# Complete classification of genus-1 simplified broken Lefschetz fibrations

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## Abstract

Broken Lefschetz fibrations are fibration structures analogous to Lefschetz fibrations in the context of near-symplectic 4-manifolds. In the first part of this thesis, we discuss the classification of smooth 4-manifolds which admit genus-1 simplified broken Lefschetz fibrations. We completely classify diffeomorphism types of total spaces of genus-1 simplified broken Lefschetz fibrations. This result is a generalization of Kas and Moishezon's classification of genus-1 Lefschetz fibrations over the sphere $S^2$.

In the second part of this thesis, we discuss hyperelliptic directed broken Lefschetz fibrations. We prove that the total space of a hyperelliptic directed broken Lefschetz fibration has an involution which preserves any fiber provided that the genus of any connected component of a regular fiber is greater than or equal to 2. We also generalize Matsumoto and Endo's local signature formulae for hyperelliptic Lefschetz fibrations to a signature formula for hyperelliptic directed broken Lefschetz fibrations.

# Contents

# Chapter 1

# Introduction

Broken Lefschetz fibrations are fibration structures on smooth 4-manifolds which we can regard as a generalization of Lefschetz fibrations. Donaldson [8] proved that symplectic 4-manifolds admit Lefschetz pencils. Conversely, Gompf [12] showed that the total space of every Lefschetz pencil admits a symplectic structure. In [1], Auroux, Donaldson and Katzarkov generalized these results to that on 4-manifolds with a near-symplectic structure (i.e. a closed 2-form which is symplectic outside a union of circles where it vanishes transversely).

Simplified broken Lefschetz fibrations are broken Lefschetz fibrations over the 2-sphere $S^2$ which satisfy several conditions on fibers and singularities. Baykur first introduced simplified broken Lefschetz fibrations in [2]. In spite of the strict conditions in the definition of simplified broken Lefschetz fibrations, it turns out that every closed oriented 4-manifold admits a simplified broken Lefschetz fibration. More strongly, every smooth map from a closed oriented 4-manifold to $S^2$ is homotopic to a simplified broken Lefschetz fibration (for details, see [31]). We can take vanishing cycles of simplified broken Lefschetz fibrations as we take those of Lefschetz fibrations. Vanishing cycles of a simplified broken Lefschetz fibration have rich information on the fibration structure. For example, Baykur [2] gave a way to obtain a Kirby diagram of the total space of a simplified broken Lefschetz fibration from vanishing cycles of the fibration. Furthermore, the author prove in this thesis that vanishing cycles of a simplified broken Lefschetz fibration determine an isomorphism class of the fibration (for details, see Theorem 2.5.2). The main purpose of this thesis is the classification of genus-1 simplified broken Lefschetz fibrations. We further generalize several results on Lefschetz fibrations to that on simplified broken Lefschetz fibrations.

We first study genus-1 simplified broken Lefschetz fibrations in Chapter 3. Kas [19] and Moishezon [26] classified diffeomorphism types of total spaces of non-trivial relatively minimal genus-1 Lefschetz fibrations over $S^2$. They proved that the total space of such a Lefschetz fibration is diffeomorphic to an elliptic surface $E(n)$ for some $n \geq 1$. Baykur and Kamada [4] and the author [14] originated the classification of diffeomorphism types of total spaces of genus-1 simplified broken Lefschetz fibrations. The author [14] classified diffeomorphism types of total spaces of genus-1 simplified broken Lefschetz fibrations under the assumption on the number of Lefschetz singularities. Behrens [5] also classified 4-manifolds admitting genus-1 simple wrinkled fibrations. Since we can change genus-1 simple wrinkled fibrations into genus-1 simplified broken Lefschetz fibrations by application of unsink deformations, we can regard the classification problem of genus-1 simple wrinkled fibrations as a part of the classification of genus-1 simplified broken Lefschetz fibrations. In this thesis, we gave the complete classification of diffeomorphism types of total spaces of genus-1 simplified broken Lefschetz fibrations (see Theorem 3.0.1). This result includes all the results above and gives the affirmative answer to Conjecture 5.3 in [14] and the negative answer to Problem 24 in [3].

We next study hyperelliptic simplified broken Lefschetz fibrations (and more generally, hyperelliptic directed broken Lefschetz fibrations) in Chapter 4. In smooth category, a hyperelliptic Lefschetz fibration is defined as a Lefschetz fibration such that all the vanishing cycles of the fibration are preserved by the hyperelliptic involution of the standard surface $\Sigma_g$ for a suitable choice of identification of a reference fiber with the surface $\Sigma_g$. Siebert and Tian [29] and Fuller [11] proved that the total space of a hyperelliptic Lefschetz fibration admits an involution which preserves any fiber of the Lefschetz fibration. In this thesis, we define hyperelliptic simplified broken Lefschetz fibrations in the same way as in the case of Lefschetz fibrations and generalize Siebert, Tian and Fuller's result on hyperelliptic Lefschetz fibrations to that on hyperelliptic simplified broken Lefschetz fibrations. We prove that the total space of a hyperelliptic simplified broken Lefschetz fibration admits an involution preserving any fiber of the fibration provided that the genus of the fibration is greater than or equal to 3 (see (i) of Theorem 4.3.1). By using the involution, we also prove that the rational homology class of the total space of a hyperelliptic simplified broken Lefschetz fibration of genus $g \geq 3$ represented by a regular fiber is not trivial (see (ii) of Theorem 4.3.1). The second statement implies that we cannot drop the assumption on a genus from the first statement. Note that we can generalize our results above to those on hyperelliptic directed broken Lefschetz fibrations (see Theorem 4.3.8).

Matsumoto defined the local signature of a Lefschetz singular fiber and proved that the signature of the total space of a Lefschetz fibration of genus-1 [23] and genus-2 [24] is equal to the sum of the local signatures of the Lefschetz singular fibers of the Lefschetz fibration. Endo [9] generalized Matsumoto's signature formulae to a formula for the signatures of hyperelliptic Lefschetz fibrations. In this thesis, we further generalize Matsumoto and Endo's signature formulae to a formula for the signatures of hyperelliptic directed broken Lefschetz fibrations as follows: we define a certain rational valued homomorphism $h_{g,c}$ on the subgroup of the hyperelliptic mapping class group which consists of mapping classes preserving $c$, where $c \subset \Sigma_g$ is a simple closed curve preserved by the hyperelliptic involution. We prove that the signature of the total space of a hyperelliptic directed broken Lefschetz fibration is equal to the sum of the local signatures of the Lefschetz singular fibers of the fibration and the values of the monodromies along the images of folds of the fibration under the homomorphisms $h_{g,d_1}, \ldots, h_{g,d_m}$, where $d_1, \ldots, d_m$ are vanishing cycles of folds (see Theorem 4.5.1).

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# Chapter 2

# Broken Lefschetz fibrations and vanishing cycles

In this chapter, we give the definition of broken Lefschetz fibrations and summarize the results on monodromies and vanishing cycles of broken Lefschetz fibrations we need in this thesis.

## 2.1 Broken Lefschetz fibrations

Let $X$ and $B$ be connected, oriented, compact, smooth manifolds of dimension 4 and 2, respectively, and $f : X \to B$ a smooth map. Assume that $f$ satisfies the condition $f^{-1}(\partial B) = \partial X$. A critical point $p \in X$ of $f$ is called an *indefinite fold singularity* if there exist real coordinates $(t, x, y, z)$ of $X$ around $p$ and $(s, w)$ of $B$ around $f(p)$ such that $f$ is locally described as follows:

$$(t, x, y, z) \mapsto (s, w) = (t, x^2 + y^2 - z^2).$$

In this paper, we will refer to this singularity as a *fold* for simplicity. A critical point $p \in X$ of $f$ is called a *Lefschetz singularity* if there exist complex coordinates $(z, w)$ of $X$ around $p$ compatible with the orientation of $X$ and $\xi$ of $B$ around $f(p)$ compatible with the orientation of $B$ such that $f$ is locally described as follows:

$$(z, w) \mapsto \xi = zw.$$

**Definition 2.1.1.** A map $f$ is called a *broken Lefschetz fibration* if $f$ satisfies the following conditions:

- the set $\mathrm{Crit}(f) \subset X$ of critical points of $f$ consists of folds and Lefschetz singularities;
- the restriction $f|_{Z_f}$ is a generic immersion, where $Z_f \subset X$ is the set of folds;
- the restriction $f|_{\mathcal{C}_f}$ is injective, where $\mathcal{C}_f \subset X$ is the set of Lefschetz singularities of $f$.

A broken Lefschetz fibration $f$ is called a *Lefschetz fibration* if the set of critical points of $f$ consists of Lefschetz singularities. All the regular fiber of a Lefschetz fibration are diffeomorphic to some genus-$g$ surface $\Sigma_g$. A Lefschetz fibration $f$ has *genus-$g$* if the genus of a regular fiber of $f$ is equal to $g$. We will refer to broken Lefschetz fibrations and Lefschetz fibrations as *BLFs* and *LFs*, respectively.

**Definition 2.1.2.** Two BLFs $f_1 : X_1 \to B_1$ and $f_2 : X_2 \to B_2$ are said to be *equivalent* if there exist diffeomorphisms $\Theta : X_1 \to X_2$ and $\theta : B_1 \to B_2$ which make the following diagram commute:

$$\begin{array}{ccc} X_1 & \xrightarrow{\Theta} & X_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ B_1 & \xrightarrow{\theta} & B_2. \end{array}$$

Let $f : X \to S^2$ be a BLF over the 2-sphere. We assume the following conditions:

(a) the restriction of $f$ to the set of singularities is injective;

(b) the image $f(Z_f)$ is a disjoint union of embedded circles parallel to the equator of $S^2$.

We put $f(Z_f) = Z_1 \amalg \cdots \amalg Z_m$, where $Z_i$ is a connected component of $f(Z_f)$. We take an embedding $\gamma : [0,1] \to S^2$ so that $\gamma$ satisfies the following properties:

1. the image $\gamma([0,1])$ is contained in the complement of $f(\mathcal{C}_f)$;
2. $\gamma$ starts at the south pole $p_s \in S^2$ and connects the south pole to the north pole $p_n \in S^2$;
3. $\gamma$ intersects each component of $f(Z_f)$ at a single point transversely.

We put $\{q_i\} = Z_i \cap \alpha([0,1])$ and $\gamma(t_i) = q_i$. We assume that $q_1, \ldots, q_m$ appear in this order when we go along $\gamma$ from $p_s$ to $p_n$ (see Figure 2.1.1). The preimage $f^{-1}(\gamma([0,1]))$ is

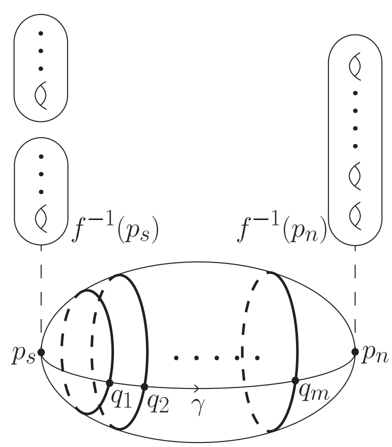


Figure 2.1.1: A path $\gamma$. The bold circles describe $f(Z_f)$.

a 3-manifold which is a cobordism between $f^{-1}(p_s)$ and $f^{-1}(p_n)$. By the definition of folds, $f^{-1}(\gamma([0, t_i + \varepsilon]))$ is obtained from $f^{-1}(\gamma([0, t_i - \varepsilon]))$ by either 1 or 2-handle attachment for each $i = 1, \ldots, m$. Thus, we obtain a handle decomposition of the cobordism $f^{-1}(\gamma([0,1]))$.

**Definition 2.1.3.** A BLF $f$ is said to be *directed* if it satisfies the following conditions:

1. $f$ satisfies the conditions (a) and (b) above;
2. all the handles of the handle decomposition of $f^{-1}(\gamma([0,1]))$ have index-1;
3. all Lefschetz singularities of $f$ are in the preimage of the component of $S^2 \setminus (Z_1 \amalg \cdots \amalg Z_m)$ which contains the point $p_n$.

In this paper, we will refer to a directed broken Lefschetz fibration as a DBLF.

**Definition 2.1.4.** A DBLF $f : X \to S^2$ is called a *simplified broken Lefschetz fibration* if the set of folds of $f$ is connected and that all the fibers of $f$ are connected. In this paper, we will refer to a simplified broken Lefschetz fibration as an SBLF.

Let $f$ be an SBLF. The set $Z_f$ is either the empty set or an embedded circle in $X$. If $Z_f$ is empty, then $f$ is an LF over $S^2$. If $Z_f$ is not empty, the image $f(Z_f)$ is an embedded circle in $S^2$. Thus, $S^2 \setminus \operatorname{Int} \nu f(Z_f)$ consists of two 2-disks $D_1$ and $D_2$, where $\nu f(Z_f)$ is a regular neighborhood of $f(Z_f)$. Furthermore, the genus of a regular fiber of the fibration $\mathrm{res} f : f^{-1}(D_1) \to D_1$ is higher than the genus of a regular fiber of the fibration $\mathrm{res} f : f^{-1}(D_2) \to D_2$ by 1. we call $f^{-1}(D_1)$ (resp. $f^{-1}(D_2)$) the *higher side* (resp. *lower side*) of $f$ and $f^{-1}(\nu f(Z_f))$ the *round cobordism* of $f$. By the definition, all the Lefschetz singularities of $f$ are in the higher side of $f$. We call the genus of a regular fiber in the higher side the *genus* of $f$.

## 2.2 Monodromy representations of Lefschetz fibrations

Let $f : X \to B$ be a genus-$g$ LF. We fix a regular value $p_0 \in B$ of $f$ and an orientation-preserving diffeomorphism $\psi_0 : f^{-1}(p_0) \to \Sigma_g$. For a loop $\gamma : (I, \partial I) \to (B \setminus f(\mathcal{C}_f), p_0)$, the pull-back $\gamma^* f = \{(t,x) \in I \times M \mid \gamma(t) = f(x)\}$ is isomorphic to the trivial $\Sigma_g$-bundle. We take a trivialization $\Psi : \gamma^* f \to I \times \Sigma_g$ so that the restriction $\Psi|_{\{0\}\times\Sigma_g}$ coincides with $\psi_0$. We put $\Psi(t,x) = (t, \psi_t(x))$. We denote by $[\psi_1 \circ \psi_0^{-1}]$ the isotopy class of the map $\psi_1 \circ \psi_0^{-1}$ and by $\mathcal{M}_g$ the mapping class group of $\Sigma_g$, that is, the set of isotopy classes of orientation-preserving diffeomorphisms. We define the map $\rho_f : \pi_1(B \setminus f(\mathcal{C}_f), p_0) \to \mathcal{M}_g$ as follows:

$$\rho_f([\gamma]) = [\psi_1 \circ \psi_0^{-1}].$$

This map is well-defined and called a *monodromy representation* of $f$. The readers should refer to [13] for details of monodromy representations of LFs.

**Remark 2.2.1.** In order to make monodromy representations of LFs homomorphisms, we define a group structure of the mapping class group $\mathcal{M}_g$ using the *opposite* multiplication to the composition as maps, that is, we define the multiplication $[f] \cdot [g]$ as $[g \circ f]$ for elements $[f], [g] \in \mathcal{M}_g$.

**Definition 2.2.2.** Let $B_i$ $(i = 1, 2)$ be a connected surface (possibly with boundary or punctures), $q_i$ a point in $B_i$ and $\rho_i : \pi_1(B_i, q_i) \to \mathcal{M}_g$ a representation. The representations $\rho_1$ and $\rho_2$ are said to be *equivalent* if there exist an element $\varphi \in \mathcal{M}_g$ and an orientation preserving diffeomorphism $h : (B_1, q_1) \to (B_2, q_2)$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(B_1, q_1) & \xrightarrow{\rho_1} & \mathcal{M}_g \\ {\scriptstyle h_*}\downarrow & & \downarrow{\scriptstyle \mathrm{conj}(\varphi)} \\ \pi_1(B_2, q_2) & \xrightarrow{\rho_2} & \mathcal{M}_g, \end{array}$$

where $\mathrm{conj}(\varphi)$ is the inner automorphism of $\mathcal{M}_g$ by the element $\varphi$.

Note that a monodromy representation of an LF depends on various choices in construction. However, the equivalent class of a monodromy representation of an LF does not depend on the choices.

**Theorem 2.2.3** ([20], [24]). *Let $f_i : X_i \to B_i$ $(i = 1, 2)$ be an LF with genus-$g$. We assume that $f_1$ satisfies one of the following conditions:*

- *the genus $g$ of $f_1$ is greater than 1;*
- *the base space $B_1$ of $f_1$ has non-empty boundary;*
- *the set $\mathrm{Crit}\, f_1$ of critical points of $f_1$ is not empty.*

*Then $f_1$ and $f_2$ are equivalent if and only if the corresponding monodromy representations $\rho_{f_1}$ and $\rho_{f_2}$ are equivalent.*

## 2.3 Surgery homomorphisms

For a simple closed curve $c \subset \Sigma_g$, we define the subgroup $\mathcal{M}_g(c)$ of the mapping class group $\mathcal{M}_g$ as follows:

$$\mathcal{M}_g(c) = \{[T] \in \mathcal{M}_g | T(c) = c\}.$$

For an element $\varphi \in \mathcal{M}_g(c)$, we take a mapping class $\Phi_c(\varphi)$ in the following way; we first take a representative $T \in \varphi$ so that $T$ preserves the curve $c$ setwise. The restriction $T|_{\Sigma_g \setminus c}$ is also a diffeomorphism. We can extend $T|_{\Sigma_g \setminus c}$ to a self-diffeomorphism of $S_c$, where $S_c$ is obtained by applying surgery to $\Sigma_g$ along $c$. We denote by $\Phi_c(\varphi)$ the isotopy class of this extended diffeomorphism. The topology of $S_c$ is determined easily as follows:

$$S_c \cong \begin{cases} \Sigma_{g-1} & (\text{if } c \text{ is type I}), \\ \Sigma_h \amalg \Sigma_{g-h} & (\text{if } c \text{ is type II}_h), \end{cases}$$

where $c$ is said to be *type* I if $c$ is non-separating, and *type* $\mathrm{II}_h$ if $c$ is separating curve which bounds a genus-$h$ surface (see Figure 2.3.1). Thus, we obtain:

$$\mathrm{Mod}(S_c) \cong \begin{cases} \mathcal{M}_{g-1} & (\text{if } c \text{ is type I}), \\ \mathcal{M}_h \times \mathcal{M}_{g-h} & (\text{if } c \text{ is type II}_h \text{ and } 2h \neq g), \\ (\mathcal{M}_h \times \mathcal{M}_h) \rtimes \mathbb{Z}/2\mathbb{Z} & (\text{if } c \text{ is type II}_h \text{ and } 2h = g), \end{cases}$$

where $\mathrm{Mod}(S_c)$ is the mapping class group of $S_c$, and in the last case, a generator of $\mathbb{Z}/2\mathbb{Z}$ is represented by a map which exchanges the components. We can prove the following lemma by the argument quite similar to that in [5].

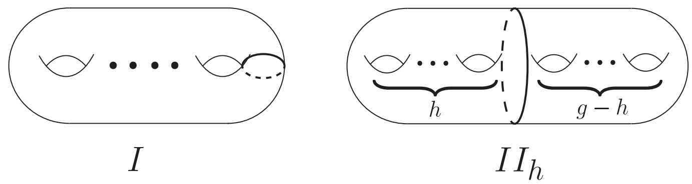


Figure 2.3.1: type I and type $\mathrm{II}_h$

**Lemma 2.3.1.** *The map* $\Phi_c : \mathcal{M}_g(c) \ni \varphi \mapsto \Phi_c(\varphi) \in \mathrm{Mod}(S_c)$ *is well-defined.*

We call the homomorphism $\Phi_c : \mathcal{M}_g(c) \to \mathrm{Mod}(S_c)$ a *surgery homomorphism* along $c$. To see the relation between surgery homomorphisms and monodromies, we consider a BLF $f : X \to S^1 \times I$ over the annulus. We assume the following conditions:

- the set $\mathrm{Crit}(f)$ consists of one component of folds;
- the restriction $f|_{\mathrm{Crit}(f)}$ is injective;
- the image $f(\mathrm{Crit}(f))$ is equal to $S^1 \times \{\frac{1}{2}\}$.

We take a point $p_0 \in S^1$. By the assumptions, we can obtain a Morse function $\mathrm{res}\, f$ : $f^{-1}(\{p_0\} \times I) \to I$ with one critical point. Suppose that the index of the critical point of this Morse function in index 1. We denote by $d \subset f^{-1}(p_0, 1)$ a attaching circle of the 2-handle. We fix an identification of $f^{-1}(p_0, 1)$ with a disjoint union $\Sigma_{g_1} \amalg \cdots \amalg \Sigma_{g_k}$. Suppose that $d$ is contained in $\Sigma_{g_1}$. The fiber $f^{-1}(p_0, 0)$ consists of $k$ components if $d$ is non-separating, and $k+1$ components if $d$ is separating. The fiber $f^{-1}(p_0, 0)$ can be identified with $S_d \amalg \Sigma_{g_2} \amalg \cdots \amalg \Sigma_{g_k}$ using the fixed identification of $f^{-1}(p_0, 1)$. We take a simple loop $\gamma_i \subset S^1 \times I \setminus f(\mathrm{Crit}(f))$ based at $(p_0, i)$ $(i = 0, 1)$ which is parallel to the boundary. We denote by $\psi_i$ the monodromy along $\gamma_i$.

**Theorem 2.3.2** (cf. [1] and [2])**.** *The element* $\psi_1$ *preserves the curve* $d$ *and is mapped to* $\psi_0$ *by the following homomorphisms:*

$$\Phi_d \times \mathrm{id}_{\mathcal{M}_{g_2}} \times \cdots \times \mathrm{id}_{\mathcal{M}_{g_k}} : \mathcal{M}_{g_1}(d) \times \mathcal{M} \to \mathrm{Mod}(S_c) \times \mathcal{M},$$

*where we put* $\mathcal{M} = \mathcal{M}_{g_2} \times \cdots \times \mathcal{M}_{g_k}$.

## 2.4 Vanishing cycles of directed broken Lefschetz fibrations

For a genus-$g$ DBLF $f : X \to S^2$, we put $f(\mathcal{C}_f) = \{p_1, \ldots, p_l\}$. We take a path $\gamma \subset S^2$ as in Section 2.1. By the definition of DBLFs, we can obtain the preimage $f^{-1}(\gamma([0, t_i - \varepsilon]))$ from the preimage $f^{-1}(\gamma([0, t_i + \varepsilon]))$ by attaching a 2-handle. We call an attaching circle $d_i \subset f^{-1}(\gamma(t_i + \varepsilon))$ of the 2-handle a *vanishing cycle* of folds.

We take paths $\gamma_1, \ldots, \gamma_l$ so that $\gamma_i$ connects $p_n$ to $p_i$ and that the paths $\gamma, \gamma_1, \ldots, \gamma_l$ are mutually disjoint except on the point $p_0$. Suppose that the indices of the paths are given so that $\gamma, \gamma_1, \ldots, \gamma_l$ appear in this order when we go around $p_0$ counterclockwise. As in the case of folds, the paths $\gamma_1, \ldots, \gamma_l$ determine *vanishing cycles* of Lefschetz singularities (the reader refer to [13], for example, for details of vanishing cycles of Lefschetz singularities). We denote by $c_i \subset f^{-1}(p_0)$ the vanishing cycle determined by $\gamma_i$. We identify the fiber $f^{-1}(p_0)$ with $\Sigma_g$ and regard the vanishing cycles as simple closed curves in $\Sigma_g$. Denote by $D_h \subset S^2$ the connected component of $S^2 \setminus \nu f(Z_f)$ which contains the point $p_n$. Since the restriction $f|_{f^{-1}(D_h)}$ is an LF, we can take a monodromy representation $\rho_f : \pi_1(D_h \setminus f(\mathcal{C}_f), p_0) \to \mathcal{M}_g$.

**Theorem 2.4.1** ([20], [24])**.** *For each reference path $\gamma_i$, denote by $a_i$ a loop in $D^2 \setminus f(\mathcal{C}_f)$ based at $p_0$ by connecting $p_0$ with a small counterclockwise circle around $p_i$ using $\gamma_i$. Then the monodromy $\rho_f(a_i)$ is equal to the right-handed Dehn twist $t_{c_i}$.*

Since the product $a_1 \cdot \cdots \cdot a_l \in \pi_1(D_h \setminus f(\mathcal{C}_f), p_0)$ is represented by a loop parallel to the boundary of $D_h$, we immediately obtain:

**Corollary 2.4.2.** *The element $t_{c_1} \cdot \cdots \cdot t_{c_l}$ is the monodromy of $f$ along a loop parallel to the boundary $\partial D_h$. Moreover, this element is contained in the kernel of the composition $\Phi_{d_1} \circ \cdots \circ \Phi_{d_m}$.*

*Proof.* The first statement is obvious, while the second statement holds since the lowest genus side of $f$ does not contain any singularities. □

## 2.5 Hurwitz cycle systems of simplified broken Lefschetz fibrations

Let $f : X \to S^2$ be a genus-$g$ SBLF with non-empty folds. As explained in the previous section, we can obtain vanishing cycles $c = d_1, c_1, \ldots, c_l$ of folds and Lefschetz singularities of $f$. We call a system of reference paths $\gamma, \gamma_1, \ldots, \gamma_l$ which give vanishing cycles $c, c_1, \ldots, c_l$ a *Hurwitz path system* and a sequence $W_f = (c; c_1, \ldots, c_l)$ a *Hurwitz cycle system* of $f$. There are two types of modifications of Hurwitz cycle systems. The first one, which we will refer to as an *elementary transformation*, is as follows:

$$(c; c_1, \ldots, c_i, c_{i+1}, \ldots, c_n) \longrightarrow (c; c_1, \ldots, c_{i+1}, t_{c_{i+1}}(c_i), \ldots, c_n).$$

It is easy to see that this modification can be realized by replacing a Hurwitz path system as described in the left side of Figure 2.5.1. The second modification, *simultaneous action by $h \in \mathcal{M}_g$*, is as follows.

$$(c; c_1, \ldots, c_n) \longrightarrow (h(c); h(c_1), \ldots, h(c_n)).$$

This modification corresponds to substitution of an identification of the reference fiber with $\Sigma_g$. Two sequences $(c; c_1, \ldots, c_l)$ and $(d; d_1, \ldots, d_l)$ of simple closed curves in $\Sigma_g$ are said to be *Hurwitz equivalent* if one can be obtained from the other by successive application of simultaneous actions, elementary transformations and their inverse. Note that for a given SBLF $f$, any sequence $W$ which is Hurwitz equivalent to $W_f$ can be realized as a Hurwitz cycle system of $f$ by replacing reference paths and an identification $f^{-1}(p_0) \cong \Sigma_g$.

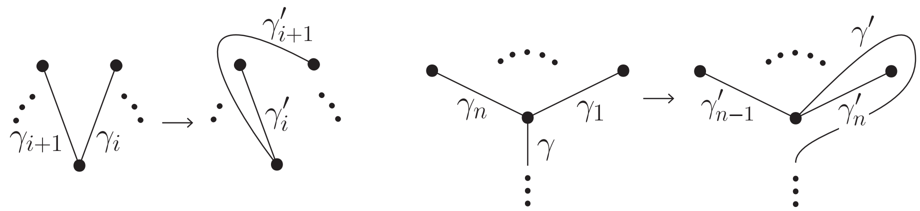


Figure 2.5.1: Left: modification of a Hurwitz path system corresponding to an elementary transformation. Right: another modification of a path system.

**Remark 2.5.1.** There is another modification of a Hurwitz cycle system which is described as follows:

$$(c; c_1, \ldots, c_n) \longrightarrow (t_{c_1}(c); c_2, \ldots, c_n, c_1).$$

It is easy to verify that this modification is induced by the modification of a Hurwitz path system described in the right side of Figure 2.5.1. Furthermore, this modification can be realized by simultaneous action by $t_{c_1}$, followed by successive application of inverse transformations of elementary transformations. This modification will play a key role in the proof of the theorem below.

**Theorem 2.5.2.** *Let $f_i : X_i \to S^2$ be an SBLF with genus $g \geq 3$ ($i = 1, 2$). The fibrations $f_1$ and $f_2$ are isomorphic if and only if the corresponding Hurwitz cycle systems $W_{f_1}$ and $W_{f_2}$ are equivalent.*

**Remark 2.5.3.** This theorem would not hold if the assumption on genera of fibrations are dropped. Indeed, there exist infinitely many SBLFs with small genera which are mutually not isomorphic but have the same Hurwitz cycle systems.

*Proof of Theorem 2.5.2.* We first prove the only if part. Suppose that $f_1$ and $f_2$ are isomorphic, and we fix diffeomorphisms $\Phi : X_1 \to X_2$ and $\varphi : S^2 \to S^2$ satisfying the condition in the definition. We take reference paths $\gamma, \gamma_1, \ldots, \gamma_n$ of the fibration $f_1$ as explained above. We denote by $W_{f_1}$ the corresponding Hurwitz cycle system of $f_1$ derived from these paths, together with an identification $\phi : f_1^{-1}(y_0) \to \Sigma_g$. We can use the paths $\varphi(\gamma), \varphi(\gamma_1), \ldots, \varphi(\gamma_n)$ and a diffeomorphism $\phi \circ \Phi^{-1} : f_2^{-1}(\varphi(y_0)) \to \Sigma_g$ to obtain a Hurwitz cycle system $W_{f_2}$ of the fibration $f_2$. It is easy to verify that $W_{f_1}$ is equal to $W_{f_2}$. Thus, all we need to prove is a Hurwitz cycle system of $f_1$ derived from different reference paths $\gamma', \gamma'_1, \ldots, \gamma'_n$ is Hurwitz equivalent to $W_{f_1}$. By the argument similar to that in the solution of Exercise 8.2.7(c) in [13], we can prove that the system $\gamma', \gamma'_1, \ldots, \gamma'_n$ can be changed into the system $\gamma, \gamma_1, \ldots, \gamma_n$ up to isotopy by successive application of the two moves in Figure 2.5.1. This completes the proof of the only if part.

We next prove the if part. By the assumption, we can take reference paths of $f_1$ and $f_2$, and identifications of reference fibers with the surface $\Sigma_g$ so that the corresponding Hurwitz cycle systems $W_{f_1}$ and $W_{f_2}$ coincide. We decompose $X_i$ into the three parts $X_i^{(r)}, X_i^{(h)}$ and $X_i^{(l)}$, that is, the preimage of a regular neighborhood of $f(Z_f)$, the highest side and the lowest side of $f_i$. The restriction $f_i|_{X_i^{(h)}}$ is an LF. By Theorem 2.2.3, the fibrations $f_1|_{X_1^{(h)}}$ and $f_2|_{X_2^{(h)}}$ are isomorphic. In particular, we can take a fiber-preserving diffeomorphism $\Phi_h : X_1^{(h)} \to$

$X_2^{(h)}$. Since there are no singularities on the boundary of the highest side, we can take an identification of the boundary $\partial X_1^{(h)}$ with the mapping torus $T(\theta) = I \times \Sigma_g/(1,x) \sim (0,\theta(x))$, where $\theta : \Sigma_g \to \Sigma_g$ is a diffeomorphism. This identification, together with a diffeomorphism $\Phi_h$, gives an identification of $\partial X_2^{(h)}$ with $T(\theta)$. We denote by $c \subset \Sigma_g$ the vanishing cycle of folds in $W_{f_1}$ (note that this cycle coincides with that in $W_{f_2}$). By Corollary 2.4.2, the isotopy class of $\theta$ is contained in $\mathcal{M}_g(c)$.

we can assume that $\theta$ preserves a regular neighborhood $\nu(c)$. For each $i = 1, 2$, we take an identification of $f_i(X_i^{(r)})$ with the annulus $I \times D^1/(1,t) \sim (0,t)$ so that the restriction $f_i|_{\partial X_i^{(h)}} : T(\theta) \to I \times \{1\}/\sim$ becomes the projection, and that the image of indefinite folds is equal to the circle $I \times \{0\}/\sim$. The following lemma can be proved easily:

**Lemma 2.5.4.** *We denote by $Z_i \subset X_i$ the set of indefinite folds of $f_i$. There exist a sufficiently small number $\varepsilon > 0$ and a diffeomorphism $\Psi_i : I \times D_\varepsilon^1 \times D_\varepsilon^2/(1,x,y_1,y_2) \sim (0,\pm x, y_1, \pm y_2) \to \nu(Z_i)$, where $D_\varepsilon^d$ is the $d$-dimensional ball with radius $\varepsilon$, which make the following diagram commute:*

$$\begin{array}{ccc} I \times D_\varepsilon^1 \times D_\varepsilon^2/\sim & \xrightarrow{\Psi_i} & \nu(Z_i) \\ \downarrow \pi & \swarrow f_i & \\ I \times [-\varepsilon,\varepsilon]/\sim & & \end{array},$$

*where $\pi$ is defined as $\pi(t,x,y_1,y_2) = (t, -x^2 + {y_1}^2 + {y_2}^2)$.*

For a positive number $s \leq 2$, we define a path $\gamma_{t,s} : [0,s] \to I \times D^1/\sim$ as $\gamma_t(x) = (t, 1-x)$. A connected component of the set $\mathrm{Sub}(S^1, \Sigma_g)$ of circles in $\Sigma_g$ is simply connected if $g \geq 2$ (see Theorem 2.7.H of [17] for example). Thus, we can take a horizontal distribution $\mathcal{H}_i$ of $f_i|_{X_i \setminus Z_i}$ so that it satisfies the following conditions:

1. in the image of $\Psi_i$, $\mathcal{H}_i$ is equal to the horizontal distribution derived from the product metric of $I \times D_\varepsilon^1 \times D_\varepsilon^2/\sim$,
2. the parallel transport $PT_{\gamma_{t,1-\frac{\varepsilon}{2}}}^{\mathcal{H}_i}$ of $\mathcal{H}_i$ along $\gamma_{t,1-\frac{\varepsilon}{2}}$ maps $\{t\} \times \nu(c)$ to the following set:
$$\pi^{-1}\left(\frac{\varepsilon}{2}\right) \cap \left\{(t,x,y_1,y_2) \in I \times D_\varepsilon^1 \times D_\varepsilon^2/\sim \ \middle|\ |x| \leq \frac{\varepsilon}{2}\right\},$$
3. the parallel transport $PT_{\gamma_{t,1-\frac{\varepsilon}{2}}}^{\mathcal{H}_1}$ is equal to the parallel transport $PT_{\gamma_{t,1-\frac{\varepsilon}{2}}}^{\mathcal{H}_2}$ under the identifications.

