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LOCALIZATION OF G-CW COMPLEXES AT A SYSTEM OF PRIMES

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1. Introduction

Let G be a compact Lie group. In [6] May, McClure and Triantafillou have studied the equivariant localization at P, a set of primes, of G-nilpotent based G-spaces. They treated the concept of a G-tower to construct the equivariant localization. Thereafter Yosimura [11,12] generalized it and its existence theorem for G-nilpotent based G-CW complexes using their methods. However since the inverse limit of G-CW complexes is generally not of the G-homotopy type of G-CW complexes, they used the G-CW approximation theorem (cf. [5], [9]). The purpose of this paper is to construct explicitly the equivariant localization after the manner of Mimura, Nishida and Toda [7]. Along this line, we generalize the notion of *P*-sequences to the equivariant one. Namely, our (ϕ, Γ) -sequences are associated with an order preserving map ϕ from $\Gamma(G)$, the set of conjugacy classes of closed subgroups of G, into the set of sets of primes and a finite subset Γ of $\Gamma(G)$. Thus our localization is a functor from the homotopy category CW^1_G of G-1-connected based G-CW complexes of G-finite type with finitely many orbit types into the homotopy category of based G-CW complexes with respect to the system of primes ϕ .

This paper is organized as follows. In §2 we construct (ϕ, Γ) -sequences. In §§3-4 we show the uniqueness of (ϕ, Γ) -sequences. Finally in §5 we establish our localization at ϕ using (ϕ, Γ) -sequences.

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2. Homotopy (ϕ, Γ) -sequences

We denote by $\Gamma(G)$ the set consisting of conjugacy classes (H) of the

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closed subgroups H of G and by $\gamma(G)$ a collection of closed subgroups of G containing precisely one subgroup from every conjugacy class in $\Gamma(G)$. For a G-space X, let $\Gamma(X)$ be the set of all the orbit types of X which are the conjugacy classes of the isotropy groups of points in X and $\gamma(X)$ the image of $\Gamma(X)$ under the canonical map $\Gamma(G) \rightarrow \gamma(G)$. A based G-CW complex is a G-CW complex with a base vertex which is left fixed by each element of G. A G-space X is said to be G-1-connected if each $X^H = \{x \in X \mid h \cdot x = x \text{ for } h \in H\}$ is 1-connected for any closed subgroup H of G. CW_G^1 denotes the category of G-1-connected based G-CW complexes X such that X has finitely many orbit types and $H_*(X^H)$ is finitely generated for any closed subgroup H of G. Let Π be the power set of all primes. We have partial orderings on $\Gamma(G)$, $(H) \leq (K)$ if H is subconjugate to K, and on Π , $P \subset Q$ if P is a subset of Q. $\phi: \Gamma(G) \rightarrow \Pi$ is said to be order preserving if $(H) \leq (K)$ implies $\phi(H) \subset \phi(K)$. Throughout this paper, for any finite subset of $\Gamma(G)$, we denote it by $\{(H_1), \dots, (H_n)\}$ so that $(H_i) > (H_j)$ implies i < j.

Let P be a set of primes. A space X is said to be P-equivalent to Y if there exists a map $f: X \to Y$ such that f induces isomorphisms of homology groups with the coefficient $\mathbb{Z}/p\mathbb{Z}$, for any $p \in P$ and with the rational coefficient.

DEFINITION 2.1. Let ϕ be a map from $\Gamma(G)$ into Π . A G-map $f: X \to Y$ is a ϕ -equivalence, if $f^H: X^H \to Y^H$, restricting f to X^H , is a $\phi(H)$ -equivalence for any $H \leq G$. Then X and Y are called ϕ -equivalent.

DEFINITION 2.2. Let ϕ be a map from $\Gamma(G)$ into Π and Γ a subset of $\Gamma(G)$ containing $\Gamma(X)$. $\{X_i, f_i\}$ is a homotopy (ϕ, Γ) -sequence of X, if

- (1) $\Gamma(X_i) \subset \Gamma$,
- (2) $f_i: X_{i-1} \rightarrow X_i$ is a ϕ -equivalence with $X_0 = X$,
- (3) for any *n*, *i*, $(H) \in \Gamma$ and prime *q* with $(q, \phi(H)) = 1$, there exists N(>i) such that

$$\pi_n((f_N, \dots, f_i)^H) \otimes \mathbb{1}_{\mathbb{Z}/q\mathbb{Z}} : \pi_n(X_{i-1}^H) \otimes \mathbb{Z}/q\mathbb{Z} \to \pi_n(X_N^H) \otimes \mathbb{Z}/q\mathbb{Z}$$

is a zero map.

