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CHARACTERIZATIONS OF CONDITIONAL EXPECTATION OPERATORS FOR L_p-VALUED FUNCTIONS ON A GENERAL MEASURE SPACE

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Introduction. Let $(\Omega, \boldsymbol{A}, \mu)$ be a measure space, where $\boldsymbol{A}$ is a σ-ring and μ is a σ-finite measure on $\boldsymbol{A}$, (X, S, λ) a measure space and E a real Banach space. We consider semi-constant-preserving contractive projections of $L_1(\Omega, \boldsymbol{A}, \mu, E)$ into itself. If $(\Omega, \boldsymbol{A}, \mu)$ is a probability space and E is a strictly-convex Banach space, then Landers and Rogge [2] proved that such operators coincide precisely with the conditional expectation operators. If $(\Omega, \boldsymbol{A}, \mu)$ is a probability space and $E=L_p(X, S, \lambda)$, where $p=1$ or ∞, then Miyadera [3] and [4] proved that such operators coincide precisely with the conditional expectation operators under some additional conditions. In this paper we deal with the case when $(\Omega, \boldsymbol{A}, \mu)$ is a general measure space, where $\boldsymbol{A}$ is a σ-ring and λ is a σ-finite measure on $\boldsymbol{A}$. Substituting constant-preserving property by semi-constant-preserving property we can prove theorems which are generalizations of characterization theorems in Landers and Rogge [2], Miyadera [3] and [4].

1. Definitions and useful Lemmas. Let $(\Omega, \boldsymbol{A}, \mu)$ be a measure space, $\boldsymbol{A}(\mu)=\{A\in\boldsymbol{A};\ \mu(A)<\infty\}$ and E *a* real Banach space with the norm $\|\ \|$. Note that E can be the class $\boldsymbol{R}$ of real numbers. Let $\boldsymbol{N}$ be the class of natural numbers. For any $A, B\in\boldsymbol{A}$ we write $A\subset B$ if $\mu(A-B)=0$ aud $A=B$ if $\mu((A-B)\cup(B-A))=0$. $A, B\in\boldsymbol{A}$ are said to be disjoint if $\mu(A\cap B)=0$. We suppose that μ is σ-finite, i.e., for any $A\in\boldsymbol{A}$ there exists a sequence of sets $\{A_n;\ n\in\boldsymbol{N}\}$ such that $A_n\in\boldsymbol{A}(\mu)$ and $A=\cup\{A_n;\ n\in\boldsymbol{N}\}$. For any $A\in\boldsymbol{A}$ we denote by I_A the indicator function of A and by $A=\emptyset$ we mean $\mu(A)=0$. Let $L_1(\Omega, \boldsymbol{A}, \mu, E)$ be the calss of E-valued Bochner integrable functions, which is a Banach space with the norm $\|\ \|_L$ defined by

$$\|f\|_L=\int\|f(\omega)\|d\mu \qquad \text{for any} \quad f\in L_1(\Omega, \boldsymbol{A}, \mu, E)\,.$$

For any $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ we denote $\{\omega;\ f(\omega)\neq 0\}$ by $s(f)$ and for any linear operator Q of $L_1(\Omega, \boldsymbol{A}, \mu, E)$ into itself we denote $S(Q)=\{A\in\boldsymbol{A}(\mu);$ there eixsts $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that $A\subset s(Q(f))\}$. For the definitions and properties of Bochner integral, see Hille and Phillips [1].

DEFINTION 1. Let $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$. For a σ-subring $\boldsymbol{B}$ of $\boldsymbol{A}$, a function g is called the conditional expectation of f given $\boldsymbol{B}$ if $g\in L_1(\Omega, \boldsymbol{B}, \mu, E)$, and

$$\int_B g d\mu = \int_B f d\mu \qquad \text{for any} \quad B\in\boldsymbol{B},$$

where the integral is the Bochner integral. We denote by $f^{\boldsymbol{B}}$ the conditional expectation of f given $\boldsymbol{B}$. For any $\phi\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ we define $\phi a\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ by $(\phi a)(\omega)=\phi(\omega)a$ for any $\omega\in\Omega$ and $a\in E$. Then it is clear that $(\phi a)^{\boldsymbol{B}}=\phi^{\boldsymbol{B}}a$.

DEFINITION 2. Let P be a linear operator of $L_1(\Omega\ \boldsymbol{A}, \mu, E)$ into itself. P is said to be *contractive* if

$$||P|| = \sup\{||P(f)||_L; f\in L_1(\Omega, \boldsymbol{A}, \mu, E) \quad \text{and} \quad ||f||_L = 1\}\leqq 1,$$

semi-constant-preserving if for any $a\in E$, $\varepsilon>0$, $A\in s(P)$ there exists $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that

$$||I_A P(f)-I_A a||_L<\varepsilon,$$

and *a projection* if $P\circ P=P$, where $(P\circ P)(f)=P(P(f))$ for any $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$.

In this paper an operator P is said to satisfy Assumption 1 if
(1) P is a semi-constant-preserving contractive projection of $L_1(\Omega, \boldsymbol{A}, \mu, E)$ into itself.

Lemma 1.1. *Let $\boldsymbol{B}$ be a σ-subring of $\boldsymbol{A}$. Then for any $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ the conditional expectation $f^{\boldsymbol{B}}$ of f given $\boldsymbol{B}$ exists uniquely up to almost everywhere and the conditional expectation operator $(\ )^{\boldsymbol{B}}$ satisfies Assumption* 1.

Proof. Let $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$. If there exists $B\in\boldsymbol{B}$ such that $s(f)\subset B$, then by a theorem in Schwartz [5] $f^{\boldsymbol{B}}$ exists uniquely up to almost everywhere and $||f^{\boldsymbol{B}}||_L\leqq||f||_L$ and $(f^{\boldsymbol{B}})^{\boldsymbol{B}}=f^{\boldsymbol{B}}$. For an arbitray $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ there exists $C\in\boldsymbol{B}$ such that

$$\int_C ||f||d\mu = \sup\{\int_B ||f||d\mu; B\in\boldsymbol{B}\}.$$

Clearly $(I_{B-C}f)(\omega)=0$ (*a.e.* ω) for any $B\in\boldsymbol{B}$. Since $s(I_C f)\subset C$, there exists $(I_C f)^{\boldsymbol{B}}$. For any $B\in\boldsymbol{B}$

$$\int_B f d\mu = \int_B I_C f d\mu+\int_{B-C} f d\mu = \int_B I_C f d\mu = \int_B (I_C f)^{\boldsymbol{B}} d\mu.$$

Therefore $(I_C f)^{\boldsymbol{B}}=f^{\boldsymbol{B}}$. The uniqueness of $f^{\boldsymbol{B}}$ is obvious from the properties of $(I_C f)^{\boldsymbol{B}}$.

$$\int ||f||d\mu \geqq \int ||I_C f||d\mu \geqq \int ||(I_C f)^B d\mu|| = \int ||f^B||d\mu ,$$

and hence $(\quad)^B$ is contractive. Since $s(f) \subset C$, $(\quad)^B$ is a projection. Next we are going to prove that $(\quad)^B$ is semi-constant-preserving. Suppose that there exist $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ and $A \in \boldsymbol{A}(\mu)$ such that $A \subset s((f)^B)$. Let $a \in E$. Write

$$B_n = \{\omega;\ ||f^B(\omega)|| > 1/n\} ,$$

then

$$s(f^B) = \cup \{B_n;\ n \in \boldsymbol{N}\} .$$

For any positive number ε there exists $n \in \boldsymbol{N}$ such that

$$||a||\mu(A - B_n) < \varepsilon .$$

Then

$$||I_A(I_{B_n}a)^B - I_A a||_L = I||_{B_n \cap A}a - I_A a||_L = ||a||\mu(A - B_n) < \varepsilon .$$

We have proved that $(\quad)^B$ is semi-constant-preserving. Q.E.D.

Lemma 1.2. *Suppose that P is a contractive projection of $L_1(\Omega, \boldsymbol{A}, \mu, R)$ into istelf and $0 \leqq P(I_A)(\omega) \leqq 1$ (a.e.ω) for any $A \in \boldsymbol{A}(\mu)$. Then there exists a σ-subring $\boldsymbol{B}$ of $\boldsymbol{A}$ such that $P = (\quad)^B$.*

For the proof see Wulbert [6].

Lemma 1.3. *Suppose that P is a contractive projection of $L_1(\Omega, \boldsymbol{A}, \mu, E)$ into itself. Then P is semi-constant-preserving and $\Omega \in s(P)$ iff P is constant-preserving in the sense used in* [2], [3] *and* [4], *i.e., $P(I_\Omega a) = I_\Omega a$ for any $a \in E$.*

Proof. First we suppose that $P(I_\Omega a) = I_\Omega a$ for any $a \in E$. It is clear that $\Omega \in s(P)$. For any $A \in s(P)$

$$||I_A P(I_\Omega a) - I_A a||_L = ||I_A a - I_A a||_L = 0 .$$

Therefore P is semi-constant-preservig.

Conversely we suppose that P is semi-constant-preserving and $\Omega \in s(P)$. For any $n \in \boldsymbol{N}$ there exists $f_n \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that

$$(2) \qquad ||P(f_n) - I_\Omega a||_L < 1/n .$$

Since P is contractive,

$$||P(f_n) - P(I_\Omega a)||_L < 1/n ,$$

and hence by (2) and arbitrariness of n

$$P(I_\Omega a) = I_\Omega a . \qquad \text{Q.E.D.}$$

In the remainder of this section we assume that Q satisfies Assumption 1.

Lemma 1.4. *Let K, $A\in \boldsymbol{A}(\mu)$, $K\cup A\in s(Q)$ and $a\in E$. Then*

$$||a-Q(I_A a)(\omega)||=||a||-||Q(I_A a)(\omega)|| \qquad (a.e.\omega) \text{ on } K.$$

Proof. Since $K\cup A\in s(Q)$ and Q is semi-constant-preserving, for any $\varepsilon>0$ there exists $f\in L_1(\Omega,\ \boldsymbol{A},\ \mu,\ E)$ such that

$$(4) \qquad ||I_{A\cup K}Q(f)-I_{A\cup K}a||_L<\varepsilon\ .$$

Since Q is a contractive projection, by using (4) twice we have

$$\begin{aligned}
&||Q(f)-Q(I_A a)||_L\leqq ||Q(f)-I_A a||_L\\
&\leqq ||I_A Q(f)-I_A a||_L+||I_{\Omega-A}Q(f)||_L\\
&\leqq \varepsilon+||I_{\Omega-A}Q(f)||_L\\
&\leqq \varepsilon+||I_A Q(f)-I_A a||_L+||I_A Q(f)||_L-||I_A a||_L+||I_{\Omega-A}Q(f)||_L\\
&\leqq 2\varepsilon+||I_A Q(f)||_L-||I_A a||_L+||I_{\Omega-A}Q(f)||_L\\
&=2\varepsilon+||Q(f)||_L-||I_A a||_L\\
&\leqq 2\varepsilon+||Q(f)||_L-||Q(I_A a)||_L\ .
\end{aligned}$$

Therefore

$$(5) \qquad ||Q(f)-Q(I_A a)||_L\leqq 2\varepsilon+||Q(f)||_L-||Q(I_A a)||_L\ .$$

Since

$$||I_{\Omega-K}Q(f)-I_{\Omega-K}Q(I_A a)||_L\geqq ||I_{\Omega-K}Q(f)||_L-||I_{\Omega-K}Q(I_A a)||_L\ ,$$

by (5) we get

$$(6) \qquad ||I_K Q(f)-I_K Q(I_A a)||_L\leqq 2\varepsilon+||I_K Q(f)||_L-||I_K Q(I_A a)||_L\ .$$

From (4) and (6) we get

$$||I_K a-I_K Q(I_A a)||_L\leqq 4\varepsilon+||I_K a||_L-||I_K Q(I_A a)||_L\ .$$

Since ε is an arbitrary positive number,

$$||I_K a-I_K Q(I_A a)||_L = ||I_K a||_L-||I_K Q(I_A a)||_L\ .$$

Therefore

$$||a-Q(I_A a)(\omega)|| = ||a||-||Q(I_A a)(\omega)|| \qquad (a.e.\omega) \text{ on } K.$$

Q.E.D.