Using the distributions, we can define a diffeomorphism $\Phi_r : X_1^{(r)} \to X_2^{(r)}$ as follows:

$$\Phi_r(w) = \begin{cases} (t,x,y) \in \nu(Z_2) & (w = (t,x,y) \in \nu(Z_1) \cap I \times D_{\frac{2\varepsilon}{3}}^1 \times D_{\frac{2\varepsilon}{3}}^2/\sim), \\ PT_{\gamma_{t,s}}^{\mathcal{H}_2}(z) \in X_2^{(r)} & (w = PT_{\gamma_{t,s}}^{\mathcal{H}_1}(z) \in X_1^{(r)}, z \in I \times (\Sigma_g \setminus \nu(c))/\sim). \end{cases}$$

It is easy to see that this map is fiber-preserving. In particular, the restriction $\Phi_r : \partial X_1^{(l)} \to \partial X_2^{(l)}$ is a fiber-preserving diffeomorphism. Since the connected component of the group $\mathrm{Diff}^+(\Sigma_{g-1})$ is contractible if $g \geq 3$, this restriction can be extended to a fiber-preserving diffeomorphism $\Phi_l : X_l^{(1)} \to X_l^{(2)}$. Combing the three diffeomorphisms $\Phi_h, \Phi_r$ and $\Phi_l$, we can obtain the desired map $\Phi$. This completes the proof of Theorem 2.5.2. □

## 2.6 Kirby diagrams of broken Lefschetz fibrations

In this section, we will give a quick review of a handle decomposition of a total space of a directed broken Lefschetz fibration which reflects its fibration structure. The readers should refer to [2] for details of them.

**Definition 2.6.1.** Let $X$ be a smooth 4-manifold and we put

$$R_i^{\pm} = I \times D^i \times D^{3-i}/((1, x_1, x_2, x_3) \sim (0, \pm x_1, x_2, \pm x_3))(i = 1, 2).$$

Let $\psi : I \times \partial D^i \times D^{3-i}/\sim \to \partial M$ be an embedding. We call $M \bigcup_{\psi} R_i^{\pm}$ a 4-manifold obtained by attaching a *round $i$-handle* and $R_i^+$(resp. $R_i^-$) *(4-dimensional) untwisted* (resp. *twisted*) *round $i$-handle.*

**Remark 2.6.2.** Both untwisted and twisted round handles are diffeomorphic to $S^1 \times D^3$, but these round handles have distinct attaching regions. The attaching region of an untwisted round $i$-handle is the trivial $S^{i-1} \times D^{3-i}$-bundle over $S^1$, while that of a twisted one is a non-trivial $S^{i-1} \times D^{3-i}$-bundle over $S^1$.

By the definition of round handles, we can regard 4-dimensional round $i$ handle attachment as $S^1$-family of 3-dimensional $i$-handle attachment. We call an attaching sphere of a $i$-handle in this family an *attaching sphere* of a round handle.

**Lemma 2.6.3** ([2]). *For $i \in \{1, 2\}$, round $i$-handle attachment is given by $i$-handle attachment followed by $(i + 1)$-handle attachment whose attaching sphere goes over the belt sphere of the $i$-handle geometrically twice, algebraically zero times if the round handle is untwisted and twice if the round handle is twisted.*

*Proof.* The handle $R_i^{\pm}$ can be decomposed into two parts $[0, \frac{1}{2}] \times D^i \times D^{3-i}$ and $[\frac{1}{2}, 1] \times D^i \times D^{3-i}$. Attachment of a round $i$-handle is equivalent to attachment of the former part followed by attachment of the latter part. It is easy to see that the former (resp. the latter) attachment can be regarded as 2-handle (resp. 3-handle) attachment. □

Let $f : X \to S^2$ be a DBLF of genus-$g$. We use the same notations $Z_1, \ldots, Z_m \subset S^2$ as in section 2.4 and denote by $\widetilde{Z_i} \subset X$ the connected component of folds of $f$ on $Z_i$. We can decompose $S^2$ into two disks $D_h, D_l$ and $m$ annuli $\nu Z_1, \ldots, \nu Z_m$. Since Lemma 2.5.4 also works for each component $\widetilde{Z_i}$, we immediately obtain:

**Lemma 2.6.4** ([2]). *The manifold $f^{-1}(D_h \amalg \nu Z_1 \amalg \cdots \amalg \nu Z_i)$ can be obtained from $f^{-1}(D_h \amalg \nu Z_1 \amalg \cdots \amalg \nu Z_{i-1})$ by attaching a round 2-handle. Moreover, an attaching circle the round 2-handle is along a vanishing cycle of $\widetilde{Z_i}$. When we regard this round 2-handle attachment as 2-handle attachment followed by 3-handle attachment, the 2-handle is attached along a vanishing cycle of $\widetilde{Z_i}$ whose framing is along the fiber.*

Using Lemma 2.6.4, we can obtain a Kirby diagram of the total space of a DBLF. Several examples of Kirby diagrams obtained in this way can be found in [2], for example. The procedure of handle decomposition above also implies the following corollary, whose proof is left to the reader.

**Corollary 2.6.5** (cf. [2]). *Let* $c_1, \ldots, c_l \subset \Sigma_g$ *and* $d_i \subset (\cdots((\Sigma_g)_{c_1})_{c_2}\cdots)_{c_{i-1}}$ $(1 \geq i \geq m)$ *be simple closed curves. Suppose that the element* $t_{c_1} \cdot \cdots \cdot t_{c_l}$ *is contained in the kernel of the map* $\Phi_{d_m} \circ \cdots \circ \Phi_{d_1}$. *Then, these exists a genus-g DBLF* $f : X \to S^2$ *such that vanishing cycles of* $f$ *obtained as in section 2.4 coincides with* $c_1, \ldots, c_l, d_1 \ldots, d_m$.

# Chapter 3

# Classification of genus-1 simplified broken Lefschetz fibrations

In this chapter, we classify total spaces of genus-1 simplified broken Lefschetz fibrations. We prove the following theorem:

**Theorem 3.0.1.** *The following* 4*-manifolds admits a relatively minimal genus-*1 *SBLF with non-empty folds and* $l \geq 0$ *Lefschetz singularities:*

- $\# k\mathbb{CP}^2 \# (l-k)\overline{\mathbb{CP}^2}$ *(*$0 \geq k \geq l-1$*);*
- $\# \frac{l}{2} S^2 \times S^2$*. Note that this manifold appears only if* $l$ *is even;*
- $S^1 \times S^3 \# S \# l\overline{\mathbb{CP}^2}$*, where* $S$ *is an* $S^2$*-bundle over* $S^2$*;*
- $L \# l\overline{\mathbb{CP}^2}$*, where* $L$ *is either of the manifolds* $L_n$ *or* $L'_n$ *(*$n \geq 2$*), which is defined by Pao [27].*

*Conversely, every* 4*-manifold which admits a relatively minimal genus-*1 *SBLF with non-empty folds is diffeomorphic to one of the manifolds above.*

## 3.1 The mapping class group of the torus

We take elements $\alpha, \beta \subset H_1(T^2;\mathbb{Z})$ so that the algebraic intersection $\alpha \cdot \beta$ is 1. We define a homomorphism $\Psi : \mathcal{M}_1 \rightarrow SL(2,\mathbb{Z})$ so that a pair $(T_*(\alpha), T_*(\beta))$ is equal to $(\alpha, \beta) \cdot {}^t\Psi([T])$ for every diffeomorphism $T : T^2 \rightarrow T^2$. It is known that $\Psi$ is an isomorphism. In the rest of this section, we identify the group $\mathcal{M}_1$ with $SL(2,\mathbb{Z})$ via this isomorphism.

Since a primitive element in $H_1(T^2;\mathbb{Z})$ uniquely determines the isotopy class of an oriented loop in $T^2$, we represent the isotopy class of a simple closed curve by its homology class (after giving some orientation). With this understood, the Dehn twist along a primitive element $\gamma = p\alpha + q\beta \in H_1(T^2;\mathbb{Z})$ makes sense and the corresponding linear representation is determined by the Picard-Lefschetz formula as follows:

$$t_\gamma = \begin{pmatrix} 1-pq & -q^2 \\ p^2 & 1+pq \end{pmatrix}.$$

In particular, we obtain $t_\alpha = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $t_\beta = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Denote these matrices by $X_1$ and $X_2$, respectively. The group $SL(2,\mathbb{Z})$ has the following finite presentation (see [22], for example):

$$SL(2,\mathbb{Z}) = < X_1, X_2 | (X_1X_2)^6, X_1X_2X_1X_2^{-1}X_1^{-1}X_2^{-1} > .$$

## 3.2 Chart descriptions of monodromy representations

Toward classification of total spaces of genus-1 SBLFs, we first prove that any genus-1 SBLF has a Hurwitz cycle system which we can easily deal with. The goal of this section is to prove the following theorem:

**Theorem 3.2.1.** *Let $f : X \to S^2$ be a relatively minimal genus-1 SBLF with non-empty folds. A Hurwitz cycle system $W_f$ is Hurwitz equivalent to the following sequence:*

$$(\alpha; S_rT(n_1,\ldots,n_s)),$$

*where $T(n_1,\ldots,n_s) = (\beta+n_1\alpha,\ldots,\beta+n_s\alpha)$ and $S_r = (\alpha,\ldots,\alpha)$ ($r$ $\alpha$'s are contained in this sequence).*

To prove this, we will introduce a graphical description of Hurwitz cycle systems.

**Definition 3.2.2.** A finite graph $\Gamma$ in $D^2$ (possibly being empty or having *hoops* that are closed edges without vertices) is called a *chart* if $\Gamma$ satisfies the following conditions:

(1) the degree of each vertex is equal to either 1, 6 or 12;

(2) each vertex in $\partial D^2$ has degree-1;

(3) each edge in $\Gamma$ is labeled 1 or 2 and is oriented;

(4) the edge from a degree-1 vertex in $\mathrm{Int}(D^2)$ is oriented toward the vertex;

(5) the six edges from a degree-6 vertex are labeled alternately with 1 and 2. Moreover, three consecutive edges are oriented toward the vertex and the other edges are oriented away from it;

(6) the twelve edges from a degree-12 vertex are labeled alternately with 1 and 2 and all the edges are oriented in the same way, oriented toward or away from the vertex (see Figure 3.2.1);

(7) an interior of each edge is contained in $\mathrm{int}D^2$;

(8) let $\{v_1, \ldots, v_n\}$ be the set of vertices in $\partial D^2$ and assume that the indices are given so that $v_1, \ldots, v_n$ appear in this order when we go along $\partial D^2$ counterclockwise. Denote by $i_k$ the label of the edge $e_i$ from $v_i$. We put $\varepsilon_k = +1$ if $e_i$ is oriented toward $v_i$ and put $\varepsilon_k = +1$ otherwise. For some $k$, the sequence $((i_k, \varepsilon_k), \ldots, (i_n, \varepsilon_n), (i_1, \varepsilon_1), \ldots, (i_{k-1}, \varepsilon_{k-1}))$ can be divided into the following subsequences:

(a) $((1, \varepsilon))$;

(b) $((i, \varepsilon), (j, \varepsilon), (i, \varepsilon), (j, \varepsilon), (i, \varepsilon), (j, \varepsilon))$, where $(\{i, j\} = \{1, 2\})$ and $\varepsilon$ is either $+1$ or $-1$.

We call such a sequence a *boundary sequence* of $\Gamma$ and two subsequences above the *unit subsequences.*

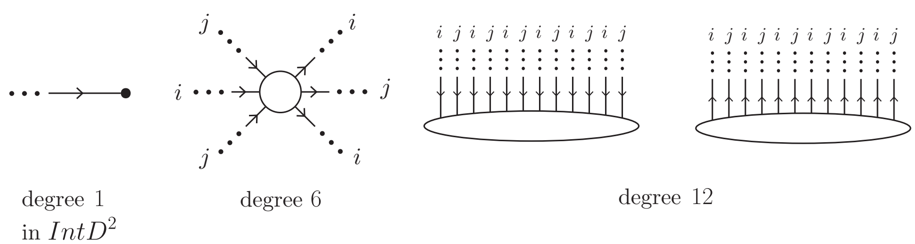


Figure 3.2.1: vertices of a chart, where $\{i, j\} = \{1, 2\}$.

An example of a chart is illustrated in Figure 3.2.2. The corresponding sequence mentioned in the condition (8) of the definition is as follows:

$$((1, -1), (1, -1), (1, -1), (1, -1), (2, 1), (1, 1), (2, 1), (1, 1), (2, 1), (1, 1)).$$

This sequence satisfies the condition (8).

For a chart $\Gamma$, we denote by $V(\Gamma)$ the set of all the vertices of $\Gamma$, and by $S_\Gamma$ the subset of $V(\Gamma)$ consisting of the degree-1 vertices in $\mathrm{Int}(D^2)$. Let $v$ be a vertex of $\Gamma$. An edge $e$ from $v$ is called an *incoming edge* of $v$ if $e$ is oriented toward $v$ and an *outgoing edge* of $v$ if $e$ is oriented away from $v$. A degree-1 or 12 vertex of a chart is *positive* (resp.*negative*) if all the edges from the vertex is outgoing edge (resp. incoming edge) of the vertex. Note that each degree-1 vertex in $\mathrm{Int}(D^2)$ is negative by the definition of charts. Among the six edges from a degree-6 vertex $v$, three consecutive edges are incoming edges of $v$ and the other edges are outgoing edges. We call the middle edge in the three incoming or outgoing edges a *middle edge* and another edge a *non-middle edge.* An edge in a chart is called a $(d_1, d_2)$-*edge* if its end points are a degree-$d_1$ vertex and a degree-$d_2$ vertex, where $d_1, d_2 \in \{1, 6, 12\}$ and $d_1 \leq d_2$. An edge in a chart is called a $(\partial, d)$-*edge* if one of its end points is in $\partial D^2$ and the other is degree-$d$ vertex, where $d \in \{1, 6, 12\}$, and we call an edge whose two end points are in $\partial D^2$ a $(\partial, \partial)$-*edge.* A $(\partial, *)$-edge is called a *boundary edge*, where $* \in \{1, 6, 12, \partial\}$. Let $((i_1, \varepsilon_1), \ldots, (i_n, \varepsilon_n))$ be a boundary sequence of $\Gamma$. A union of six vertices and edges which correspond to a subsequence $((i, \varepsilon), (j, \varepsilon), (i, \varepsilon), (j, \varepsilon), (i, \varepsilon), (j, \varepsilon))$ is called a *boundary comb* of $\Gamma$.

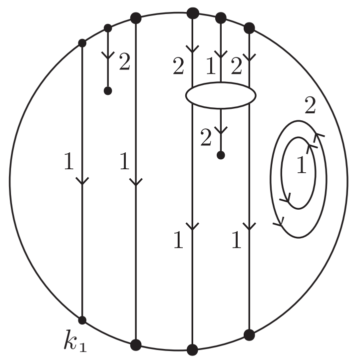


Figure 3.2.2: An example of a chart.

Let $\Gamma$ be a chart in $D^2$. A path $\eta : [0,1] \to D^2$ is said to be *in general position* with respect to $\Gamma$ if $\eta([0,1])$ is away from any vertices of $\Gamma$ and intersects edges of $\Gamma$ transversely. We put $\eta([0,1]) \cap \Gamma = \{p_1, \ldots, p_n\}$. Assume that $p_1, \ldots, p_n$ appear in this order when we go along $\eta$ from $\eta(0)$ to $\eta(1)$. We denote by $k_i \in \{1,2\}$ a label of the edge of $\Gamma$ which goes through $p_i$ and by $\varepsilon_i$ the sign of the intersection between $\eta$ and $\Gamma$ at $p_i$. We call a word $w_\Gamma(\eta) = X_{k_1}^{\varepsilon_1} \cdot \cdots \cdot X_{k_n}^{\varepsilon_n}$ the *intersection word* of $\eta$ with respect to $\Gamma$. We regard this word as an element of $SL(2,\mathbb{Z})$ by identifying the letters $X_1, X_2$ with the matrices defined in section 3.1.

**Definition 3.2.3.** Let $\Gamma$ be a chart in $D^2$. We take a point $p_0 \in D^2 \setminus V(\Gamma)$. We define a homomorphism

$$\rho_\Gamma : \pi_1(D^2 \setminus S_\Gamma, p_0) \to SL(2,\mathbb{Z})$$

in the following way: for an element $\xi \in \pi_1(D^2 \setminus S_\Gamma, p_0)$, we choose a representative path$\eta$ : $[0,1] \to D^2 \setminus S_\Gamma$ of $\xi$ which is in general position with respect to $\Gamma$. Then we put $\rho_\Gamma(\xi) = w_\Gamma(\eta)$. We call the homomorphism $\rho_\Gamma$ a *monodromy representation associated with* $\Gamma$. We can prove $\rho_\Gamma$ is well-defined by the same way as in the proof of Lemma 12 of [18].

Since the monodromy along the boundary of the highest side of a genus-1 SBLF preserves a vanishing cycle of folds, we can assume that the monodromy is equal to $\pm X_1^m$ for some integer $m$ after changing an identification of the reference fiber with $T^2$. Thus, the following lemma can be proved in the same way as in the proof of Theorem 15 of [18].

**Lemma 3.2.4.** *For any genus-1 SBLF $f : X \to S^2$, there exists a chart $\Gamma$ in $D^2$ such that the monodromy representation of the highest side of $f$ is equal to the monodromy representation associated with $\Gamma$ up to equivalence.*

We next introduce several moves of charts which do not change the associated monodromy representations. The following lemma can be obtained in a similar way to the proof of Lemma 16 in [18].

**Lemma 3.2.5.** *Let $\Gamma_1$ and $\Gamma_2$ be charts and $E \subset D^2$ a 2-disk. We assume that $E$ contains no degree-1 vertices of $\Gamma_1$ and $\Gamma_2$ in $\mathrm{Int}(D^2)$, that $\Gamma_1$ coincides with $\Gamma_2$ outside of $E$ and that the complement $D^2 \setminus E$ is path connected. Then, the monodromy representation associated with $\Gamma_1$ is equal to that associated with $\Gamma_2$.*

**Definition 3.2.6.** Suppose that two charts $\Gamma_1$ and $\Gamma_2$ are in the situation of Lemma 3.2.5. We say that $\Gamma_1$ is obtained from $\Gamma_2$ by a *CI-move* in $E$. A typical CI-move is described in Figure 3.2.3, which is called a *channel change*.

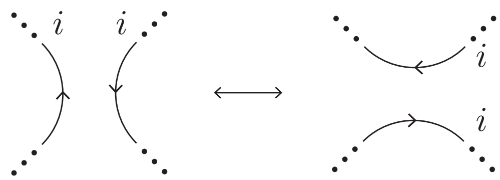


Figure 3.2.3: a channel change

**Lemma 3.2.7.** *Let $\Gamma_1$ and $\Gamma_2$ be charts and $E \subset D^2$ a 2-disk. We assume that $\Gamma_1$ is different from $\Gamma_2$ in $E$ as described in Figure 3.2.4, that $\Gamma_1$ coincides with $\Gamma_2$ outside of $E$ and that the complement $D^2 \setminus E$ is path connected. Then, the monodromy representation associated with $\Gamma_1$ is equal to that associated with $\Gamma_2$ up to equivalence.*

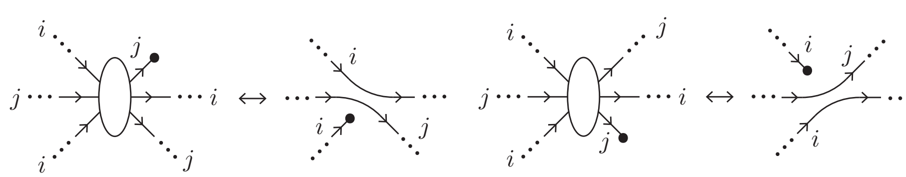

Figure 3.2.4: CII-moves

We omit a proof of Lemma 3.2.7 since it is quite similar to that of Lemma 18 of [18].

**Definition 3.2.8.** Suppose that two charts $\Gamma_1$ and $\Gamma_2$ are in the situation of Lemma 3.2.7. We say that $\Gamma_1$ is obtained from $\Gamma_2$ by a *CII-move* in $E$.

By a *C-move*, we mean a CI-move, a CII-move or an isotopic deformation in $D^2$. Two charts are said to be *C-move equivalent* if one can be obtained from the other by successive application of C-moves. Note that monodromy representations associated with C-move equivalent charts are equivalent by Lemma 3.2.5 and Lemma 3.2.7.

**Lemma 3.2.9.** *Let $\Gamma$ be a chart. By successive application of C-moves, we can change $\Gamma$ into a chart which has no degree-*12 *vertices.*

*Proof.* We choose a decomposition of the boundary sequence of $\Gamma$ into the unit subsequences. Let $v_1$ and $v_2$ be consecutive vertices in $\partial D^2$ which are not contained in the same boundary comb. We denote by $S$ the connected component of $\partial D^2 \setminus (\partial D^2 \cap \Gamma)$ between $v_1$ and $v_2$. We can move all the degree-12 vertices in $\Gamma$ into a region of $\partial D^2 \setminus \Gamma$ containing $S$ by using CI-moves illustrated in Figure 12 of [18]. By the CI-move illustrated in Figure 3.2.5, we can eliminate all the degree-12 vertices in $\Gamma$. This completes the proof of Lemma 3.2.9. □

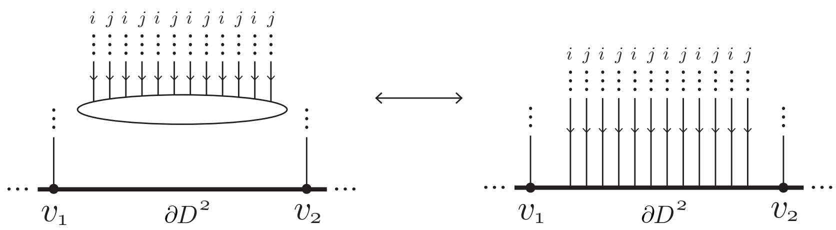


Figure 3.2.5: CI-move used to get rid of degree-12 vertices. The bold lines represent $\partial D^2$.

**Lemma 3.2.10.** *Let $\Gamma$ be a chart. By successive application of C-moves, we can change $\Gamma$ into a chart $\Gamma'$ such that each $(1,6)$-edge $e$ in $\Gamma'$ satisfies the following conditions:*

*(i) $e$ is a middle edge;*

*(ii) the label of $e$ is $2$;*

*(iii) let $K$ be the connected component of $D^2 \setminus \Gamma'$ whose closure contains $e$. The set $K \cap \partial D^2$ is not empty.*

The idea of the proof of Lemma 3.2.10 is similar to that of the proof of Lemma 22 in [18]. However, the two proofs are slightly different because of the difference of the definition of charts. Thus, we give the full proof below.

*Proof.* Let $n(\Gamma)$ be the sum of the number of degree-6 vertices and the number of $(1,6)$-edges in $\Gamma$. The proof proceeds by induction on $n(\Gamma)$.

If $n(\Gamma) = 0$, the conclusion of Lemma 3.2.10 holds since $\Gamma$ has no $(1,6)$-edges. We assume that $n(\Gamma) > 0$ and there exists a $(1,6)$-edge which does not satisfy at least one of the conditions *(i)*, *(ii)* or *(iii)* of Lemma 3.2.10.

*Case.1*: Suppose that $\Gamma$ has a non-middle $(1,6)$-edge. Let $v$ be a degree-6 vertex which is an end point of a $(1,6)$-edge. We can apply a CII-move around $v$ and eliminate this vertex. The number $n(\Gamma)$ decreases and the conclusion holds by the induction hypothesis.

*Case.2*: Suppose that $\Gamma$ has a middle $(1,6)$-edge $e$ whose label is 1. Let $v_0$ and $v_1$ the end points of $e$ whose degrees are 1 and 6, respectively. We denote by $K$ the connected component of $D^2 \setminus \Gamma$ whose closure contains $v_0$, $v_1$ and $e$. We take a sequence $f_1, \ldots, f_m$ of edges of $\Gamma$ with signs as in the proof of Lemma 22 of [18]. For each $f_i$, we take a letter $w(f_i) = X_k^{\varepsilon}$, where $k$ is equal to the label of the edge $f_i$ and $\varepsilon$ is equal to the sign of $f_i$. Note that both $f_1$ and $f_m$ are equal to $e$ and the sign of $f_1$ is negative, while the sign of $f_m$ is positive, since the vertex $v_0$ is negative.

*Case.2.1*: There exists a consecutive pair $f_i$ and $f_{i+1}$ such that two edges share a vertex and

$$(w(f_i), w(f_{i+1})) = (X_1^{-1}, X_2^{-1}).$$

*Case.2.2*: There exists a consecutive pair $f_i$ and $f_{i+1}$ such that two edges share a vertex and

$$(w(f_i), w(f_{i+1})) = (X_2, X_1).$$

*Case.2.3*: The set $K \cap \partial D^2$ is empty.

If one of the above three cases occurs, then the conclusion holds by the same argument as that in Lemma 22 of [18].

*Case.2.4*: Suppose that $K \cap \partial D^2$ is not empty. Then one of the edges $f_1, \ldots, f_m$ is a boundary edge. By Cases 2.1 and 2.2, we can assume that $(w(f_i), w(f_{i+1}))$ is not equal to either of the subsequences $(X_1^{-1}, X_2^{-1})$ and $(X_2, X_1)$ if $f_i$ and $f_{i+1}$ share a vertex. Let $f_k$ be a boundary edge with the smallest index. By the assumption above, $w(f_k)$ is equal to either $X_1^{-1}$ or $X_2$. If $w(f_k) = X_1^{-1}$, we can decrease the number of $(1,6)$-edges by applying C-moves illustrated in Figure 3.2.6. Thus, the conclusion holds by the induction hypothesis. Suppose

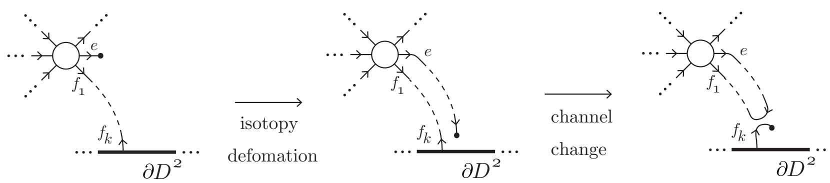


Figure 3.2.6: The bold line in the figure describes $\partial D^2$.

that $w(f_k) = X_2$. One of $f_{k+1}, \ldots, f_m$ is a boundary edge but not a $(\partial, 1)$-edge. Let $f_l$ be such an edge with the smallest index.

*Case.2.4.1*: Suppose that $w(f_l) = X_1$. Then we can decrease the number of $(1,6)$-edges by applying C-moves illustrated in Figure3.2.7 and the conclusion holds by the induction hypothesis.

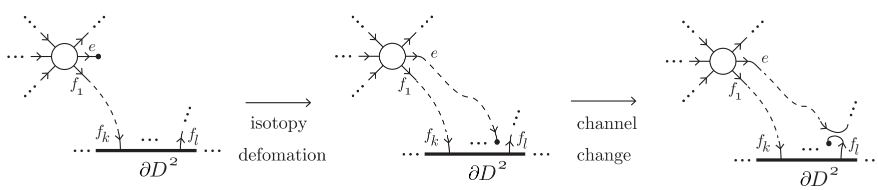


Figure 3.2.7:

*Case. 2.4.2*: Suppose that $w(f_l) = X_2$. We fix a decomposition of the boundary sequence of $\Gamma$ into the unit subsequences. It is easy to see that a boundary comb which contains $f_k$ is distinct from a boundary comb which contains $f_l$. Thus, we can apply C-moves as shown in Figure 3.2.8 and the conclusion holds by induction.

*Case.2.4.3*: Suppose that $w(f_l) = X_2^{-1}$. If both $f_k$ and $f_l$ were contained in a same boundary comb, there would be at least one $(\partial, 1)$-edge between $f_k$ and $f_l$. However, all the degree-1 edges are negative. This contradiction says that a boundary comb that contains $f_k$ is distinct from a boundary comb that contains $f_l$. Thus, we can apply C-moves similar to that we used in Case.2.4.2 and the conclusion holds by induction hypothesis.

*Case.2.4.4*: Suppose that $w(f_l) = X_1^{-1}$. If each $f_{l+1}, \ldots, f_m$ were not a boundary edge, then $(w(f_l), \ldots, w(f_m))$ would be equal to $(X_1^{-1}, X_2, X_1^{-1}, X_2, \ldots)$. This contradicts $w(f_m) = X_1$. Thus, at least one of $f_{l+1}, \ldots, f_m$ is a boundary edge. Let $f_{k'}$ be such an edge with the smallest index. The word $w(f_{k'})$ is equal to either $X_1^{-1}$ or $X_2$. If $w(f_{k'}) = X_1^{-1}$, the conclusion holds

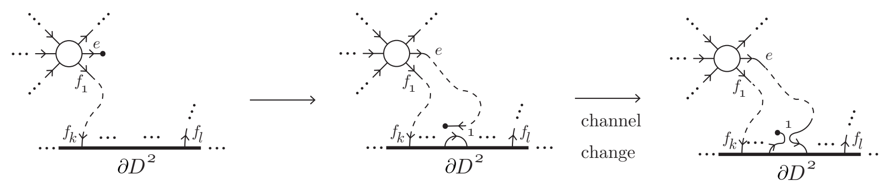


Figure 3.2.8: We first apply CI-move between the two boundary comb which contain $f_k$ and $f_l$, respectively, and we obtain a new 1-labeled $(\partial, \partial)$-edge. Then we move $v_0$ near this edge by isotopy deformation and apply a channel change.

by the above argument. If $w(f_{k'}) = X_2$, one of four cases above occurs for $f_{k'}$. When one of the former three cases occurs, the conclusion holds by the same argument. When Case.2.4.4 occurs for $f_{k'}$, we can take $f_{k''}$ again as we take $f_{k'}$. We can repeat the above argument and the conclusion holds since $m$ is finite.

*Case.3*: Suppose that $\Gamma$ has a middle $(1, 6)$-edge whose label is 2 which does not satisfy the condition *(iii)* in Lemma 3.2.10. We define $K$ as we defined in Case.2. By the assumption, $K \cap \partial D^2$ is empty. Thus, we can prove the conclusion by the same argument as that in Cases 2.1, 2.2 and 2.3.

Combining the conclusions of Cases.1, 2 and 3, we complete the proof of Lemma 3.2.10. □

*Proof of Theorem 3.2.1.* By Lemma 3.2.4, we can take a chart $\Gamma$ such that the associated monodromy representation $\rho_\Gamma$ is equal to the monodromy representation of the higher side of $f$ up to inner automorphisms of $SL(2, \mathbb{Z})$. We first remove all the degree-12 vertices in $\Gamma$ by applying Lemma 3.2.9. By applying Lemma 3.2.10, we change the chart $\Gamma$ into a chart such that all the $(1, 6)$-edges satisfy conditions *(i)*, *(ii)* and *(iii)* in Lemma 3.2.10. In the process of the proof of Lemma 3.2.10, the number of degree-12 vertices does not increase. Thus, the chart obtained by the above process has no degree-12 vertices. Let $\{v_1, \ldots, v_m\}$ be the set of degree-1 vertices of $\Gamma$ in $\partial D^2$. We choose the indices of $v_i$ so that $v_1, \ldots, v_m$ appear in this order when we go along $\partial D^2$ counterclockwise. We further assume that $v_1$ and $v_m$ are not contained in the same boundary comb. We denote by $e_i$ a boundary edge whose end point is $v_i$. We put $S_\Gamma = \{p_1, \ldots, p_n\}$. Let $K_i$ be a connected component of $D^2 \setminus \Gamma$ whose closure contains $p_i$. By the assumption on $\Gamma$, each $p_i$ is an end point of either $(1, 6)$-edge or $(\partial, 1)$-edge. For each $p_i$ which is an end point of $(1, 6)$-edge, we choose a connected component $E_i$ of $K_i \cap \partial D^2$. We denote the two points of $\partial E_i$ by $v_{k_i}$ and $v_{k_i+1}$, where $k_i \in \{1, \ldots, m\}$ and $v_{m+1} = v_1$. Let $V$ be a sufficiently small collar neighborhood of $\partial D^2$ in $D^2$ and $p_0$ a point in $V \cap K$, where $K$ is a connected component of $D^2 \setminus \Gamma$ whose closure contains a connected component of $\partial D^2 \setminus \{v_1, \ldots, v_m\}$ between $v_m$ and $v_1$. We take embedded paths $A_i$ $(i = 1, \ldots, n)$ in $D^2$ starting from $p_0$ as follows:

(a) if $i \neq j$, then $A_i \cap A_j = \{p_0\}$;

(b) if $p_i$ is an end point of a $(\partial, 1)$-edge $e_j$, then $A_i$ starts from $p_0$, travels in $V$ across the edges $e_1, \ldots, e_{j-1}$, goes into $K_i$ and ends at $p_i$;

(c) if $p_i$ is an end point of a $(1,6)$-edge, then $A_i$ starts from $p_0$, travels in $V$ across the edges $e_1, \dots, e_{k_i-1}$, goes into $K_i$ and ends at $p_i$.

