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PROJECTIVE MODULES OVER SIMPLE REGULAR RINGS

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Recently K.R. Goodearl and D. Handelman [6] have studied simple regular rings from the point of view of dimension-like functions. They have shown that there exists a unique dimension function on the lattice of principal right ideals of a simple, regular and directly finite ring satisfying the comparability axiom. In this note we study some structures of projective modules over such a ring by making use of the dimension function.

In the section 1 we show that if there exists a dimension function on the lattice of principal right ideals of a regular ring, then this can be extended to a function on the set of all projective modules.

In the section 2 we investigate some structures of projective modules over a simple, regular and directly finite ring satisfying the comparability axiom and show that a directly finite projective module is isomorphic to a direct sum of a finitely generated free module and a projective right ideal, and a directly infinite projective module is a free module.

In the final section directly finite, regular and right self-injective rings are investigated. We show that this ring is a finite direct product of simple rings if and only if any non-singular directly finite injective right module is a finitely generated module.

Throughout this paper a ring R is an associative ring with identity and modules are unitary right R-modules.

1. Dimension functions

For any (Von Neumann) regular ring R, let $L(R)$ be the lattice of principal right ideals and $P(R)$ ($FP(R)$) the set of all projective (finitely generated projective) R-modules. We denote by $M \lesssim N$ the fact that M is isomorphic to a submodule of N for two modules M, N. In particular if R is regular, then $A \lesssim P$ for A in $FP(R)$ and P in $P(R)$ if and only if A is isomorphic to a direct summand of P [8, Lemma 4].

Definition [6, p. 807]. A *dimension function* D on $L(R)$ is a function from $L(R)$ into non-negative real numbers satisfying the following conditions;

(1) $D(J)=0$ if and only if $J=0$
(2) $D(R)=1$
(3) if $J\lesssim K$, then $D(J)\leqq D(K)$
(4) if $J\oplus K\in L(R)$, then $D(J\oplus K)=D(J)+D(K)$.

I. Halperin [7] proved that if a dimension function D exists on $L(R)$, then D can be uniquely extended to a function on $FP(R)$. We shall show that this function D can be moreover extended to a function on $P(R)$ by making use of the following lemma.

Lemma 1.1 [10]. *For any projective module P over a regular ring, P is isomorphic to a direct sum of principal right ideals and any two direct sum decompositions of P have an isomorphic refinement.*

Let P be in $P(R)$. From now on, by $P=\oplus_{J\in\mathfrak{M}}J$ we denote the fact that there exists a set $\mathfrak{M}$ of independent non-zero submodules isomorphic to some principal right ideal and P is a direct sum of the members of $\mathfrak{M}$. We put $D^*(P)=\sup\{\sum_{J\in\mathfrak{M}'}D(J)$; any finite subset $\mathfrak{M}'$ of $\mathfrak{M}\}$ for any P in $P(R)$ and any decomposition $P=\oplus_{J\in\mathfrak{M}}J$. If the above supremum is not convergent, we put $D^*(P)=\infty$. Now we shall prove that $D^*(P)$ does not depend on the decomposition of P. Let $P=\oplus_{K\in\mathfrak{N}}K$ be another decomposition. It is sufficient to prove that two numbers a, b defined by $\mathfrak{M}$ and $\mathfrak{N}$ coincide when $\mathfrak{N}$ is a refinement of $\mathfrak{M}$. For any J in $\mathfrak{M}$, there exists a finite subset $\mathfrak{N}'$ of $\mathfrak{N}$ such that $J=\oplus_{K\in\mathfrak{N}'}K$. Hence we have $a\leqq b$. Conversely for any finite subset $\mathfrak{N}'$ of $\mathfrak{N}$ and any K in $\mathfrak{N}'$, there exists some J in $\mathfrak{M}$ such that K is a direct summand of J. Therefore there exists a finite subset $\mathfrak{M}'$ of $\mathfrak{M}$ such that $\sum_{K\in\mathfrak{N}'}D(K)\leqq\sum_{J\in\mathfrak{M}'}D(J)$and so we have $b\leqq a$.

