LOCALIZATION OF G-CW COMPLEXES AT A
SYSTEM OF PRIMES

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1. Introduction

Let $G$ be a compact Lie group. In [6] May, McClure and Triantafillou have studied the equivariant localization at $P$, a set of primes, of $G$-nilpotent based $G$-spaces. They treated the concept of a $G$-tower to construct the equivariant localization. Thereafter Yosimura [11,12] generalized it and its existence theorem for $G$-nilpotent based $G$-CW complexes using their methods. However since the inverse limit of $G$-CW complexes is generally not of the $G$-homotopy type of $G$-CW complexes, they used the $G$-CW approximation theorem (cf. [5], [9]). The purpose of this paper is to construct explicitly the equivariant localization after the manner of Mimura, Nishida and Toda [7]. Along this line, we generalize the notion of $P$-sequences to the equivariant one. Namely, our $(\phi, \Gamma)$-sequences are associated with an order preserving map $\phi$ from $\Gamma(G)$, the set of conjugacy classes of closed subgroups of $G$, into the set of sets of primes and a finite subset $\Gamma$ of $\Gamma(G)$. Thus our localization is a functor from the homotopy category $\mathcal{CW}_G^1$ of $G$-1-connected based $G$-CW complexes of $G$-finite type with finitely many orbit types into the homotopy category of based $G$-CW complexes with respect to the system of primes $\phi$.

This paper is organized as follows. In §2 we construct $(\phi, \Gamma)$-sequences. In §§3–4 we show the uniqueness of $(\phi, \Gamma)$-sequences. Finally in §5 we establish our localization at $\phi$ using $(\phi, \Gamma)$-sequences.

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2. Homotopy $(\phi, \Gamma)$-sequences

We denote by $\Gamma(G)$ the set consisting of conjugacy classes $(H)$ of the
closed subgroups $H$ of $G$ and by $\gamma(G)$ a collection of closed subgroups of $G$ containing precisely one subgroup from every conjugacy class in $\Gamma(G)$. For a $G$-space $X$, let $\Gamma(X)$ be the set of all the orbit types of $X$ which are the conjugacy classes of the isotropy groups of points in $X$ and $\gamma(X)$ the image of $\Gamma(X)$ under the canonical map $\Gamma(G) \to \gamma(G)$. A based $G$-$CW$ complex is a $G$-$CW$ complex with a base vertex which is left fixed by each element of $G$. A $G$-space $X$ is said to be $G$-1-connected if each $X^H = \{x \in X \mid h \cdot x = x \text{ for } h \in H\}$ is 1-connected for any closed subgroup $H$ of $G$. $\mathcal{CW}_0^G$ denotes the category of $G$-1-connected based $G$-$CW$ complexes $X$ such that $X$ has finitely many orbit types and $H_*(X^H)$ is finitely generated for any closed subgroup $H$ of $G$. Let $\Pi$ be the power set of all primes. We have partial orderings on $\Gamma(G)$, $(H) \leq (K)$ if $H$ is subconjugate to $K$, and on $\Pi$, $P \subseteq Q$ if $P$ is a subset of $Q$. $\phi: \Gamma(G) \to \Pi$ is said to be order preserving if $(H) \leq (K)$ implies $\phi(H) \subseteq \phi(K)$. Throughout this paper, for any finite subset of $\Gamma(G)$, we denote it by $\{(H_1), \ldots, (H_n)\}$ so that $(H_i) > (H_j)$ implies $i < j$.

Let $P$ be a set of primes. A space $X$ is said to be $P$-equivalent to $Y$ if there exists a map $f: X \to Y$ such that $f$ induces isomorphisms of homology groups with the coefficient $\mathbb{Z}/p\mathbb{Z}$, for any $p \in P$ and with the rational coefficient.

**Definition 2.1.** Let $\phi$ be a map from $\Gamma(G)$ into $\Pi$. A $G$-map $f: X \to Y$ is a $\phi$-equivalence, if $f^H: X^H \to Y^H$, restricting $f$ to $X^H$, is a $\phi(H)$-equivalence for any $H \leq G$. Then $X$ and $Y$ are called $\phi$-equivalent.

**Definition 2.2.** Let $\phi$ be a map from $\Gamma(G)$ into $\Pi$ and $\Gamma$ a subset of $\Gamma(G)$ containing $\Gamma(X)$. $\{X_i, f_i\}$ is a homotopy $(\phi, \Gamma)$-sequence of $X$, if

1. $\Gamma(X_i) \subseteq \Gamma$,
2. $f_i: X_{i-1} \to X_i$ is a $\phi$-equivalence with $X_0 = X$,
3. for any $n$, $i$, $(H) \in \Gamma$ and prime $q$ with $(q, \phi(H)) = 1$, there exists $N(i > n)$ such that
   $$\pi_n((f_n^H, \ldots, f_i^H)) \otimes \mathbb{Z}/q\mathbb{Z}: \pi_n(X_i^H) \otimes \mathbb{Z}/q\mathbb{Z} \to \pi_n(X_n^H) \otimes \mathbb{Z}/q\mathbb{Z}$$
   is a zero map.

