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LOCALIZATION OF G-CW COMPLEXES AT A SYSTEM OF PRIMES

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1. Introduction

Let G be a compact Lie group. In [6] May, McClure and Triantafillou have studied the equivariant localization at P, a set of primes, of G-nilpotent based G-spaces. They treated the concept of a G-tower to construct the equivariant localization. Thereafter Yosimura [11,12] generalized it and its existence theorem for G-nilpotent based G-CW complexes using their methods. However since the inverse limit of G-CW complexes is generally not of the G-homotopy type of G-CW complexes, they used the G-CW approximation theorem (cf. [5], [9]). The purpose of this paper is to construct explicitly the equivariant localization after the manner of Mimura, Nishida and Toda [7]. Along this line, we generalize the notion of P-sequences to the equivariant one. Namely, our (ϕ, Γ)-sequences are associated with an order preserving map ϕ from $\Gamma(G)$, the set of conjugacy classes of closed subgroups of G, into the set of sets of primes and a finite subset Γ of $\Gamma(G)$. Thus our localization is a functor from the homotopy category $\mathcal{CW}_G^1$ of G-1-connected based G-CW complexes of G-finite type with finitely many orbit types into the homotopy category of based G-CW complexes with respect to the system of primes ϕ.

This paper is organized as follows. In §2 we construct (ϕ, Γ)-sequences. In §§3-4 we show the uniqueness of (ϕ, Γ)-sequences. Finally in §5 we establish our localization at ϕ using (ϕ, Γ)-sequences.

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2. Homotopy (ϕ, Γ)-sequences

We denote by $\Gamma(G)$ the set consisting of conjugacy classes (H) of the closed subgroups H of G and by $\gamma(G)$ a collection of closed subgroups of G containing precisely one subgroup from every conjugacy class in $\Gamma(G)$. For a G-space X, let $\Gamma(X)$ be the set of all the orbit types of X which are the conjugacy classes of the isotropy groups of points in X and $\gamma(X)$ the image of $\Gamma(X)$ under the canonical map $\Gamma(G)\to\gamma(G)$. A based G-CW complex is a G-CW complex with a base vertex which is left fixed by each element of G. A G-space X is said to be G-1-connected if each $X^H=\{x\in X \mid h\cdot x=x \text{ for } h\in H\}$ is 1-connected for any closed subgroup H of G. $\mathcal{CW}_G^1$ denotes the category of G-1-connected based G-CW complexes X such that X has finitely many orbit types and $H_*(X^H)$ is finitely generated for any closed subgroup H of G. Let Π be the power set of all primes. We have partial orderings on $\Gamma(G)$, $(H)\leqq(K)$ if H is subconjugate to K, and on Π, $P\subset Q$ if P is a subset of Q. $\phi\colon\Gamma(G)\to\Pi$ is said to be order preserving if $(H)\leqq(K)$ implies $\phi(H)\subset\phi(K)$. Throughout this paper, for any finite subset of $\Gamma(G)$, we denote it by $\{(H_1),\cdots,(H_n)\}$ so that $(H_i)>(H_j)$ implies $i<j$.

Let P be a set of primes. A space X is said to be P-equivalent to Y if there exists a map $f\colon X\to Y$ such that f induces isomorphisms of homology groups with the coefficient $\boldsymbol{Z}/p\boldsymbol{Z}$, for any $p\in P$ and with the rational coefficient.

DEFINITION 2.1. Let ϕ be a map from $\Gamma(G)$ into Π. A G-map $f\colon X\to Y$ is a *ϕ-equivalence,* if $f^H\colon X^H\to Y^H$, restricting f to X^H, is a $\phi(H)$-equivalence for any $H\leqq G$. Then X and Y are called *ϕ-equivalent.*

DEFINITION 2.2. Let ϕ be a map from $\Gamma(G)$ into Π and Γ a subset of $\Gamma(G)$ containing $\Gamma(X)$. $\{X_i,f_i\}$ is a *homotopy* (ϕ,Γ)-*sequence* of X, if

(1) $\Gamma(X_i)\subset\Gamma$,

(2) $f_i\colon X_{i-1}\to X_i$ is a ϕ-equivalence with $X_0=X$,

(3) for any n, i, $(H)\in\Gamma$ and prime q with $(q,\phi(H))=1$, there exists $N(>i)$ such that

$$\pi_n((f_N,\cdots,f_i)^H)\otimes 1_{\boldsymbol{Z}/q\boldsymbol{Z}}\colon \pi_n(X_{i-1}^H)\otimes\boldsymbol{Z}/q\boldsymbol{Z}\to\pi_n(X_N^H)\otimes\boldsymbol{Z}/q\boldsymbol{Z}$$

is a zero map.