For example, the paths $A_1, \dots, A_n$ are as shown in Figure3.2.9 for the charts described in Figure3.2.2. Let $a_i$ be an element of $\pi_1(D^2 \setminus S_\Gamma, p_0)$ which is represented by a curve obtained

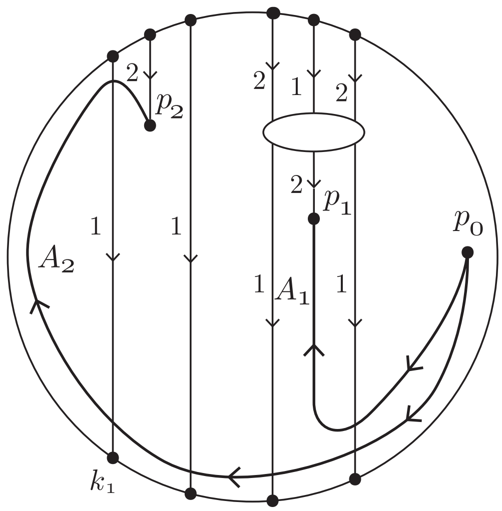


Figure 3.2.9: Examples of paths $A_1, \dots, A_n$ determined by the condition (a) and the constructions (b) and (c).

by connecting counterclockwise circle around $p_i$ to the base point $p_0$ by using $A_i$. It is sufficient to prove that each $\rho_\Gamma(a_i)$ is equal to either $X_1$ or $X_1^{-n} X_2 X_1^n$, where $n$ is an integer.

*Case.1*: Suppose that $p_i$ is an end point of $(1,6)$-edge and $e_{k_i}$ is not contained in a boundary comb. Then the intersection word of $A_i$ is equal to $X_1^n$. Thus, $\rho_\Gamma(a_i)$ is equal to $X_1$ if the label of the $(1,6)$-edge is 1 and $X_1^n X_2 X_1^{-n}$ if the label of the $(1,6)$-edge is 2.

*Case.2*: Suppose that $p_i$ is an end point of $(\partial, 1)$-edge and the edge is not contained in a boundary comb. Then the intersection word of $A_i$ is equal to $X_1^n$ and the conclusion holds.

*Case.3*: Suppose that $p_i$ is an end point of $(1,6)$-edge and $e_{k_i}$ is contained in a boundary comb. Let $e_l$ and $e_{l+6}$ be two edges at the end of the boundary comb which contains $e_{k_i}$. Then one of 24 cases illustrated in Figure3.2.10 occurs.

The intersection word of a path which starts from $p_0$, travels in $V$ across the edges $e_1, \dots, e_{l-1}$, ends near the boundary comb is equal to $X_1^n$, where $n$ is an integer. Since the label of the $(1,6)$-edge which contains $p_i$ as an end point is 2, $\rho_\Gamma(a_i)$ is calculated as follows:

$$\rho_\Gamma(a_i) = \begin{cases} X_1^{n+1} X_2 X_1^{-n-1} & \text{(if (1), (2), (22) or (23) occurs),} \\ X_1^{n-1} X_2 X_1^{-n+1} & \text{(if (5), (6), (10), (11), (13) or (14) occurs),} \\ X_1^n X_2 X_1^n & \text{(if (7), (12), (17), (18), (19) or (24) occurs),} \\ X_1 & \text{(otherwise).} \end{cases}$$

For each case, the conclusion holds.

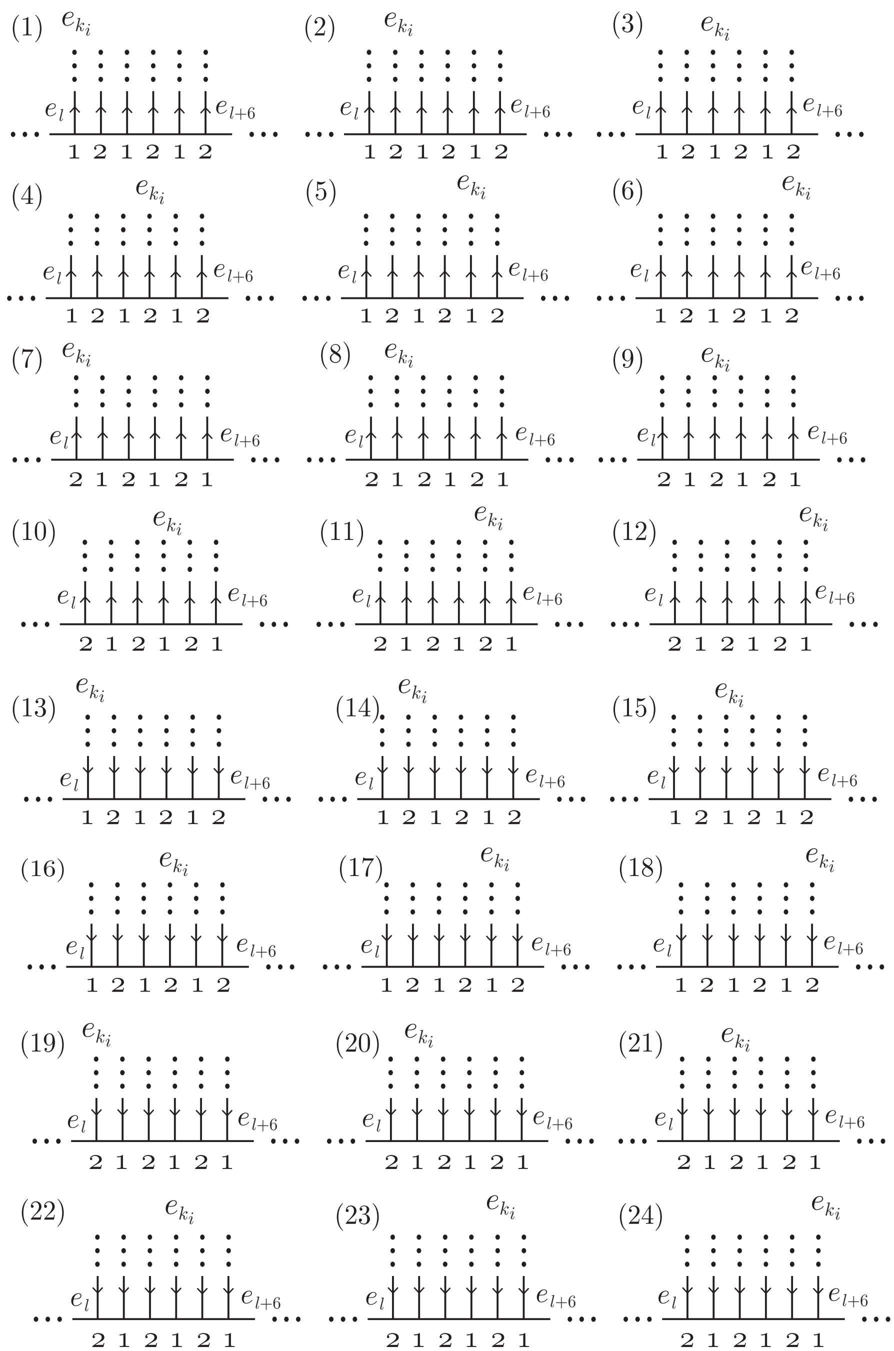

Figure 3.2.10: 24 cases about $e_{k_i}$ and the boundary comb containing $e_{k_i}$.

*Case.4*: Suppose that $p_i$ is an end point of $(\partial, 1)$-edge $e_j$ and $e_j$ is contained in a boundary comb. Let $e_l$ and $e_{l+6}$ be two edges at the end of the boundary comb which contains $e_j$. Since the degree-1 vertex $p_i$ is negative, one of 12 cases illustrated in Figure3.2.11 occurs. We assume that the intersection word of a path which starts from $p_0$, travels in $V$ across the edges $e_1, \dots, e_{l-1}$, ends near the boundary comb is equal to $X_1^n$, where $n$ is an integer. By using the relation $X_1X_2X_1X_2^{-1}X_1^{-1}X_2^{-1} = (X_1X_2)^6 = E$, $\rho_\Gamma(a_i)$ is calculated as follows:

$$\rho_\Gamma(a_i) = \begin{cases} X_1 & \text{(if (1), (4), (9) or (12) occurs),} \\ X_1^{n+1}X_2X_1^{-n-1} & \text{(if (2) or (5) occurs),} \\ X_1^nX_2X_1^{-n} & \text{(if (3), (6), (7) or (10) occurs),} \\ X_1^{n-1}X_2X_1^{-n+1} & \text{(otherwise).} \end{cases}$$

For each cases, the conclusion holds.

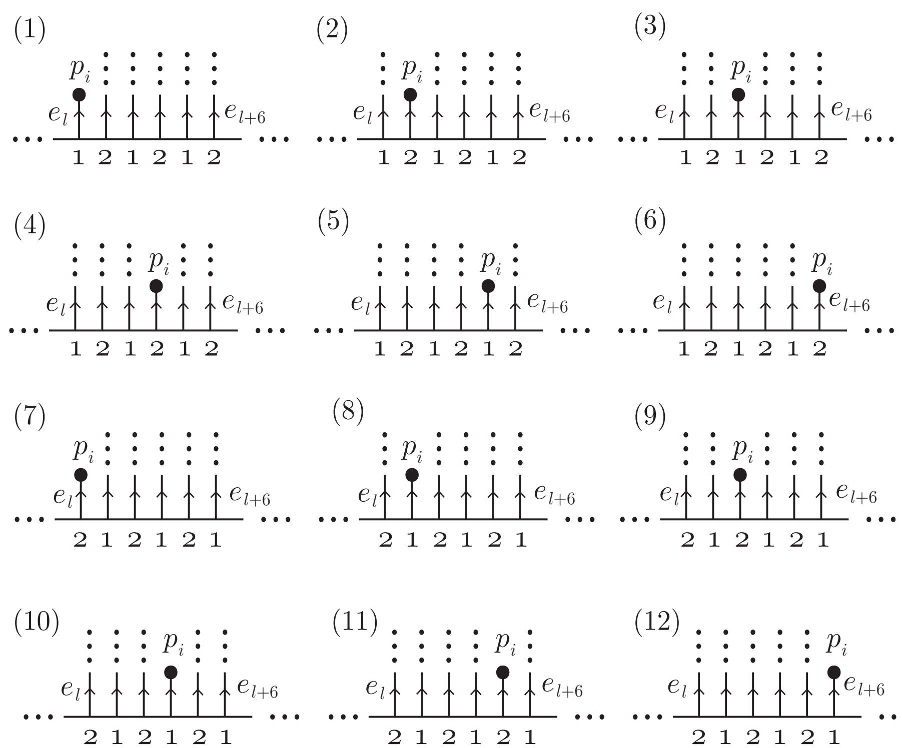

Figure 3.2.11: 12 cases about $e_j$ and the boundary comb containing $e_j$.

Combining the conclusions we obtain in Cases.1, 2, 3 and 4, we complete the proof of Theorem 3.2.1. □

## 3.3 Examples of genus-1 simplified broken Lefschetz fibrations

For a sequence $W = (c; c_1, \dots, c_n)$ of simple closed curves in $T^2$, we denote by $w(W)$ a product $t_{c_1} \cdot \dots \cdot t_{c_n}$ of Dehn twists. By Corollary 2.6.5, if a sequence $W = (c; c_1, \dots, c_n)$ satisfies the condition $w(W) \in \mathrm{Ker}(\Phi_c)$, there exists a genus-1 SBLF $f : X \to S^2$ such that a Hurwitz cycle system of $f$ coincides with the sequence $W$. Furthermore, the condition $w(W) \in \mathrm{Ker}(\Phi_c)$ is equivalent to the condition $t_{c_1} \cdot \dots \cdot t_{c_n} \in \mathcal{M}_1(c)$ (or the condition $t_c w(W) t_c^{-1} = w(W)$) since the mapping class group of $S^2$ is trivial. In this section, we give some examples of sequences of simple closed curves in $T^2$ which satisfy the condition above and determine what 4-manifolds may admit genus-1 SBLFs whose Hurwitz cycle systems coincide with these examples.

**Proposition 3.3.1.** *We define sequences $S_r$ and $T_s$ of simple closed curves as follows:*

$$\begin{aligned} S_r &= (\alpha; \alpha, \dots, \alpha) \quad (r+1\ \alpha\text{'s are contained}), \\ T_s &= (\alpha; \beta + n_1\alpha, \dots, \beta + n_s\alpha), \end{aligned}$$

*where $n_1 = 2s - 3$, $n_s = -2s + 3$ and $n_i = 2s - 6 + 4(i-1)$ $(i = 2, \dots, s-1)$. Then $t_\alpha w(S_r) t_{\alpha^{-1}} = w(S_r)$ and $t_\alpha w(T_s) t_\alpha^{-1} = w(T_s)$. In particular, these sequences are Hurwitz systems of some genus-1 SBLF.*

*Proof.* It is obvious that $w(S_r)$ is equal to ${X_1}^r$ and in particular the statement for $w(S_r)$ holds. We prove $w(T_s) = (-1)^{s+1} X_1^{-5s+6}$ by induction on $s$. Since $(X_1X_2)^3 = -E$ and $X_1X_2X_1 = X_2X_1X_2$, we obtain:

$$\begin{aligned} X_2{X_1}^2X_2{X_1}^2 &= X_2X_1(X_2X_1X_2)X_1 \\ &= -E. \end{aligned}$$

Thus, $w(T_2)$ and $w(T_3)$ are computed as follows:

$$\begin{aligned} w(T_2) &= (X_1^{-1}X_2X_1)(X_1X_2X_1^{-1}) \\ &= X_1^{-1}(X_2{X_1}^2X_2)X_1^{-1} \\ &= X_1^{-1}(-X_1^{-2})X_1^{-1} \\ &= -X_1^{-4}. \\ w(T_3) &= (X_1^{-3}X_2{X_1}^3)X_2({X_1}^3X_2X_1^{-3}) \\ &= X_1^{-3}X_2X_1({X_1}^2X_2{X_1}^2)X_1X_2X_1^{-3} \\ &= X_1^{-3}X_2X_1(-X_2^{-1})X_1X_2X_1^{-3} \\ &= -X_1^{-3}(X_2X_1X_2^{-1})X_1X_2X_1^{-3} \\ &= -X_1^{-3}(X_1^{-1}X_2X_1)X_1X_2X_1^{-3} \\ &= -X_1^{-4}(X_2{X_1}^2X_2)X_1^{-3} \\ &= X_1^{-9}. \end{aligned}$$

By the definition of $T_s$, $w(T_s)$ is represented by $w(T_{s-2})$ as follows:

$$w(T_s) = (X_1^{-2s+3}X_2X_1^{2s-3})(X_1^{-2s+6}X_2X_1^{2s-6})(X_1^{-2s+7}X_2^{-1}X_1^{2s-7})w(T_{s-2})$$

$$\begin{aligned}
&(X_1^{2s-7}X_2^{-1}X_1^{-2s+7})(X_1^{2s-6}X_2X_1^{-2s+6})(X_1^{2s-3}X_2X_1^{-2s+3})\\
=&X_1^{-2s+3}X_2{X_1}^2(X_1X_2X_1)X_2^{-1}X_1^{2s-7}w(T_{s-2})\\
&X_1^{2s-7}X_2^{-1}(X_1X_2X_1){X_1}^2X_2X_1^{-2s+3}\\
=&X_1^{-2s+3}(X_2{X_1}^2X_2)X_1^{2s-6}w(T_{s-2})X_1^{2s-6}(X_2{X_1}^2X_2)X_1^{-2s+3}\\
=&X_1^{-2s+3}(-X_1^{-2})X_1^{2s-6}w(T_{s-2})X_1^{2s-6}(-X_1^{-2})X_1^{-2s+3}\\
=&X_1^{-5}w(T_{s-2})X_1^{-5}.
\end{aligned}$$

Thus the conclusion holds by the induction hypothesis. This completes the proof of Proposition 3.3.1. □

**Theorem 3.3.2.** *Let $f: X \to S^2$ be a genus-1 SBLF. Suppose that $W_f$ is equal to $S_r$. Then $X$ is diffeomorphic to one of the following 4-manifolds:*

*(1)* $\#r\overline{\mathbb{CP}^2}$*;*

*(2)* $L\#r\overline{\mathbb{CP}^2}$*;*

*(3)* $S^1 \times S^3 \# S \# r\overline{\mathbb{CP}^2}$*,*

*where $S$ is either of the manifolds $S^2 \times S^2$ and $S^2 \tilde{\times} S^2$ and $L$ is either of the manifolds $L_n$ and $L'_n$ (for some $n \geq 2$).*

Before proving Theorem 3.3.2, we review the definition and some properties of $L_n$ and $L'_n$. For more details, see [27]. Let $N_0$ and $N_1$ be 4-manifolds diffeomorphic to $D^2 \times T^2$. The boundaries of $N_0$ and $N_1$ are $\partial D^2 \times T^2$. Let $(t, x, y)$ be a coordinate of $\mathbb{R}^3$. We identify $\partial D^2 \times T^2$ with $\mathbb{R}^3/\mathbb{Z}^3$. The group $GL(3, \mathbb{Z})$ naturally acts on $\mathbb{R}^3$ and this action descends to an action on the lattice $\mathbb{Z}^3$. Thus, $GL(3, \mathbb{Z})$ acts on $\partial D^2 \times T^2$. For an element $A$ of $GL(3, \mathbb{Z})$, we denote by $f_A$ a self-diffeomorphism of $\partial D^2 \times T^2$ defined as follows:

$$f_A([t, x, y]) = [(t, x, y)^t A].$$

We define elements $A_n$ and $A'_n$ of $GL(3, \mathbb{Z})$ as follows:

$$A_n = \begin{pmatrix} 0 & 1 & 1 \\ 0 & n & n-1 \\ 1 & n & 0 \end{pmatrix},\ A'_n = \begin{pmatrix} 0 & 1 & 1 \\ 0 & n & n-1 \\ 1 & n-1 & 0 \end{pmatrix}.$$

We denote by $S^1_x$ and $S^1_y$ circles with coordinates $x$ and $y$, respectively, and take an embedded ball $D^3$ in $D^2 \times S^1_y$. The manifold $D^3 \times S^1_x$ is contained in $(D^2 \times S^1_y) \times S^1_x = D^2 \times T^2 = N_0$. We define $L_n$ and $L'_n$ as follows:

$$\begin{aligned}
L_n &= S^2 \times D^2 \cup_{\mathrm{id}} (N_0 \setminus (\mathrm{Int}\, D^3 \times S^1)) \cup_{f_{A_n}} N_1,\\
L'_n &= S^2 \times D^2 \cup_{\mathrm{id}} (N_0 \setminus (\mathrm{Int}\, D^3 \times S^1)) \cup_{f_{A'_n}} N_1,
\end{aligned}$$

where we identify $S^{m-1}$ with $\partial D^m$.

**Remark 3.3.3.** The original definitions of $L_n$ and $L'_n$ are different from the definition given above. However, both two definitions are equivalent (c.f. Lemma V.7 in [27]). Note that

Auroux, Donaldson and Katzarkov also mentioned these manifolds in Example 1 of section 8.2 of [1], although they did not state that the manifolds they gave in [1] are the manifolds $L_n$ and $L'_n$. Indeed, $N_1$ (resp. $N_0 \setminus (\mathrm{int}B^3 \times S^1)$, $D^2 \times S^2$) in our paper corresponds to $X_-$ (resp. $W$, $X_+$) in [1].

We next take handle decompositions of $L_n$ and $L'_n$. Since $N_1$ is $D^2 \times T^2$, we can draw a Kirby diagram of $N_1$ as in the left side of Figure 3.3.1. The coordinate $(t, x, y)$ is also described as in Figure 3.3.1. The manifold $B^3 \times S^1$ has a handle decomposition consisting of

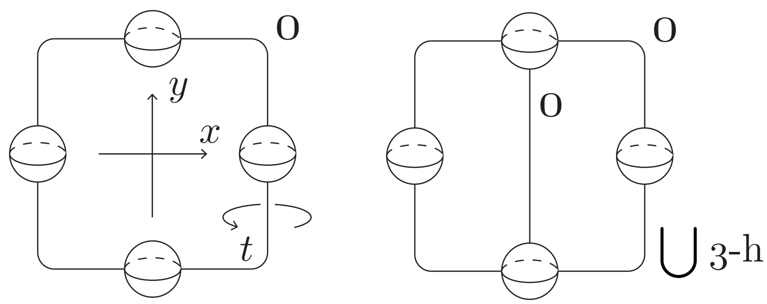


Figure 3.3.1: The left diagram is a Kirby diagram of $N_1$, while the right one is a diagram of $(N_0 \setminus (\mathrm{int}B^3 \times S^1)) \cup_{f_{A_n}} N_1$, which is also a diagram of $(N_0 \setminus (\mathrm{int}B^3 \times S^1)) \cup_{f_{A'_n}} N_1$. $t$ represents the coordinate of $\partial D^2$, while $x$ and $y$ represent the coordinates of $T^2$.

a 0-handle and a 1-handle. Thus, we can decompose $N_0 \setminus (\mathrm{int}B^3 \times S^1)$ as follows:

$$N_0 \setminus (\mathrm{int}B^3 \times S^1) = \partial N_0 \times I \cup (\text{2-handle}) \cup (\text{3-handle}).$$

Let $C_1 \subset \partial N_0$ be an attaching circle of the 2-handle. By the construction of the decomposition, we obtain:

$$C_1 = \{[t, 0, 0] \in \partial N_0 | t \in [0, 1]\}.$$

Since $f_{A_n}([t, 0, 0]) = f_{A'_n}([t, 0, 0]) = [0, 0, t]$, an attaching circle of the 2-handle is in a regular fiber and along $y$-axis in the diagram of $N_1$. Since $f_{A_n}([t, 0, \varepsilon]) = f_{A'_n}([t, 0, \varepsilon]) = [\varepsilon, (n-1)\varepsilon, t]$ for sufficiently small $\varepsilon > 0$, the framing of the 2-handle is along a regular fiber. Thus, we can draw Kirby diagrams of $(N_0 \setminus (\mathrm{int}B^3 \times S^1)) \cup_{f_{A_n}} N_1$ and $(N_0 \setminus (\mathrm{int}B^3 \times S^1)) \cup_{f_{A'_n}} N_1$ as shown in the right side of Figure 3.3.1. We can decompose $D^2 \times S^2$ as follows:

$$D^2 \times S^2 = \partial D^2 \times S^2 \times I \cup (\text{2-handle}) \cup (\text{4-handle}).$$

Let $C_2 \subset \partial N_0$ be the image under $h$ of the attaching circle of the 2-handle of $D^2 \times S^2$. After moving $C_2$ by isotopy in $N_0$, we obtain:

$$C_2 = \{[0, t, \delta] \in \partial N_0 | t \in [0, 1]\},$$

where $\delta > 0$ is sufficiently small. The framing of the 2-handle is $\{[0, t, \delta'] \in \partial N_0 | t \in [0, 1]\}$, where $\delta' > \delta$ is sufficiently small. Since $f_{A_n}([0, t, \delta]) = [t + \delta, nt + (n-1)\delta, nt]$, we can describe the attaching circle of the 2-handle of $D^2 \times S^2$ contained in $L_n$ and the knot representing the framing of the 2-handle in the diagram described in Figure 3.3.1. Eventually, we can draw a

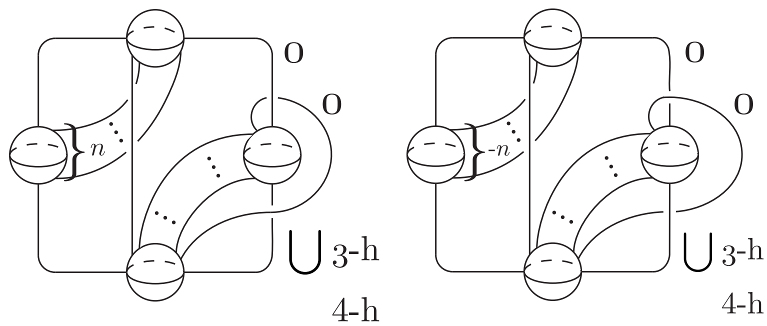


Figure 3.3.2: Left: A Kirby diagram of $L_n$ for $n \geq 0$. Right: A Kirby diagram of $L_n$ for $n < 0$.

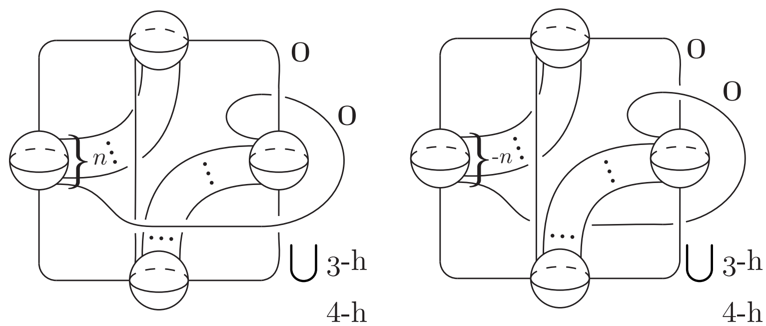


Figure 3.3.3: Left: A Kirby diagram of $L'_n$ for $n \geq 0$. Right: A Kirby diagram of $L'_n$ for $n < 0$.

Kirby diagram of $L_n$ as shown in Figure 3.3.2. Similarly, we can draw a Kirby diagram of $L'_n$ as shown in Figure 3.3.3. By the diagrams of $L_n$ and $L'_n$ described in Figure 3.3.2 and Figure 3.3.3, both $L_n$ and $L'_n$ admit genus-1 SBLFs without Lefschetz singularities. We can easily prove by Kirby calculus that $L_{-n}$ (resp. $L'_{-n}$) is diffeomorphic to $L_n$ (resp. $L'_n$).

*Proof of Theorem 3.3.2.* The higher side of $f$ is obtained by attaching $r$ 2-handles to a trivial $T^2$ bundle over $D^2$. Each attaching circle of the 2-handle is in a regular fiber and isotopic to a simple closed curve $\alpha$. Since $w(W_f) = {X_1}^r$, a 2-handle of a round 2-handle is attached along $\alpha$ in a regular fiber of the boundary of the higher side. We obtain $X$ by attaching a 2-handle and a 4-handle to the 4-manifold obtained by successive handle attachment to $D^2 \times T^2$. If the attaching circle of the 2-handle of $D^2 \times S^2$ goes through the 1-handle that the 2-handle of the round handle goes through, we can slide the 2-handle of $D^2 \times S^2$ to the 2-handle of the round handle so that the 2-handle of $D^2 \times S^2$ does not go through the 1-handle. Thus, a Kirby diagram of $X$ is one of the diagrams in Figure 3.3.4, where $n$ and $l$ are integers. It is obvious that the two 4-manifolds illustrated in Figure 3.3.4 are diffeomorphic to each other. We denote by $X_{n,l}$ the 4-manifold illustrated in Figure 3.3.4. Note that $l$ framed knot in Figure 3.3.4 represents a 2-handle of $D^2 \times S^2$ and the attachment of the lower side depends only on the parity of $l$. In particular, $X_{n,l}$ and $X_{n,l'}$ are diffeomorphic if $l \equiv l'$ (mod 2).

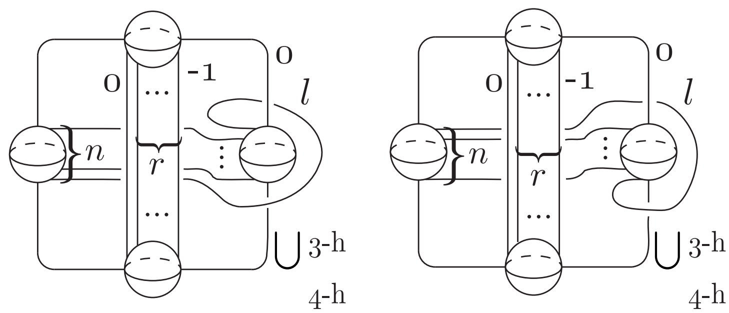


Figure 3.3.4: A Kirby diagram of a genus-1 SBLF with Hurwitz cycle system $S_r$. Framings of $r$ 2-handles parallel to the 2-handle of the round 2-handle are all $-1$.

We change a Kirby diagram of $X_{n,l}$ as shown in Figure 3.3.5. We first slide $r$ 2-handles representing Lefschetz singularities to the 2-handle of the round 2-handle. We next slide the 2-handle of $D^2 \times T^2$ to the 2-handle of the round 2-handle and move this 2-handle so that the attaching circle of the 2-handle does not go through 1-handles.

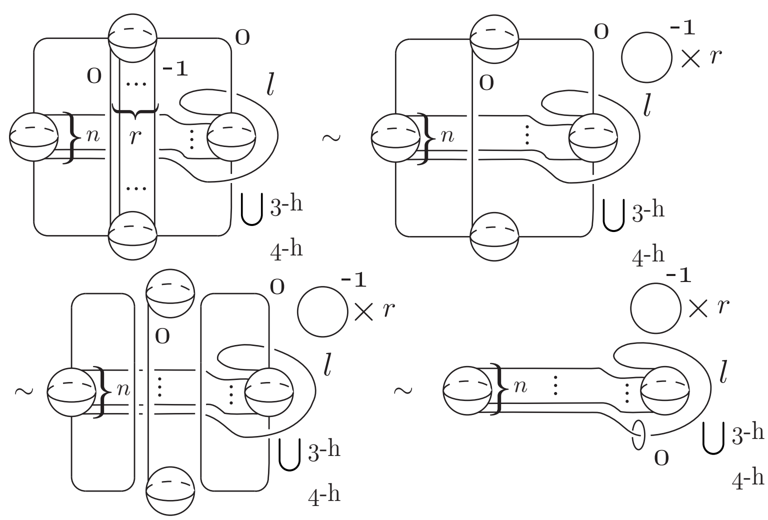


Figure 3.3.5: A Kirby diagram of $X_{n,l}$.

The diagram of $X_{0,l}$ consists of $r$ 2-handles with $(-1)$-framing, a 1-handle and the Hopf link such that the two components have 0 and $l$-framing, respectively. Thus, we obtain:

$$X_{0,l} = S^1 \times S^3 \# S \# r\overline{\mathbb{CP}^2},$$

where $S$ is equal to $S^2 \times S^2$ if $l$ is even and $S^2 \tilde{\times} S^2$ if $l$ is odd.

The diagram of $X_{1,l}$ has two canceling pairs of handles. By canceling these pairs, we can change the diagram of $X_{1,l}$ into the diagram which has only $r$ 2-handles with $(-1)$-framing. Thus, we obtain:

$$X_{1,l} = \# r\overline{\mathbb{CP}^2}.$$

We can change the diagrams of $L_n$ and $L'_n$ illustrated in Figure 3.3.2 and 3.3.3 as shown in Figure 3.3.6. The upper three diagrams in Figure 3.3.6 describe $L_n$, where $n \geq 0$. We first

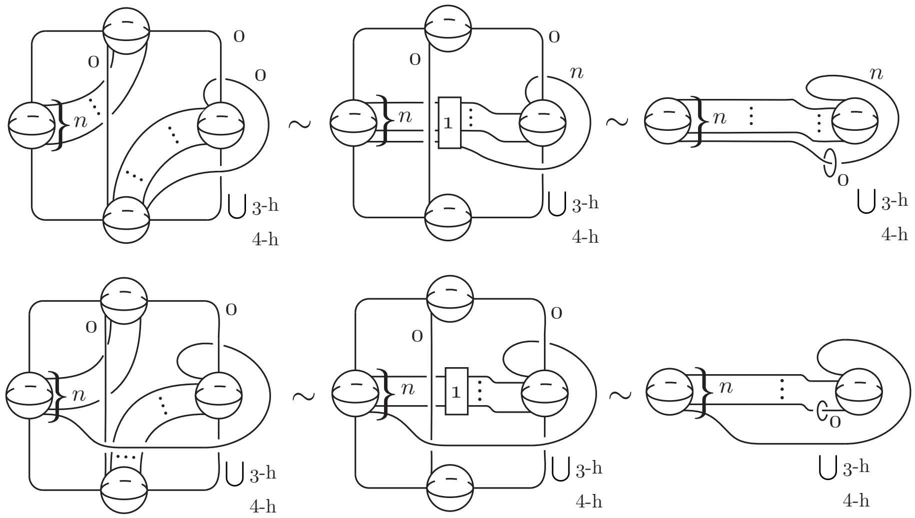


Figure 3.3.6: The upper three diagrams describe $L_n$, while the lower three diagrams describe $L'_n$.

slide 2-handle of $D^2 \times S^2$ to the 2-handle of $N_0$. Then we slide the 2-handle of $D^2 \times T^2$ to the 2-handle of $N_0$ and eliminate a canceling pair. The lower three diagrams in Figure 3.3.6 describe $L'_n$, where $n \geq 0$. We can apply the same move as above to the far left diagram and we obtain the far right diagram. Eventually, we obtain:

$$X_{n,l} = L\#r\overline{\mathbb{CP}^2},$$

where $L$ is either $L_n$ or $L'_n$. This completes the proof of Theorem 3.3.2. □

**Theorem 3.3.4.** *Let $f : X \to S^2$ be a genus-1 SBLF. Suppose that $W_f = T_s$. Then $X$ is diffeomorphic to $S\#(s-2)\mathbb{CP}^2$, where $S$ is either of the manifolds $S^2 \times S^2$ and $S^2\tilde{\times}S^2$.*

*Proof.* We prove the statement by induction on $s$. Let $X_s$ be a total space of genus-1 SBLF $f$ with $W_f = T_s$. We first look at the manifold $X_2$. We can draw a Kirby diagram of $X_2$ as shown in Figure 3.3.7. We slide the 2-handles representing Lefschetz singularities and the 2-handle of $D^2 \times T^2$ to the 2-handle of the round handle. Then we eliminate the obvious canceling pair and slide the $(-2)$-framed knot and the $l$-framed knot to the 0-framed knot. We can change the $l$-framed knot and the 0-framed meridian of this into the Hopf link by using the 0-framed meridian. We can obtain the last diagram of Figure 3.3.7 by canceling two pairs of handles. Eventually, we can prove that $X_2$ is diffeomorphic to $S^2 \times S^2$ if $m$ is even or $S^2\tilde{\times}S^2$ if $m$ is odd.