We denote by $\mathcal{DCW}_0^G$ the subcategory consisting of finite dimensional $G$-$CW$ complexes in $\mathcal{CW}_0^G$.

**Lemma 2.3.** Let $\phi$ be an order preserving map from $\Gamma(G)$ into $\Pi$. If $X$ is a $G$-1-connected based $G$-$CW$ complex, then for any $(H) \in \Gamma(G)$, $j \geq 2$, and $q$ prime to $\phi(H)$, there exists a $G$-1-connected based $G$-$CW$ complex $Y = Y(X, H, j, q)$ and a based $G$-map $f: X \to Y$ such that

1. $\Gamma(Y) = \Gamma(X) \cup \{(H)\}$,
2. $f$ is a $\phi$-equivalence,
3. $\pi_j(f^H) \otimes \mathbb{Z}/q\mathbb{Z} = 0$. 


Further if $X$ is in $\mathcal{DCW}_G$, then $Y$ is in $\mathcal{DCW}_G$.

Proof. Let $\{\beta_i\} : S^j \to X^\mu$ be the generators of $\pi_j(X^\mu) \otimes \mathbb{Z}/q\mathbb{Z}$. We define two based $G$-maps as follows:

$$
\beta : Z \to X \quad \beta(gH \wedge z) = g(\vee \beta_i)(z), \\
\alpha : Z \to Z \quad \alpha(gH \wedge z) = gH \wedge \check{q}(z),
$$

where $Z = (G/H)^j \wedge (\vee S^j)$ and $\check{q} : \vee S^j \to \vee S^j$ is a map of degree $q$. Put $Y = Z \vee X$, which is the pushout of $G$-CW complexes constructed as in [7].

$$
\begin{array}{ccc}
Z & \xrightarrow{\alpha} & Z \\
\downarrow{\beta} & & \downarrow{\beta} \\
X & \xrightarrow{f} & Y
\end{array}
$$

Since $j \geq 2$, $Y$ is a $G$-1-connected based $G$-CW complex with $\Gamma(Y) = \Gamma(X) \cup \{(H)\}$. Note that $Y^K = Z^K \vee X^K$ for any $K \leq G$ and $f^K$ is a homotopy equivalence unless $(K) \leq (H)$. The conditions follow from the elementary properties of the pushout diagram.

**Theorem 2.4.** Let $\phi$ be an order preserving map from $\Gamma(G)$ into $\Pi$ and $\Gamma$ a finite subset of $\Gamma(G)$ containing $\Gamma(X)$. If $X$ is a $G$-1-connected based $G$-CW complex with finitely many orbit types, then there exists a homotopy $(\phi, \Gamma)$-sequence $\{X_i, f_i\}$ of $X$ such that if $X$ is in $\mathcal{DCW}_G$, then each $X_i$ is in $\mathcal{DCW}_G$.

Proof. Put $M = (N - \{1\}) \times (N - \{1\}) \cup \{(0, 0)\}$. We define an order on $M$ by $(a, i) < (b, j)$ if $a + i < b + j$, or $a + i = b + j$ and $a < b$. Assume that there exists a sequence of $\phi$-equivalences $X = X_{(0,0)} \to X_{(2,2)} \to X_{(3,2)} \to \cdots \to X_{(m,k)}$ such that for any $(l', r')$ and $(l, r)$ in $M$ with $(l', r') < (l, r) \leq (m, k)$,

1. $X_{(l', r')}$ is $G$-1-connected and its orbit type is the same as $\Gamma$ except for $X_{(0,0)}$,
2. for any $(H) \in \Gamma$,

$$
\pi_i(X_{(l', r')}) \to X_{(l, r)} \otimes \mathbb{Z}/r\mathbb{Z} = 0, \quad \text{if } (r, \phi(H)) = 1.
$$

Let $\Gamma = \{(H_1), \ldots, (H_s)\}$ and $(j, q)$ be the next to $(m, k)$ in $M$. We put $Y_0 = X_{(m, k)}$ and for $0 < i \leq r$,

$$
Y_i = \begin{cases} 
Y(Y_{i-1}, H_i, j, q) & \text{if } (q, \phi(H_i)) = 1, \\
Y_{i-1} & \text{otherwise}.
\end{cases}
$$

Then we take $X_{(j, q)} = Y_i$, which satisfies the conditions (1) and (2). The proof is completed.
3. The transmission of $P$-equivalences

Let $P$ be a set of primes and denote by $C_P$ the class of finite abelian groups without $P$-torsion. A homomorphism $f$ of abelian groups is called a mod $C_P$ isomorphism if the kernel and the cokernel of $f$ belong to $C_P$. We set $\Theta(K, L, A, B; M) = \{ g \in G \mid g^{-1}(aKa^{-1} \cup bLb^{-1}) \subseteq M \}$ for some $a \in A$ and $b \in B$ for closed subgroups $K, L, M$ of $G$ and closed subspaces $A, B$ of $G$. This set is empty unless $(K) \subseteq (M)$ or $(L) \subseteq (M)$. In this section we abuse $f$ with any restriction of a map $f$ and assume that any subspace of $G$ which we treat is closed.