We denote by $\mathcal{FDCW}_G^1$ the subcategory consisting of finite dimensional G-CW complexes in $\mathcal{CW}_G^1$.

Lemma 2.3. *Let ϕ be an order preserving map from $\Gamma(G)$ into Π. If X is a G-1-connected based G-CW complex, then for any $(H)\in\Gamma(G)$, $j\geqq 2$, and q prime to $\phi(H)$, there exists a G-1-connected based G-CW complex $Y=Y(X,H,j,q)$ and a based G-map $f\colon X\to Y$ such that*

(1) $\Gamma(Y)=\Gamma(X)\cup\{(H)\}$,

(2) *f is a ϕ-equivalence,*

(3) $\pi_j(f^H)\otimes 1_{\boldsymbol{Z}/q\boldsymbol{Z}}=0$.

Further if X is in $\mathcal{FDCW}^1_G$, then Y is in $\mathcal{FDCW}^1_G$.

Proof. Let $\{\beta_i\}: S^j \to X^H$ be the generators of $\pi_j(X^H)\otimes \boldsymbol{Z}/q\boldsymbol{Z}$. We define two based G-maps as follows:

$$\begin{array}{ll} \beta\colon Z \to X & \beta(gH\wedge z) = g(\vee \beta_i)(z)\,, \\ \alpha\colon Z \to Z & \alpha(gH\wedge z) = gH\wedge \hat{q}(z)\,, \end{array}$$

where $Z=(G/H)^+\wedge(\vee S^j)$ and $\hat{q}\colon \vee S^j \to \vee S^j$ is a map of degree q. Put $Y=Z\vee_Z X$, which is the pushout of G-CW complexes constructed as in [7].

$$\begin{array}{ccc} Z & \xrightarrow{\alpha} & Z \\ \downarrow{\scriptstyle\beta} & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Since $j\geqq 2$, Y is a G-1-connected based G-CW complex with $\Gamma(Y)=\Gamma(X)\cup\{(H)\}$. Note that $Y^K=Z^K\vee_{Z^K} X^K$ for any $K\leqq G$ and f^K is a homotopy equivalence unless $(K)\leqq(H)$. The conditions follow from the elementary properties of the pushout diagram.

Theorem 2.4. *Let ϕ be an order preserving map from $\Gamma(G)$ into Π and Γ a finite subset of $\Gamma(G)$ containing $\Gamma(X)$. If X is a G-1-connected based G-CW complex with finitely many orbit types, then there exists a homotopy (ϕ, Γ)-sequence $\{X_i, f_i\}$ of X such that if X is in $\mathcal{FDCW}^1_G$, then each X_i is in $\mathcal{FDCW}^1_G$.*

Proof. Put $M=(\boldsymbol{N}-\{1\})\times(\boldsymbol{N}-\{1\})\cup\{(0,0)\}$. We define an order on M by $(a,i)<(b,j)$ if $a+i<b+j$, or $a+i=b+j$ and $a<b$. Assume that there exists a sequence of ϕ-equivalences $X=X_{(0,0)}\to X_{(2,2)}\to X_{(2,3)}\to X_{(3,2)}\to\cdots\to X_{(m,k)}$ such that for any (l',r') and (l,r) in M with $(l',r')<(l,r)\leqq(m,k)$,

(1) $X_{(l,r)}$ is G-1-connected and its orbit type is the same as Γ except for $X_{(0,0)}$,

(2) for any $(H)\in\Gamma$,

$$\pi_l(X^H_{(l',r')} \to X^H_{(l,r)})\otimes \boldsymbol{Z}/r\boldsymbol{Z} = 0\,, \qquad \text{if} \quad (r, \phi(H)) = 1\,.$$

Let $\Gamma=\{(H_1),\cdots,(H_s)\}$ and (j,q) be the next to (m,k) in M. We put $Y_0=X_{(m,k)}$ and for $0<i\leqq r$,

$$Y_i = \begin{cases} Y(Y_{i-1}, H_i, j, q) & \text{if} \quad (q, \phi(H_i)) = 1\,, \\ Y_{i-1} & \text{otherwise.} \end{cases}$$

Then we take $X_{(j,q)} = Y_s$, which satisfies the conditions (1) and (2). The proof is completed.

3. The transmission of P-equivalences

Let P be a set of primes and denote by $\mathcal{C}_P$ the class of finite abelian groups without P-torsion. A homomorphism f of abelian groups is called a mod $\mathcal{C}_P$ isomorphism if the kernel and the cokernel of f belong to $\mathcal{C}_P$. We set $\Theta(K, L, A, B; M)=\{g\in G \mid g^{-1}(aKa^{-1}\cup bLb^{-1})g\subset M$ for some $a\in A$ and $b\in B\}$ for closed subgroups K, L, M of G and closed subspaces A, B of G. This set is empty unless $(K)\leqq(M)$ or $(L)\leqq(M)$. In this section we abuse f with any restriction of a map f and assume that any subspace of G which we treat is closed.