Suppose that $s$ is greater than or equal to 3. $f$ has the configuration of reference paths and the corresponding vanishing cycles as described in the far left of Figure 3.3.8. Since two vanishing cycles of folds determined by the dashed arcs in the far left of Figure 3.3.8 intersect at a single point transversely, we can apply fold merge to $f$ as described in the figure. We

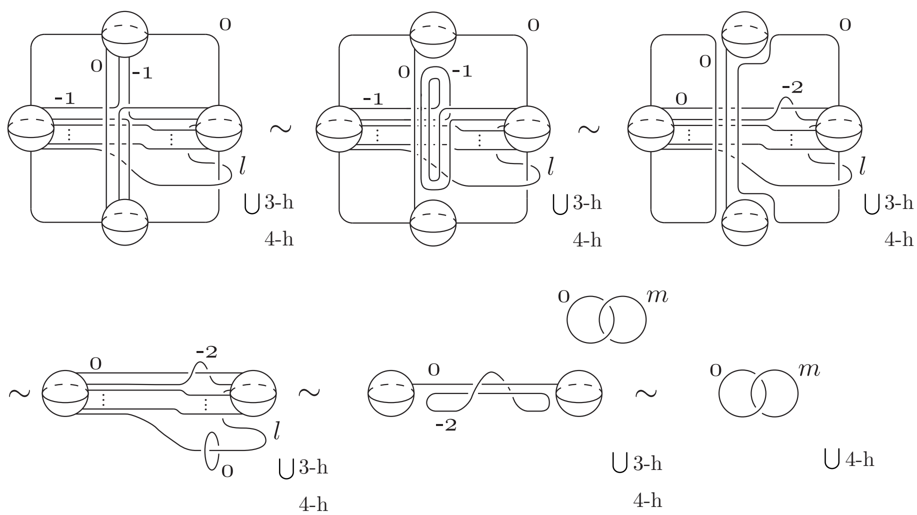


Figure 3.3.7: A diagram of a SBLF whose Hurwitz system is $T_2$.

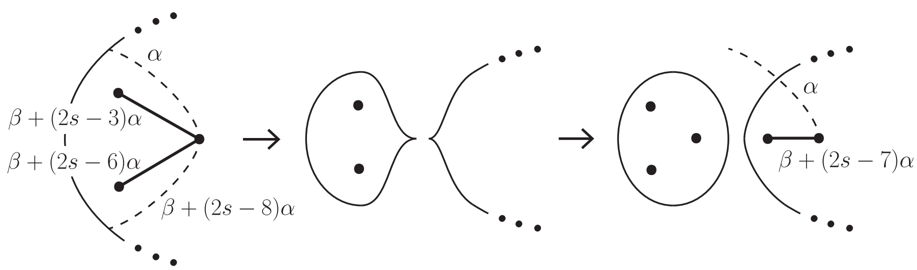


Figure 3.3.8: Left: configuration of reference paths. Right: configuration of the image of singularities after applying several homotopies. Four dots in this picture are the image of Lefschetz singularities.

can further apply unsink to $f$ and the configuration of the image of singularities of $f$ are changed as in Figure 3.3.8. It is easy to verify that we can apply wrinkle on the preimage of a disk which contains the circular image of folds and the three Lefschetz critical values in the circle. By application of wrinkle, the folds and three cusps are changed into an achiral Lefschetz singularity with null-homotopic vanishing cycle (the reader should refer to [21] or [6] for details of several homotopies in the above argument). Eventually, we can obtain a genus-1 SBLF $f' : X' \to S^2$ so that $X$ is diffeomorphic to $X'\#\mathbb{CP}^2$. Furthermore, a Hurwitz cycle system of $f'$ is as follows:

$$(\alpha; \beta + (2s-7)\alpha, \beta + (2s-10)\alpha, \beta + (2s-14)\alpha, \ldots, \beta + (-2s+6)\alpha, \beta + (-2s+3)\alpha).$$

This sequence is Hurwitz equivalent to the sequence $T_{s-1}$. Thus, the conclusion holds by induction hypothesis and this completes the proof of Theorem 3.3.4. □

We can obtain the following corollary by the same argument as in the proof of Theorem 3.3.4:

**Corollary 3.3.5.** *Let $f : X \to S^2$ be a genus-1 SBLF with non-empty folds. Suppose that $W_f$ is Hurwitz equivalent to the following sequence:*

$$(\alpha; \beta, \beta - 3\alpha, c_1, \ldots, c_l),$$

*where $c_i$ is an element in $H_1(T^2; \mathbb{Z})$. Then there exists a genus-1 SBLF $f' : X' \to S^2$ which satisfies the following conditions;*

1. *$W_{f'}$ is Hurwitz equivalent to $(\alpha; \beta - 4\alpha, c_1, \ldots, c_l)$,*
2. *$X$ is diffeomorphic to $X' \# \overline{\mathbb{CP}}^2$.*

## 3.4 Further properties of Hurwitz cycle systems

We define matrices $A$ and $B$ in $SL(2, \mathbb{Z})$ as follows:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}.$$

Both $X_1$ and $X_2$ are represented by $A$ and $B$ as follows:

$$X_1 = ABA\,,\; X_2 = BA^2.$$

Since $X_1$ and $X_2$ generate the group $SL(2, \mathbb{Z})$, the matrices $A$ and $B$ also generate $SL(2, \mathbb{Z})$. Let $a$, $b$, $x_1$ and $x_2$ be elements of $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{\pm E\}$ represented by $A$, $B$, $X_1$ and $X_2$, respectively, where we denote by $E$ the unit matrix. Then $PSL(2, \mathbb{Z})$ has the following finite presentation (see [22]):

$$PSL(2, \mathbb{Z}) = < a, b | a^3, b^2 > .$$

In particular, $PSL(2, \mathbb{Z})$ is isomorphic to the free product $\mathbb{Z}/3 * \mathbb{Z}/2$. The sequence $(w_1, \ldots, w_n)$ of elements of $PSL(2, \mathbb{Z})$ is called *reduced* if the set $\{w_i, w_{i+1}\}$ is equal to either of the sets $\{a, b\}$ and $\{a^2, b\}$ for each $i \in \{1, \ldots, n-1\}$.

**Lemma 3.4.1** ([22])**.** *For every element $g$ of $PSL(2, \mathbb{Z})$, there exists a reduced sequence $(w_1, \ldots, w_n)$ of $PSL(2, \mathbb{Z})$ such that $g = w_1 \cdot \cdots \cdot w_n$. Moreover, such a sequence is unique.*

Let $W = (\alpha; c_1, \ldots, c_l)$ be a Hurwitz cycle system of some genus-1 SBLF. By Theorem 3.2.1, we can assume that $W$ is equal to $(\alpha; S_r T(n_1, \ldots, n_s))$. The element $t_{\beta + n_i \alpha}$ is equal to $X_1^{-n_i} X_2 X_1^{n_i}$. The product $w(W) = t_{c_1} \cdot \cdots \cdot t_{c_l}$ is represented by $X_1$ and $X_2$ as follows:

$$w(W) = X_1^r X_1^{-n_1} X_2 X_1^{n_1 - n_2} X_2 \cdots X_1^{n_{s-1} - n_s} X_2 X_1^{n_s}.$$

By Corollary 2.4.2, $w(W)$ preserves the curve $\alpha$. In particular, $w(W)$ is equal to $t_\alpha^i (t_\alpha t_\beta)^{3j}$ for some integers $i, j \in \mathbb{Z}$. Since $(t_\alpha t_\beta)^{3j}$ is equal to $(-E)^j$, we can obtain the following relation in $PSL(2, \mathbb{Z})$:

$$x_2 x_1^{n_1 - n_2} x_2 \cdots x_2 x_1^{n_{s-1} - n_s} x_2 = x_1^k,$$

where $k$ is some integer.

**Lemma 3.4.2.** *Suppose that each $n_i - n_{i+1}$ is not equal to either 1, 2 or 3. Then $x_2 {x_1}^{n_1-n_2} \cdot \cdots \cdot {x_1}^{n_{s-1}-n_s} x_2$ is equal to $bS$ or $a^2ba^2bS$, where $S = w_1 \cdot \cdots \cdot w_k$ and $(w_1, \ldots, w_k)$ is a reduced sequence such that $w_1 = a$ or $a^2$.*

*Proof.* We prove this statement by induction on $s$.

We first look at the case $s = 2$. $x_2 {x_1}^{n_1-n_2} x_2$ is calculated as follows:

$$x_2 {x_1}^{n_1-n_2} x_2 = \begin{cases} ba^2 \cdot a(ba^2)^{n_1-n_2-1} ba \cdot ba^2 & (\text{if } n_1 - n_2 \geq 4), \\ ba^2 \cdot a^2(ba)^{-n_1+n_2-1} ba^2 \cdot ba^2 & (\text{if } n_1 - n_2 \leq 0), \end{cases}$$
$$= \begin{cases} a^2ba^2(ba^2)^{n_1-n_2-3} baba^2 & (\text{if } n_1 - n_2 \geq 4), \\ (ba)^{-n_1+n_2} ba^2ba^2 & (\text{if } n_1 - n_2 \leq 0). \end{cases}$$

Thus, the statement holds.

We then look at the general case. By the induction hypothesis, we obtain:

$$x_2 {x_1}^{n_2-n_3} \cdot \cdots \cdot {x_1}^{n_{s-1}-n_s} x_2 = bS \text{ or } a^2ba^2bS,$$

where $S$ is the product of a reduced sequence starting from $a$ or $a^2$. We denote by $S'$ the words above. We can calculate $x_2 {x_1}^{n_1-n_2}$ as follows:

$$x_2 {x_1}^{n_1-n_2} = \begin{cases} ba^2 \cdot a(ba^2)^{n_1-n_2-1} ba & (\text{if } n_1 - n_2 \geq 4), \\ ba^2 \cdot a^2(ba)^{-n_1+n_2-1} ba^2 & (\text{if } n_1 - n_2 \leq 0), \end{cases}$$
$$= \begin{cases} a^2ba^2(ba^2)^{n_1-n_2-4} ba^2ba & (\text{if } n_1 - n_2 \geq 4), \\ (ba)^{-n_1+n_2} ba^2 & (\text{if } n_1 - n_2 \leq 0), \end{cases}$$

Thus, we obtain:

$$x_2 {x_1}^{n_1-n_2} \cdot \cdots \cdot {x_1}^{n_{s-1}-n_s} x_2$$
$$= \begin{cases} a^2ba^2(ba^2)^{n_1-n_2-4} ba^2babS & (\text{if } n_1 - n_2 \geq 4 \text{ and } S' = bS), \\ a^2ba^2(ba^2)^{n_1-n_2-4} ba^2baa^2ba^2bS & (\text{if } n_1 - n_2 \geq 4 \text{ and } S' = a^2ba^2bS), \\ (ba)^{-n_1+n_2} ba^2bS & (\text{if } n_1 - n_2 \leq 0 \text{ and } S' = bS), \\ (ba)^{-n_1+n_2} ba^2a^2ba^2bS & (\text{if } n_1 - n_2 \leq 0 \text{ and } S' = a^2ba^2bS), \end{cases}$$
$$= \begin{cases} a^2ba^2(ba^2)^{n_1-n_2-4} ba^2babS & (\text{if } n_1 - n_2 \geq 4 \text{ and } S' = bS), \\ a^2ba^2(ba^2)^{n_1-n_2-4} babS & (\text{if } n_1 - n_2 \geq 4 \text{ and } S' = a^2ba^2bS), \\ (ba)^{-n_1+n_2} ba^2bS & (\text{if } n_1 - n_2 \leq 0 \text{ and } S' = bS), \\ (ba)^{-n_1+n_2} baba^2bS & (\text{if } n_1 - n_2 \leq 0 \text{ and } S' = a^2ba^2bS). \end{cases}$$

This completes the proof of Lemma 3.4.2. □

**Corollary 3.4.3.** *Assume $s$ is greater than 1. There exists an integer $i \in \{1, \ldots, s-1\}$ such that $n_i - n_{i+1}$ is equal to either 1, 2 or 3.*

*Proof.* Suppose that $n_i - n_{i+1}$ is not equal to either 1, 2 or 3 for any $i \in \{1, \ldots, s-1\}$. By Lemma 3.4.2, $w(W)$ is not trivial. The element $x_1^k$ is equal to $a(ba^2)^{k-1}ba$ if $k \geq 1$ or $a^2(ba)^{-k+1}$. This contradicts the condition $w(W) = x_1^k$ for some $k$. □

## 3.5 Classification

*Proof of Theorem 3.0.1.* We first prove that each 4-manifold in the statement of Theorem 3.0.1 admits a genus-1 SBLF. We proved that the manifolds $S$, $\#r\overline{\mathbb{CP}^2}$, $S^1 \times S^3 \# S \# r\overline{\mathbb{CP}^2}$ and $L\#r\overline{\mathbb{CP}^2}$ admit genus-1 SBLFs in the proof of Theorem 3.3.2. By Corollary 2.6.5, there exists a genus-1 SBLF $f : X(l,k) \to S^2$ with Hurwitz cycle system $(\alpha; S_{l-k-1}T_{k+1})$ for any $l \geq 3$ and $k \in [1, l-1]$. It is easy to verify that $X(l,k)$ is diffeomorphic to $\#k\mathbb{CP}^2\#(l-k)\overline{\mathbb{CP}^2}$. Thus, it is sufficient to construct a genus-1 SBLF on $\#kS^2 \times S^2$. A diagram in Figure 3.5.1 represents the total space of a genus-1 SBLF $f_k : X_k \to S^2$ whose Hurwitz cycle system is equal to $(\alpha; kT_2)$. It is easy to prove that the manifold $M_k$ is diffeomorphic to $\#kS^2 \times S^2$.

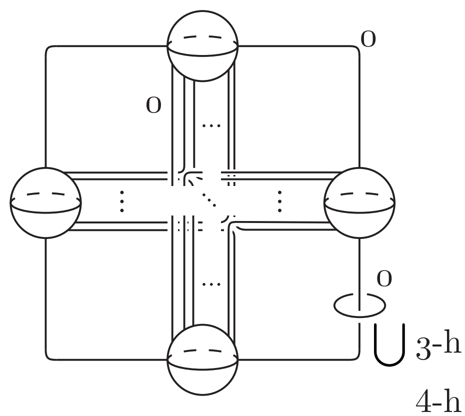


Figure 3.5.1: A Kirby diagram of $X_k$. All the 2-handles derived from Lefschetz singularities have $(-1)$-framing.

The details are left to the readers.

We next prove that the total space of a genus-1 SBLF $f$ is diffeomorphic to one of the manifolds in the statement of Theorem 3.0.1. Denote by $l$ the number of Lefschetz singularities of $f$. We will prove the statement by induction on $l$. The statement for the case $l = 0$ has already been proved in the proof of Theorem 3.3.2. We assume $l > 0$ and $W_f$ is equal to $(\alpha; S_rT(n_1, \ldots, n_s))$.

We first consider the case $X$ is not simply connected. In this case, it is easy to verify that $s = 0$ and therefore, there exists a genus-1 SBLF $f : X' \to S^2$ without Lefschetz singularities such that $X$ is diffeomorphic to $X'\#r\overline{\mathbb{CP}^2}$. Thus, we can deduce the conclusion from induction hypothesis.

We next consider the case $X$ is simply connected. If $r$ is not equal to 0, we can reduce the number of Lefschetz singularities by blowing down $X$ and the conclusion holds by induction hypothesis. Assume that $r$ is equal to 0 and $s = l$. By Lemma 3.4.2, there exists a number $i \in \{1, \ldots, l-1\}$ such that $n_i - n_{i+1}$ is equal to either $1, 2$ or $3$. If $n_i - n_{i+1}$ is equal to 1, then $W_f$ is Hurwitz equivalent to the following sequence:

$$(\alpha; S_1T(n_1 + 1, \ldots, n_{i-1} + 1, n_i, n_{i+2} \ldots, n_l))$$

since the sequence $T(n_i, n_{i+1})$ is Hurwitz equivalent to $S_1T(n_i)$. Thus, the conclusion holds by induction hypothesis. If $n_i - n_{i+1}$ is equal to 2, then $W_f$ is Hurwitz equivalent to the following sequence:

$$(\alpha; T(n_i, n_{i+1})T(n_1 - 4, \ldots, n_{i-1} - 4, n_{i+2} \ldots, n_l))$$

since the composition $t_{\beta+n_i\alpha}t_{\beta+n_{i+1}\alpha}$ is equal to $(t_\alpha t_\beta)^3 t_\alpha^{-4}$. By the argument similar to that in the proof of Lemma 6.13 in [16], we can prove that there exists a genus-1 SBLF $f' : X' \to S^2$ such that $X$ is diffeomorphic to $X'\#S$, where $S$ is an $S^2$-bundle over $S^2$. Thus, the conclusion holds by induction hypothesis. If $n_i - n_{i+1}$ is equal to 3, then $W_f$ is Hurwitz equivalent to $(\alpha; T(0, -3)W)$, where $W$ is some sequence which consists of $l-2$ simple closed curves. We can apply Corollary 3.3.5 to this fibration and the conclusion holds by the induction hypothesis. This completes the proof of Theorem 3.0.1. □

# Chapter 4

# Hyperelliptic broken Lefschetz fibrations

## 4.1 Preliminaries

### 4.1.1 Hyperelliptic mapping class groups

Let $\Sigma_g$ be a closed oriented surface of genus $g \geq 1$. Denote by $\iota : \Sigma_g \to \Sigma_g$ an involution described in Figure 4.1.1. Let $C(\iota)$ denote the centralizer of $\iota$ in the diffeomorphism

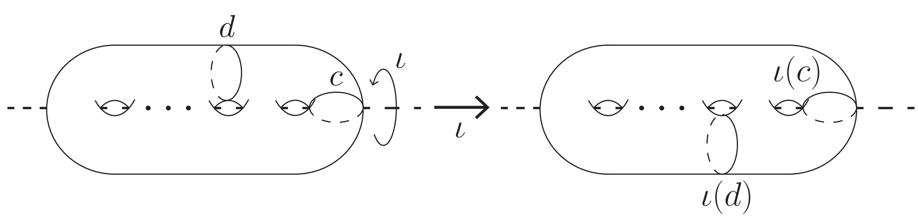


Figure 4.1.1: the hyperelliptic involution on the surface $\Sigma_g$.

group $\mathrm{Diff}_+ \Sigma_g$, and endow $C(\iota) \subset \mathrm{Diff}_+ \Sigma_g$ with the relative topology. The inclusion homomorphism $C(\iota) \to \mathrm{Diff}_+ \Sigma_g$ induces a natural homomorphism $\pi_0 C(\iota) \to \mathcal{M}_g$ between their path-connected components.

**Theorem 4.1.1** (Birman-Hilden [7]). *When $g \geq 2$, the homomorphism*

$$\pi_0 C(\iota) \to \mathcal{M}_g$$

*is injective.*

Denote the image of this homomorphism by $\mathcal{H}_g$ for $g \geq 1$. This group is called the *hyperelliptic mapping class group*. In fact, they showed the above result in more general settings later. See [7] for details.

In the following, we review some properties of the hyperelliptic mapping class group. Let $S$ be a 2-disk or a 2-sphere. For a positive integer $n$ and distinct points $\{p_i\}_{i=1}^{n}$ in $\mathrm{Int}\, S$, Denote by $\mathrm{Diff}_+(S, \partial S, \{p_1, p_2, \cdots, p_n\})$ the group defined by

$$\begin{aligned}&\mathrm{Diff}_+(S, \partial S, \{p_1, p_2, \cdots, p_n\})\\ &= \{T \in \mathrm{Diff}_+ S \mid T|_{\partial S} = \mathrm{id}_{\partial S}, T(\{p_1, p_2, \cdots, p_n\}) = \{p_1, p_2, \cdots, p_n\}\}.\end{aligned}$$

Denote by $\mathcal{M}_0^n$ or $\mathcal{M}_{0,1}^n$ its mapping class group when $S = S^2$ or $S = D^2$, respectively. Let $D_i$ be a disk in $\mathrm{Int}\, S$ which includes $p_i$ and $p_{i+1}$ but is disjoint from all $p_j$ for $j \neq i, i+1$, and denote by $\nu(\partial D_i)$ a neighborhood of the boundary $\partial D_i$ in $D_i$. Choose a diffeomorphism $T_i \in \mathrm{Diff}_+(S, \partial S, \{p_1, p_2, \cdots, p_n\})$ such that $T_i|_{D_i}$ interchanges the points $p_i$ and $p_{i+1}$, $T_i|_{X - \mathrm{Int}\, D_i}$ is the identity map, and $T_i^2$ is isotopic to the Dehn twist along $\partial D_i$ (see Birman-Hilden p.87-88 for details). The mapping class group $\mathcal{M}_0^n$ and $\mathcal{M}_{0,1}^n$ is generated by $\{\sigma_i\}_{i=1}^{n-1}$, where $\sigma_i$ is the mapping class represented by the diffeomorphism $T_i$.

Identifying the quotient space $\Sigma_g / \langle \iota_g \rangle$ with $S^2$, let $p_1, \ldots, p_{2g+2} \subset S^2$ be the branched points of the quotient map $\Sigma_g \to \Sigma_g / \langle \iota_g \rangle$. By the definition, any diffeomorphism $T$ in $C(\iota_g)$ satisfies $T\iota_g(x) = \iota_g T(x)$ for $x \in \Sigma_g$. Hence, there exists a unique diffeomorphism $\bar{T} \in \mathrm{Diff}_+ S^2$ such that the diagram

$$\begin{array}{ccc} \Sigma_g & \xrightarrow{T} & \Sigma_g \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ S^2 & \xrightarrow{\bar{T}} & S^2 \end{array}$$

commutes. Moreover, it satisfies $\bar{T}(\{p_1, p_2, \cdots, p_{2g+2}\}) = \{p_1, p_2, \cdots, p_{2g+2}\} \subset S^2$. By the above diagram, we can define

$$\mathcal{P}_g : \mathcal{H}_g^s \to \mathcal{M}_0^{2g+2}$$

by $\mathcal{P}_g([T]) = [\bar{T}]$.

**Theorem 4.1.2** (Birman-Hilden [7])**.** *Let $g \geq 1$. the sequence*

$$1 \longrightarrow \langle \iota_g \rangle \longrightarrow \mathcal{H}_g^s \xrightarrow{\mathcal{P}_g} \mathcal{M}_0^{2g+2} \longrightarrow 1$$

*is exact.*

They showed the homomorphism $\mathcal{P}_g : \mathcal{H}_g^s \to \mathcal{M}_0^{2g+2}$ maps the Dehn twist $t_{c_i}$ to $\sigma_i$ in [7]. Furthermore, they proved:

**Proposition 4.1.3.** *Let $g \geq 1$. The group $\mathcal{H}_g^s$ is generated by $\{t_{c_1}, \cdots, t_{c_{2g+1}}\}$, where $c_i$ is a simple closed curve in $\Sigma_g$ described in Figure 4.1.2.*

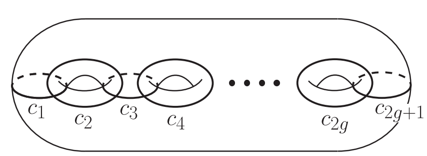


Figure 4.1.2: simple closed curves $c_1, \cdots, c_{2g+1}$

### 4.1.2 Hyperelliptic fibrations

A Lefschetz fibration is said to be *hyperelliptic* if we can take an identification of the fiber of a base point with the closed oriented surface so that the image of the monodromy representation of the fibration is contained in the hyperelliptic mapping class group. Thus, it is natural to generalize this definition to directed (and especially simplified) BLFs as follows: Let $f : M \to S^2$ be a DBLF. We use the same notations as those we use in Sections 2.1 and 2.4. We put $r_i = \alpha\left(\frac{t_i+t_{i+1}}{2}\right)$ $(i = 1, \dots, m-1)$ and $r_m = p_n$. We can regard the vanishing cycle $d_i$ we took in Section 2.4 a simple closed curve in $f^{-1}(r_i)$. Once we fix an identification of $f^{-1}(r_m)$ with $\Sigma_{g_1} \amalg \cdots \amalg \Sigma_{g_k}$, we obtain an involution $\iota_i$ on $f^{-1}(r_i)$ induced by the hyperelliptic involution on $f^{-1}(r_m)$ since we can identify $f^{-1}(r_{i-1}) \setminus \{\text{two points}\}$ with $f^{-1}(r_i) \setminus d_i$ by using $\alpha$. $f$ is said to be *hyperelliptic* if it satisfies the following conditions for a suitable identification of $f^{-1}(r_m)$ with $\Sigma_{g_1} \amalg \cdots \amalg \Sigma_{g_k}$:

- the image of the monodromy representation of the Lefschetz fibration $\operatorname{res} f : f^{-1}(D_h) \to D_h$ is contained in the group $\mathcal{H}_g$;
- $d_i$ is preserved by the involution $\iota_i$ up to isotopy.

In this paper, we will call a hyperelliptic SBLF HSBLF for short.

**Remark 4.1.4.** Every SBLF whose genus is less than or equal to 2 is hyperelliptic since $\mathcal{H}_g = \mathcal{M}_g$ and all simple closed curves in $\Sigma_g$ are preserved by $\iota$ if $g \leq 2$.

### 4.1.3 Meyer's signature cocycle and the local signature

It is known that, for an HLF $f : X \to \Sigma$ over a closed oriented surface $\Sigma$, the signature $\operatorname{Sign} X$ is described as the sum of invariants of the singular fiber germs in $X$. We review this invariant.

Let $\varphi, \psi$ be elements in the mapping class group $\mathcal{M}_g$. We denote by $E_{\varphi,\psi}$ a $\Sigma_g$-bundle over a pair of pants $S^2 - \amalg_{i=1}^{3} \operatorname{Int} D^2$ whose monodromies along $\alpha$ and $\beta$ in Figure 4.1.3 are $\varphi$ and $\psi$, respectively.

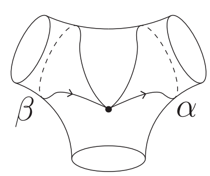


Figure 4.1.3: paths $\alpha$ and $\beta$

**Theorem 4.1.5** (Meyer [25])**.** *Define a 2-cochain $\tau_g : \mathcal{M}_g \times \mathcal{M}_g \to \mathbb{Z}$ of the mapping class group by $\tau_g(\varphi, \psi) = -\operatorname{Sign} E_{\varphi,\psi}$. Then, $\tau_g$ is a 2-cocycle, and the order of its homology class is as follows.*

1. *The order of $[\tau_1] \in H^2(\mathcal{M}_1; \mathbb{Z})$ is 3,*
2. *The order of $[\tau_2] \in H^2(\mathcal{M}_2; \mathbb{Z})$ is 5,*

*3. When $g \geq 3$, $[\tau_g] \neq 0 \in H^2(\mathcal{M}_g;\mathbb{Q})$.*

**Proposition 4.1.6** (Endo [9]). *If we restrict $\tau_g$ to $\mathcal{H}_g$, the order of $[\tau_g] \in H^2(\mathcal{H}_g;\mathbb{Z})$ is $2g+1$.*

Since $\tau_g$ represents a trivial homology class in $H^2(\mathcal{H}_g;\mathbb{Q})$, there exists a cobounding function $\phi_g : \mathcal{H}_g \to \mathbb{Q}$ of it. Furthermore, since $H_1(\mathcal{H}_g;\mathbb{Q})$ is trivial, this cobounding function $\phi_g$ is unique.

**Lemma 4.1.7** (Endo [9]). *Let $f : X \to \Sigma$ be a $\Sigma_g$-bundle over a compact oriented surface $\Sigma$. Assume that the image of the monodromy representation $\pi_1(\Sigma, p_0) \to \mathcal{M}_g$ is in $\mathcal{H}_g$ if we choose a suitable identification $f^{-1}(p_0) \cong \Sigma_g$. Let $\{\partial_j\}_{j=1}^n$ denote the boundary components of $\Sigma$, and give orientations coming from $\Sigma$. Then, we have*

$$\operatorname{Sign} X = -\sum_{j=1}^{l} \phi(\psi_j),$$

*where $\psi_j \in \mathcal{H}_g$ is the monodromy along $\partial_j$.*

Using this function, he generalized the local signature of LFs of genus 1 [23] and of genus 2 [24] constructed by Matsumoto. Let $f : X \to \Sigma$ be an HLF of genus $g$ over a closed oriented surface $\Sigma$, and $p_1, \cdots, p_l$ the image of the set of Lefschetz singularities under $f$. For the Lefschetz singular fiber $f^{-1}(p_j)$, define a rational number $\sigma_{\mathrm{loc}}(f^{-1}(p_j))$ by

$$\sigma_{\mathrm{loc}}(f^{-1}(p_j)) = -\phi_g(\varphi_j) + \operatorname{Sign}(f^{-1}\nu(p_j)),$$

where $\varphi_j \in \mathcal{H}_g$ is the monodromy along $\partial\nu(p_j)$. He computed the values for Lefschetz singular fibers as follows:

**Lemma 4.1.8** (Endo [9]). *We call a Lefschetz singular fiber is type* I *or type* $\mathrm{II}_h$ *if the vanishing cycle is type* I *or type* $\mathrm{II}_h$*, respectively. Then we have*

$$\sigma_{\mathrm{loc}}(\mathrm{I}) = -\frac{g+1}{2g+1}, \quad \sigma_{\mathrm{loc}}(\mathrm{II}_h) = \frac{4h(g-h)}{2g+1} - 1.$$

Furthermore, Endo gave the following formula for signatures of HLFs.

**Theorem 4.1.9** (Endo [9]). *Let $f : X \to \Sigma$ be an HLF as above. Then we have*

$$\operatorname{Sign} X = \sum_{i=1}^{l} \sigma_{\mathrm{loc}}(f^{-1}(p_j)).$$

## 4.2 A subgroup $\mathcal{H}_g(c)$

Let $c$ be an essential simple closed curve in the surface $\Sigma_g$ which is preserved by the involution $\iota \in \mathrm{Diff}_+ \Sigma_g$ as a set. Let $\mathcal{H}_g(c)$ denote the subgroup of the hyperelliptic mapping class group defined by $\mathcal{H}_g(c) := \mathcal{H}_g \cap \mathcal{M}_g(c)$. As introduced in Theorem 4.1.1, the hyperelliptic mapping class group $\mathcal{H}_g$ is isomorphic to the group consisting of the path-connected components of $C(\iota)$. Hence, the group $\mathcal{H}_g(c)$ consists of the mapping classes which can be represented by both of elements in the centralizer $C(\iota)$ and elements in $\mathrm{Diff}_+(\Sigma_g, c)$. Let $\mathcal{H}_g^s(c)$ denote the subgroup of $\pi_0 C(\iota)$ defined by $\mathcal{H}_g^s(c) := \{[T] \in \pi_0 C(\iota) \,|\, T(c) = c\}$. In this section, we will prove the following lemma.

**Lemma 4.2.1.** *Assume that $g$ is greater than or equal to 1. The natural map $\pi_0 C(\iota) \to \mathcal{H}_g$ in Theorem 4.1.1 restricts to a surjective map between the groups $\mathcal{H}_g^s(c)$ and $\mathcal{H}_g(c)$. Furthermore, this restriction is an isomorphism if $g \geq 2$.*

To prove the lemma, It is enough to show that the homomorphism maps $\mathcal{H}_g^s(c)$ onto $\mathcal{H}_g(c)$. Let $[T]$ be a mapping class in $\mathcal{H}_g(c)$. We can choose a representative $T : \Sigma_g \to \Sigma_g$ in the centralizer $C(\iota)$. Since it is isotopic to some diffeomorphism on $\Sigma_g$ which preserves the curve $c$, the curve $T(c)$ is isotopic to $c$.

We call an isotopy $L_0 : \Sigma_g \times [0,1] \to \Sigma_g$ is symmetric if $L_0(*, t) \in C(\iota)$ for any $t \in [0,1]$. In the following, we will construct a symmetric isotopy $L : \Sigma_g \times [0,1] \to \Sigma_g$ satisfying

$$L(*, 0) = T, \text{ and } L(c, 1) = c \subset \Sigma_g.$$

It indicates that $L(*, 1)$ represents an element in $\mathcal{H}_g^s(c)$, and $[L(*, 1)] = [T] \in \pi_0 C(\iota)$. Hence, we see that the homomorphism $\mathcal{H}_g^s(c) \to \mathcal{H}_g(c)$ is surjective.

To construct the symmetric isotopy $L : \Sigma_g \times [0,1] \to \Sigma_g$, we need a proposition, so called the bigon criterion.

**Proposition 4.2.2** (Farb-Margalit Proposition 1.7 [10])**.** *Let $S$ be a compact surface. The geometric intersection number of two transverse simple closed curves in $S$ is minimal if and only if they do not form a bigon.*

We may assume that the curves $c$ and $T(c)$ are transverse by changing the diffeomorphism $T$ in terms of some symmetric isotopy. Since $c$ and $T(c)$ are isotopic, the minimal intersection number of them is 0. Hence, there exist bigons such that each of their boundaries is the union of an arc of $c$ and an arc of $T(c)$. Choose an innermost bigon $\Delta$ among them.