Now D^* is a function from $P(R)$ into non-negative real numbers or ∞, and from the definition and by Lemma 1.1, we can easily prove the following properties;

(1) if $P\lesssim Q$ in $P(R)$, then $D^*(P)\leqq D^*(Q)$
(2) if $P\oplus Q\in P(R)$, then $D^*(P\oplus Q)=D^*(P)+D^*(Q)$.

2. Projective modules

First we recall some definitions and some results in [6].

DEFINITION. A ring R is *directly finite* if $xy=1$ implies $yx=1$ for x, y in R. A module M is *directly finite* if $End_R(M)$ is directly finite. A ring R (a module M) is *directly infinite* if it is not directly finite. It is easily seen that a module M is directly finite if and only if M is not isomorphic to a proper direct summand of itself. A regular ring R satisfies the *comparability axiom* if we have either $J\lesssim K$ or $K\lesssim J$ for all J, K in $L(R)$. For a cardinal number α and a module M, αM denotes a direct sum of α copies of M.

NOTE. Throughout this section R is a simple, regular and directly finite ring satisfying the comparability axiom. In this case, any finitely generated projective R-module is directly finite by [6, Corollary 3.10].

EXAMPLE [6, pp. 815, 831 and 832]. Let F be a field and R_n the full matrix ring of degree 2^n over F. Let $f_n: R_n \to R_{n+1}$ be a diagonal homomorphism, i.e., $x \to \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$, and let R be a direct limit of $\{R_n, f_n\}$. This ring R is a simple, regular and directly finite ring which satisfies the comparability axiom and which is not artinian. Further R is neither left nor right self-injective.

Lemma 2.1 [6, Theorem 3.13 and Proposition 3.14]. *Let J be in $L(R)$. We put $D(J) = sup\{mn^{-1}; m \geqq 0, n > 0, mR \lesssim nJ\}$. Then D is a unique dimension function on $L(R)$. Further, for all J, K in $L(R)$, we have $J \lesssim K$ if and only if $D(J) \leqq D(K)$.*

From now on, let D^* be the extension of the dimension function D as in the section 1. We consider projective modules over R from the point of view of D^*.

Lemma 2.2 *Let A, B in $FP(R)$. $A \lesssim B$ if and only if $D^*(A) \leqq D^*(B)$. In particular, $A \cong B$ if and only if $D^*(A) = D^*(B)$.*

Proof. We have $A \lesssim B$ or $B \lesssim A$ by [6, Lemma 3.7]. Then the proof of the first property is easy. If $D^*(A) = D^*(B)$, then $A \lesssim B$ and $B \lesssim A$. Hence A is isomorphic to a direct summand of itself. Then $A \cong B$, because A is directly finite.

The next is a key lemma for Theorem 2.4.

Lemma 2.3. *For P in $P(R)$ and A in $FP(R)$, $P \lesssim A$ if and only if $D^*(P) \leq D^*(A)$.*

Proof. By the definition, "only if" part is trivial. We assume $D^*(P) \leqq D^*(A)$ and $P = \oplus_{J \in \mathfrak{M}} J$. First we know $\mathfrak{M}$ is a countable set, because for each positive integer n, the set $\mathfrak{M}_n = \{J;\ D(J) > n^{-1}\}$ is a finite set and $\mathfrak{M} = \cup_n \mathfrak{M}_n$. Now put $\mathfrak{M} = \{J_n; n = 1, 2, \cdots\}$ and $P_n = \oplus_1^n J_i$, then we have $P = \cup_n P_n$. For each n, we can choose a monomorphism $f_n: P_n \to A$ by Lemma 2.2, because $D^*(P_n) \leqq D^*(A)$. If we construct monomorphism $g_n: P_n \to A$ for each n such that g_{n+1} is an extension of g_n, then we have $P \lesssim A$. Put $g_1 = f_1$ and assume we have g_k for all $k \leqq n$. We have decompositions $A = g_n(P_n) \oplus Q_n = f_{n+1}(P_n) \oplus f_{n+1}(J_{n+1}) \oplus Q_{n+1}$ for some submodules Q_n, Q_{n+1}, because homomorphism g_n, f_{n+1} split. Then we have $Q_n \cong f_{n+1}(J_{n+1}) \oplus Q_{n+1}$ by [6, Theorem 3.9] and so we choose a monomorphism $h: f_{n+1}(J_{n+1}) \to Q_n$. Consequently $g_{n+1} = g_n \oplus h f_{n+1}: P_{n+1} \to A$ is an extension of g_n.

