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REGULAR G_a INVARIANTS

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1. Introduction

Let G_a denote the additive group of complex numbers, and X a complex affine variety. By an action of G_a on X we will mean an algebraic action. It is well known (e.g. [4]) that every such action can be realized as the exponential of some locally nilpotent derivation D of the coodinate ring $\mathbf{C}[X]$ and that every locally nilpotent derivation gives rise to an action. The ring C_0 of G_a invariants in $\mathbf{C}[X]$ is equal to the ring of constants of the generating derivation.

Given an action $\sigma\colon G_a \times X \to X$, let $\overline{\sigma}\colon G_a \times X \to X \times X$ denote the graph morphism and $\hat{\sigma}\colon \mathbf{C}[X] \to \mathbf{C}[X, t]$ (resp. $\tilde{\sigma}\colon \mathbf{C}[X \times X] \to \mathbf{C}[X, t]$) denote the induced maps on coordinate rings.

The action is said to be proper if $\overline{\sigma}$ is a proper morphism (i.e. if $\mathbf{C}[X, t]$ is integral over the image of $\tilde{\sigma}$). The action is said to be equivariantly trivial if there is a variety Y for which X is a G_a equivariantly isomorphic to $Y \times G_a$, the action on $Y \times G_a$ being given by $g * (y, h) = (y, g + h)$. The action is locally trivial if there are affine varieties Y_i and a cover of X by G_a stable affine open subsets X_i on which the action is equivariantly trivial. Equivariant triviality of an action on X is equivalent with the existence of a regular function $s \in \mathbf{C}[X]$ for which $Ds = 1$. Such a function is called a slice and, if one exists, $\mathbf{C}[X] = C_0[s]$. Local triviality is equivalent with the intersection of the kernel and the image of D generating the unit ideal in $\mathbf{C}[X]$.

Locally trivial actions are proper, and proper actions on $\mathbf{C}^n$ are locally trivial provided $\mathbf{C}[X]$ is a flat ring extension of C_0 [4, Theorem 2.8]. It was also shown there, for $X = \mathbf{C}^n$, that properness is equivalent with surjectivity of $\tilde{\sigma}$. It had been believed e.g. [13] that proper G_a actions on normal varieties are locally trivial, until an example of a proper action on $\mathbf{C}^5$ which is not locally trivial was produced [5]. In that example, C_0 is affine, but the associated variety Y has a line of singularities. The fibers of the morphism $\mathbf{C}^5 \to Y$ over the singular points are all two dimensional. The first example of a locally trivial but not equivariantly trivial G_a action on complex affine space was discovered by Winkelmann [18]. In that example, C_0 is affine and regular and, at this writing, no example of a locally trivial action on $\mathbf{C}^n$ with non regular ring of invariants is known.

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As a consequence of the main result of the paper, Theorem 2.1, singularities in the variety associated to C_0 are shown to be the only obstruction to local triviality of a proper action. Moreover, the structure of the morphism $\mathbf{C}^n \to Y$ is elucidated in some cases where Y is singular. Finally, the main result leads to a generalization from $n = 3$ to arbitrary n of an algorithm given in [12] to determine whether a set of $n-1$ elements of $\mathbf{C}[x_1, \ldots, x_n]$ is part of a set of variables.

2. Smooth points of Y

Denote the polynomial ring $\mathbf{C}[x_1 \ldots, x_n]$ by $\mathbf{C}^{[n]}$. Suppose that R is an affine subring of $\mathbf{C}^{[n]}$ whose quotient field has transcendence degree $n-1$ over $\mathbf{C}$, and let S be a multiplivatively closed subset of R. If, for some $\{f_1, \ldots, f_{n-1}\} \subset R$, $\{df_1, \ldots, df_{n-1}\}$ generates the module of differentials $\Omega_{S^{-1}R/\mathbf{C}}$, then $\Omega_{S^{-1}C^{[n]}/S^{-1}R}$ is the quotient of the free module generated by $\{dx_i \mid 1 \le i \le n\}$ by the submodule generated by $\{df_i \mid 1 \le i \le n-1\}$. Thus the first Fitting ideal of $\Omega_{S^{-1}C^{[n]}/S^{-1}R}$ is generated by the $(n-1)\times(n-1)$ minors of the Jacobian matrix $[\partial f_i/\partial x_j]$ [10, Sec. 20.2]. In particular, $\Omega_{S^{-1}C^{[n]}/S^{-1}R}$ is free of rank one if and only if the first Fitting ideal is $S^{-1}C^{[n]}$.