**Lemma 3.1.** Let $X$ be a $G$-space, $K, L$ closed subgroups of $G$. Let $A$ be a subspace of $G$ and $\gamma$ a subset of $\gamma(G)$ containing $\gamma(X)$. Then $A \cdot X^K \cap X^L = \bigcup_{H \in G} \Theta(K, L, A, \{ \epsilon \}; H) \cdot X^H$, where $\epsilon$ is the unit element of $G$. In particular $A \cdot X^K = \bigcup_{H \in G} \Theta(K, A, \{ \epsilon \}; H) \cdot X^H$.

Proof. We denote by $X^{(H)}$ the subspace of $X$ consisting of points whose isotropy groups are $H$. Then $A \cdot X^K = A \cdot (\bigcup_{H \in G} G \cdot X^{(H)} \cap X^K) = \bigcup_{H \in G} A \cdot \{ g \in G \mid g^{-1}Kg \subseteq H \} \cdot X^{(H)} = \bigcup_{H \in G} \Theta(K, A, \{ \epsilon \}; H) \cdot X^{(H)}$. Since $\Theta(K, A, \{ \epsilon \}; H) \cdot X^H \cap G \cdot X^{(H)} \subseteq \Theta(K, A, \{ \epsilon \}; N) \cdot X^{(N)}$, we have $A \cdot X^K = \bigcup_{H \in G} \Theta(K, A, \{ \epsilon \}; H) \cdot X^H$. Similarly $A \cdot X^K \cap X^L = \bigcup_{H \in G} \Theta(K, L, A, \{ \epsilon \}; H) \cdot X^H$.

**Proposition 3.2.** Let $X$ be a $G$-space, $K, L$ closed subgroups of $G$. Let $A, B$ be subspaces of $G$ and $\gamma$ a subset of $\gamma(G)$ containing $\gamma(X)$. Then $A \cdot X^K \cap B \cdot X^L = \bigcup_{H \in G} \Theta(K, L, A, B; H) \cdot X^H$.

**Theorem 3.3.** Let $f: X \to Y$ be a $G$-map between $G$-CW complexes with finitely many orbit types. Suppose that $f_*: H_*(X^H) \to H_*(Y^H)$ is a mod $C_P$ isomorphism for any $H$ in $\gamma(X) \cup \gamma(Y)$. Then any closed subspaces $A_1, \ldots, A_r$ of $G$ and any closed subgroups $K_1, \ldots, K_r$ of $G$ (for any $r$),

$$f_*: H_*(\bigcup_{i=1}^r A_i \cdot X^{K_i}) \to H_*(\bigcup_{i=1}^r A_i \cdot Y^{K_i})$$

is a mod $C_P$ isomorphism.

Proposition 3.2 means that Theorem 3.3 implies that if $f_*: H_*(X^H) \to H_*(Y^H)$ is a mod $C_P$ isomorphism for any $H$ in $\gamma(X) \cup \gamma(Y)$, then $f_*: H_*(X^K) \to H_*(Y^K)$ is a mod $C_P$ isomorphism for any $K \subseteq G$. We need some lemmas to show the above theorem.

We set $X^{>K} = \{ x \in X \mid G_x \nsubseteq K \}$ and $X^{>(K)} = G \cdot X^{>K} = \{ x \in X \mid (G_x \nsubseteq (K) \} for any $K \subseteq G$. Note that $A \cdot X^{>K} = \bigcup_{H \in G} \Theta(K, H, A, G; H) \cdot X^H$.

**Lemma 3.4.** Let $X$ be a $G$-CW complex and $A$ a closed subspace of $G$. ...
Then for any $K \leq G$, $(A \cdot X^K, A \cdot X^K \cap X^{> (K)})$ is an NDR-pair.