Lemma 1.5. *Let $A\in s(Q)$ and $a\in E$. Then for any positive number ε there exist $f\in L_1(\Omega,\ \boldsymbol{A},\ \mu,\ E)$ and $B\in s(Q)$ such that*

$$B \subset s(Q(f)),$$
$$\|I_A a - I_B a\|_L < \varepsilon,$$
$$\|I_{s(Q(f))} Q(I_B a) - Q(I_A a)\|_L < \varepsilon,$$
$$\|I_{\Omega - s(Q(f))} Q(I_B a)\|_L < \varepsilon,$$

and

$$\|a - Q(I_B a)(\omega)\| = \|a\| - \|Q(I_B a)(\omega)\| \qquad (a.e.\omega) \text{ on } s(Q(f)).$$

Proof. For any $\varepsilon > 0$ we can choose a positive number δ such that $4\delta < \varepsilon$. Since Q is semi-constant-preserving, there exists $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that

$$(7) \qquad \|I_A Q(f) - I_A a\|_L < \delta.$$

Write $B = A \cap s(Q(f))$. Therefore

$$(8) \qquad \|I_A a - I_B a\|_L = \|I_A a - I_{A \cap s(Q(f))} a\|_L$$
$$= \|I_{A - s(Q(g))} a\|_L = \|I_{\Omega - s(Q(f))}(I_A Q(f) - I_A a)\|_L < \delta < \varepsilon.$$

Since Q is contractive, by (8) and the triangle inequality

$$\|I_{s(Q(f))} Q(I_B a) - Q(I_A a)\|_L$$
$$\leqq \|I_{s(Q(f))} Q(I_B a) - I_{s(Q(f))} Q(I_A a)\|_L + \|I_{\Omega - s(Q(f))} Q(I_A a)\|_L$$
$$\leqq \|I_B a - I_A a\|_L + \|I_{\Omega - s(Q(f))} Q(I_A a)\|_L$$
$$< \delta + \|I_{\Omega - s(Q(f))} Q(I_A a)\|_L$$
$$= \delta + \|I_{\Omega - s(Q(f))} Q(I_A a) - Q(f)\|_L - \|Q(f)\|_L,$$

where the last equality comes from the fact that

$$\|I_{\Omega - s(Q(f))} Q(I_A a) - Q(f)\|_L = \|I_{\Omega - s(Q(f))} Q(I_A a)\|_L + \|Q(f)\|_L.$$

By the triangle inequality and the fact that Q is contractive,

$$\delta + \|I_{\Omega - s(Q(f))} Q(I_A a) - Q(f)\|_L - \|Q(f)\|_L$$
$$\leqq \delta + \|I_{\Omega - s(Q(f))} Q(I_A a) - Q(f) + I_{s(Q(f))} Q(I_A a)\|_L + \|I_{s(Q(f))} Q(I_A a)\|_L - \|Q(f)\|_L$$
$$\leqq \delta + \|Q(I_A a) - Q(f)\|_L + \|I_{s(Q(f))} Q(I_A a)\|_L - \|Q(f)\|_L$$
$$\leqq \delta + \|I_A a - Q(f)\|_L + \|I_A a\|_L - \|Q(f)\|_L.$$

By (7)

$$\delta + \|I_A a - Q(f)\|_L + \|I_A a\|_L - \|Q(f)\|_L$$
$$\leqq 3\delta + \|I_A Q(f) - Q(f)\|_L + \|I_A Q(f)\|_L - \|Q(f)\|_L = 3\delta < \varepsilon.$$

We have proved that

$$\|I_{s(Q(f)} Q(I_B a) - Q(I_A a)\|_L < 3\delta < \varepsilon,$$

and hence by (8)

$$\begin{aligned}||I_{\Omega-s(Q(f))}Q(I_B a)||_L &= ||Q(I_B a)-I_{s(Q(f))}Q(I_B a)||_L \\ &\leqq ||Q(I_B a)-Q(I_A a)||_L+||Q(I_A a)-I_{s(Q(f))}Q(I_B a)||_L \\ &\leqq ||I_B a-I_A a||_L+3\delta<\delta+3\delta<\varepsilon .\end{aligned}$$

There exists a sequence $\{K_n;\, n\in \boldsymbol{N}\}$ such that $K_n\in \boldsymbol{A}(\mu)$ and $s(Q(f))=\cup\{K_n;\, n\in \boldsymbol{N}\}$. Since $B\cup K_n\in s(Q)$ for any $n\in \boldsymbol{N}$, by Lemma 1.4

$$||a-Q(I_B a)(\omega) = ||a||-||Q(I_B a)(\omega)|| \qquad (a.e.\omega) \text{ on } K_n .$$

Therefore

$$||a-Q(I_B a)(\omega)|| = ||a||-||Q(I_B a)(\omega)|| \qquad (a.e.\omega) \text{ on } s(Q(f)) .$$

Q.E.D.

For any $A\in \boldsymbol{A}(\mu)$ let

$$k = \sup\{\mu(C);\, C\in \boldsymbol{A},\, C\subset A \text{ and } \mu(C\cap D)=0 \qquad \text{for any} \quad D\in s(Q)\} .$$

Then there exists $E\in \boldsymbol{A}$ such that $E\subset A$, $\mu(E\cap D)=0$ for any $D\in s(Q)$ and $\mu(E)=k$. We write $N_Q(A)=E$. Clearly for any $A\in \boldsymbol{A}$ $N_Q(A)$ is unique up to sets of measure zero. When just one operator Q is under discussion, we omit the letter Q from symbols and write N instead of N_Q.

Lemma 1.6. *Let $A_n, B_m\in A(\mu)$ for any $n, m\in \boldsymbol{N}$ and $\cup\{A_n;\, n\in \boldsymbol{N}\}\subset \cup\{B_m;\, m\in \boldsymbol{N}\}$. Then $\cup\{N(A_n);\, n\in \boldsymbol{N}\}\subset\cup\{N(B_m);\, m\in \boldsymbol{N}\}$.*

Proof. For any $n, m\in \boldsymbol{N}$ $N(A_n)\cap B_m\in \boldsymbol{A}(\mu)$, $N(A_n)\cap B_m\subset B_m$ and $(N(A_n)\cap B_m)\cap D=\emptyset$ for any $D\in s(Q)$, and hence $N(A_n)\cap B_m\subset N(B_m)$. Therefore

$$\cup\{N(A_n);\, n\in \boldsymbol{N}\} = \cup\{N(A_n)\cap B_m;\, n, m\in \boldsymbol{N}\}\subset\cup\{N(B_m);\, m\in \boldsymbol{N}\} .$$

Q.E.D.

We can define $N(A)$ for any $A\in \boldsymbol{A}$, even if $\mu(A)=\infty$. Let $A_n\in \boldsymbol{A}(\mu)$ such that $A=\cup\{A_n;\, n\in \boldsymbol{N}\}$ and let $N(A)=\cup\{N(A_n);\, n\in \boldsymbol{N}\}$. By Lemma 1.6 $N(A)$ is independent of the choice of the sequence $\{A_n;\, n\in \boldsymbol{N}\}$. For any $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ let $N(f)=I_{N(s(f))}f$, then N is a mapping of $L_1(\Omega, \boldsymbol{A}, \mu, E)$ into itself.

Lemma 1.7. *Let $A, B\in \boldsymbol{A}$ with $A\subset B$ and $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$. Then $N(A)=N(B)\cap A$, $N(A)\subset N(B)$, $N(N(A))=N(A)$ and $N(s(f))=s(N(f))$.*

Proof. We can choose sequences $\{A_n;\, n\in \boldsymbol{N}\}$ and $\{C_m;\, m\in \boldsymbol{N}\}$ such that $A_n, C_m\in \boldsymbol{A}(\mu)$ for any $n, m\in \boldsymbol{N}$ and $A=\cup\{A_n;\, n\in \boldsymbol{N}\}$ and $B-A=\cup\{C_m;\, m\in \boldsymbol{N}\}$. By the definition of N we have $N(B)\cap A=(\cup\{N(A_n)\cup N(C_m);\, n, m\in \boldsymbol{N}\})\cap A=\cup\{N(A_n);\, n\in \boldsymbol{N}\}=N(A)$, and hence $N(A)\subset N(B)$. Since $N(A)\subset A$, $N(N(A))=N(A)\cap N(A)=N(A)$. $N(f)=I_{N(s(f))}f$, and hence $s(N(f))=N(s(f))$. Q.E.D.

Lemma 1.8. *The family $\{N(A);\ A\in \boldsymbol{A}\}$ is a σ-subring of A.*

Proof. Let $A, B, A_n \in \boldsymbol{A}$ for any $n\in \boldsymbol{N}$ and let $C=\cup\{A_n;\ n\in\boldsymbol{N}\}\cup A\cup B$. Since $A, B, A-B\subset C$, by Lemma 1.7 $N(A)-N(B)=(A\cap N(C))-(B\cap N(C))=(A-B)\cap N(C)=N(A-B)$. $\cup\{A_n;\ n\in\boldsymbol{N}\}\subset C$, and hence $N(\cup\{A_n;\ n\in\boldsymbol{N}\})=\cup\{A_n;\ n\in\boldsymbol{N}\}\cap N(C)=\cup\{A_n\cap N(C);\ n\in\boldsymbol{N}\}=\cup\{N(A_n);\ n\in\boldsymbol{N}\}$. Q.E.D.

Lemma 1.9. *The operator N of $L_1(\Omega, \boldsymbol{A}, \mu, E)$ into itself is a contractive projection and $\|f-N(f)\|_L\leqq\|f\|_L$ for any $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$.*

Proof. First we will show that N is a linear operator. Since $s(af)=s(f)$ for any $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ and $a\in R$ with $a\neq 0$,

$$N(af)=I_{N(s(af))}af=aI_{N(s(f))}f=aN(f)\,.$$

For any $f, g\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ let $C=s(f)\cup s(g)$. Since $s(f), s(g), s(f+g)\subset C$, by Lemma 1.7 and the definition of N

$$\begin{aligned}N(f+g)&=I_{N(s(f+g))}(f+g)=I_{N(C)\cap s(f+g)}(f+g)=I_{N(C)}(f+g)\\&=I_{N(C)}f+I_{N(C)}g=I_{N(C)\cap s(f)}f+I_{N(C)\cap s(g)}g=N(f)+N(g)\,.\end{aligned}$$

Next we are going to show that N is a contractive projection. By Lemma 1.7

$$(9)\qquad s(N(f))=N(s(f))\,.$$

By (9) and Lemma 1.7

$$\begin{aligned}N\circ N(f)&=I_{N(s(N(f)))}N(f)=I_{N(N(s(f)))}N(f)\\&=I_{N(s(f))}N(f)=I_{s(N(f))}N(f)=N(f)\,,\end{aligned}$$

and hence N is a projection.

$$\|N(f)\|_L=\|I_{N(s(f))}f\|_L\leqq\|f\|_L\,,$$

and hence N is contractive.

$$\|f-N(f)\|_L=\|f-I_{N(s(f))}f\|_L\leqq\|f\|_L\,. \qquad \text{Q.E.D.}$$

We define an operator Q^* of $L_1(\Omega, \boldsymbol{A}, \mu, E)$ into itself by $Q^*(f)=(Q-Q\circ N)(f)=Q(f-N(f))$ for any $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$. Since N is linear, Q^* is a linear operator.

Let $\boldsymbol{C}$ be a σ-subring of $\boldsymbol{A}$ and $\boldsymbol{P}$ the conditional expectation operator given $\boldsymbol{C}$. For any $A\in\boldsymbol{A}$ and $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ we denote $s(P)$, $N_P(A)$ and $N_P(f)$ by $s((\ )^{\boldsymbol{C}})$, $N_{\boldsymbol{C}}(A)$ and $N_{\boldsymbol{C}}(f)$ respectively. Let $\boldsymbol{A}_{\boldsymbol{C}}=\{N_{\boldsymbol{C}}(A);\ A\in\boldsymbol{A}\}$, then by Lemma 1.8 $\boldsymbol{A}_{\boldsymbol{C}}$ is σ-subring of $\boldsymbol{A}$. Note that for any $D\in\boldsymbol{A}$ we have $D\in s(\boldsymbol{P})$ iff there exists $C\in\boldsymbol{C}$ such that $D\subset C$.

