+^1} \mathfrak{g}_{-\alpha}\,.$$

Then $\mathfrak{g}_1$ is a reductive Lie subalgebra of $\mathfrak{u}$. Let W (resp. W_1) be the Weyl group of $\mathfrak{g}$ (resp. $\mathfrak{g}_1$). Then W_1 may be regarded as a subgroup of W. We know that there exists the unipue involutive element $\kappa \in W$ such that $\kappa\Delta_+ = \Delta_-$. Put $\Phi_s = s\Delta_- \cap \Delta_+$ for $s \in W$. When we put $-\square = \{-\alpha \mid \alpha \in \square\}$ and $s\square = \{s\alpha \mid \alpha \in \square\}$ for a subset $\square \subset \Delta$ and $s \in W$, we clearly have $\Phi_{s\kappa} = s\Delta_+ \cap \Delta_+$, $\Delta_+ = \Phi_s \cup \Phi_{s\kappa}$ (disjoint sum) and $\Phi_{s^{-1}} = -s^{-1}\Phi_s$. For an element $s \in W$, the index of s is defined by $n(s) = |\Phi_s|$, $|\cdot|$ denoting the cardinal number of the set. Note that $n(s^{-1}) = n(s)$ for $s \in W$. Next put $\Psi = \Delta_+ - \Delta_+^1$ and define the subset W^1 of W as $W^1 = \{s \in W \mid \Phi_s \subset \Psi\}$. Then the map $W_1 \times W^1 \ni (s, t) \rightsquigarrow st \in W$ is bijective, that is, the subset W^1 is a representative system of the right coset space $W_1 \backslash W$ (Kostant [3]). If $s \in W_1$, s preserves Ψ and $\Delta_+^1 \cup -\Delta_+^1$ and $s(\Delta_+^1) = \Delta_+^1$ if and only if s is the unit element. It follows that $n(s) \leqq n(t)$ for $s \in W^1$ and for any $t \in W_1 s$, where $n(s) = n(t)$ holds only if $s = t$.

Next put $\mathfrak{m} = \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$, and let M be the Lie subgroup of G corresponding to $\mathfrak{m}$. Then $\mathfrak{m}$ is a maximal nilpotent subalgebra of $\mathfrak{g}$. Let $N_G(H)$ be the normalizer in G of the fixed Cartan subgroup H, and identify the Weyl group W with the factor group $N_G(H)/H$. Then the double coset MsU depends only on the class of $s \in N_G(H)$ modulo H since $H \subset U \cap N_G(H)$. According to Kostant [4, §6], we have then the generalized Bruhat decomposition $G = \bigcup_{s \in W} Ms^{-1}U$, and $Ms^{-1}U = Mt^{-1}U$ if and only if $s \in W_1 t$. Therefore we have a disjoint union $G = \bigcup_{s \in W^1} Ms^{-1}U$. For the homogeneous space $X = G/U$, this amounts to $X = \bigcup_{s \in W^1} Ms^{-1} \cdot o$ (disjoint union), where o is the origin $\{U\}$ of X. Put $V_s = Ms^{-1} \cdot o$, the M-orbit of a point $s^{-1} \cdot o \in X$ for $s \in W$, so that $X = \bigcup_{s \in W^1} V_s$. Furthermore let $\mathfrak{m}_s^+$, $\mathfrak{m}_s^-$ be the nilpotent Lie subalgebras defined by $\mathfrak{m}_s^+ = \sum_{\alpha \in \Phi_s} \mathfrak{g}_\alpha$, $\mathfrak{m}_s^- = \sum_{\beta \in \Phi_s} \mathfrak{g}_{-\beta}$, and M_s^+, M_s^- be the corresponding Lie subgroups in G respectively. Then, if $s \in W^1$, we can express

$$V_s = M_{s^{-1}}^+ s^{-1} \cdot o = s^{-1} M_s^- \cdot o\,, \tag{1.1}$$

and moreover the map

$$\mathfrak{m}_{s^{-1}}^+ \ni X \rightsquigarrow (\exp X) s^{-1} \cdot o \in V_s$$

is biholomorphic. Therefore the complex submanifold V_s is a complex $n(s)$-cell for $s \in W^1$, which is called a *generalized Schubert cell.*

Thus a kählerian C-space X admits an analytic cell decomposition, and $\{V_s\}_{s \in W^1}$ is moreover a CW-complex [5]. Notice further that the submanifold V_s is locally closed in X both in the sense of Zariski and Hausdorff, and the closure $\bar{V}_s$ is a union of V_s and some Schubert cells whose dimensions are properly lower than that of V_s. The same letter $\bar{V}_s$ denoting also the cycle which the Schubert variety $\bar{V}_s$ carries, the set $\{\bar{V}_s\}_{s \in W^1}$ forms a basis of the integral homology group of X ($\bar{V}_s \in H_{2n(s)}(X, \boldsymbol{Z})$).