Proof. Since $X$ is compactly generated and $G \cdot X^K$ is closed in $X$, $G \cdot X^K$ is compactly generated. Since $A$ is closed in $G$, $A \cdot X^K$ is closed in $G \cdot X^K$ and $A \cdot X^K$ is compactly generated (See [8]). We denote by $(A \cdot X^K)_a$ the union of $A \cdot X^K \cup X^{> (K)}$ and $G$-cells of $G$-dimension $\leq n$ in $G \cdot X^{< K}$ and put $(A \cdot X^K)_a = (G \cdot X^K)_a \cap A \cdot X^K$. Then $(\bigcup A \cdot NK/K \times D^*_\gamma, \bigcup A \cdot NK/K \times S^{* - 1})$ and $((A \cdot X^K)_a, (A \cdot X^K)_{n-1} \cup X^{> (K)})$ are relatively homeomorphic, where $NK$ is the normalizer of $K$ in $G$, and $D^*_\gamma$ and $S^{* - 1}$ be copies of the $n$-disk and $(n - 1)$-sphere respectively. Since the former is an NDR-pair, so is the latter.

The next proposition is due to Proposition 2 in [6].

**Proposition 3.5.** Let $f: Y \rightarrow Z$ be a $G$-map between $G$-spaces and $P$ a set of primes. If $H_\ast(f \I L)$ is a mod $C_P$ isomorphism for any $L \leq G$, then also $H_\ast(f/M): H_\ast(Y/M) \rightarrow H_\ast(Z/M)$ is a mod $C_P$ isomorphism for any $M \leq G$.

Now we start to prove Theorem 3.3. By Proposition 3.2, we can assume that $r = n$ and $L_i = H_i$, where $\gamma(X) \cup \gamma(Y) = \{H_1, \ldots, H_s\}$. We show the assertion by induction on the maximal number of the suffixes of $H_i$; assuming that the assertion is true for any $j_1, \ldots, j_i$ with $j_1 < \cdots < j_i < s$, we shall show that the assertion is true for any $j_1, \ldots, j_q$ with $j_1 < \cdots < j_q < s + 1$. First we shall show that $f_\ast: H_\ast(A \cdot X^H) \rightarrow H_\ast(A \cdot Y^H)$ is a mod $C_P$ isomorphism for any closed subspace $A$ of $G$. Put $A = A \cdot NH$, $H = H_\ast$ for short. By Proposition 3.2 and our assumption, $f_\ast: H_\ast(X^H) \rightarrow H_\ast(Y^H)$ and $f_\ast: H_\ast(A \cdot X^H) \rightarrow H_\ast(A \cdot Y^H)$ are mod $C_P$ isomorphisms. We consider $A \times X^H$ as an $NH$-space via the $NH$-action $n \cdot (a, x) = (a \cdot n^{-1}, n \cdot x)$. Then by Proposition 3.5 $(1 \times f)_\ast: H_\ast(A \times X^H) \rightarrow H_\ast(A \times Y^H)$ is a mod $C_P$ isomorphism. Similarly $(1 \times f)_\ast: H_\ast(A \times X^{> H}) \rightarrow H_\ast(A \times Y^{> H})$ is a mod $C_P$ isomorphism. Since $A \cdot X^{< H}$ is homeomorphic to $A \times X^{< H}$, $H_\ast(A \cdot X^{< H})$ is isomorphic to $H_\ast(A \cdot X^H, A \cdot X^{> H})$ and $f_\ast: H_\ast(A \cdot X^H, A \cdot X^{> H}) \rightarrow H_\ast(A \cdot Y^H, A \cdot Y^{> H})$ is a mod $C_P$ isomorphism. Thus so is $f_\ast: H_\ast(A \cdot X^H) \rightarrow H_\ast(A \cdot Y^H)$. Let $j_1, \ldots, j_q$ be any integers with $j_1 < \cdots < j_q < s + 1$ (for any $q$). By comparing two Mayer-Vietoris exact sequences for $X$ and $Y$, we obtain that $f_\ast: H_\ast(\bigcup_{i=1}^m A_i \cdot X^{H_i}) \rightarrow H_\ast(\bigcup_{i=1}^m A_i \cdot Y^{H_i})$ is a mod $C_P$ isomorphism. This completes the proof.

**Corollary 3.6.** Let $f: X \rightarrow Y$ be a $G$-map between $G$-CW complexes with finitely many orbit types. Suppose that for any $H$ in $\gamma(X) \cup \gamma(Y)$, there exists a set of primes $P(H)$ such that $f^H$ is a $P(H)$-equivalence. If $\phi$ is the map from $\Gamma(G)$ into $\Pi$ defined by $\phi(K) = \bigcap_{(H \leq K) \in \gamma} P(H)$, then $f$ is a $\phi$-equivalence.
In the same manner as the proof of Theorem 3.3, we have

**Proposition 3.7.** Let $X$ be a $G$-CW complex with finitely many orbit types. If $H_\bullet(X^K)$ is finitely generated for any $H \in \gamma(X)$, then $H_\bullet(X^K)$ is finitely generated for any $K \leq G$.