Lemma 1.10. *Let $\boldsymbol{C}$ be a σ-subring of $\boldsymbol{A}$. Then*

$$(\ )^{\boldsymbol{C}}\circ N_{\boldsymbol{C}} = N_{\boldsymbol{C}}\circ(\ )^{\boldsymbol{C}}$$

Proof. Let $P=(\ )^{\boldsymbol{C}}$ and $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$. By the definition of $N_{\boldsymbol{C}}$ for any $A\in\boldsymbol{A}$ and $D\in s((\ ))^{\boldsymbol{C}}=s(P)$ we have $N_{\boldsymbol{C}}(A)\cap D=\emptyset$. $D\in s(P)$ iff there exists $C\in\boldsymbol{C}$ such that $D\subset C$, and hence for any $A\in\boldsymbol{A}$ and $C\in\boldsymbol{C}$

$$N_{\boldsymbol{C}}(A)\cap C=\emptyset\,. \tag{10}$$

$(N_{\boldsymbol{C}}(f))^{\boldsymbol{C}}=(I_{N_{\boldsymbol{C}}(s(f))}f)^{\boldsymbol{C}}=0$, since by (10) $N_{\boldsymbol{C}}(s(f))\cap C=\emptyset$ for any $C\in\boldsymbol{C}$. $s(f^{\boldsymbol{C}})\in\boldsymbol{C}$, and hence by (10) we have

$$N_{\boldsymbol{C}}(s(f^{\boldsymbol{C}}))=N_{\boldsymbol{C}}(s(f^{\boldsymbol{C}}))\cap s(f^{\boldsymbol{C}})=\emptyset\,.$$

Therefore

$$N_{\boldsymbol{C}}(f^{\boldsymbol{C}})=I_{N_{\boldsymbol{C}}(s(f^{\boldsymbol{C}}))}f^{\boldsymbol{C}}=0\,.$$ Q.E.D.

Lemma 1.11. *Operators Q, Q^* and N satisfy the conditions $N\circ Q=Q^*\circ N=0$, $Q^*\circ Q=Q$, $Q^*\circ Q^*=Q^*$ and $s(Q)=s(Q^*)$.*

Proof. By the definition of N we have $\mu(N(s(Q(f)))=0$, and hence

$$N\circ Q(f)=I_{N(s(Q(f)))}Q(f)=0\,. \tag{11}$$

By Lemma 1.9 N is a projection, i.e., $N\circ N=N$, and hence by the definition of Q^*

$$Q^*\circ N=(Q-Q\circ N)\circ N=Q\circ N-Q\circ N\circ N=0\,.$$

By (11)

$$Q^*\circ Q=(Q-Q\circ N)\circ Q=Q\circ Q-Q\circ(N\circ Q)=Q\circ Q=Q\,,$$

and hence

$$Q^*\circ Q^*=Q^*\circ(Q-Q\circ N)=(Q^*\circ Q^*)-(Q^*\circ Q)\circ N=Q-Q\circ N=Q^*\,.$$

By the definition of Q^* for any $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$

$$Q^*(f)=Q(f-N(f))\,, \tag{12}$$

and by the preceding part of this lemma $Q=Q^*\circ Q$, and hence

$$Q(f)=Q^*\circ Q(f)\,. \tag{13}$$

By (12) and (13) we have $s(Q)=s(Q^*)$. Q.E.D.

Lemma 1.12. *Q^* is semi-constant-preserving contractive projection and $Q(I_A a)=Q^*(I_A a)$ for any $A\in s(Q^*)$ and $a\in E$.*

Proof. Let $a\in E$, $\varepsilon>0$ and $A\in s(Q^*)$. By Lemma 1.11 $A\in s(Q)$, and hence by the fact that Q is semi-constant-preserving we can choose $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that

$$||I_A Q(f)-I_A a||_L<\varepsilon .$$

By Lemma 1.11

$$Q(f) = Q^*\circ Q(f) ,$$

and hence

$$||I_A Q^*\circ Q(f)-I_A a||_L<\varepsilon .$$

Therefore Q^* is semi-constant-preserving. Since $A\in s(Q)$, $N(A)=\emptyset$. Therefore by Lemma 1.9

$$Q^*(I_A a) = Q(I_A a-N(I_A a)) = Q(I_A a) .$$

$||Q^*(f)||_L=||Q(f-N(f))||_L\leqq||f-N(f)||_L\leqq||f||_L$, and hence Q^* is contractive. By Lemma 1.11 $Q^*\circ Q^*=Q^*$, and hence Q^* is a projection. Q.E.D.

Lemma 1.13. *For any $A\in\boldsymbol{A}(\mu)$ there exists a pairwise disjoint sequence $\{A_n\in s(Q);\ n\in\boldsymbol{N}\}$ such that*

$$A-N(A) = \cup\{A_n;\ n\in\boldsymbol{N}\} .$$

Proof. Let $k=\sup\{\mu(C);\ C\in\boldsymbol{A},\ C\subset A$ and there exists $C_n\in s(Q)$ for each $n\in\boldsymbol{N}$ such that $C\subset\cup\{C_n;\ n\in\boldsymbol{N}\}\}$. Then there exist $D\in\boldsymbol{A}$ and $D_n\in s(Q)$ for any $n\in\boldsymbol{N}$ such that $D\subset A$, $D\subset\cup\{D_n;\ n\in\boldsymbol{N}\}$ and $\mu(D)=k$. By the definition of k we have $\mu((A-D)\cap E)=0$ for any $E\in s(Q)$, and hence by Lemma 1.6 we have $A-D\subset N(A)$. Therefore

$$A-N(A)\subset D\subset\cup\{D_n;\ n\in\boldsymbol{N}\} .$$

Write $A_n=A\cap(D_n-\cup\{D_i;\ i\leqq n-1\})$. Since $A_n\in s(Q)$, $\mu(A_n\cap N(A))=0$. Hence the sequence $\{A_n;\ n\in\boldsymbol{N}\}$ consists of pairwise disjoint elements of $s(Q)$ and

$$A-N(A) = \cup\{A_n;\ n\in\boldsymbol{N}\} . \qquad \text{Q.E.D.}$$

In the remainder of this paper we assume that (S, X, λ) is a measure space, where S is a σ-ring and λ is a measure on S, and for any $K\in S$ we denote by J_K the indicator function of K. For any $K, H\in S$ we write $K\subset H$ if $\lambda(K-H)=0$, $K=\emptyset$ if $\lambda(K)=0$. K and H are said to be disjoint if $K\cap H=\emptyset$. For any real-valued measurable function $a(x)$, $b(x)$ on X we write $a\leqq b$ if $a(x)\leqq b(x)$ (a.e.x), i.e., $\lambda(\{x;\ a(x)>b(x)\})=0$ and $a=b$ if $a(x)=b(x)$ (a.e.x).

2. Lemmas for L_p-valued functions, where $1<p<\infty$. Let λ be a σ-finite measure on S. Throughout this section we assume that $E=L_p(X, S, \lambda, R)$ with $1<p<\infty$,

$$||a|| = (\int |a(x)|^p d\lambda)^{1/p} \qquad \text{for any} \quad a \in E$$

and that Q satisfies Assumption 1. (See (1).)

Lemma 2.1. *If $a, b \in E$ and $||a+b|| = ||a|| + ||b||$, then there exists a real number k such that $a = kb$ or $b = ka$.*

For the proof see Yosida [7] pp. 33 and 34.

Lemma 2.2. *Let $A \in s(Q)$, then there exists $\psi \in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that $Q(I_A a) = \psi a$ for any $a \in E$ and $0 \leqq \psi(\omega) \leqq 1$ (a.e.ω).*

Proof. By Lemma 1.5 for any $n \in \boldsymbol{N}$ there exist $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ and $B \in s(Q)$ such that

$$||I_{s(Q(f))} Q(I_B a) - Q(I_A a)||_L < 1/n\,, \tag{14}$$

and

$$||a - Q(I_B a)(\omega)|| = ||a|| - ||Q(I_B a)(\omega)|| \qquad (a.e.\omega) \text{ on } s(Q(f))\,.$$

Therefore by Lemma 2.1 there exists $\psi_n \in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that

$$I_{s(Q(f))} Q(I_B a) = \psi_n a$$

and

$$0 \leqq \psi_n(\omega) \leqq 1 \qquad (\text{a.e.}\omega)\,, \tag{15}$$

and hence by (14) we have

$$||Q(I_A a) - \psi_n a||_L < 1/n\,. \tag{16}$$

Since by (16) ψ_n is a Cauchy sequence, there exists $\psi \in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that

$$||\psi - \psi_n||_L \to 0 \qquad \text{as} \quad n \to \infty\,. \tag{17}$$

By (16) and (17) we have

$$Q(I_A a) = \psi a\,.$$

By (15) $0 \leqq \psi(\omega) \leqq 1$ (a.e.ω). Cleary ψ is independent of the choice of $a \in E$, since Q is a linear operator. Q.E.D.

3. Lemmas for L_1-valued functions. Let S be a σ-algebra and $S(\lambda) = \{K;\ K \in S \text{ and } \lambda(K) < \infty\}$.

DEFINITION 3. A measure space (X, S, λ) is said to be licalizable if any nonempty collection $\mathcal{V} \subset S(\lambda)$ has $\sup \mathcal{V} \in S$, in the sense that for any $K \in \mathcal{V}$, $\lambda(K - \sup \mathcal{V}) = 0$ and that if $H_1 \in S$ and $\lambda(K - H_1) = 0$ for any $K \in \mathcal{V}$, then

$\lambda(\sup \mathcal{V}-H_1)=0$.

DEFINITION 4. We say that a measure space (X, S, λ) has the finite subset property if for any $K\in S$ with $\lambda(K)>0$, there is $H\in S$ such that $H\subset K$ and $0<\lambda(H)<\infty$.

DEFINITION 5. A class $\{f(x, K); K\in S(\lambda)\}$ of real-valued S-measureable functions on (X, S, λ) is called a cross-section if $f(x, K)=0$ on K^c and for any $K, H\in S(\lambda)$ $J_{K\cap H}(x)f(x, K)=J_{K\cap H}(x)f(x, H)$ (a.e.x).

Lemma 3.1. *Suppose that a measure space (X, S, λ) is localizable. Then for any corss-section $\{f(x, K); K\in S(\lambda)\}$ there exists a real-valued S-measurable function f such that $J_K(x)f(x)=f(x, K)$ (a.e.x) for any $K\in S(\lambda)$.*

For the proof see Zaanen [8].

DEFINITION 6. Let T be a one-to-one transformation of (X, S, λ) into itself. Then T is called a bounded measurable transformation if T is a measurable transformation and there exists a positive number k such that $\lambda(T^{-1}(A))\leqq k\lambda(A)$ for any $A\in S$.

DEFINITION 7. Let $\mathfrak{T}$ be a class of bounded measurable transformations T of X onto X such that $T^{-1}(S(\lambda)=S(\lambda)$ for any $T\in\mathfrak{T}$. Then $(X, S, \lambda, \mathfrak{T})$ is said to be ergodic if $A\in S$ and $\lambda(A\Delta T^{-1}(A))=0$ for any $T\in\mathfrak{T}$ imply $\lambda(A)=0$ or $\lambda(A^c)=0$.

Lemma 3.2. *If $(X, S, \lambda, \mathfrak{T})$ is an ergodic space, then for any bounded measurable function f on X, $f(x)=f(T(x))$ for any $T\in\mathfrak{T}$ imply that $f(x)=const.$*

For the proof see Miyadera [3].

Throughout this section we assume that $(X, S, \lambda, \mathfrak{T})$ is an ergodic localizable measure space with the finite subset property, $E=L_1(X, S, \lambda, R)$ with the norm

$$||a|| = \int |a(x)|\, d\lambda \qquad \text{for any} \quad a\in E$$

and Q satisfies Assumption 1. Let

$$E^+ = \{a;\ a\in E \text{ and } a(x)\geqq 0 \text{ (a.e.x.)}\}\ .$$

For any $a\in E$ we write $0\leqq a$ if $a\in E^+$. For a real-valued measurable function $a(x)$, it is clear that $a(T(x))$ is also measurable, because of the measurability of T. If, in addition, $a\in E$, then $a(T(x))\in E$. We shall write $T(a)(x)=a(T(x))$, and remark that T can be regarded as a bounded operator of E into istelf in the sense that there exists a real number k such that $||T(a)||\leqq k||a||$ for any $a\in E$.

DEFINITION 8. Let Q be a transformation of $L_1(\Omega, A, \mu, E)$ into itself. Then Q is said to be covariant under $\mathfrak{T}$ if $Q(\psi T(a))(\omega)=T(Q(\psi(a)(\omega))$ (a.e.ω) for any $\psi\in L_1(\Omega, \boldsymbol{A}, \mu, R)$, $a\in E$ and $T\in\mathfrak{T}$.