We shall determine the structures of projective modules over a simple, regular and directly finite ring satisfying the comparability axiom.

Theorem 2.4. *Let R be a simple, regular and directly finite ring satisfying the comparability axiom. For a projective R-module P, the following conditions are equivalent.*

(1) *P is directly finite.*

(2) $D^*(P)<\infty$

(3) *P has a decomposition $P\simeq nR\oplus J$ for some integer $n\geqq 0$ and some right ideal J.*

(4) *$P\lesssim tR$ for some integer $t>0$.*

Proof. (1)$\Rightarrow$(2). We assume $D^*(P)=\infty$. Put $P=\oplus_{J\in\mathfrak{M}}J$, then there exists a sequence of finite subsets $\mathfrak{M}_i$ $(i=1,2,\cdots)$ of $\mathfrak{M}$ such that $\mathfrak{M}_i\cap\mathfrak{M}_j=\phi$ if $i\neq j$ and $D^*(\oplus_{J\in\mathfrak{M}_i}J)\geqq 1$ for each i. Put $P_i=\oplus_{J\in\mathfrak{M}_i}J$, then we have $R\lesssim P_i$ by Lemma 2.2 and so we have $P_i=R_i\oplus Q_i$, where $R_i\simeq R$. $F=\oplus_1^\infty R_i$ is a direct summand of P and $2F\simeq F$. This contradicts that every direct summand of P is also directly finite.

(2)$\Rightarrow$(3). We choose non-negative integer n such that $n<D^*(P)\leqq n+1$. If $n=0$, then we have $P\lesssim R$ by Lemma 2.3. If n is positive, the first inequality implies that $nR\lesssim P$ from the definition of D^* and by Lemma 2.2. Then we have $P=P_1\oplus P_2$, where $P_1\simeq nR$. $D^*(P_2)=D^*(P)-D^*(P_1)\leqq 1$ implies $P_2\lesssim R$ by Lemma 2.3.

(3)$\Rightarrow$(4) It is trivial.

(4)$\Rightarrow$(1) If P is directly infinite, then there exists a set $\{P_i\}_1^\infty$ of independent non-zero cyclic submodules of P such that $P_i\simeq P_j$ for all i, j. Then $D^*(\oplus_1^\infty P_i)=\infty$. This contradicts $D^*(P)\leqq t$.

REMARK. A right ideal of R is projective if and only if it is countably generated. Further any right ideal has a projective submodule as an essential one [4, Lemmas 12 and 13].

The next three results follow to the advice of K. Oshiro.

Lemma 2.5. *Let P and Q be countably generated but not finitely generated projective R-modules. If $D^*(P)=D^*(Q)$, then $P\simeq Q$.*

Proof. Since P and Q are not finitely generated, we put $P=\oplus_1^\infty P_n$ and $Q=\oplus_1^\infty Q_m$, where each P_n and Q_m are isomorphic to some non-zero members of $L(R)$. We prove that there exist two increasing sequences $1=n(1)<n(2)<\cdots$, $1\leqq m(1)<m(2)<\cdots$, of positive integers and two sets $\{A_i\}_1^\infty$, $\{B_i\}_1^\infty$ of independent non-zero submodules of P satisfying, for each i