We denote by \mathcal{FDCW}_{G}^{1} the subcategory consisting of finite dimensional G-CW complexes in CW_{G}^{1} .

Lemma 2.3. Let ϕ be an order preserving map from $\Gamma(G)$ into Π . If X is a G-1-connected based G-CW complex, then for any $(H) \in \Gamma(G)$, $j \ge 2$, and q prime to $\phi(H)$, there exists a G-1-connected based G-CW complex Y =Y(X, H, j, q) and a based G-map $f: X \rightarrow Y$ such that

- (1) $\Gamma(Y) = \Gamma(X) \cup \{(H)\},\$
- (2) f is a ϕ -equivalence,
- (3) $\pi_j(f^H) \otimes \mathbb{1}_{\mathbb{Z}/q\mathbb{Z}} = 0.$

Further if X is in \mathfrak{FDCW}^1_G , then Y is in \mathfrak{FDCW}^1_G .

Proof. Let $\{\beta_i\}: S^j \to X^H$ be the generators of $\pi_j(X^H) \otimes \mathbb{Z}/q\mathbb{Z}$. We define two based G-maps as follows:

$$\begin{aligned} \beta \colon Z \to X & \beta(gH \wedge z) = g(\vee \beta_i)(z) \,, \\ \alpha \colon Z \to Z & \alpha(gH \wedge z) = gH \wedge \hat{q}(z) \,, \end{aligned}$$

where $Z = (G/H)^+ \land (\lor S^j)$ and $\hat{q}: \lor S^j \to \lor S^j$ is a map of degree q. Put $Y = Z \lor X$, which is the pushout of G-CW complexes constructed as in [7].

$$\begin{array}{cccc}
z & \xrightarrow{\alpha} & z \\
\downarrow \beta & \downarrow \\
x & \xrightarrow{f} & Y
\end{array}$$

Since $j \ge 2$, Y is a G-1-connected based G-CW complex with $\Gamma(Y) = \Gamma(X) \cup \{(H)\}$. Note that $Y^{\kappa} = Z^{\kappa} \bigvee X^{\kappa}$ for any $K \le G$ and f^{κ} is a homotopy equivalence unless $(K) \le (H)$. The conditions follow from the elementary properties of the pushout diagram.

Theorem 2.4. Let ϕ be an order preserving map from $\Gamma(G)$ into Π and Γ a finite subset of $\Gamma(G)$ containing $\Gamma(X)$. If X is a G-1-connected based G-CW complex with finitely many orbit types, then there exists a homotopy (ϕ, Γ) -sequence $\{X_i, f_i\}$ of X such that if X is in \mathcal{FDCW}_G^1 , then each X_i is in \mathcal{FDCW}_G^1 .

Proof. Put $M=(N-\{1\})\times(N-\{1\})\cup\{(0,0)\}$. We define an order on M by (a,i)<(b,j) if a+i<b+j, or a+i=b+j and a<b. Assume that there exists a sequence of ϕ -equivalences $X=X_{(0,0)}\rightarrow X_{(2,2)}\rightarrow X_{(3,2)}\rightarrow \cdots \rightarrow X_{(m,k)}$ such that for any (l', r') and (l, r) in M with $(l', r')<(l, r)\leq (m, k)$,

- (1) $X_{(l,r)}$ is G-1-connected and its orbit type is the same as Γ except for $X_{(0,0)}$,
- (2) for any $(H) \in \Gamma$,

$$\pi_l(X_{(l',r')}^H \rightarrow X_{(l,r)}^H) \otimes \mathbb{Z}/r\mathbb{Z} = 0, \quad \text{if} \quad (r, \phi(H)) = 1.$$

Let $\Gamma = \{(H_1), \dots, (H_s)\}$ and (j, q) be the next to (m, k) in M. We put $Y_0 = X_{(m,k)}$ and for $0 < i \leq r$,

$$Y_i = \begin{cases} Y(Y_{i-1}, H_i, j, q) & \text{if } (q, \phi(H_i)) = 1, \\ Y_{i-1} & \text{otherwise.} \end{cases}$$

Then we take $X_{(j,q)} = Y_s$, which satisfies the conditions (1) and (2). The proof is completed.