Let $\alpha$ be the arc $c \cap \partial\Delta$ and $\beta$ the arc $T(c) \cap \partial\Delta$, respectively. Since $\Delta$ is a bigon, the endpoints of them coincide. Denote them by $\{x_1, x_2\} \subset \partial\Delta$.

**Lemma 4.2.3.**

$$\operatorname{Int}\Delta \cap (T(c) \cup c) = \emptyset$$

*Proof.* If the set $\operatorname{Int}\Delta \cap c$ is non-empty, there exists an arc of $c$ in $\Delta$ which forms an bigon with the arc $\beta$. Since the bigon $\Delta$ is innermost, it is a contradiction. We can also show that $\operatorname{Int}\Delta \cap T(c) = \emptyset$ in the same way. □

Note that the bigon $\iota(\Delta)$ is also innermost. By Lemma 4.2.3, we have $\Delta \cap \iota(\Delta) = \partial\Delta \cap \partial\iota(\Delta)$.

**Lemma 4.2.4.**

$$\partial\Delta \cap \partial\iota(\Delta) \subset \{x_1, x_2\}.$$

*Proof.* Since $\partial\alpha = \partial\beta = \alpha \cap \beta = \{x_1, x_2\}$, it suffices to show that $\operatorname{Int}\alpha \cap \partial\iota(\Delta) = \operatorname{Int}\beta \cap \partial\iota(\Delta) = \emptyset$. Since $\alpha \cap T(c) = \{x_1, x_2\}$, we have $\operatorname{Int}\alpha \cap \iota(\beta) = \emptyset$. Next, we will show that $\operatorname{Int}\alpha \cap \operatorname{Int}\iota(\alpha) = \emptyset$. We assume $\operatorname{Int}\alpha \cap \operatorname{Int}\iota(\alpha) \neq \emptyset$. Since $c$ is simple and contains $\alpha$ and $\iota(\alpha)$, $\alpha$ and $\iota(\alpha)$ must coincide. In particular, we have $\partial\alpha = \partial\iota(\alpha)$. So $\beta \cup \iota(\beta)$ forms a simple closed curve, and this curve is null-homotopic because both of the arcs $\beta$ and $\iota(\beta)$ are homotopic to $\alpha = \iota(\alpha)$ relative to their boundaries. Since $T(c)$ is simple and contains $\beta$ and $\iota(\beta)$, $T(c)$ and $\beta \cup \iota(\beta)$ must coincide. This contradicts that $T(c)$ is essential. In the same way, we can show that $\operatorname{Int}\beta \cap \partial\iota(\Delta) = \emptyset$. □

Let $\Sigma_g^\iota$ denote the fixed point set of the involution $\iota$ on $\Sigma_g$.

**Lemma 4.2.5.** *If $c$ is non-separating, the set $c \cap \Sigma_g^\iota$ consists of 2 points, and*

$$c \cap \Sigma_g^\iota = T(c) \cap \Sigma_g^\iota.$$

*If $c$ is separating,*

$$c \cap \Sigma_g^\iota = T(c) \cap \Sigma_g^\iota = \emptyset.$$

*Proof.* Endow the curves $c$ and $T(c)$ with arbitrary orientations.

First, consider the case when $c$ is a non-separating simple closed curve. In this case, the curve $T(c)$ is also non-separating. They represent nontrivial homology classes in $H_1(\Sigma_g; \mathbf{Z})$. Since the involution $\iota$ acts on $H_1(\Sigma_g; \mathbf{Z})$ by $-1$, it changes the orientations of $c$ and $T(c)$. Hence, both of the sets $c \cap \Sigma_g^\iota$ and $T(c) \cap \Sigma_g^\iota$ consist of 2 points.

We will show that $T(c) \cap \Sigma_g^\iota = c \cap \Sigma_g^\iota$. Since $c$ and $T(c)$ are isotopic, the Dehn twists $t_c$ and $t_{T(c)}$ represent the same element in $\mathcal{H}_g$. The mapping classes $\Psi([t_c])$ and $\Psi([t_{T(c)}])$ in $\mathcal{M}_0^{2g+2}$ permute the branched points $p(c \cap \Sigma_g^\iota)$ and $p(T(c) \cap \Sigma_g^\iota)$, respectively. Hence, the sets $p(c \cap \Sigma_g^\iota)$ and $p(T(c) \cap \Sigma_g^\iota)$ coincide. It shows that $c \cap \Sigma_g^\iota = T(c) \cap \Sigma_g^\iota$.

Next, let $c$ be a separating simple closed curve. Since $\iota$ preserves the orientations of the subsurfaces bounded by $c$ or $T(c)$, it also preserves the orientation of $c$ and $T(c)$. In general, if an involution acts on a circle preserving its orientation, it does not have a fixed point. Hence, we have $c \cap \Sigma_g^\iota = T(c) \cap \Sigma_g^\iota = \emptyset$. □

*Proof of Lemma 4.2.1.* Let $c$ be a non-separating curve. By Lemma 4.2.5, the geometric intersection number of $c$ and $T(c)$ is at least 2. Hence, there is an innermost bigon $\Delta$. By Lemma 4.2.4, the cardinality $\sharp(\Delta \cap \iota(\Delta))$ is equal to 0, 1, or 2 as shown in Figure 4.2.1.

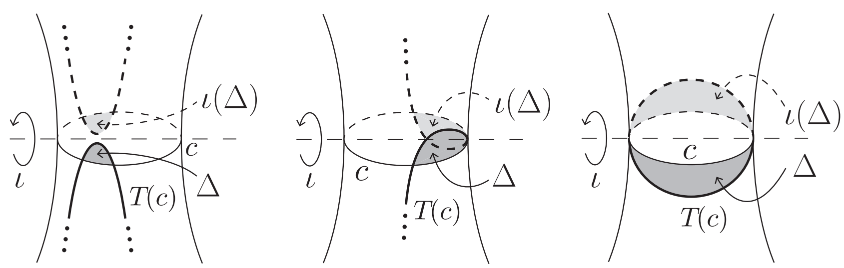


Figure 4.2.1: Left: $\sharp(\Delta \cap \iota(\Delta)) = 0$. Center: $\sharp(\Delta \cap \iota(\Delta)) = 1$. Right: $\sharp(\Delta \cap \iota(\Delta)) = 2$. The bold curves describe the curves $T(c)$.

Firstly, assume that $\sharp(\Delta \cap \iota(\Delta)) = 0$. In this case, there is a symmetric isotopy $L_1 : \Sigma_g \times [0,1] \to \Sigma_g$ such that $L_1(*, 0)$ is the identity, and $L_1(*, 1)$ collapses the bigon $\Delta$ as in Figure 4.2.2. Therefore, we can decrease the geometric intersection number of $c$ and $T(c)$ by 4 by replacing the diffeomorphism $T$ by $L_1(*, 1)T$.

Secondly, assume that $\sharp(\Delta \cap \iota(\Delta)) = 1$. In this case, we also have a symmetric isotopy $L_2 : \Sigma_g \times [0,1] \to \Sigma_g$ which decreases the geometric intersection number by 2 as in Figure 4.2.3. Note that $\Delta \cap \iota(\Delta)$ is a branched point, and $L_2(*, t)$ fixes it for any $t \in [0,1]$.

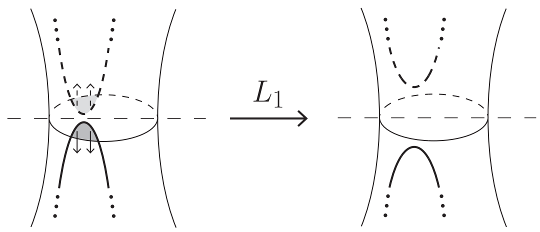


Figure 4.2.2: An isotopy $L_1$.

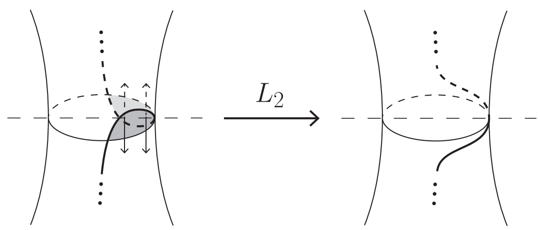


Figure 4.2.3: An isotopy $L_2$.

After replacing the diffeomorphism $T$ in these two cases, the branch points $\{x_1, x_2\}$ remains in $c \cap T(c)$. Hence, if we repeat to replace $T$, the case when $\sharp(\Delta \cap \iota(\Delta)) = 2$ will definitely occur. In this case, there is a symmetric isotopy $L_3 : \Sigma_g \times [0,1] \to \Sigma_g$ such that

$$L_3(*, 0) \text{ is the identity map,}$$
$$L_3(\beta, 1) = \alpha,$$
$$L_3(\iota(\beta), 1) = \iota(\alpha),$$

as in Figure 4.2.4. It indicates that $L_3(*, 1)T$ preserves the curve $c$. By combining these isotopies, we have obtained a desired symmetric isotopy.

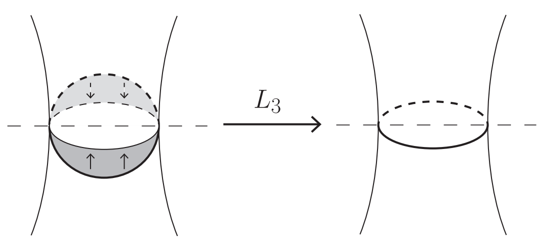


Figure 4.2.4: An isotopy $L_3$.

Next, let $c$ be a separating curve. If the geometric intersection number of $c$ and $T(c)$ is 0, the curves $c$ and $T(c)$ bound an annulus $A$. Since $\iota$ acts on $A$ without fixed points, $A/\langle\iota\rangle$ is also an annulus. Hence, we can make a symmetric isotopy which moves $T(c)$ to $c$.

Assume that the geometric intersection number is not 0. Since we have $c \cap \Sigma_g^\iota = T(c) \cap \Sigma_g^\iota = \emptyset$, the cardinality $\sharp(\Delta \cap \iota(\Delta)) \neq 1$. By Lemma 4.2.4, we have $\sharp(\Delta \cap \iota(\Delta)) = 0$ or 2. By the same argument as in the case when $c$ is non-separating, we can collapse the bigons $\Delta$ and

$\iota(\Delta)$. □

## 4.3 An involution on hyperelliptic broken Lefschetz fibrations

In this section, we prove the following theorem:

**Theorem 4.3.1.** *Let $f : X \to S^2$ be a genus-$g$ hyperelliptic simplified broken Lefschetz fibration. We assume that $g$ is greater than or equal to 3.*

*(i) Let $s$ be the number of Lefschetz singularities of $f$ whose vanishing cycles are separating. Then there exists an involution*

$$\omega : M \to M$$

*such that the fixed point set of $\omega$ is the union of (possibly unorientable) surfaces and $s$ isolated points. Moreover, $\omega$ can be extended to an involution*

$$\overline{\omega} : X\#s\overline{\mathbb{CP}^2} \to X\#s\overline{\mathbb{CP}^2}$$

*so that $X\#s\overline{\mathbb{CP}^2}/\overline{\omega}$ is diffeomorphic to $S\#2s\overline{\mathbb{CP}^2}$, where $S$ is $S^2$-bundle over $S^2$, and that the quotient map*

$$/\overline{\omega} : X\#s\overline{\mathbb{CP}^2} \to X\#s\overline{\mathbb{CP}^2}/\overline{\omega} \cong S\#2s\overline{\mathbb{CP}^2}$$

*is a double branched covering.*

*(ii) A regular fiber $F$ of the fibration $f$ represents a non-trivial rational homology class of $X$, that is, $[F] \neq 0$ in $H_2(X;\mathbb{Q})$.*

*Proof of (i) in Theorem 4.3.1.* Let $f : X \to S^2$ be genus-$g \geq 3$ HSBLF, $c_i \subset \Sigma_g$ $(i = 1, \dots, l)$ a vanishing cycle of a Lefschetz singularity of $f$ and $c \subset \Sigma_g$ a vanishing cycle of indefinite folds of $f$. We assume that $c_1, \dots, c_n$ and $c$ are preserved by the involution $\iota : \Sigma_g \to \Sigma_g$. By the argument in Section 2.6, we can decompose $X$ as follows:

$$X = D^2 \times \Sigma_g \cup (h_1^2 \amalg \cdots \amalg h_n^2) \cup R^2 \cup D^2 \times \Sigma_{g-1},$$

where $h_i^2 = D^2 \times D^2$ is the 2-handle attached along the simple closed curve $\{p_i\} \times c_i \in \partial D^2 \times \Sigma_g$ and $R^2$ is a round 2-handle. We first prove existence of an involution $\omega$ by using the above decomposition.

**Step 1**: We define an involution $\omega_1$ on $D^2 \times \Sigma_g$ as follows:

$$\begin{array}{rccc} \omega_1 = \mathrm{id} \times \iota : & D^2 \times \Sigma_g & \longrightarrow & D^2 \times \Sigma_g \\ & \cup\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\shortmid \\ & (z, x) & \longmapsto & (z, \iota(x)). \end{array}$$

In the following steps, we will define an involution on each component in the above decomposition of $X$ which is compatible with the involution $\omega_1$.

**Step 2**: We next define an involution $\omega_{2,i}$ on the 2-handle $h_i^2$ attached along $\{q_i\} \times c_i \subset \partial D^2 \times \Sigma_g$. We will abuse the notation by denoting the attaching circle $\{q_i\} \times c_i$ by $c_i$.

We take a tubular neighborhood $\nu c_i$ in $\{q_i\} \times \Sigma_g$ and an identification

$$\nu c_i \cong S^1 \times [-1, 1]$$

so that $c_i$ corresponds to the circle $S^1 \times \{0\}$ under the identification. We assume that the standard orientation of $S^1 \times [-1, 1]$ coincides with that of $\{q_i\} \times \Sigma_g$. We take a sufficiently small neighborhood $I_{q_i}$ of $q_i$ in $\partial D^2$ as follows:

$$I_{q_i} = \{q_i \cdot \exp(\sqrt{-1}\theta) \in \partial D^2 | \theta \in [-\varepsilon_1, \varepsilon_1]\},$$

where $\varepsilon_1 > 0$ is a sufficiently small number. We further identify the neighborhood $I_{q_i}$ with the unit interval $[-1, 1]$ by using the following map:

$$\begin{array}{ccc} [-1,1] & \xrightarrow{\sim} & I_{q_i} \\ \cup & & \cup \\ s & \longmapsto & q_i \cdot \exp(\sqrt{-1}\varepsilon_1 s). \end{array}$$

We regard $I_{q_i} \times [-1, 1]$ as the subset of $\mathbb{C}$ by the following embedding:

$$\begin{array}{ccc} I_{q_i} \times [-1,1] & \hookrightarrow & \{z \in \mathbb{C} |\, |\mathrm{Re}\, z| \leq 1, |\mathrm{Im}\, z| \leq 1\} \\ \cup & & \cup \\ (s,t) & \longmapsto & s + t\sqrt{-1}. \end{array}$$

We put $J = \{z \in \mathbb{C} |\, |\mathrm{Re}\, z| \leq 1, |\mathrm{Im}\, z| \leq 1\}$. The orientation of $\partial D^2 \times \Sigma_g$ is opposite to the natural orientation of $J \times S^1$. Thus, the attaching map of the 2-handle $h_i^2$ is described as follows:

$$\begin{array}{cccc} \varphi_i : & \partial D^2 \times D^2 & \longrightarrow & J \times S^1 \subset \partial D^2 \times \Sigma_g \\ & \cup & & \cup \\ & (w_1, w_2) & \longmapsto & (\varepsilon_2 w_2 w_1, w_1), \end{array}$$

where $\varepsilon_2 > 0$ is a sufficiently small number. Note that the map $\varphi_i$ is orientation-preserving if we give the natural orientation of $\partial D^2 \times D^2$.

**Case 2.1**: If $c_i$ is non-separating, we can take a tubular neighborhood $\nu c_i \cong S^1 \times [-1, 1]$ so that the involution $\omega_1$ acts on $\nu c_i$ as follows:

$$\begin{array}{cccc} \omega_1|_{\nu c_i} : & S^1 \times [-1,1] & \longrightarrow & S^1 \times [-1,1] \\ & \cup & & \cup \\ & (z,t) & \longmapsto & (\overline{z}, -t). \end{array}$$

Since the involution $\omega_1 : D^2 \times \Sigma_g \to D^2 \times \Sigma_g$ preserves the first component, $\omega_1$ acts on $I_{q_i} \times \nu c_i \cong J \times S^1$ as follows:

$$\begin{array}{cccc} \omega_1|_{J \times S^1} : & J \times S^1 & \longrightarrow & J \times S^1 \\ & \cup & & \cup \\ & (z_1, z_2) & \longmapsto & (\overline{z_1}, \overline{z_2}). \end{array}$$

We define an involution $\omega_{2,i}$ on the 2-handle $h_i^2$ as follows:

$$\begin{array}{cccc} \omega_{2,i}: & D^2\times D^2 & \longrightarrow & D^2\times D^2 \\ & \Uparrow & & \Uparrow \\ & (w_1,w_2) & \longmapsto & (\overline{w_1},\overline{w_2}). \end{array}$$

Then the following diagram commutes:

$$\begin{array}{ccc} \partial D^2\times D^2 & \xrightarrow{\omega_{2,i}} & \partial D^2\times D^2 \\ {\scriptstyle\varphi_i}\downarrow & & \downarrow{\scriptstyle\varphi_i} \\ J\times S^1 & \xrightarrow{\omega_1} & J\times S^1. \end{array}$$

Thus, we can define an involution $\omega_1\cup\omega_{2,i}$ on the manifold $D^2\times\Sigma_g\cup_{\varphi_i}h_i^2$.

**Case 2.2**: If $c_i$ is separating, we can take a tubular neighborhood $\nu c_i\cong S^1\times[-1,1]$ so that the involution $\omega_1$ acts on $\nu c_i$ as follows:

$$\begin{array}{cccc} \omega_1|_{\nu c_i}: & S^1\times[-1,1] & \longrightarrow & S^1\times[-1,1] \\ & \Uparrow & & \Uparrow \\ & (z,t) & \longmapsto & (-z,t). \end{array}$$

Then $\omega_1$ acts on $I_{q_i}\times\nu c_i\cong J\times S^1$ as follows:

$$\begin{array}{cccc} \omega_1|_{J\times S^1}: & J\times S^1 & \longrightarrow & J\times S^1 \\ & \Uparrow & & \Uparrow \\ & (z_1,z_2) & \longmapsto & (z_1,-z_2). \end{array}$$

We define an involution $\omega_{2,i}$ on the 2-handle $h_i^2$ as follows:

$$\begin{array}{cccc} \omega_{2,i}: & D^2\times D^2 & \longrightarrow & D^2\times D^2 \\ & \Uparrow & & \Uparrow \\ & (w_1,w_2) & \longmapsto & (-w_1,-w_2). \end{array}$$

Then the following diagram commutes:

$$\begin{array}{ccc} \partial D^2\times D^2 & \xrightarrow{\omega_{2,i}} & \partial D^2\times D^2 \\ {\scriptstyle\varphi_i}\downarrow & & \downarrow{\scriptstyle\varphi_i} \\ J\times S^1 & \xrightarrow{\omega_1} & J\times S^1. \end{array}$$

Thus, we can define an involution $\omega_1\cup\omega_{2,i}$ on the manifold $D^2\times\Sigma_g\cup_{\varphi_i}h_i^2$.

Combining Case 2.1 and Case 2.2, we can define the involution $\tilde{\omega}_2$ on the 4-manifold $X_h=D^2\times\Sigma_g\cup(h_1^2\amalg\cdots\amalg h_n^2)$ as follows:

$$\tilde{\omega}_2=\omega_1\cup(\omega_{2,1}\cup\cdots\cup\omega_{2,n}).$$

Before giving an involution on the round 2-handle, we look at the $\Sigma_g$-bundle structure of $\partial X_h$. The projection $\pi_h:\partial X_h\to\partial D^2$ of this bundle is described as follows:

- for an element $(z,x)\in\partial D^2\times\Sigma_g\setminus(\amalg\,\mathrm{Int}\,\varphi_i(\partial D^2\times D^2))$, $\pi_h$ is defined as follows:

$$\pi_h(z,x)=z,$$

- for an element $(w_1, w_2) \in D^2 \times \partial D^2 \subset \partial h_i^2$, $\pi_h$ is defined as follows:

$$\pi_h(w_1, w_2) = q_i \cdot \exp(\sqrt{-1}\varepsilon_1\varepsilon_2(\operatorname{Re} w_1 \operatorname{Re} w_2 - \operatorname{Im} w_1 \operatorname{Im} w_2).$$

Indeed, the map $\pi_h$ is well-defined. To see this, we only need to verify the following equation:

$$q_i \cdot \exp(\sqrt{-1}\varepsilon_1\varepsilon_2(\operatorname{Re} w_1 \operatorname{Re} w_2 - \operatorname{Im} w_1 \operatorname{Im} w_2)) = p_1 \circ \varphi_i(w_1, w_2),$$

where $(w_1, w_2) \in D^2 \times \partial D^2 \subset \partial h_i^2$ and $p_1 : J \times S^1 \to I_{q_i}$ is the projection. $p_1 \circ \varphi_i(w_1, w_2)$ is calculated as follows:

$$\begin{aligned} p_1 \circ \varphi_i(w_1, w_2) &= p_1(\varepsilon_2 w_2 w_1, w_1) \\ &= q_i \cdot \exp(\sqrt{-1}\varepsilon_1 \operatorname{Re}(\varepsilon_2 w_2 w_1)) \\ &= q_i \cdot \exp(\sqrt{-1}\varepsilon_1\varepsilon_2(\operatorname{Re} w_1 \operatorname{Re} w_2 - \operatorname{Im} w_1 \operatorname{Im} w_2)) \end{aligned}$$

This implies that the definition of $\pi_k$ above makes sense.

**Lemma 4.3.2.** *The involution $\tilde{\omega}_2$ preserves the fibers of $\pi_h$. Moreover, there exists a lift $V$ of the vector field $\frac{d}{d\theta}\exp(\sqrt{-1}\theta)$ by the map $\pi_h$ which is compatible with the involution $\tilde{\omega}_2$, that is,*

$$\tilde{\omega}_{2*}(V) = V.$$

*Proof of Lemma 4.3.2.* It is easy to verify that $\tilde{\omega}_2$ preserves the fibers of $\pi_h$ by direct calculation. The details of this are left to the readers.

To prove existence of a lift $V$, we construct $V$ explicitly. We define a vector field $V_1$ on $\partial D^2 \times \Sigma_g \setminus (\amalg \varphi_i(\partial D^2 \times D^2)$ as follows:

$$V_1(\exp(\sqrt{-1}\theta_0), x) = \frac{d}{d\theta}\exp(\sqrt{-1}\theta)\bigg|_{\theta=\theta_0} \in T_{(\exp(\sqrt{-1}\theta_0), x)}(\partial D^2 \times \Sigma_g),$$

for a point $(\exp(\sqrt{-1}\theta_0), x) \in \partial D^2 \times \Sigma_g \setminus (\amalg \operatorname{Int} \varphi_i(\partial D^2 \times D^2)$. The vector field $V_1$ is described in $J \times S^1$ as follows:

$$V_1(s + t\sqrt{-1}, z) = \frac{1}{\varepsilon_1}\frac{\partial}{\partial s}\bigg|_s \in T_{(s+t\sqrt{-1}, z)}(J \times S^1).$$

We also define a vector field $V_2$ on $D^2 \times \partial D^2 \subset \partial h_i^2$ as follows:

$$V_2(w_1, w_2) = \frac{\varrho(|w_1|^2)}{\varepsilon_1\varepsilon_2 |w_1|^2}\left(x_1\frac{\partial}{\partial x_2} - y_1\frac{\partial}{\partial y_2}\right) + \frac{1 - \varrho(|w_1|^2)}{\varepsilon_1\varepsilon_2}\left(x_2\frac{\partial}{\partial x_1} - y_2\frac{\partial}{\partial y_1}\right),$$

where $w_i = x_i + y_i\sqrt{-1}$ and $\varrho : [0, 1] \to [0, 1]$ is a monotone increasing smooth function which satisfies the following conditions:

- $\varrho(t) = 0$ for $t \in \left[0, \frac{1}{3}\right]$;
- $\varrho(t) = 1$ for $t \in \left[\frac{2}{3}, 1\right]$.

For $(w_1, w_2) \in \partial D^2 \times \partial D^2$, $d\varphi_i(V_2(w_1, w_2))$ is calculated as follows:

$$\begin{aligned}
&d\varphi_i(V_2(w_1, w_2)) \\
=&d\varphi_i\left(\frac{1}{\varepsilon_1\varepsilon_2}\left(x_1\frac{\partial}{\partial x_2} - y_1\frac{\partial}{\partial y_2}\right)\right) && (\because |w_1| = 1) \\
=&\frac{1}{\varepsilon_1\varepsilon_2}x_1 d\varphi_i\left(\frac{\partial}{\partial x_2}\right) - \frac{1}{\varepsilon_1\varepsilon_2}y_1 d\varphi_i\left(\frac{\partial}{\partial y_2}\right) \\
=&\frac{1}{\varepsilon_1}x_1\left(x_1\frac{\partial}{\partial s} + y_1\frac{\partial}{\partial t}\right) - \frac{1}{\varepsilon_1}y_1\left(-y_1\frac{\partial}{\partial s} + x_1\frac{\partial}{\partial t}\right) \\
=&\frac{1}{\varepsilon_1}({x_1}^2 + {y_1}^2)\frac{\partial}{\partial s} \\
=&V_1(\varphi_i(w_1, w_2)).
\end{aligned}$$

Hence, we can define a vector field $V = V_1 \cup V_2$ on the manifold $\partial X_h$. Moreover, it can be shown that $V_1$ and $V_2$ is a lift of the vector field $\frac{d}{d\theta}\exp(\sqrt{-1}\theta)$ by the map $\pi_h$. Thus, the vector field $V$ is a lift of $\frac{d}{d\theta}\exp(\sqrt{-1}\theta)$. We can show that the vector field $V$ is compatible with the involution $\tilde{\omega}_2$ by direct calculation. This completes the proof of Lemma 4.3.2. □

We choose a base point $q_0 \in \partial D^2 \setminus (\amalg I_{q_i})$ and define a map $\Theta_V : f^{-1}(q_0) \to f^{-1}(q_0)$ as follows:

$$\Theta_V(x) = c_{V,x}(2\pi),$$

where $c_{V,x}$ is the integral curve of the vector field $V$ constructed in Lemma 4.3.2 which satisfies $c_{V,x}(0) = x$. We identify $f^{-1}(q_0)$ with the surface $\Sigma_g$ via the projection onto the second component. Then the map $\Theta_V$ is contained in the centralizer $C(\iota) \subset \mathrm{Diff}_+ \Sigma_g$ since the vector field $V$ is compatible with $\tilde{\omega}_2$. The isotopy class represented by $\Theta_V$ is the monodromy of the boundary of $X_h$. In particular, this class is contained in the group $\mathcal{H}_g(c)$. By Lemma 4.2.1, there exists an isotopy $H_t : \Sigma_g \to \Sigma_g$ satisfying the following conditions:

- $H_0 = \Theta_V$;
- $H_1$ preserves the curve $c$ as a set;
- for each level $t$, $H_t$ is in the centralizer $C(\iota)$.

We obtain the following isomorphism of $\Sigma_g$-bundles:

$$\partial X_h \cong [0,1] \times \Sigma_g/((1,x) \sim (0, H_1(x))).$$

We identify the above $\Sigma_g$-bundles via the isomorphism. Under this identification, the involution $\tilde{\omega}_2$ acts on $\partial X_h$ as follows:

$$\tilde{\omega}_2(t, x) = (t, \iota(x)),$$

where $(t, x)$ is an element in $[0,1] \times \Sigma_g/((1,x) \sim (0, H_1(x))) \cong \partial X_h$.

**Step 3**: In this step, we define an involution $\omega_3$ on the round 2-handle $R^2$. Since $c$ is non-separating and $c$ is preserved by $\iota$, $c$ contains two fixed points of the involution $\iota$. We denote these points by $v_1$ and $v_2$. We can take a tubular neighborhood $\nu c \cong S^1 \times [-1,1]$ in $\Sigma_g$ so that the involution $\iota$ acts on $\nu c$ as follows:

$$\iota(z,t) = (\overline{z}, -t).$$

By perturbing the map $H_1$, we can assume that $H_1$ preserves the neighborhood $\nu c$. Since the genus of the fibration $f$ is not equal to 1, the attaching region of the round 2-handle $R^2$ is $[0,1] \times \nu c/((1,x) \sim (0, H_1(x)))$.

**Case 3.1**: If $H_1$ preserves the orientation of $c$ and two points $v_1$ and $v_2$, then the round handle $R^2$ is untwisted and the restriction $H_1|_{\nu c}$ is described as follows:

$$H_1(z,t) = (z,t),$$

where $(z,t) \in S^1 \times [-1,1] \cong \nu c$. Moreover, the attaching map of the round handle is described as follows:

$$\begin{array}{cccc} \varphi: & [0,1] \times \partial D^2 \times D^1/\sim & \longrightarrow & [0,1] \times S^1 \times [-1,1]/\sim \\ & \uplus & & \uplus \\ & (s,z,t) & \longmapsto & (s,z,t), \end{array}$$

where $[0,1] \times \partial D^2 \times D^1$ is the attaching region of $R^2$ and $[0,1] \times S^1 \times [-1,1] \cong [0,1] \times \nu c$ is the subset of $\partial X_h$. We define an involution $\omega_3$ on the round handle as follows:

$$\begin{array}{cccc} \omega_3: & [0,1] \times D^2 \times D^1/\sim & \longrightarrow & [0,1] \times D^2 \times D^1/\sim \\ & \uplus & & \uplus \\ & (s,z,t) & \longmapsto & (s,\overline{z},-t), \end{array}$$

Then the following diagram commutes:

$$\begin{array}{ccc} [0,1] \times \partial D^2 \times D^1 & \xrightarrow{\omega_3} & [0,1] \times \partial D^2 \times D^1 \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ [0,1] \times S^1 \times [-1,1] & \xrightarrow{\tilde{\omega}_2} & [0,1] \times S^1 \times [-1,1]. \end{array}$$

Therefore, we obtain an involution $\tilde{\omega}_3 = \tilde{\omega}_2 \cup \omega_3$ on $X_h \cup X_r = X_h \cup R^2$.

**Case 3.2**: If $H_1$ preserves the orientation of $c$ but does not preserve two points $v_1$ and $v_2$, then the round handle $R^2$ is untwisted and the restriction $H_1|_{\nu c}$ is described as follows:

$$H_1(z,t) = (-z,t),$$

The attaching map of the round handle is described as follows:

$$\begin{array}{cccc} \varphi: & [0,1] \times \partial D^2 \times D^1/\sim & \longrightarrow & [0,1] \times S^1 \times [-1,1]/\sim \\ & \uplus & & \uplus \\ & (s,z,t) & \longmapsto & (s,\exp(\pi\sqrt{-1}s)z,t). \end{array}$$

We define an involution $\omega_3$ on the round handle as follows:

$$\begin{array}{cccc} \omega_3: & [0,1] \times D^2 \times D^1/\sim & \longrightarrow & [0,1] \times D^2 \times D^1/\sim \\ & \uplus & & \uplus \\ & (s,z,t) & \longmapsto & (s,\exp(-2\pi\sqrt{-1}s)\overline{z},-t), \end{array}$$

Then we can define an involution $\tilde{\omega}_3 = \tilde{\omega}_2 \cup \omega_3$ on $X_h \cup X_r = X_h \cup R^2$ by the same reason as in Case 3.1.

**Case 3.3**: If $H_1$ does not preserve the orientation of $c$ but preserves two points $v_1$ and $v_2$, then the round handle $R^2$ is twisted and the restriction $H_1|_{\nu c}$ is described as follows:

$$H_1(z,t) = (\overline{z}, -t),$$

where $(z,t) \in S^1 \times [-1,1] \cong \nu c$. Moreover, the attaching map of the round handle is described as follows:

$$\begin{array}{cccc} \varphi : & [0,1] \times \partial D^2 \times D^1/\sim & \longrightarrow & [0,1] \times S^1 \times [-1,1]/\sim \\ & \Psi & & \Psi \\ & (s,z,t) & \longmapsto & (s,z,t). \end{array}$$

We define an involution $\omega_3$ on the round handle as follows:

$$\begin{array}{cccc} \omega_3 : & [0,1] \times D^2 \times D^1/\sim & \longrightarrow & [0,1] \times D^2 \times D^1/\sim \\ & \Psi & & \Psi \\ & (s,z,t) & \longmapsto & (s,\overline{z},-t), \end{array}$$

Then we can define an involution $\tilde{\omega}_3 = \tilde{\omega}_2 \cup \omega_3$ on $X_h \cup X_r = X_h \cup R^2$.

**Case 3.4**: If $H_1$ preserves neither the orientation of $c$ nor two points $v_1$ and $v_2$, then the round handle $R^2$ is twisted and the restriction $H_1|_{\nu c}$ is described as follows:

$$H_1(z,t) = (-\overline{z}, -t),$$

where $(z,t) \in S^1 \times [-1,1] \cong \nu c$. Moreover, the attaching map of the round handle is described as follows:

$$\begin{array}{cccc} \varphi : & [0,1] \times \partial D^2 \times D^1/\sim & \longrightarrow & [0,1] \times S^1 \times [-1,1]/\sim \\ & \Psi & & \Psi \\ & (s,z,t) & \longmapsto & (s,\exp(\pi\sqrt{-1}s)z,t). \end{array}$$

We define an involution $\omega_3$ on the round handle as follows:

$$\begin{array}{cccc} \omega_3 : & [0,1] \times D^2 \times D^1/\sim & \longrightarrow & [0,1] \times D^2 \times D^1/\sim \\ & \Psi & & \Psi \\ & (s,z,t) & \longmapsto & (s,\exp(-2\pi\sqrt{-1}s)\overline{z},-t), \end{array}$$

Then we can define an involution $\tilde{\omega}_3 = \tilde{\omega}_2 \cup \omega_3$ on $X_h \cup X_r = X_h \cup R^2$.

