<table>
<thead>
<tr>
<th>Title</th>
<th>Generalizations of Nakayama ring. V. (Left serial rings with (*,2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Harada, Manabu</td>
</tr>
<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 24(2) P.373-P.389</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1987</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/5155">https://doi.org/10.18910/5155</a></td>
</tr>
<tr>
<td>DOI</td>
<td>10.18910/5155</td>
</tr>
</tbody>
</table>
We have studied a left serial algebra over an algebraically closed field with \((*, n)\) as right modules in [4] and further investigated an artinian left serial ring \(R\) with \((*, 1)\) in [7], when \(eJ/eJ^2\) is square-free for each primitive idempotent \(e\), where \(J\) is the Jacobson radical of \(R\). On the other hand, we have given a characterization of a certain artinian ring with \((*, 3)\) in [6].

For a left serial ring \(R\), we shall obtain, in the second section of this paper, a characterization of \(R\) with \((*, 1)\) (Theorem 1), and one of \(R\) with \((*, 2)\) (Theorem 2) in the third section. We shall study hereditary rings with \((*, 2)\) in the forthcoming paper.

In order to give a complete study of a left serial ring with \((*, 1)\), we need deep properties of a division ring (much more difficult than Artin problem, see (#)).

We shall use the same terminologies given in [7] and every ring \(R\) is a both-sided artinian ring with identity, unless otherwise stated.

1. Left serial rings

In this section, we assume that \(R\) is a left serial ring. Then 
\[ eJ = \sum_k \oplus A_k, \]
where the \(A_k\) are hollow right \(R\)-modules by [8], Corollary 4.2. We shall describe this situation as the following diagram:

\[
\begin{array}{cccc}
A_1 & \rightarrow & A_2 & \rightarrow & \cdots & \rightarrow & A_n & \rightarrow & eJ \\
| & & | & & | & & | & & \\
A_{11} & \rightarrow & A_{1a_1} & \rightarrow & A_{21} & \rightarrow & \cdots & \rightarrow & eJ^2 \\
| & & | & & | & & | & & \\
A_{1} & \rightarrow & B_2 & \rightarrow & \cdots & \rightarrow & N_{1} & \rightarrow & eJ \\
| & & | & & | & & | & & \\
A_{21} & \rightarrow & A_{21} & \rightarrow & B_{21} & \rightarrow & \cdots & \rightarrow & eJ^2 \\
\end{array}
\]

or

\[
\begin{array}{cccc}
A_1 & \rightarrow & B_2 & \rightarrow & \cdots & \rightarrow & N_1 & \rightarrow & eJ \\
| & & | & & | & & | & & \\
A_{21} & \rightarrow & A_{21} & \rightarrow & B_{21} & \rightarrow & \cdots & \rightarrow & eJ^2 \\
\end{array}
\]
where \( A, B, \cdots \) are hollow modules. (cf. [3], §2).

Let \( e \) be a primitive idempotent and put \( \Delta = eRe/Je \), and for a submodule \( A \) of \( eR \), \( \Delta (A) = \{ x | x \in eRe, xA \subset A \} \), where \( x \) is the coset of \( x \) in \( \Delta \). Then \( \Delta (A) \) is a division subring of \( \Delta \) (see [1]). It is clear that \( \Delta (A) = \Delta (A) = \{ x | x \in eRe, xA \subset A \} \) provided \( A \) is hollow; \( \overline{A} = A \mid J(A) \).

Let \( A_i \supset A_{ii} \) be as in the diagram above. We put \( R = R/J(t > i) \) and \( A_{ii} = (A_{ii} + eJ^t) / eJ^t \). Then we can express \( A_{ii} = A_{ii} \) as a direct sum \( A_{ii} \oplus C \), where \( C \subset eJ^t - A_{ii} \) (see the diagram above). Let \( p \) and \( q \) be the projections of \( A_{ii} + eJ^t \) to \( A_{ii} \) and \( C \) respectively. We can define \( \Delta (A_{ii}) \) and \( \Delta (A_{ii}) \). Since \( eRe/eJe \cong (eRe/eJ^t)e/eJ^t \), \( \Delta (A_{ii}) \) is canonically contained in \( \Delta (A_{ii}) \). Conversely, let \( x \) be an element in \( \Delta \) such that \( x(A_{ii} + eJ^t) \subset A_{ii} + eJ^t \). Put \( f = qx_i | A_{ii} \) and \( f \) is in \( \text{Hom}_R(A_{ii}, eJ^t) \), where \( x_i \) means the left-sided multiplication of \( x \). Let \( A_{ii} = ar \) and \( ag = a \) for some primitive idempotent \( g \). Since \( b = f(a) = f(a)g \), there exists \( d \) in \( eJe \) such that \( da = b \) (note \( i > t \)), since \( R \) is left serial. Then \( x_i | A_{ii} = (px_i + qx_i) | A_{ii} = px_i | A_{ii} + f = px_i | A_{ii} + d_i | A_{ii} \) and \( px_i | A_{ii} \in \text{Hom}_R(A_{ii}, A_{ii}) \). Hence \( (x-d) = x \in \Delta (A_{ii}) \). Thus we have (from now on \( A_{ii} \) means always a hollow module in the diagram above).

**Lemma 1.** Let \( R \) be a left serial ring, and let \( A_{ii} \) and \( A_{ii} \) be as above. Then \( \Delta (A_{ii}) = \Delta (A_{ii}) \).

**Lemma 2.** Let \( R \) be a left serial ring. Let \( A_{ii} \) contain \( A_{j} \) and \( A_{jk} \). Then \( \Delta (A_{j}) \subset \Delta (A_{ii}) \), and if \( f : A_{ji} \cong A_{jk} \), there exists a unit \( g \) in \( eRe \) which induces \( f \) and \( gA_{ii} = A_{ii} \).

Proof. Assume \( f : A_{ji} \cong A_{jk} \). There exists a unit \( x \) in \( eRe \) such that \( xA_{ji} = A_{jk} \) from [7], Lemma 2, and \( x_i \) induces \( f \), since \( R \) is left serial. For \( x \), we employ the similar argument given in the proof of Lemma 1. Let \( eJ^t = A_{ii} \oplus E \) and \( p, q \) the projections. Consider \( qx_i | A_{ii} \) (\( = g \)). Since \( g(A_{ji}) = qx_{j}A_{ji} = qA_{ji} = 0 \), \( g \) is not a monomorphism. Hence \( g = d_i \) for some \( d \) in \( eJe \) and so \( (x-d)A_{ii} \subset A_{ii} \). Hence \( (x-d) \) induces \( f \). If we put \( k = 1 \) in the above, we obtain the first half of the lemma.

2. \((*, 1)\)

First we recall the definition of \((*, n)\)

\((*, n)\) Every maximal submodule of a direct sum of \( n \) hollow modules is also a direct sum of hollow modules [5].

We shall study, in this section, left serial rings \( R \) with \((*, 1)\). We obtained a characterization of a left serial ring with \((*, 1)\), when \( eJ/eJ^2 \) is square-free, i.e., \( \overline{A} \cong \overline{B} \cong \cdots \cong \overline{N} \) in [7], Theorem. Hence we may consider \( eR \) satisfying \( A_1 \cong B_1 \).
Let $R$ be left serial. Assume that $A_i \approx B_i$ and $(\ast, 1)$ holds. Then, for any submodules $C_i \supset D_i$ in $A_i$ such that $C_i/D_i$ is simple and $f; C_i/D_i \approx C_2/D_2$, $f$ or $f^{-1}$ is extendible to an element $g$ in $\text{Hom}_R(A_i/D_i, A_i/D_2)$ or $\text{Hom}_R(A_i/D_2, A_i/D_i)$.

**Proof.** There exists a unit element $u$ in $eRe$ such that $B_1 = uA_1$. Put $C_1 = uC_2, D_1 = uD_2$ and $f' = u_i f$. Then $f'$ (or $f^{-1} u_i ^{-1}$) is extendible to an element $g'$ in $\text{Hom}_R(A_i/D_i, B_i/D_1)$ (or $\text{Hom}_R(B_i/D_2, A_i/D_i)$) by [6], Theorem 4. Then $g = u_i ^{-1} g'$ (or $g = g' u_i$) is the desired extension of $f$ (or $f^{-1}$).

**Proposition 1.** Let $R, A_i$ and $B_i$ be as in Lemma 3. If there are three non-zero hollow modules $A_{i1}, A_{i2}, A_{i3}$ for some $i$, they are isomorphic to one another.