4. Uniqueness of the $(\phi, \Gamma)$-sequences

**Lemma 4.1.** Let $(Y, X)$ be a $G$-CW pair with $\Gamma(Y-X) = \{(H)\}$, $Z$ a $G$-space and $f: X \to Z$ a $G$-map. $f$ can be extended over $Y$ as a $G$-map if and only if $f^H: X^H \to Z^H$ can be extended over $Y^H$ as a WH-map.

This proof is quite obvious and omitted here.

**Definition 4.2.** Let $\{X_i, f_i\}$ and $\{Y_i, h_i\}$ be homotopy $(\phi, \Gamma)$-sequences of $X$ and $Y$ respectively, and $k: X \to Y$ a based $G$-map. A **morphism** $\{k_i\}$ from $\{X_i, f_i\}$ into $\{Y_i, h_i\}$ covering $k$ is defined as follows: For any $i$, there exist $\sigma(i) (\geq \sigma(i-1))$ and $G$-maps $k_i: X_i \to Y_{\tau(i)}$ such that $k_0 = k$, and $k_{i+1} \circ f_{i+1}$ and $\sigma(i+1,0) \circ k_i$ is $G$-homotopic, where $k_{i+1} \circ f_{i+1} = \sigma(i+1) \circ k_i$.

**Definition 4.3.** Let $\{k_i\}$ and $\{k_i'\}$ be two morphisms between homotopy $(\phi, \Gamma)$-sequences: $\{X_i, f_i\} \to \{Y_i, h_i\}$. $\{k_i\}$ and $\{k_i'\}$ are said to be $G$-homotopic, if there exists a morphism $\{H_i\}: \{X_i \wedge I^+, f_i \wedge 1\} \to \{Y_i, h_i\}$ covering the $G$-homotopy $k \simeq k'$ such that

1. $H_i: X_i \wedge I^+ \to Y_{\tau(i)}$,
2. $\tau(i) \geq \max(\tau(i-1), \sigma(i), \sigma'(i))$,
3. $H_i(, 0) = k_i$ and $H_i(, 1) = k'_i$ in $Y_{\tau(i)}$,
4. $H_{i+1} \circ f_{i+1} \wedge 1 = h_{i+1,0} \circ H_i$ rel. $X_i \wedge I^+$.

**Lemma 4.4.** Let $A$ be a finite abelian group with the order $q$. If $f: B \to C$ be a homomorphism which induces a zero homomorphism $f \otimes 1: B \otimes \mathbb{Z}/q\mathbb{Z} \to C \otimes \mathbb{Z}/q\mathbb{Z}$, then $\text{Ext}(1, f): \text{Ext}(A, B) \to \text{Ext}(A, C)$ is a zero homomorphism.

Proof. Since $A$ is finite, $\text{Ext}(A, B)$ is isomorphic to $A \otimes B$ and so $\text{Ext}(1, f)$ is zero.

For $K \leq G$ we denote by $W_0 K$ the identity component of $WK = NK/K$.

**Theorem 4.5.** Let $\Gamma$ be a subset of $\Gamma(G)$ containing $\Gamma(X) \cup \Gamma(Y)$ and $\phi$ an order preserving map from $\Gamma(G)$ into $\Pi$. Let $\{X_i\}$ and $\{Y_i\}$ be $(\phi, \Gamma)$-sequences of $X$ and $Y$ in $C\Psi_b$ respectively. For any based $G$-map $k: X \to Y$, if $X$ is finite dimensional, then there exists a morphism $\{k_i\}: \{X_i\} \to \{Y_i\}$ between $(\phi, \Gamma)$-sequences covering $k$. Further it is unique up to $G$-homotopy.

Proof. Put $\Gamma(X_i) = \{(H_i), \cdots, (H_i)\}$. We assume that there exists a based
G-map $k_{i,j}: \bigcup_{j=1}^{j} G \cdot X_i^{H} \cup X_{i-1} \rightarrow Y_{\sigma(i,j)}$ extending $k_{i-1}$ and $k_{i-1}$. Put $L = H_{i+1}$ and $Z = X_{i}^{L} \cup X_{i-1}^{L}$ for short. Then the obstruction to extend $k_{i,j}$ over $X_{i}^{L}$ lies in $H_{*+1}^{*}(X_{i}^{L}, Z; \pi_{*}(\sigma_{(i,j)}))$ \([1]\), where $\pi_{*}$ is an $\mathcal{O}_{WL}$-group \([2]\) satisfying $\pi_{*}(A)(WL/K) = \pi_{*}(A^{K})$. There exists a functorial universal coefficients spectral sequence \([6]\) which converges to the above group and satisfies

$$E_{2}^{p,q} = \text{Ext}_{\mathcal{O}_{WL}}(H_{*}(X_{i}^{L}, Z), \pi_{*}(\sigma_{(i,j)})),$$

where $H_{*}$ is an $\mathcal{O}_{WL}$-group satisfying $H_{*}(A)(WL/K) = H_{*}(A^{K}/W_{0}K)$. Consider the exact sequence of the triad $(X_{i}^{L}, Z, X_{i}^{L})$.