Eventually, we obtain the involution $\tilde{\omega}_3$ on $X_h \cup X_r$ in any cases. We next look at $\Sigma_{g-1}$-bundle structure of $\partial(X_h \cup X_r)$. The projection $\pi_r : \partial(X_h \cup X_r) \to [0,1]/\{0,1\}$ of this bundle is described as follows:

$$\begin{aligned} \pi_r(s,x) &= s \quad \big((s,x) \in \big([0,1] \times \Sigma_g/(1,x) \sim (0,H_1(x))\big) \setminus ([0,1] \times \nu c/\sim)\big); \\ \pi_r(s,z,t) &= s \quad \big((s,z,t) \in [0,1] \times D^2 \times \partial D^1\big). \end{aligned}$$

Indeed, it is easy to show that $\pi_r$ is well-defined.

**Lemma 4.3.3.** *The involution $\tilde{\omega}_3$ preserves the fibers of $\pi_r$. Moreover, there exists a lift $\tilde{V}$ of the vector field $\dfrac{d}{ds}$ on $[0,1]/\{0,1\}$ by the map $\pi_r$ which is compatible with the involution $\tilde{\omega}_3$.*

*Proof of Lemma 4.3.3.* It is obvious that the involution $\tilde{\omega}_3$ preserves the fibers of $\pi_r$. We construct $\tilde{V}$ as we do in Lemma 4.3.2. We define a vector field $\tilde{V}_1$ on $\big([0,1]\times\Sigma_g/\sim\big)\setminus\big([0,1]\times\nu c/\sim\big)$ as follows:

$$\tilde{V}_1(s,x)=\frac{d}{ds}.$$

We first consider the case $H_1$ preserves the points $v_1$ and $v_2$. In this case, we define a vector field $\tilde{V}_2$ on the round handle $R^2$ as follows:

$$\tilde{V}_2(s,z,t)=\frac{d}{ds},$$

where $(s,z,t)\in[0,1]\times D^2\times\partial D^1\subset\partial R^2$. It is easy to verify the equality $d\varphi\left(\frac{d}{ds}\right)=\frac{d}{ds}$. Hence, we can define vector field $\tilde{V}=\tilde{V}_1\cup\tilde{V}_2$ on $\partial(X_h\cup X_r)$. It is obvious that $\tilde{V}$ is a lift of the vector field $\frac{d}{ds}$ on $[0,1]/\{0,1\}$ by $\pi_r$ and is compatible with the involution $\tilde{\omega}_3$.

We next consider the case $H_1$ does not preserve the points $v_1$ and $v_2$. In this case, we define a vector field $\tilde{V}_2$ on $R^2$ as follows:

$$\tilde{V}_2(s,x+y\sqrt{-1},t)=\frac{d}{ds}+\pi y\frac{\partial}{\partial x}-\pi x\frac{\partial}{\partial y},$$

where $(s,x+y\sqrt{-1},t)\in[0,1]\times D^2\times\partial D^1\subset\partial R^2$. The differential $d\varphi(\tilde{V}_2(s,x+\sqrt{-1}y,t))$ is calculated as follows:

$$\begin{aligned}
&d\varphi(\tilde{V}_2(s,x+\sqrt{-1}y,t))\\
=&d\varphi\left(\frac{d}{ds}+\pi y\frac{\partial}{\partial x}-\pi x\frac{\partial}{\partial y}\right)\\
=&\left(\frac{d}{ds}+\pi(-x\sin\pi s-y\cos\pi s)\frac{d}{dx}+\pi(x\cos\pi s-y\sin\pi s)\frac{d}{dy}\right)\\
&\quad+\pi y\left(\cos\pi s\frac{d}{dx}+\sin\pi s\frac{d}{dy}\right)-\pi x\left(-\sin\pi s\frac{d}{dx}+\cos\pi s\frac{d}{dy}\right)\\
=&\frac{d}{ds}\\
=&\tilde{V}_1(\varphi(s,x+\sqrt{-1}y,t)).
\end{aligned}$$

Hence, we can define a vector field $\tilde{V}=\tilde{V}_1\cup\tilde{V}_2$ on $\partial(X_h\cup X_r)$. It is obvious that $\tilde{V}$ is a lift of the vector field $\frac{d}{ds}$ on $[0,1]/\{0,1\}$ by $\pi_r$. To verify that $\tilde{V}$ is compatible with the involution $\tilde{\omega}_3$, we need to prove that the following equation holds for any points $x\in\partial(X_h\cup X_r)$:

$$d\tilde{\omega}_3(\tilde{V}(x))=\tilde{V}(\tilde{\omega}_3(x)).$$

If $x$ is contained in $[0,1]\times\Sigma_g/\sim\setminus\big([0,1]\times\nu c/\sim\big)$, the above equation can be proved easily. If $x=(s,x+\sqrt{-1}y,t)\in[0,1]\times D^2\times\partial D^1\subset\partial R^2$, then $d\tilde{\omega}_3(\tilde{V}(x))$ is calculated as follows:

$$\begin{aligned}
&d\tilde{\omega}_3(\tilde{V}(x))\\
=&d\tilde{\omega}_3(\frac{d}{ds}+\pi y\frac{\partial}{\partial x}-\pi x\frac{\partial}{\partial y})\\
=&\left(\frac{d}{ds}+2\pi(-x\sin 2\pi s-y\cos 2\pi s)\frac{\partial}{\partial x}+2\pi(-x\cos 2\pi s+y\sin 2\pi s)\frac{\partial}{\partial y}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\pi y\left(\cos 2\pi s\frac{\partial}{\partial x}-\sin 2\pi s\frac{\partial}{\partial y}\right)-\pi x\left(-\sin 2\pi s\frac{\partial}{\partial x}-\cos 2\pi s\frac{\partial}{\partial y}\right)\\
=&\frac{d}{ds}+\pi(-x\sin 2\pi s-y\cos 2\pi s)\frac{\partial}{\partial x}+\pi(-x\cos 2\pi s+y\sin 2\pi s)\frac{\partial}{\partial y}\\
=&\tilde{V}(\tilde{\omega}_3(x)).
\end{aligned}
$$

Thus, $\tilde{V}$ is compatible with the involution $\tilde{\omega}_3$. This completes the proof of Lemma 4.3.3. □

We define the map $\Theta_{\tilde{V}}:\pi_r^{-1}(0)\to\pi_r^{-1}(0)$ as follows:

$$
\begin{array}{rccc}
\Theta_{\tilde{V}}: & \pi_r^{-1}(0) & \longrightarrow & \pi_r^{-1}(0)\\
 & \cup & & \cup\\
 & x & \longmapsto & c_{\tilde{V},x}(1),
\end{array}
$$

where $c_{\tilde{V},x}$ is the integral curve of $\tilde{V}$ starting at $x$. We identify the fiber $\pi_r^{-1}(0)$ with the surface $\Sigma_{g-1}$. The map $\Theta_{\tilde{V}}$ is contained in the centralizer $C(\iota)$ since $\tilde{V}$ is compatible with $\tilde{\omega}_3$. Moreover, $\Theta_{\tilde{V}}$ is isotopic to the identity map. By Theorem 4.1.1, we can take an isotopy $\tilde{H}_t:\Sigma_{g-1}\to\Sigma_{g-1}$ which satisfies the following conditions:

- $\tilde{H}_0=\Theta_{\tilde{V}}$;
- $\tilde{H}_1$ is the identity map;
- $\tilde{H}_t$ is contained in the centralizer $C(\iota)$.

Note that such an isotopy may not be taken if the condition $g\geq 3$ is dropped. Indeed, the map $\pi_0 C(\iota)\to\mathcal{M}_1$ induced by the inclusion is not injective.

By using the isotopy $\tilde{H}_t$, we obtain the following isomorphism of $\Sigma_{g-1}$-bundle:

$$\partial(X_h\cup X_r)\cong[0,1]\times\Sigma_{g-1}/(1,x)\sim(0,x).$$

The involution $\tilde{\omega}_3$ acts on $[0,1]\times\Sigma_{g-1}/(1,x)\sim(0,x)$ via the above isomorphism as follows:

$$\tilde{\omega}_3(s,x)=(s,\iota(x)).$$

**Step 4**: We define an involution $\omega_4$ on $D^2\times\Sigma_{g-1}$ as follows:

$$\omega_4(z,x)=(z,\iota(x)),$$

where $(z,x)\in D^2\times\Sigma_{g-1}$. Let $\Phi:[0,1]\times\Sigma_{g-1}/\sim\to\partial D^2\times\Sigma_{g-1}$ be the attaching map of the lower side. Since the genus of the fibration $f$ is greater than 2, we can assume that $\Phi$ is given by $\Phi(s,x)=(\exp(2\pi\sqrt{-1}s),x)$. In particular, the following diagram commutes:

$$
\begin{array}{ccc}
[0,1]\times\Sigma_{g-1}/\sim & \xrightarrow{\tilde{\omega}_3} & [0,1]\times\Sigma_{g-1}/\sim\\
\Phi\big\downarrow & & \big\downarrow\Phi\\
\partial D^2\times\Sigma_{g-1} & \xrightarrow{\omega_4} & \partial D^2\times\Sigma_{g-1}.
\end{array}
$$

Hence, we obtain an involution $\omega=\tilde{\omega}_3\cup\omega_4$ on $X$.

We next look at the fixed point set of $\omega$. The involution $\omega$ is equal to $\mathrm{id} \times \iota$ on $D^2 \times \Sigma_g$. Thus, we obtain:

$$X^\omega \cup D^2 \times \Sigma_g = D^2 \times \{v_1, \ldots, v_{2g+2}\},$$

where $v_1, \ldots, v_{2g+2} \in \Sigma_g$ are the fixed points of $\iota$. Note that $X^\omega \cup D^2 \times \Sigma_g$ has the natural orientation derived from the orientation of $D^2$.

The involution $\omega$ acts on the 2-handle $h_i^2 = D^2 \times D^2$ as follows:

$$\omega(w_1, w_2) = \begin{cases} (\overline{w_1}, \overline{w_2}) & (c_i\text{:non-separating}), \\ (-w_1, -w_2) & (c_i\text{:separating}), \end{cases}$$

where $(w_1, w_2) \in D^2 \times D^2$. Thus, the fixed point set $h_i^{2\omega}$ is equal to $(D^2 \cap \mathbb{R}) \times (D^2 \cap \mathbb{R})$ if $c_i$ is non-separating, and is equal to $\{(0,0)\}$ if $c_i$ is separating. Furthermore, if $c_i$ is non-separating, we can give an orientation to $(D^2 \cap \mathbb{R}) \times (D^2 \cap \mathbb{R})$ which is compatible with the orientation of $D^2 \times \{v_1, \ldots, v_{2g+2}\}$. Hence, the fixed point set ${X_h}^\omega$ is the union of the oriented surfaces and the $s$ points, where $s$ is the number of Lefschetz singularities of $f$ whose vanishing cycle is separating.

The involution $\omega$ acts on the round 2-handle $R^2$ in the following way:

- if $H_1$ preserves the two points $v_1$ and $v_2$, then $\omega(s, z, t)$ is equal to $(s, \overline{z}, -t)$ for $(s, z, t) \in R^2 = [0,1] \times D^2 \times D^1 / \sim$;
- if $H_1$ does not preserve the two points $v_1$ and $v_2$, then $\omega(s, z, t)$ is equal to $(s, \exp(-2\pi\sqrt{-1}s)\overline{z}, -t)$ for $(s, z, t) \in R^2 = [0,1] \times D^2 \times D^1 / \sim$.

The fixed point set $R^{2\omega}$ is equal to $[0,1] \times (D^2 \cap \mathbb{R}) \times \{0\} / \sim$ if $H_1$ preserves the two points $v_1$ and $v_2$, and ${R_2}^\omega$ is equal to the following set otherwise:

$$\{(s, z, 0) \in R^2 \mid z = r\exp(-\pi\sqrt{-1}s), r \in [-1, 1]\}.$$

In particular, the fixed point set $R^{2\omega}$ is equal to the annulus or the Möbius band. As explained in the previous paragraph, we can give an orientation of the 2-dimensional part of ${X_h}^\omega$ in the canonical way. It is easy to see that any orientation of $R^{2\omega}$ is not compatible with this canonical orientation of ${X_h}^\omega$. In particular, even if $R^{2\omega}$ is the annulus, the 2-dimensional part of the fixed point set $(X_h \cup X_r)^\omega$ may not be orientable. Indeed, this part is orientable if and only if $R^{2\omega}$ is the annulus, and there is a connected component in ${X_h}^\omega$ whose boundary contains only one component of $\partial R^{2\omega}$.

The involution $\omega$ is equal to $\mathrm{id} \times \iota$ on $D^2 \times \Sigma_{g-1}$. Thus, the fixed point set $(D^2 \times \Sigma_{g-1})^\omega$ is equal to $D^2 \times \{\tilde{v}_1, \ldots, \tilde{v}_{2g}\}$, where $\{\tilde{v}_1, \ldots, \tilde{v}_{2g}\}$ is the set of the fixed points of $\iota$. Eventually, $X^\omega$ is the union of the closed surfaces and the $s$ points. The 2-dimensional part of $X^\omega$ is orientable if and only if that of $(X_h \cup X_r)^\omega$ is orientable. This completes the proof of the statement in Theorem 4.3.1 on the fixed point set of $\omega$.

We next extend the involution $\omega$ to the manifold $X \# s\overline{\mathbb{CP}^2}$. We assume that the curves $c_{k_1}, \ldots, c_{k_s}$ are separating. We construct the manifold $X \# s\overline{\mathbb{CP}^2}$ by blowing up $X$ $s$ times at $(0,0) \in h_{k_i}^2$ $(i = 1, \ldots, s)$. We can obtain a natural decomposition of $X \# s\overline{\mathbb{CP}^2}$ as follows:

$$D^2 \times \Sigma_g \cup (h_1^2 \amalg \overset{\hat{k_1}, \ldots, \hat{k_s}}{\cdots\cdots} \amalg h_n^2) \cup (\tilde{h}_{k_1} \amalg \cdots \amalg \tilde{h}_{k_s}) \cup R^2 \cup D^2 \times \Sigma_{g-1},$$

where $\tilde{h}_{k_i} = \{((w_1, w_2), [l_1 : l_2]) \in D^2 \times D^2 \times \mathbb{CP}^1 \mid w_1 l_2 - w_2 l_1 = 0\} \cong h_{k_i} \# \overline{\mathbb{CP}^2}$. We define an involution $\overline{\omega}$ on $X \# s\overline{\mathbb{CP}^2}$ as follows:

$$\overline{\omega}(x) = \omega(x) \ \ (x \in X \# s\overline{\mathbb{CP}^2} \setminus (\tilde{h}_{k_1} \amalg \cdots \amalg \tilde{h}_{k_s})),$$
$$\overline{\omega}((w_1, w_2), [l_1 : l_2]) = ((-w_1, -w_2), [l_1 : l_2]) \ \ (((w_1, w_2), [l_1 : l_2]) \in \tilde{h}_{k_i}).$$

It is obvious that $\overline{\omega}$ is an extension of $\omega$. The fixed point set of $\overline{\omega}$ is the union of the 2-dimensional part of $X^{\omega}$ and $s$ 2-spheres.

We next prove that $X \# s\overline{\mathbb{CP}^2}/\overline{\omega}$ is diffeomorphic to $S \# 2s\overline{\mathbb{CP}^2}$, where $S$ is an $S^2$-bundle over $S^2$. Since $\Sigma_g/\iota$ is diffeomorphic to $S^2$, it is easy to see that $D^2 \times \Sigma_g/\overline{\omega}$ is diffeomorphic to $D^2 \times S^2$. Thus, the manifold $X \# s\overline{\mathbb{CP}^2}$ is obtained by attaching $h_j/\overline{\omega}$ $(j \neq k_1, \ldots, k_s)$, $\tilde{h}_{k_i}/\overline{\omega}$, $R^2/\overline{\omega}$ and $D^2 \times \Sigma_{g-1}/\overline{\omega} \cong D^2 \times S^2$ to $D^2 \times S^2$.

**Lemma 4.3.4.** *Suppose that $c_i$ is non-separating. Then,*

$$(D^2 \times \Sigma_g \cup_{\varphi_i} h_i^2)/\overline{\omega} \cong D^2 \times S^2.$$

*Proof of Lemma 4.3.4.* If we identify $h_i^2 = D^2 \times D^2$ with $D^4$, then $\overline{\omega}$ is equal to the covering transformation of the double covering $D^4 \to D^4$ branched at the unknotted 2-disk in $D^4$. In particular, we obtain $h_i^2/\overline{\omega}$ is diffeomorphic to $D^4$. Moreover, the attaching region of $h_i^2$ corresponds to the 3-disk in $\partial D^4$ under the diffeomorphism. Denote by $\overline{\varphi_i} : h_i^2/\overline{\omega} \to \partial D^2 \times \Sigma_g/\overline{\omega}$ the embedding induced by $\varphi_i$. We obtain:

$$\begin{aligned}(D^2 \times \Sigma_g \cup_{\varphi_i} h_i^2)/\overline{\omega} &\cong (D^2 \times \Sigma_g/\overline{\omega}) \cup_{\overline{\varphi_i}} h_i^2/\overline{\omega} \\ &\cong D^2 \times S^2 \natural D^4 \\ &\cong D^2 \times S^2.\end{aligned}$$

This completes the proof of Lemma 4.3.4. □

**Lemma 4.3.5.** *For each $i \in \{1, \ldots, s\}$, $(D^2 \times \Sigma_g \cup_{\varphi_i} \tilde{h}_{k_i}^2)/\overline{\omega} \cong D^2 \times S^2 \# 2\overline{\mathbb{CP}^2}$.*

*Proof of Lemma 4.3.5.* By eliminating the corner of $D^2 \times D^2$, we identify $\tilde{h}_{k_i}^2$ with the following manifold:

$$H = \{((w_1, w_2), [l_1 : l_2]) \in D^4 \times \mathbb{CP}^1 \mid w_1 l_2 - w_2 l_1 = 0\}.$$

Under this identification, the attaching region of $\tilde{h}_{k_i}^2$ corresponds to the tubular neighborhood of the circle $\{((w_1, 0), [1 : 0]) \in \partial H \mid |w_1| = 1\}$ in $\partial H$. Let $p_2 : H \to \mathbb{CP}^1$ be the projection onto the second component. The map $p_2$ is the $D^2$-bundle over the 2-sphere with Euler number $-1$. We define $D_1, D_2 \subset \mathbb{CP}^1$, and local trivializations $\psi_1$ and $\psi_2$ of $p_2$ as follows:

$$D_1 = \{[l_1 : l_2] \in \mathbb{CP}^1 \mid |l_1| \geq |l_2|\},$$
$$D_2 = \{[l_1 : l_2] \in \mathbb{CP}^1 \mid |l_2| \geq |l_1|\},$$

$$\begin{array}{cccc} \psi_1 : & D^2 \times D^2 & \longrightarrow & p_2^{-1}(D_1) \\ & \cup & & \cup \\ & (w_1, w_2) & \longmapsto & \left( \dfrac{w_2}{\sqrt{1 + |w_1|^2}} (1, w_1), [1, w_1] \right), \end{array}$$

$$\begin{array}{rccc} \psi_2: & D^2\times D^2 & \longrightarrow & p_2^{-1}(D_2) \\ & \cup & & \cup \\ & (w_1,w_2) & \longmapsto & \left(\dfrac{w_2}{\sqrt{1+|w_1|^2}}(w_1,1),[w_1,1]\right). \end{array}$$

Denote $p_2^{-1}(D_1)$ and $p_2^{-1}(D_2)$ by $H_1$ and $H_2$, respectively. We identify $H_1$ and $H_2$ with $D^2\times D^2$ by the above trivializations. The manifold $H$ can be identified with $D^2\times D^2\cup_\Psi D^2\times D^2$, where $\Psi=\psi_1^{-1}\circ\psi_2:(w_1,w_2)\longmapsto(\frac{1}{w_1},w_1w_2)$. Under the identification, the attaching region of $H$ corresponds to $\partial D^2\times D^2\subset\partial H_1$.

We define $\tilde{H}=\tilde{H}_1\cup_{\tilde{\Psi}}\tilde{H}_2$, where $\tilde{H}_i=D^2\times D^2$ $(i=1,2)$ and $\tilde{\Psi}:\partial D^2\times D^2\to\partial D^2\times D^2$ is a diffeomorphism defined as follows:

$$\tilde{\Psi}(w_1,w_2)=(\frac{1}{w_1},{w_1}^2w_2).$$

We can define $\mathcal{P}:H\to\tilde{H}$ as follows:

$$\mathcal{P}(w_1,w_2)=\begin{cases}(w_1,{w_2}^2)\in\tilde{H}_1 & ((w_1,w_2)\in H_1),\\ (w_1,{w_2}^2)\in\tilde{H}_2 & ((w_1,w_2)\in H_2).\end{cases}$$

The map $\mathcal{P}$ is a double branched covering branched at the 0-section of $\tilde{H}$ as a $D^2$-bundle. Moreover, $\tilde{\omega}$ is the non-trivial covering transformation of $\mathcal{P}$. Thus, we obtain $H/\tilde{\omega}$ is diffeomorphic to $\tilde{H}$.

Since the attaching region of $H$ is mapped by $\mathcal{P}$ to $D^2\times\partial D^2\subset\partial\tilde{H}_1$, we can regard $\tilde{H}_1$ and $\tilde{H}_2$ as 2-handles. Thus, $(D^2\times\Sigma_g\cup_{\varphi_i}\tilde{h}^2_{k_i})/\overline{\omega}$ is obtained by attaching the 2-handles $\tilde{H}_1$ and $\tilde{H}_2$ to $D^2\times S^2$. To prove the statement, we look at the attaching maps of $\tilde{H}_1$ and $\tilde{H}_2$.

We take an identification $\nu c_{k_i}\cong J\times S^1$ as we take in Step 2 of the construction of $\omega$. The attaching map $\varphi_{k_i}$ of the 2-handle $h^2_{k_i}$ satisfies $\varphi_{k_i}(w_1,w_2)=(\varepsilon_2w_2w_1,w_1)$. Since the manifold $H$ is obtained by eliminating the corner of $\tilde{h}^2_{k_i}$, the attaching map of $H_1$ is described as follows:

$$\begin{array}{rcccc} \Phi: & \partial H_1\supset & D^2\times\partial D^2 & \longrightarrow & J\times S^1 \\ & & \cup & & \cup \\ & & (w_1,w_2) & \longmapsto & (\varepsilon_2{w_2}^2w_1,w_2). \end{array}$$

For an element $(z_1,z_2)\in J\times S^1$, the image $\overline{\omega}(z_1,z_2)$ is equal to $(z_1,-z_2)$. Thus, the manifold $J\times S^1/\overline{\omega}$ is diffeomorphic to $J\times S^1$ and the quotient map $/\overline{\omega}:J\times S^1\to J\times S^1/\overline{\omega}\cong J\times S^1$ satisfies the equality $/\overline{\omega}(z_1,z_2)=(z_1,{z_2}^2)$. The attaching map $\tilde{\Phi}:D^2\times\partial D^2\to J\times S^1$ of $\tilde{H}_1$ satisfies the equality $\tilde{\Phi}(w_1,w_2)=(\varepsilon_2w_2w_1,w_2)$. It is easy to see that the attaching circle of $\tilde{H}_1$ is equal to the circle $c_{k_i}/\overline{\omega}$. Moreover, the framing of $\tilde{\Phi}$ is $-1$ relative to the framing along $\{*\}\times S^2\subset\partial D^2\times S^2$.

By the definition of $\tilde{\Psi}$, the attaching circle of $\tilde{H}_2$ is equal to the belt circle of $\tilde{H}_1$, which is isotopic to the meridian of the attaching circle of $\tilde{H}_1$. In particular, there exists the natural framing of the attaching circle of $\tilde{H}_2$ which is represented by the meridian of the attaching circle of $\tilde{H}_1$ parallel to the attaching circle of $\tilde{H}_2$. Since the Euler number of $\tilde{H}$ as a $D^2$-bundle is equal to $-2$, the framing of the attaching map $\tilde{\Psi}$ is equal to $-2$ relative to the natural framing. Therefore, we can draw a Kirby diagram of $(D^2\times\Sigma_g\cup_{\varphi_i}\tilde{h}^2_{k_i})/\overline{\omega}$ as shown in Figure 4.3.1. It is obvious that this manifold is diffeomorphic to $D^2 \times S^2 \# 2\overline{\mathbb{CP}^2}$ and this completes the proof of Lemma 4.3.5. □

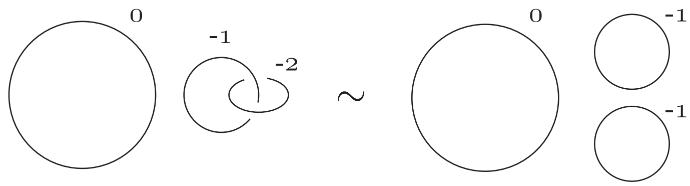


Figure 4.3.1: the $(-1)$-framed knot describes $\tilde{H}_1$, while the $(-2)$-framed knot describes $\tilde{H}_2$.

By applying the arguments in Lemma 4.3.4 and 4.3.5 successively, we can prove that $X_h \# s\overline{\mathbb{CP}^2}/\varpi$ is diffeomorphic to $D^2 \times S^2 \# 2s\overline{\mathbb{CP}^2}$.

**Lemma 4.3.6.** $((X_h \cup X_r)\# s\overline{\mathbb{CP}^2})/\varpi \cong D^2 \times S^2 \# 2s\overline{\mathbb{CP}^2}$.

*Proof of Lemma 4.3.6.* We can decompose $R^2$ into two components as follows:

$$R^2 = \left[0, \frac{1}{2}\right] \times D^2 \times D^1 \cup \left[\frac{1}{2}, 1\right] \times D^2 \times D^1.$$

Denote $\left[0, \frac{1}{2}\right] \times D^2 \times D^1$ and $\left[\frac{1}{2}, 1\right] \times D^2 \times D^1$ by $R_1$ and $R_2$, respectively. It is easy to see that $R_i/\varpi$ is diffeomorphic to $D^4$ and $R_i$ is the double covering of $D^4 \cong R_i/\varpi$ branched at the unknotted 2-disk.

The attaching region of $R_1$ is equal to $[0, \frac{1}{2}] \times \partial D^2 \times D^1$. The quotient $[0, \frac{1}{2}] \times \partial D^2 \times D^1/\varpi$ is a 3-ball in $\partial D^4 \cong \partial R_1$. Thus, we obtain:

$$\begin{aligned}(X_h \cup R_1)/\varpi &\cong X_h/\varpi \cup R_1/\varpi \\ &\cong D^2 \times S^2 \# 2s\overline{\mathbb{CP}^2} \natural D^4 \\ &\cong D^2 \times S^2 \# 2s\overline{\mathbb{CP}^2}.\end{aligned}$$

The attaching region of $R_2$ is equal to $\left[\frac{1}{2}, 1\right] \times \partial D^2 \times D^1 \cup \{\frac{1}{2}, 1\} \times D^2 \times D^1$. The quotient $\left[\frac{1}{2}, 1\right] \times \partial D^2 \times D^1/\varpi$ is a 3-ball $D_0$ in $\partial D^4 \cong \partial R_2$, while $\{\frac{1}{2}, 1\} \times D^2 \times D^1/\varpi$ is a disjoint union of two 3-balls $D_1 \amalg D_2$ in $\partial D^4$. Both of the intersections $D_0 \cap D_1$ and $D_0 \cap D_2$ are 2-disks in $\partial D_0$. Eventually, the attaching region of $R_2$ is a 3-ball in $\partial D^4$. Thus, we can prove $(X_h \cup R_1 \cup R_2)/\varpi$ is diffeomorphic to $D^2 \times S^2 \# 2s\overline{\mathbb{CP}^2}$. This completes the proof of Lemma 4.3.6. □

It is easy to see that $D^2 \times \Sigma_{g-1}/\varpi$ is diffeomorphic to $D^2 \times S^2$, and is attached to $(X_h \cup X_r)/\varpi$ so that the following diagram commutes:

$$\begin{array}{ccccc}(X_h \cup X_r)/\varpi \supset & S^1 \times S^2 & \longrightarrow & \partial D^2 \times S^2 & \subset D^2 \times \Sigma_{g-1}/\varpi \\ & \downarrow & & \downarrow & \\ & S^1 & \longrightarrow & \partial D^2, & \end{array}$$

where the upper horizontal arrow in the diagram represents the attaching map, the lower horizontal arrow represents the identity map, and vertical arrows represent the projection onto the first component (in other word, the attaching map is a bundle map as a $S^2$-bundle over $S^1$). In particular, we obtain:

$$X\#s\overline{\mathbb{CP}^2}/\overline{\omega} \cong S\#2s\overline{\mathbb{CP}^2}.$$

It is obvious that the quotient map $/\overline{\omega} : X\#s\overline{\mathbb{CP}^2} \to S\#2s\overline{\mathbb{CP}^2}$ is a double branched covering. This completes the proof of the statement (i) in Theorem 4.3.1. □

*Proof of (ii) in Theorem 4.3.1.* Let $F_h \subset X$ be a regular fiber in the higher side of $f$. It is easy to see that $F_h$ represents the same rational homology class of $X$ as that represented by $F$. Let $\omega : X \to X$ be the involution constructed in the proof of (i) in Theorem 4.3.1. If $f$ has no indefinite fold singularities, then the 2-dimensional part of the fixed point set $X^\omega$ of the involution $\omega$ is an orientable surface and the algebraic intersection number between this part and $F_h$ is equal to $2g+2$, especially is non-zero. Thus, the statement (ii) in Theorem 4.3.1 holds.

Suppose that $f$ has indefinite fold singularities. We first prove that $F_h$ represents a non-trivial rational homology class of $X_h \cup X_r$. To prove this, we construct an element $\mathcal{S}$ in the group $H_2(X_h \cup X_r, \partial(X_h \cup X_r); \mathbb{Q})$ such that $[F_h] \cdot \mathcal{S} \neq 0$. Let $\tilde{S}$ be the intersection between the 2-dimensional part of $X^\omega$ and $X_h$, which is the union of compact oriented surfaces. We use the notations $H_1$, $c$, $v_1$, $v_2$ and $R^2$ as we used in the proof of (i) in Theorem 4.3.1.

**Case 1**: If the map $H_1$ preserves the orientation of $c$ and two points $v_1$ and $v_2$, then $R^2$ is untwisted and $\tilde{S} \cap R^2 = \{(s, \pm 1, 0) \in R^2 \mid s \in [0,1]\}$ is a disjoint union of two circles. We define four annuli $A_1$, $A_2$, $A_3$ and $A_4$ as follows:

$$\begin{aligned} A_1 =& \{(s,t,0) \in R^2 \mid s \in [0,1], t \in [0,1]\}, \\ A_2 =& \{(s,t,0) \in R^2 \mid s \in [0,1], t \in [-1,0]\}, \\ A_3 =& \{(s,0,t) \in R^2 \mid s \in [0,1], t \in [0,1]\}, \\ A_4 =& \{(s,0,t) \in R^2 \mid s \in [0,1], t \in [-1,0]\}. \end{aligned}$$

The union $S = \tilde{S} \cup A_1 \cup A_2 \cup A_3 \cup A_4$ represents the homology class of the pair $(X_h \cup X_r, \partial(X_h \cup X_r))$ after giving suitable orientations to the annuli $A_1$, $A_2$, $A_3$ and $A_4$. We denote this class by $\mathcal{S}$. It is easy to verify that the intersection number $\mathcal{S} \cdot [F_h]$ is equal to $2g+2$, especially is non-zero.

**Case 2**: If the map $H_1$ preserves the orientation of $c$ but does not preserve two points $v_1$ and $v_2$, then $R^2$ is untwisted and $\tilde{S} \cap R^2 = \{(s, \pm \exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0,1]\}$ is a circle. We define three annuli $A_5$, $A_6$ and $A_7$ as follows:

$$\begin{aligned} A_5 =& \{(s, t\exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0,1], t \in [0,1]\} \\ & \cup \{(s, -t\exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0,1], t \in [0,1]\}, \\ A_6 =& \{(s,0,t) \in R^2 \mid s \in [0,1], t \in [0,1]\}, \\ A_7 =& \{(s,0,t) \in R^2 \mid s \in [0,1], t \in [-1,0]\}. \end{aligned}$$

The union $S = \tilde{S} \cup A_5 \cup A_6 \cup A_7$ represents the homology class of the pair $(X_h \cup X_r, \partial(X_h \cup X_r))$ after giving suitable orientations to the annuli $A_5$, $A_6$ and $A_7$. We denote this class by $\mathcal{S}$. It is easy to verify that the intersection number $\mathcal{S} \cdot [F_h]$ is equal to $2g+2$, especially is non-zero.

