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| Title        | Projective modules over simple regular rings                              |
| Author(s)    | Kado, Jiro  |
| Citation     | Osaka Journal of Mathematics. 1979, 16(2), p. 405-412                     |
| Version Type | VoR   |
| URL          | <a href="https://doi.org/10.18910/5144">https://doi.org/10.18910/5144</a> |
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## PROJECTIVE MODULES OVER SIMPLE REGULAR RINGS

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(Received July 7, 1978)  
(Revised October 30, 1978)

Recently K.R. Goodearl and D. Handelman [6] have studied simple regular rings from the point of view of dimension-like functions. They have shown that there exists a unique dimension function on the lattice of principal right ideals of a simple, regular and directly finite ring satisfying the comparability axiom. In this note we study some structures of projective modules over such a ring by making use of the dimension function.

In the section 1 we show that if there exists a dimension function on the lattice of principal right ideals of a regular ring, then this can be extended to a function on the set of all projective modules.

In the section 2 we investigate some structures of projective modules over a simple, regular and directly finite ring satisfying the comparability axiom and show that a directly finite projective module is isomorphic to a direct sum of a finitely generated free module and a projective right ideal, and a directly infinite projective module is a free module.

In the final section directly finite, regular and right self-injective rings are investigated. We show that this ring is a finite direct product of simple rings if and only if any non-singular directly finite injective right module is a finitely generated module.

Throughout this paper a ring  $R$  is an associative ring with identity and modules are unitary right  $R$ -modules.

### 1. Dimension functions

For any (Von Neumann) regular ring  $R$ , let  $L(R)$  be the lattice of principal right ideals and  $P(R)$  ( $FP(R)$ ) the set of all projective (finitely generated projective)  $R$ -modules. We denote by  $M \lesssim N$  the fact that  $M$  is isomorphic to a submodule of  $N$  for two modules  $M, N$ . In particular if  $R$  is regular, then  $A \lesssim P$  for  $A$  in  $FP(R)$  and  $P$  in  $P(R)$  if and only if  $A$  is isomorphic to a direct summand of  $P$  [8, Lemma 4].

DEFINITION [6, p. 807]. A *dimension function*  $D$  on  $L(R)$  is a function from  $L(R)$  into non-negative real numbers satisfying the following conditions;

- (1)  $D(J)=0$  if and only if  $J=0$
- (2)  $D(R)=1$
- (3) if  $J \lesssim K$ , then  $D(J) \leq D(K)$
- (4) if  $J \oplus K \in L(R)$ , then  $D(J \oplus K) = D(J) + D(K)$ .

I. Halperin [7] proved that if a dimension function  $D$  exists on  $L(R)$ , then  $D$  can be uniquely extended to a function on  $FP(R)$ . We shall show that this function  $D$  can be moreover extended to a function on  $P(R)$  by making use of the following lemma.

**Lemma 1.1** [10]. *For any projective module  $P$  over a regular ring,  $P$  is isomorphic to a direct sum of principal right ideals and any two direct sum decompositions of  $P$  have an isomorphic refinement.*