Theorem 2.1. *Let D be a locally nilpotent derivation of $\mathbf{C}^{[n]}$ whose associated G_a action is fixed point free. Suppose that the ring of invariants C_0 is finitely generated and let Y denote the associated affine variety. If $y \in Y$ is a smooth point defined by a maximal ideal m of C_0, and $S \equiv C_0 - m$, then $\Omega_{S^{-1}C^{[n]}/S^{-1}C_0}$ is free of rank one.*

Proof. Since y is a smooth point there are $g_i \in C_0$, $1 \le i \le n-1$, which generate the maximal ideal of C_{0_m}. These elements define a C_0 derivation D_1 of $\mathbf{C}^{[n]}$ by

$$D_1 f \equiv \det \mathrm{Jac}(f, g_1, \ldots, g_{n-1}).$$

Since the action generated by D is free, the $\mathbf{C}^{[n]}$ module of derivations of $\mathbf{C}^{[n]}$ over C_0 is free of rank one [9, Prop. 2.1]. Thus there are $a_0,\ a_1 \in \mathbf{C}^{[n]}$ with $a_0 D = a_1 D_1$. The fixed point freeness of the action generated by D is equivalent with $(Dx_1, \ldots, Dx_n) = \mathbf{C}^{[n]}$. It follows that a_1 divides a_0 and thus D_1 is a $\mathbf{C}^{[n]}$ multiple of D.

It is well known that $\Omega \equiv \Omega_{C_{0_m}/\mathbf{C}}$ is generated by the dg_i. By the remarks above, the $D_1(x_i)$, i.e. the $(n-1)\times(n-1)$ minors of $[\partial g_i/\partial x_j]$, generate the first Fitting ideal of Ω', the module of differentials of $S^{-1}\mathbf{C}^{[n]}$ over C_{0_m}. According to [3, Cor. 3.9], this ideal is contained in no height one prime ideal. Thus D_1 is a multiple of D by a unit in $S^{-1}C^{[n]}$. Since the Dx_i generate the unit ideal in $\mathbf{C}^{[n]}$, the $D_1 x_i$ generate the unit ideal in $S^{-1}\mathbf{C}^{[n]}$. Thus the first Fitting ideal is $S^{-1}\mathbf{C}^{[n]}$ and Ω' is free of rank 1. □

Corollary 2.2. *With conditions as in the theorem, suppose in addition that Y is smooth. Then the morphism $\mathbf{C}^n \to Y$ induced by the ring inlusion $C_0 \hookrightarrow \mathbf{C}^{[n]}$ is smooth of relative dimension* 1.

Proof. Since $\Omega_{\mathbf{C}^{[n]}/C_0}$ is finitely presented, it suffices by the Quillen-Suslin theorem to show that it is locally free as a $\mathbf{C}^{[n]}$ module. But this follows immediately from the theorem. □

It should be noted that the fixed point freeness assumption is essential. If D is the locally nilpotent derivation of $\mathbf{C}[x, y, z]$ given by

$$D: x \mapsto y \mapsto z \mapsto 0$$

then the ring of invariants for the associated action is well known to be $\mathbf{C}[z, 2xz - y^2]$. Since $d(2xz - y^2) = 2z\,dx - 2y\,dy + 2x\,dz$ and $[z - y, x]$ is not a unimodular row over $\mathbf{C}[x, y, z]$, the module of differentials of $\mathbf{C}[x, y, z]$ over $\mathbf{C}[z, 2xz - y^2]$ is not free.

Corollary 2.3. *If G_a acts on $\mathbf{C}^n$ without fixed points and C_0 is affine and regular, then the action is locally trivial if and only if it is proper.*

Proof. By the previous corollary, $\mathbf{C}^n \to Y$ is smooth and therefore flat. The result then follows from [4, Theorem 2.8]. □

The next application of Theorem 2.1 generalizes a criterion for locally triviality in [8]. The notion of GICO morphism was introduced by Miyanishi in [14].