Lenma 3.3. *Let $A\in s(Q)$ and $K\in S(\lambda)$. Then*

$$0\leqq Q(I_A J_K)(\omega)\leqq J_K \qquad (\text{a.e.}\omega)\,.$$

Proof. By Lemma 1.5 for an arbitrary positive real number ε there exist $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ and $B\in s(Q)$ such that

$$(18) \qquad ||I_{s(Q(f))}Q(I_B J_K)-Q(I_A J_K)||_L<\varepsilon$$

and

$$||J_K-Q(I_B J_K)(\omega)|| = ||J_K||-||Q(I_B J_K)(\omega)|| \qquad (\text{a.e.w}) \text{ on } s(Q(f))\,.$$

By the definition of the norm $||\quad||$

$$(19) \qquad \int|J_K-Q(I_B J_K)(\omega)|\,d\lambda = \int|J_K|\,d\lambda-\int|Q(I_B J_K)(\omega)|\,d\lambda \qquad (a.e.\omega) \text{ on } a\ s(Q(f))\,,$$

which shows that

$$(20) \qquad 0\leqq I_{s(Q(f))}Q(I_B J_K)(\omega)\leqq J_K \qquad (a.e.\omega)\,.$$

Since ε is an arbitrary number, by (18) and (20) we have

$$0\leqq Q(I_A J_K)(\omega)\leqq J_K \qquad (a.e.\omega)\,. \qquad \text{Q.E.D.}$$

Lemma 3.4. *Let $A\in s(Q)$. Suppose that Q is covariant under $\mathfrak{T}$. Then there exists $\psi\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that $Q(I_A a)=\psi a$ for $a\in E$ and $0\leqq\psi(\omega)\leqq 1$ $(a.e.\omega)$.*

Proof. Let $C\in\boldsymbol{A}(\mu)$. For any $K\in S(\lambda)$ write

$$e(K)=\int_C Q(I_A J_K)d\mu\in E\,.$$

By Lemma 3.3 for any $K\in S(\lambda)$

$$(21) \qquad 0\leqq e(K)\leqq J_K\,\mu(C)\,.$$

By (21) for any $K, H\in S(\lambda)$

$$\begin{aligned} J_{K\cap H}e(K) &= J_{K\cap H}(e(K\cap H)+e(K-H)) = J_{K\cap H}e(K\cap H) \\ &= J_{K\cap H}(e(K\cap H)+e(H-K)) = J_{K\cap H}e(H)\,, \end{aligned}$$

and hence $\{e(K);\, K\in S(\lambda)\}$ is a cross section. By Lemma 3.1 there exists a real-valued S-measurable function b on X such that

$$(22) \qquad J_K b = e(K) \qquad \text{for any} \quad K \in S(\lambda).$$

Since Q is covariant under $\mathfrak{T}$, for any $T \in \mathfrak{T}$

$$(23) \qquad J_{T^{-1}(K)} T(b) = T(J_K b) = T\left(\int_C Q(I_A J_K) d\mu\right)$$
$$= \int_C T(Q(I_A J_K)) d\mu = \int_C Q(I_A T(J_K)) d\mu = \int_C Q(I_A J_{T^{-1}(K)}) d\mu$$
$$= J_{T^{-1}(K)} b.$$

Since $(X, S, \lambda, \mathfrak{T})$ is ergodic, by the definition 7 $S(\lambda) = T^{-1}(S(\lambda))$. K is an arbitrary element of $S(\lambda)$, and hence (23) implies that $J_K T(b) = J_K b$ for any $K \in S(\lambda)$. By the finite subset property of (X, S, λ)

$$(24) \qquad T(b) = b.$$

By (21) and (22) b is a positive bounded function on X, and hence by Lemma 3.2 and (24) there exists a positive number $k(C)$ depending on C and A but not depending on K such that

$$b = J_X k(C).$$

Therefore for any $C \in \boldsymbol{A}(\mu)$

$$\int_C Q(I_A J_K) d\mu = J_K k(C).$$

Since μ is σ-finite, we can define a real-valued measure k on $\boldsymbol{A}$ by

$$J_K k(C) = \int_C Q(I_A J_K) d\mu \qquad \text{for any} \quad C \in \boldsymbol{A}.$$

Note that this integral is the Bochner integral, and hence $J_K k(C) \in E$. Therefore $0 \leqq k(C) < \infty$. Since k is absolutely continuous in the usual sense with respect to μ, there exists $\psi \in L_1(\Omega, \boldsymbol{A}, \mu, R)$, which may vary with A, such that

$$k(C) = \int_C \psi d\mu \qquad \text{for any} \quad C \in \boldsymbol{A}.$$

Thereofre for any $C \in \boldsymbol{A}$

$$\int_C Q(I_A J_K) d\mu = \int_C \psi J_K d\mu,$$

and hence

$$Q(I_A J_K) = \psi J_K.$$

By Lemma 3.3 $0 \leqq \psi(\omega) \leqq 1$ $(a.e.\omega)$. Since $k(\ )$ is independent of the choice of

K, so is ψ. Any $a\in E$ can be approximated by a sequence of simple functions, and hence we have for any $a\in E$

$$Q(I_A a) = \psi a . \qquad t.\text{E.D.}$$

4. Lemmas for L_∞-valued functions. Throughout this section we assume that $E=L_\infty(X, S, \lambda, R)$, for $a\in E$

$$||a|| = \text{ess. sup}\{|a(X)|\,;\, x\in X\}$$

and Q satisfies Assujption 1. Let

$$E^+ = \{a;\, a\in E \quad \text{and} \quad a(x)\geqq 0 \qquad (\text{a.e.x})\} .$$

Lemma 4.1. *For any $A\in s(Q)$ and $K\in S$,*

$$||Q(I_A J_K)(\omega)||\leqq 1 \qquad (a.e.\omega)$$

and

$$J_K Q(I_A J_K)(\omega)\in E^+ \qquad (a.e.\omega) .$$

Proof. For any arbitrary positive number ε by Lemma 1.5 there exist $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ and $B\in s(Q)$ such that

$$(25) \qquad ||I_{s(Q(f))}Q(I_B J_K)-(I_A J_K)||_L<\varepsilon$$

and

$$||J_K-Q(I_B J_K)(\omega)|| = ||J_K||-||Q(I_B J_K)(\omega)|| \qquad (\text{a.e.}\omega) \text{ on } s(Q(f)) .$$

Therefore

$$(26) \qquad ||I_{s(Q(f))}Q(I_B J_K)(\omega)||\leqq 1 \qquad (\text{a.e.}\omega)$$

and

$$(27) \qquad I_{s(Q(f))}J_K Q(I_B J_K)(\omega)\in E^+ \qquad (\text{a.e.}\omega) .$$

By (25), (26) and (27) we have

$$||Q(I_A J_K)(\omega)||\leqq 1 \qquad (\text{a.e.}\omega)$$

and

$$J_K Q(I_A J_K)(\omega)\in E^+ \qquad (\text{a.e.}\omega).$$

Lemma 4.2. *Let A, $B\in s(Q)$ and $A\subset B$. Suppose that there exists a pairwise disjoint class $\{K, L, M\}$ such that $\lambda(K)>0$ and $\lambda(L\cup M)>0$, where L can be a set of measure zero. Then for any natural number k*

$$(28) \qquad \mu(B)\geqq\int_B ||Q(I_A J_K)+J_L+(-1)^k J_M||d\mu-\int_{\Omega-B}||Q(I_A J_K)||d\mu .$$

Proof. Since Q is semi-constant-preserving, for an arbitrary positive number δ there exist $f, g \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that

$$\|I_B Q(f) - I_B J_M\|_L < \delta \tag{29}$$

and

$$\|I_B Q(g) - I_B J_L\| < \delta . \tag{30}$$

Write

$$\varepsilon = \int_{\Omega - B} \|Q(I_A J_K)\| d\mu . \tag{31}$$

Therefore by (29), (30), (31) and the relation $A \subset B$

$$\begin{aligned}
\mu(B) &= \int_B \|I_A J_K + J_L + (-1)^k J_M\| d\mu \\
&\geqq \int_B \|I_A J_K + Q(g) + (-1)^k Q(f)\| d\mu - 2\delta \\
&= \int \|I_A J_K + Q(g) + (-1)^k Q(f)\| d\mu \\
&\quad - \int_{\Omega - B} \|I_A J_K + Q(g) + (-1)^k Q(f)\| d\mu - 2\delta \\
&\geqq \int_B \|Q(I_A J_K) + Q(g) + (-1)^k Q(f)\| d\mu \\
&\quad + \int_{\Omega - B} \|Q(I_A J_K) + Q(g) + (-1)^k Q(f)\| d\mu \\
&\quad - \int_{\Omega - B} \|I_A J_K + Q(g) + (-1)^k Q(f)\| d\mu - 2\delta \\
&\geqq \int_B \|Q(I_A J_K) + Q(g) + (-1)^k Q(f)\| d\mu \\
&\quad + \int_{\Omega - B} \|Q(g) + (-1)^k Q(f)\| d\mu - \int_{\Omega - B} \|Q(g) + (-1)^k Q(f)\| d\mu - 2\delta - \varepsilon \\
&= \int_B \|Q(I_A J_K) + Q(g) + (-1)^k Q(f)\| d\mu - 2\delta - \varepsilon \\
&\geqq \int_B \|Q(I_A J_K) + J_L + (-1)^k J_M\| d\mu - 4\delta - \varepsilon .
\end{aligned}$$

We have proved (28), since δ is an arbitrary number. Q.E.D.

Lemma 4.3 *Let K and L be disjoint elements of S which are of positive meaaure. Then for any $A \in s(Q)$*

$$\int J_L Q(I_A J_K) d\mu = 0 .$$

Proof. Suppose that there exists a positive real number ε such that

(32) $$\|\int J_L Q(I_A J_K)d\mu\|>7\varepsilon\,.$$

By Lemma 1.5 there exist $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ and $B\in A(\mu)$ such that $B\subset s(Q(f))$,

(33) $$\|I_{s(Q(f))}Q(I_B J_K)-Q(I_A J_K)\|_L<\varepsilon$$

and

(34) $$\|I_{\Omega-s(Q(f))}Q(I_B J_K)\|_L<\varepsilon\,.$$

By (32) and (33)

(35) $$\|\int I_{s(Q(f))}J_L Q(I_B J_K)d\mu\|>6\varepsilon\,.$$

By (34) and (35) we can choose $C\in \boldsymbol{A}(\mu)$ such that $C\subset s(Q(f))$,

(36) $$\|I_{\Omega-C}Q(I_B J_K)\|_L<2\varepsilon$$

and

(37) $$\|\int I_C J_L Q(I_B J_K)d\mu\|>5\varepsilon\,.$$

By (37) and the definition of the norm $\|\ \|$ there exist $M\in S$ and a natural number k such that $M\subset L$,

(38) $$(-1)^k\int I_C J_M Q(I_B J_K)d\mu\in E^+$$

and

(39) $$\|\int I_C J_M Q(I_B J_K)d\mu\|>5\varepsilon\,.$$

$B\cup C\subset s(Q(f))$, and hence $B\cup C\in s(Q)$. By (36) we have

(40) $$\int_{\Omega-(B\cup C)}\|Q(I_B J_K)\|d\mu<2\varepsilon$$

and

(41) $$\int_{B-C}\|Q(I_B J_K)\|d\mu<2\varepsilon\,.$$

K and M are disjoint, and hence by Lemma 4.2, (38), (39), (40) and (41)

$$\begin{aligned}\mu(B\cup C)&=\int_{B\cup C}\|I_B J_K+(1-{}^{k})J_M\|d\mu\\ &\geqq\int_{B\cup C}\|Q(I_B J_K)+(-1)^k J_M\|d\mu-2\varepsilon\end{aligned}$$

$$\geqq \int_{B\cup C} \|J_M Q(I_B J_K) + (-1)^k J_M\| d\mu - 2\varepsilon$$
$$\geqq \int_{B\cup C} \|I_C J_M Q(I_B J_K) + (-1)^k J_M\| d\mu - 4\varepsilon$$
$$\geqq \left\| \int_C J_M Q(I_B J_K) d\mu + (-1)^k \mu(B\cup C) J_M \right\| - 4\varepsilon$$
$$= \left\|(-1)^k \int_C J_M Q(I_B J_K) d\mu\right\| + \mu(B\cup C) - 4\varepsilon$$
$$> 5\varepsilon + \mu(B\cup C) - 4\varepsilon = \mu(B\cup C) + \varepsilon\,,$$

which is a contradiction. Therefore

$$\int J_L Q(I_A J_K) d\mu = 0\,. \qquad \text{Q.E.D.}$$

Lemma 4.4. *Suppose that* $f, g, h \in L_1(\Omega, \boldsymbol{A}, \mu, R)$, $f(\omega) \geqq 0$, $g(\omega) \geqq 0$ *and* $h(\omega) \geqq 0$ *(a.e.ω). Then we have*

$$\int (g \vee h) d\mu \leqq \int ((f \vee h) + (f \vee g - g) + (f \vee g - f)) d\mu\,.$$

Proof.

$$\int (g \vee h) d\mu \leqq \int (f + |f-g|) \vee h d\mu \leqq \int ((f \vee h) + |f-g|) d\mu$$
$$= \int ((f \vee h) + (f \vee g - g) + (f \vee g - f)) d\mu\,. \qquad \text{Q.E.D.}$$

DEFINITION 9. A class of subsets $\{K, L, M\}$ is said to be a *partition* of X if K, L and M are pairwise disjoint and $\lambda(K) > 0$, $\lambda(L) > 0$, $\lambda(M) > 0$ and $K \cup L \cup M = X$ (a.e.x).

