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Osaka J. Math.
40 (2003), 279–285

EXISTENCE OF BASE-POINT-FREE PENCILS OF DEGREE $g-1$ ON BI-ELLIPTIC CURVES

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(Received August 28, 2001)

1. Introduction

It has been known that a smooth complex projective irreducible algebraic curve X of genus $g \geq 4$ has a base-point-free and complete pencil of degree $g-1$ unless X is hyperelliptic. One way of proving this result is to reduce the problem into a few special cases and then check the validity of the statement for the following three classes of curves:

(i) X is trigonal.

(ii) X is a smooth plane quintic.

(iii) X is a bi-elliptic curve, i.e. X is a double cover of an elliptic curve E.

As it turned out, the cases (i) and (ii) were relatively easy to handle; cf. [6, Beispiel 3]. However, for the case of bi-elliptic curves, some of the proofs which appeared in the literature does not seem to be complete, which has been already pointed out in [4]. For example, in the proof of [5, Theorem 5] the author obtained a plane model of a bi-elliptic curve of degree $g+1$ with a singular point s of certain high multiplicity. He then proceeded to exhibit the existence of another singular point by using a well-known formula for the geometric genus of a singular plane curve. Unfortunately, the singular point different from s could be infinitely singular points lying over s. Therefore the projection method used in [5] to obtain a complete and base-point-free pencil of degree $g-1$ which is cut out by lines through the other singular point does not work well if the singular point s of high multiplicity is not an ordinary singular point. Incidentally, the same objection applies to the proof of Shokurov [7]. A proof due to J. Harris, which was sketched in [2, Chapter VIII; Exercise D and F], seems to be the only complete proof without a gap which appeared in the literature as far as the author knows. On the other hand, the proof of Harris uses the so-called enumerative method as well as several advanced results in Brill-Noether theory and hence one needs a quite a bit of heavy duty machinery for a proof of this seemingly simple fact; indeed the proof in [2] shows the reducibility of $W^1_{g-1}(C)$, which is a much harder problem and the existence of the base-point-free pencil follows as a corollary.

Partially supported by BK21, Graduate Student Support Program (Seoul National University). During the period when this note was prepared for publication, the author was enjoying the hospitality of Mathematics Department of University of Erlangen.

The purpose of this paper is to provide a simple and easy geometric proof of the following theorem of J. Harris only using elementary tools using an idea from [4, Appendix].

Theorem 1 (J. Harris [2]). *Let X be a bi-elliptic curve of genus $g \geq 6$. Then X has a base-point-free and complete pencil g^1_{g-1}.*

Using a similar method, we also provide a relatively simpler proof of the fact that there exists a complete base-point-free pencil g^1_{g-2} on a curve X of genus g which is a double cover of a curve of genus 2 if $g \geq 11$. It should be remarked that the result has been known already; cf. [4, Appendix] for $g \geq 13$ and [3] for $g \geq 11$. Note that the proof in [3] uses enumerative method whereas the proof in [4, Appendix] uses only simple and geometric arguments.

Therefore our main purpose is to assure the readers that geometric arguments can be pushed forward even beyond the range of $g \geq 13$ in [4] so that one gets the same genus bound as in [3].

We use standard notation for divisors, linear series and line bundles on algebraic curves following [2]. As usual, g^r_d is an r-dimensional linear series of degree d on X, which may be possibly incomplete. If D is a divisor on X, we write $|D|$ for the associated complete linear series on X. By K we denote a canonical divisor on X, and $|K|$ is the canonical linear series on X. A base-point-free g^r_d on X defines a morphism $f\colon X \longrightarrow \mathbb{P}^r$ onto a non-degenerate irreducible (possibly singular) curve in $\mathbb{P}^r$. We close this section by recalling the following well-known result which will be used in the next section.

Proposition 1 ([2, Chapter III-Exercise F]). *Let L be a line bundle of degree $d \geq 2g+2$ on a smooth curve X of genus g. Let*

$$\varphi_L\colon X \to \mathbb{P}^{d-g}$$

be the embedding induced by L. Then $\varphi_L(X)$ is the intersection of quadrics.

2. Proof of Theorem 1

Let $\pi\colon X \longrightarrow E$ be the two sheeted map onto an elliptic curve E; note that such π is unique (up to isomorphism) by Castelnuovo-Severi inequality [1, Thorem 3.5]. We break up the proof into several steps as follows.

STEP 1. The canonical image of X lies on a cone of degree $g-1$.

Since X is non-hyperelliptic, we may identify X with its canonical image $\varphi_K(X)$ in $\mathbb{P}^{g-1}$.