**Case 3**: If the map $H_1$ does not preserve the orientation of $c$ but preserves two points $v_1$ and $v_2$, then $R^2$ is twisted and $\tilde{S} \cap R^2 = \{(s, \pm 1, 0) \in R^2 \mid s \in [0,1]\}$ is a disjoint union of two circles. We define three annuli $A_8$, $A_9$ and $A_{10}$ as follows:

$$\begin{aligned} A_8 =& \{(s,t,0) \in R^2 \mid s \in [0,1], t \in [0,1]\}, \\ A_9 =& \{(s,t,0) \in R^2 \mid s \in [0,1], t \in [-1,0]\}, \\ A_{10} =& \{(s,0,t) \in R^2 \mid s \in [0,1], t \in [0,1]\} \\ & \cup \{(s,0,t) \in R^2 \mid s \in [0,1], t \in [-1,0]\}. \end{aligned}$$

The union $S = \tilde{S} \cup A_8 \cup A_9 \cup A_{10}$ represents the homology class of the pair $(X_h \cup X_r, \partial(X_h \cup X_r))$ after giving suitable orientations to the annuli $A_8$, $A_9$ and $A_{10}$. We denote this class by $\mathcal{S}$. It is easy to verify that the intersection number $\mathcal{S} \cdot [F_h]$ is equal to $2g+2$, especially is non-zero.

**Case 4**: If the map $H_1$ preserves neither the orientation of $c$ nor two points $v_1$ and $v_2$, then $R^2$ is twisted and $\tilde{S} \cap R^2 = \{(s, \pm \exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0,1]\}$ is a circle. We define two annuli $A_{11}$ and $A_{12}$ as follows:

$$\begin{aligned} A_{11} =& \{(s, t\exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0,1], t \in [0,1]\} \\ & \cup \{(s, -t\exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0,1], t \in [0,1]\}, \\ A_{12} =& \{(s,0,t) \in R^2 \mid s \in [0,1], t \in [0,1]\}, \\ & \cup \{(s,0,t) \in R^2 \mid s \in [0,1], t \in [-1,0]\}. \end{aligned}$$

The union $S = \tilde{S} \cup A_{11} \cup A_{12}$ represents the homology class of the pair $(X_h \cup X_r, \partial(X_h \cup X_r))$ after giving suitable orientations to the annuli $A_{11}$ and $A_{12}$. We denote this class by $\mathcal{S}$. It is easy to verify that the intersection number $\mathcal{S} \cdot [F_h]$ is equal to $2g+2$, especially is non-zero.

Eventually, we can construct the element $\mathcal{S}$ satisfying the desired condition in any cases. Thus, we complete to prove $[F_h]$ is not trivial in $H_2(X_h \cup X_r; \mathbb{Q})$.

We are now ready to prove the statement (ii) in Theorem 4.3.1. There exists the following exact sequence which is the part of the Meyer-Vietoris exact sequence:

$$H_2(S^1 \times \Sigma_{g-1}; \mathbb{Q}) \xrightarrow{i_1 \oplus i_2} H_2(X_h \cup X_r; \mathbb{Q}) \oplus H_2(D^2 \times \Sigma_{g-1}; \mathbb{Q}) \xrightarrow{j_1 - j_2} H_2(M; \mathbb{Q}).$$

Suppose that $(j_1 - j_2)([F_h], 0) = [F_h] = 0$. There exists an element $\mu \in H_2(S^1 \times \Sigma_{g-1}; \mathbb{Q})$ which satisfies the equality $(i_1 \oplus i_2)(\mu) = ([F_h], 0)$. By a Künneth formula, we obtain the following isomorphism:

$$H_2(S^1 \times \Sigma_{g-1}; \mathbb{Q}) \cong H_2(\Sigma_{g-1}; \mathbb{Q}) \oplus \left(H_1(\Sigma_{g-1}; \mathbb{Q}) \otimes H_1(S^1; \mathbb{Q})\right).$$

Since the map $i_2 : H_2(S^1 \times \Sigma_{g-1}; \mathbb{Q}) \to H_2(D^2 \times \Sigma_{g-1}; \mathbb{Q}) \cong H_2(\Sigma_{g-1}; \mathbb{Q})$ is regarded as the projection onto the first component via the above isomorphism, The element $\mu$ is contained in $H_1(\Sigma_{g-1}; \mathbb{Q}) \otimes H_1(S^1; \mathbb{Q})$. The involution $\omega$ acts on the component $H_2(\Sigma_{g-1}; \mathbb{Q})$ trivially and on the component $H_1(\Sigma_{g-1}; \mathbb{Q}) \otimes H_1(S^1; \mathbb{Q})$ by multiplying $-1$. Thus, we obtain:

$$\omega_*(\mu) = -\mu.$$

The composition $i_1 \circ \omega_*$ is equal to $\omega_* \circ i_1$ since $i_1$ is induced by the inclusion map. Thus, we obtain:

$$\begin{aligned}[F_h] &= \omega_*([F_h]) \\ &= \omega_* \circ i_1(\mu) \\ &= i_1 \circ \omega_*(\mu) \\ &= i_1 \circ (-\mu) = -[F_h].\end{aligned}$$

This means that $2[F_h] = 0$ in $H_2(X_h \cup X_r; \mathbb{Q})$. This contradicts $[F_h] \neq 0$. Therefore, we obtain $[F_h] \neq 0$ in $H_2(M; \mathbb{Q})$ and this completes the proof of the statement. □

**Remark 4.3.7.** By the argument similar to that in the proof of Theorem 4.3.1, we can generalize Theorem 4.3.1 to DBLFs as follows:

**Theorem 4.3.8.** *Let $f : X \to S^2$ be an HDBLF. Suppose that the genus of every connected component of fiber of $f$ is greater than or equal to 2.*

*(i) Let $s_1$ be the number of Lefschetz singularities of $f$ whose vanishing cycles are separating. We define $s_2$ as follows:*

$$s_2 = max\{s \in \mathbb{N} \mid f^{-1}(x) \text{ has } s \text{ components. } x \in S^2\}.$$

*Then, there exists an involution*

$$\omega : X \to X$$

*such that the fixed point set of $\omega$ is the union of (possibly non-orientable) surfaces and $s_1$ isolated points. Moreover, the involution $\omega$ can be extended to an involution*

$$\overline{\omega} : X \# s_1 \overline{\mathbb{CP}^2} \to X \# s_1 \overline{\mathbb{CP}^2}$$

*such that $X \# s_1 \overline{\mathbb{CP}^2} / \overline{\omega}$ is diffeomorphic to $\# s_2 S \# 2 s_1 \overline{\mathbb{CP}^2}$, where $S$ is $S^2$-bundle over $S^2$, and the quotient map*

$$/\overline{\omega} : X \# s_1 \overline{\mathbb{CP}^2} \to X \# s_1 \overline{\mathbb{CP}^2} / \overline{\omega} \cong \# s_2 S \# 2 s_1 \overline{\mathbb{CP}^2}$$

*is the double branched covering.*

*(ii) Let $F \in X$ be a regular fiber of $f$. Then $F$ represents a non-trivial rational homology class of $X$.*

We leave the details of the proof of Theorem 4.3.8 to the readers.

## 4.4 A generating set of $\mathcal{H}_g(c)$

In this section, we investigate the abelianization and a generating set of the group $\mathcal{H}_g(c)$. In the last paragraphs of Subsection 4.4.1 and Subsection 4.4.2, we will prove the following proposition:

**Proposition 4.4.1.** *Assume that $g$ is greater than or equal to 1.*

1. *Let $c$ be a non-separating simple closed curve of type I in Figure 2.3.1. The group $\mathcal{H}_g(c)$ is generated by*
$$\{t_{c_1}, \cdots, t_{c_{2g-1}}, t_{c_{2g+1}}, \iota_g\}.$$
2. *Let $1 \leq h \leq g-1$, and $c$ a separating simple closed curve of type $\mathrm{II}_h$ in Figure 2.3.1. The group $\mathcal{H}_g(c^{\mathrm{ori}})$ is generated by*
$$\{t_{c_1}, t_{c_2}, \cdots, t_{c_{2h}}, t_{c_{2h+2}}, t_{c_{2h+3}}, \cdots, t_{c_{2g+1}}\}.$$

### 4.4.1 When $c$ is non-separating

First, consider the case when $c$ is type I. For simplicity, we choose $c$ as in Figure 2.3.1. Let $\gamma \in \Sigma_g / \langle \iota_g \rangle$ be the projection of the curve $c$ by $p : \Sigma_g \to \Sigma_g / \langle \iota_g \rangle$. Identifying $\Sigma_g / \langle \iota_g \rangle$ with $S^2$, define a group $\mathcal{M}_0^{2g}(\gamma)$ by

$$\mathcal{M}_0^{2g}(\gamma) = \{[T] \in \mathcal{M}_0^{2g+2} \,|\, T(\gamma) = \gamma\}.$$

For a diffeomorphism $T \in C(\iota_g)$, we have a diffeomorphism

$$\bar{T} \in \mathrm{Diff}_+(S^2, p_1, p_2, \cdots, p_{2g+1}, p_{2g+2})$$

defined by $pT = \bar{T}p$ as in Section 4.1.1. Moreover, if $T \in C(\iota_g)$ preserves $c$ setwise, $\bar{T}$ also preserves the path $\gamma$ setwise. Hence, the image $\mathcal{P}_g(\mathcal{H}_g^s(c))$ is contained in $\mathcal{M}_0^{2g}(\gamma)$. Conversely, if $\bar{T} \in \mathrm{Diff}_+(S^2, p_1, p_2, \cdots, p_{2g+1}, p_{2g+2})$ preserves the path $\gamma$ setwise, there is a diffeomorphism $T \in C(\iota_g)$ such that $T(c) = c$ and $pT = \bar{T}p$. Thus, we have $\mathcal{P}_g(\mathcal{H}_g^s(c)) = \mathcal{M}_0^{2g}(\gamma)$. Consider the exact sequence obtained by restricting the homomorphism $\mathcal{P}_g : \mathcal{H}_g^s \to \mathcal{M}_0^{2g+2}$ in Theorem 4.1.2 to $\mathcal{H}_g^s(c)$.

**Lemma 4.4.2.** *For $g \geq 1$, the exact sequence*

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathcal{H}_g^s(c) \xrightarrow{\mathcal{P}_g} \mathcal{M}_0^{2g}(\gamma) \longrightarrow 1$$

*splits. In particular, we have $\mathcal{H}_g^s(c) \cong \mathbb{Z}/2\mathbb{Z} \times \mathcal{M}_0^{2g}(\gamma)$.*

*Proof.* Define a map $\lambda : \mathcal{H}_g^s(c) \to \mathbb{Z}/2\mathbb{Z}$ by $\lambda(\varphi) = 0$ if $\varphi_*[c] = [c] \in H_1(\Sigma_g; \mathbb{Z})$, and $\lambda(\varphi) = 1$ if $\varphi_*[c] = -[c] \in H_1(\Sigma_g; \mathbb{Z})$. Then, $\lambda$ is a homomorphism, and satisfies $\lambda([\iota_g]) = 1 \in \mathbb{Z}/2\mathbb{Z}$. Thus, it induces a splitting of the exact sequence. □

Let $s : \partial D^2 \to \partial D^2$ denote the half-rotation of the circle. Let $\mathcal{M}_{0,\mathrm{half}}^{2g}$ denote the group which consists of the path-connected components of $\{T \in \mathrm{Diff}_+(D^2, p_1, p_2, \cdots, p_{2g}) \,|\, T|_{\partial D^2} = s \text{ or } \mathrm{id}_{\partial D^2}\}$.

**Lemma 4.4.3.** *Assume that $g$ is greater than or equal to 1. The group $\mathcal{M}_0^{2g}(\gamma)$ is isomorphic to $\mathcal{M}_{0,\mathrm{half}}^{2g}$.*

*Proof.* Let $\mathcal{M}_0^{2g}(\gamma^{\mathrm{ori}})$ be a subgroup of $\mathcal{M}_0^{2g}(\gamma)$ consists of mapping classes which preserve the orientation of the path $\gamma$. First, we prove the isomorphism

$$\mathcal{M}_0^{2g}(\gamma^{\mathrm{ori}}) \cong \mathcal{M}_{0,1}^{2g}.$$

Let $\mathrm{Diff}_+(S^2, \{p_1, \cdots, p_{2g+2}\}, [\gamma])$ be the group which consists of orientation-preserving diffeomorphisms $T : S^2 \to S^2$ satisfying the following conditions:

- $T(\{p_1, \cdots, p_{2g+2}\}) = \{p_1, \cdots, p_{2g+2}\}$;
- there exists a closed neighborhood $\nu(\gamma)$ of $\gamma$ where $T|_{\nu(\gamma)}$ is the identity map.

Let $T$ be a representative of a mapping class in $\mathcal{M}_0^{2g}(\gamma^{\mathrm{ori}})$. Using the isotopy extension theorem, we can change $T$ into a diffeomorphism satisfying the conditions above by some isotopy. Moreover, we can also prove that

$$\mathcal{M}^{2g}(\gamma^{\mathrm{ori}}) \cong \pi_0 \operatorname{Diff}_+(S^2, \{p_1, \cdots, p_{2g+2}\}, [\gamma]),$$

using the isotopy extension theorem. Similarly, we denote by

$$\operatorname{Diff}_+(S^2 - \operatorname{Int} D^2, p_1, \cdots, p_{2g}, [\partial D^2])$$

a group of orientation-preserving diffeomorphisms $T : S^2 - \operatorname{Int} D^2 \to S^2 - \operatorname{Int} D^2$ such that there exists a closed neighborhood $\nu(\partial D^2)$ on which $T|_{\nu(\partial D^2)}$ is the identity map. We can also show that

$$\mathcal{M}_{0,1}^{2g} \cong \pi_0 \operatorname{Diff}_+(S^2 - \operatorname{Int} D^2, p_1, \cdots, p_{2g}, [\partial D^2]).$$

Separate the circle $\partial D^2$ into two arcs $\alpha : [0,1] \to \partial D^2$ and $\beta : [0,1] \to \partial D^2$ such that $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. If we identify $\alpha(t)$ and $\beta(t)$ in $S^2 - \operatorname{Int} D^2$, the quotient space is diffeomorphic to $S^2$. Choose an identification $L$ of the $(2g+3)$-tuples

$$\begin{aligned} &(S^2 - \operatorname{Int} D^2/(\alpha(t) \sim \beta(t)), p_1, \cdots, p_{2g}, \alpha(0), \alpha(1)) \\ \cong &(S^2, p_1, \cdots, p_{2g}, p_{2g+1}, p_{2g+2}). \end{aligned}$$

Since a diffeomorphism $T \in \operatorname{Diff}_+(S^2 - \operatorname{Int} D^2)$ satisfying $T|_{\nu(\partial D^2)} = \mathrm{id}_{\nu(\partial D^2)}$ induces a diffeomorphism $\bar{T}$ of $S^2 - \operatorname{Int} D^2/(\alpha(t) \sim \beta(t))$, we have the isomorphism $\mathcal{M}_{0,1}^{2g} \cong \mathcal{M}^{2g}(\gamma^{\mathrm{ori}})$ defined by $[T] \mapsto [L\bar{T}L^{-1}]$.

Next, we prove $\mathcal{M}_0^{2g}(\gamma) \cong \mathcal{M}_{0,\mathrm{half}}^{2g}$. Choose a diffeomorphism $r \in \operatorname{Diff}_+(S^2 - \operatorname{Int} D^2)$ such that $r\alpha(t) = \beta(1-t)$ and $r(\{p_1, \cdots, p_{2g}\}) = \{p_1, \cdots, p_{2g}\}$. It induces a diffeomorphism $\bar{r} \in \operatorname{Diff}_+ S^2$ such that $\bar{r}(\{p_1, \cdots, p_{2g}\}) = \{p_1, \cdots, p_{2g}\}$, $\bar{r}(p_{2g+1}) = p_{2g+2}$, and $\bar{r}(p_{2g+2}) = p_{2g+1}$. Consider the group consisting of diffeomorphisms $T$ of $S^2$ such that $T(\{p_1, \cdots, p_{2g+2}\}) = \{p_1, \cdots, p_{2g+2}\}$, and $T|_{\nu(\gamma)}$ is equal to $\bar{r}|_{\nu(\gamma)}$ or $\mathrm{id}_{\nu(\gamma)}$ for some closed neighborhood $\nu(\gamma)$ instead of $\operatorname{Diff}_+(S^2, \{p_1, \cdots, p_{2g+2}\}, [\gamma])$. In the same way, consider the group consisting of diffeomorphisms $T$ of $S^2 - \operatorname{Int} D^2$ such that $T(\{p_1, \cdots, p_{2g}\}) = \{p_1, \cdots, p_{2g}\}$, and $T|_{\nu(\partial D^2)}$ is equal to $r|_{\nu(\partial D^2)}$ or $\mathrm{id}_{\nu(\partial D^2)}$ instead of the group

$$\operatorname{Diff}_+(S^2 - \operatorname{Int} D^2, p_1, \cdots, p_{2g}, [\partial D^2]).$$

Then, we have the isomorphism between their path-connected components, similarly. Thus, we have $\mathcal{M}^{2g}(\gamma) \cong \mathcal{M}_{0,\mathrm{half}}^{2g}$. □

We can define a homomorphism $\mathcal{M}_{0,\mathrm{half}}^{2g} \to \langle s \rangle$ by mapping $[T]$ to $T|_{\partial D^2}$, where $\langle s \rangle$ is the cyclic group of order 2 generated by $s$. Then, the kernel is the subgroup $\mathcal{M}_{0,1}^{2g}$.

**Lemma 4.4.4.** *For $g \geq 1$, the exact sequence*

$$1 \longrightarrow \mathcal{M}_{0,1}^{2g} \longrightarrow \mathcal{M}_{0,\mathrm{half}}^{2g} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

*splits.*

*Proof.* We may assume $p_1, \cdots, p_{2g}$ are arranged in the disk as in Figure 4.4.1. Consider an involution $\mu \in \mathrm{Diff}_+(D^2, p_1, \cdots, p_{2g})$ which rotates the disk 180 degrees and interchanges the points $p_i$ and $p_{g+i}$ for $i = 1, \cdots, g$. Define a homomorphism $j : \mathbb{Z}/2\mathbb{Z} \to \mathcal{M}_{0,\mathrm{half}}^{2g}$ by $j(1) = \mu$.

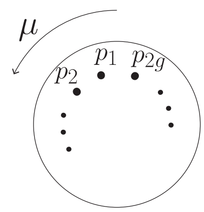


Figure 4.4.1: $p_1, \cdots, p_{2g}$ in $D^2$

This induces the splitting of the above exact sequence. □

**Lemma 4.4.5.** *Let $g \geq 1$, and $c$ a non-separating simple closed curve such that $\iota_g(c) = c$. Then, we have*

$$H_1(\mathcal{H}_g^s(c); \mathbb{Z}) = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2.$$

*Proof.* By Lemma 4.4.2 and Lemma 4.4.4, we have

$$H_1(\mathcal{H}_g^s(c); \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus H_1(\mathcal{M}_0^{2g}(\gamma); \mathbb{Z}), \ H_1(\mathcal{M}_{0,\mathrm{half}}^{2g}; \mathbb{Z}) \cong H_1(\mathcal{M}_{0,1}^{2g}; \mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}.$$

We showed $\mathcal{M}_0^{2g}(\gamma) \cong \mathcal{M}_{0,\mathrm{half}}^{2g}$ in Lemma 4.4.3, and it is known that $H_1(\mathcal{M}_{0,1}^{2g}; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ (see, for example, Section 9.1.3 and 9.2 of [10]). Hence, we have $H_1(\mathcal{H}_g^s(c); \mathbb{Z}) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$. □

Consider the case when $g = 1$. As is well-known, the group $\mathcal{H}_1$ coincides with $\mathcal{M}_1$. Hence, $\mathcal{H}_1(c)$ also coincides with $\mathcal{M}_1(c)$. If $c = c_3$ in Figure 4.1.2, the group $\mathcal{M}_1(c)$ is described as

$$\mathcal{M}_1(c) = \left\{ \begin{pmatrix} \epsilon & n \\ 0 & \epsilon \end{pmatrix} \in \mathrm{SL}(2; \mathbb{Z}) \,\middle|\, \epsilon \in \{\pm 1\}, n \in \mathbb{Z} \right\}.$$

By mapping $[T] \in \mathcal{M}_1(c)$ to $\epsilon \in \mathbb{Z}/2\mathbb{Z}$, we have a split exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{M}_1(c) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$

Thus, we have $H_1(\mathcal{H}_1(c); \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Combining Lemma 4.4.5, Lemma 4.2.1, and the case when $g = 1$ as above, we have:

**Lemma 4.4.6.** *Let $c$ be a non-separating simple closed curve such that $\iota_g(c) = c$. Then, we have*

$$H_1(\mathcal{H}_g(c); \mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2 & \text{when } g \geq 2, \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{when } g = 1. \end{cases}$$

*Proof of Proposition 4.4.1 (i).* Let $\sigma \in \mathcal{M}_{0,\mathrm{half}}^{2g}$ denote the half twist along $\partial D^2$. By the exact sequence in Lemma 4.4.4, the group $\mathcal{M}_{0,\mathrm{half}}^{2g}$ is generated by $\{\sigma_1, \cdots, \sigma_{2g-1}, \sigma\}$. By Theorem 2 of [7], we have $\mathcal{P}_g(t_{c_i}) = \sigma_i$ for $i = 1, \cdots, 2g$ and $\mathcal{P}_g(t_{c_{2g+1}}) = \sigma$. By the exact sequence in Lemma 4.4.2, the group $\mathcal{H}_g(c)$ is generated by $t_{c_i}$ for $i = 1, 2, \cdots, 2g-1, 2g+1$ and $\iota_g$. □

### 4.4.2 When $c$ is separating

Next, consider the case when $c$ is type $\mathrm{II}_h$. For simplicity, we choose $c$ as in Figure 2.3.1.

As we will see in Section 4.5.1, when the vanishing cycle of $Z_i$ in the hyperelliptic directed BLF is separating, the image of the monodromy representation along $\partial_0 A_i$ is contained in $\mathcal{H}_g(c^{\mathrm{ori}})$. Hence, we only consider the group $\mathcal{H}_g(c^{\mathrm{ori}})$ in this section instead of $\mathcal{H}_g(c)$. Of course, if $g \neq 2h$, we have $\mathcal{H}_g(c) = \mathcal{H}_g(c^{\mathrm{ori}})$ since any diffeomorphism of $\Sigma_g$ which preserves $c$ setwise acts trivially on $\pi_0(\Sigma_g - c)$.

First, consider the case when $h = 0, g$. For any diffeomorphism $T$ of $\Sigma_g$, we can change $T$ so that it preserves $c$ setwise by some isotopy. Thus, we have $\mathcal{H}_g(c^{\mathrm{ori}}) = \mathcal{H}_g$.

In the following, we only consider the case $1 \leq h \leq g-1$. Choose a disk $D$ in $\Sigma_g - \bigcup_{i=1}^{2g} c_i$ so that $\iota_g(D) = D$, where $c_i$ is the simple closed curve in Figure 4.1.2. Denote by $\Sigma_{g,1}$ the subsurface $\Sigma_g - \operatorname{Int} D$, and by $\iota_{g,1}$ the restriction of $\iota_g$ to $\Sigma_{g,1}$. The mapping class group $\mathcal{M}_{g,1}$ of $\Sigma_{g,1}$ is defined by $\mathcal{M}_{g,1} = \pi_0 \operatorname{Diff}_+(\Sigma_{g,1}, \partial\Sigma_{g,1})$, where $\operatorname{Diff}_+(\Sigma_{g,1}, \partial\Sigma_{g,1})$ is the diffeomorphism group of $\Sigma_{g,1}$ with $C^\infty$ topology which fixes the boundary pointwise.

We identify the subsurfaces of $\Sigma_g$ bounded by $c$ with $\Sigma_{h,1}$ and $\Sigma_{g-h,1}$ so that $\iota_g|_{\Sigma_{h,1}} = \iota_{h,1}$ and $\iota_g|_{\Sigma_{g-h,1}} = \iota_{g-h,1}$. For $T_1 \in \operatorname{Diff}_+(\Sigma_{h,1}, \partial\Sigma_{h,1})$ and $T_2 \in \operatorname{Diff}_+(\Sigma_{g-h,1}, \partial\Sigma_{g-h,1})$, the diffeomorphism $T_1 \cup T_2 \in \operatorname{Diff}_+ \Sigma_g$ preserves the curve $c$. Hence, we can define a map

$$\Psi : \mathcal{M}_{h,1} \times \mathcal{M}_{g-h,1} \to \mathcal{M}_g(c^{\mathrm{ori}})$$

by $\Psi([T_1], [T_2]) = [T_1 \cup T_2]$. This is a well-defined homomorphism.

Define a subgroup $\mathcal{H}_{g,1}$ of $\mathcal{M}_{g,1}$ by $\mathcal{H}_{g,1} = \{[T] \in \mathcal{M}_{g,1} \mid \iota_{g,1} T \iota_{g,1}^{-1} = T\}$. Apparently, the image $\Psi(\mathcal{H}_{h,1} \times \mathcal{H}_{g-h,1})$ is contained in the subgroup $\mathcal{H}_g(c^{\mathrm{ori}}) \subset \mathcal{M}_g(c^{\mathrm{ori}})$.

**Lemma 4.4.7.** *Let $g \geq 2$. When $1 \leq h \leq g-1$, the sequence*

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{H}_{h,1} \times \mathcal{H}_{g-h,1} \xrightarrow{\Psi} \mathcal{H}_g(c^{\mathrm{ori}}) \longrightarrow 1$$

*is exact.*

*Proof.* By Theorem 3.18 in [10], we have

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{M}_{h,1} \times \mathcal{M}_{g-h,1} \xrightarrow{\Psi} \mathcal{M}_g(c^{\mathrm{ori}}) \longrightarrow 1.$$

The kernel of $\Psi$ is generated by $(t_{\partial\Sigma_1}, t_{\partial\Sigma_2}^{-1})$, and it is contained in $\mathcal{H}_{h,1} \times \mathcal{H}_{g-h,1}$. Thus, we only need to prove $\Psi(\mathcal{H}_{h,1} \times \mathcal{H}_{g-h,1}) = \mathcal{H}_g(c^{\mathrm{ori}})$.

Let $\varphi$ be a mapping class in $\mathcal{H}_g(c)$. By Lemma 4.2.1, we can choose a representative $T \in \operatorname{Diff}_+ \Sigma_g$ of $\varphi$ satisfying $T\iota_g = \iota_g T$ and $T(c) = c$. Using some isotopy, we may assume $T|_c$ is the identity map. Then, $T|_{\Sigma_{h,1}}$ and $T|_{\Sigma_{g-h,1}}$ represent mapping classes in $\mathcal{H}_{h,1}$ and $\mathcal{H}_{g-h,1}$, respectively. Since $\Psi([T|_{\Sigma_{h,1}}], [T|_{\Sigma_{g-h,1}}]) = [T]$, we obtain $\Psi(\mathcal{H}_{h,1} \times \mathcal{H}_{g-h,1}) = \mathcal{H}_g(c^{\mathrm{ori}})$. □

We define a group $C(\iota_{g,1})$ as follows:

$$C(\iota_{g,1}) = \{T \in \operatorname{Diff}_+(\Sigma_{g,1}, \partial\Sigma_{g,1}) \mid \iota_{g,1} T \iota_{g,1}^{-1} = T\}.$$

We have the homomorphism $\mathcal{P}_{g,1} : \pi_0(C(\iota_{g,1})) \to \mathcal{M}_{0,1}^{2g+1}$ defined by $[T] \mapsto [\bar{T}]$ in the same way as $\mathcal{P}_g : \mathcal{H}_g^s \to \mathcal{M}_0^{2g+2}$ in Subsection 4.1.1. Since any isotopy of $\operatorname{Diff}_+(D^2, \partial D^2, \{p_1, \cdots, p_{2g+1}\})$

can be lifted to an isotopy of $C(\iota_{g,1})$, $\mathrm{Ker}(\mathcal{P}_{g,1})$ is represented by the deck transformation $\iota_{g,1}$ or $\mathrm{id}_{\Sigma_{g,1}}$. Since $C(\iota_{g,1})$ does not contain $\iota_{g,1}$, the kernel of the homomorphism $\mathcal{P}_{g,1}$ is trivial. Furthermore, $\mathcal{P}_{g,1} : \pi_0 C(\iota_{g,1}) \to \mathcal{M}_{0,1}^{2g+1}$ is an isomorphism since $\mathcal{M}_{0,1}^{2g+1}$ is generated by $\{\sigma_i\}_{i=1}^{2g}$ and $\mathcal{P}_{g,1}(t_{c_i}) = \sigma_i$ for $i = 1, \cdots, 2g$.

**Lemma 4.4.8.** *For $g \geq 1$, the natural homomorphism $\pi_0 C(\iota_{g,1}) \to \mathcal{H}_{g,1}$ is an isomorphism.*

*Proof.* By the definition of $\mathcal{H}_{g,1}$, the natural homomorphism $\pi_0(C(\iota_{g,1})) \to \mathcal{H}_{g,1}$ is surjective. Hence, it suffices to show the injectivity.

Embed $\Sigma_{g,1}$ in $\Sigma_{g+1}$ so that $\iota_{g+1}|_{\Sigma_{g,1}} = \iota_{g,1}$. For a diffeomorphism $T$ of $\Sigma_{g,1}$, we can extend $T$ to a diffeomorphism $\tilde{T}$ of $\Sigma_{g+1}$ by the identity map on $\Sigma_{g+1} \setminus \Sigma_{g,1}$. Thus, we have homomorphisms $\pi_0(C(\iota_{g,1})) \to \pi_0(C(\iota_{g+1}))$ and $\mathcal{H}_{g,1} \to \mathcal{H}_{g+1}$ defined by $[T] \mapsto [\tilde{T}]$. By gluing a disk with three marked points to $D^2$, we can also define a homomorphism $\mathcal{M}_{0,1}^{2g+1} \to \mathcal{M}_0^{2g+4}$ in the same way. By Theorem 3.18 in [10], the latter homomorphism is injective.

If we consider $(\Sigma_{g+1} \setminus \mathrm{Int}\,\Sigma_{g,1})/\langle \iota_{g+1} \rangle$ as a disk with three marked points, we have a commutative diagram

$$\begin{array}{ccccc}
\mathcal{M}_{0,1}^{2g+1} & \xleftarrow{\mathcal{P}_{g,1}} & \pi_0 C(\iota_{g,1}) & \longrightarrow & \mathcal{H}_{g,1} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M}_0^{2g+4} & \xleftarrow{\mathcal{P}_{g+1}} & \pi_0 C(\iota_{g+1}) & \longrightarrow & \mathcal{H}_{g+1}.
\end{array}$$

The left side shows that $\pi_0 C(\iota_{g,1}) \to \pi_0 C(\iota_{g+1})$ is injective. By Theorem 4.1.1, the right side shows that $\pi_0 C(\iota_{g,1}) \to \mathcal{H}_{g,1}$ is also injective. □

**Lemma 4.4.9.** *Let $g \geq 2$ and $1 \leq h \leq g-1$. Let $c$ be a separating simple closed curve which bounds subsurfaces of genus $h$ and $g-h$ and satisfies $\iota_g(c) = c$. Then, we have*

$$H_1(\mathcal{H}_g(c^{\mathrm{ori}}); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z},$$

*where $d = \gcd(4h(2h+1), 4(g-h)(2g-2h+1))$.*

*Proof.* Since $\mathcal{H}_{h,1} \cong \mathcal{M}_{0,1}^{2h+1}$, we have $H_1(\mathcal{H}_{h,1}; \mathbb{Z}) \cong \mathbb{Z}$. By the chain relation (see, for example, Proposition 4.12 in [10]), the mapping class $(t_{c_1} \cdots t_{c_{2h}})^{4h+2} \in \mathcal{H}_{h,1}$ coincides with the Dehn twist $t_{\partial\Sigma_{h,1}}$ along the boundary. In the same way, we have $(t_{c_1} \cdots t_{c_{2(g-h)}})^{4(g-h)+2} = t_{\partial\Sigma_{g-h,1}} \in \mathcal{H}_{g-h,1}$.

The kernel of the homomorphism $\mathcal{H}_{h,1} \times \mathcal{H}_{g-h,1} \to \mathcal{H}_g(c^{\mathrm{ori}})$ is the cyclic group generated by $(t_{\partial\Sigma_{h,1}}, t_{\partial\Sigma_{g-h,1}}^{-1})$. Hence, we have

$$\begin{aligned}
&H_1(\mathcal{H}_g(c^{\mathrm{ori}}); \mathbb{Z}) \\
\cong&(\mathbb{Z} \oplus \mathbb{Z})/\langle (4h(2h+1), -4(g-h)(2g-2h+1)) \rangle \\
\cong&\mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}.
\end{aligned}$$

□

*Proof of Proposition 4.4.1 (ii).* As explained in the paragraph before Lemma 4.4.8, $\mathcal{H}_{g,1}$ is generated by $t_{c_1}, \cdots, t_{c_{2g}}$. Thus, $\mathcal{H}_g(c)$ is generated by the following set by Lemma 4.4.7:

$$\{t_{c_i} \mid i = 1, 2, \ldots, 2h, 2h+2, 2h+3, \ldots, 2g+2\}.$$

This completes the proof of Proposition 4.4.1 (ii). □

## 4.5 Localization of the signature

In this section, we generalize Theorem 4.1.9 for a signature formula for HDBLFs. To give the statement of the generalization precisely, we first introduce a homomorphism

$$h_{g,c} : \mathcal{H}_g(c) \to \mathbb{Q},$$

which we will define in Subsection 4.5.3. We will also calculate the value of generators of $\mathcal{H}_g(c)$ given in Proposition 4.4.1 (see Proposition 4.5.6).

