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Recently K.R. Goodearl and D. Handelman [6] have studied simple regular rings from the point of view of dimension-like functions. They have shown that there exists a unique dimension function on the lattice of principal right ideals of a simple, regular and directly finite ring satisfying the comparability axiom. In this note we study some structures of projective modules over such a ring by making use of the dimension function.

In the section 1 we show that if there exists a dimension function on the lattice of principal right ideals of a regular ring, then this can be extended to a function on the set of all projective modules.

In the section 2 we investigate some structures of projective modules over a simple, regular and directly finite ring satisfying the comparability axiom and show that a directly finite projective module is isomorphic to a direct sum of a finitely generated free module and a projective right ideal, and a directly infinite projective module is a free module.

In the final section directly finite, regular and right self-injective rings are investigated. We show that this ring is a finite direct product of simple rings if and only if any non-singular directly finite injective right module is a finitely generated module.

Throughout this paper a ring $R$ is an associative ring with identity and modules are unitary right $R$-modules.

1. Dimension functions

For any (Von Neumann) regular ring $R$, let $L(R)$ be the lattice of principal right ideals and $P(R)$ ($FP(R)$) the set of all projective (finitely generated projective) $R$-modules. We denote by $M \leq N$ the fact that $M$ is isomorphic to a submodule of $N$ for two modules $M, N$. In particular if $R$ is regular, then $A \leq P$ for $A$ in $FP(R)$ and $P$ in $P(R)$ if and only if $A$ is isomorphic to a direct summand of $P$ [8, Lemma 4].

Definition [6, p. 807]. A dimension function $D$ on $L(R)$ is a function from $L(R)$ into non-negative real numbers satisfying the following conditions;
(1) \( D(J) = 0 \) if and only if \( J = 0 \)
(2) \( D(R) = 1 \)
(3) if \( J \leq K \), then \( D(J) \leq D(K) \)
(4) if \( J \oplus K \in L(R) \), then \( D(J \oplus K) = D(J) + D(K) \).

I. Halperin [7] proved that if a dimension function \( D \) exists on \( L(R) \), then \( D \) can be uniquely extended to a function on \( FP(R) \). We shall show that this function \( D \) can be moreover extended to a function on \( P(R) \) by making use of the following lemma.

**Lemma 1.1** [10]. *For any projective module \( P \) over a regular ring, \( P \) is isomorphic to a direct sum of principal right ideals and any two direct sum decompositions of \( P \) have an isomorphic refinement.*

Let \( P \) be in \( P(R) \). From now on, by \( P = \bigoplus_{J \in \mathfrak{R}} J \) we denote the fact that there exists a set \( \mathfrak{R} \) of independent non-zero submodules isomorphic to some principal right ideal and \( P \) is a direct sum of the members of \( \mathfrak{R} \). We put \( D^*(P) = \sup \{ \sum_{J \in \mathfrak{R}} D(J) ; \text{any finite subset } \mathfrak{R}' \text{ of } \mathfrak{R} \} \) for any \( P \) in \( P(R) \) and any decomposition \( P = \bigoplus_{J \in \mathfrak{R}} J \). If the above supremum is not convergent, we put \( D^*(P) = \infty \). Now we shall prove that \( D^*(P) \) does not depend on the decomposition of \( P \). Let \( P = \bigoplus_{K \in \mathfrak{R}} K \) be another decomposition. It is sufficient to prove that two numbers \( a, b \) defined by \( \mathfrak{R} \) and \( \mathfrak{R} \) coincide when \( \mathfrak{R} \) is a refinement of \( \mathfrak{R} \). For any \( J \) in \( \mathfrak{R} \), there exists a finite subset \( \mathfrak{R}' \) of \( \mathfrak{R} \) such that \( J = \bigoplus_{K \in \mathfrak{R}'} K \). Hence we have \( a \leq b \). Conversely for any finite subset \( \mathfrak{R}' \) of \( \mathfrak{R} \) and any \( K \) in \( \mathfrak{R}' \), there exists some \( J \) in \( \mathfrak{R} \) such that \( K \) is a direct summand of \( J \). Therefore there exists a finite subset \( \mathfrak{R}' \) of \( \mathfrak{R} \) such that \( D^*(P) = \sum_{J \in \mathfrak{R}'} D(J) \) and so we have \( b \leq a \).