For $r_i \in E$, let $\pi^*(r_i) = p_i + \overline{p}_i\,; i = 1, 2$. Then for the effective divisor

$$D = \pi^*(r_1 + r_2) = p_1 + \overline{p}_1 + p_2 + \overline{p}_2 \in g^1_4 = \pi^*(g^1_2) = \pi^*(|r_1 + r_2|),$$

$\dim \overline{D} = 2$ by the geometric version of Riemann-Roch theorem; i.e. D spans a 2-plane in $\mathbb{P}^{g-1}$. Therefore for any $r, r' \in E$, the two lines spanned by $\pi^*(r)$ and $\pi^*(r')$ must intersect. Since X is non-degenerate in $\mathbb{P}^{g-1}$, all the lines spanned by $\pi^*(r)$, $r \in E$ pass through a point $v \in \mathbb{P}^{g-1}$. Let

$$S_{g-1} = \bigcup_{r \in E} \overline{\pi^*(r)},$$

which is a cone with vertex v containing the canonical image of X. Furthermore, one sees easily that $v \notin \varphi_K(X)$; if $v \in \varphi_K(X)$, then the divisor $v + \pi^*(r_1)$ with $\pi(v) \neq r_1$ is a trisecant line hence $v + \pi^*(r_1)$ moves in a pencil which is contradictory to the Castelnuovo-Severi inequality.

Let $H \cong \mathbb{P}^{g-2}$ be a hyperplane in $\mathbb{P}^{g-1}$ not passing through v and φ be the projection away from v to H. By our construction, E is isomorphic to the hyperplane section $H \cap S_{g-1}$, which we use the same symbol E for simplicity. A hyperplane section $H_E = E \cap \mathbb{P}^{g-3} \subset H = \mathbb{P}^{g-2}$ of E is the image under φ of the intersection $\varphi_K(X) \cap \langle H_E, v\rangle$, where $\langle H_E, v\rangle$ is the hyperplane in $\mathbb{P}^{g-1}$ spanned by H_E and v. Since the projection φ is indeed the degree two morphism $\pi\colon X \longrightarrow E$,

$$\deg E = \deg(H_E) = \frac{1}{2}\deg(\varphi_K(X) \cap \langle H_E, v\rangle) = \frac{1}{2}(2g-2) = g-1,$$

and hence $\deg S_{g-1} = g - 1$.

STEP 2. There is a sequence of birational maps $\{\varphi_i\}_{0 \le i \le g-4}$ with the following properties.
(1) $\varphi_0 = \varphi_K\colon X \longrightarrow \mathbb{P}^{g-1}$ is the canonical map of X.
(2) For $1 \le i \le g-4$, $\varphi_i\colon \mathbb{P}^{g-i} \dashrightarrow \mathbb{P}^{g-1-i}$ is a projection away from a point and restricted to $(\varphi_{i-1} \circ \cdots \circ \varphi_0)(X)$, $\varphi_i\colon (\varphi_{i-1} \circ \cdots \circ \varphi_0)(X) \dashrightarrow \mathbb{P}^{g-1-i}$ is still birational onto its image.
(3) $(\varphi_i \circ \cdots \circ \varphi_0)(X)$ has only one singular point for every $2 \le i \le g-4$.
(4) $(\varphi_{g-4} \circ \cdots \circ \varphi_0)(X)$ lies on a cubic cone in $\mathbb{P}^3$.

Choose a point $p_1 \in X_{g-1} := \varphi_0(X)$ and let φ_1 be the projection away from p_1 onto $\mathbb{P}^{g-2}$. Let $q_{g-1} := \overline{p}_1$ be the conjugate point of p_1 with respect to π and take $X_{g-2} := \varphi_1(X_{g-1})$. The image of S_{g-1} under the projection φ_1 is also a cone of degree $g-2$ with vertex $q_{g-2} := \varphi_1(q_{g-1}) = \varphi_1(v)$, which is denoted by S_{g-2}. Now we take a general point p_2 in X_{g-2} and let φ_2 be the projection away from p_2 onto $\mathbb{P}^{g-3}$. Applying this process repeatedly, we can obtain $\{(\varphi_i, S_{g-1-i}, X_{g-1-i}, p_i, q_{g-1-i})\colon i =$