Lemma 4.5. *Suppose that $A \in s(Q)$ and $K \in S$. If we can choose $L, M \in S$ such that $X = K \cup L \cup M$ (a.e.x), $\lambda(L) > 0$, $\lambda(M) > 0$ and $\lambda(L \cap M) = 0$, then $J_{L\cup M} Q(I_A J_K) = 0$. (Note that K may be a set of measure zero.)*

Proof. Suppose that

$$\mu(\{\omega;\ \|J_L Q(I_A J_K)\| > 0\}) > 0\,.$$

Then there exist positive real numbers δ and ε such that

$$\mu(\{\omega;\ \|J_L Q(I_A J_K)\| > 4\delta\}) > 3\varepsilon\,.$$

Let

$$F = \{\omega;\ \|J_L Q(I_A J_K)\| > 4\delta\}\,,$$

then $\mu(F) > 3\varepsilon$. By Lemma 1.5 there exist $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ and $B \in s(Q)$

such that $B\subset s(Q(f))$,

$$\|I_{\Omega-s(Q(f))}Q(I_BJ_K)\|_L<\varepsilon\delta \tag{42}$$

and

$$\|Q(I_BJ_K)-Q(I_AJ_K)\|_L<\varepsilon\delta\ . \tag{43}$$

By (42) we can choose $C\in \boldsymbol{A}(\mu)$ such that $C\subset s(Q(f))$ and

$$\|I_{\Omega-C}Q(I_BJ_K)\|_L<\varepsilon\delta\ .$$

Let

$$D=\{\omega;\ \|J_LQ(I_BJ_K)\|>3\delta\}\ .$$

Then by (43)

$$\delta\mu(F-D)\leqq\int_{F-D}\|Q(I_BJ_K)-Q(I_AJ_K)\|d\mu<\varepsilon\delta\ ,$$

and hence $\mu(F-D)<\varepsilon$. Since $\mu(F)>3\varepsilon$, $\mu(D)>2\varepsilon$. Therefore

$$\int_D\|J_LQ(I_BJ_K)\|d\mu>6\varepsilon\delta\ . \tag{44}$$

Then by (42) and (44)

$$\int_{D\cap s(Q(f))}\|J_LQ(I_BJ_K)\|d\mu>6\varepsilon\delta-\varepsilon\delta=5\varepsilon\delta\ .$$

Let $E=(D\cap s(Q(f)))\cup C\cup B$, then $E\subset s(Q(f))$,

$$\|I_EJ_LQ(I_BJ_K)\|_L>5\varepsilon\delta\ . \tag{45}$$

and

$$\|I_{\Omega-E}Q(I_BJ_K)\|_L<\varepsilon\delta\ . \tag{46}$$

By Lemma 4.2, Lemma 4.3 and (46) for any $k\in \boldsymbol{N}$

$$\begin{aligned}
(47)\quad \mu(E)&=\int_E\|I_BJ_K+J_M+(-1)^kJ_L\|d\mu\\
&\geqq\int_E\|Q(I_BJ_K)+J_M+(-1)^kJ_L\|d\mu-\varepsilon\delta\\
&\geqq\int_E\|J_MQ(I_BJ_K)+J_M\|\vee\|J_LQ(I_BJ_K)+(-1)^kJ_L\|d\mu-\varepsilon\delta\\
&\geqq\int\|J_MQ(I_BJ_K)+I_EJ_M\|\vee\|J_LQ(I_BJ_K)+(-1)^kI_EJ_L\|d\mu-2\varepsilon\delta\\
&\geqq\int\|J_MQ(I_BJ_K)+I_EJ_M\|d\mu\wedge\int\|J_LQ(I_BJ_K)+(-1)^kI_EJ_L\|d\mu-2\varepsilon\delta\\
&\geqq\left\|\int J_MQ(I_BJ_K)d\mu+\mu(E)J_M\right\|\vee\left\|\int J_LQ(I_BJ_K)d\mu+(-1)^k\mu(E)J_L\right\|-2\varepsilon\delta
\end{aligned}$$

$$=||\mu(E)J_M||\wedge||(-1)^k\mu(E)J_L||-2\varepsilon\delta = \mu(E)-2\varepsilon\delta\,,$$

where the last equation comes from the fact that $M\neq\emptyset$ and $L\neq\emptyset$. Therefore by Lemma 4.4, (47) and (45)

$$\mu(E)+4\varepsilon\delta\geqq\int ||J_L Q(I_B J_K)+I_E J_L||\vee||J_L Q(I_B J_K)-I_E J_L||d\mu$$
$$=\int(||J_L Q(I_B J_K)||+I_E)d\mu\geqq\mu(E)+5\varepsilon\delta\,,$$

which is a contradiction. Therefore

(48) $$J_L Q(I_A J_K)=0\,.$$

Similarly we can prove

(49) $$J_M Q(I_A J_K)=0\,.$$

By (48) and (49) we have

$$J_{L\cup M}Q(I_A J_K)=0\,.$$ Q.E.D.

Lemma 4.6. *Suppose that $A\in s(Q)$ and there exists a partition $\{K, L, M\}$ of X. Then there exists $\psi\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that $0\leqq\psi(\omega)\leqq1$ (a.e.ω) and $Q(I_A a)=\psi a$ for any $a\in E$.*

Proof. By Lemma 1.5 for any arbitrary number $\varepsilon>0$ there exist $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ and $B\in s(Q)$ such that

(50) $$||I_{s(Q(f))}Q(I_B J_X)-Q(I_A J_X)||_K<\varepsilon$$

and

(51) $$||J_X-Q(I_B J_X)(\omega)||=||J_X||-||Q(I_B J_X)(\omega)|| \quad \text{(a.e.}\omega\text{) on } s(Q(f))\,,$$

and hence

$$Q(I_B J_X)(\omega)=||Q(I_B J_X)(\omega)||J_X \text{ (a.e.x) on } s(Q(f))\,,$$

which implies

(52) $$I_{s(Q(f))}Q(I_B J_X)=||Q(I_B J_X)||I_{s(Q(f))}J_X\,.$$

$||Q(I_B J_X)||I_{s(Q(f))}\in L_1(\Omega, \boldsymbol{A}, \mu, R)$, and hence by (50) and (52) there exists $\psi\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that

(53) $$Q(I_A J_X)=\psi J_X\,.$$

By (51) $0\leqq\psi(\omega)\leqq1$ (a.e.ω). Let $N\in S$ and $\lambda(N)>0$. If $\lambda(K\cap N)>0$, then by the assumption that $\{K, L, M\}$ is a partition of X and Lemma 4.5 we have

$$J_{N\cap K}Q(I_AJ_L)=0\,,\quad J_{N\cap K}Q(I_AJ_M)=0\,,\quad J_{N\cap K}Q(I_AJ_{K-N})=0$$

and

$$J_{X-(N\cap K)}Q(I_AJ_{N\cap K})=0\,.$$

Therefore by (53)

$$Q(I_AJ_{N\cap K})=J_{N\cap K}Q(I_AJ_{N\cap K})=J_{N\cap K}Q(I_AJ_X)=\psi J_{N\cap K}\,. \tag{54}$$

If $\lambda(K\cap N)=0$, then (54) is trivial. Similarly we can prove that

$$Q(I_AJ_{N\cap L})=\psi J_{N\cap L} \tag{55}$$

and

$$Q(I_AJ_{N\cap M})=\psi J_{N\cap M}\,. \tag{56}$$

Therefore by (54), (55) and (56) we have $Q(I_AJ_N)=\psi J_N$ and ψ is independent of the choice of N. Since N is an arbitrary element of S and any $a\in E$ can be approximated by a sequence of simple functions, we have for any $a\in E$

$$Q(I_Aa)=\psi a\,. \qquad \text{Q.E.D.}$$

5. Semi-constant-preserving contracttve projections and conditional expectations. In this section an operator Q is said to satisfy Assumption 2 if

(57) for any $A\in s(Q)$ there exists $\psi\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that $0\leqq\psi(\omega)\leqq 1$ (a.e.ω) and $Q(I_Aa)=\psi a$ for any a$\in E$, where ψ is independent of the choice of a.

In Section 2, Section 3 and Section 4 we used the following conditions (58), (59) and (60) respectively.

(58) $E=L_p(X, S, \lambda, R)$, where $1<p<\infty$.

(59) $E=L_1(X, S, \lambda, R)$, where $(X, S, \lambda, \mathcal{G})$ is an ergodic licalizable measure space and Q is covariant under $\mathcal{G}$.

(60) $E=L_\infty(X, S, \lambda, R)$ and there exists a partition $\{K, L, M\}$ of X.

If Q satisfies Assumption 1 (See (1).) and one of the conditions (58), (59) and (60) is satisfied, then by Lemma 2.2, Lemma 3.4 and Lemma 4.6 Q satisfies Assumption 2.

Lemma 5.1. *Suppose that Q satisfies Assumption* 1 *and Assumption* 2, *then for any $\psi\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ there exists $\phi\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that for any $a\in E$*

$$Q^*(\psi a)=\phi a$$

and

$$\phi(\omega)\geqq 0 \ \ (a.e.\omega) \ \ if \ \ \phi(\omega)\geqq 0 \ \ (a.e.\omega)\,.$$

Proof. It is sufficient to prove this Lemma for $\psi=I_A$ with $A\in\boldsymbol{A}(\mu)$. By Lemma 1.13 there exists a sequence $\{A_n;\, n\in\boldsymbol{N}\}$ of pairwise disjoint elements of $s(Q)$ such that

$$A-N(A)=\cup\{A_n;\, n\in\boldsymbol{N}\}\ .$$

By (57) for any n there exists $\phi_n\in L_1(\Omega,\ \boldsymbol{A},\ \mu,\ R)$ such that for any $a\in E$

$$Q(I_{A_n}a)=\phi a_n\ .$$

Since Q is contractive,

$$||\phi_n||_L||a||=||\phi_n a||_L\leqq||I_{A_n}a||_L=\mu(A_n)||a||\ ,$$

and hence

$$\sum\{||\phi_n||_L;\, n\in\boldsymbol{N}\}\leqq\mu(A)\ .$$

Therefore by writting $\phi=\sum\{\phi_n;\, n\in\boldsymbol{N}\}$ we have $\phi\in L_1(\Omega, \boldsymbol{A}, \mu, R)$. $Q^*(I_Aa)=\sum\{Q(I_{A_n}a);\, n\in\boldsymbol{N}\}=\phi a$ for any $a\in E$. Q.E.D.