**Proof.** First we shall show $A_{i1} \approx A_{i2}$. Put $C_1 = A_{i1} \otimes A_{i3}$ and $C_2 = A_{i2} \otimes A_{i3}$. Considering $R/J^{i+1}$ from [3], Lemma 1, we may assume that the $A_{ij}$ are simple. Now $f; C_1 \approx A_{i3} \approx C_2 \approx A_{i2}$. Then by Lemma 3, there exists an element $x$ in $eRe$ which induces $f$ or $f^{-1}$, i.e., $f(a + A_{i3}) = xa + A_{i2}$ for $a \in A_i$. Since $C_1, C_2$ are contained in $e J^{i}$ but not in $e J^{i+1}$, $x$ is a unit, and $xA_i = A_{i2}$ (or $xA_{i2} = A_{i1}$) from the argument of the proof of [4], Theorem 3. Therefore $A_{i1} \approx A_{i2}$. Since $R$ is left serial and $A_{ij}$ are hollow, $A_{i1} \approx A_{i2}$ from [7], Lemma 2.

Let $\Delta \supset \Delta_1$ be division rings. $[\ ]_r ([\ ]_l)$ means the dimension of $\Delta$ over $\Delta_1$ as a right (left) $\Delta_1$-module.

**Proposition 2.** Let $A_i, B_i$ be as in Lemma 3. Then for $A_{i1} \supset A_{i2}$ $[\Delta(A_{ij})]: \Delta(A_{ij}) = |A_{i1}J^{j-i+1}|/|A_{i1}J^{j-i}|$, except $A_{i1}J^{j-i} = A_{i2} \oplus A_{i2}$ and $A_{i1} \approx A_{i2}$ (in the exceptional case $\Delta(A_{ij}) = \Delta(A_{ij})$, cf. Example 2 below).

**Proof.** We may assume from Lemma 1 and [3], Lemma 1 that $J^{j-i+1} = 0$, and hence $A_{i1}J^{j-i+1} = 0$, and so $A_{ij}$ is simple. Let $A_{ij} = aR$ and $\{e, \delta_2, \delta_3, \ldots, \delta_k\}$ be a linearly independent set in $\Delta_i = \Delta(A_{ij})$ over $\Delta_i = \Delta(A_{ij})$ such that $\delta_i A_{i1} \subset A_{i1}$ for all $k$. We shall show $A_{i1} + \delta_2 A_{i1} + \delta_3 A_{i1} + \cdots + \delta_k A_{i1} = A_{i1} \oplus \delta_2 A_{i1} + \delta_2 A_{i2} \oplus \cdots \oplus \delta_k A_{i1}$. If $(A_{i1} + \delta_2 A_{i1} + \cdots + \delta_k A_{i1}) \cap \delta_i A_{i1} = 0, \delta_i A_{i1} \subset A_{i1} \oplus \cdots + \delta_i A_{i1}$, since $\delta_i A_{i1}$ is simple. Then $\delta_i a = a_1 + \delta_2 a_2 + \cdots + \delta_k a_k$, where $a_j \in A_{i1}$. The mapping; $a \rightarrow a_i$ gives an endomorphism of $A_{i1}$. Hence $a_i = k_i a$ for some $k_i \in A_{i1}$ by Lemma 2. Accordingly $\delta_i = \delta_i A_{i1} + \delta_i A_{i2} + \cdots + \delta_i A_{i1}$, since $J^{j-i+1} = 0$, a contradiction. From the similar argument we can show that $\{A_{i1}, \delta_2 A_{i1}, \ldots, \delta_k A_{i1}\}$ is independent. Hence $[\Delta(A_{ij})]: \Delta(A_{ij}) = |A_{i1}J^{j-i}|$. Assume $|A_{i1}J^{j-i}| > 3$. Then by Proposition 1 $A_{i1}J^{j-i} = A_{i1} \oplus A_{i2} \oplus \cdots \oplus A_{ip}$, for $2 < k < p$. There exists $x_p$ in $D_i (x_p \in eRe)$ such that $x_p A_{i1} = x_p A_{i1} = A_{i1}$. We shall show that $\{e, x_2, \ldots, x_p\}$ is linearly independent.
over $\Delta_j$. Assume $x_j = x_{j-1} + x_{j-2} + \cdots + x_1 + k$, where $k \in eRe$.

Since $J A_j = 0$, $A_j = x_j A_j = x_j A_{j-1} + x_{j-1} A_{j-1} + \cdots + x_1 A_1$, a contradiction. Hence $|A_i J^{-i}| \leq |\Delta(A_i)|$. Finally assume $|A_i J^{-i}| \leq 2$. If $A_j \approx A_j$, we have the same result. If $A_j \approx A_j$, $p \leq 2$ from Proposition 1, and $\Delta_i = \Delta_j$ from the initial argument. If $A_2 = \cdots = A_i = 0$, it is clear that $\Delta_i = \Delta_i$. Hence $|\Delta(A_j)| = 1$.

We consider the situation in Proposition 2 and $J^{*+1} = 0$. Let $A_i J^{-k} = \sum_{j=1}^k A_{j-1}$. If $p \geq 3$, $A_i = A_j$ for all $j$ by Proposition 1. Put $\Delta_i = \Delta(A_i)$ and $A_i = A_i$. Then $[\Delta_i : \Delta_i] = p$ by Proposition 2. Further $A_i J^{-k} = A_i \oplus \delta_i A_i \oplus \cdots \oplus \delta_i A_i$, where $A_i = A R$, and every simple submodule in $A_i J^{-k}$ is of a form $A_i A_i$ for some $\delta$ in $\Delta_i$. Now we shall identify $A_i J^{-k} = \Delta_i A_i \oplus \delta_i A_i \oplus \cdots \oplus \delta_i A_i$ with $A_i = A_i \oplus \delta_i A_i \oplus \cdots \oplus \delta_i A_i$, i.e., $\text{Hom}_R(A_i, A_i J^{-k}) = A_i (A_i = A_i J^{-k})$ as left $A_i$ right $\Delta_i$-modules. Let $T_1 \supset T_1$ and $S_1 \supset S_2$ be submodules in $A_i J^{-k}$ such that $f: T_1 / T_2 \approx S_1 / S_2$ and $|T_1| = |S_1| (|T_1| < |S_1|), |T_1 / T_2| = 1$. Then $f$ is extendible to an element $h$ in $\text{Hom}_R(A_i, A_i / S_2)$. Since $S_1, T_1$ are contained in $A_i J^{-k}$, $h$ is given by a unit element $x$ in $eRe$. As given in the proof of Lemma 2, $(x+j) A_i$ is in $\text{Hom}_R(A_i, A_i)$ for some $j$ in $eRe$. Since $J T_2 = 0$, $x+j$ induces $f$, and $x+j \in A_i$ means $(x+j) T_2 = S_2 ((x+j) T_2 \subset S_2)$ and $f(t_1 + T_2) = (x+j) t_1 + S_2$ for any $t_1$ in $T_1$. We translate the above fact to $A_i = \text{Hom}_R(A_i, A_i J^{-k})$.

For any $\Delta_i$-subspace $V_1, V_2$ in $\Delta_i$ with $|V_1| = |V_2| (|V_1| \leq |V_2|)$ and $V_i = x V_i (\forall V_i \in \Delta_i)$, there exists $x$ in $\Delta_i$ such that $x V_1 = V_2 (x V_1 \subset V_2)$ and $x v_1 \equiv v_2 (\mod V_2)$.

**Lemma 4.** Let $\Delta \supset \Delta_1$ be division rings. Assume that $(\#)$ holds for $\Delta$ and $\Delta_1$. Then $[\Delta : \Delta_1] \leq 2$.

**Proof.** We may assume $\Delta = \Delta_1$. Let $\delta$ be a fixed element in $\Delta - \Delta_1$ and $\delta'$ an element in $\Delta - \Delta_1$. Put $V_1 = V_2 = \Delta_1$, $v_1 = \delta$ and $v_2 = \delta' y$ for any $y \in \Delta_1$ in $(\#)$. Then there exists $x$ in $\Delta_1$ such that $x \delta = \delta' y + z$ for some $z$ in $\Delta_1$. Hence $\delta' \Delta_1 \subset \Delta_1 \oplus \Delta_1 \delta$. Since $\delta'$ is arbitrary, $\Delta_1 = \Delta_1 \oplus \Delta_1 \delta$, and so $[\Delta : \Delta_1] \leq 2$.

**Proposition 3.** Let $R, A_1$ and $B_1$ be as in Lemma 3. Then for $A_i \supset A_j$, $\Delta(A_i)$ and $\Delta(A_j)$ satisfy $(\#)$ and so $[\Delta(A_i) : \Delta(A_j)] \leq 2$.

**Proof.** It is clear by Proposition 2 that if $A_i \approx A_j$, $\Delta(A_i) = \Delta(A_j)$. If $A_i \approx A_j \approx \cdots \approx A_j$, Proposition 1, where $t = [\Delta(A_i) : \Delta(A_j)]$. Then $\Delta(A_i)$ and $\Delta(A_j)$ satisfy $(\#)$ from the remark before Lemma 4. Hence $[\Delta(A_i) : \Delta(A_j)] \leq 2$ from Lemma 4.
Corollary 4. Let $A_1$ and $B_1$ be as above. Assume either $\Delta(A_1)$ is commutative or $R$ is an algebra over a field with finite dimension. Then $A_1J^{i-1} = A_{i+1} \oplus A_{i+2}$ for all $i \geq 2$, i.e., $[\Delta(A_1) : \Delta(A_{i+1})]_F = 2$.