3. The transmission of *P*-equivalences

Let P be a set of primes and denote by C_P the class of finite abelian groups without P-torsion. A homomorphism f of abelian groups is called a mod C_P isomorphism if the kernel and the cokernel of f belong to C_P . We set $\Theta(K, L, A, B; M) = \{g \in G | g^{-1}(aKa^{-1} \cup bLb^{-1})g \subset M \text{ for some } a \in A \text{ and } b \in B\}$ for closed subgroups K, L, M of G and closed subspaces A, B of G. This set is empty unless $(K) \leq (M)$ or $(L) \leq (M)$. In this section we abuse f with any restriction of a map f and assume that any subspace of G which we treat is closed.

Lemma 3.1. Let X be a G-space, K, L closed subgroups of G. Let A be a subspace of G and γ a subset of $\gamma(G)$ containing $\gamma(X)$. Then $A \cdot X^{\kappa} \cap X^{L} = \bigcup_{H \in \gamma} \Theta(K, L, A, \{e\}; H) \cdot X^{H}$, where e is the unit element of G. In particular $A \cdot X^{\kappa} = \bigcup_{H \in \gamma} \Theta(K, \{e\}, A, \{e\}; H) \cdot X^{H}$.

Proof. We denote by $X^{\langle H \rangle}$ the subspace of X consisting of points whose isotropy groups are H. Then $A \cdot X^{\kappa} = A \cdot (\bigcup_{H \in Y} G \cdot X^{\langle H \rangle} \cap X^{\kappa}) = \bigcup_{H \in Y} A \cdot \{g \in G \mid g^{-1}Kg \subset H\} \cdot X^{\langle H \rangle} = \bigcup_{H \in Y} \Theta(K, \{e\}, A, \{e\}; H) \cdot X^{\langle H \rangle}$. Since $\Theta(K, \{e\}, A, \{e\}; H) \cdot X^{H} \cap G \cdot X^{\langle N \rangle} \subset \Theta(K, \{e\}, A, \{e\}; N) \cdot X^{\langle N \rangle}$, we have $A \cdot X^{\kappa} = \bigcup_{H \in Y} \Theta(K, \{e\}, A, \{e\}; H) \cdot X^{H}$. Similarly $A \cdot X^{\kappa} \cap X^{L} = \bigcup_{H \in Y} \Theta(K, L, A, \{e\}; H) \cdot X^{H}$.

Proposition 3.2. Let X be a G-space, K, L closed subgroups of G. Let A, B be subspaces of G and γ a subset of $\gamma(G)$ containing $\gamma(X)$. Then $A \cdot X^{\kappa} \cap B \cdot X^{L} = \bigcup_{H \in \Upsilon} \Theta(K, L, A, B; H) \cdot X^{H}$.

Theorem 3.3. Let $f: X \to Y$ be a G-map between G-CW complexes with finitely many orbit types. Suppose that $f_*: H_*(X^H) \to H_*(Y^H)$ is a mod C_P isomorphism for any H in $\gamma(X) \cup \gamma(Y)$. Then any closed subspaces A_1, \dots, A_r of G and any closed subgroups K_1, \dots, K_r of G (for any r),

$$f_*\colon H_*(\bigcup_{i=1}^r A_i \cdot X^{\kappa_i}) \to H_*(\bigcup_{i=1}^r A_i \cdot Y^{\kappa_i})$$

is a mod C_P isomorphism.

Proposition 3.2 means that Theorem 3.3 implies that if $f_*: H_*(X^H) \to H_*(Y^H)$ is a mod \mathcal{C}_P isomorphism for any H in $\gamma(X) \cup \gamma(Y)$, then $f_*: H_*(X^K) \to H_*(Y^K)$ is a mod \mathcal{C}_P isomorphism for any $K \leq G$. We need some lemmas to show the above theorem.

We set $X^{>\kappa} = \{x \in X \mid G_x > K\}$ and $X^{>(\kappa)} = G \cdot X^{>\kappa} = \{x \in X \mid (G_x) > (K)\}$ for any $K \leq G$. Note that $A \cdot X^{>\kappa} = \bigcup_{\substack{H \in \Upsilon(X) \\ \langle K \rangle < \langle H \rangle}} \Theta(K, H, A, G; H) \cdot X^H$.

Lemma 3.4. Let X be a G-CW complex and A a closed subspace of G.

Then for any $K \leq G$, $(A \cdot X^{\kappa}, A \cdot X^{\kappa} \cap X^{>(\kappa)})$ is an NDR-pair.