Now \( D^* \) is a function from \( P(R) \) into non-negative real numbers or \( \infty \), and from the definition and by Lemma 1.1, we can easily prove the following properties;

(1) if \( P \leq Q \) in \( P(R) \), then \( D^*(P) \leq D^*(Q) \)
(2) if \( P \oplus Q \in P(R) \), then \( D^*(P \oplus Q) = D^*(P) + D^*(Q) \).

2. **Projective modules**

First we recall some definitions and some results in [6].

**Definition.** A ring \( R \) is *directly finite* if \( xy = 1 \) implies \( yx = 1 \) for \( x, y \) in \( R \). A module \( M \) is *directly finite* if \( \text{End}_R(M) \) is directly finite. A ring \( R \) (a module \( M \)) is *directly infinite* if it is not directly finite. It is easily seen that a module \( M \) is directly finite if and only if \( M \) is not isomorphic to a proper direct summand of itself. A regular ring \( R \) satisfies the *comparability axiom* if we have either \( J \leq K \) or \( K \leq J \) for all \( J, K \) in \( L(R) \). For a cardinal number \( \alpha \) and...
a module $M$, $\alpha M$ denotes a direct sum of $\alpha$ copies of $M$.

**NOTE.** Throughout this section $R$ is a simple, regular and directly finite ring satisfying the comparability axiom. In this case, any finitely generated projective $R$-module is directly finite by [6, Corollary 3.10].

**EXAMPLE** [6, pp. 815, 831 and 832]. Let $F$ be a field and $R_n$ the full matrix ring of degree $2^n$ over $F$. Let $f_n: R_n \to R_{n+1}$ be a diagonal homomorphism, i.e., $x \to (\overbrace{x \ldots x}^n)$, and let $R$ be a direct limit of $\{R_n, f_n\}$. This ring $R$ is a simple, regular and directly finite ring which satisfies the comparability axiom and which is not artinian. Further $R$ is neither left nor right self-injective.

**Lemma 2.1** [6, Theorem 3.13 and Proposition 3.14]. Let $J$ be in $L(R)$. We put $D(J) = \inf\{mn^{-1}; m \geq 0, n > 0, mR \subseteq nJ\}$. Then $D$ is a unique dimension function on $L(R)$. Further, for all $J, K$ in $L(R)$, we have $J \leq K$ if and only if $D(J) \leq D(K)$.

From now on, let $D^*$ be the extension of the dimension function $D$ as in the section 1. We consider projective modules over $R$ from the point of view of $D^*$.

**Lemma 2.2** Let $A, B$ in FP($R$). $A \leq B$ if and only if $D^*(A) \leq D^*(B)$. In particular, $A \approx B$ if and only if $D^*(A) = D^*(B)$.

Proof. We have $A \leq B$ or $B \leq A$ by [6, Lemma 3.7]. Then the proof of the first property is easy. If $D^*(A) = D^*(B)$, then $A \leq B$ and $B \leq A$. Hence $A$ is isomorphic to a direct summand of itself. Then $A \approx B$, because $A$ is directly finite.

The next is a key lemma for Theorem 2.4.

**Lemma 2.3.** For $P$ in $P(R)$ and $A$ in FP($R$), $P \leq A$ if and only if $D^*(P) \leq D^*(A)$.

Proof. By the definition, “only if” part is trivial. We assume $D^*(P) \leq D^*(A)$ and $P = \bigoplus_{J \in \mathcal{M}} J$. First we know $\mathcal{M}$ is a countable set, because for each positive integer $n$, the set $\mathcal{M}_n = \{J; D(J) > n^{-1}\}$ is a finite set and $\mathcal{M} = \bigcup_n \mathcal{M}_n$. Now put $\mathcal{M} = \{J_n; n = 1, 2, \ldots\}$ and $P_n = \bigoplus J_n$, then we have $P = \bigcup_n P_n$. For each $n$, we can choose a monomorphism $f_n: P_n \to A$ by Lemma 2.2, because $D^*(P_n) \leq D^*(A)$. If we construct monomorphism $g_n: P_n \to A$ for each $n$ such that $g_{n+1}$ is an extension of $g_n$, then we have $P \leq A$. Put $g_1 = f_1$ and assume we have $g_k$ for all $k \leq n$. We have decompositions $A = g_k(P_n) \oplus Q_n = f_{n+1}(P_n) \oplus f_{n+1}(J_{n+1}) \oplus Q_{n+1}$ for some submodules $Q_n, Q_{n+1}$ because homomorphism $g_n, f_{n+1}$ split. Then we have $Q_n \leq f_{n+1}(J_{n+1}) \oplus Q_{n+1}$ by [6, Theorem 3.9] and so we choose a monomorphism $h: f_{n+1}(J_{n+1}) \to Q_n$. Consequently $g_{n+1} = g_n \oplus hf_{n+1}: P_{n+1} \to A$ is an extension of $g_n$. 