Lemma 3.1. *Let X be a G-space, K, L closed subgroups of G. Let A be a subspace of G and γ a subset of $\gamma(G)$ containing $\gamma(X)$. Then $A\cdot X^K\cap X^L=\bigcup_{H\in\gamma}\Theta(K, L, A, \{e\}; H)\cdot X^H$, where e is the unit element of G. In particular $A\cdot X^K=\bigcup_{H\in\gamma}\Theta(K, \{e\}, A, \{e\}; H)\cdot X^H$.*

Proof. We denote by $X^{\langle H\rangle}$ the subspace of X consisting of points whose isotropy groups are H. Then $A\cdot X^K=A\cdot(\bigcup_{H\in\gamma}G\cdot X^{\langle H\rangle}\cap X^K)=\bigcup_{H\in\gamma}A\cdot\{g\in G \mid g^{-1}Kg\subset H\}\cdot X^{\langle H\rangle}=\bigcup_{H\in\gamma}\Theta(K, \{e\}, A, \{e\}; H)\cdot X^{\langle H\rangle}$. Since $\Theta(K, \{e\}, A, \{e\}; H)\cdot X^H\cap G\cdot X^{\langle N\rangle}\subset\Theta(K, \{e\}, A, \{e\}; N)\cdot X^{\langle N\rangle}$, we have $A\cdot X^K=\bigcup_{H\in\gamma}\Theta(K, \{e\}, A, \{e\}; H)\cdot X^H$. Similarly $A\cdot X^K\cap X^L=\bigcup_{H\in\gamma}\Theta(K, L, A, \{e\}; H)\cdot X^H$.

Proposition 3.2. *Let X be a G-space, K, L closed subgroups of G. Let A, B be subspaces of G and γ a subset of $\gamma(G)$ containing $\gamma(X)$. Then $A\cdot X^K\cap B\cdot X^L=\bigcup_{H\in\gamma}\Theta(K, L, A, B; H)\cdot X^H$.*

Theorem 3.3. *Let $f\colon X\to Y$ be a G-map between G-CW complexes with finitely many orbit types. Suppose that $f_*\colon H_*(X^H)\to H_*(Y^H)$ is a* mod $\mathcal{C}_P$ *isomorphism for any H in $\gamma(X)\cup\gamma(Y)$. Then any closed subspaces $A_1, \cdots, A_r$ of G and any closed subgroups $K_1, \cdots, K_r$ of G (for any r),*

$$f_*\colon H_*(\bigcup_{i=1}^{r} A_i\cdot X^{K_i})\to H_*(\bigcup_{i=1}^{r} A_i\cdot Y^{K_i})$$

is a mod $\mathcal{C}_P$ *isomorphism.*

Proposition 3.2 means that Theorem 3.3 implies that if $f_*\colon H_*(X^H)\to H_*(Y^H)$ is a mod $\mathcal{C}_P$ isomorphism for any H in $\gamma(X)\cup\gamma(Y)$, then $f_*\colon H_*(X^K)\to H_*(Y^K)$ is a mod $\mathcal{C}_P$ isomorphism for any $K\leqq G$. We need some lemmas to show the above theorem.

We set $X^{>K}=\{x\in X \mid G_x>K\}$ and $X^{>(K)}=G\cdot X^{>K}=\{x\in X \mid (G_x)>(K)\}$ for any $K\leqq G$. Note that $A\cdot X^{>K}=\bigcup_{\substack{H\in\gamma(X)\\(K)<(H)}}\Theta(K, H, A, G; H)\cdot X^H$.

Lemma 3.4. *Let X be a G-CW complex and A a closed subspace of G.*

Then for any $K \leqq G$, $(A \cdot X^K, A \cdot X^K \cap X^{>(K)})$ *is an NDR-pair.*

Proof. Since X is compactly generated and $G \cdot X^K$ is closed in X, $G \cdot X^K$ is compactly generated. Since A is closed in G, $A \cdot X^K$ is closed in $G \cdot X^K$ and $A \cdot X^K$ is compactly generated (See [8]). We denote by $(G \cdot X^K)_n$ the union of $A \cdot X^K \cup X^{>(K)}$ and G-cells of G-dimension$\leqq n$ in $G \cdot X^{\langle K \rangle}$ and put $(A \cdot X^K)_n = (G \cdot X^K)_n \cap A \cdot X^K$. Then $(\coprod_\iota A \cdot NK/K \times D^n_\iota, \coprod_\iota A \cdot NK/K \times S^{n-1}_\iota)$ and $((A \cdot X^K)_n, (A \cdot X^K)_{n-1} \cup X_n^{>(K)})$ are relatively homeomorphic, where NK is the normalizer of K in G, and D^n_ι and S^{n-1}_ι be copies of the n-disk and $(n-1)$-sphere respectively. Since the former is an NDR-pair, so is the latter.