Our aim is to determine the intersection matrix with respect to this basis.

2. Reduction of Theorem I to Theorem I′; dual cells

Recall that we put $V_s = Ms^{-1}\cdot o$ for $s \in W$. Then $V_s = V_t$ if and only if $s \in W_1 t$, and by the results in §1 we have $\dim_c V_s \leqslant n(s)$ for $s \in W$, the equality holding if and only if $s \in W^1$. Under this notation, we have

$$\dim_c V_s + \dim_c V_{s\kappa} = n \qquad \text{for} \qquad s \in W$$

where $n = \dim_c X$ and κ is the element introduced in §1 (Takeuchi [5]). We see as in the case of V_s that if $s \in W^1$

$$V_{s\kappa} = \exp\Big(\sum_{\beta \in \kappa s^{-1}(\Psi - \Phi_s)} \mathfrak{g}_{-\beta}\Big)\kappa s^{-1}\cdot o\,. \tag{2.1}$$

Theorem I in the introduction can now be stated as:

Theorem I. *Let $\{\bar{V}_s\}_{s\in W^1}$ be the set of the generalized Schubert cycles on a kählerian C-space. Then we have $\bar{V}_s \cdot \bar{V}_{s\kappa} = 1$ and $\bar{V}_s \cdot \bar{V}_t = 0$ for any $t \in W^1$ such that $n(t) = n - n(s)$ and $\bar{V}_t \neq \bar{V}_s$.*

This theorem can be paraphrased by means of "*dual*" cells $*V_s$ as follows (cf. Takeuchi [5]). Put $\mathfrak{m}^- = \sum_{\alpha\in\Delta_+} \mathfrak{g}_{-\alpha}$ whose Lie subgroup is denoted by M^-; put $\mathfrak{n}_s^- = \sum_{\beta \in s^{-1}\Psi \cap \Delta_+} \mathfrak{g}_{-\beta}$ whose Lie subgroup is denoted by N_s^-; and finally put $*V_s = M^- s^{-1}\cdot o$ (the M^--orbit of $s^{-1}\cdot o \in X$). Then from (2.1) and the definition of $\kappa \in W$, we see that if $s \in W^1$,

$$*V_s = N_s^- s^{-1}\cdot o = \kappa V_{s\kappa}\,. \tag{2.2}$$

Therefore we have $\dim_c V_s + \dim_c *V_s = |\Psi| = n$, $\dim_c *V_s = n - n(s)$ and $s^{-1}\cdot o \in V_s \cap *V_s$ for $s \in W^1$. Moreover we know that if $*V_t \subset *\bar{V}_s - *V_s$ for $s, t \in W^1$, then $n(t) > n(s)$ by the same reason as for $\bar{V}_s$. On the other hand, a dual Schubert cycle $*\bar{V}_s$ is homologous to the Schubert cycle $\bar{V}_{s\kappa}$. In fact, this follows from (2.2) since κ is homotopic to the identity because of the connectedness of G. Hence Theorem I is reduced to

Theorem I′. *For Schubert varieties $\bar{V}_s$ and $*\bar{V}_s$, we have*

i) $\bar{V}_s \cap *\bar{V}_s = \{s^{-1}\cdot o\}$ for $s \in W^1$ *and they mutually intersect transversally,*

ii) $\bar{V}_s \cap *\bar{V}_t = \phi$ *for* $s, t \in W^1$

such that $n(s) = n(t)$ and $s \neq t$.

REMARK. Theorem I′, i) was proved by Takeuchi [5] through the Borel-Weil imbedding. Here we shall prove i) in another way, together with ii).