(1) $\oplus_{n(i)+1}^{n(i+1)}P_j=B_i\oplus A_{i+1}$

(2) $\oplus_{m(i-1)+1}^{m(i)}Q_t\simeq A_i\oplus B_i$

where $A_1=P_1$ and $m(0)=0$.
First we choose integers $1\leqq m(1)$, $1<n(2)$ such that $D^*(P_1)<D^*(\oplus_1^{m(1)}Q_t)\leqq D^*(\oplus_1^{n(2)}P_j)$. Then, by Lemma 2.2, we have $P_1\oplus X\cong\oplus_1^{m(1)}Q_t$ and $\oplus_1^{m(1)}Q_t\oplus Y\cong\oplus_1^{n(2)}P_j$, for some modules X, Y. Then we have $X\oplus Y\cong\oplus_2^{n(2)}P$ by [6, Theorem 3.9]. Put $n(1)=1$, $A_1=P_1$ and $B_1\oplus A_2=\oplus_2^{n(2)}P_j$, where $B_1\cong X$ and $A_2\cong Y$. Next we assume that there exist two increasing sequences, $n(1)<\cdots<n(k+1)$, $m(1)<\cdots<m(k)$ and two sets $\{A_i\}_1^{k+1}$, $\{B_i\}_1^k$ of independent non-zero submodules of P satisfying the properties (1) and (2). Since $\oplus_1^k(A_i\oplus B_i)\cong\oplus_1^{m(k)}Q_t$ and $D^*(P)=D^*(Q)$, then $D^*(A_{k+1}\oplus(\oplus_{n(k+1)+1}^{\infty}P_i))=D^*(\oplus_{m(k)+1}^{\infty}Q_t)$. We can take positive integers $m(k+1)$, $n(k+2)$ such that $m(k)<m(k+1)$, $n(k)<n(k+2)$ and $D^*(A_{k+1})<D^*(\oplus_{m(k)+1}^{m(k+1)}Q_t)\leqq D^*(A_{k+1}\oplus(\oplus_{n(k+1)+1}^{n(k+2)}P_j))$. Then, again by Lemma 2.2, we obtain $A_{k+1}\oplus X'\cong\oplus_{m(k)+1}^{m(k+1)}Q_t$ and $\oplus_{m(k)+1}^{m(k+1)}Q_t\oplus Y'\cong A_{k+1}\oplus(\oplus_{n(k+1)+1}^{n(k+2)}P_j)$, for some modules X', Y'. Since $A_{k+1}\oplus X'\oplus Y'\cong A_{k+1}\oplus(\oplus_{n(k+1)+1}^{n(k+2)}P_j)$, then we have a decomposition $\oplus_{n(k+1)+1}^{n(k+2)}P_j=B_{k+1}\oplus A_{k+2}$, where $B_{k+1}\cong X'$ and $A_{k+2}\cong Y'$, by [6, Theorem 3.9]. By the above procedure, we can construct independent non-zero submodules $A_1, B_1, A_2, B_2, \cdots$ which satisfy the properties (1) and (2). Since each P_n is contained in $B_i\oplus A_{i+1}$ for some i, then $P=\oplus_1^{\infty}(A_i\oplus B_i)$. On the other hand we have $Q=\oplus_1^{\infty}(\oplus_{m(i-1)+1}^{m(i)}Q_t)$. Therefore we conclude that $P\cong Q$.

REMARK. The result obtained by applying Lemma 2.5 for P, Q in $P^*(R)$ means that the Grothendieck group generated by the isomorphism classes of directly finite projective R-modules is isomorphic to some subgroup of the additive group of $\boldsymbol{R}$. (Cf. [2, Corollaries. 10.14 and 10.16]).