Let  $P$  be in  $P(R)$ . From now on, by  $P = \bigoplus_{J \in \mathfrak{M}} J$  we denote the fact that there exists a set  $\mathfrak{M}$  of independent non-zero submodules isomorphic to some principal right ideal and  $P$  is a direct sum of the members of  $\mathfrak{M}$ . We put  $D^*(P) = \sup \{ \sum_{J \in \mathfrak{M}'} D(J) \}$ ; any finite subset  $\mathfrak{M}'$  of  $\mathfrak{M}$  for any  $P$  in  $P(R)$  and any decomposition  $P = \bigoplus_{J \in \mathfrak{M}'} J$ . If the above supremum is not convergent, we put  $D^*(P) = \infty$ . Now we shall prove that  $D^*(P)$  does not depend on the decomposition of  $P$ . Let  $P = \bigoplus_{K \in \mathfrak{N}} K$  be another decomposition. It is sufficient to prove that two numbers  $a, b$  defined by  $\mathfrak{M}$  and  $\mathfrak{N}$  coincide when  $\mathfrak{N}$  is a refinement of  $\mathfrak{M}$ . For any  $J$  in  $\mathfrak{M}$ , there exists a finite subset  $\mathfrak{N}'$  of  $\mathfrak{N}$  such that  $J = \bigoplus_{K \in \mathfrak{N}'} K$ . Hence we have  $a \leq b$ . Conversely for any finite subset  $\mathfrak{N}'$  of  $\mathfrak{N}$  and any  $K$  in  $\mathfrak{N}'$ , there exists some  $J$  in  $\mathfrak{M}$  such that  $K$  is a direct summand of  $J$ . Therefore there exists a finite subset  $\mathfrak{M}'$  of  $\mathfrak{M}$  such that  $\sum_{K \in \mathfrak{N}'} D(K) \leq \sum_{J \in \mathfrak{M}'} D(J)$  and so we have  $b \leq a$ .

Now  $D^*$  is a function from  $P(R)$  into non-negative real numbers or  $\infty$ , and from the definition and by Lemma 1.1, we can easily prove the following properties;

- (1) if  $P \lesssim Q$  in  $P(R)$ , then  $D^*(P) \leq D^*(Q)$
- (2) if  $P \oplus Q \in P(R)$ , then  $D^*(P \oplus Q) = D^*(P) + D^*(Q)$ .

## 2. Projective modules

First we recall some definitions and some results in [6].

**DEFINITION.** A ring  $R$  is *directly finite* if  $xy=1$  implies  $yx=1$  for  $x, y$  in  $R$ . A module  $M$  is *directly finite* if  $End_r(M)$  is directly finite. A ring  $R$  (a module  $M$ ) is *directly infinite* if it is not directly finite. It is easily seen that a module  $M$  is directly finite if and only if  $M$  is not isomorphic to a proper direct summand of itself. A regular ring  $R$  satisfies the *comparability axiom* if we have either  $J \lesssim K$  or  $K \lesssim J$  for all  $J, K$  in  $L(R)$ . For a cardinal number  $\alpha$  and

a module  $M$ ,  $\alpha M$  denotes a direct sum of  $\alpha$  copies of  $M$ .

NOTE. Throughout this section  $R$  is a simple, regular and directly finite ring satisfying the comparability axiom. In this case, any finitely generated projective  $R$ -module is directly finite by [6, Corollary 3.10].

EXAMPLE [6, pp. 815, 831 and 832]. Let  $F$  be a field and  $R_n$  the full matrix ring of degree  $2^n$  over  $F$ . Let  $f_n: R_n \rightarrow R_{n+1}$  be a diagonal homomorphism, i.e.,  $x \rightarrow \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ , and let  $R$  be a direct limit of  $\{R_n, f_n\}$ . This ring  $R$  is a simple, regular and directly finite ring which satisfies the comparability axiom and which is not artinian. Further  $R$  is neither left nor right self-injective.

**Lemma 2.1** [6, Theorem 3.13 and Proposition 3.14]. *Let  $J$  be in  $L(R)$ . We put  $D(J) = \sup\{mn^{-1}; m \geq 0, n > 0, mR \lesssim nJ\}$ . Then  $D$  is a unique dimension function on  $L(R)$ . Further, for all  $J, K$  in  $L(R)$ , we have  $J \lesssim K$  if and only if  $D(J) \leq D(K)$ .*

From now on, let  $D^*$  be the extension of the dimension function  $D$  as in the section 1. We consider projective modules over  $R$  from the point of view of  $D^*$ .