Lemma 5.2. *If Q satisfies Assumption 1 and Assumption 2, then for any $f\in L_1(\Omega,\ \boldsymbol{A},\ \mu,\ E)$ there exists $\psi\in L_1(\Omega,\ \boldsymbol{A},\ \mu,\ R)$ such that $\psi(\omega)\geqq 0$ (a.e.ω) and $s(Q^*(\psi a))\supset s(Q^*(f))$ (a.e.ω) for any non-zero element a of E.*

Proof. First we suppose that f is a simple function and $f=I_{A_1}a_1+\cdots+I_{A_n}a_n$, where $A_i\in\boldsymbol{A}(\mu)$, $A_i\cap A_j=\emptyset$ $(i\neq j)$ and $a_i\in E$ for $i=1, 2, \cdots, n$. By Lemma 5.1 there exists $\phi_i\in L_1(\Omega,\ \boldsymbol{A},\ \mu,\ R)$ for any i such that $\phi_i(\omega)\geqq 0$ (a.e.ω) and $Q^*(I_{A_i}a_i)=\phi_i a_i$. Let $\psi=I_{A_1\cup\cdots\cup A_n}$ and a an arbitrary non-zero element of E, then

$$s(Q^*(f))=s(\phi_1a_1+\cdots+\phi_n a_n)\subset s(\phi_1 a+\cdots+\phi_n a)=s(Q^*(\psi a))\ .$$

For an arbitrary $f\in L_1(\Omega,\ \boldsymbol{A},\ \mu,\ E)$ and $n\in\boldsymbol{N}$ there exists a simple function $f_n\in L_1(\Omega,\ \boldsymbol{A},\ \mu,\ E)$ such that

$$||f-f_n||_L<1/n\ . \tag{58}$$

In the preceding part of this proof we have proved that for any f_n there exists $\psi_n\in L_1(\Omega,\ \boldsymbol{A},\ \mu,\ R)$ such that

$$s(Q^*(f_n))\subset s(Q^*(\psi_n a)) \tag{59}$$

and

$$\psi_n(\omega)\geqq 0 \qquad \text{(a.e.}\omega)\ .$$

Let

$$\psi=\sum\{(\psi_n/(2^n||\psi_n||_L));\, n\in\boldsymbol{N}\}\ .$$

Then

$$s(Q^*(\psi a))=\cup\{s(Q^*(\psi_n a));\, n\in\boldsymbol{N}\}\ . \tag{60}$$

By (58), (59) and (60) and the fact that Q^* is contractive

$$(61) \qquad \int_{s(Q^*(f))-s(Q^*(\psi a))} \|Q^*(f)\| d\mu \leqq \int_{s(Q^*(f))-\cup\{s(Q^*(f_n));\, n\in N\}} \|Q^*(f)\| d\mu$$

$$= \int_{s(Q^*(f))-\cup\{s(Q^*(f_n));\, n\in N\}} \|Q^*(f)-Q^*(f_n)\| d\mu \leqq \|f-f_n\|_L < 1/n .$$

Since $\|Q^*(f)(\omega)\|>0$ for any $\omega\in s(Q^*(f))-s(Q^*(\psi a))$ and n is an arbitrary number, (61) implies that

$$\mu(s(Q^*(f))-s(Q^*(\psi a))) = 0 . \qquad \text{Q.E.D.}$$

Lemma 5.3. *Suppose that Q satisfiw satisfies Assumption 1 and Assumption 2 and $A_n \in s(Q)=s(Q^*)$ for any $n\in \boldsymbol{N}$. If $\cup\{A_n;\, n\in \boldsymbol{N}\}\in \boldsymbol{A}(\mu)$, then $\cup\{A_n;\, n\in \boldsymbol{N}\}\in s(Q)=s(Q^*)$.*

Proof. Since $A_n\in s(Q^*)$, by the definition of $s(Q^*)$ there exists $f_n\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that $A_n\subset s(Q^*(f_n))$. Therefore by Lemma 5.1 and 5.2 there exist ψ_n, $\phi_n\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ and $a\in E$ such that $\psi_n(\omega)\geqq 0$ (a.e.ω), $\phi_n(\omega)\geqq 0$ (a.e.ω), $Q^*(\psi_n a)=\phi_n a$ and

$$s(Q^*(f_n))\subset s(Q^*(\psi_n a)) = s(\phi_n) ,$$

where we can assume that $\|\psi_n\|_L=1/2^n$. Q^* is contractive, and hence $\|\phi_n\|_L\leqq 1/2^n$.

Write $\psi=\sum\{\psi_n;\, n\in \boldsymbol{N}\}$ and $\phi=\sum\{\phi_n;\, n\in \boldsymbol{N}\}$. Then ψ, $\phi\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ and

$$s(Q^*(\psi a)) = s(\phi) = \cup\{s(\phi_n);\, n\in \boldsymbol{N}\} .$$

Therefore $\cup\{A_n;\, n\in \boldsymbol{N}\}\subset s(Q^*(\psi a))$. Since $\cup\{A_n;\, n\in \boldsymbol{N}\}\in \boldsymbol{A}(\mu)$, by the definition of $s(Q^*)$ $\cup\{A_n;\, n\in \boldsymbol{N}\}\in s(Q^*)$. Q.E.D.

The following lemma is more delicate than Lemma 5.1.

Lemma 5.4. *Suppose that Q satisfies Assumption 1 and Assumption 2. Then for any $A\in \boldsymbol{A}(\mu)$ there exists $\psi\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that $0\leqq\psi(\omega)\leqq 1$ (a.e.ω) and $Q^*(I_A a)=\psi a$ for any $a\in E$.*

Proof. Let $A\in \boldsymbol{A}(\mu)$. Then by Lemma 1.13 there exists a sequence $\{A_n;\, n\in \boldsymbol{N}\}$ such that $A_n\in s(Q)$ and

$$A-N(A) = \cup\{A_n;\, n\in \boldsymbol{N}\} .$$

By Lemma 5.3 $\cup\{A_n;\, n\in \boldsymbol{N}\}\in s(Q)$, and hence

$$A-N(A)\in s(Q) .$$

By Assumption 2 there exists $\psi\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that $0\leqq\psi(\omega)\leqq 1$ (a.e.ω)

and

$$Q(I_{A-N(A)}a) = \psi a .$$

Therefore

$$Q^*(I_A a) = Q(I_{A-N(A)}a) = \psi a . \qquad \text{Q.E.D.}$$

Lemma 5.5. *If Q satisfies Assumption* 1 *and Assumption* 2, *then there exists a σ-subring $\boldsymbol{B}$ of $\boldsymbol{A}$ such that*

(i) $$Q^*(f) = f^{\boldsymbol{B}} ,$$

(ii) $$N_Q(f) = N_{\boldsymbol{B}}(f)$$

and

(iii) $$Q(f) \in L_1(\Omega, \boldsymbol{B}, \mu, E) \qquad \textit{for any} \quad f \in L_1(\Omega, \boldsymbol{A}, \mu, E) .$$

Proof. (i) By Lemma 5.4 for any $\psi \in L_1(\Omega, \boldsymbol{A}, \mu, R)$ there exists $\phi \in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that

$$Q^*(\psi a) = \phi a \qquad \text{for any} \quad a \in E ,$$

and that $0 \leqq \phi(\omega) \leqq 1$ (a.e.ω) if $\psi = I_A$ for some $A \in \boldsymbol{A}(\mu)$. If we fix a, Q^* can be regarded as an operator of $L_1(\Omega, \boldsymbol{A}, \mu, R)$ into itself, which satisfies the assumption of Lemma 1.2. Therefore there exists a σ-subring $\boldsymbol{B}$ of $\boldsymbol{A}$ such that $Q^*(\psi a) = \psi^{\boldsymbol{B}} a$ for any $\psi \in L_1(\Omega, \boldsymbol{A}, \mu, R)$ and any $a \in E$. Since any $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ can be approximated by simple functions, $Q^*(f) = f^{\boldsymbol{B}}$ for any $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$.

(ii) It is sufficient to show that $s(Q) = s((\ )^{\boldsymbol{B}})$. If $A \in s(Q)$ then there exists $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that

$$A \subset s(Q(f)) . \tag{62}$$

By Lemma 1.11 and the preceding part of this proof

$$Q(f) = Q^*(Q(f)) = Q(f)^{\boldsymbol{B}} . \tag{63}$$

By (62) and (63) we have $A \in s((\ )^{\boldsymbol{B}})$. On the other hand if $A \in s((\ )^{\boldsymbol{B}})$, then there exists $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that

$$A \subset s(f^{\boldsymbol{B}}) . \tag{64}$$

By the definition of $Q^*$and the preceding part of this Lemma

$$f^{\boldsymbol{B}} = Q^*(f) = Q(f - N_Q(f)) . \tag{65}$$

By (64) and (65) we have $A \in s(Q)$.

(iii) Since $Q(f) = Q^*(Q(f)) = Q(f^{\boldsymbol{B}})$, $Q(f) \in L_1(\Omega, \boldsymbol{B}, \mu, E)$ Q.E.D.

Theorem 1. (i) *If Q satisfies Assumption* 1 *and Assumption* 2, *then there* *exists a σ-subring $\boldsymbol{B}$ of $\boldsymbol{A}$ such that $Q(f)=f^{\boldsymbol{B}}+Q(N_Q(f))=f^{\boldsymbol{B}}+Q(N_{\boldsymbol{B}}(f))$ for any $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$.*

(ii) *If there exists a σ-subring $\boldsymbol{B}$ of $\boldsymbol{A}$ and a contractive linear operator $\boldsymbol{P}$ of $L_1(\Omega, \boldsymbol{A}_{\boldsymbol{B}}, \mu, E)$ into $L_1(\Omega, \boldsymbol{B}, \mu, E)$, then the operator defined by $Q(f)=f^{\boldsymbol{B}}+P(N_{\boldsymbol{B}}(f))$ for any $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ satisfies Assumption* 1 *and Assumption* 2.

Proof. (i) By Lemma 5.5 and the definitions of Q^*, N_Q and $N_{\boldsymbol{B}}$ there exists a σ-subring $\boldsymbol{B}$ of $\boldsymbol{A}$ such that

$$Q(f) = Q^*(f)+Q(N_Q(f)) = f^{\boldsymbol{B}}+Q(N_{\boldsymbol{B}}(f))\,.$$

(ii) By the fact that $P(f)\in L_1(\Omega, \boldsymbol{B}, \mu, E)$ for any $f\in L_1(\Omega, \boldsymbol{A}_{\boldsymbol{B}}, \mu, E)$ and properties of operators $(\ )^{\boldsymbol{B}}$ and $N_{\boldsymbol{B}}$ and Lemma 1.10 we have

$$(\ )^{\boldsymbol{B}}\circ \boldsymbol{P} = \boldsymbol{P}\,, \tag{66}$$

$$N_{\boldsymbol{B}}\circ \boldsymbol{P} = 0\,, \tag{67}$$

$$(\ )^{\boldsymbol{B}}\circ N_{\boldsymbol{B}} = 0\,, \tag{68}$$

and

$$N_{\boldsymbol{B}}\circ(\ )^{\boldsymbol{B}} = 0\,, \tag{69}$$

which imply that

$$Q\circ(\ )^{\boldsymbol{B}} = (\ )^{\boldsymbol{B}}\circ(\ )^{\boldsymbol{B}}+P\circ N_{\boldsymbol{B}}\circ(\ )^{\boldsymbol{B}} = (\ )^{\boldsymbol{B}}\,. \tag{70}$$

By (66), (67) and (69)

$$\begin{aligned} Q\circ Q(f) &= (f^{\boldsymbol{B}}+P(N_{\boldsymbol{B}}(f)))^{\boldsymbol{B}}+P(N_{\boldsymbol{B}}(f^{\boldsymbol{B}}+P(N_{\boldsymbol{B}}(f)))) \\ &= f^{\boldsymbol{B}}+P(N_{\boldsymbol{B}}(f)) = Q(f)\,. \end{aligned}$$

Therefore Q is a projection.
By (68) and the fact that $(\ )^{\boldsymbol{B}}$ and $\boldsymbol{P}$ are contractive

$$\begin{aligned} ||Q(f)||_L &\leqq ||f^{\boldsymbol{B}}||_L+||P(N_{\boldsymbol{B}}(f))||_L = ||f^{\boldsymbol{B}}-(N_{\boldsymbol{B}}(f))^{\boldsymbol{B}}||_L+||P(N_{\boldsymbol{B}}(f))||_L \\ &\leqq ||f-N_{\boldsymbol{B}}(f)||_L+||N_{\boldsymbol{B}}(f)||_L \\ &= ||I_{s(f)-N_{\boldsymbol{B}}(s(f))}f||_L+||I_{N_{\boldsymbol{B}}(s(f))}f||_L = ||f||_L\,, \end{aligned}$$

and hence Q is contractive.

Next we are going to show that Q is semi-constant-preserving and satisfies Assumption 2.
Let $A\in s(Q)$, $a\in E$ and $\varepsilon>0$. By the definition of $s(Q)$ there exists $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that $A\subset s(Q(f))$. By Lemma 5.5 $Q(f)\in L_1(\Omega, \boldsymbol{B}, \mu, E)$, and hence

$$A\subset s(Q(f)) = s((Q(f))^{\boldsymbol{B}})\,. \tag{71}$$

Conditional expectation operators are semi-constant-preserving, and hence by (71) there exists $g\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that

$$\|I_A g^B - I_A a\|_L < \varepsilon . \tag{72}$$

By (70) and (72)

$$\|I_A Q(g^B) - I_A a\|_L < \varepsilon ,$$

which implies that Q is semi-constant-preserving. Since by (71) and the definition of N_B $N_B(I_A a)=0$,

$$Q(I_A a) = (I_A a)^B + P(N_B(I_A a)) = (I_A a)^B = (I_A)^B a ,$$

and hence Q satisfies Assumption 2. Q.E.D.