The next proposition is due to Proposition 2 in [6].

Proposition 3.5. *Let* $f: Y \to Z$ *be a G-map between G-spaces and P a set of primes. If* $H_*(f^L)$ *is a* mod $\mathcal{C}_P$ *isomorphism for any* $L \leqq G$, *then also* $H_*(f/M)$: $H_*(Y/M) \to H_*(Z/M)$ *is a* mod $\mathcal{C}_P$ *isomorphism for any* $M \leqq G$.

Now we start to prove Theorem 3.3. By Proposition 3.2, we can assume that $r=n$ and $L_i = H_i$, where $\gamma(X) \cup \gamma(Y) = \{H_1, \cdots, H_n\}$. We show the assertion by induction on the maximal number of the suffixes of H_i; assuming that the assertion is true for any $j_1, \cdots, j_t$ with $j_1 < \cdots < j_t < s$, we shall show that the assertion is true for any $j_1, \cdots, j_q$ with $j_1 < \cdots < j_q < s+1$. First we shall show that $f_*: H_*(A \cdot X^{H_s}) \to H_*(A \cdot Y^{H_s})$ is a mod $\mathcal{C}_P$ isomorphism for any closed subspace A of G. Put $\tilde{A} = A \cdot NH_s$ and $H = H_s$ for short. By Proposition 3.2 and our assumption, $f_*: H_*(X^{>H}) \to H_*(Y^{>H})$ and $f_*: H_*(A \cdot X^{>H}) \to H_*(A \cdot Y^{>H})$ are mod $\mathcal{C}_P$ isomorphisms. We consider $\tilde{A} \times X^H$ as an NH-space via the NH-action $n \cdot (a, x) = (a \cdot n^{-1}, n \cdot x)$. Then by Proposition 3.5 $(1 \underset{NH}{\times} f)_*$: $H_*(\tilde{A} \underset{NH}{\times} X^H) \to H_*(\tilde{A} \underset{NH}{\times} Y^H)$ is a mod $\mathcal{C}_P$ isomorphism. Similarly $(1 \underset{NH}{\times} f)_*: H_*(\tilde{A} \underset{NH}{\times} X^{>H}) \to H_*(\tilde{A} \underset{NH}{\times} Y^{>H})$ is a mod $\mathcal{C}_P$ isomorphism. Since $A \cdot X^{\langle H \rangle}$ is homeomorphic to $\tilde{A} \underset{NH}{\times} X^{\langle H \rangle}$, $H_*(A \cdot X^H, A \cdot X^{>H})$ is isomorphic to $H_*(\tilde{A} \underset{NH}{\times} X^H, \tilde{A} \underset{NH}{\times} X^{>H})$ and $f_*: H_*(A \cdot X^H, A \cdot X^{>H}) \to H_*(A \cdot Y^H, A \cdot Y^{>H})$ is a mod $\mathcal{C}_P$ isomorphism. Thus so is $f_*: H_*(A \cdot X^H) \to H_*(A \cdot Y^H)$. Let $j_1, \cdots, j_q$ be any integers with $j_1 < \cdots < j_q < s+1$ (for any q). By comparing two Mayer-Vietoris exact sequences for X and Y, we obtain that $f_*: H_*(\bigcup_{i=1}^q A_i \cdot X^{H_{j_i}}) \to H_*(\bigcup_{i=1}^q A_j \cdot Y^{H_{j_i}})$ is a mod $\mathcal{C}_P$ isomorphism. This completes the proof.

Corollary 3.6. *Let* $f: X \to Y$ *be a G-map between G-CW complexes with finitely many orbit types. Suppose that for any* H *in* $\gamma(X) \cup \gamma(Y)$, *there exists a set of primes* $P(H)$ *such that* f^H *is a* $P(H)$*-equivalence. If* ϕ *is the map from* $\Gamma(G)$ *into* Π *defined by* $\phi(K) = \bigcap_{\substack{(H) \in \Gamma(X) \cup \Gamma(Y) \\ (H) \geqq (K)}} P(H)$, *then* f *is a* ϕ*-equivalence.*

In the same manner as the proof of Theorem 3.3, we have

Proposition 3.7. *Let X be a G-CW complex with finitely many orbit types. If $H_*(X^H)$ is finitely generated for any $H\in\gamma(X)$, then $H_*(X^K)$ is finitely generated for any $K\leqq G$.*

4. Uniqueness of the (ϕ, Γ)-sequences

Lemma 4.1. *Let (Y, X) be a G-CW pair with $\Gamma(Y-X)=\{(H)\}$, Z a G-space and $f: X\to Z$ a G-map. f can be extended over Y as a G-map if and only if $f^H: X^H\to Z^H$ can be extended over Y^H as a WH-map.*

This proof is quite obvious and omitted here.