3. Proof of Theorem I′

Lemma 1 (Kostant [4], Lemma 6.2). *Let* $\mathfrak{n}$ *be a nilpotent Lie algebra. Assume* $\mathfrak{n}=\mathfrak{n}_1+\mathfrak{n}_2$ *is a linear direct sum where* $\mathfrak{n}_1$ *and* $\mathfrak{n}_2$ *are Lie subalgebras. Let* N *be a simply connected Lie group corresponding to* $\mathfrak{n}$, *and* N_i, $i=1,2$, *the subgroups corresponding to* $\mathfrak{n}_i$, $i=1, 2$. *Then the map*

$$\varphi : N_1\times N_2\ni(n_1, n_2) \rightsquigarrow n_1 n_2\in N$$

is a bijection.

In case of complex Lie algebras the above map clearly becomes biholomorphic.

Now let $\mathfrak{n}^-=\sum_{\alpha\in\Psi}\mathfrak{g}_{-\alpha}$ and N^- the corresponding Lie subgroup. Put $\mathcal{N}_s=s^{-1}N^-\cdot o$. Then $\mathcal{N}_s$ is a complex n-cell which is biholomorphic both to N^- and to $\mathfrak{n}^-$.

Lemma 2. *Let* $s\in W^1$ *and let* $M^+_{s^{-1}}$, N^-_s *be the nilpotent Lie groups defined in* §§ 1, 2. *For the complex cell* $\mathcal{N}_s$ *defined above, we have*

$$\mathcal{N}_s = M^+_{s^{-1}}\cdot N^-_s s^{-1}\cdot o$$

and the map

$$M^+_{s^{-1}}\times N^-_s\in(m, n) \rightsquigarrow mns^{-1}\cdot o\in\mathcal{N}_s$$

is biholomorphic.

Proof. Write $\mathcal{N}_s=s^{-1}N^-\cdot o=(s^{-1}N^-s)s^{-1}\cdot o$. The Lie algebra corresponding to the nilpotnet group $s^{-1}N^-s$ has the form

$$\mathrm{Ad}\, s^{-1}(\mathfrak{n}^-) = \sum_{\alpha\in s^{-1}\Psi}\mathfrak{g}_{-\alpha}\,.$$

On the other hand, recall the Lie algebras of $M^+_{s^{-1}}$ and N^-_s are expressed as

$$\mathfrak{m}^+_{s^{-1}} = \sum_{\beta\in-\Phi_{s^{-1}}}\mathfrak{g}_{-\beta}$$
$$\mathfrak{n}^-_s = \sum_{\gamma\in s^{-1}\Psi\cap\Delta_+}\mathfrak{g}_{-\gamma}$$

Now we see

$$-\Phi_{s^{-1}} = -(s^{-1}\Delta_-\cap\Delta_+) = s^{-1}\Delta_+\cap\Delta_- = s^{-1}\Psi\cap\Delta_-$$

since $s\in W^1$. Therefore we have a linear direct sum

$$\mathrm{Ad}\, s^{-1}(\mathfrak{n}^-) = \mathfrak{m}^+_{s^{-1}}+\mathfrak{n}^-_s.$$

Lemma 2 follows then from Lemma 1. q.e.d.

Recall (1.1), (2.2) for a Schubert cell V_s and its dual $*V_s$: we have

$$V_s = Ms^{-1}\cdot o = M^+_{s^{-1}}s^{-1}\cdot o\,,$$
$$*V_s = M^-s^{-1}\cdot o = N^-_s s^{-1}\cdot o\,,$$

if $s\in W^1$. Therefore from Lemma 2 we see:

REMARK. A cell $\mathfrak{N}_s$ is imbedded in X as a direct product $V_s\times {}^*V_s$ with its origin $V_s\cap {}^*V_s=\{s^{-1}\cdot o\}$ if $s\in W^1$.