Proof. Since X is compactly generated and $G \cdot X^{K}$ is closed in X, $G \cdot X^{K}$ is compactly generated. Since A is closed in G, $A \cdot X^{K}$ is closed in $G \cdot X^{K}$ and $A \cdot X^{K}$ is compactly generated (See [8]). We denote by $(G \cdot X^{K})_{n}$ the union of $A \cdot X^{K} \cup X^{>(K)}$ and G-cells of G-dimension $\leq n$ in $G \cdot X^{\langle K \rangle}$ and put $(A \cdot X^{K})_{n} = (G \cdot X^{K})_{n} \cap A \cdot X^{K}$. Then $(\coprod_{i} A \cdot NK/K \times D_{i}^{n}, \coprod_{i} A \cdot NK/K \times S_{i}^{n-1})$ and $((A \cdot X^{K})_{n}, (A \cdot X^{K})_{n-1} \cup X_{n}^{>(K)})$ are relatively homeomorphic, where NK is the normalizer of K in G, and D_{i}^{n} and S_{i}^{n-1} be copies of the *n*-disk and (n-1)-sphere respectively. Since the former is an NDR-pair, so is the latter.

The next proposition is due to Proposition 2 in [6].

Proposition 3.5. Let $f: Y \to Z$ be a G-map between G-spaces and P a set of primes. If $H_*(f^L)$ is a mod C_P isomorphism for any $L \leq G$, then also $H_*(f|M)$: $H_*(Y|M) \to H_*(Z|M)$ is a mod C_P isomorphism for any $M \leq G$.

Now we start to prove Theorem 3.3. By Proposition 3.2, we can assume that r=n and $L_i=H_i$, where $\gamma(X)\cup\gamma(Y)=\{H_1,\dots,H_n\}$. We show the assertion by induction on the maximal number of the suffixes of H_i ; assuming that the assertion is true for any j_1, \dots, j_t with $j_1 < \dots < j_t < s$, we shall show that the assertion is true for any j_1, \dots, j_q with $j_1 < \dots < j_q < s+1$. First we shall show that $f_*: H_*(A \cdot X^{H_s}) \rightarrow H_*(A \cdot Y^{H_s})$ is a mod \mathcal{C}_P isomorphism for any closed subspace A of G. Put $\tilde{A}=A\cdot NH_s$ and $H=H_s$ for short. By Proposition 3.2 and our assumption, $f_*: H_*(X^{>H}) \to H_*(Y^{>H})$ and $f_*: H_*(A \cdot X^{>H}) \to H_*(A \cdot Y^{>H})$ are mod \mathcal{C}_P isomorphisms. We consider $\tilde{A} \times X^H$ as an NH-space via the NH-action $n \cdot (a, x) = (a \cdot n^{-1}, n \cdot x).$ Then by Proposition 3.5 $(1 \underset{N^H}{\times} f)_* \colon H_*(\tilde{A} \underset{N^H}{\times} X^H) \to$ $H_*(\tilde{A} \underset{NH}{\times} Y^H)$ is a mod \mathcal{C}_P isomorphism. Similarly $(1 \underset{NH}{\times} f)_*: H_*(\tilde{A} \underset{NH}{\times} X^{>H}) \rightarrow$ $H_*(A \times Y^{>H})$ is a mod \mathcal{C}_P isomorphism. Since $A \cdot X^{\langle H \rangle}$ is homeomorphic to $\tilde{A} \underset{\scriptscriptstyle \nabla H}{\times} \tilde{X^{\langle H \rangle}}, H_*(A \cdot X^H, A \cdot X^{>H})$ is isomorphic to $H_*(\tilde{A} \underset{\scriptscriptstyle NH}{\times} X^H, \tilde{A} \underset{\scriptscriptstyle NH}{\times} X^{>H})$ and $f_*: H_*(A \cdot X^H, A \cdot X^{>H}) \rightarrow H_*(A \cdot Y^H, A \cdot Y^{>H})$ is a mod \mathcal{C}_P isomorphism. Thus so is $f_*: H_*(A \cdot X^H) \to H_*(A \cdot Y^H)$. Let j_1, \dots, j_q be any integers with $j_1 < \dots < j_q < s+1$ (for any q). By comparing two Mayer-Vietoris exact sequences for X and Y, we obtain that $f_*: H_*(\bigcup_{i=1}^q A_i \cdot X^{H_j}) \to H_*(\bigcup_{i=1}^q A_j \cdot Y^{H_j})$ is a mod \mathcal{C}_P isomorphism. This completes the proof.