**Lemma 2.2** *Let  $A, B$  in  $FP(R)$ .  $A \lesssim B$  if and only if  $D^*(A) \leq D^*(B)$ . In particular,  $A \cong B$  if and only if  $D^*(A) = D^*(B)$ .*

Proof. We have  $A \lesssim B$  or  $B \lesssim A$  by [6, Lemma 3.7]. Then the proof of the first property is easy. If  $D^*(A) = D^*(B)$ , then  $A \lesssim B$  and  $B \lesssim A$ . Hence  $A$  is isomorphic to a direct summand of itself. Then  $A \cong B$ , because  $A$  is directly finite.

The next is a key lemma for Theorem 2.4.

**Lemma 2.3.** *For  $P$  in  $P(R)$  and  $A$  in  $FP(R)$ ,  $P \lesssim A$  if and only if  $D^*(P) \leq D^*(A)$ .*

Proof. By the definition, "only if" part is trivial. We assume  $D^*(P) \leq D^*(A)$  and  $P = \bigoplus_{J \in \mathfrak{M}} J$ . First we know  $\mathfrak{M}$  is a countable set, because for each positive integer  $n$ , the set  $\mathfrak{M}_n = \{J; D(J) > n^{-1}\}$  is a finite set and  $\mathfrak{M} = \bigcup_n \mathfrak{M}_n$ . Now put  $\mathfrak{N} = \{J_n; n = 1, 2, \dots\}$  and  $P_n = \bigoplus_1^n J_i$ , then we have  $P = \bigcup_n P_n$ . For each  $n$ , we can choose a monomorphism  $f_n: P_n \rightarrow A$  by Lemma 2.2, because  $D^*(P_n) \leq D^*(A)$ . If we construct monomorphism  $g_n: P_n \rightarrow A$  for each  $n$  such that  $g_{n+1}$  is an extension of  $g_n$ , then we have  $P \lesssim A$ . Put  $g_1 = f_1$  and assume we have  $g_k$  for all  $k \leq n$ . We have decompositions  $A = g_n(P_n) \oplus Q_n = f_{n+1}(P_n) \oplus f_{n+1}(J_{n+1}) \oplus Q_{n+1}$  for some submodules  $Q_n, Q_{n+1}$ , because homomorphism  $g_n, f_{n+1}$  split. Then we have  $Q_n \cong f_{n+1}(J_{n+1}) \oplus Q_{n+1}$  by [6, Theorem 3.9] and so we choose a monomorphism  $h: f_{n+1}(J_{n+1}) \rightarrow Q_n$ . Consequently  $g_{n+1} = g_n \oplus hf_{n+1}: P_{n+1} \rightarrow A$  is an extension of  $g_n$ .

We shall determine the structures of projective modules over a simple, regular and directly finite ring satisfying the comparability axiom.

**Theorem 2.4.** *Let  $R$  be a simple, regular and directly finite ring satisfying the comparability axiom. For a projective  $R$ -module  $P$ , the following conditions are equivalent.*

- (1)  $P$  is directly finite.
- (2)  $D^*(P) < \infty$
- (3)  $P$  has a decomposition  $P \cong nR \oplus J$  for some integer  $n \geq 0$  and some right ideal  $J$ .
- (4)  $P \lesssim tR$  for some integer  $t > 0$ .

Proof. (1) $\Rightarrow$ (2). We assume  $D^*(P) = \infty$ . Put  $P = \bigoplus_{J \in \mathfrak{M}} J$ , then there exists a sequence of finite subsets  $\mathfrak{M}_i$  ( $i=1, 2, \dots$ ) of  $\mathfrak{M}$  such that  $\mathfrak{M}_i \cap \mathfrak{M}_j = \emptyset$  if  $i \neq j$  and  $D^*(\bigoplus_{J \in \mathfrak{M}_i} J) \geq 1$  for each  $i$ . Put  $P_i = \bigoplus_{J \in \mathfrak{M}_i} J$ , then we have  $R \lesssim P_i$  by Lemma 2.2 and so we have  $P_i = R_i \oplus Q_i$ , where  $R_i \cong R$ .  $F = \bigoplus_1^\infty R_i$  is a direct summand of  $P$  and  $2F \cong F$ . This contradicts that every direct summand of  $P$  is also directly finite.