Lemma 3. *Let $s, t\in W^1$. Assume $n(s)\leqslant n(t)$ for the indices of s, t. If $V_s\cap {}^*V_t\neq\phi$ for the Schubert cells, then we have $s=t$.*

Proof. Let $p\in V_s\cap {}^*V_t$ and consider the $M^+_{t^{-1}}$-orbit of p. Then $M^+_{t^{-1}}\cdot p\subset V_s$ since $M^+_{t^{-1}}\subset M$ and $p\in V_s$. Here we see $V_s=M\cdot p=M^+_{s^{-1}}\cdot p$, and let $H_p=\{m\in M;\ m\cdot p=p\}$ be the isotropy subgroup of M at p. From the argument in § 1, we have then a linear direct sum

$$\mathfrak{m}=\mathfrak{m}^+_{s^{-1}}+\mathfrak{h}_p\,.$$

On other hand, we have also a linear direct sum

$$\mathfrak{m}=\mathfrak{m}^+_{t^{-1}}+\mathfrak{h}_p\,.$$

In fact, since $p\in {}^*V_t$, the group $M^+_{t^{-1}}$ acts effectively on the space $M^+_{t^{-1}}\cdot p$ because of the definition of ${}^*V_t$ and Lemma 2. Therefore we have $M^+_{t^{-1}}\cap H_p=\{1\}$, which shows the above formula since $\dim_c \mathfrak{m}^+_{t^{-1}}=n(t)\geqslant n(s)=\dim_c \mathfrak{m}^+_{t^{-1}}$ by the assumption. Using Lemma 1, we have $V_s=M^+_{t^{-1}}\cdot p$ from the above two linear direct sums. Therefore $s^{-1}\cdot o\in\mathfrak{N}_t$. Now $\mathfrak{N}_t=t^{-1}\,{}^*V_e$ where $e\in W^1$ is the unit element. Thus $s^{-1}\cdot o\in\mathfrak{N}_t$ implies $ts^{-1}\cdot o\in {}^*V_e$. Therefore $ts^{-1}\in W_1$ because of the M^--orbit decomposition. Hence $s=t$ since $s, t\in W^1$ and $s\in W_1\,t$, which proves the lemma.

Proof of Theorem I′

For i) we shall show $\bar V_s\cap {}^*\bar V_s=\{s^{-1}\cdot o\}$ if $s\in W^1$. We get $V_s\cap {}^*V_s=\{s^{-1}\cdot o\}$ by Remark to Lemma 2. Let $V_{s'}\subset\bar V_s-V_s$, ${}^*V_{s''}\subset {}^*\bar V_s-{}^*V_s$. Then $n(s')<n(s)<n(s'')$. Hence $V_{s'}\cap {}^*\bar V_s=\phi$, ${}^*V_{s''}\cap\bar V_s=\phi$ from Lemma 3. The transversal intersection property being clear, these prove Theorem I′, i). For ii) we shall show ${}^*\bar V_s\cap {}^*\bar V_t=\phi$ for $s, t\in W^1$ such that $s\neq t$ and $n(s)=n(t)$. Assume $\bar V_s\cap {}^*\bar V_t\neq\phi$. Then there exist $s', t'\in W^1$ such that $V_{s'}\subset\bar V_s$, ${}^*V_{t'}\subset {}^*\bar V_t$ and $V_{s'}\cap {}^*V_{t'}\neq\phi$. Notice $n(s')\leqslant n(s)=n(t)\leqslant n(t')$ from the inclusion relations. It follows from Lemma 3 that $s'=t'$, which gives $n(s')=n(s)$ and $n(t)=n(t')$. Since the boundary of a Schubert cell consists of strictly lower-dimensional Schubert cells we have $s'=s$, $t'=t$ and hence $s=t$, which proves Theorem I′, ii).

Thus we have completed the proof of Theorem I′, hence Theorem I.

4. The case of Tits' real R-spaces

In this section, we point out that an analogue to Theorem I holds for some kind of real algebraic homogeneous spaces (Tits' real R-spaces in the terminology of [5]) which are real point sets of kählerian C-spaces. Henceforth we use the notations in Takeuchi [5], Chapt. I, §4–§6, and so they are slightly different from the ones which appeared previously. For the details of Tits' real R-spaces, we refer also to the above paper.