Proof. From the assumption and Proposition 3, $[\Delta(A_1) : \Delta(A_{i+1})]_F = 2$.

Proposition 5. Let $A_i$, $B_i$ be as in Lemma 3. Assume $J(A_{j-1}) = A_{j-1} \oplus A_{j+1} \oplus \cdots \oplus A_{j+p}$. If $p \geq 2$, $A_{i+1}$ is uniserial for all $k$.

Proof. Assume that $J(A_{j-1})$ is not uniserial, i.e., $J(A_{j-1}) = A_{j+1} \oplus A_{j+2} \oplus \cdots$ for $j > i + 1$. We shall divide ourselves into two cases.

i) $A_{i+1} \cong A_{i+2}$. Then $p \leq 2$ by Proposition 1, and $A_{i+1}J^{-i-1} = 0$ by assumption: $A_1 \cong B_1$, Proposition 1 and [7], Lemma 3. Put $D_1 = A_1 \oplus J(A_2)$, $D_2 = A_1 \oplus \cdots \oplus J(A_p)$, $C_1 = A_1 + D_1$ and $C_2 = A_2 + D_2$. Then $f: C_1/D_1 \cong A_1 \cong C_2/D_2$. Since $(*)_1$ is satisfies, $f$ or $f^{-1}$ is extendible to $x$ for some $x$ in $eRe$ by Lemma 3. Being $f(A_1J^1 + D_1) = A_1J^1 + D_2$, $x$ is a unit. Hence $xD_1 \subset D_2$ or $xD_2 \subset D_1$ (see the proof of [4], Theorem 3). However, by [7], Lemma 3, it is impossible.

ii) $A_{i+1} \cong A_{i+2} \cong \cdots \cong A_{i+p}$. Then $A_{j+1} \cong A_{j+2}$. By Proposition 1. Since $A_{i+1} \cong A_{i+2}$, $\Delta(A_1) = \Delta(A_{i+1})$ by Proposition 2. Similarly $\Delta(A_j) = \Delta(A_{j+1})$. Hence $[\Delta(A_1) : \Delta(A_{i+1})], = [\Delta(A_1) : \Delta(A_j)], = [\Delta(A_j) : \Delta(A_{j+1})], = 2$ by Proposition 3 and Lemma 4. However $\Delta(A_1) \supset \Delta(A_{i+1}) \supset \Delta(A_{j+1})$ by Lemma 2, which is impossible.

We shall give the structure of $A_1$. From Propositions 1 and 5 we obtain the following diagrams (a) and (b').

Assume $t \geq 3$ and $J(A_{i+1}) = A_{i+1} \cong 0$. Put $D_1 = A_{i+1} \oplus A_{i+2}$, $D_2 = A_{i+1} \oplus A_{i+2} \oplus A_{i+3}$, $C_1 = A_{i+1} + D_1$ and $C_2 = A_{i+1} + D_2$. Then $C_1/D_1 \cong A_{i+1} \cong C_2/D_2$. However, $xD_1 \subset D_2$. Hence we obtain a contradiction as above. Thus we
Lemma 5. Let $R$ be left serial. Then in the diagram (a), any two distinct simple sub-factor modules (e.g. $A_i/A_{i+1}$, $A_{i+1}/A_{i+1}$) are not isomorphic to one another.

Proof. Assume $A_k \approx A_{k'}$ for $k \leq i-1$ and $p \geq i$. Put $A_k = a_kR$, $A_{k'} = a_{k'}R$ and $a_kg = a_k$, $a_{k'}g = a_{k'}$ for a primitive idempotent $g$. Since $A_k \approx B_k$, $A_{k'} \approx B_{k'}$ and $A_{k'} \approx B_{k'} = b_{k'}R$; $b_{k'}g = b_{k'}g$. Then there exists $d$ in $B$ such that $da_k = b_{k'}$ by [7], Lemma 2, and $d \in T(e^{f_{*}}g)$. Since $0 \neq b_{k'} \in J^{*}g$, $db_k \in T(e^{f_{*}}g)$. Let $db_k = x_1 + x_2$; $x_j = x_jg \in B_{ij}$ $(j=1, 2)$. Assume $x_j \in T(e^{f_{*}}g)$. Then $b_{k'} = x_2u$ for some unit $u$ in $eRe$, and so $d(a_k - b_{k'}) = -x_1u$. Hence $-x_1u = -x_1ug \in T(B_{k'})$. Accordingly, $B_{k'} \approx B_{k'}$, which contradicts [7], Lemma 3. Therefore $x_2 \in T(e^{f_{*}}g)$, and so $x_1 = x_1g \in T(e^{f_{*}}g)$. Again we obtain the same contradiction from [7], Lemma 3. Thus $A_k \approx A_{k'}$. We can use the same argument for other cases (note that, for the case $A_k \approx A_{k'}$, $(k < k' < i-1)$, use [7], Lemma 7).

Lemma 6. Assume that $R$ is a left serial ring. Then in (b) we have the same situation as in Lemma 5 for simple sub-factor modules between $A_i$ and $J(A_{i-1})$. Further $\Delta(A_i)$ and $\Delta(A_{i+1})$ satisfy (#), provided ($*1$) holds. For (b2) any two of simple sub-factor modules between $A_i$ and $J(A_{i-1})$ (and of $A_{i+1}$) are not isomorphic to one another, respectively. (Some simple sub-factor modules between $A_i$ and $J(A_{i-1})$ may be isomorphic to one of $A_{i+1}$.)

Proof. The first halves of (b1) and (b2) are obtained from the argument similarly to Lemma 5. The last one of (b1) is clear from Proposition 3.

Lemma 7. Let $R$ be left serial, and consider the diagram (a). Let $C_i \supseteq D_1$ and $C_i \supseteq D_2$ be submodules in $A_i$ such that $f: C_i/D_2 \approx C_i/D_2$ and $|C_i/D_1| = 1$. Then $f$ or $f^{-1}$ is extendible to an element in $\text{Hom}_R(A_i/D_1, A_i/D_2)$ or $\text{Hom}_R(A_i/D_2, A_i/D_1)$.

Proof. We may assume $C_i = c_iR + D_i$ and $c_ig = c_i$ for $i = 1, 2$. If $c_i \in T(A_i)$ $(k \leq i-1)$, $C_i = A_k$ and $D_i = J(C_i) = A_{i+1}$. Then $c_2 \in T(A_k)$ by Lemma 5. Hence there exists a unit $d$ in $eRe$ such that $dc_i = c_2$. We may
assumed $dA_1 = A_1$ by Lemma 2. Then $dD_1 = dA_4 J \subseteq C_2 J = D_2$. Therefore $d_1$ is an extension of $f$. Thus we may assume that $J(A_{i-1})$ contains $C_1$ and $C_2$. From Lemma 5 every submodule in $J(A_{i-1})$ is standard (see the definition before Lemma 10 below). Let $C_i = A_{ji} \oplus A_{k2}, \ D_i = A_{ji+1} \oplus A_{k2}$. Since $C_1/D_1 = C_2/D_2, \ C_2 = A_{ji} \oplus A_{k2}, \ D_2 = A_{ji+1} \oplus A_{k2}$. If $k \leq k'$ (resp. $k \geq k'$), $f$ is extendable to an element $d_1$ in $\text{Hom}_R(A_i/D_i, A_i/D_i)$ as above by Lemmas 2 and 5.

**Lemma 8.** Let $R$ be left serial. In the diagram (b1), we assume that $\Delta(A_1)$ and $\Delta(A_{i1})$ satisfy $(\#)$. Further we assume $[\Delta(A_1): \Delta(A_{i1})]=2$ in (b2). Then we obtain the same result as in Lemma 7.

Proof. Let $c_j$ be as in the proof of Lemma 7. If $c_j$ is in $T(A_{si})$ $(s_j \leq i-1)$, then $C_i = C_2 = A_{si}$ and $D_i = D_2 = A_{si+1}$ by Lemma 6. Hence we can prove the lemma as in the proof of Lemma 7. Similarly if $C_1 = A_{si}$ and $C_2$ is contained in $J(A_{i-1})$, we can easily prove the lemma, since $D_1 = J(C_1)$. Therefore we may assume $J(A_{i-1})$ contains $C_1$ and $C_2$.

(b1) Since $C_i$ is in $J(A_{i-1})$, we have the lemma from $(\#)$.