6. $\boldsymbol{R}^2$-valued case. Let $E=L_\infty(X, S, \lambda, R)$. If we cannot choose K, L and M such that $\{K, L, M\}$ is a partition of X, then $E\cong R$ with the norm $\|x\|=|x|$ for $x\in R$ or $E\cong R^2$ with the norm $\|(x, y)\|=|x|\vee|y|$ for $(x, y)\in R^2$. If $E\cong R$, then we can use Lemma 2.2. Therefore our next aim is to consider the case when $E\cong R^2$. Throughout this section we assume that $E=R^2$ with the norm $\|(x, y)\|=|x|\vee|y|$ for $(x, y)\in R^2$. Note that for any $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ there exist $f_1, f_2\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that $f(\omega)=(f_1(\omega), f_2(\omega))$. Throughout this section we assume that Q is a linear operator of $L_1(\Omega, \boldsymbol{A}, \mu, E)$ into itself.

Lemma 6.1. *Let Q satisfy Assumption 1 and $A\in s(Q)$. If $Q((I_A, I_A))=(f_1, f_2)$ and $Q((I_A, -I_A))=(g_1, g_2)$, then $f_1=f_2$, $g_1=-g_2$, $0\leqq f_1(\omega)\leqq 1$ (a.e.ω) and $0\leqq g_1(\omega)\leqq 1$ (a.e.ω).*

Proof. By Lemma 1.5 for any $\varepsilon>0$ there exist $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ and $B\in \boldsymbol{A}(\mu)$ such that $B\subset s(Q(f))$,

$$\|I_{s(Q(f))} Q(I_B(1, 1)) - Q(I_A(1, 1))\|_L < \varepsilon \tag{73}$$

and

$$\begin{aligned} &\|(1, 1) - Q(I_B(1, 1))(\omega)\| \\ &= \|(1, 1)\| - \|Q(I_B(1, 1))(\omega)\| \qquad \text{(a.e.}\omega\text{) on } s(Q(f)) . \end{aligned} \tag{74}$$

Let $(h_1, h_2)=I_{s(Q(f))}Q(I_B(1, 1))$. Then by (74)

$$\|(1, 1)-(h_1, h_2)\| = \|(1, 1)\| - \|(h_1, h_2)\| ,$$

and hence we have

$$|1-h_1(\omega)| \vee |1-h_2(\omega)| = 1 - |h_1(\omega)| \vee |h_2(\omega)| ,$$

which shows that $h_1=h_2$, $0\leqq h_1(\omega)\leqq 1$ (a.e.ω). Therefore by (73)

$$||(f_1, f_2)-(h_1, h_1)||_L<\varepsilon ,$$

which shows that

$$f_1=f_2\,, \quad 0\leqq f_1(\omega)\leqq 1 \qquad (\text{a.e.}\omega)\,,$$

since ε is an arbitrary number.
Similarly we can prove that $g_1=-g_2$ and $0\leqq g_1(\omega)\leqq 1$. Q.E.D.

If an operator Q satisfies Assumption 1, then by Lemma 6.1 we can define linear operator Q_1 and Q_2 of $L_1(\Omega, \boldsymbol{A}, \mu, R)$ into itself by

(75) $$Q^*(f, f)=(Q_1(f), Q_1(f))$$

and

(76) $$Q^*(f, -f)=(Q_2(f), -Q_2(f))\,.$$

Then by the definitions of Q_1 and Q_2

(77) $$\begin{aligned} Q^*(f, g) &= (1/2)Q^*(f+g+f-g, f+g-(f-g)) \\ &= (1/2)(Q_1(f+g)+Q_2(f-g), Q_1(f+g)-Q_2(f-g))\,. \end{aligned}$$

Lemma 6.2. *Let Q satisfy Assumption* 1. *Then Q_1 and Q_2 are contractive projections and for any $A\in s(Q)$ and $\varepsilon>0$ there exist $f, g\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that*

(78) $$||I_A Q_1(f)-I_A||_L<\varepsilon$$

and

(79) $$||I_A Q_2(g)-I_A||_L<\varepsilon\,.$$

In particular Q_1 and Q_2 are semi-constant-preserving.

Proof. Let $A\in s(Q)$ and $\varepsilon>0$. By Lemma 1.1 Q^* is a semi-constant-preserving contractive projection, and hence Q_1 and Q_2 are contractive projections and there exist $f', g'\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that

(80) $$||I_A Q^*(f', g')-(I_A, I_A)||_L<\varepsilon\,.$$

By (77)

$$\begin{aligned} \int_A |Q_1((f'+g')/2)+Q_2((f'-g')/2)-1| \vee |Q_1((f'+g')/2) \\ -Q_2((f'-g')/2)-1|\,d\mu<\varepsilon\,, \end{aligned}$$

which implies that

$$\int_A |Q_1((f'+g')/2)-1|\,d\mu<\varepsilon\,,$$

and by writing $f=(f'+g')/2$ we have

$$(78) \qquad ||I_A Q_1(f)-I_A||_L<\varepsilon\,.$$

Similarly we can prove that

$$(79) \qquad ||I_A Q_2(g)-I_A||_L<\varepsilon\,.$$

Clearly $s(Q)=s(Q^*)\supset s(Q_1)$, $s(Q_2)$, and hence by (78) and (79) Q_1 and Q_2 are semi-constant-preserving. Q.E.D.

Since Q_1 and Q_2 are operators of $L_1(\Omega, \boldsymbol{A}, \mu, R)$ into itself we can use the result of Section 1 and Section 2 for Q_1 and Q_2.

Lemma 6.3. *Let Q satisfy Assumption 1. Then there exist σ-subrings $\boldsymbol{B}$ and $\boldsymbol{C}$ of $\boldsymbol{A}$ such that for any $f\in L_1(\Omega, \boldsymbol{A}, \mu, R)$*

$$Q_1(f)=f^{\boldsymbol{B}}\,,$$
$$Q_2(f)=f^{\boldsymbol{C}}$$

and

$$N_{\boldsymbol{B}}(A)=N_{\boldsymbol{C}}(A)=N_Q(A) \qquad \text{for any} \quad A\in\boldsymbol{A}(\mu)\,.$$

Proof. By Lemma 6.2 Q_1 and Q_2 are semi-constant-preserving contractive projections of $L_1(\Omega, \boldsymbol{A}, \mu, R)$ into itself, and hence by Lemma 2.2 and Theorem 1 there exist σ-subrings $\boldsymbol{B}$ and $\boldsymbol{C}$ such that for any $f\in L_1(\Omega, \boldsymbol{A}, \mu, R)$

$$(81) \qquad Q_1(f)=f^{\boldsymbol{B}}+Q_1(N_{Q_1}(f))\,,$$

$$(82) \qquad Q_2(f)=f^{\boldsymbol{C}}+Q_2(N_{Q_2}(f))\,,$$

$$(83) \qquad N_{Q_1}(f)=N_{\boldsymbol{B}}(f)$$

and

$$(84) \qquad N_{Q_2}(f)=N_{\boldsymbol{C}}(f)\,.$$

Let $A\in s(Q)$. By (78) and (79) for any $n\in N$ there exist $f_n, g_n\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that

$$||I_A Q_1(f_n)-I_A||_L<1/n$$

and

$$||I_A Q_2(g_n)-I_A||_L<1/n\,.$$

Therefore

$$\mu(A-s(Q_1(f_n)))<1/n$$

and

$$\mu(A-s(Q_2(g_n)))<1/n\,.$$

Write $A_n=A\cap s(Q_1(f_n))$. Then $A_n\in s(Q_1)$ and

$$(85) \qquad A = \cup \{A_n;\ n \in \boldsymbol{N}\} \qquad (\text{a.e.}\omega)\,.$$

By Lemma 2.2 and Lemma 6.2 Q_1 satisfies Assumption 1 and Assumption 2, and hence by (85) and Lemma 5.3 $A \in s(Q_1)$. Since A is an arbitrary element of $s(Q)$, we have proved that $s(Q) \subset s(Q_1)$. By the definition of Q_1 and Lemma 1.11 $s(Q_1) \subset s(Q^*) = s(Q)$. Therefore we have

$$(86) \qquad s(Q) = s(Q_1)\,.$$

Similarly we can prove that

$$(87) \qquad s(Q) = s(Q_2)\,.$$

By (86) and (87) togehter with (83) and (84) we have

$$(88) \qquad N_Q(A) = N_{Q_1}(A) = N_{Q_3}(A) = N_{\boldsymbol{B}}(A) = N_{\boldsymbol{C}}(A)\,.$$

By Lemma 1.11 $Q^* \circ N_Q = 0$, and hence by (75) and (76)

$$(89) \qquad Q_1 \circ N_Q = 0$$

and

$$(90) \qquad Q_2 \circ N_Q = 0\,.$$

By (81), (82), (88), (89) and (90)

$$Q_1(f) = f^{\boldsymbol{B}}$$

and

$$Q_2(f) = f^{\boldsymbol{C}} \qquad \text{for any} \quad f \in L_1(\Omega, \boldsymbol{A}, \mu, R)\,. \qquad \text{Q.E.D.}$$

By (77) and Lemma 6.3 we have

$$(91) \qquad Q^*(f, g) = (1/2)(f^{\boldsymbol{B}} + g^{\boldsymbol{B}} + f^{\boldsymbol{C}} - g^{\boldsymbol{C}},\ f^{\boldsymbol{B}} + g^{\boldsymbol{B}} - f^{\boldsymbol{C}} + g^{\boldsymbol{C}})\,.$$

Let us denote the operator, expressed in the right hand side of the above formula, by $F(\boldsymbol{B}, \boldsymbol{C})$.

Lemma 6.4. *For any σ-subrings $\boldsymbol{B}$ and $\boldsymbol{C}$ with $N_{\boldsymbol{B}} = N_{\boldsymbol{C}}$ the operator $F(\boldsymbol{B}, \boldsymbol{C})$ satisfies Assumption* 1.

Proof. It is clear that $F(\boldsymbol{B}, \boldsymbol{C}) \circ F(\boldsymbol{B}, \boldsymbol{C}) = F(\boldsymbol{B}, \boldsymbol{C})$, and hence $F(\boldsymbol{B}, \boldsymbol{C})$ is a projection. Next we are going to show that $F(\boldsymbol{B}, \boldsymbol{C})$ is semi-constant-preserving. Let $A \subset s(F(\boldsymbol{B}, \boldsymbol{C})(f, g))$ for some $f, g \in L_1(\Omega, \boldsymbol{A}, \mu, R)$ and $a = (a_1, a_2) \in E$. Then by the definition of $F(\boldsymbol{B}, \boldsymbol{C})$ we can choose sequences $\{B_n \in \boldsymbol{B}(\mu);\ n \in \boldsymbol{N}\}$ and $\{C_n \in \boldsymbol{C}(\mu);\ n \in \boldsymbol{N}\}$ such that

$$s(F(\boldsymbol{B}, \boldsymbol{C})(f, g)) \subset \cup \{B_n;\ n \in \boldsymbol{N}\} \cup \{C_n;\ n \in \boldsymbol{N}\}\,.$$

Then $A \subset \cup \{B_n;\ n \in \boldsymbol{N}\} \cup \{C_n;\ n \in \boldsymbol{N}\}$. By the definition of $N_{\boldsymbol{C}}$ we have

$N_{\boldsymbol{C}}(A)\cap C_n=\emptyset$ for any $n\in\boldsymbol{N}$, and hence

$$N_{\boldsymbol{C}}(A)\subset\cup\{B_n;\ n\in\boldsymbol{N}\}\ .$$

Since $N_{\boldsymbol{B}}(A)=N_{\boldsymbol{C}}(A)$, $N_{\boldsymbol{B}}(A)=N_{\boldsymbol{C}}(A)\subset\cup\{B_n;\ n\in\boldsymbol{N}\}$. By the definition of $N_{\boldsymbol{B}}$ we have $N_{\boldsymbol{B}}(A)\cap B_n=\emptyset$ for any $n\in\boldsymbol{N}$, and hence

$$N_{\boldsymbol{C}}(A)=N_{\boldsymbol{B}}(A)=\emptyset \qquad (\text{a.e.}\omega)\ . \tag{92}$$

Therefore by (92) and the definitions of $N_{\boldsymbol{B}}(A)$ and $N_{\boldsymbol{C}}(A)$ for any $\varepsilon>0$ there exist $B\in\boldsymbol{B}(\mu)$ and $C\in\boldsymbol{C}(\mu)$ such that

$$\mu(A-B)<\varepsilon/||a|| \tag{93}$$

and

$$\mu(A-C)<\varepsilon/||a||\ . \tag{94}$$

By (93), (94) and the fact that $I_B(I_{B\cup C})^{\boldsymbol{B}}=I_B$ and $I_C(I_{B\cup C})^{\boldsymbol{C}}=I_C$ we have

$$\begin{aligned}
&||I_A F(\boldsymbol{B},\boldsymbol{C})(I_{B\cup C}a_1,\ I_{B\cup C}a_2)-I_A(a_1,\ a_2)||_L\\
&=||(1/2)(I_A(I_{B\cup C})^{\boldsymbol{B}}(a_1+a_2,\ a_1+a_2)+I_A(I_{B\cup C})^{\boldsymbol{C}}(a_1-a_2,\ -a_1+a_2))\\
&\quad -I_A(a_1,\ a_2)||_L\\
&\leqq||(1/2)(I_AI_B(I_{A\cup C})^{\boldsymbol{B}}(a_1+a_2,\ a_1+a_2)+I_AI_C(I_{B\cup C})^{\boldsymbol{C}}(a_1-a_2,\ -a_1+a_2))\\
&\quad -I_A(a_1,\ a)_2||_L+2\varepsilon\\
&=||(1/2)(I_AI_B(a_1+a_2,\ a_1+a_2)+I_AI_C(a_1-a_2,\ -a_1+a_2))-I_A(a_1,\ a_2)||_L+2\varepsilon\\
&\leqq||(1/2)(I_A(a_1+a_2,\ a_1+a_2)+I_A(a_1-a_2,\ -a_1+a_2))-I_A(a_1,\ a_2)||_L+4\varepsilon\\
&=4\varepsilon\ ,
\end{aligned}$$

and hence $F(\boldsymbol{B},\boldsymbol{C})$ is semi-constant-preserving, since ε is an arbitrary number.