We shall determine the structures of protective modules over a simple, regular and directly finite ring satisfying the comparability axiom.

**Theorem 2.4.** Let $R$ be a simple, regular and directly finite ring satisfying the comparability axiom. For a projective $R$-module $P$, the following conditions are equivalent.

1. $P$ is directly finite.
2. $D^*(P) < \infty$
3. $P$ has a decomposition $P = nR \oplus J$ for some integer $n \geq 0$ and some right ideal $J$.
4. $P \sim tR$ for some integer $t > 0$.

**Proof.** (1)$\Rightarrow$(2). We assume $D^*(P) = \infty$. Put $P = \bigoplus_{J \in \mathfrak{M}} J$, then there exists a sequence of finite subsets $\mathfrak{M}_i (i=1,2,\cdots)$ of $\mathfrak{M}$ such that $\mathfrak{M}_i \cap \mathfrak{M}_j = \emptyset$ if $i \neq j$ and $D^*(\bigoplus_{J \in \mathfrak{M}_i} J) \geq 1$ for each $i$. Put $P_i = \bigoplus_{J \in \mathfrak{M}_i} J$, then we have $R \leq P_i$ by Lemma 2.2 and so we have $P_1 = R_1 \oplus Q_1$, where $R_1 \simeq R$. $F = \bigoplus_{1}^\infty R_i$ is a direct summand of $P$ and $2F \simeq F$. This contradicts that every direct summand of $P$ is also directly finite.

(2)$\Rightarrow$(3). We choose non-negative integer $n$ such that $n < D^*(P) \leq n + 1$. If $n=0$, then we have $P \leq R$ by Lemma 2.3. If $n$ is positive, the first inequality implies that $nR \leq P$ from the definition of $D^*$ and by Lemma 2.2. Then we have $P = P_1 \oplus P_2$, where $P_1 \simeq nR$. $D^*(P_2) - D^*(P_1) \leq 1$ implies $P_2 \leq R$ by Lemma 2.3.

(3)$\Rightarrow$(4) It is trivial.

(4)$\Rightarrow$(1) If $P$ is directly infinite, then there exists a set $\{P_i\}_1^\infty$ of independent non-zero cyclic submodules of $P$ such that $P_i \simeq P_j$ for all $i, j$. Then $D^*(\bigoplus_{1}^\infty P_i) = \infty$. This contradicts $D^*(P) \leq 1$.

**Remark.** A right ideal of $R$ is projective if and only if it is countably generated. Further any right ideal has a projective submodule as an essential one [4, Lemmas 12 and 13].

The next three results follow to the advice of K. Oshiro.

**Lemma 2.5.** Let $P$ and $Q$ be countably generated but not finitely generated projective $R$-modules. If $D^*(P) = D^*(Q)$, then $P \simeq Q$.

**Proof.** Since $P$ and $Q$ are not finitely generated, we put $P = \bigoplus_{1}^\infty P_n$ and $Q = \bigoplus_{1}^\infty Q_m$, where each $P_n$ and $Q_m$ are isomorphic to some non-zero members of $L(R)$. We prove that there exist two increasing sequences $1 = n(1) < n(2) < \cdots, 1 \leq m(1) < m(2) < \cdots$, of positive integers and two sets $\{A_i\}_1^\infty, \{B_i\}_1^\infty$ of independent non-zero submodules of $P$ satisfying, for each $i$

1. $\bigoplus_{a+i}^{n+i} P_j \simeq B_i \oplus A_{i+1}$
2. $\bigoplus_{a+i}^{n+i} Q_j \simeq A_i \oplus B_i$
where $A_1 = P_1$ and $m(0) = 0$.