Corollary 3.6. Let $f: X \to Y$ be a G-map between G-CW complexes with finitely many orbit types. Suppose that for any H in $\gamma(X) \cup \gamma(Y)$, there exists a set of primes P(H) such that f^H is a P(H)-equivalence. If ϕ is the map from $\Gamma(G)$ into Π defined by $\phi(K) = \bigcap_{\substack{(B) \in \Gamma(X) \cup \Gamma(Y) \\ (B) \subset K}} P(H)$, then f is a ϕ -equivalence. T. SUMI

In the same manner as the proof of Theorem 3.3, we have

Proposition 3.7. Let X be a G-CW complex with finitely many orbit types. If $H_*(X^H)$ is finitely generated for any $H \in \gamma(X)$, then $H_*(X^K)$ is finitely generated for any $K \leq G$.

4. Uniqueness of the (ϕ, Γ) -sequences

Lemma 4.1. Let (Y, X) be a G-CW pair with $\Gamma(Y-X) = \{(H)\}, Z$ a G-space and $f: X \rightarrow Z$ a G-map. f can be extended over Y as a G-map if and only if $f^{H}: X^{H} \rightarrow Z^{H}$ can be extended over Y^{H} as a WH-map.

This proof is quite obvious and omitted here.

DEFINITION 4.2. Let $\{X_i, f_i\}$ and $\{Y_i, h_i\}$ be homotopy (ϕ, Γ) -sequences of X and Y respectively, and $k: X \to Y$ a based G-map. A morphism $\{k_i\}$ from $\{X_i, f_i\}$ into $\{Y_i, h_i\}$ covering k is defined as follows: For any i, there exist $\sigma(i) (\geq \sigma(i-1))$ and G-maps $k_i: X_i \to Y_{\sigma(i)}$ such that $k_0 = k$, and $k_{i+1} \circ f_{i+1}$ and $h_{\sigma(i+1,i)} \circ k_i$ is G-homotopic, where $h_{\sigma(i+1,i)} = h_{\sigma(i+1)} \circ \cdots \circ h_{\sigma(i)+1}$.

DEFINITION 4.3. Let $\{k_i\}$ and $\{k'_i\}$ be two morphisms between homotopy (ϕ, Γ) -sequences: $\{X_i, f_i\} \rightarrow \{Y_i, h_i\}$. $\{k_i\}$ and $\{k'_i\}$ are said to be *G*homotopic, if there exists a morphism $\{H_i\}: \{X_i \wedge I^+, f_i \wedge 1\} \rightarrow \{Y_i, h_i\}$ covering the *G*-homotopy $k \underset{g}{\simeq} k'$ such that

- (1) $H_i: X_i \wedge I^+ \to Y_{\tau(i)},$
- (2) $\tau(i) \ge \max(\tau(i-1), \sigma(i), \sigma'(i)),$
- (3) $H_i(, 0) = k_i$ and $H_i(, 1) = k'_i$ in $Y_{\tau(i)}$.
- (4) $H_{i+1}\circ(f_i\wedge 1) \simeq h_{\tau(i+1,i)}\circ H_i$ rel. $X_i\wedge I^+$.

Lemma 4.4. Let A be a finite abelian group with the order q. If $f: B \rightarrow C$ be a homomorphism which induces a zero homomorphism $f \otimes 1: B \otimes \mathbb{Z}/q\mathbb{Z} \rightarrow C \otimes \mathbb{Z}/q\mathbb{Z}$, then $\text{Ext}(1, f): \text{Ext}(A, B) \rightarrow \text{Ext}(A, C)$ is a zero homomorphism.

Proof. Since A is finite, Ext(A, B) is isomorphic to $A \otimes B$ and so Ext(1, f) is zero.

For $K \leq G$ we denote by W_0K the identity component of WK = NK/K.

Theorem 4.5. Let Γ be a subset of $\Gamma(G)$ containing $\Gamma(X) \cup \Gamma(Y)$ and ϕ an order preserving map from $\Gamma(G)$ into Π . Let $\{X_i\}$ and $\{Y_i\}$ be (ϕ, Γ) -sequences of X and Y in CW^1_G respectively. For any based G-map $k: X \to Y$, if X is finite dimensional, then there exists a morphism $\{k_i\}: \{X_i\} \to \{Y_i\}$ between (ϕ, Γ) -sequences covering k. Further it is unique up to G-homotopy.