Definition 4.2. Let $\{X_i, f_i\}$ and $\{Y_i, h_i\}$ be homotopy (ϕ, Γ)-sequences of X and Y respectively, and $k: X\to Y$ a based G-map. A *morphism* $\{k_i\}$ from $\{X_i, f_i\}$ into $\{Y_i, h_i\}$ covering k is defined as follows: For any i, there exist $\sigma(i)$ $(\geqq\sigma(i-1))$ and G-maps $k_i: X_i\to Y_{\sigma(i)}$ such that $k_0=k$, and $k_{i+1}\circ f_{i+1}$ and $h_{\sigma(i+1,i)}\circ k_i$ is G-homotopic, where $h_{\sigma(i+1,i)}=h_{\sigma(i+1)}\circ\cdots\circ h_{\sigma(i)+1}$.

Definition 4.3. Let $\{k_i\}$ and $\{k_i'\}$ be two morphisms between homotopy (ϕ, Γ)-sequences: $\{X_i, f_i\}\to\{Y_i, h_i\}$. $\{k_i\}$ and $\{k_i'\}$ are said to be *G-homotopic,* if there exists a morphism $\{H_i\}: \{X_i\wedge I^+, f_i\wedge 1\}\to\{Y_i, h_i\}$ covering the G-homotopy $k\underset{G}{\simeq}k'$ such that

(1) $H_i: X_i\wedge I^+\to Y_{\tau(i)}$,
(2) $\tau(i)\geqq\max(\tau(i-1), \sigma(i), \sigma'(i))$,
(3) $H_i(\ ,0)=k_i$ and $H_i(\ ,1)=k_i'$ in $Y_{\tau(i)}$.
(4) $H_{i+1}\circ(f_i\wedge 1)\underset{G}{\simeq}h_{\tau(i+1,i)}\circ H_i$ rel. $X_i\wedge \dot{I}^+$.

Lemma 4.4. *Let A be a finite abelian group with the order q. If $f: B\to C$ be a homomorphism which induces a zero homomorphism $f\otimes 1: B\otimes \boldsymbol{Z}/q\boldsymbol{Z}\to C\otimes \boldsymbol{Z}/q\boldsymbol{Z}$, then* $\mathrm{Ext}(1,f)$: $\mathrm{Ext}(A, B)\to\mathrm{Ext}(A, C)$ *is a zero homomorphism.*

Proof. Since A is finite, $\mathrm{Ext}(A, B)$ is isomorphic to $A\otimes B$ and so $\mathrm{Ext}(1,f)$ is zero.

For $K\leqq G$ we denote by W_0K the identity component of $WK=NK/K$.

Theorem 4.5. *Let Γ be a subset of $\Gamma(G)$ containing $\Gamma(X)\cup\Gamma(Y)$ and ϕ an order preserving map from $\Gamma(G)$ into Π. Let $\{X_i\}$ and $\{Y_i\}$ be (ϕ, Γ)-sequences of X and Y in $\mathcal{CW}_G^1$ respectively. For any based G-map $k: X\to Y$, if X is finite dimensional, then there exists a morphism $\{k_i\}: \{X_i\}\to\{Y_i\}$ between (ϕ, Γ)-sequences covering k. Further it is unique up to G-homotopy.*

Proof. Put $\Gamma(X_i)=\{(H_1), \cdots, (H_r)\}$. We assume that there exists a based

G-map $k_{i,j}\colon \bigcup_{k=1}^{j} G\cdot X_i^{H_k}\cup X_{i-1}\to Y_{\sigma(i,j)}$ extending $k_{i,j-1}$ and k_{i-1}. Put $L=H_{j+1}$ and $Z=X_{i-1}^L\cup X_i^{>L}$ for short. Then the obstruction to extend $k_{i,j}^L$ over X_i^L lies in $H_{WL}^{*+1}(X_i^L, Z; \underline{\pi}_*(Y_{\sigma(i,j)}^L))$ ([1]), where $\underline{\pi}_*$ is an $\mathcal{O}_{WL}$-group ([2]) satisfying $\underline{\pi}_*(A)(WL/K)=\pi_*(A^K)$. There exists a functorial universal coefficients spectral sequence ([6]) which converges to the above group and satisfies

$$E_2^{*,*}=\mathrm{Ext}_{\mathcal{O}_{WL}}(\underline{H}_*(X_i^L, Z), \underline{\pi}_*(Y_{\sigma(i,j)}^L)),$$

where $\underline{H}_*$ is an $\mathcal{O}_{WL}$-group satisfying $\underline{H}_*(A)(WL/K)=H_*(A^K/W_0K)$. Consider the exact sequence of the triad $(X_i^L; Z, X_{i-1}^L)$.