DEFINITION 1. Let $\phi: X \to Y$ be a morphism of affine varieties. Then ϕ is GICO over Y provided that for any height one prime ideal p of $\mathbf{C}[Y]$ and prime ideal P of $\mathbf{C}[X]$ minimal over $p\mathbf{C}[X]$, defining a codimension one subvariety T of X, the field $\mathbf{C}(\overline{\phi T})$ is algebraically closed in $\mathbf{C}(T)$.

Suppose that a G_a action on the affine variety X has finitely generated ring of invariants C_0. With Y denoting the affine variety with coordinate ring C_0, the action is said to be GICO if the morphism $X \to Y$ induced by the inclusion $C_0 \subset \mathbf{C}[X]$ is GICO. It should be noted that if X is factorial, i.e. $\mathbf{C}[X]$ is a ufd, then $\mathbf{C}[Y]$ is a factorially closed subring of $\mathbf{C}[X]$, hence also a unique factorization domain. Thus we are concerned with the extension of the quotient field of $C_0/(p)$ to the quotient field of $\mathbf{C}[X]/p\mathbf{C}[X]$ for all principal prime ideals (p) of C_0.

For a G_a action on a factorial affine variety, the GICO condition is easily seen to be equivalent to the condition that the intersection of C_0 and image of the generating derivation, which is an ideal of C_0, is contained in no height one prime ideal of C_0. In [8], it was shown that GICO actions on $\mathbf{C}^n$ with regular invariants are locally trivial, with the added hypothesis that the morphism $\mathbf{C}^n \to Y$ has open image. It was also shown there that proper actions on factorial affine varieties are GICO. Testing the GICO condition seems to be difficult, while properness is very easy to check. On the other hand, no actions which are fixed point free but not GICO are known to the authors.

In light of Cor. 2.2, the hypothesis that the image of $\mathbf{C}^n \to Y$ is open can be dropped:

Corollary 2.4. *A GICO action on $\mathbf{C}^n$ is locally trivial provided that C_0 is finitely generated and regular.*

3. Nonregular invariants

Consider a GICO action on $X = \mathbf{C}^n$ generated by the locally nilpotent derivation D with finitely generated invariant ring C_0. Let $\pi\colon \mathbf{C}^n \to Y$ as above be the morphism induced by the ring inclusion $C_0 \subset \mathbf{C}^{[n]}$, and let I denote the ideal $C_0 \cap \operatorname{im} D$. Denote by Z the closed subset of Y defined by I, observing that every irreducible component of Z has codimension at least two and that $\pi|_{X-\pi^{-1}Z}\colon X - \pi^{-1}Z \to Y - Z$ is a principal G_a bundle.

Recall the following lemma of Miyanishi [14, Section 2].

Lemma 3.1. *Let (O, M) e a regular local ring of dimension $n \geq 2$ and let A be a factorial, finitely generated O domain with $O \hookrightarrow A$. Let $f\colon X \to Y$ be the morphism induced by the ring inclusion, where $X = \mathbf{Spec}\, A$ and $Y = \mathbf{Spec}\, O$. Let $U = Y - \{M\}$. Assume that $f_U\colon f^{-1}(U) \to U$ is an $\mathbf{A}^1$ bundle. Then either $X \cong \mathbf{A}^1 \times Y$ or $f^{-1}(\{M\}) = \emptyset$ (the latter is only possible if $n = 2$).*

This lemma applies to the investigation of the dimensions of fibers of $\mathbf{C}^n \to Y$ over singular points when Y is not regular but the action is geometrically irreducible in codimension one (GICO). All of the pathological examples known to the authors, in particular the proper but not locally trivial action in [5] and the nonproper twin triangular actions investigated in [7], satisfy the hypothesis of the following theorem.