Proof. Put $\Gamma(X_i) = \{(H_1), \dots, (H_r)\}$. We assume that there exists a based

G-map $k_{i,j}$: $\bigcup_{k=1}^{j} G \cdot X_i^{H_k} \cup X_{i-1} \rightarrow Y_{\sigma(i,j)}$ extending $k_{i,j-1}$ and k_{i-1} . Put $L = H_{j+1}$ and $Z = X_{i-1}^{L} \cup X_i^{>L}$ for short. Then the obstruction to extend $k_{i,j}^{L}$ over X_i^{L} lies in $H_{WL}^{*+1}(X_i^L, Z; \underline{\pi}_*(Y_{\sigma(i,j)}^L))$ ([1]), where $\underline{\pi}_*$ is an \mathcal{O}_{WL} -group ([2]) satisfying $\underline{\pi}_*(A) (WL/K) = \pi_*(A^K)$. There exists a functorial universal coefficients spectral sequence ([6]) which converges to the above group and satisfies

$$E_2^{*,*} = \operatorname{Ext}_{\mathcal{O}_{WL}}(\underline{H}_*(X_i^L, Z), \underline{\pi}_*(Y_{\sigma(i,j)}^L)),$$

where \underline{H}_* is an \mathcal{O}_{WL} -group satisfying $\underline{H}_*(A)(WL/K) = H_*(A^K/W_0K)$. Consider the exact sequence of the triad $(X_i^L; Z, X_{i-1}^L)$.

For $(K) \geq (L)$, since $H_*(X_i^K, X_{i-1}^K) \in \mathcal{C}_{\phi(K)}$ and $\phi(K) \supset \phi(L)$, $H_*(X_i^K, X_{i-1}^K) \in \mathcal{C}_{\phi(L)}$. By Corollary 3.6 we obtain that the below group of the above diagram is in $\mathcal{C}_{\phi(L)}$. Hence for any $K \leq WL$, $H_*((X_i^L, Z)^K) \in \mathcal{C}_{\phi(L)}$ and by Proposition 3.5 $\underline{H}_*(X_i^L, Z) (WL/K) \in \mathcal{C}_{\phi(L)}$. Then there exists $\sigma(i, j+1) \ (\geq \sigma(i, j))$ such that $\pi_*(Y_{\sigma(i,j)}^L) \otimes \mathbb{Z}/q\mathbb{Z} \to \pi_*(Y_{\sigma(i,j+1)}^L) \otimes \mathbb{Z}/q\mathbb{Z}$ is a zero homomorphism for the order q of $\underline{H}_*(X_i^L, Z) (WL/eL)$. By Lemma 4.4

$$\operatorname{Ext}_{\mathcal{O}_{WL}}(\underline{H}_{*}(X_{i}^{L}, Z), \underline{\pi}_{*}(Y_{\sigma(i,j)}^{L})) \to \operatorname{Ext}_{\mathcal{O}_{WL}}(\underline{H}_{*}(X_{i}^{L}, Z), \underline{\pi}_{*}(Y_{\sigma(i,j+1)}^{L}))$$

is a zero homomorphism (Note that $H_*((X_i^L, Z)^K)=0$ if $K \neq eL$). Hence the obstruction is vanished and $k_{i,j}^L$ can be extended over $X_i^L \cup X_{i-1}^L$ as a WL-map, and $k_{i,j}$ can be extended over $\bigcup_{k=1}^{j+1} G \cdot X_i^{H_k} \cup X_{i-1}$. Then we may take $\sigma(i)=\sigma(i,r)$ and $k_i=k_{i,r}$.

For two morphisms $\{k_i\}$, $\{k'_i\}$ covering k, $H_0: X_0 \wedge I^+ \to Y_0$ is given by $H_0(x, t) = k(x)$ and $H_i: X_i \times 0 \cup X_i \times 1 \to Y_{\tau(i)}$ is defined by k_i and k'_i . By making use of the above method, we have a homotopy combining $\{k_i\}$ and $\{k'_i\}$ This completes the proof.

DEFINITION 4.6. $\{X_i, f_i\}$ is *G*-homotopy equivalent to $\{Y_i, h_i\}$, if there exist morphisms $k_i: \{X_i, f_i\} \rightarrow \{Y_i, h_i\}$ and $k'_i: \{Y_i, h_i\} \rightarrow \{X_i, f_i\}$ such that morphisms $\{k'_{\sigma(i)} \circ k_i\}$ and $\{k_{\sigma'(i)} \circ k'\}$ cover 1_X and 1_Y respectively.

By Theorem 4.5 we immediately have

Corollary 4.7. Let Γ be a subset of $\Gamma(G)$ containing $\Gamma(X)$ and ϕ an order preserving map from $\Gamma(G)$ into Π . Then a homotopy (ϕ, Γ) -sequence $\{X_i\}$ of X in \mathcal{FDCW}^{l}_{G} is unique up to G-homotopy type.