(b2) Let $J(A_{i-1}) = A_{i1} \oplus A_{i2} \to C_i \supseteq D_i$ be submodules with $|C_i/D_i| = 1$. Let $p_j$ be the projection of $J(A_{i-1})$ to $A_{ij}$. We shall show for $C (=C_1)$ and $D (=D_1)$ that there exists a unit $x$ in $eRe$ such that

1) $xA_1 = A_1$ and $xC = A_{i1} \oplus A_{i2} \supseteq xD = A_{i1} \oplus A_{i2}$.

First we remark the following fact: for $C = A_{i1} \oplus A_{i2}$, there exists a unit $y$ in $eRe$ such that $yA_1 = A_1$ and $yC = A_{i1}\oplus A_{i2}$. i) $t \geq r$. There exists $y$ in $eRe$ such that $yA_1 = A_1$ and $yA_{i1} = A_{i1}$ by Lemma 6. Since $yA_{i2} \neq A_{i2}, \ p_j(yA_{i2}) \neq 0$, and so $p_j' y(A_{i2}) = A_{i1}$ by Lemma 6. Hence $yC = A_{i1} \oplus A_{i2}$.

ii) $t < r$. Take a unit $y'$ such that $y'A_{i2} = A_{i1}$ and $y'A_1 = A_1$.

Put $D' = D \cap A_{ji}$ and $D' = p_j(D)$ $(j = 1, 2)$. Then $g': D' \supseteq D' = p_j(D)$ $(j = 1, 2)$. Let $D' = A_{k1}, \ D' = A_{k2}, \ D' = A_{k-1}$ and $D' = A_{k-2}$. We may assume $k \leq s$ from the remark (actually $k = s$ by Lemma 6). There exists $\alpha$ in $eRe$ such that $x_1$ induces $g$. Hence $xD' \subseteq D' \subsetneq D'$. Putting $\alpha = e + x, \ (D' \oplus D') \supseteq D' \oplus D'$ and $\alpha(A_{k-1} \oplus A_{k-2}) \subseteq \alpha A_{k-1} \oplus A_{k-2} \oplus D' = D'$. We may assume $k \leq s$ from the remark (actually $k = s$ by Lemma 6). Hence $\alpha = e + x_1$. $\alpha(A_{k-1} \oplus A_{k-2}) \subseteq \alpha A_{k-1} \oplus A_{k-2} \oplus D' = D'$. $\alpha$ is clearly a unit, and so $\alpha^{-1} A_{k-1} \oplus D' = A_{k-1} \oplus A_{k-2}$. Now $\alpha^{-1} A_{k-1} \oplus D' = A_{k-1} \oplus A_{k-2}$, where $k' \equiv 0, 1$. Since $|C/D| = 1, \ \alpha^{-1} C$ is one of the following: $A_{k-1} \oplus A_{k-2}, \ A_{k-1} \oplus A_{k-2}$ and $(e+y)A_{k-1} \oplus \alpha^{-1} D$ (in the last case $k' = s$), where $y \in eRe$ and $yA_{k-1} = A_{k-2}$ and $k \leq s$, we obtain (1) from the initial remark.

Next we assume that $C_1 \supset D_1$ are of the form (1). Put $C_i = A_{i1} \oplus A_{i2}$ and $D_i = A_{i1} \oplus A_{i2}$ for $i = 1, 2$. Since $f: C_i/D_i \approx C_i/D_i, \ k_1 = k_2$ (by Lemma 6). We shall divide ourselves to the following cases:

(a) $k \leq \min(s_1, s_2)$. We may assume $s_1 \geq s_2$. Let $A_{k-1} = aR$. Then there
exists a unit z in eRe such that \( f(a+D_1) = za+D_2 \) and \( zA_{k-11} = A_{k-11} \), \( zA_1 = A_1 \) by Lemma 2. Since \( k \leq s_2 \leq s_1 \), \( zD_1 = z(A_{k+1} \oplus A_{s_2}) \subseteq A_{k+1} \oplus A_{s_2} = D_2 \). Hence \( z_1 \) is an extension of \( f \).

(\( \beta \) \) \( s_2 \leq k \leq s_1 \) \( (s_1 \leq k \leq s_2) \). We obtain the same result as in (\( \alpha \)). (Take \( f^{-1} \).

(\( \gamma \) \) \( k < \max(s_1, s_2) \). We may assume \( s_1 \geq s_2 \). Let \( A_{k-12} = aR \) and \( \delta A_{12} = A_{11} \) for some unit \( \delta \) by Lemma 2. Then \( A_{k-11} = \delta aR \) and \( f(\delta a+D_1) = \delta wa+D_2 \) for some \( w \) with \( wA_{1} = A_1 \) and \( wA_{k-12} = A_{k-12} \). Since \( [\Delta(A_1) : \Delta(A_{11})] = 2 \), there exist \( y_1 \) and \( y_2 \) in \( eRe \) such that \( \delta w = \bar{y}_1 + \bar{y}_2 \) and \( y_j A_{12} = A_{12} \) and \( y_j A_1 = A_1 \) for \( j = 1, 2 \). i.e., \( \delta w = \bar{y}_1 + \bar{y}_2 \beta + j \). \( x \in eFe \). Then \( jA_1 = (\delta w - \bar{y}_1 - \bar{y}_2 \beta)A_1 \subseteq A_1 \), and so \( y_2(\delta a) = (\delta w - \bar{y}_1 - j)a = \delta w a - (\bar{y}_1 + j)a \equiv \delta w a \pmod{D_1} \) and \( y_2 A_1 \subseteq D_2 \), since \( s_2 \leq s_1 \leq k \) and \( j \in eFe \). Hence \( y_2 \) is an extension of \( f \).

Finally we consider the general case. Let \( f : C_1/D_1 \to C_2/D_2 \) be as before.

Then there exist \( u_1, u_2 \) in \( eRe \) as in (1). Take

\[
\begin{align*}
 f' : (A_{k_1} \oplus A_{s_2})/(A_{k_1} \oplus A_{s_2}) & \to C_1/D_1 \to C_2/D_2 \to C_2/D_2
\end{align*}
\]

Applying the above argument to \( f' \), we can find \( v \) in \( eRe \) such that \( v_1 \) induces \( f' \) (or \( f^{-1} \)) and \( vA_1 = A_1 \). Therefore \( (u_1, v_1, u_2^{-1}) \) \( ((u_2, v_2^{-1}) \) induces \( f \) (or \( f^{-1} \)).

Thus we obtain

**Theorem 1.** Let \( R \) be a left serial ring, and \( eJ = A_1 \oplus B_1 \oplus \cdots \oplus N_1 \) a direct sum of hollow modules. Then \((\ast, 1)\) holds for any hollow right \( R \)-module if and only if the following conditions are satisfied:

1) If \( A_1 \approx B_1 \), \( A_1 \) has the structure of (a), (b) or (b) such that (\#) holds for \( \Delta(A_1) \) and \( \Delta(A_{11}) \) if \( t \geq 3 \) in (b), and \( [\Delta(A_1) : \Delta(A_{11})] = 2 \) if \( t = 2 \) in (b) and (b).

2) The condition in [7], Theorem is satisfied.

**Proof.** If \( A_1 \approx B_1 \), we obtain 2). Assume \( A_1 \approx B_1 \). We have studied an isomorphism \( f : C_1/D_2 \approx C_2/D_2 \) for submodules \( C_1 \supseteq D_1 \) in \( A_1 \). If \( C_2 \) is a submodule of \( B_1 \), \( xC_2 \) is a submodule in \( A_1 \), where \( xB_1 = A_1 \) for some unit \( x \). Then using the manner given in the proof of Lemma 8, we can extend \( f \) to an element in \( \text{Hom}_R(A_1/D_1, B_1/D_2) \) or \( \text{Hom}_R(B_1/D_2, A_1/D_1) \).

**Proposition 6.** Let \( R \) be as above. Assume \( A_1 \approx B_1 \approx \cdots \approx N_1 \) for each primitive idempotent. Then \((\ast, 1)\) holds for any hollow right \( R \)-module if and only if 1) in Theorem 1 holds.

**Remark.** If \( R \) is left serial, \( eR \) has the structure in § 1. Under this assumption, for a fixed primitive idempotent \( e \), we have studied a problem: when is \( eJ/K \) a direct sum of hollow modules for any submodule \( K \)? Hence Theorem 1 gives a characterization of such \( e \), provided \( R \) is left serial. This remark
is applicable to the next section, in particular to Proposition 7 below. We shall give some algebras concerning Theorem and Propositions.