(2) $\Rightarrow$ (3). We choose non-negative integer  $n$  such that  $n < D^*(P) \leq n+1$ . If  $n=0$ , then we have  $P \lesssim R$  by Lemma 2.3. If  $n$  is positive, the first inequality implies that  $nR \lesssim P$  from the definition of  $D^*$  and by Lemma 2.2. Then we have  $P = P_1 \oplus P_2$ , where  $P_1 \cong nR$ .  $D^*(P_2) = D^*(P) - D^*(P_1) \leq 1$  implies  $P_2 \lesssim R$  by Lemma 2.3.

(3) $\Rightarrow$ (4) It is trivial.

(4) $\Rightarrow$ (1) If  $P$  is directly infinite, then there exists a set  $\{P_i\}_1^\infty$  of independent non-zero cyclic submodules of  $P$  such that  $P_i \cong P_j$  for all  $i, j$ . Then  $D^*(\bigoplus_1^\infty P_i) = \infty$ . This contradicts  $D^*(P) \leq t$ .

REMARK. A right ideal of  $R$  is projective if and only if it is countably generated. Further any right ideal has a projective submodule as an essential one [4, Lemmas 12 and 13].

The next three results follow to the advice of K. Oshiro.

**Lemma 2.5.** *Let  $P$  and  $Q$  be countably generated but not finitely generated projective  $R$ -modules. If  $D^*(P) = D^*(Q)$ , then  $P \cong Q$ .*

Proof. Since  $P$  and  $Q$  are not finitely generated, we put  $P = \bigoplus_1^\infty P_n$  and  $Q = \bigoplus_1^\infty Q_m$ , where each  $P_n$  and  $Q_m$  are isomorphic to some non-zero members of  $L(R)$ . We prove that there exist two increasing sequences  $1 = n(1) < n(2) < \dots$ ,  $1 \leq m(1) < m(2) < \dots$ , of positive integers and two sets  $\{A_i\}_1^\infty$ ,  $\{B_i\}_1^\infty$  of independent non-zero submodules of  $P$  satisfying, for each  $i$

- (1)  $\bigoplus_{n(i)+1}^{n(i+1)} P_j = B_i \oplus A_{i+1}$
- (2)  $\bigoplus_{m(i-1)+1}^{m(i)} Q_j \cong A_i \oplus B_i$

where  $A_1 = P_1$  and  $m(0) = 0$ .