5. Localization of G-CW complexes

Let X and Y be in \mathcal{FDCW}_{c}^{l} , Γ a finite subset of $\Gamma(G)$ containing $\Gamma(X) \cup \Gamma(Y)$ and ϕ an order preserving map from $\Gamma(G)$ into Π . The localization of X at (ϕ, Γ) , denoted by $X_{(\phi, \Gamma)}$, is defined to be the based G-CW complex constructed by the "telescope construction" of the homotopy (ϕ, Γ) -sequence $\{X_i, f_i\}$ of X, that is,

$$X_{(\phi,\Gamma)} = \bigvee (X_i \wedge I^+) / (x_i, 1) \sim (f_{i+1}(x_i), 0) .$$

By Theorem 4.5 a G-map $f: X \to Y$ induces a G-map $f_{(\phi,\Gamma)}: X_{(\phi,\Gamma)} \to Y_{(\phi,\Gamma)}$, which is unique up to G-homotopy. By Corollary 4.7, $X_{(\phi,\Gamma)}$ is determined uniquely up to G-homotopy type. Now let X be in $C\mathcal{W}_{G}^{1}$. $(X^{n})_{(\phi,\Gamma)}$ is uniquely determined up to G-homotopy type, where X^{n} is the G-n-skeleton of X. Also there is a natural G-map $(X^{n})_{(\phi,\Gamma)} \to (X^{n+1})_{(\phi,\Gamma)}$ induced from the inclusion $X^{n} \to X^{n+1}$. Then we put $X_{(\phi,\Gamma)} = \lim_{i \to \infty} (X^{n})_{(\phi,\Gamma)}$, which is determined uniquely up to G-homotopy type. If $f: X \to Y$ be a G-cellular map, then it induces $(f^{n})_{(\phi,\Gamma)}$, which is unique up to G-homotopy. Thus we obtain a G-map $f_{(\phi,\Gamma)}: X_{(\phi,\Gamma)} \to Y_{(\phi,\Gamma)}$, extending f.

Here we see a relation between our localizations. If $\phi(K) \subset \eta(K)$ for any $(K) \in \Gamma(G)$, then we write $\phi \subset \eta$.

Proposition 5.1. Let X be in CW^1_G , Γ , Υ and Δ finite subsets of $\Gamma(G)$ containing $\Gamma(X)$, and ϕ , η and μ order preserving maps from $\Gamma(G)$ into Π .

- (1) If, $\phi \subset \eta$ then there exists a ϕ -equivalence $j_{(\phi,\eta,\Gamma)}: X_{(\eta,\Gamma)} \rightarrow X_{(\phi,\Gamma)}$.
- (2) If $\Gamma \subset \Upsilon$, then there exists a ϕ -equivalence $j_{(\phi,\Gamma,\Gamma)}: X_{(\phi,\Gamma)} \rightarrow X_{(\phi,\Gamma)}$.
- (3) If $\eta(H)$ is the set of all primes for any $(H) \in \Gamma$, then $X_{(\eta,\Gamma)} = X$ and $j_{(\phi,\eta,\Gamma)}$ coincides with the canonical inclusion.
- (4) For $\phi \subset \eta \subset \mu$, $j_{(\phi,\eta,\Gamma)} \circ j_{(\eta,\mu,\Gamma)} \simeq j_{(\phi,\mu,\Gamma)}$.
- (5) For $\Gamma \subset \Upsilon \subset \Delta$, $j_{(\phi, \mathbf{r}, \Lambda)} \circ j_{(\phi, \Gamma, \mathbf{r})} \simeq j_{(\phi, \Gamma, \Lambda)}$.

Proof. (3) is clear from our construction. Otherwise, we may consider the obstruction theory appeared in the proof of Theorem 4.5.

Choose a bijection $x: N \to \Gamma(G)$ such that x(1) = (G). We define finite subsets Γ_n of $\Gamma(G)$ by $\Gamma_1 = \Gamma(X)$ and $\Gamma_n = \Gamma_{n-1} \cup \{x(n)\}$. We put $X_{\phi} = \varinjlim X_{(\phi, \Gamma_n)}$ and $f_{\phi} = \varinjlim f_{(\phi, \Gamma_n)}$ for a G-map $f: X \to Y$. Then we have

Theorem 5.2. Let ϕ , η and μ be order preserving maps from $\Gamma(G)$ into Π with $\phi \subset \eta \subset \mu$.

- (1) For X in CW_G^1 , there exists a localization X_{ϕ} , which is determined uniquely up to G-homotopy type, and a ϕ -equivalence $j_X : X \rightarrow X_{\phi}$.
- (2) There exists a ϕ -equivalence $j_{\phi,\eta}: X_{\eta} \to X_{\phi}$ which coincides with j_X if $\eta(K)$

is the set of all primes for any $(K) \in \Gamma(G)$, and satisfies that $j_{\phi,\eta} \circ j_{\eta,\mu} \cong j_{\phi,\mu}$.