1 Let $L\supset K'\supset K$ be fields with $[L: K']=[K': K]=2$. Let $L=K'+K'u$ and $K'=K+Kv$. We construct a similar example to ones in [4].

$$e_1 R = e_1 L + e_1 J$$

$$(12)K'+B \cong (12)uK'+uB$$  $e_1 J$

$$(12)(23)K \cong (12)(23)vK$$  $$(12)(23)uK \cong (12)(23)uvK$$  $e_1 J^3$

$$e_2 R = e_2 K' + e_2 J$$

$$e_3 R = e_3 K$$

$$(23)K \cong (23)vK$$

$$(23)K \cong (23)vK$$

$$(23)K \cong (23)vK$$

$$(23)K \cong (23)vK$$

where $B=(12)(23)K \oplus (12)(23)vK$ and $l'e_i = e_i l$ for any $l$ in $L$, $k'e_i = e_i k'$ for any $k'$ in $K'$. Then $R=\sum_{i=1}^3 e_i R$ is a left serial algebra. Further we can show from Theorem 1 that $(\ast, 1)$ holds for any hollow right $R$-module $((12)(23)K \cong (12)(23)vK \cong (12)(23)uvK)$. This example shows that [7], Lemma 6 is not true if $i=j$.

2

$$e_1 R = e_1 K' + e_1 J$$

$$(12)K+B \cong (12)\nu K+\nu B$$  $e_1 J$

$$(12)(23)K \cong (12)(24)K$$  $$(12)(23)vK \cong (12)(24)vK$$  $e_1 J^3$

$$e_2 R = e_2 K + e_2 J$$

$$e_3 R = e_3 K$$

$$e_4 R = e_4 K$$

$$(23)K \cong (24)K$$

where $B=(12)(23)K \oplus (12)(24)K$ and $k'e_1 = e_i k'$ for any $k'$ in $K'$. Then $R=\sum_{i=1}^4 e_i R$ is a left serial algebra with $(\ast, 1)$ $(12)(23)K \cong (12)(24)K$.

3 In Example 1, we replace $K'$ by an extension $K'_0$ over $K$ ($K'_0 = K(v)$ and $[K'_0: K] \geq 3$). We add further semisimple modules $(12)(23)\nu^2 K \oplus (12)(23)\nu^3 K \oplus \cdots$ to $B$ and $(23)\nu^2 K \oplus (23)\nu^3 K \oplus \cdots$ to $e_2 R$. Then $(\ast, 1)$ does not hold by Corollary 4.
We shall give a characterization of left serial rings with \((\ast, 2)\).

**Proposition 7.** Let \(R\) be a right artinian ring and \(e\) a fixed primitive idempotent. Assume that \((\ast, 2)\) holds for any two hollow modules of form \(eR/K\). Then \(eJ\) is a direct sum of uniserial modules.

**Proof.** Since \(eR\oplus eJ\) is a maximal submodule of \(eR\oplus eJ\) by assumption, where the \(A_i\) are hollow. We shall show by induction that \(A_i/A_iJ^*\) is uniserial for all \(i\). If \(k=0\), \(A_1/A_1J^*=0\). Assume that \(A_i/A_iJ^*\) is uniserial for all \(i\). Let \(A_mJ^*/A_mJ^{*+1}=B_{m1}\oplus B_{m2}\oplus \cdots \oplus B_{m_m}\), where the \(B_{mj}\) are simple. We shall show \(s_m=1\). Otherwise, \(\overline{B}_{m1} \neq 0\) and \(\overline{B}_{m2} \neq 0\). Put \(B^*_j=\sum_{i=1}^{m-1} A_iJ^*/B_j\), where \(A_mJ^{*+1} \subset B_j \subset A_mJ^*\) for \(j=1, 2\) and \(B_i/A_iJ^{*+1}=\overline{B}_{m1}\oplus B_{m2}\oplus \cdots \oplus \overline{B}_{m_m}\), and \(D=eR/B_j\oplus eR/B^*_j\). We shall show, in this case, that \(D\) does not satisfy \((\ast, 2)\). Contrarily assume that \(D\) satisfies \((\ast, 2)\). Then \(D\) contains a maximal submodule \(M\) with a direct summand \(M_1\) isomorphic to \(\overline{eR}=eR/(B_j^* \cap B^*_j)\) where \(j \in eJe\) by \([3]\), Lemma 3. Since \(eJ^{*+1} \supset B^*_j \supset eJ^{*+2}\) and \(jB^*_j \subseteq eJ^{*+2}\), \((e+j)B^*_j=B^*_j\). Hence \(M_1=eR/(B^*_j \cap B^*_j)\) \((=\overline{eR})\). We shall denote \(A_i/A_iJ^*(i=m)\) and \(A_m/B_j\) by \(\overline{A}_i\) and \(\overline{A}_m\), respectively, where \(B_j/A_jJ^{*+1}=\sum_{i=1}^{m} \overline{B}_{mj}\). Let \(M=M_1 \oplus M^*\) and \(|\overline{A}_1|=n_1\) and \(|\overline{A}_m|=n_m+1\), where \(n_1 \leq n_m\) and \(n_m=1+1\). Then \(|\overline{eR}|=|M_1| =\sum_{i=1}^{n} n_i+2\) and \(|D|=2 \sum_{i=1}^{n} n_i+2\). Put \(\overline{D}=D/J(D)\supset \overline{M}=M/J(D)\). We note that \(\overline{M}=(\overline{e}+\overline{e})eR/eJ\) in \(\overline{D}\) (see \([3]\), Lemma 3). Since \(|\overline{D}|=2\), \(\overline{M}\) is a simple module. Now \(M^*=\sum_{i=1}^{n} M_i; M_i\) are hollow by \((\ast, 2)\). If \(\overline{M}_2=(M_2+J(D))/J(D)=\overline{M}, eR/B^*_j\) is an epimorphic image of \(M_2\) by the remark above. Then \(|M_2| \geq |\overline{eR}|-1\) and so \(|M_1| \geq |M_1|+|M_2| \geq |D|\), a contradiction. Hence \(M^* \subset J(D)\). Let \(\varphi\) be the given isomorphism of \(\overline{eR}\) to \(M_1\). It is clear that \(\varphi(\overline{eJ}) \subset J(D)\), and hence

\[
(2) \quad J(D) = \varphi(\overline{eJ}) \oplus M^*
\]
(note \(M \supset J(D)\)). Put \(Q=A_1 \oplus \cdots \oplus A_{m-1}\), and \(\overline{eJ}=Q \oplus A_m\). Then

\[
(3) \quad J(D) = Q_1 \oplus Q_2 \oplus Q_3 \oplus L_2,
\]
where \(Q_1=Q_2=Q, L_1=Q_m \oplus \overline{B}_{m1}\) and \(L_2=Q_m \oplus \overline{B}_{m2}\). From \(3\) \(\varphi(Q) = \{q+0+q+0|q \in Q\}\). Hence

\[
(4) \quad J(D) = \varphi(Q) \oplus L_1 \oplus Q_2 \oplus L_2.
\]

On the other hand, \(\text{Soc}(\varphi(\overline{A}_m))=\text{Soc}(L_1) \oplus \text{Soc}(L_2)\), and \(\text{Soc}(\varphi(\overline{eJ}))=\text{Soc}(\varphi(Q))\).
\( \oplus \text{Soc}(\varphi(A_n)) \). Let \( p \) be the projection of \( J(D) \) onto \( Q_2 \) in (4). Then \( p|\text{Soc}(M^*) \) is a monomorphism from the above observation (note \( \text{soc}(M^*) \cap \text{Soc}(\varphi(\widetilde{e}J))=0 \)), and hence so is \( p|M^* \). Hence \( |M^*| \leq |Q_2| = \sum_{i=1}^{n} n_i \). Therefore \( |M| = |M_1| + |M^*| \leq \sum_{i=1}^{n} n_i + 2 + \sum_{i=1}^{n} n_i = 2 \sum_{i=1}^{n} n_i + 2 - n_m \leq 2 \sum_{i=1}^{n} n_i + 1 = |D| - 1 \) (note \( n_m = n + 1 \geq 2 \)), which is a contradiction. Hence \( A_mJ^*/A_mJ^{*+1} \) is simple.

The following lemma is substantially due to T. Sumioka [9].

**Lemma 9.** Let \( R \) be left serial and \( eJ \) a direct sum of uniserial modules \( A_i \) and \( A'_i \), i.e., \( eJ = \bigoplus A_i = \bigoplus A'_i \). Let \( d' \) be an element in \( efe \) such that \( d'A_{i_1} = A'_{i_1} \) for \( A_{i_1} \subset A_i \) and \( A'_{i_1} \subset A'_i \). Then there exists \( d \) in \( A_i \cap efe \) such that \( d|A_{i_1} = d'|A'_{i_1} \). Further for such \( d \) \( dA_i = 0 \) (\( i \neq 1 \)).

**Proof.** Put \( A_{i+a} = a_aR, A_{i+a} = a_aR \) and \( A'_{i+a} = a'_aR \) (\( d'_{a_a} = a'_a \)). Assume that \( a_ag = a_a \) and \( a'_bg = a'_b \) for a primitive idempotent \( g \). Let \( d' = \sum d'_a, d'_a \in A'_a \). Since \( A'_1 \supset A_1 \Rightarrow a'_a = d'a_a = \sum d'_aa_a, a'_a \neq d'_aa_a \). Put \( d = d'|A_i \cap efe \). Since \( da_a = a'_a, \ d \in T(J^R_a) \). Assume \( da_i \neq 0 \) for some \( A_i = a_aR \) (\( i \neq 1 \)). Then \( da_i \) is an element in \( T(A'_1 - \varphi) \), which is a contradiction to [7], Lemma 7. Hence \( da_i = 0 \) for \( i \neq 1 \).