First we choose integers $1 \leq m(1), 1 < n(2)$ such that

\[ D^*(P_1) < D^*(\oplus_{\downarrow 1}^{\uparrow 1} Q_1) \leq D^*(\oplus_{\downarrow 2}^{\uparrow 2} P_j) \]

Then, by Lemma 2.2, we have $P_1 \oplus X \cong \oplus_{\downarrow 1}^{\uparrow 1} Q_1$ and $\oplus_{\downarrow 1}^{\uparrow 1} Q_1 \oplus Y \cong \oplus_{\downarrow 2}^{\uparrow 2} P_j$, for some modules $X, Y$. Then we have $X \oplus Y \cong \oplus_{\downarrow 2}^{\uparrow 2} P_j$ by [6, Theorem 3.9]. Put $n(1) = 1, A_1 = P_1$ and $B_1 \oplus A_2 = \oplus_{\downarrow 2}^{\uparrow 2} P_j$, where $B_1 \cong X$ and $A_2 \cong Y$. Next we assume that there exist two increasing sequences, $n(1) < \cdots < n(k+1), m(1) < \cdots < m(k)$ and two sets $\{A_j\}_{j=1}^{k+1}, \{B_j\}_{j=1}^{k}$ of independent non-zero submodules of $P$ satisfying the properties (1) and (2). Since $\oplus_{\downarrow 1}^{\uparrow 1} (A_i \oplus B_i) \cong \oplus_{\downarrow 1}^{\uparrow 1} Q_i$ and $D^*(P) = D^*(Q)$, then $D^*(A_{k+1} \oplus (\oplus_{\downarrow 1}^{\uparrow 1} Q_i)) = D^*(\oplus_{\downarrow 1}^{\uparrow 1} Q_i)$. We can take positive integers $m(k+1), m(k+1) < n(k+2)$ such that $n(k) < m(k+1), n(k) < n(k+2)$ and $D^*(A_{k+1}) < D^*(\oplus_{\downarrow 1}^{\uparrow 1} Q_i) \leq D^*(A_{k+1} \oplus (\oplus_{\downarrow 1}^{\uparrow 1} Q_i))$. Then again by Lemma 2.2, we obtain $A_{k+1} \oplus X' \cong \oplus_{\downarrow 1}^{\uparrow 1} Q_i$, and $\oplus_{\downarrow 1}^{\uparrow 1} Q_i \oplus Y' \cong A_{k+1} \oplus (\oplus_{\downarrow 1}^{\uparrow 1} Q_i) \oplus Y' \cong A_{k+1} \oplus (\oplus_{\downarrow 1}^{\uparrow 1} Q_i) \oplus Y'. \] Since $A_{k+1} \oplus X' \oplus Y' \cong A_{k+1} \oplus (\oplus_{\downarrow 1}^{\uparrow 1} Q_i) \oplus Y'$, then we have a decomposition $\oplus_{\downarrow 1}^{\uparrow 1} Q_i = A_{k+1} \oplus A_{k+2}$, where $A_{k+1} \cong X'$ and $A_{k+2} \cong Y'$, by [6, Theorem 3.9]. By the above procedure, we can construct independent non-zero submodules $A_1, B_1, A_2, B_2, \cdots$ which satisfy the properties (1) and (2). Since each $P_n$ is contained in $B_i \oplus A_{i+1}$ for some $i$, then $P = \bigoplus_{\downarrow 1}^{\uparrow 1} (A_i \oplus B_i)$. On the other hand we have $Q = \bigoplus_{\downarrow 1}^{\uparrow 1} Q_i$. Therefore we conclude that $P \cong Q$.

**Remark.** The result obtained by applying Lemma 2.5 for $P, Q$ in $P^*(R)$ means that the Grothendieck group generated by the isomorphism classes of directly finite projective $\Lambda$-modules is isomorphic to some subgroup of the additive group of $\mathbb{R}$. (Cf. [2, Corollaries. 10.14 and 10.16]).

**Theorem 2.6.** Let $R$ be a simple, regular and directly finite ring satisfying the comparability axiom. Any directly infinite projective $\Lambda$-modules is a free $\Lambda$-module.