$$\cdots \rightarrow H_{*}(X_{i}^{L}, Y_{i}^{L}) \rightarrow H_{*}(X_{i}^{L}, Z) \rightarrow H_{*+1}(Z, X_{i}^{L}) \rightarrow \cdots$$

For $(K) \geq (L)$, since $H_{*}(X_{i}^{L}, X_{i}^{L}) \in C_{*}(K)$ and $\phi(K) \supseteq \phi(L)$, $H_{*}(X_{i}^{K}, X_{i}^{K}) \in C_{*}(L)$. By Corollary 3.6 we obtain that the below group of the above diagram is in $C_{*}(L)$. Hence for any $K \subseteq WL$, $H_{*}(X_{i}^{L}, Z)^{K} \subseteq C_{*}(L)$ and by Proposition 3.5 $H_{*}(X_{i}^{L}, Z)(WL/K) \subseteq C_{*}(L)$. Then there exists $\sigma(i,j+1)$ such that $\pi_{*}(\sigma_{(i,j+1)}) \otimes Z/qZ \rightarrow \pi_{*}(\sigma_{(i,j+1)}) \otimes Z/qZ$ is a zero homomorphism for the order $q$ of $H_{*}(X_{i}^{L}, Z)(WL/eL)$. By Lemma 4.4

$$\text{Ext}_{\mathcal{O}_{WL}}(H_{*}(X_{i}^{L}, Z), \pi_{*}(\sigma_{(i,j+1)})) \rightarrow \text{Ext}_{\mathcal{O}_{WL}}(H_{*}(X_{i}^{L}, Z), \pi_{*}(\sigma_{(i,j+1)}))$$

is a zero homomorphism (Note that $H_{*}(X_{i}^{L}, Z)^{e} = 0$ if $K \neq eL$). Hence the obstruction is vanished and $k_{i,j}$ can be extended over $X_{i}^{L} \cup X_{i-1}^{L}$ as a $WL$-map, and $k_{i,j}$ can be extended over $\bigcup_{j=1}^{j+1} G \cdot X_{i}^{H} \cup X_{i-1}$. Then we may take $\sigma(i) = \sigma(i, r)$ and $k_{i} = k_{i,r}$.

For two morphisms $\{k_{i}\}, \{k'_{i}\}$ covering $k$, $H_{0}: X_{0} \cap I^{+} \rightarrow Y_{0}$ is given by $H_{0}(x, t) = k(x)$ and $H_{1}: \bigcup_{i=1}^{i=1} G \cdot X_{i}^{H} \cup X_{i-1} \rightarrow Y_{\sigma(i)}$ is defined by $k_{i}$ and $k'_{i}$. By making use of the above method, we have a homotopy combining $\{k_{i}\}$ and $\{k'_{i}\}$. This completes the proof.

**Definition 4.6.** $\{X_{i}, f_{i}\}$ is $G$-homotopy equivalent to $\{Y_{i}, h_{i}\}$, if there exist morphisms $k_{i}: \{X_{i}, f_{i}\} \rightarrow \{Y_{i}, h_{i}\}$ and $k'_{i}: \{Y_{i}, h_{i}\} \rightarrow \{X_{i}, f_{i}\}$ such that morphisms $k_{i} \circ k'_{i}$ and $k'_{i} \circ k_{i}$ cover $1_{X}$ and $1_{Y}$ respectively.

By Theorem 4.5 we immediately have

**Corollary 4.7.** Let $\Gamma$ be a subset of $\Gamma(G)$ containing $\Gamma(X)$ and $\phi$ an order preserving map from $\Gamma(G)$ into $\Pi$. Then a homotopy $(\phi, \Gamma)$-sequence $\{X_{i}\}$ of $X$ in $\varphi DCW_{0}$ is unique up to $G$-homotopy type.
5. Localization of \( G \)-CW complexes

Let \( X \) and \( Y \) be in \( C\Gamma(G)^\circ \), \( \Gamma \) a finite subset of \( \Gamma(G) \) containing \( \Gamma(X) \cup \Gamma(Y) \) and \( \phi \) an order preserving map from \( \Gamma(G) \) into \( \Pi \). The localization of \( X \) at \( (\phi, \Gamma) \), denoted by \( X_{(\phi, \Gamma)} \), is defined to be the based \( G \)-CW complex constructed by the "telescope construction" of the homotopy \( \phi \)-sequence \( \{X_i, f_i\} \) of \( X \), that is,

\[
X_{(\phi, \Gamma)} = \bigvee_i (X_i \wedge I^+) / (f_i \circ (\mathbb{1}, 1) - (f_{i+1}(x), 0)).
\]