Let \( M = \bigoplus_{i=1}^{t} N_i \). For \( N_i \subset M_i, i=1, 2, \ldots, t \), we call \( \bigoplus_{i=1}^{t} N_i \) a standard submodule of \( M \) (with respect to the decomposition \( \bigoplus_{i=1}^{t} M_i \)).

**Lemma 10** ([9], Lemma 3.3) Let \( R \) be a left serial ring such that \( eJ \) is a direct sum of uniserial modules \( A_i \). Then every submodule in \( eJ \) is a standard submodule with respect to some direct decomposition of \( eJ \), whose direct summands are all uniserial.

**Proposition 8.** Let \( R \) be left serial and \( eJ \) a direct sum of uniserial modules. Then (\( *, 2 \)) holds for any direct sum of two hollow modules of form \( eR/K \).

**Proof.** We may consider a maximal submodule \( M' \) in \( D' = eR/E_1 \oplus eR/E_2 \), where \( E_i \) are submodules in \( eJ \). There exists a maximal submodule \( M \) in \( D = eR \oplus eJ \) such that \( M \supset E_1 \oplus E_2 \) and \( M(E_1 \oplus E_2) = M' \). From [0], Theorem 2 there exists a decomposition \( D = eR(f) \oplus eJ \) such that \( M = eR(f) \oplus eJ \), where \( f \in \text{Hom}_e(eR, eR) \). Since \( E_2 \subset 0 \oplus eJ \), \( D/E_2 = eR(f) \oplus eJ/E_2 \). Hence \( M' = M/(E_1 \oplus E_2) = (eR(f) \oplus eJ/E_2)/\varphi(E_1) \), where \( \varphi; E_1 \to eR(f) \oplus eJ/E_2 \) is the natural mapping. Accordingly, since \( eR \approx eR(f) \), we may show for submodules \( X_i \) in \( eJ \) (\( i=1, 2 \)) and \( Y \) in \( D^* = eR/X_1 \oplus eJ/X_2 \)

(5) \( D^*/Y \) is a direct sum of hollow modules.

First assume \( X_1 \subseteq eJ \). Let \( S' \) be a submodule in \( eJ \oplus eJ \) such that \( (Y) \supset S' \supset X_1 \oplus X_2 \) and \( S'/(X_1 \oplus X_2) (= S) \) is simple. We shall show
(6) \[ D^*/S \cong eR/X_1 \oplus eJ/X_2, \]
where \( X_1 \subset eR \) and \( X_2 \subset eJ \).

Put \( X_1 = A_{a_1} \oplus \cdots \oplus A_{a_m}, \)
\( X_2 = A_{a_1} \oplus \cdots \oplus A_{a_n} \) by Lemma 10, where \( eJ = \sum_{i=1}^{n} A_i \), \( A_{a_1} \subset A_1 \) and \( A_{a_2} \subset A_2 \). Then \( S \subset A_1 \oplus A_{a_1} \oplus \cdots \oplus A_{a_n} \oplus A_{a_1} \oplus A_{a_2} \oplus \cdots \oplus A_{a_n} \). If \( S \subset \sum_{i=1}^{n} A_i \oplus A_{a_1} \oplus A_{a_2} \) \( D^*/S = eR/X_1 \oplus eJ/X_2 \). Since \( eJ/X_2 \) is a direct sum of uniserial modules by Lemma 10, \( D^*/S \) is a direct sum of hollow modules. We obtain the same result for a case \( S \subset \sum_{i=1}^{m} A_i \).

Let \( p_i : eJ/X_1 \oplus eJ/X_2 \to A_i/A_{a_i} \) and \( q_i : eJ/X_1 \oplus eJ/X_2 \to A_i/A_{a_i} \) be the projections. We shall show (6) by induction on \( t \), where \( t = \) (the number of \( \{ p_i \) and \( q_i | p_i(S) \neq 0 \) and \( q_i(S) \neq 0 \}) \). If \( t = 1 \), we are done from the observation above. Now we may assume that \( S = \{ s_1 + f_1(s_1) + \cdots + f_m(s_1) + f_i(s_1) + \cdots + f_{m_i}(s_1) | s_1 \in A_{a_1} \oplus A_{a_2} \oplus \cdots \oplus A_{a_n} \} \). From the above assumption, we may assume \( f_1 \neq 0 \). If \( \alpha_1 = \beta_1 \), then there exists a unit \( x \) in \( eRe \) such that \( x \mid A_{a_1} \oplus A_{a_2} \to A_{a_3} / A_{a_1} \). Accordingly \( xA_{a_1} = A_{a_1} \) and so

(7) \[ x_1 (= h) \in \text{Hom}_R (A_i/A_{a_1}, eR/X_1). \]

Next assume \( \alpha_1 > \beta_1 \) or \( \alpha_1 < \beta_1 \). In the former case we obtain \( d \) in \( eJ \) as the above \( x \). Let \( \alpha_1 < \beta_1 \). Then there exists \( d' \) in \( eJ \) such that \( d' \mid A_{a_1-1} \). Further, since \( d'eR \subset A_i \), we may assume \( d' \in A_i \) and \( d'A_k = 0 \) for \( k \neq 1 \). From Lemma 9, we may assume \( d' \in \text{Hom}_R (A_{a_1-1} \oplus A_{a_2} \oplus A_{a_3} / A_{a_1}, A_i/A_{a_1} / A_{a_2} \). \)

The above assumption, we may assume \( f_1 \neq 0 \). If \( \alpha_1 = \beta_1 \), then there exists a unit \( x \) in \( eRe \) such that \( x \mid A_{a_1-1} \oplus A_{a_2} \to A_{a_3} / A_{a_1} \). Accordingly \( xA_{a_1} = A_{a_1} \) and so

(8) \[ d' (= h') \in \text{Hom}_R (eR/X_1, A_i/A_{a_1}). \]

Case (7)

(9) \[ eR/X_1 \oplus eJ/X_2 = eR/X_1 \oplus (A_i/A_{a_1})(h) \oplus \sum_{i=2}^{n} A_i/A_{a_i}. \]

Then \( S \subset (\sum_{i=1}^{m} p_i + \sum q_i)(S) \), where \( p_i \) and \( q_i \) are the projections of (9). It is clear that (the number of \( \{ p_i, q_i \} \) (the number of \( \{ p_i, q_i \} \) )

Case (8)

(10) \[ eR/X_1 \oplus eJ/X_2 = (eR/X_1)(h') \oplus eJ/X_2. \]

Then \( S \subset (\sum_{i=1}^{m} p_i + \sum q_i)(S) \). Hence we obtain the same situation. If \( X_1 = eJ \), \( eR/X_1 \) is simple. This is a special case in the above argument. In case (9), since (\( A_i/A_{a_1} )(h) \cong A_i/A_{a_1} \), we obtain the isomorphism \( f_1 : eR/X_1 \oplus (A_i/A_{a_1})(h) \oplus \sum A_i/A_{a_i} \to eR/X_1 \oplus eJ/X_2 \). Similarly in case (10) we have \( f_2 : (eR/X_1)(h') \oplus eJ/X_2 \to eR/X_1 \oplus eJ/X_2 \). Then (the number of \( \{ p_i, q_i | p_i(f_i(S)) \neq 0, q_i(f_i(S)) \neq 0 \})
(the number of \( q_i, p_i \mid \{ p_i(S) \neq 0, q_i(S) = 0 \} \) for \( k = 1, 2 \) (note \( f(J((eR/X_i) (h')) = J(eR/X_i)) \). Further \( D^*/S \approx f_0(D^*)/f_0(S) = D^*/f_0(S) \). Therefore (6) holds by induction on \( t \). If we take a chain \( Y = S_{k+1} \supset S_k \supset \cdots \supset S_1 \supset X_1 \oplus X_2 = S_0 \) such that \( S_i/S_{i+1} \) is simple, we can show (5).

From the above proof and Proposition 7 we have

**Theorem 2.** Let \( R \) be a left serial ring and \( e \) a primitive idempotent. Then the following conditions are equivalent:

1) \((*, 2)\) holds for a direct sum of any two hollow right \( R \)-modules of form \( eR/K \).
2) \( eJ \) is a direct sum of uniserial modules.
3) Every factor module of \( eR \oplus eJ \) is a direct sum of hollow modules (direct sum of a hollow module and uniserial modules).
4) Every factor module of \( eR \oplus eJ^{(n)} \) is a direct sum of hollow modules, where \( eJ^{(n)} \) is a direct sum of \( n \)-copies of \( eJ \).