$$\begin{array}{c} \cdots\to H_*(X_i^L, X_{i-1}^L)\to H_*(X_i^L, Z)\to H_{*-1}(Z, X_{i-1}^L)\to\cdots \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow\cong \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad H_{*-1}(X_i^{>L}, X_{i-1}^{>L}) \end{array}$$

For $(K)\geqq(L)$, since $H_*(X_i^K, X_{i-1}^K)\in\mathcal{C}_{\phi(K)}$ and $\phi(K)\supset\phi(L)$, $H_*(X_i^K, X_{i-1}^K)\in\mathcal{C}_{\phi(L)}$. By Corollary 3.6 we obtain that the below group of the above diagram is in $\mathcal{C}_{\phi(L)}$. Hence for any $K\leqq WL$, $H_*((X_i^L, Z)^K)\in\mathcal{C}_{\phi(L)}$ and by Proposition 3.5 $\underline{H}_*(X_i^L, Z)(WL/K)\in\mathcal{C}_{\phi(L)}$. Then there exists $\sigma(i, j+1)$ ($\geqq\sigma(i,j)$) such that $\pi_*(Y_{\sigma(i,j)}^L)\otimes\boldsymbol{Z}/q\boldsymbol{Z}\to\pi_*(Y_{\sigma(i,j+1)}^L)\otimes\boldsymbol{Z}/q\boldsymbol{Z}$ is a zero homomorphism for the order q of $\underline{H}_*(X_i^L, Z)(WL/eL)$. By Lemma 4.4

$$\mathrm{Ext}_{\mathcal{O}_{WL}}(\underline{H}_*(X_i^L, Z), \underline{\pi}_*(Y_{\sigma(i,j)}^L))\to\mathrm{Ext}_{\mathcal{O}_{WL}}(\underline{H}_*(X_i^L, Z), \underline{\pi}_*(Y_{\sigma(i,j+1)}^L))$$

is a zero homomorphism (Note that $H_*((X_i^L, Z)^K)=0$ if $K\neq eL$). Hence the obstruction is vanished and $k_{i,j}^L$ can be extended over $X_i^L\cup X_{i-1}^L$ as a WL-map, and $k_{i,j}$ can be extended over $\bigcup_{k=1}^{j+1} G\cdot X_i^{H_k}\cup X_{i-1}$. Then we may take $\sigma(i)=\sigma(i,r)$ and $k_i=k_{i,r}$.

For two morphisms $\{k_i\}$, $\{k_i'\}$ covering k, $H_0\colon X_0\wedge I^+\to Y_0$ is given by $H_0(x,t)=k(x)$ and $H_i\colon X_i\times 0\cup X_i\times 1\to Y_{\tau(i)}$ is defined by k_i and k_i'. By making use of the above method, we have a homotopy combining $\{k_i\}$ and $\{k_i'\}$ This completes the proof.

DEFINITION **4.6.** $\{X_i, f_i\}$ is *G-homotopy equivalent* to $\{Y_i, h_i\}$, if there exist morphisms $k_i\colon \{X_i, f_i\}\to\{Y_i, h_i\}$ and $k_i'\colon \{Y_i, h_i\}\to\{X_i, f_i\}$ such that morphisms $\{k'_{\sigma(i)}\circ k_i\}$ and $\{k_{\sigma'(i)}\circ k'\}$ cover 1_X and 1_Y respectively.

By Theorem 4.5 we immediately have

Corollary 4.7. *Let Γ be a subset of $\Gamma(G)$ containing $\Gamma(X)$ and ϕ an order preserving map from $\Gamma(G)$ into Π. Then a homotopy (ϕ, Γ)-sequence $\{X_i\}$ of X in $\mathcal{FDCW}_G^1$ is unique up to G-homotopy type.*

5. Localization of *G-CW* complexes

Let X and Y be in $\mathcal{FDCW}_G^1$, Γ a finite subset of $\Gamma(G)$ containing $\Gamma(X)\cup\Gamma(Y)$ and ϕ an order preserving map from $\Gamma(G)$ into Π. The localization of X at (ϕ, Γ), denoted by $X_{(\phi,\Gamma)}$, is defined to be the based G-CW complex constructed by the "telescope construction" of the homotopy (ϕ, Γ)-sequence $\{X_i, f_i\}$ of X, that is,

$$X_{(\phi,\Gamma)} = \bigvee_i (X_i \wedge I^+)/(x_i, 1)\sim(f_{i+1}(x_i), 0)\,.$$