First we choose integers  $1 \leq m(1), 1 < n(2)$  such that  $D^*(P_1) < D^*(\bigoplus_1^{m(1)} Q_i) \leq D^*(\bigoplus_1^{n(2)} P_j)$ . Then, by Lemma 2.2, we have  $P_1 \oplus X \cong \bigoplus_1^{m(1)} Q_i$  and  $\bigoplus_1^{m(1)} Q_i \oplus Y \cong \bigoplus_1^{n(2)} P_j$ , for some modules  $X, Y$ . Then we have  $X \oplus Y \cong \bigoplus_2^{n(2)} P$  by [6, Theorem 3.9]. Put  $n(1) = 1, A_1 = P_1$  and  $B_1 \oplus A_2 = \bigoplus_2^{n(2)} P_j$ , where  $B_1 \cong X$  and  $A_2 \cong Y$ . Next we assume that there exist two increasing sequences,  $n(1) < \dots < n(k+1), m(1) < \dots < m(k)$  and two sets  $\{A_i\}_1^{k+1}, \{B_i\}_1^k$  of independent non-zero submodules of  $P$  satisfying the properties (1) and (2). Since  $\bigoplus_1^k (A_i \oplus B_i) \cong \bigoplus_1^{m(k)} Q_i$  and  $D^*(P) = D^*(Q)$ , then  $D^*(A_{k+1} \oplus (\bigoplus_{n(k+1)+1}^\infty P_i)) = D^*(\bigoplus_{m(k)+1}^\infty Q_i)$ . We can take positive integers  $m(k+1), n(k+2)$  such that  $m(k) < m(k+1), n(k) < n(k+2)$  and  $D^*(A_{k+1}) < D^*(\bigoplus_{m(k)+1}^{m(k+1)} Q_i) \leq D^*(A_{k+1} \oplus (\bigoplus_{n(k+1)+1}^{n(k+2)} P_j))$ . Then, again by Lemma 2.2, we obtain  $A_{k+1} \oplus X' \cong \bigoplus_{m(k)+1}^{m(k+1)} Q_i$  and  $\bigoplus_{m(k)+1}^{m(k+1)} Q_i \oplus Y' \cong A_{k+1} \oplus (\bigoplus_{n(k+1)+1}^{n(k+2)} P_j)$ , for some modules  $X', Y'$ . Since  $A_{k+1} \oplus X' \oplus Y' \cong A_{k+1} \oplus (\bigoplus_{n(k+1)+1}^{n(k+2)} P_j)$ , then we have a decomposition  $\bigoplus_{n(k+1)+1}^{n(k+2)} P_j = B_{k+1} \oplus A_{k+2}$ , where  $B_{k+1} \cong X'$  and  $A_{k+2} \cong Y'$ , by [6, Theorem 3.9]. By the above procedure, we can construct independent non-zero submodules  $A_1, B_1, A_2, B_2, \dots$  which satisfy the properties (1) and (2). Since each  $P_n$  is contained in  $B_i \oplus A_{i+1}$  for some  $i$ , then  $P = \bigoplus_1^\infty (A_i \oplus B_i)$ . On the other hand we have  $Q = \bigoplus_1^\infty (\bigoplus_{m(i-1)+1}^{m(i)} Q_i)$ . Therefore we conclude that  $P \cong Q$ .

REMARK. The result obtained by applying Lemma 2.5 for  $P, Q$  in  $P^*(R)$  means that the Grothendieck group generated by the isomorphism classes of directly finite projective  $R$ -modules is isomorphic to some subgroup of the additive group of  $\mathbf{R}$ . (Cf. [2, Corollaries. 10.14 and 10.16]).

**Theorem 2.6.** *Let  $R$  be a simple, regular and directly finite ring satisfying the comparability axiom. Any directly infinite projective  $R$ -modules is a free  $R$ -module.*

Proof. By Theorem 2.4 and Lemma 2.5, we already see that every directly infinite, countably generated projective  $R$ -module is isomorphic to  $\aleph_0 R$ . Thus we shall show that every directly infinite projective  $R$ -module can be expressed as a direct sum of directly infinite, countably generated submodules. Let  $P = \bigoplus_{\alpha \in I} P_\alpha$  be a directly infinite projective  $R$ -module, where each  $P_\alpha$  is isomorphic to some non-zero  $J$  in  $L(R)$ . Let  $\mathfrak{B}$  be the set of all countably infinite subsets of  $I$ . We consider the family consisting of all subsets  $\mathfrak{F}$  of  $\mathfrak{B}$  satisfying the following properties;

- (1) each members of  $\mathfrak{F}$  is pairwise disjoint
- (2)  $D^*(\bigoplus_{\alpha \in K} P_\alpha) = \infty$  for each  $K$  in  $\mathfrak{F}$ .

Since this family is a inductively ordered set using the inclusion relation, there exists a maximal member  $\mathfrak{F}$  by Zorn's Lemma. Put  $I^* = \bigcup_{K \in \mathfrak{F}} K$ . If  $I^* = I$ , then our proof is complete. Next we consider the case that  $I^* \neq I$ . First we shall show that  $D^*(\bigoplus_{\alpha \in I^{**}} P_\alpha) < \infty$ , where  $I^{**}$  is the complement of  $I^*$ . Other-