We shall study further structures of \( R \) with \((*, 2)\) when \( eJ \) is square-free.

**Lemma 11.** Let \( R \) be a left serial ring. Let \( \alpha = e + d \ (d \in eJ) \) be a unit in \( eRe \). Assume \( A_i \cong A_j \) if \( i \neq j \). Then if \( \alpha A_i \neq A_i, \alpha A_i = A_i \) for \( i \neq j \), where \( eJ = \bigoplus_i A_i \) and the \( A_i \) are uniserial.

**Proof.** From [7], Lemma 5 \( d \in A_j \) for some \( j \). Since \( \alpha A_i \neq A_i, j \neq 1 \), and so \( dA_i \neq 0 \). Therefore \( dA_k = 0 \) for \( k \neq 1 \) by Lemma 9.

**Proposition 9.** Let \( R \) be left serial. Assume that \( eJ \) is a direct sum of uniserial modules \( A_i; eJ = \bigoplus_i A_i \) and that \( eJ \) is square-free. Let \( X \) be a submodule of \( eJ \). Then there exist uniquely \( k \) and \( k' \) (not depending on \( X \)) such that \( X = \alpha \bigoplus_i A_i = A_i \oplus \cdots \oplus A_{k-i+k'} \oplus A_{k+i+k'+\cdots+n_i} \), where \( A_{j+i} \subseteq A_j \) and \( \alpha A_k \subseteq A_k \oplus A_{k'} \). Further all \( A_i \) except \( A_k \) are characteristic and the number of hollow modules of form \( eR/K \) is finite up to isomorphism.

**Proof.** Let \( eJ = \bigoplus_i A_i \) be as in the proposition. Assume that a sub-factor module of \( A_i \) is isomorphic to one of \( A_2 \). Then from [7], Lemma 2 there exists \( d \) in \( A_2 \) (or \( A_1 \)) which induces this isomorphism. If we have the same situation between \( A_i \) and \( A_j \), we obtain \( d' \) in \( A_i \) (or \( A_j \)). Then \( i \neq 2 \) by assumption and [7], Lemma 4. Since \( A_2 \) is uniserial, \( \text{Soc}(A_2) \cong A_{k+i}/A_{k+i+1} \approx A_{j+k}/A_{k+1} \) for some \( k \) and \( s \). Hence \( j = 1 \) by [7], Lemmas 2 and 4. Therefore, for \( j \neq 1, 2 \), any sub-factor modules of \( A_j \) are not isomorphic to any one of \( A_k \) for all \( k \neq j \). Put \( F_1 = A_1 \oplus A_2 \) and \( F_2 = \bigoplus_{i \neq 1, 2} A_i \). Then we can easily show by induction on \( m \) that every submodule of \( F_2 \) is standard. Further from
the argument after (1) in the proof of Lemma 8, every submodule of \( F_x \) is of a form \( \alpha(A_{1k} \oplus A_{2k}) \); \( \alpha = e + d, d \in A_2 \). Let \( p_i \) be the projection of \( eJ \) onto \( F_i \), and \( X \) a submodule of \( eJ \). Put \( X^{(j)} = p_j(X) \) and \( X^{(j)} = X \cap F_j \). Assume \( X^{(0)} = X, \) and \( X^{(0)} = \alpha(A_{1k} \oplus A_{2k}) \). \( A_1 \oplus A_2 = \alpha^{-1}(A_1 \oplus A_2) \supset \alpha^{-1}X^{(0)} \supset \alpha^{-1}X = A_{1k} \oplus A_{2k} \). Hence some simple sub-factor module \( T \) of \( X^{(0)} \) is isomorphic to one of \( A_1 \) or \( A_2 \). Since \( X^{(0)} / X^{(0)} = X^{(0)} / X^{(2)}, \) \( T \) is isomorphic to a sub-factor module of \( X^{(0)} / X^{(2)} \). On the other hand, every submodule of \( F_2 \) is standard, and so \( T \) is isomorphic to a sub-factor module of some \( A_j (j \geq 3) \), which is impossible from the initial observation. Hence \( X^{(0)} = X^{(0)} \), and \( X = X^{(0)} \oplus X^{(0)} = \alpha(A_{1k} \oplus A_{2k}) \oplus \sum_{j \geq 3} \oplus A_{jk} = \alpha(\sum_{i=1}^{n} A_{ik}) \) by Lemma 11. The remaining part is clear from the above.

**Lemma 12.** Let \( R \) be a right artinian ring with \((*, 2)\). Let \( D \) be a direct sum of two hollow modules and \( M \) a maximal submodule of \( D \). Then \( M \) has the following decomposition: \( M = M_1 \oplus M_2; M_1 \) is a hollow module not contained in \( J(D) \) and \( J(D) = J(M_1) \oplus M_2 \).

**Proof.** Let \( D = e_1 R/E \oplus e_2 R/E' \). If \( e_1 R \cong e_2 R, M = eR/E \oplus e'J/E' \) (or \( eR \cong e'R \), we can obtain the lemma for any \( M \) similarly to (2) in the proof of Proposition 7.

For two integers \( \alpha(1) \) and \( \alpha(2) \), we denote \( \max \{\alpha(1), \alpha(2)\} \) (resp. \( \min \{\alpha(1), \alpha(2)\} \)) by \( \alpha \) (resp. \( \alpha \)). If \( R \) is a right artinian ring with \((*, 2)\),

\[
(11) \quad eJ = \sum_{i=1}^{n} A_i; \text{the } A_i \text{ are uniserial}
\]

from Proposition 7.

**Proposition 10.** Let \( R \) be a left serial ring with \((*, 2)\) and let \( eJ \) and \( A_i \) be as above. We assume that \( eJ \) is square-free. Put \( E_i = A_{i_1} \oplus \cdots \oplus A_{n_i} \); \( A_{i_1} \subset A_i \) for \( i = 1, 2 \) and all \( k \). Then every maximal submodule \( M \) of \( D = eR/E \oplus eR/E' \) is isomorphic to \( eR/(A_{i_1} \oplus A_{i_2} \oplus \cdots \oplus A_{i_k}) \oplus A_1 / A_{i_1} \oplus A_2 / A_{i_2} \oplus \cdots \oplus A_n / A_{i_k} \) unless \( M = eR/E \oplus eJ/E' \) or \( eR \cong e'R \).

**Proof.** We may assume that \( R \) is basic. Assume \( \overline{M} = (\bar{e} + \bar{\alpha})eRe/eJ \). \( 0 \neq \bar{\alpha} \in eRe/eJ \). Then \( A_1 / A_{i_1} \oplus \cdots \oplus A_n / A_{i_n} \oplus (A_1 / A_{i_1} \oplus \cdots \oplus A_n / A_{i_n}) = J(D) = eJ/(E_1 \cap (\alpha + j)E_2) \oplus M_2 \) by Lemma 12 and [3], Lemma 3. On the other hand, \( E_1 \cap (\alpha + j)E_2 = \gamma(A_{i_1} \oplus \cdots \oplus A_{i_n}) \) by Proposition 9. Hence \( eJ/E_1 \cap (\alpha + j)E_2 \approx A_1 / A_{i_1} \oplus \cdots \oplus A_n / A_{i_n} \). Since \( eJ \) is square-free, either \( A_1 / A_{i_1} \approx A_1 / A_{i_2} \) or \( A_1 / A_{i_1} \). Therefore \( \alpha(3) = \alpha(1) \) or \( \alpha(2) \). Further \( A_{i_1} \oplus \cdots \oplus A_{i_n} \) implies \( A_{i_1} \oplus \cdots \oplus A_{i_n} \). Considering the projection of \( eJ \) to \( A_i \), we obtain \( \alpha(3) \approx \alpha(1) \) (note \( A_i \approx \gamma A_i \)).
Similarly \( \alpha_i(3) \geq \alpha_i(2) \), and so \( \alpha_i(3) = \alpha_i \). Therefore \( M_2 \approx \sum_{i=1}^n A_i / A_i x_i \).

**Corollary 11.** Let \( R \) be as above. Then the number of isomorphism classes of maximal submodules in a direct sum of (fixed) two hollow modules is at most three.

**Remark.** Assume in (11) that \( eJ \) is not square-free. Then we can show, by direct computation, the following fact:

Let \( D = eR / (A_1 \oplus eR / (A_2 \oplus A_3) \). Then \( D \) contains the following maximal submodules:

\[ eJ / A_1 \oplus eR / (A_2 \oplus A_3), \quad eR / A_1 \oplus eJ / (A_2 \oplus A_3), \quad eR / A_1 \oplus A_2 / A_2, \] and \( eR / A_1 \oplus A_2 / A_2 \). (cf. the proof of [6], Lemma 3). Therefore Corollary 11 characterizes almost left serial rings with \((*, 2)\) and \( eJ \) being square-free.