By Theorem 4.5 a \( G \)-map \( f: X \to Y \) induces a \( G \)-map \( f_{(\phi, \Gamma)}: X_{(\phi, \Gamma)} \to Y_{(\phi, \Gamma)} \), which is unique up to \( G \)-homotopy. By Corollary 4.7, \( X_{(\phi, \Gamma)} \) is determined uniquely up to \( G \)-homotopy type. Now let \( X \) be in \( C\Gamma(G)^\circ \). \( X_{(\phi, \Gamma)} \) is uniquely determined up to \( G \)-homotopy type, where \( X^\ast \) is the \( G \)-\( n \)-skeleton of \( X \). Also there is a natural \( G \)-map \( (X^\ast)_{(\phi, \Gamma)} \to (X^{n+1})_{(\phi, \Gamma)} \) induced from the inclusion \( X^\ast \to X^{n+1} \). Then we put \( X_{(\phi, \Gamma)} = \lim_{\to} (X^\ast)_{(\phi, \Gamma)} \), which is determined uniquely up to \( G \)-homotopy type. If \( f: X \to Y \) be a \( G \)-cellular map, then it induces \( (f^\ast)_{(\phi, \Gamma)} \), which is unique up to \( G \)-homotopy. Thus we obtain a \( G \)-map \( f_{(\phi, \Gamma)}: X_{(\phi, \Gamma)} \to Y_{(\phi, \Gamma)} \), extending \( f \).

Here we see a relation between our localizations. If \( \phi(K) \subset \eta(K) \) for any \( (K) \in \Gamma(G) \), then we write \( \phi \subset \eta \).

**Proposition 5.1.** Let \( X \) be in \( C\Gamma(G)^\circ \), \( \Gamma \), \( \mathcal{T} \) and \( \Delta \) finite subsets of \( \Gamma(G) \) containing \( \Gamma(X) \), and \( \phi \), \( \eta \) and \( \mu \) order preserving maps from \( \Gamma(G) \) into \( \Pi \).

1. If \( \phi \subset \eta \) then there exists a \( \phi \)-equivalence \( j_{(\phi, \eta, \Gamma)}: X_{(\eta, \Gamma)} \to X_{(\phi, \Gamma)} \).
2. If \( \Gamma \subset \mathcal{T} \), then there exists a \( \phi \)-equivalence \( j_{(\phi, \mathcal{T}, \Gamma)}: X_{(\phi, \Gamma)} \to X_{(\phi, \mathcal{T})} \).
3. If \( \eta(H) \) is the set of all primes for any \( (H) \in \Gamma \), then \( X_{(\eta, \Gamma)} = X \) and \( j_{(\phi, \eta, \Gamma)} \) coincides with the canonical inclusion.
4. For \( \phi \subseteq \mu \subseteq \eta \), \( j_{(\phi, \mu, \Gamma)} \circ j_{(\mu, \eta, \Gamma)} = j_{(\phi, \eta, \Gamma)} \).
5. For \( \Gamma \subset \mathcal{T} \subset \Delta \), \( j_{(\phi, \mathcal{T}, \Delta)} \circ j_{(\phi, \Gamma, \mathcal{T})} = j_{(\phi, \Gamma, \Delta)} \).

**Proof.** (3) is clear from our construction. Otherwise, we may consider the obstruction theory appeared in the proof of Theorem 4.5.

Choose a bijection \( x: \mathcal{N} \to \Gamma(G) \) such that \( x(1) = (G) \). We define finite subsets \( \Gamma_n \) of \( \Gamma(G) \) by \( \Gamma_1 = \Gamma(X) \) and \( \Gamma_n = \Gamma_{n-1} \cup \{x(n)\} \). We put \( X_\phi = \lim_{\to} X_{(\phi, \Gamma_n)} \) and \( f_\phi = \lim_{\to} f_{(\phi, \Gamma_n)} \) for a \( G \)-map \( f: X \to Y \). Then we have

**Theorem 5.2.** Let \( \phi \), \( \eta \) and \( \mu \) be order preserving maps from \( \Gamma(G) \) into \( \Pi \) with \( \phi \subset \eta \subset \mu \).

1. For \( X \) in \( C\Gamma(G)^\circ \), there exists a localization \( X_\phi \), which is determined uniquely up to \( G \)-homotopy type, and a \( \phi \)-equivalence \( j_X: X \to X_\phi \).
2. There exists a \( \phi \)-equivalence \( j_{\phi, \eta}: X_\eta \to X_\phi \) which coincides with \( j_X \) if \( \eta(K) \subset \phi(K) \subset (K) \in \Gamma(G) \).
is the set of all primes for any \((K) \in \Gamma(G)\), and satisfies that \(j_\phi \circ j_\mu = j_\phi \mu\).