Next we are going to show that $F(\boldsymbol{B},\boldsymbol{C})$ is contractive. Since

$$|x|\vee|y|=(1/2)(|x+y|+|x-y|) \qquad \text{for any} \quad x,y\in\boldsymbol{R}\ ,$$

$$\begin{aligned}
||F(\boldsymbol{B},\boldsymbol{C})(f,g)||_L&=(1/2)\int|f^{\boldsymbol{B}}+g^{\boldsymbol{B}}+f^{\boldsymbol{C}}-g^{\boldsymbol{C}}|\vee|f^{\boldsymbol{B}}+g^{\boldsymbol{B}}-f^{\boldsymbol{C}}+g^{\boldsymbol{C}}|d\mu\\
&=(1/2)\int(|f^{\boldsymbol{B}}+g^{\boldsymbol{B}}|+|f^{\boldsymbol{C}}-g^{\boldsymbol{C}}|)d\mu\\
&\leqq(1/2)\int(|f+g|+|f-g|)d\mu\\
&=\int|f|\vee|g|d\mu=||(f,g)||_L\ ,
\end{aligned}$$

which shows that $F(\boldsymbol{B},\boldsymbol{C})$ is contractive. Q.E.D.

Obviously $L(\boldsymbol{B},\boldsymbol{C})=\{F(\boldsymbol{B},\boldsymbol{C})(f,g);\ (f,g)\in L_1(\Omega,\ \boldsymbol{A},\ \mu,\ E)\}$ is a normed linear subspace of $L_1(\Omega,\ \boldsymbol{A},\ \mu,\ E)\}$.

Theorem 2. *Let Q be a linear operator of $L_1(\Omega, \boldsymbol{A}, \mu, E)$ into istelf. Then Q satisfies Assumption* 1 *if and only if there exist σ-subrings $\boldsymbol{B}$ and $\boldsymbol{C}$ of $\boldsymbol{A}$ with $N_{\boldsymbol{B}}=N_{\boldsymbol{C}}$ (As a consequence $\boldsymbol{A}_{\boldsymbol{B}}=\boldsymbol{A}_{\boldsymbol{C}}$.) and a contractive operator P of $L_1(\Omega, \boldsymbol{A}_{\boldsymbol{B}}, \mu, E)$ into $L(\boldsymbol{B}, \boldsymbol{C})$ such that for any $f, g \in L_1(\Omega, \boldsymbol{A}, \mu, R)$*

$$Q(f, g) = (1/2)(f^{\boldsymbol{B}}+g^{\boldsymbol{B}}+f^{\boldsymbol{C}}-g^{\boldsymbol{C}}, f^{\boldsymbol{B}}+g^{\boldsymbol{B}}-f^{\boldsymbol{C}}+g^{\boldsymbol{C}})+P(N_{\boldsymbol{B}}(f, g)) .$$

Proof. Suppose that Q satisfies Assumption 1. Then by Lemma 6.3 and the definitions of Q^* and N_Q we have

$$N_{\boldsymbol{B}} = N_{\boldsymbol{C}} = N_Q \tag{95}$$

and

$$\begin{aligned} Q(f, g) &= Q^*(f, g)+Q(N_Q(f, g)) \\ &= (1/2)(f^{\boldsymbol{B}}+g^{\boldsymbol{B}}+f^{\boldsymbol{C}}-g^{\boldsymbol{C}}, f^{\boldsymbol{B}}+g^{\boldsymbol{B}}-f^{\boldsymbol{C}}+g^{\boldsymbol{C}})+Q(N_{\boldsymbol{B}}(f, g)) . \end{aligned} \tag{96}$$

By (95) $\boldsymbol{A}_{\boldsymbol{B}}=\boldsymbol{A}_{\boldsymbol{C}}$, and hence

$$N_{\boldsymbol{B}}(f, g) \in L_1(\Omega, \boldsymbol{A}_{\boldsymbol{B}}, \mu, E) . \tag{97}$$

By Lemma 1.11 and Lemma 6.3 for any $f, g \in L_1(\Omega, \boldsymbol{A}, \mu, R)$

$$Q(f, g) = Q^* \circ Q(f, g) = F(\boldsymbol{B}, \boldsymbol{C}) \circ Q(f, g) \in L(\boldsymbol{B}, \boldsymbol{C}) . \tag{98}$$

Denote by P the restriction of Q to $L_1(\Omega, \boldsymbol{A}_{\boldsymbol{B}}, \mu, E)$, then by (96), (97) and (98) P is a contractive operator of $L_1(\Omega, \boldsymbol{A}, \mu, E)$ into $L(\boldsymbol{B}, \boldsymbol{C})$ and

$$Q(f, g) = (1/2)(f^{\boldsymbol{B}}+g^{\boldsymbol{B}}+f^{\boldsymbol{C}}-g^{\boldsymbol{C}}, f^{\boldsymbol{B}}+g^{\boldsymbol{B}}-f^{\boldsymbol{C}}+g^{\boldsymbol{C}})+P(N_{\boldsymbol{B}}(f, g)) .$$

Conversely suppose that there exist σ-subrings $\boldsymbol{B}$ and $\boldsymbol{C}$ of $\boldsymbol{A}$ with $N_{\boldsymbol{B}}=N_{\boldsymbol{C}}$ and a contractive operator P of $L_1(\Omega, \boldsymbol{A}_{\boldsymbol{B}}, \mu, E)$ into $L(\boldsymbol{B}, \boldsymbol{C})$ such that

$$Q(f, g) = F(\boldsymbol{B}, \boldsymbol{C})(f, g)+P(N_{\boldsymbol{B}}(f, g)) .$$

Let $A \in s(Q)$, $a \in E$ and $\varepsilon > 0$. Since $F(\boldsymbol{B}, \boldsymbol{C}) \circ F(\boldsymbol{B}, \boldsymbol{C})=F(\boldsymbol{B}, \boldsymbol{C})$,

$$F(\boldsymbol{B}, \boldsymbol{C})(f, g) = (f, g) \qquad \text{for any} \quad (f, g) \in L(\boldsymbol{B}, \boldsymbol{C}) . \tag{99}$$

Since $P(f, g) \in L(\boldsymbol{B}, \boldsymbol{C})$, by (99) we have

$$F(\boldsymbol{B}, \boldsymbol{C}) \circ P = P . \tag{100}$$

By the definition of $N_{\boldsymbol{B}}$ and $N_{\boldsymbol{C}}$ and the condition that $N_{\boldsymbol{B}}=N_{\boldsymbol{C}}$ we have

$$\begin{aligned} N_{\boldsymbol{B}} \circ (\ )^{\boldsymbol{C}} &= N_{\boldsymbol{C}} \circ (\ )^{\boldsymbol{C}} = 0 , \\ N_{\boldsymbol{C}} \circ (\ )^{\boldsymbol{B}} &= N_{\boldsymbol{B}} \circ (\ )^{\boldsymbol{B}} = 0 , \\ (\ )^{\boldsymbol{B}} \circ N_{\boldsymbol{C}} &= (\ )^{\boldsymbol{C}} \circ N_{\boldsymbol{C}} = 0 \end{aligned}$$

and

$$(\ )^{\boldsymbol{C}} \circ N_{\boldsymbol{B}} = (\ )^{\boldsymbol{C}} \circ N_{\boldsymbol{C}} = 0 ,$$

and hence by the definition and properties of $F(\boldsymbol{B}, \boldsymbol{C})$ and $\boldsymbol{P}$ we have

$$N_{\boldsymbol{B}}\circ F(\boldsymbol{B}, \boldsymbol{C}) = N_{\boldsymbol{C}}\circ F(\boldsymbol{B}, \boldsymbol{C}) = 0\,, \tag{101}$$

$$N_{\boldsymbol{B}}\circ P = N_{\boldsymbol{C}}\circ P = 0 \tag{102}$$

and

$$F(\boldsymbol{B}, \boldsymbol{C})\circ N_{\boldsymbol{B}} = F(\boldsymbol{B}, \boldsymbol{C})\circ N_{\boldsymbol{C}} = 0\,. \tag{103}$$

For convenience's sake we denote $F(\boldsymbol{B}, \boldsymbol{C})$ by F. By Lemma 6.4 and (100)

$$F\circ Q = F\circ(F+P\circ N_{\boldsymbol{B}}) = F\circ F+F\circ P\circ N_{\boldsymbol{B}} = F+\boldsymbol{P}\circ N_{\boldsymbol{B}} = Q\,. \tag{104}$$

By (101), (102) and (104)

$$Q\circ Q = F\circ Q+\boldsymbol{P}\circ N_{\boldsymbol{B}}\circ(F+\boldsymbol{P}\circ N_{\boldsymbol{B}}) = Q+P\circ N_{\boldsymbol{B}}\circ F+\boldsymbol{P}\circ N_{\boldsymbol{B}}\circ P\circ N_{\boldsymbol{B}} = Q\,,$$

which shows that Q is a projection. By (103) and the fact that F and P are contractive we have

$$\begin{aligned} \|Q(f, g)\|_L &= \|F(f, g)+P\circ N_{\boldsymbol{B}}(f, g)\|_L \\ &= \|F((f, g)-N_{\boldsymbol{B}}(f, g))+F\circ N_{\boldsymbol{B}}(f, g)+P\circ N_{\boldsymbol{B}}(f, g)\|_L \\ &\leqq \|F((f, g)-N_{\boldsymbol{B}}(f, g))\|_L+\|P\circ N_{\boldsymbol{B}}(f, g)\|_L \\ &\leqq \|(f, g)-N_{\boldsymbol{B}}(f, g)\|_L+\|N_{\boldsymbol{B}}(f, g)\|_L = \|(f, g)\|_L\,, \end{aligned}$$

which implies that Q is contractive. Next we are going to show that Q is semi-constant-preserving. Let $A\in a(Q)$, $a\in E$ and $\varepsilon>0$. Then there exist $f, g\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that $A\subset s(Q(f, g))$. By (104)

$$A\subset s(Q(f, g)) = s(F\circ Q(f, g))\,,$$

and hence $A\in s(F)$. By Lemma 6.4 there exist $f', g'\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that

$$\|I_A F(\boldsymbol{B}, \boldsymbol{C})(f', g')-I_A a\|_L<\varepsilon\,. \tag{105}$$

By Lemma 6.4 and (101)

$$Q\circ F = (F+\boldsymbol{P}\circ N_{\boldsymbol{B}})\circ F = F\circ F+\boldsymbol{P}\circ N_{\boldsymbol{B}}\circ F = F+0 = F\,,$$

and hence by (105)

$$\|I_A Q(F(\boldsymbol{B}, C)(f', g'))-I_A a\|_L<\varepsilon\,,$$

which shows that Q is semi-constant-preserving. Q.E.D.

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