Theorem 2.6. *Let R be a simple, regular and directly finite ring satisfying the comparability axiom. Any directly infinite projective R-modules is a free R-module.*

Proof. By Theorem 2.4 and Lemma 2.5, we already see that every directly infinite, countably generated projective R-module is isomorphic to $\aleph_0 R$. Thus we shall show that every directly infinite projective R-module can be expressed as a direct sum of directly infinite, countably generated submodules. Let $P=\oplus_{\alpha\in I}P_\alpha$ be a directly infinite projective R-module, where each P_α is isomorphic to some non-zero J in $L(R)$. Let $\mathfrak{B}$ be the set of all countably infinite subsets of I. We consider the family consisting of all subsets $\mathfrak{F}$ of $\mathfrak{B}$ satisfying the following properties;

(1) each members of $\mathfrak{F}$ is pairwise disjoint

(2) $D^*(\oplus_{\alpha\in K}P_\alpha)=\infty$ for each K in $\mathfrak{F}$.

Since this family is a inductively ordered set using the inclusion relation, there exists a maximal member $\mathfrak{F}$ by Zorn's Lemma. Put $I^*=\cup_{K\in\mathfrak{F}}K$. If $I^*=I$, then our proof is complete. Next we consider the case that $I^*\neq I$. First we shall show that $D^*(\oplus_{\alpha\in I^{**}}P_\alpha)<\infty$, where I^{**} is the complement of I^*. Otherwise we can take a countably infinite subset I' of I^{**} such that $D^*(\oplus_{\alpha\in I'}P_\alpha) =\infty$. Then the set $\mathfrak{F}\cup\{I'\}$ is strictly greater than $\mathfrak{F}$. This is a contradiction. By the proof of Lemma 2.3, we see that I^{**} is a countable set. Choose one member K' of $\mathfrak{F}$, and put $\mathfrak{F}'=\mathfrak{F}-\{K'\}$, and $K''=K'\cup I^{**}$. Then K'' is a countably infinite set and $D^*(\oplus_{\alpha\in K''}P_\alpha)=\infty$. The decomposition $P=(\oplus_{K\in\mathfrak{F}'}(\oplus_{\alpha\in K}P_\alpha))\oplus(\oplus_{\alpha\in K''}P_\alpha)$ is a desired one.

DEFINITION [5, p. 174]. Let A be a module. If $A=0$, define $\mu(A)=0$. If $A\neq 0$, define $\mu(A)$ to be the smallest infinite cardinal number α such that $\alpha A\not\lesssim A$.

Proposition 2.7. *Let P and S be projective modules which are not finitely generated. If $P\lesssim S$ and $S\lesssim P$, then $P\simeq S$.*

Proof. Since $D^*(P)=D^*(S)$ by the definition of D^*, then they are both directly finite or both directly infinite by Theorem 2.4. If P and S are directly finite, then they are countably generated by the proof of Lemma 2.3. Thus we have $P\simeq S$ by Lemma 2.5. If P and S are directly infinite, then $P\simeq\alpha R$ and $S\simeq\beta R$ for some infinite cardinal numbers α, β by Theorem 2.6. We can assume $\alpha\leq\beta$. Let Q be the maximal ring of quotients of R and we use the notation $E(A)$ to stand for an injective hull of a module A. Since $P\lesssim S$ and $S\lesssim P$, then $E(P)\simeq E(S)$ by [1, Corollary]. On the other hand, $E(P)\simeq E(\alpha Q)$ and $E(S)\simeq E(\beta Q)$ and also Q is a prime ring because it satisfies the comparability. Therefore, by [5, Theorem 6.32], $\max\{\alpha', \mu(Q)\}=\mu(E(P))=\mu(E(S))=\max\{\beta', \mu(Q)\}$, where α' and β' are the successores of α and β. Thus, if $\alpha<\beta$, then it must hold that $(\aleph_1\leqq)\alpha'<\beta'\leqq\mu(Q)$. Since $\aleph_1<\mu(Q)$, $\aleph_1 Q\lesssim Q$. Therefore let $\{A_\tau\}_{\tau\in I}$ be a independent set of principal right ideals of Q such that $A_\tau\simeq Q$ for each τ in I and the cardinality of I is $\aleph_1$. Then $\{A_\tau\cap R\}_{\tau\in I}$ is a independent set of non-zero right ideals of R. This contradicts the fact that there is no uncountable direct sum of non-zero right ideals of R. Consequently we must have $\alpha=\beta$ and hence $P\simeq S$.