**Lemma 13.** Let \( R \) be a left serial ring. Assume that \( eJ \) is square-free and \( eJ \) is a direct sum of uniserial modules; \( eJ = \sum_{i=1}^n A_i \). Let \( x \) be a unit in \( eRe \) and \( xA_i = A_i \). Then there exists \( d \) in \( eJe \) such that \((x + d)A_i = A_i \) for all \( i \).

**Proof.** Let \( p_i \) be the projection of \( eJ \) onto \( A_i \), and \( A_j = a_j R \) for \( j = 1, 2, \ldots, m \). Since \( eJ \) is square-free, \( p_i x A_i \subset J(A_i) \) for \( i \neq 1 \). Hence \( p_i x A_i = (d_i) \), for some \( d_i \) in \( J(A_i) \) by [7], Lemma 2. By assumption and [7], Lemma 4, only one \( d_i \), say \( d_2 \), is non-zero, since \( xA_2 = A_2 \). Similarly for \( j \neq 1 \) and \( i \neq j \), \( p_j x A_j = (d_j) \), for some \( d_j \). Then \( d_k = 0 \) (\( k \neq 2 \)) by [7], Lemma 4. Assume \( d_2 \neq 0 \). Since \( d_2 \neq 0 \), \( 0 + d_2 a_2 R \subset d_2 a_2 R \). Let \( d_2 a_2 = d_2 a_2 \) (and \( a_2 g = a_2 \) and \( a_2 g = a_2 \) for a primitive idempotent \( g \)). Hence there exist non-zero three elements \( a_2 g, a_2 g \) and \( d_2 a_2 g \). This is a contradiction to [7], Lemma 5. Hence \( xA_j = A_j \) (\( j \neq 1, 2 \)). If \( xA_3 = A_3 \), we obtain again a contradiction to [7], Lemmas 2 and 4. Finally, since \( 0 + d_2 A_2 \subset A_2 \), \( d_2 A_j = 0 \) for \( j \neq 1 \) from Lemma 9. Therefore \((x + d_2)A_i = A_i \) for all \( i \).

From Proposition 10 we know the form of maximal submodules in \( eR / E_1 \oplus eR / E_2 \) up to isomorphism, provided \((**, 2)\) holds and \( eJ \) is square-free. We shall show explicitly such an isomorphism. Let \( eJ = A_1 \oplus A_2 \oplus \cdots \oplus A_n \) be a direct sum of uniserial submodules. Put \( E_i = A_j / A_j x_j \) for \( i = 1, 2 \), where \( A_j x_j \subset A_j \). Set \( D = eR / E_1 \oplus eR / E_2 \) and let \( M \) be a maximal submodule in \( D \). Put \( M^* = eR / (A_1 A_2 \oplus \cdots \oplus A_n \oplus A_1 x_1 \oplus A_2 x_2 \oplus \cdots \).
\( \oplus A_s/A_s. \) and \( \bar{D} = D[J(D)] \Rightarrow \bar{M} = M[J(D)] \). We may assume \( \bar{M} = (\bar{c} + \bar{e}k)\Delta \) (cf. [2], p. 93), where \( k \equiv 0 \in \Delta \) (\( R \) is basic). From Lemma 13, we may assume \( kA_i = A_i \) for all \( i \). We define a mapping \( \varphi : M^* \rightarrow D \) by setting for \( x \in eR, a_i \in A_i, \)

\[
(12) \quad \varphi(x(A_{1a_1} \oplus \cdots \oplus A_{n a_n}) + (a_1 + A_{1a_1}) + \cdots + (a_n + A_{n a_n}))
\]

\[
= (x + a_1d_{\delta a_1(1)} + \cdots + a_n d_{\delta a_n(1)}) + (A_{1a_1(1)} \oplus \cdots \oplus A_{n a_n(1)})
\]

\[
+ (kx + a_1d'_{\delta a_1(2)} + \cdots + a_n d'_{\delta a_n(2)}) + (A_{1a_2(2)} \oplus \cdots \oplus A_{n a_n(2)}),
\]

where the \( \delta, \delta' \) are Kronecker deltas such that \( \delta_{\delta a_n(2)} = 0 \) provided \( \alpha_i(1) = \alpha_i(2) \).

Since \( (A_{1a_1(1)} \oplus \cdots \oplus A_{n a_n(1)}) \cap (A_{1a_2(2)} \oplus \cdots \oplus A_{n a_n(2)}) = A_{1a_1(1)} \oplus \cdots \oplus A_{n a_n(2)} \), \( \varphi \) is an \( R \)-homomorphism. \( (\varphi(M^*) + J(D))/J(D) = \bar{M} \) means \( \varphi(M^*) \subset M \), and so \( \varphi(M^*) = M \), since \( |M^*| = |S| - 1 = |M| \).

Finally we shall give a property of a right artinian ring with \((*, 2)\). Put \( P = \sum_{k=1}^i A_k \) and \( Q = \sum_{k=1}^i A_k \) in (11). Assume \( \bar{A}_k \approx \bar{A}_{k'} \) for all \( k, k' \) such that \( k \leq i < k' \).

**Proposition 12.** Let \( R, P \) and \( Q \) be as above. Let \( L \) be a direct summand of \( ej \) such that \( L/LJ \approx P/PJ \). Then there exists a unit \( \alpha = e + j \) \((j \in eJ)\) such that \( \alpha P = L \).

Proof. From the assumption \( L/LJ \approx P/PJ \) and Krull-Remak-Schmidt theorem, \( L \approx P \). We apply the exchange property of \( L \) to \( eJ = P \oplus Q \). Then \( eJ = L \oplus P' \oplus Q' \), where \( P' \subset P \) and \( Q' \subset Q \). Since no one of indecomposable direct summands of \( L \) is isomorphic to any one in \( Q \), \( eJ = L \oplus Q \). Put \( D = eR/P \oplus eR/L \). We shall employ the similar argument to the proof of Proposition 7. From [3], Lemma 3 and its proof, \( D \) contains a maximal submodule \( M \) such that \( M = M_1 \oplus M^* \) with \( M_1 \approx eR/K, \) where \( K = P \cap \alpha L, \) \( \alpha = e + j \). Now

\( (13) \quad J(D) = Q_1 \oplus Q_2, \) where \( Q_i \approx Q \).

Further, as in the proof of Proposition 7,

\( J(D) = \varphi(eJ/K) \oplus M^* \), \( \varphi : eR/K \rightarrow D \) is the given injection. On the other hand, \( \varphi((Q + K)/K) = Q_1(f) \), where \( f : Q_1 \rightarrow Q_2 \). Hence

\( (14) \quad J(D) = \varphi((Q + K)/K) \oplus Q_2 \) and \( \varphi(P/K) \subset Q_2 \).

Let \( p \) be the projection of \( J(D) \) onto \( Q_2 \) in (14), and \( x \) an element in \( p(Soc(M^*)) \cap \varphi(P/K) \); \( x = p(y) \) for some \( y \) in \( Soc(M^*) \). Then \( y = (1 - p)y + py \) and \( (1 - p)y \in \varphi((Q + K)/K) \). Hence \( y \in \varphi(eJ/Q) \cap M^* = 0 \), and so \( x = 0 \). Similarly, we know \( p \mid Soc(M^*) \) is a monomorphism. Hence

\( (15) \quad p(M^*) \oplus (P/K) \subset Q_2 \) and \( p(M^*) \approx M^* \).

Now \( |M| = |M_1| + |M^*| = |eR/K| + |M^*| = 1 + |Q| + |P/K| + |M^*| \leq \)
1 + |Q| + |Q_2| = |D| - 1 = |M| from (15). Hence $p(M^*) \oplus \varphi(P/K) = Q_s = \sum_{k=1}^{n} A_k$ and so $\varphi(P/K)$ is isomorphic to a direct sum of some $A_k (k \geq i+1)$ by Krull-Remak-Schmidt theorem. On the other hand, $A_i \cong A_k$ for $s \leq i < k$, and hence $P=K=P \cap aL$. Therefore $aL=P$.

**Example 4.** Let $Q$ be the field of rationals. We regard $Q(\sqrt{\omega}-1)$ as a $Q$-space. Then we can directly compute that $V=Q \oplus Q(\sqrt{-1}+\sqrt{\omega}-1)$ is not transferred to a standard submodule of $L=Q \oplus Q\alpha \oplus Q\alpha^2 \oplus Q\alpha^3$ by a unit, where $\alpha=\sqrt{-1}$. Hence

$$(L \quad L)$$

$$(0 \quad Q)$$

is a left serial ring with $(*, 2)$ by [3], Proposition 3, however $(0, V)$ is not transferred to a standard submodule of a decomposition $eI=(0, Q) \oplus (0, Q\alpha) \oplus (0, Q\alpha^2) \oplus (0, Q\alpha^3)$, (cf. Lemma 10 and Proposition 9).

**References**


Department of Mathematics
Osaka City University
Sugimoto-3, Sumiyoshi-ku
Osaka 558, Japan