$1, \ldots, g-4\}$ as follows;

$$\begin{array}{ccccccc} & \varphi_1 & & \varphi_2 & & \varphi_{g-4} & \\ \mathbb{P}^{g-1} & \dashrightarrow & \mathbb{P}^{g-2} & \dashrightarrow & \cdots & \dashrightarrow & \mathbb{P}^3 \\ \cup & & \cup & & & & \cup \\ S_{g-1} & \dashrightarrow & S_{g-2} & \dashrightarrow & \cdots & \dashrightarrow & S_3 \\ \cup & & \cup & & & & \cup \\ X_{g-1} & \dashrightarrow & X_{g-2} & \dashrightarrow & \cdots & \dashrightarrow & X_3 \end{array}$$

where φ_i is the projection away from a general point $p_i \in X_{g-i}$ onto a hyperplane, $q_{g-2-i} = \varphi_i(q_{g-1-i})$, $S_{g-2-i} = \varphi_i(S_{g-1-i})$ and $X_{g-2-i} = \varphi_i(X_{g-1-i})$. Note that S_{g-1-i} is a cone with vertex q_{g-1-i} of degree $g-1-i$, X_{g-1-i} is a curve of degree $2g-2-i$ and $\text{mult}_{q_{g-1-i}} X_{g-1-i} = i$; X_{g-1-i} is the image of the morphism induced by $|K - p_1 - \cdots - p_i|$, $\dim |K - p_1 - \cdots - p_i - \overline{p}_1 - \cdots - \overline{p}_i| = \dim |K - p_1 - \cdots - p_i| - 1$ and $|K - p_1 - \cdots - p_i - \overline{p}_1 - \cdots - \overline{p}_i|$ is base-point-free. In particular the image of X_4 in $\mathbb{P}^3$ under φ_{g-4} lies on a cubic cone S_3.

Let $E_k := S_k \cap H$ where $H \cong \mathbb{P}^{k-1}$ is a hyperplane not passing through the vertex q_k of $S_k \subset \mathbb{P}^k$. Since S_{g-1} is a cone over the elliptic curve $E \subset \mathbb{P}^{g-2}$ of degree $g-1$ and S_k is obtained by successive projections, we easily see that $\deg E_k = \deg S_k = k$ and $E_k \cong E$, i.e. $g(E_k) = 1$. Applying Proposition 1 to the hyperplane bundle on $E_k \subset \mathbb{P}^{k-1}$, E_k is cut out by quadrics in H and hence S_k is also cut out by quadrics in $\mathbb{P}^k$ for $k \geq 4$.

Note that, for $k \geq 3$, any singular point of X_k different from q_k may only arise from a trisecant line of $X_{k+1} \subset S_{k+1} \subset \mathbb{P}^{k+1}$ other than rulings of the cone S_{k+1}. Since S_{k+1} is cut out by quadrics for $k \geq 3$, we see that there is no trisecant line of X_{k+1} other than rulings of S_{k+1}. Therefore X_k has no singular point other than q_k for $k = 3, \ldots, g-3$.

STEP 3. X is birational to a plane curve $X_2 \subset \mathbb{P}^2$ of degree $g+1$ with ordinary singular point of multiplicity $g-3$.

The projection away from a general point $p_{g-3} \in X_3$, denoted by φ_{g-3}, gives a birational map from X_3 onto $X_2 := \varphi_{g-3}(X_3)$ in $\mathbb{P}^2$. Note that $\deg \varphi_{g-3}(X_3) = \deg X_3 - 1 = g+1$ and the point $q_2 := \varphi_{g-3}(q_3)$ is singular point with multiplicity $g-3$. We observe that q_2 being an ordinary singular point is equivalent to

$$|K - p_1 - \cdots - p_{g-3} - \overline{p}_1 - \cdots - \overline{p}_{g-3} - \overline{p_i} - \overline{p_j}| = \emptyset \tag{1}$$

for all distinct $i, j \in \{1, 2, \ldots, g-3\}$. Therefore in order to show that q_2 is an ordinary singular point, we need to choose the points $p_1, \ldots, p_{g-4} \in X$ properly in Step 2 as well as p_{g-3} which satisfy the condition (1). We now set

$$T_{ij} := \{(p_1, \ldots, p_{g-3}) \in X^{g-3} \colon \dim |K - p_1 - \cdots - p_{g-3} - \overline{p_1} - \cdots - \overline{p}_{g-3} - \overline{p}_i - \overline{p}_j| \geq 0\}$$

for distinct $i,\ j \in \{1, 2, \ldots, g-3\}$ and $T := \bigcup T_{ij}$. Since T_{ij} is closed in the $(g-3)$-fold product X^{g-3}, so is T. Therefore it is sufficient to show that each T_{ij} is a proper closed subset in X^{g-3}; then any $(p_1, \ldots, p_{g-3}) \in X^{g-3} \backslash T$ satisfies the condition (1). Accordingly, without loss of generality, we assume $(i, j) = (1, 2)$ and proceed as follows.