Let G be a reductive irreducible real algebraic linear group, whose complexification $\tilde{G}$ is a reductive connected complex algebraic linear group. An algebraic subgroup U of G is called *parabolic* when its complexification $\tilde{U}$ is parabolic in $\tilde{G}$. For such U, the real algebraic homogeneous space $X=G/U$ is called a *Tits' real R-space.* Now, suppose a Tits' real R-space $X=G/U$ be given. Then its complexification, the kählerian C-space $\tilde{X}=\tilde{G}/\tilde{U}$ can be $\tilde{G}$-equivariantly imbedded in a complex projective space $\boldsymbol{P}^N(\boldsymbol{C})$ so that the injection $\tilde{X}\to\boldsymbol{P}^N(\boldsymbol{C})$ is defined over the field of real numbers $\boldsymbol{R}$. Here, the Tits' real R-space X is the set of $\boldsymbol{R}$-rational points in $\tilde{X}$; this means that with respect to the real projective space $\boldsymbol{P}^N(\boldsymbol{R})$ naturally inbedded in $\boldsymbol{P}^N(\boldsymbol{C})$, we may regard $X=\tilde{X}\cap\boldsymbol{P}^N(\boldsymbol{R})$. A Tits' real R-space X admits a cell decomposition compatible with the kählerian C-space $\tilde{X}$. In more detail, W being the Weyl group of G, there exists a subset W^1 of W similar to the complex case, which gives rise to a cell decomposition

$$X=\bigcup_{s\in W^1} Ms^{-1}\cdot o=\bigcup_{s\in W^1} V_s\,,$$

where M is the real form ($\boldsymbol{R}$-rational point set) of a suitable maximal unipotent algebraic subgroup $\tilde{M}$ of $\tilde{G}$. Here V_s is homeomorphic to $\boldsymbol{R}^{n(s)}$ and there exists a (complex) Schubert cell $\tilde{V}_\sigma$ ($\tilde{M}$-orbit) of the complexification $\tilde{X}$ such that V_s is the set of $\boldsymbol{R}$-rational points of $\tilde{V}_\sigma$. Moreover the above cell decomposition gives a minimal cell decomposition modulo 2 of X. Let $\kappa\in W$ be the unique involutive element which transforms all the (real) positive roots to the negative roots. Then we can define the correspondence of the real Schubert cycles $\bar{V}_s \rightsquigarrow \bar{V}_{s\kappa}$ where $\dim_{\boldsymbol{R}} V_s+\dim_{\boldsymbol{R}} V_{s\kappa}=\dim_{\boldsymbol{R}} X$ and $\bar{V}_s$ is the closure of V_s in X. We can now state a generalization of Takeuchi's theorem [5, Chapt, I, Th. 18]:

Theorem II. *For (real) Schubert cycles $\{\bar{V}_s\}_{s\in W^1}$ of a Tits' real R-space X, the correspondence $\bar{V}_s \rightsquigarrow \bar{V}_{s\kappa}$ gives the Poincaré duality modulo 2 of X.*

REMARK. Th. 18 in [5] is for real *symmetric* R-spaces (of Nagano type), which also generalizes Ehresmann [2].

Proof. We can define a dual cell $*V_s$ for V_s similar to the complex case. Moreover we recall that there exists a (complex) Schubert cell $\tilde{V}_\sigma$, and a dual cell $*\tilde{V}_\sigma$ in $\tilde{X}$ such that a real Schubert cell V_s (resp. $*V_s$) is the set of $\boldsymbol{R}$-rational points of $\tilde{V}_\sigma$ (resp. $*\tilde{V}_\sigma$). We call $\tilde{V}_\sigma$ (resp. $*\tilde{V}_\sigma$) a complexification of V_s (resp.

$*V_s$).

Let $\bar{V}_s \cap *\bar{V}_t \neq \phi$ and $n(s)=n(t)$. Let $\tilde{V}_\sigma$, $*\tilde{V}_\tau$ be the complexifications of V_s, $*V_t$, and assume $\sigma, \tau \in \tilde{W}^1$ (the canonically defined subset of the Weyl group of $\tilde{G}$, denoted previously by W^1). Then $n(\sigma)=n(\tau)$ and $\bar{\tilde{V}}_\sigma \cap *\bar{\tilde{V}}_\tau \neq \phi$, where $\bar{\tilde{V}}_\sigma$, $*\bar{\tilde{V}}_\tau$ are the closures in $\tilde{X}$. Therefore $\sigma=\tau$ from Theorem I$'$, which gives $V_s=V_t$, i.e. $s=t$ if $s, t \in W^1$. It follows from Theorem I$'$ that $\bar{V}_s \cap *\bar{V}_s=\{s^{-1}\cdot o\}$ and they intersect transversally since $\bar{V}_s \cap *\bar{V}_s \subset \bar{\tilde{V}}_\sigma \cap *\bar{\tilde{V}}_\sigma$. Thus Theorem II is proved.

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