Let $f : X \to S^2$ be an HDBLF. We use the same notations as those we use in Sections 2.1 and 2.4. The boundary $\partial \nu Z_i$ has two components. We denote by $\partial_h \nu Z_i$ the component of $\partial \nu Z_i$ whose preimage contains vanishing cycles of folds (the right side of $Z_i$ in Figure 2.1.1). There is a unique component of $f^{-1}(\partial_h \nu Z_i)$ which contains vanishing cycles of folds. Let $g_i$ be the genus of a fiber in this component and we identify the fiber with $\Sigma_{g_i}$. We can regard $d_i$ with a simple closed curve in $\Sigma_{g_i}$. Denote by $\varphi_i$ the restriction of a monodromy of $f$ along $\partial_h \nu Z_i$ to the component $\Sigma_g$. By the definition of HDBLFs, this element in contained in the group $\mathcal{H}_{g_i}(d_i)$. The purpose of this section is to prove the following theorem:

**Theorem 4.5.1.** *Let $f : X \to S^2$ be an HDBLF as above. Then, we have*

$$\operatorname{Sign} X = \sum_{i=1}^{m} h_{g_i,d_i}(\varphi_i) + \sum_{j=1}^{l} \sigma_{\mathrm{loc}}(f^{-1}(p_j)).$$

### 4.5.1 Signatures of round cobordisms

We use the same notation as in Sections 2.1 and 2.4. . Let $f : X \to S^2$ be an DBLF. Lemma 2.6.4 implies that the manifold $f^{-1}(\nu Z_i)$ can be obtained by attaching a round 2-handle to a surface bundle over an annulus. Moreover, when the vanishing cycle $d_i$ is a separating curve, the monodromy $\varphi$ is in $\mathcal{M}_{g_i}(d_i^{\mathrm{ori}})$. This is because, if $\varphi$ changes the orientation of $d_i$, the monodromy along $\partial_h \nu Z_i$ permutes the component of the fiber. Inductively, the monodromy along $\partial_h \nu Z_1$ permutes the component of the fiber. However, since $f^{-1}(D_l)$ is a trivial surface bundle over a disk, the image of this monodromy under the map $\Phi_{d_1}$ must be trivial.

For a mapping class $\varphi \in \mathcal{M}_g(c)$ represented by $T \in \operatorname{Diff}_+ \Sigma_g$ satisfying $T(c) = c$, define a mapping torus $V_\varphi$ by $V_\varphi = \Sigma_g \times [0,1]/((0, T(x)) \sim (1, x))$. We can identify $f^{-1}(\partial_h \nu Z_i)$ with $V_\varphi$ for some $\varphi \in \mathcal{M}_{g_i}(d_i)$. Let $R = I \times D^2 \times D^1/\sim$ be a round 2-handle which is untwisted if $\varphi$ preserves the orientation of $d_i$ and is twisted otherwise. Choose an embedding

$$j : I \times \partial D^2 \times D^1/\sim \to V_\varphi$$

such that $j(0, \partial D^2, 0) = c \times \{0\} \subset V_\varphi$ and $p_1 \circ j(t, x, s) = t$, By Lemma 2.6.4, we can obtain the following diffeomorphism:

$$f^{-1}(\nu Z_i) \cong (V_\varphi \times [0,1]) \cup_j R.$$

Note that the isotopy class of the attaching map $j : I \times \partial D^2 \times D^1/\sim \to V_\varphi \times \{0\}$ is unique if the genus $g$ is greater than or equal to 2. Eventually, we obtain:

**Lemma 4.5.2.**

$$\operatorname{Sign} f^{-1}(\nu Z_i) = \operatorname{Sign}((V_\varphi \times [0,1]) \cup_j R).$$

### 4.5.2 Wall's non-additivity formula

In [28], Sato defined a class function $m : \mathcal{M}_{g,2} \to \mathbb{QP}^1$. We review this function, and calculate the signature of the compact 4-manifold $(V_\varphi \times [0,1]) \cup_j R$ in Section 4.5.1.

For a mapping class $\varphi = [T] \in \mathcal{M}_{g,2}$, let $V'_\varphi = \Sigma_{g,2} \times [0,1]/(0, T(x)) \sim (1, x)$ be its mapping torus. Choose points $x_1$ and $x_2$ in each boundary component of $\Sigma_{g,2}$, and define a continuous map by $l_i : S^1 \to V'_\varphi$ by $l_i(t) = (t, x_i)$ for $i = 1, 2$. Let $\partial_1$ and $\partial_2$ be the two boundary components of $\Sigma_{g,2}$. Denote by $e_1$, $e_2$, $e_3$, and $e_4$ the homology classes $[l_1]$, $[l_2]$, $[\partial_1 \times \{0\}]$, and $[\partial_2 \times \{0\}]$, respectively. Then, for some $p, q \in \mathbb{Q}$, the set $\{e_1 + e_2, p(e_3 - e_4) + qe_1\}$ forms a basis of $\mathrm{Ker}(H_1(\partial V'_\varphi; \mathbb{Q}) \to H_1(V'_\varphi; \mathbb{Q}))$. The element $[p : q] \in \mathbb{QP}^1$ is unique, and we can define a function $m : \mathcal{M}_{g,2} \to \mathbb{QP}^1$ by $m(\varphi) = [p : q]$. Since it satisfies $m(\varphi t_{\partial_1} t_{\partial_2}^{-1}) = m(\varphi)$, it induces the class function on $\mathcal{M}_g(c^{\mathrm{ori}})$. For simplicity, we also denote it by $m : \mathcal{M}_g(c^{\mathrm{ori}}) \to \mathbb{QP}^1$.

Define a map $s : \mathcal{M}_g(c) \to \mathbb{Z}$ by $s(\varphi) = \mathrm{Sign}((V_\varphi \times [0,1]) \cup R)$. We can write the signature $s(\varphi)$ with the function $m : \mathcal{M}_g(c^{\mathrm{ori}}) \to \mathbb{QP}^1$ as follows:

**Lemma 4.5.3.** *Let $\varphi \in \mathcal{M}_g(c)$. Then, we have*

$$s(\varphi) = \begin{cases} \mathrm{sign}(m(\varphi)), & \textit{if } c : \textit{non-separating}, \varphi \textit{ preserves the orientation of } c, \\ 0, & \textit{otherwise.} \end{cases}$$

*Proof.* We apply Wall's nonadditivity Formula to the pasting of the round 2-handle. First, we review his formula. Let $X_-$, $X_0$, and $X_+$ be compact 3-manifolds, and let $Y_-$ and $Y_+$ be compact 4-manifolds such that

$$\partial X_- = \partial X_+ = \partial X_+ = Z, \quad \partial Y_- = X_- \cup X_0, \quad \partial Y_+ = X_+ \cup X_0.$$

We denote by $Y$ and $X$ the compact 4-manifold $Y = Y_- \cup Y_+$ and the space $X = X_- \cup X_0 \cup X_+$, respectively. Suppose that $Y$ is oriented inducing orientations of $Y_-$ and $Y_+$. Orient the other manifolds so that

$$\begin{aligned} \partial_*[Y_-] &= [X_0] - [X_-], \\ \partial_*[Y_+] &= [X_+] - [X_0], \\ \partial_*[X_-] = \partial_*[X_+] &= \partial_*[X_0] = [Z]. \end{aligned}$$

We define vector spaces $V$, $A$, $B$, and $C$ as follows:

$$\begin{aligned} V &= H_1(Z; \mathbb{Q}), \\ A &= \mathrm{Ker}(H_1(Z; \mathbb{Q}) \to H_1(X_-; \mathbb{Q})), \\ B &= \mathrm{Ker}(H_1(Z; \mathbb{Q}) \to H_1(X_0; \mathbb{Q})), \\ C &= \mathrm{Ker}(H_1(Z; \mathbb{Q}) \to H_1(X_+; \mathbb{Q})). \end{aligned}$$

On the vector space $W = B \cap (C + A)/((B \cap C) + (B \cap A))$, Wall defined a symmetric bilinear map $\Psi : W \times W \to \mathbb{Q}$ as follows. Let $I : H_1(Z; \mathbb{Q}) \times H_1(Z; \mathbb{Q}) \to \mathbb{Q}$ denote the intersection form, and $b, b' \in B \cap (C + A)$. Since $b' \in B \cap (C + A)$, there exist $c' \in C$ and $a' \in A$ such that $a' + b' + c' = 0$. Then, define a map $\Psi : W \times W \to \mathbb{Q}$ by $\Psi'([b], [b']) = I(b, c')$. He showed that this map is well-defined and symmetric. Denote by $\mathrm{Sign}(V; B, C, A)$ the signature of this symmetric bilinear form. His signature formula is:

**Theorem 4.5.4** (Wall [30])**.**

$$\operatorname{Sign} Y = \operatorname{Sign} Y_- + \operatorname{Sign} Y_+ - \operatorname{Sign}(V; B, C, A).$$

Next, we apply his formula to our settings. We should let $Y_-$ and $Y_+$ denote the manifolds

$$Y_- = R = I \times D^2 \times D^1/\sim \text{ and } Y_+ = V_\varphi \times [0,1],$$

respectively. The rest of the manifolds are

$$\begin{gathered}\partial Y_- = (I \times \partial D^2 \times D^1/\sim) \cup (I \times D^2 \times \partial D^1/\sim),\\ \partial Y_+ = (V_\varphi \times \{1\}) \amalg (V_\varphi \times \{0\}),\\ X_0 = I \times \partial D^2 \times D^1/\sim, \quad X_- = I \times D^2 \times \partial D^1/\sim,\\ X_+ = (V_\varphi \times \{1\}) \amalg (V_\varphi \times \{0\} - \operatorname{Int} j(X_0)), \quad Z = I \times \partial D^2 \times \partial D^1/\sim .\end{gathered}$$

Consider the case when $T|_{\nu(c)} = id$. Choose a point $x$ in $\partial D^2$. Define continuous maps $f_i : S^1 \to S^1 \times \partial D^2 \times \partial D^1$ by $f_i(t) = (t, x, (-1)^i)$ for $i = 1, 2$. The set consisting of the homology classes $e_1 = [\partial D^2 \times \{-1\}]$, $e_2 = [\partial D^2 \times \{1\}]$, $e_3 = [f_1]$, and $e_4 = [f_2]$ in $H_1(Z; \mathbb{Q})$ forms a basis.

When $c$ is separating, we have $A = C = \mathbb{Q}e_1 \oplus \mathbb{Q}e_2$. Hence, we obtain $W = (B \cap (C + A))/((B \cap C) + (B \cap A)) = 0$. When $c$ is non-separating, $\operatorname{Sign}(V_\varphi \times [0,1] \cup R)$ is calculated in Lemma 3.4 of [28].

Consider the case when $T$ does not preserve the orientation of $c$. In this case, the curve $c$ is non-separating. Define a continuous map $f : S^1 \to I \times \partial D^2 \times \partial D^1/\sim$ by

$$f(t) = \begin{cases} (2t, x, -1, ) & \text{when } 0 \le t \le \dfrac{1}{2}, \\ (2t-1, x, 1, ) & \text{when } \dfrac{1}{2} \le t \le 1. \end{cases}$$

The set of homology classes consisting of $e_1 = [\partial D^2 \times \{-1\}]$ and $e_2 = [f]$ in $H_1(Z; \mathbb{Q})$ forms a basis. In this case, $A = B = \mathbb{Q}e_1$. Hence, we have $W = 0$. □

### 4.5.3 The homomorphism $h_{g,c}$

Let $c$ be a simple closed curve in $\Sigma_g$. Since the neighborhood $\nu(c)$ of $c$ is diffeomorphic to $\partial D^2 \times [-1,1]$, we obtain a manifold $L(c) = \Sigma_g \times [0,1] \cup_{\nu(c)} (D^2 \times [-1,1])$ by gluing $D^2 \times [-1,1]$ along $\nu(c)$. This is diffeomorphic to a fiber of the projection $(V_\varphi \times [0,1]) \cup R \to S^1$. We denote $\tilde{V}_\varphi = (V_\varphi \times [0,1]) \cup R$ in the following.

Let $\varphi$ and $\psi$ be mapping classes in $\mathcal{M}_g(c)$. For example, by gluing the $L(c)$-bundles $\tilde{V}_\varphi \times [0,1]$ and $\tilde{V}_\psi \times [0,1]$ on an annulus, we obtain a $L(c)$-bundle over $S^2 - \amalg_{i=1}^3 \operatorname{Int} D^2$ whose fiberwise boundary is $E_{\varphi,\psi} \amalg -E_{\Phi(\varphi),\Phi(\psi)}$ and the whole boundary is

$$(E_{\varphi,\psi} \amalg -E_{\Phi(\varphi),\Phi(\psi)}) \cup_{\partial E_{\varphi,\psi} \amalg -\partial E_{\Phi(\varphi),\Phi(\psi)}} (-\tilde{V}_\varphi \amalg -\tilde{V}_\psi \amalg -\tilde{V}_{(\varphi\psi)^{-1}}).$$

Hence, we have

$$\operatorname{Sign} E_{\varphi,\psi} - \operatorname{Sign} E_{\Phi(\varphi),\Phi(\psi)} - \operatorname{Sign} \tilde{V}_\varphi - \operatorname{Sign} \tilde{V}_\psi - \operatorname{Sign} \tilde{V}_{(\varphi\psi)^{-1}} = 0.$$

If we rewrite it by Meyer's signature cocycle and the function $s : \mathcal{M}_g(c) \to \mathbb{Z}$, we have

$$-\tau_g(\varphi, \psi) + \Phi^*\tau_{g-1}(\varphi, \psi) - \delta s(\varphi, \psi) = 0 \in C^2(\mathcal{M}_g(c); \mathbb{Z})(c\text{:type I}),$$
$$-\tau_g(\varphi, \psi) + \Phi^*(\tau_h \times \tau_{g-h})(\varphi, \psi) - \delta s(\varphi, \psi) = 0 \in C^2(\mathcal{M}_g(c^{\mathrm{ori}}); \mathbb{Z})(c\text{:type II}_h).$$

If we restrict the Meyer cocycles to $\mathcal{H}_g$, we have $\tau_g = \delta\phi_g \in C^2(\mathcal{H}_g; \mathbb{Q})$, and $\tau_{g-1} = \delta\phi_{g-1} \in C^2(\mathcal{H}_{g-1}; \mathbb{Q})$. Thus, we have proved:

**Lemma 4.5.5.** *When $c$ is type* I*, define a function $h_{g,c} : \mathcal{H}_g(c) \to \mathbb{Q}$ by*

$$h_{g,c}(\varphi) = s(\varphi) + \phi_g(\varphi) - \Phi^*\phi_{g-1}(\varphi).$$

*When $c$ is type* $\mathrm{II}_h$*, define $h_{g,c} : \mathcal{H}_g(c^{\mathrm{ori}}) \to \mathbb{Q}$ by*

$$h_{g,c}(\varphi) = s(\varphi) + \phi_g(\varphi) - \Phi^*(\phi_h \times \phi_{g-h})(\varphi).$$

*Then, both of these maps are homomorphisms.*

The values of generators of $\mathcal{H}_g(c)$ given in Proposition 4.4.1 under the map $h_{g,c}$ is calculated as follows:

**Proposition 4.5.6.** *Suppose that the genus $g$ is greater than or equal to* 1*.*

1. *Let $c$ be a non-separating simple closed curve of type* I *in Figure 2.3.1. The values of the homomorphism $h_{g,c} : \mathcal{H}_g(c) \to \mathbb{Q}$ are*

$$h_{g,c}(\iota_g) = 0,$$
$$h_{g,c}(t_{c_i}) = -\frac{1}{4g^2-1} \quad \text{for } i = 1\cdots, 2g-1,$$
$$h_{g,c}(t_{c_{2g+1}}) = -\frac{g}{2g+1}.$$

2. *Let $0 \le h \le g$, and $c$ a separating simple closed curve of type* $\mathrm{II}_h$ *in Figure 2.3.1. When $1 \le h \le g-1$, the values of the homomorphism $h_{g,c} : \mathcal{H}_g(c^{\mathrm{ori}}) \to \mathbb{Q}$ are*

$$h_{g,c}(t_{c_i}) = \frac{g+1}{2g+1} - \frac{h+1}{2h+1} \quad \text{for } i = 1, \cdots, 2h,$$
$$h_{g,c}(t_{c_i}) = \frac{g+1}{2g+1} - \frac{g-h+1}{2(g-h)+1} \quad \text{for } i = 2h+2, \cdots, 2g.$$

   *When $h = 0, g$, the homomorphism $h_{g,c}$ is the zero map.*

*Proof.* First, consider the case when the vanishing cycle $c$ is type I in Figure 2.3.1. Since $h_{g,c}$ is a homomorphism, we have $h_{g,c}(\iota_g) = 0$. The mapping classes $t_{c_i}$ for $i = 1, 2, \cdots, 2g-1$ are mutually conjugate in $\mathcal{H}_g(c)$. Therefore, we have $h_{g,c}(t_{c_1}) = \cdots = h_{g,c}(t_{c_{2g-1}})$. By the chain relation, we have $(t_{c_1} \cdots t_{c_{2g-1}})^{2g} = t_{c_{2g+1}}^2$. Thus, we obtain $h_{g,c}(t_{c_{2g+1}}) = g(2g-1)h_{g,c}(t_{c_1})$. Hence, it suffices to show that $h_{g,c}(\sigma_{2g+1}) = -g/(2g+1)$.

In Lemma 3.3 of [9], Endo showed that $\phi_g(t_{c_{2g+1}}) = (g+1)/(2g+1)$. Since $\Phi(t_{c_{2g+1}}) = 1 \in \mathcal{M}_{g-1}$, we have $\Phi^*\phi_{g-1}(t_{c_{2g+1}}) = 0$. By Lemma 4.5.3, we have

$$s(t_{c_{2g+1}}) = \operatorname{sign} m(t_{c_{2g+1}}) = \operatorname{sign}([1 : -1]) = -1.$$

Thus, we obtain $h_{g,c}(t_{c_{2g+1}}) = -g/(2g+1)$.

Next, consider the case when the vanishing cycle $c$ is type $\mathrm{II}_h$ in Figure 2.3.1. When $1 \le h \le g-1$, this follows from Lemma 3.3 of [9] since $s(t_{c_i}) = 0$. When $h = 0, g$, $h_{g,c}$ is the zero map since $H^1(\mathcal{H}_g(c); \mathbf{Q}) = H^1(\mathcal{H}_g; \mathbf{Q}) = 0$. □

### 4.5.4 A local signature formula

In this subsection, we prove Theorem 4.5.1.

*Proof of Theorem 4.5.1.* We prepare the hyperelliptic mapping class group of the non-connected surface $f^{-1}(r_i)$, where the monodromy of it along $\partial_h \nu Z_i$ lies. Identify $f^{-1}(r_i)$ with some standard surface $S_i = \Sigma_{n_i(1)} \amalg \cdots \amalg \Sigma_{n_i(k_i)}$, where $n_i(1), \cdots, n_i(k_i)$ are non-negative integers. We may assume that the action on $f^{-1}(r_i)$ induced by $\iota_g$ in Subsection 4.1.2 coincides with $\iota_{n_i(1)} \amalg \cdots \amalg \iota_{n_i(k_i)}$, and the vanishing cycle $d_i$ lies in $\Sigma_{n_i(1)}$ $(n_i(1) = g_i)$. Define groups $\mathcal{H}_{S_i}$ and $\mathcal{H}_{S_i}(d_i)$ by $\mathcal{H}_{S_i} = \mathcal{H}_{n_i(1)} \times \cdots \mathcal{H}_{n_i(k_i)}$ and $\mathcal{H}_{S_i}(d_i) = \mathcal{H}_{n_i(1)}(d_i) \times \mathcal{H}_{n_i(2)} \times \cdots \mathcal{H}_{n_i(k_i)}$, respectively. By the definition HDBLFs, the monodromy $\tilde{\varphi}_m$ along $\partial_h \nu Z_m$ is contained in $\mathcal{H}_{S_m}(d_m)$. As stated in Section 4.5.1, the monodromy $\tilde{\varphi}_{m-1}$ of $f^{-1}(r_{m-1})$ along $\partial_h \nu Z_{m-1}$ is the image of $\tilde{\varphi}_m \in \mathcal{H}_{S_m}(d_m)$ under $\Phi_{d_m} : \mathcal{H}_{S_m}(d_m) \to \mathcal{H}_{S_{m-1}}$. By Theorem 2.3.2, it is contained in $\mathcal{H}_{S_{m-1}}(d_{m-1})$. Define a natural homomorphism $\Phi_{S_i} : \mathcal{H}_{S_i}(d_i) \to \mathcal{H}_{S_{i-1}}$ by $\Phi_{S_i}(x_1, x_2, \cdots, x_{k_i}) = (\Phi_{d_i}(x_1), x_2, \cdots, x_{k_i})$, for $i = 1, \cdots, m$. Inductively, the monodromy $\tilde{\varphi}_i$ along $\partial_h \nu Z_i$ is contained in $\mathcal{H}_{S_i}(d_i)$, and $\tilde{\varphi}_{i-1} = \Phi_{S_i}(\tilde{\varphi}_i)$.

By the Novikov additivity, we have

$$\begin{aligned}\operatorname{Sign} X &= \sum_{i=1}^{m} \operatorname{Sign} f^{-1}(\nu Z_i) + \operatorname{Sign} f^{-1}(D_l) + \operatorname{Sign} f^{-1}(D_h) \\ &= \sum_{i=1}^{m} \operatorname{Sign} f^{-1}(\nu Z_i) + \operatorname{Sign} f^{-1}(D_h - \coprod_{j=1}^{l} \operatorname{Int} \nu(p_j)) \\ &\quad + \sum_{j=1}^{l} \operatorname{Sign} f^{-1}(\nu(p_j)).\end{aligned}$$

Define the Meyer function $\phi_{S_i} : \mathcal{H}_{S_i} \to \mathbb{Q}$ by $\phi_{S_i}(x_1, \cdots, x_{k_i}) = \sum_{j=1}^{k_i} \phi_{S_i}(x_j)$. We take a loop $a_j$ around the image $p_j \in D_h$ as in Theorem 2.4.1 Let $\psi_j \in \mathcal{H}_g$ denote the monodromy along the loop $a_j$.

We denote by $M_i$ the component of $f^{-1}(\partial_h \nu Z_i)$ which contains vanishing cycles of $\widetilde{Z_i}$. By Lemma 4.1.7 and Lemma 4.5.3, we have

$$\operatorname{Sign} X = \sum_{i=1}^{m} s(\varphi_i) + \left( -\phi_g(\tilde{\varphi}_m^{-1}) - \sum_{j=1}^{l} \phi_g(\psi_j) \right) + \sum_{j=1}^{l} \operatorname{Sign} f^{-1}(\nu(p_j)),$$

where $\varphi_i$ is the monodromy of $M_i$ along $\partial_h \nu Z_i$. Since $f^{-1}(D_l)$ is a trivial bundle, we have $\tilde{\varphi}_0 = 1 \in \mathcal{H}_{S_0}$. Since $\Phi_{S_i}(\tilde{\varphi}_i) = \tilde{\varphi}_{i-1} \in \mathcal{H}_{S_{i-1}}(d_{i-1})$, we have

$$\sum_{i=1}^{m} (\phi_{S_i}(\tilde{\varphi}_i) - \Phi_{S_i}^* \phi_{S_{i-1}}(\tilde{\varphi}_i)) = \phi_g(\tilde{\varphi}_m).$$

Since the Meyer function has the property $\phi_g(\varphi^{-1}) = -\phi_g(\varphi)$ (see [9]) for any $\varphi \in \mathcal{H}_g$, we obtain

$$\operatorname{Sign} X = \sum_{i=1}^{m} \left( s(\varphi_i) + \phi_{S_i}(\tilde{\varphi}_i) - \Phi_{S_i}^* \phi_{S_{i-1}}(\tilde{\varphi}_i) \right)$$

$$+\sum_{j=1}^{l}(-\phi_g(\psi_j)+\operatorname{Sign} f^{-1}(\nu(p_j))).$$

By the definition of $\Phi_{S_i}$, we have

$$\begin{aligned}&\phi_{S_i}(x_1,\cdots,x_{k_i})-\Phi_{S_{i-1}}^*\phi_{S_{i-1}}(x_1,\cdots,x_{k_{i-1}})\\ =&\begin{cases}\phi_{g_i}(x_1)-\Phi_{d_i}^*\phi_{g_i}(x_1), & (d_i\text{:nonseparating}),\\ \phi_{g_i}(x_1)-\Phi_{d_i}^*(\phi_h\times\phi_{g_i-h})(x_1), & (d_i \text{ bounds subsurfaces of genus } h).\end{cases}\end{aligned}$$

Thus, we have

$$\operatorname{Sign} X=\sum_{i=1}^{m}h_{g_i,d_i}(\varphi_i)+\sum_{j=1}^{l}\sigma_{\mathrm{loc}}(f^{-1}(p_j)).$$

This completes the proof of Theorem 4.5.1. □

### 4.5.5 Examples of calculation of signatures

Let $c_1,\ldots,c_{2g+1}\subset\Sigma_g$ be simple closed curves described in Figure 4.1.2.

**Example 4.5.7.** As shown in the proof of Theorem 1.4 in [15], there exists an SBLF $f_{g,n}: X_{g,n}\to S^2$ which has the following Hurwitz cycle system:

$$(c_{2g+1};(c_{2g},\ldots,c_2,c_1,c_1,c_2,\ldots,c_{2g})^{2n}).$$

By the definition of $f_{g,n}$, it is hyperelliptic. We denote by $p_1,\ldots,p_{8gn}\in S^2$ the critical values of $f_{g,n}$. By using the formula in Theorem 4.5.1, the signature of $X_{g,n}$ can be calculated as follows:

$$\begin{aligned}\operatorname{Sign} X_{g,n}&=\sum_{i=1}^{8gn}\sigma_{\mathrm{loc}}(f_{g,n}^{-1}(p_i))+h((t_{c_{2g}}\cdots t_{c_2}t_{c_1}^2t_{c_2}\cdots t_{c_{2g}})^{2n})\\ &=8gn\cdot\frac{-g-1}{2g+1}+h(t_{c_{2g+1}}^{-4n})\\ &=\frac{-8g^2n-8gn}{2g+1}+(-4n)\cdot\frac{-g}{2g+1}\\ &=-4gn.\end{aligned}$$

It is easy to see that $X_{g,n}$ is simply connected and that the Euler characteristic of $X_{g,n}$ is $8gn-4g+6$. As shown in [15], $X_{g,n}$ is spin if and only if both of the integers $g$ and $n$ are even. Thus, by Freedman's theorem, $X_{g,n}$ is homeomorphic to $\#\frac{gn}{4}E(2)\#(\frac{5gn}{4}-2g+2)S^2\times S^2$ if both $g$ and $n$ are even and $\#(2gn-2g+2)\mathbb{CP}^2\#(6gn-2g+2)\overline{\mathbb{CP}^2}$ otherwise.

**Example 4.5.8.** As shown in the proof of Theorem 1.4 in [15], there exists an SBLF $\tilde{f}_{g,n}:\tilde{X}_{g,n}\to S^2$ which has the following Hurwitz cycle system:

$$(c_{2g+1};(c_{2g},\ldots,c_2,c_1,c_1,c_2,\ldots,c_{2g})^{2n},(c_1,\ldots,c_{2g-2})^{2(2g-1)n}).$$

By the definition of $\tilde{f}_{g,n}$, it is hyperelliptic. We denote by $\tilde{p}_1,\ldots,\tilde{p}_{8g^2n-4gn+4n}\in S^2$ the critical values of $\tilde{f}_{g,n}$. By using the formula in Theorem 4.5.1, the signature of $\tilde{X}_{g,n}$ can be calculated as follows:

$$\begin{aligned}
\operatorname{Sign} \tilde{X}_{g,n} = & \sum_{i=1}^{8g^2n-4gn+4n} \sigma_{\mathrm{loc}}(\tilde{f}_{g,n}^{-1}(\tilde{p}_i)) \\
& + h((t_{c_{2g}} \cdots t_{c_2} {t_{c_1}}^2 t_{c_2} \cdots t_{c_{2g}})^{2n} \cdot (t_{c_1} \cdots t_{c_{2g-2}})^{2(2g-1)n}) \\
= & (8g^2n - 4gn + 4n) \cdot \frac{-g-1}{2g+1} + 2n \cdot h(t_{c_{2g+1}}^{-2} \cdot \iota_g) \\
& + 2(2g-1)n \cdot h(t_{c_1} \cdots t_{c_{2g-2}}) \\
= & \frac{-8g^3n + 4g^2n - 4gn - 8g^2n + 4gn - 4n}{2g+1} - 4n \cdot \frac{-g}{2g+1} \\
& + 2(2g-1)n(2g-2) \cdot \frac{-1}{4g^2-1} \\
= & -4g^2n.
\end{aligned}$$

It is easy to see that $\tilde{X}_{g,n}$ is simply connected, and that the Euler characteristic of $\tilde{X}_{g,n}$ is $8g^2n - 4gn + 4n - 4g + 6$. As shown in [15], $\tilde{X}_{g,n}$ is spin if and only if $g$ is even. Thus, we can easily determine the homeomorphism type of $\tilde{X}_{g,n}$ as in Example 4.5.7.

# Bibliography


[1] D. Auroux, S. K. Donaldson and L. Katzarkov, *Singular Lefschetz pencils*, Geom. Topol. **9**(2005), 1043–1114

[2] R. İ. Baykur, *Topology of broken Lefschetz fibrations and near-symplectic 4-manifolds*, Pacific J. Math. **240**(2009), 201–230

[3] R. İ. Baykur, *Broken Lefschetz fibrations and smooth structures on 4-manifolds*, Geom. Topol. Monogr., **12**(2012), 9–34

[4] R. İ. Baykur and S. Kamada, *Classification of broken Lefschetz fibrations with small fiber genera*, preprint

[5] S. Behrens, *On* 4*-manifolds, folds and cusps*, preprint

[6] S. Behrens and K. Hayano, *Vanishing Cycles and Homotopies of Wrinkled Fibrations*, preprint

[7] J. S. Birman and H. M. Hilden, *On the mapping class groups of closed surfaces as covering spaces*, In Adevances in the theory of Riemann surfaces (Proc. Conf., Stony Brook, N.Y., 1969), 85–115, Ann. of Math. Studies, No. 66. Princeton Univ. Press, Princeton, N.J., 1971.

[8] S. K. Donaldson, *Lefschetz pencils on symplectic manifolds*, J. Differential Geom. **53**(1999), no.2, 205–236

[9] H. Endo, *Meyer's signature cocycle and hyperelliptic fibrations*, Math. Ann. **316**(2000), no. 2, 237–257.

[10] B. Farb and D. Margalit, *A Primer on Mapping Class Groups*, Princeton University Press, 2011

[11] T. Fuller, *Hyperelliptic Lefschetz fibrations and branched covering spaces*, Pacific J. Math. **196**(2000), 369–393

[12] R. E. Gompf, *Toward a topological characterization of symplectic manifolds*, J. Symplectic Geom. **2**(2004), no.2, 177–206

[13] R. E. Gompf and A.I.Stipsicz, *4-Manifolds and Kirby Calculus*, Graduate Studies in Mathematics **20**, American Mathematical Society, 1999

[14] K. Hayano, *On genus-*1 *simplified broken Lefschetz fibrations*, Algebr. Geom. Topol. **11**(2011), 1267–1322

[15] K. Hayano, *A note on sections of broken Lefschetz fibrations*, to appear in Bull. London Math. Soc.

[16] K. Hayano, *Modification rule of monodromies in $R_2$-move*, preprint

[17] N. V. Ivanov, *Mapping class groups*, Handbook of geometric topology, North-Holland, Amsterdam, 2002, 523–633

[18] S. Kamada, Y. Matsumoto, T. Matumoto, K. Waki, Chart description and a new proof of the classification theorem of genus one Lefschetz fibrations, J.Math. Soc. Japan **57**(2005), no.2 537–555

[19] A. Kas, *On the deformation types of regular elliptic surfaces*, Complex Analysis and Algebraic Geometry, 1977, 107–111

[20] A. Kas, *On the handlebody decomposition associated to a Lefschetz fibration*, Pacific J. Math. **89**(1980), 89–104

[21] Y. Lekili, *Wrinkled fibrations on near-symplectic manifolds*, Geom. Topol. **13**(2009), 277–318

[22] W. Magnus, A.Karrass, D.Solitar, *Combinatorial group theory.* Presentations of groups in terms of generators and relations. Second revised edition. Dover Publications, Inc., New York, 1976

[23] Y. Matsumoto, *On 4–manifolds fibered by tori II*, Proc. Japan Acad. **59**(1983), 100–103.

[24] Y. Matsumoto, *Lefschetz fibrations of genus two - a topological approach* -, Proceedings of the 37th Taniguchi Symposium on Topology and Teichmüller Spaces, (S. Kojima, et. al., eds.), World Scientific, 1996, 123–148

[25] W. Meyer, *Die Signatur von Flächenbündeln*, Math. Ann. **201**(1973), no. 3, 239–264.

[26] B. Moishezon, *Complex surfaces and connected sums of complex projective planes*, Lecture Notes in Math. **603**, Springer-Verlag, 1977.

[27] P. S. Pao, *The topological structure of 4-manifold with effective torus actions. I*, Amer. Math. Soc. **227**(1977), 279–317

[28] M. Sato, *A class function on the mapping class group of an orientable surface and the Meyer cocycle*, Algebr. Geom. Topol. **8**(2008), no. 3, 1647–1665.

[29] B. Siebert and G. Tian, *On hyperelliptic $C^\infty$ Lefschetz fibrations of four-manifolds*, Commun. Contemp. Math. **1**(1999), no. 2, 255–280

[30] C.T.C. Wall, *Non-additivity of the signature*, Invent. Math. **7**(1969), no. 3, 269–274.

[31] J. D. Williams, *The h–principle for broken Lefschetz fibrations*, Geom. Topol. **14**(2010), no.2, 1015–1063