(3) For a based \(G\)-map \(f: X \to Y\) in \(CW_G\), there exists a based \(G\)-map \(f_\phi: X_\phi \to Y_\phi\), unique up to \(G\)-homotopy, such that the following diagram is \(G\)-homotopy commutative.

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow j_X & & \downarrow j_Y \\
X_\phi & \xrightarrow{f_\phi} & Y_\phi
\end{array}
\]

Next we shall study some elementary properties of our localizations. Let \(H^*(X)\) and \(Z_\phi\) be \((\cdot)\)-\(G\)-groups defined by \(H_*(X; Z)\) and \(Z_\phi(G/H) = Z_\phi(\phi(H))\), the integer localized at \(\phi(H)\). The proof of the next proposition is analogous to that of Theorem 2.5 [7].

**Proposition 5.3.** Let \(X\) be in \(CW_G\) and \(\phi\) an order preserving map from \(\Gamma(G)\) into \(\Pi\).

1. \(H_*(X_\phi) = H_*(X) \otimes Z_\phi\). Moreover \((j_X)_*\) is equivalent to \(1 \otimes \iota: H_*(X) \to H_*(X) \otimes Z_\phi\), where \(\iota\) is the natural inclusion.

2. \(\pi_*(X_\phi) = \pi_*(X) \otimes Z_\phi\). Moreover \((j_X)_*\) is equivalent to \(1 \otimes \iota: \pi_*(X) \to \pi_*(X) \otimes Z_\phi\).

By the equivariant version of Whitehead Theorem ([5]) obviously we obtain the following proposition.

**Proposition 5.4.** Let \(X\) be in \(CW_G\) and \(Y\) a based \(G\)-CW complex. Let \(\phi\) be an order preserving map from \(\Gamma(G)\) into \(\Pi\). If there exists a based \(G\)-map \(f: Y \to X\) which induces an isomorphism \(\pi_*(Y^K) \to \pi_*(X^K) \otimes Z_\phi(K)\) for any \((K) \in \Gamma(G)\), then \(X_\phi\) is the same \(G\)-homotopy type as \(Y\).

**Theorem 5.5.** Let \(\phi\) be an order preserving map from \(\Gamma(G)\) into \(\Pi\). The localization at \(\phi\) has the following properties:

1. The correspondence \(X \to X_\phi\) is a functor from the homotopy category \(CW_G\) to the homotopy category of \(G\)-1-connected based \(G\)-CW complexes.

2. A based \(G\)-map \(f: X \to Y\) in \(CW_G\) is a \(\phi\)-equivalence if and only if \(f_\phi\) is a \(G\)-homotopy equivalence.

A \(G\)-space \(Z\) is called \(\phi\)-local if \(\pi_*(Z)\) is a \(Z_\phi\)-module. By the obstruction theory, we easily see the following.

**Theorem 5.6.** Let \(Z\) be any \(\phi\)-local \(G\)-space. If \(f: X \to Y\) is a \(\phi\)-equivalence between \(G\)-CW complexes \(X\) and \(Y\), then \(f^*: [Y, Z]_G \to [X, Z]_G\) is a bijection.
Corollary 5.7. For $\phi \subset \eta, j_{\phi, \eta}* : [X_\phi, X_\phi]_G \rightarrow [X_\eta, X_\phi]_G$ is a bijection. In particular $j_x* : [X_\phi, X_\phi]_G \rightarrow [X, X_\phi]_G$ is a bijection.

Corollary 5.8. If $X$ is in $C^{W_1}_G$, then an arbitrary $G$-map $f : X_\eta \rightarrow Y_\eta$ induces $f_\phi : X_\phi \rightarrow Y_\phi$ such that the following diagram commutes up to $G$-homotopy:

$$
\begin{array}{ccc}
X_\eta & \xrightarrow{f} & Y_\eta \\
\downarrow j_{\phi, \eta} & & \downarrow j_{\phi, \eta} \\
X_\phi & \xrightarrow{f_\phi} & Y_\phi
\end{array}
$$

We may consider both $(X_\phi)_\eta$ and $(X_\eta)_\phi$ as $X_\phi \times Y_\phi$. For $X, Y$ in $C^{W_1}_G$ we consider the $(G \times G)$-$CW$ complex $X \times Y$ as a $G$-space via the diagonal $G$-action. By [3] this $G$-space has a $G$-homotopy type of $G$-$CW$ complexes and might admit infinitely many orbit types. But we may consider the localization of it at $\phi$ as $X_\phi \times Y_\phi$. When $P$ is a set of primes (a constant system), the $(G \times G)$-localization $(X \times Y)_P$ is $(G \times G)$-homotopy equivalent to $X_P \times Y_P$.

References


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