Theorem 3.2. *Consider a GICO action of G_a on $X = \mathbf{C}^n$ and assume that C_0 is affine and Cohen Macauley defining the affine variety Y. Assume also that the singular locus W of Y has dimension strictly less than the minimum of the dimensions of the irredicible components of Z. Then either the action is locally trivial (i.e. $\pi^{-1}(Z) = \emptyset$) or $\pi(\pi^{-1}(Z)) \subset W$. In the latter case, fibers over points in W are either empty or have dimension strictly greater than 1.*

Proof. Assume that the action is not locally trivial, so that the image of π has nonempty intersection with some irreducible component Z_1 of Z. Let p be a prime ideal of C_0 defining Z_1, and note that by assumption C_{0_p} is a regular local ring. Set $S = C_0 - p$. By Miyanishi's lemma, applied to $(O, M) = (C_{0_p}, pC_{0_p})$ and $A = S^{-1}\mathbf{C}^{[n]}$, we see that either A has a slice or the height of p is equal to 2 and $pA = A$.

The first case leads to a contradiction: If A has a slice, write it as h/k with h, $k \in \mathbf{C}^{[n]}$, $k \in S$. From $1 = D(h/k)$ one readily concludes that k and $D(k)$ have a common factor, which is impossible for a locally nilpotent derivation, unless $D(k) = 0$. It follows that $k = D(h) \in C_0 \cap \operatorname{im}(D) \subset p$, a contradiction.

As a consequence of the second case, $\pi(\pi^{-1}(Z_1))$ is not dense in Z_1. Since the dimension of X is one more than the dimension of Y, and Y is normal, a theorem of Chevalley [1] implies that nonempty fibers over points of Z_1 must have dimension strictly greater than 1.

If Y is smooth at $y \in Z_1 \cap \operatorname{im}(\pi)$ then Theorem 2.1 shows that there are open neighborhoods V of y in Y and U of $\pi^{-1}(y)$ so that $\pi|_U : U \to V$ is smooth. But a smooth morphism is open and therefore has dense image in Z_1. Since the image of π is not dense in Z_1 no such y exists. $\square$

4. Extendibility to a coordinate system

In [16], Rabier gives a simple algorithm to determine if $z \in \mathbf{C}[x, y]$ is a variable, i.e. if there is an f with $\mathbf{C}[z, f] = \mathbf{C}[x, y]$. In [17, Cor. p. 160], a criterion for an element to be a variable in $\mathbf{C}[x, y, z]$ is given, and in [12], van den Essen gives an algorithm to determine if two elements f, g are part of a coordinate system for $\mathbf{C}^3$. We say that a set $\{f_1, \ldots, f_{n-1}\} \subset \mathbf{C}^{[n]}$ is part of a coordinate system for $\mathbf{C}^n$ if f_n exists so that $\mathbf{C}[f_1, \ldots, f_{n-1}, f_n] = \mathbf{C}^{[n]}$. We extend the method in [12] to give an algorithm to decide whether $n-1$ polynomials are part of a coordinate system for $\mathbf{C}^n$. The algorithm is based on the following theorem:

Theorem 4.1. *A set of polynomials $\{y_1, \ldots, y_{n-1}\}$ is part of a coordinate system for $\mathbf{C}^n$ if and only if $\mathbf{C}[y_1, \ldots, y_{n-1}]$ is the ring of invariants for a proper G_a action on $\mathbf{C}^n$. In this case, the action is generated by the derivation*

$$D : h \mapsto \lambda \det \operatorname{Jac}(y_1, \ldots, y_{n-1}, h)$$

for some $\lambda \in \mathbf{C}^*$.

Proof. If $\mathbf{C}^{[n]} = \mathbf{C}[y_1, \ldots, y_{n-1}, w]$ then the derivation $D : y_i \mapsto 0$, $w \mapsto 1$ generates the desired G_a action. It is straightforward to verify that $D(h) = \lambda \partial h/\partial w$ where $\lambda = \det \operatorname{Jac}(y_1, \ldots, y_{n-1}, w) \in \mathbf{C}^*$.