3. Directly finite, regular and right self-injective ring

Lemma 3.1 [3, Lemma 5′ and 6, Proposition 1.4]. *A prime, directly finite, regular and right self-injective ring is a simple ring satisfying the comparability axiom.*

Proposition 3.2. *Let R be a directly finite, regular and right self-injective ring. Then R is a finite direct product of simple rings if and only if any non-singular directly finite injective R-module is finitely generated.*

Proof. First we shall prove that "only if" part. There exists a set $\{e_i\}_1^n$ of orthogonal central idempotents such that $\sum_1^n e_i=1$ and each e_iR is a simple ring. Let M be a non-singular directly finite injective R-module. There exists a projective R-module P such that P is an essential submodule of M, because any non-singular finitely generated R-module is a projective and injective module (cf. [9, Theorem 2.7]). M is directly finite, and so P is also directly finite. Put $P_i = Pe_i$ for each i, then each P_i is also a directly finite projective module as an e_iR-module. Therefore there exists a positive integer t such that $P_i \lesssim t(e_iR)$ for all i by Lemma 3.1 and Theorem 2.4. Thus $P \lesssim tR$, because $P = \oplus_1^n P_i$. This monomorphism can be extended to be monomorphism from M into tR. Then M is isomorphic to a direct summand of tR. Conversely we assume that R can be decomposed into no finite direct product of prime rings. Then R itself is not prime. Hence there exist non-zero two-sided ideals A, B such that $AB=0$. Let A', B' be the injective hull of A, B in R, then they are also two-sided ideals and generated by central idempotents by [3, Lemma 1]. Since R is semi-prime, $A \cap B = 0$. Then $A' \cap B' = 0$. Hence there exist orthogonal central idempotents $\{e_i\}_1^3$ such that $\sum_1^3 e_i = 1$. By the assumption, at least one of e_iR, say e_jR, is not prime. Use the same argument for the ring e_jR, then there exists another set $\{e_i'\}_1^5$ of orthogonal central idempotents of R such that $\sum_1^5 e_i' = 1$. Repeating these procedures, we obtain a countably infinite set $\{e_n\}_1^\infty$ of orthogonal non-zero central idempotents. If $\oplus_1^\infty e_nR$ is not essential in R_R, we choose some central idempotent f which generates the injective hull of $\oplus_1^\infty e_nR$ and we consider $\{e_n, 1-f\}_1^\infty$. Therefore we may assume that $\oplus_1^\infty e_nR$ is essential in R_R. Since R_R is injective and $\oplus_1^\infty e_nR$ is a two-sided ideal, $R \cong End_R(\oplus_1^\infty e_nR)$. $End_R(\oplus_1^\infty e_nR) \cong \prod_n End_R(e_nR) \cong \prod_n e_nR$, because $Hom_R(e_nR, e_mR)=0$ for $n \neq m$ and each e_n is a central idempotent. Consequently $R \cong \prod_n e_nR$ by the mapping: $r \to (e_nr)$. We put $M_n = n(e_n\mathrm{R})$ for each n and we consider the R-module $\mathrm{M} = \prod_n M_n$. This is obviously a non-singular injective R-module. We also know that it is directly finite, because $End_R(M) \cong \prod_n End_R(M_n)$ and $End_R(M_n)$ is directly finite for all n. By the assumption, there exists a positive integer t such that $M \lesssim tR$. Now we choose an integer m which is larger than t. That $M_m \lesssim tR \cong \prod_n t(e_nR)$ implies that $M_m \lesssim t(e_mR)$, because $Hom_R(M_m, t(e_nR))=0$ for all $n \neq m$. This contradicts that M_m is directly finite. Hence R is a finite direct product of prime rings. Prime directly finite regular right self-injective rings are simple by Lemma 3.1, and so we have proved.

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