(3) For a based G-map f: X→Y in CW¹_G, there exists a based G-map f_φ: X_φ→Y_φ, unqiue up to G-homotopy, such that the following diagram is G-homotopy commutative.



Next we shall study some elementary properties of our localizations. Let $\underline{\underline{H}}_{*}(X)$ and \underline{Z}_{ϕ} be \mathcal{O}_{G} -groups defined by $\underline{\underline{H}}_{*}(X)(G/H) = H_{*}(X^{H}; \mathbb{Z})$ and $\underline{Z}_{\phi}(G/H) = \mathbb{Z}_{\phi(H)}$, the integer localized at $\phi(H)$. The proof of the next proposition is analogous to that of Theorem 2.5 [7].

Proposition 5.3. Let X be in CW_G^1 and ϕ an order preserving map from $\Gamma(G)$ into Π .

- (1) $\underline{\underline{H}}_{*}(X_{\phi}) \simeq \underline{\underline{H}}_{*}(X) \otimes \underline{Z}_{\phi}$. Moreover $(j_{X})_{*}$ is equivalent to $1 \otimes \iota \colon \underline{\underline{H}}_{*}(X) \rightarrow \underline{\underline{H}}_{*}(X) \otimes \underline{Z}_{\phi}$, where ι is the natural inclusion.
- (2) $\underline{\pi}_*(X_{\phi}) \simeq \underline{\pi}_*(X) \otimes \underline{Z}_{\phi}$. Moreover $(j_X)_*$ is equivalent to $1 \otimes \iota : \underline{\pi}_*(X) \rightarrow \pi_*(X) \otimes \underline{Z}_{\phi}$.

By the equivariant version of Whitehead Theorem ([5]) obviously we obtain the following proposition.

Proposition 5.4. Let X be in CW_G^1 and Y a based G-CW complex. Let ϕ be an order preserving map from $\Gamma(G)$ into Π . If there exists a based G-map $f: Y \to X$ which induces an isomorphism $\pi_*(Y^K) \to \pi_*(X^K) \otimes \mathbb{Z}_{\phi(K)}$ for any $(K) \in \Gamma(G)$, then X_{ϕ} is the same G-homotopy type as Y.

Theorem 5.5. Let ϕ be an order preserving map from $\Gamma(G)$ into Π . The localization at ϕ has the following properties:

- (1) The correspondence $X \rightarrow X_{\phi}$ is a functor from the homotopy category $C\mathcal{W}_{G}^{1}$ to the homotopy category of G-1-connected based G-CW complexes.
- (2) A based G-map $f: X \to Y$ in CW_G^1 is a ϕ -equivalence if and only if f_{ϕ} is a G-homotopy equivalence.

A G-space Z is called ϕ -local if $\pi_*(Z)$ is a \mathbb{Z}_{ϕ} -module. By the obstruction theory, we easily see the following.

Theorem 5.6. Let Z be any ϕ -local G-space. If $f: X \rightarrow Y$ is a ϕ -equivalence between G-CW complexes X and Y, then $f^*: [Y, Z]_G \rightarrow [X, Z]_G$ is a bijection.

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Corollary 5.7. For $\phi \subset \eta$, $j_{\phi,\eta}^* : [X_{\phi}, X_{\phi}]_G \to [X_{\eta}, X_{\phi}]_G$ is a bijection. In particular $j_X^* : [X_{\phi}, X_{\phi}]_G \to [X, X_{\phi}]_G$ is a bijection.

Corollary 5.8. If X is in CW_G^1 , then an arbitrary G-map $f: X_\eta \to Y_\eta$ induces $f_{\phi}: X_{\phi} \to Y_{\phi}$ such that the following diagram commutes up to G-homotopy:



We may consider both $(X_{\phi})_{\eta}$ and $(X_{\eta})_{\phi}$ as $X_{\phi \cap \eta}$. For X, Y in CW_G^1 we consider the $(G \times G)$ -CW complex $X \times Y$ as a G-space via the diagonal G-action. By [3] this G-space has a G-homotopy type of G-CW complexes and might admit infinitely many orbit types. But we may consider the localization of it at ϕ as $X_{\phi} \times Y_{\phi}$. When P is a set of primes (a constant system), the $(G \times G)$ -localization $(X \times Y)_P$ is $(G \times G)$ -homotopy equivalent to $X_P \times Y_P$.

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