By Theorem 4.5 a G-map $f\colon X\to Y$ induces a G-map $f_{(\phi,\Gamma)}\colon X_{(\phi,\Gamma)}\to Y_{(\phi,\Gamma)}$, which is unique up to G-homotopy. By Corollary 4.7, $X_{(\phi,\Gamma)}$ is determined uniquely up to G-homotopy type. Now let X be in $\mathcal{CW}_G^1$. $(X^n)_{(\phi,\Gamma)}$ is uniquely determined up to G-homotopy type, where X^n is the G-n-skeleton of X. Also there is a natural G-map $(X^n)_{(\phi,\Gamma)}\to(X^{n+1})_{(\phi,\Gamma)}$ induced from the inclusion $X^n\to X^{n+1}$. Then we put $X_{(\phi,\Gamma)}=\varinjlim(X^n)_{(\phi,\Gamma)}$, which is determined uniquely up to G-homotopy type. If $f\colon X\to Y$ be a G-cellular map, then it induces $(f^n)_{(\phi,\Gamma)}$, which is unique up to G-homotopy. Thus we obtain a G-map $f_{(\phi,\Gamma)}\colon X_{(\phi,\Gamma)}\to Y_{(\phi,\Gamma)}$, extending f.

Here we see a relation between our localizations. If $\phi(K)\subset\eta(K)$ for any $(K)\in\Gamma(G)$, then we write $\phi\subset\eta$.

Proposition 5.1. *Let X be in $\mathcal{CW}_G^1$, Γ, Υ and Δ finite subsets of $\Gamma(G)$ containing $\Gamma(X)$, and ϕ, η and μ order preserving maps from $\Gamma(G)$ into Π.*

(1) *If, $\phi\subset\eta$ then there exists a ϕ-equivalence $j_{(\phi,\eta,\Gamma)}\colon X_{(\eta,\Gamma)}\to X_{(\phi,\Gamma)}$.*
(2) *If $\Gamma\subset\Upsilon$, then there exists a ϕ-equivalence $j_{(\phi,\Gamma,\Upsilon)}\colon X_{(\phi,\Gamma)}\to X_{(\phi,\Upsilon)}$.*
(3) *If $\eta(H)$ is the set of all primes for any $(H)\in\Gamma$, then $X_{(\eta,\Gamma)}=X$ and $j_{(\phi,\eta,\Gamma)}$ coincides with the canonical inclusion.*
(4) *For $\phi\subset\eta\subset\mu$, $j_{(\phi,\eta,\Gamma)}\circ j_{(\eta,\mu,\Gamma)}\underset{G}{\simeq}j_{(\phi,\mu,\Gamma)}$.*
(5) *For $\Gamma\subset\Upsilon\subset\Delta$, $j_{(\phi,\Upsilon,\Delta)}\circ j_{(\phi,\Gamma,\Upsilon)}\underset{G}{\simeq}j_{(\phi,\Gamma,\Delta)}$.*

Proof. (3) is clear from our construction. Otherwise, we may consider the obstruction theory appeared in the proof of Theorem 4.5.

Choose a bijection $x\colon \boldsymbol{N}\to\Gamma(G)$ such that $x(1)=(G)$. We define finite subsets Γ_n of $\Gamma(G)$ by $\Gamma_1=\Gamma(X)$ and $\Gamma_n=\Gamma_{n-1}\cup\{x(n)\}$. We put $X_\phi=\varinjlim X_{(\phi,\Gamma_n)}$ and $f_\phi=\varinjlim f_{(\phi,\Gamma_n)}$ for a G-map $f\colon X\to Y$. Then we have

Theorem 5.2. *Let ϕ, η and μ be order preserving maps from $\Gamma(G)$ into Π with $\phi\subset\eta\subset\mu$.*

(1) *For X in $\mathcal{CW}_G^1$, there exists a localization X_ϕ, which is determined uniquely up to G-homotopy type, and a ϕ-equivalence $j_X\colon X\to X_\phi$.*
(2) *There exists a ϕ-equivalence $j_{\phi,\eta}\colon X_\eta\to X_\phi$ which coincides with j_X if $\eta(K)$*

is the set of all primes for any $(K)\in\Gamma(G)$, *and satisfies that* $j_{\phi,\eta}\circ j_{\eta,\mu}\underset{G}{\simeq}j_{\phi,\mu}$.

(3) *For a based G-map* $f\colon X\to Y$ *in* $\mathcal{CW}^1_G$, *there exists a based G-map* $f_\phi\colon X_\phi\to Y_\phi$, *unqiue up to G-homotopy, such that the following diagram is G-homotopy commutative.*

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ \downarrow{\scriptstyle j_X} & & \downarrow{\scriptstyle j_Y} \\ X_\phi & \xrightarrow{\ f_\phi\ } & Y_\phi \end{array}$$

Next we shall study some elementary properties of our localizations. Let $\underline{\underline{H}}_*(X)$ and $\underline{\boldsymbol{Z}}_\phi$ be $\mathcal{O}_G$-groups defined by $\underline{\underline{H}}_*(X)(G/H)=H_*(X^H;\boldsymbol{Z})$ and $\underline{\boldsymbol{Z}}_\phi(G/H)=\boldsymbol{Z}_{\phi(H)}$, the integer localized at $\phi(H)$. The proof of the next proposition is analogous to that of Theorem 2.5 [7].