CLAIM. For distinct $p_1,\ p_2 \in X$ such that $\pi(p_1) \neq \pi(p_2)$, $|p_1+2\overline{p}_1+p_2+2\overline{p}_2| = g^1_6$.

For this we consider $|2p_1+2p_2+2\overline{p}_1+2\overline{p}_2|$ and let $\pi(p_i) = r_i \in E$, $i = 1, 2$. Since X cannot be hyperelliptic $\dim|2p_1+2p_2+2\overline{p}_1+2\overline{p}_2| = \dim|\pi^*(2r_1+2r_2)| = 3$ by Clifford's theorem. Note that the linear series $|2p_1+2p_2+2\overline{p}_1+2\overline{p}_2|$ induces the double covering $\pi\colon X \longrightarrow E$. Therefore, $\dim|p_1+2\overline{p}_1+p_2+2\overline{p}_2| = \dim|2p_1+2p_2+2\overline{p}_1+2\overline{p}_2| - 2 = 1$ since $\pi(p_1) \neq \pi(p_2)$ and this finishes the proof of the claim.

By the claim, $\dim|K-p_1-p_2-2\overline{p}_1-2\overline{p}_2| = g-6$ and therefore we may choose general points $p_3, \ldots, p_{g-4} \in X$ so that $\dim|K-p_1-p_2-2\overline{p}_1-2\overline{p}_2-p_3-\cdots-p_{g-4}| = 0$. Finally we take a point $p_{g-3} \in X$ such that $p_{g-3} \notin |K-p_1-p_2-2\overline{p}_1-2\overline{p}_2-p_3-\cdots-p_{g-4}|$ and p_{g-3} is not a conjugate point of p_i for any $i = 1, \ldots, g-4$. Then $(p_1, \ldots, p_{g-3}) \notin T_{12}$ and this shows that T_{12} is a proper closed subset of X^{g-3}.

STEP 4. The plane curve X_2 constructed in Step 3 has another singular point with multiplicity 2.

Since q_2 is a singular point of multiplicity $g-3$, we have

$$g \leq p_a(X_2) - \frac{(g-3)(g-4)}{2} = \frac{g(g-1)}{2} - \frac{(g-3)(g-4)}{2} = 3g-6.$$

Note that $g < 3g-6$ for $g \geq 6$. Since q_2 is an ordinary singular point, it follows that there exist another singular point, say $q_0 \in X_2$ besides q_2. Suppose that $\operatorname{mult}_{q_0} X_2 \geq 3$. Recall that X_k has only one singular point q_k for every $k = 3, \ldots, g-3$. Therefore the singular point $q_0 \in X_2$ with $\operatorname{mult}_{q_0} X_2 \geq 3$ arises from at least a 4-secant line passing through p_{g-3} other than ruling of the cone S_3. Since S_3 is a cubic cone, this is impossible. Therefore we have $\operatorname{mult}_{q_0} X_2 = 2$ and the pencil of lines through q_0 cuts out base-point-free and complete g^1_{g-1} on X. □

3. Double covering of a curve of genus 2

We will provide a simpler proof of a result in [3] by using a similar argument we used in the previous section. This will also improve the genus bound in [4, Appendix] ($g \geq 11$ compared with the bound $g \geq 13$ in [4]). The proof given in [4, Appendix] consists of two parts. In the first part it is shown that there exists a plane model of degree g with a singular point s of multiplicity $g-6$, where everything works well even with the assumption $g \geq 11$. In the second part it is shown that s is an ordinary singularity and the restricted assumption $g \geq 13$ is required when monodromy argument is used. Accordingly, we only need to argue that s is still an ordinary singularity under a slightly wider range $g \geq 11$.

Theorem 2. *Let X be a double cover of a genus-2-curve C of genus $g \geq 11$. Then X has a complete and base-point-free pencil g^1_{g-2} of degree $g-2$.*

Proof. Let $f\colon X \to C$ be the double covering over a curve C of genus 2. We note that such a covering is unique by the Castelnuovo-Severi inequality and the assumption $g \geq 11$. We briefly recall several facts which were already shown in the first part of the proof in [4, Appendix]. The series $|K - g^1_4|$ is very ample for the unique $g^1_4 = f^*(|K_C|) = f^*(g^1_2)$. For a general choice of $p_1, \dots, p_{g-6} \in X$, the series $|K - g^1_4 - p_1 - \cdots - p_{g-6}|$ induces a singular plane model Γ of X of degree g. Denoting the conjugate points of $p_1, \dots, p_{g-6}$ by $\overline{p}_1, \dots, \overline{p}_{g-6}$, the series $|K - g^1_4 - p_1 - \cdots - p_{g-6} - \overline{p}_1 - \cdots - \overline{p}_{g-6}|$ is a base-point-free g^1_6 and hence there is a singularity $s \in \Gamma$ with multiplicity $g-6$. To show that s is an ordinary singularity, it is enough to prove that

$$(1) \qquad |K - g^1_4 - p_1 - \cdots - p_{g-6} - \overline{p}_1 - \cdots - \overline{p}_{g-6} - \overline{p}_i - \overline{p}_j| = \emptyset$$

for $1 \leq i < j \leq g-6$.