**Proof.** By Theorem 2.4 and Lemma 2.5, we already see that every directly infinite, countably generated projective $\Lambda$-module is isomorphic to $\mathbb{R}_\Lambda$. Thus we shall show that every directly infinite projective $\Lambda$-module can be expressed as a direct sum of directly infinite, countably generated submodules. Let $P = \bigoplus_{\alpha} P_\alpha$ be a directly infinite projective $\Lambda$-module, where each $P_\alpha$ is isomorphic to some non-zero $J$ in $L(\Lambda)$. Let $\mathcal{B}$ be the set of all countably infinite subsets of $I$. We consider the family consisting of all subsets $\mathcal{F}$ of $\mathcal{B}$ satisfying the following properties;

1. each members of $\mathcal{F}$ is pairwise disjoint
2. $D^*(\bigoplus_{\alpha \in K} P_\alpha) = \infty$ for each $K$ in $\mathcal{F}$.

Since this family is a inductively ordered set using the inclusion relation, there exists a maximal member $\mathcal{F}$ by Zorn’s Lemma. Put $I^* = \bigcup_{K \in \mathcal{F}} K$. If $I^* = I$, then our proof is complete. Next we consider the case that $I^* \neq I$. First we shall show that $D^*(\bigoplus_{\alpha \in I^* \setminus I} P_\alpha) < \infty$, where $I^{**}$ is the complement of $I^*$. Other-
wise we can take a countably infinite subset \( I' \) of \( I^{**} \) such that \( D^*(\bigoplus_{a \in I'} P_a) = \infty \). Then the set \( \mathcal{F} \cup \{I'\} \) is strictly greater than \( \mathcal{F} \). This is a contradiction.

By the proof of Lemma 2.3, we see that \( I^{**} \) is a countable set. Choose one member \( K' \) of \( \mathcal{F} \), and put \( F'=\mathcal{F}-\{K'\} \), and \( K''=K' \cup I^{**} \). Then \( K'' \) is a countably infinite set and \( D^*(\bigoplus_{a \in K''} P_a) = \infty \). The decomposition \( P = (\bigoplus_{K \in \mathcal{F}} (\bigoplus_{a \in K} P_a)) \oplus (\bigoplus_{a \in K''} P_a) \) is a desired one.

**Definition** [5, p. 174]. Let \( A \) be a module. If \( A=0 \), define \( \mu(A)=0 \). If \( A \neq 0 \), define \( \mu(A) \) to be the smallest infinite cardinal number \( \alpha \) such that \( \alpha A \preceq A \).

**Proposition 2.7.** Let \( P \) and \( S \) be projective modules which are not finitely generated. If \( P \preceq S \) and \( S \preceq P \), then \( P \cong S \).

**Proof.** Since \( D^*(P) = D^*(S) \) by the definition of \( D^* \), then they are both directly finite or both directly infinite by Theorem 2.4. If \( P \) and \( S \) are directly finite, then they are countably generated by the proof of Lemma 2.3. Thus we have \( P \cong S \) by Lemma 2.5. If \( P \) and \( S \) are directly infinite, then \( P \cong \alpha R \) and \( S \cong \beta R \) for some infinite cardinal numbers \( \alpha, \beta \) by Theorem 2.6. We can assume \( \alpha \leq \beta \).

Let \( Q \) be the maximal ring of quotients of \( R \) and we use the notation \( E(A) \) to stand for an injective hull of a module \( A \). Since \( P \preceq S \) and \( S \preceq P \), then \( E(P) \cong E(S) \) by [1, Corollary]. On the other hand, \( E(P) \cong E(\alpha Q) \) and \( E(S) \cong E(\beta Q) \) and also \( Q \) is a prime ring because it satisfies the comparability. Therefore, by [5, Theorem 6.32], \( \max\{\alpha', \mu(Q)\} = \mu(E(P)) = \mu(E(S)) = \max\{\beta', \mu(Q)\} \), where \( \alpha' \) and \( \beta' \) are the successors of \( \alpha \) and \( \beta \). Thus, if \( \alpha < \beta \), then it must hold that \( (n_1, 1) < \alpha' < \beta' \leq \mu(Q) \). Since \( n_1 < \mu(Q) \), \( n_1 Q \subseteq Q \). Therefore let \( \{A_{\tau}\}_{\tau \in I} \) be an independent set of principal right ideals of \( Q \) such that \( A_{\tau} \subseteq Q \) for each \( \tau \) in \( I \) and the cardinality of \( I \) is \( n_1 \). Then \( \{A_{\tau} \cap R\}_{\tau \in I} \) is a independent set of non-zero right ideals of \( R \). This contradicts the fact that there is no uncountable direct sum of non-zero right ideals of \( R \). Consequently we must have \( \alpha = \beta \) and hence \( P \cong S \).