Proposition 5.3. *Let* X *be in* $\mathcal{CW}^1_G$ *and* ϕ *an order preserving map from* $\Gamma(G)$ *into* Π.

(1) $\underline{\underline{H}}_*(X_\phi)\cong\underline{\underline{H}}_*(X)\otimes\underline{\boldsymbol{Z}}_\phi$. *Moreover* $(j_X)_*$ *is equivalent to* $1\otimes\iota\colon\underline{\underline{H}}_*(X)\to\underline{\underline{H}}_*(X)\otimes\underline{\boldsymbol{Z}}_\phi$, *where* ι *is the natural inclusion.*

(2) $\underline{\pi}_*(X_\phi)\cong\underline{\pi}_*(X)\otimes\underline{\boldsymbol{Z}}_\phi$. *Moreover* $(j_X)_*$ *is equivalent to* $1\otimes\iota\colon\underline{\pi}_*(X)\to\underline{\pi}_*(X)\otimes\underline{\boldsymbol{Z}}_\phi$.

By the equivariant version of Whitehead Theorem ([5]) obviously we obtain the following proposition.

Proposition 5.4. *Let* X *be in* $\mathcal{CW}^1_G$ *and* Y *a based G-CW complex. Let* ϕ *be an order preserving map from* $\Gamma(G)$ *into* Π. *If there exists a based G-map* $f\colon Y\to X$ *which induces an isomorphism* $\pi_*(Y^K)\to\pi_*(X^K)\otimes\boldsymbol{Z}_{\phi(K)}$ *for any* $(K)\in\Gamma(G)$, *then* X_ϕ *is the same G-homotopy type as* Y.

Theorem 5.5. *Let* ϕ *be an order preserving map from* $\Gamma(G)$ *into* Π. *The localization at* ϕ *has the following properties*:

(1) *The correspondence* $X\to X_\phi$ *is a functor from the homotopy category* $\mathcal{CW}^1_G$ *to the homotopy category of G-1-connected based G-CW complexes.*

(2) *A based G-map* $f\colon X\to Y$ *in* $\mathcal{CW}^1_G$ *is a* ϕ*-equivalence if and only if* f_ϕ *is a G-homotopy equivalence.*

A G-space Z is called ϕ-local if $\underline{\pi}_*(Z)$ is a $\underline{\boldsymbol{Z}}_\phi$-module. By the obstruction theory, we easily see the following.

Theorem 5.6. *Let* Z *be any* ϕ*-local G-space. If* $f\colon X\to Y$ *is a* ϕ*-equivalence between G-CW complexes* X *and* Y, *then* $f^*\colon[Y,Z]_G\to[X,Z]_G$ *is a bijection.*

Corollary 5.7. *For* $\phi\subset\eta$, $j_{\phi,\eta}{}^*\colon [X_\phi, X_\phi]_G\to[X_\eta, X_\phi]_G$ *is a bijection. In particular* $j_X{}^*\colon [X_\phi, X_\phi]_G\to[X, X_\phi]_G$ *is a bijection.*

Corollary 5.8. *If* X *is in* $\mathcal{CW}_G^1$, *then an arbitrary G-map* $f\colon X_\eta\to Y_\eta$ *induces* $f_\phi\colon X_\phi\to Y_\phi$ *such that the following diagram commutes up to G-homotopy:*

$$\begin{array}{ccc} X_\eta & \xrightarrow{f} & Y_\eta \\ \downarrow{\scriptstyle j_{\phi,\eta}} & & \downarrow{\scriptstyle j_{\phi,\eta}} \\ X_\phi & \xrightarrow{f_\phi} & Y_\phi \end{array}$$

We may consider both $(X_\phi)_\eta$ and $(X_\eta)_\phi$ as $X_{\phi\cap\eta}$. For X, Y in $\mathcal{CW}_G^1$ we consider the $(G\times G)$-CW complex $X\times Y$ as a G-space via the diagonal G-action. By [3] this G-space has a G-homotopy type of G-CW complexes and might admit infinitely many orbit types. But we may consider the localization of it at ϕ as $X_\phi\times Y_\phi$. When P is a set of primes (a constant system), the $(G\times G)$-localization $(X\times Y)_P$ is $(G\times G)$-homotopy equivalent to $X_P\times Y_P$.

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