wise we can take a countably infinite subset  $I'$  of  $I^{**}$  such that  $D^*(\bigoplus_{\alpha \in I'} P_\alpha) = \infty$ . Then the set  $\mathfrak{F} \cup \{I'\}$  is strictly greater than  $\mathfrak{F}$ . This is a contradiction. By the proof of Lemma 2.3, we see that  $I^{**}$  is a countable set. Choose one member  $K'$  of  $\mathfrak{F}$ , and put  $\mathfrak{F}' = \mathfrak{F} - \{K'\}$ , and  $K'' = K' \cup I^{**}$ . Then  $K''$  is a countably infinite set and  $D^*(\bigoplus_{\alpha \in K''} P_\alpha) = \infty$ . The decomposition  $P = (\bigoplus_{K \in \mathfrak{F}'} (\bigoplus_{\alpha \in K} P_\alpha)) \oplus (\bigoplus_{\alpha \in K''} P_\alpha)$  is a desired one.

**DEFINITION** [5, p. 174]. Let  $A$  be a module. If  $A = 0$ , define  $\mu(A) = 0$ . If  $A \neq 0$ , define  $\mu(A)$  to be the smallest infinite cardinal number  $\alpha$  such that  $\alpha A \not\leq A$ .

**Proposition 2.7.** *Let  $P$  and  $S$  be projective modules which are not finitely generated. If  $P \leq S$  and  $S \leq P$ , then  $P \cong S$ .*

*Proof.* Since  $D^*(P) = D^*(S)$  by the definition of  $D^*$ , then they are both directly finite or both directly infinite by Theorem 2.4. If  $P$  and  $S$  are directly finite, then they are countably generated by the proof of Lemma 2.3. Thus we have  $P \cong S$  by Lemma 2.5. If  $P$  and  $S$  are directly infinite, then  $P \cong \alpha R$  and  $S \cong \beta R$  for some infinite cardinal numbers  $\alpha, \beta$  by Theorem 2.6. We can assume  $\alpha \leq \beta$ . Let  $Q$  be the maximal ring of quotients of  $R$  and we use the notation  $E(A)$  to stand for an injective hull of a module  $A$ . Since  $P \leq S$  and  $S \leq P$ , then  $E(P) \cong E(S)$  by [1, Corollary]. On the other hand,  $E(P) \cong E(\alpha Q)$  and  $E(S) \cong E(\beta Q)$  and also  $Q$  is a prime ring because it satisfies the comparability. Therefore, by [5, Theorem 6.32],  $\max\{\alpha', \mu(Q)\} = \mu(E(P)) = \mu(E(S)) = \max\{\beta', \mu(Q)\}$ , where  $\alpha'$  and  $\beta'$  are the successors of  $\alpha$  and  $\beta$ . Thus, if  $\alpha < \beta$ , then it must hold that  $(\aleph_1 \leq) \alpha' < \beta' \leq \mu(Q)$ . Since  $\aleph_1 < \mu(Q)$ ,  $\aleph_1 Q \leq Q$ . Therefore let  $\{A_\tau\}_{\tau \in I}$  be a independent set of principal right ideals of  $Q$  such that  $A_\tau \cong Q$  for each  $\tau$  in  $I$  and the cardinality of  $I$  is  $\aleph_1$ . Then  $\{A_\tau \cap R\}_{\tau \in I}$  is a independent set of non-zero right ideals of  $R$ . This contradicts the fact that there is no uncountable direct sum of non-zero right ideals of  $R$ . Consequently we must have  $\alpha = \beta$  and hence  $P \cong S$ .