3. **Directly finite, regular and right self-injective ring**

**Lemma 3.1** [3, Lemma 5' and 6, Proposition 1.4]. A prime, directly finite, regular and right self-injective ring is a simple ring satisfying the comparability axiom.

**Proposition 3.2.** Let \( R \) be a directly finite, regular and right self-injective ring. Then \( R \) is a finite direct product of simple rings if and only if any non-singular directly finite injective \( R \)-module is finitely generated.

**Proof.** First we shall prove that "only if" part. There exists a set \( \{e_i\}^n_{i=1} \) of orthogonal central idempotents such that \( \sum_{i=1}^{n} e_i = 1 \) and each \( e_i R \) is a simple
ring. Let $M$ be a non-singular directly finite injective $R$-module. There exists a projective $R$-module $P$ such that $P$ is an essential submodule of $M$, because any non-singular finitely generated $R$-module is a projective and injective module (cf. [9, Theorem 2.7]). $M$ is directly finite, and so $P$ is also directly finite. Put $P_i = Pe_i$ for each $i$, then each $P_i$ is also a directly finite projective module as an $e_iR$-module. Therefore there exists a positive integer $t$ such that $P_i \leq t(e_iR)$ for all $i$ by Lemma 3.1 and Theorem 2.4. Thus $P \leq tR$, because $P = \bigoplus_i P_i$. This monomorphism can be extended to be monomorphism from $M$ into $tR$. Then $M$ is isomorphic to a direct summand of $tR$. Conversely we assume that $R$ can be decomposed into no finite direct product of prime rings. Then $R$ itself is not prime. Hence there exist non-zero two-sided ideals $A$, $B$ such that $AB = 0$. Let $A'$, $B'$ be the injective hull of $A$, $B$ in $R$, then they are also two-sided ideals and generated by central idempotents by [3, Lemma 1]. Since $R$ is semi-prime, $A \cap B = 0$. Then $A' \cap B' = 0$. Hence there exist orthogonal central idempotents $\{e_i\}_1^\infty$ such that $\sum e_i = 1$. By the assumption, at least one of $e_iR$, say $e_jR$, is not prime. Use the same argument for the ring $e_jR$, then there exists another set $\{e'_i\}_1^\infty$ of orthogonal central idempotents of $R$ such that $\sum e'_i = 1$. Repeating these procedures, we obtain a countably infinite set $\{e_i\}_1^\infty$ of orthogonal non-zero central idempotents. If $\bigoplus e_iR$ is not essential in $R$, we choose some central idempotent $f$ which generates the injective hull of $\bigoplus e_iR$ and we consider $\{e_i, 1-f\}_1^\infty$. Therefore we may assume that $\bigoplus e_iR$ is essential in $R$. Since $R$ is injective and $\bigoplus e_iR$ is a two-sided ideal, $R \cong \text{End}_R(\bigoplus e_iR)$. $\text{End}_R(\bigoplus e_iR) \cong \prod e_i \text{End}_R(e_iR) \cong \prod e_i e_jR$, because $\hom_R(e_iR, e_jR) = 0$ for $i \neq j$ and each $e_i$ is a central idempotent. Consequently $R \cong \prod e_i e_jR$ by the mapping: $r \mapsto (e_i r)$. We put $M_n = n(e_iR)$ for each $n$ and we consider the $R$-module $M = \prod_n M_n$. This is obviously a non-singular injective $R$-module. We also know that it is directly finite, because $\text{End}_R(M) \cong \prod \text{End}_R(M_n)$ and $\text{End}_R(M_n)$ is directly finite for all $n$. By the assumption, there exists a positive integer $t$ such that $M \leq tR$. Now we choose an integer $m$ which is larger than $t$. That $M_m \leq tR \cong \prod t(e_iR)$ implies that $M_m \leq t(e_mR)$, because $\hom_R(M_m, t(e_mR)) = 0$ for all $n \neq m$. This contradicts that $M_m$ is directly finite. Hence $R$ is a finite direct product of prime rings. Prime directly finite regular right self-injective rings are simple by Lemma 3.1, and so we have proved.

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References