Keeping these in mind, we now proceed as follows.
We let $T_{ij} := \{(p_1, \dots, p_{g-6}) \in X^{g-6} : \dim|K - g^1_4 - p_1 - \cdots - p_{g-6} - \overline{p}_1 - \cdots - \overline{p}_{g-6} - \overline{p}_i - \overline{p}_j| \geq 0\} \subset X^{g-6}$ for distinct i, j and $T := \bigcup T_{ij}$. Since T_{ij} is closed in the $(g-6)$-fold product X^{g-6}, so is T. Therefore it is enough to show that each T_{ij} is a proper closed subset in X^{g-6}; then any $(p_1, \dots, p_{g-6}) \in X^{g-6} \backslash T$ satisfies the condition (1). Accordingly, without loss of generality, we assume $(i, j) = (1, 2)$.

CLAIM. For any p_1 and $p_2 \in X$ with $f(p_1) \neq f(p_2)$, $|g^1_4 + p_1 + 2\overline{p}_1 + p_2 + 2\overline{p}_2| = g^2_{10}$.

To demonstrate the validity of the claim, we recall the well-known Riemann-Hurwitz relation for double coverings. Let C be a curve of genus h and let $f\colon X \to C$ be a double covering. Let $R \subset C$ be a branch locus of f. Then we have

$$(2) \qquad f_*(\mathcal{O}_X) \cong \mathcal{O}_C \oplus \mathcal{S} \quad \text{and} \quad \mathcal{S}^{\otimes 2} \cong \mathcal{O}_C(-R).$$

In our case, $h = 2$ and $\deg \mathcal{S} = 3 - g \leq -8$. Let $\pi(p_i) = \pi(\overline{p}_i) = r_i \in C$, $i = 1, 2$ and we consider $\mathcal{O}_X(g^1_4 + 2p_1 + 2p_2 + 2\overline{p}_1 + 2\overline{p}_2)$. By (2) and the projection formula, we have

$$\begin{aligned} h^0(X, \mathcal{O}_X(g^1_4 + 2p_1 + 2p_2 + 2\overline{p}_1 + 2\overline{p}_2)) &= h^0(X, \mathcal{O}_X(f^*(g^1_2 + 2r_1 + 2r_2))) \\ &= h^0(C, f_*\mathcal{O}_X(f^*(g^1_2 + 2r_1 + 2r_2))) \\ &= h^0(C, \mathcal{O}_C(g^1_2 + 2r_1 + 2r_2)) + h^0(C, \mathcal{O}_C(g^1_2 + 2r_1 + 2r_2) \otimes \mathcal{S}) = 5. \end{aligned}$$

Note that the linear series $|g^1_4 + 2p_1 + 2p_2 + 2\overline{p}_1 + 2\overline{p}_2|$ induces the double covering

$f : X \longrightarrow C$. Therefore,

$$\dim |g_4^1 + p_1 + 2\overline{p}_1 + p_2 + 2\overline{p}_2| = \dim |g_4^1 + 2p_1 + 2p_2 + 2\overline{p}_1 + 2\overline{p}_2| - 2 = 2$$

since $f(p_1) \neq f(p_2)$ and this finishes the proof of the claim.

By the claim, $|K - g_4^1 - p_1 - p_2 - 2\overline{p}_1 - 2\overline{p}_2| = g_{2g-12}^{g-9}$ and hence we can choose $p_3, \dots, p_{g-7} \in X$ such that $\dim |K - g_4^1 - p_1 - 2\overline{p}_1 - p_2 - 2\overline{p}_2 - p_3 - \cdots - p_{g-7}| = 0$. Finally, we take a point $p_{g-6} \in X$ such that $p_{g-6} \notin |K - g_4^1 - p_1 - 2\overline{p}_1 - p_2 - 2\overline{p}_2 - p_3 - \cdots - p_{g-7}|$ and p_{g-6} is not conjugate to p_i, for any $i = 1, \dots, g-7$. Therefore $(p_1, \dots, p_{g-6}) \notin T_{12}$ and it follows that T_{12} is proper and closed in X^{g-6}. □

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