### 3. Directly finite, regular and right self-injective ring

**Lemma 3.1** [3, Lemma 5' and 6, Proposition 1.4]. *A prime, directly finite, regular and right self-injective ring is a simple ring satisfying the comparability axiom.*

**Proposition 3.2.** *Let  $R$  be a directly finite, regular and right self-injective ring. Then  $R$  is a finite direct product of simple rings if and only if any non-singular directly finite injective  $R$ -module is finitely generated.*

*Proof.* First we shall prove that “only if” part. There exists a set  $\{e_i\}_1^n$  of orthogonal central idempotents such that  $\sum_{i=1}^n e_i = 1$  and each  $e_i R$  is a simple

ring. Let  $M$  be a non-singular directly finite injective  $R$ -module. There exists a projective  $R$ -module  $P$  such that  $P$  is an essential submodule of  $M$ , because any non-singular finitely generated  $R$ -module is a projective and injective module (cf. [9, Theorem 2.7]).  $M$  is directly finite, and so  $P$  is also directly finite. Put  $P_i = Pe_i$  for each  $i$ , then each  $P_i$  is also a directly finite projective module as an  $e_iR$ -module. Therefore there exists a positive integer  $t$  such that  $P_i \lesssim t(e_iR)$  for all  $i$  by Lemma 3.1 and Theorem 2.4. Thus  $P \lesssim tR$ , because  $P = \bigoplus_1^n P_i$ . This monomorphism can be extended to be monomorphism from  $M$  into  $tR$ . Then  $M$  is isomorphic to a direct summand of  $tR$ . Conversely we assume that  $R$  can be decomposed into no finite direct product of prime rings. Then  $R$  itself is not prime. Hence there exist non-zero two-sided ideals  $A, B$  such that  $AB=0$ . Let  $A', B'$  be the injective hull of  $A, B$  in  $R$ , then they are also two-sided ideals and generated by central idempotents by [3, Lemma 1]. Since  $R$  is semi-prime,  $A \cap B = 0$ . Then  $A' \cap B' = 0$ . Hence there exist orthogonal central idempotents  $\{e_i\}_1^3$  such that  $\sum_1^3 e_i = 1$ . By the assumption, at least one of  $e_iR$ , say  $e_jR$ , is not prime. Use the same argument for the ring  $e_jR$ , then there exists another set  $\{e'_i\}_1^5$  of orthogonal central idempotents of  $R$  such that  $\sum_1^5 e'_i = 1$ . Repeating these procedures, we obtain a countably infinite set  $\{e_n\}_1^\infty$  of orthogonal non-zero central idempotents. If  $\bigoplus_1^\infty e_nR$  is not essential in  $R_R$ , we choose some central idempotent  $f$  which generates the injective hull of  $\bigoplus_1^\infty e_nR$  and we consider  $\{e_n, 1-f\}_1^\infty$ . Therefore we may assume that  $\bigoplus_1^\infty e_nR$  is essential in  $R_R$ . Since  $R_R$  is injective and  $\bigoplus_1^\infty e_nR$  is a two-sided ideal,  $R \cong \text{End}_R(\bigoplus_1^\infty e_nR)$ .  $\text{End}_R(\bigoplus_1^\infty e_nR) \cong \prod_n \text{End}_R(e_nR) \cong \prod_n e_nR$ , because  $\text{Hom}_R(e_nR, e_mR) = 0$  for  $n \neq m$  and each  $e_n$  is a central idempotent. Consequently  $R \cong \prod_n e_nR$  by the mapping:  $r \rightarrow (e_n r)$ . We put  $M_n = n(e_nR)$  for each  $n$  and we consider the  $R$ -module  $M = \prod_n M_n$ . This is obviously a non-singular injective  $R$ -module. We also know that it is directly finite, because  $\text{End}_R(M) \cong \prod_n \text{End}_R(M_n)$  and  $\text{End}_R(M_n)$  is directly finite for all  $n$ . By the assumption, there exists a positive integer  $t$  such that  $M \lesssim tR$ . Now we choose an integer  $m$  which is larger than  $t$ . That  $M_m \lesssim tR \cong \prod_n t(e_nR)$  implies that  $M_m \lesssim t(e_mR)$ , because  $\text{Hom}_R(M_m, t(e_nR)) = 0$  for all  $n \neq m$ . This contradicts that  $M_m$  is directly finite. Hence  $R$  is a finite direct product of prime rings. Prime directly finite regular right self-injective rings are simple by Lemma 3.1, and so we have proved.

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