Conversely, a proper G_a action is fixed point free [4, Theorem 2.3], so that Corollary 2.3 shows that the action is locally trivial. But since the ring of invariants is a polynomial ring, [6, Theorem 3.3] shows that the action is conjugate to a translation. Thus, with D denoting the derivation generating the action, there is an element $w \in \mathbf{C}^{[n]}$ with $D(w) = 1$. From [19, Proposition 2.1], it follows that $\mathbf{C}^{[n]} = \mathbf{C}[y_1, \ldots, y_{n-1}, w]$. □

This theorem yields an extension of the algorithm in [12] to decide if $\{y_1 \ldots, y_{n-1}\}$ is part of a coordinate system for $\mathbf{C}^n$. Given $\{y_1, \ldots, y_{n-1}\}$,

1. Define a derivation D on $\mathbf{C}^{[n]}$ by

$$D(\theta) = \det \mathrm{Jac}(\theta, y_1, \ldots, y_{n-1}).$$

If $(D(x_1), \ldots, D(x_n))$ is not the unit ideal of $\mathbf{C}^{[n]}$, then D cannot generate a fixed point free G_a action, and therefore $\{y_1, \ldots, y_{n-1}\}$ is not part of a coordinate system.

2. Check whether D is locally nilpotent. As in [12], calculate

$$N = \max_{1 \leq i \leq n} \{[\mathbf{C}(x_1, \ldots, x_n) : \mathbf{C}(y_1, \ldots, y_{n-1}, x_i)]\}$$

using the algorithm in [15, Lemma 2.3]. D is locally nilpotent if and only if $D^N x_i = 0$ for each i. If D is not locally nilpotent, then $\{y_1, \ldots, y_{n-1}\}$ is not a part of a coordinate system.

3. Check that $\mathbf{C}[y_1, \ldots, y_{n-1}] = C_0$, the ring of invariants. While it is not known a priori that the ring of invariants for a fixed point free G_a action on $\mathbf{C}^n$ is finitely generated for $n > 3$, the algorithm in [11] can be modified to determine whether this ring is $\mathbf{C}[y_1, \ldots, y_{n-1}]$. By steps 1 and 2, $0 \neq D^r x_i \in C_0$ for some r, i. If $D^r x_i$ is not in $\mathbf{C}[y_1, \ldots, y_{n-1}]$, then $\mathbf{C}[y_1, \ldots, y_{n-1}] \neq C_0$ and $\{y_1, \ldots, y_{n-1}\}$ is not part of a coordinate system.

If $D^r x_i$ is in $\mathbf{C}[y_1 \ldots, y_{n-1}]$, set $s = D^{r-1} x_i / D^r x_i$, noting that $Ds = 1$. Following algorithm in [11] calculate $v_j = \exp(-zD) x_j|_{z=s}$ for each j. Set $c_j = [D^r(x_i)]^{n_j} v_j$ where n_j is the least exponent e for which $[D^r(x_i)]^e v_j \in C_0$. Use the subalgebra membership algorithm [2] to determine whether the $c_j \in \mathbf{C}[y_1, \ldots, y_{n-1}]$. If any c_j does not lie in $\mathbf{C}[y_1, \ldots, y_{n-1}]$, then $\{y_1, \ldots, y_{n-1}\}$ is not part of a coordinate system. If $c_j \in \mathbf{C}[y_1, \ldots, y_{n-1}]$ for each j, then

$$\mathbf{C}[y_1, \ldots, y_{n-1}] \subset C_0 \subset \mathbf{C}\left[y_1, \ldots, y_{n-1}, \frac{1}{D^r(x_i)}\right].$$

Assuming these inclusions, the algorithm in [11] constructs an increasing chain of subrings of C_0 beginning with $\mathbf{C}[y_1, \ldots, y_{n-1}]$ which eventually reaches C_0 if this ring is finitely generated. For our purposes, it suffices to construct the first such subring [11]. If it is properly larger than $\mathbf{C}[y_1, \ldots, y_{n-1}]$ then $\{y_1, \ldots, y_{n-1}\}$ is not part of a coordinate system. Otherwise, $\mathbf{C}[y_1, \ldots, y_{n-1}] = C_0$.

4. Check that the G_a action is proper. In view of [4, Theorem 2.8] this is equivalent to

$$t \in \mathbf{C}[x_1, \ldots, x_n, \exp(tD)x_1, \ldots, \exp(tD)x_n],$$

and requires one application of the subalgebra membership algorithm to decide.

If the action is not proper, then $\{y_1, \ldots, y_{n-1}\}$ is not part of a coordinate system. Otherwise, Theorem 4.1 shows that $\{y_1, \ldots, y_{n-1}\}$ is part of a coordinate system.

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