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Complete classification of genus-1 simplified broken Lefschetz fibrations

Kenta Hayano

Abstract

Broken Lefschetz fibrations are fibration structures analogous to Lefschetz fibrations in the context of near-symplectic 4-manifolds. In the first part of this thesis, we discuss the classification of smooth 4-manifolds which admit genus-1 simplified broken Lefschetz fibrations. We completely classify diffeomorphism types of total spaces of genus-1 simplified broken Lefschetz fibrations. This result is a generalization of Kas and Moishezon's classification of genus-1 Lefschetz fibrations over the sphere S^2.

In the second part of this thesis, we discuss hyperelliptic directed broken Lefschetz fibrations. We prove that the total space of a hyperelliptic directed broken Lefschetz fibration has an involution which preserves any fiber provided that the genus of any connected component of a regular fiber is greater than or equal to 2. We also generalize Matsumoto and Endo's local signature formulae for hyperelliptic Lefschetz fibrations to a signature formula for hyperelliptic directed broken Lefschetz fibrations.

Contents

Chapter 1

Introduction

Broken Lefschetz fibrations are fibration structures on smooth 4-manifolds which we can regard as a generalization of Lefschetz fibrations. Donaldson [8] proved that symplectic 4-manifolds admit Lefschetz pencils. Conversely, Gompf [12] showed that the total space of every Lefschetz pencil admits a symplectic structure. In [1], Auroux, Donaldson and Katzarkov generalized these results to that on 4-manifolds with a near-symplectic structure (i.e. a closed 2-form which is symplectic outside a union of circles where it vanishes transversely).

Simplified broken Lefschetz fibrations are broken Lefschetz fibrations over the 2-sphere S^2 which satisfy several conditions on fibers and singularities. Baykur first introduced simplified broken Lefschetz fibrations in [2]. In spite of the strict conditions in the definition of simplified broken Lefschetz fibrations, it turns out that every closed oriented 4-manifold admits a simplified broken Lefschetz fibration. More strongly, every smooth map from a closed oriented 4-manifold to S^2 is homotopic to a simplified broken Lefschetz fibration (for details, see [31]). We can take vanishing cycles of simplified broken Lefschetz fibrations as we take those of Lefschetz fibrations. Vanishing cycles of a simplified broken Lefschetz fibration have rich information on the fibration structure. For example, Baykur [2] gave a way to obtain a Kirby diagram of the total space of a simplified broken Lefschetz fibration from vanishing cycles of the fibration. Furthermore, the author prove in this thesis that vanishing cycles of a simplified broken Lefschetz fibration determine an isomorphism class of the fibration (for details, see Theorem 2.5.2). The main purpose of this thesis is the classification of genus-1 simplified broken Lefschetz fibrations. We further generalize several results on Lefschetz fibrations to that on simplified broken Lefschetz fibrations.

We first study genus-1 simplified broken Lefschetz fibrations in Chapter 3. Kas [19] and Moishezon [26] classified diffeomorphism types of total spaces of non-trivial relatively minimal genus-1 Lefschetz fibrations over S^2. They proved that the total space of such a Lefschetz fibration is diffeomorphic to an elliptic surface $E(n)$ for some $n \geq 1$. Baykur and Kamada [4] and the author [14] originated the classification of diffeomorphism types of total spaces of genus-1 simplified broken Lefschetz fibrations. The author [14] classified diffeomorphism types of total spaces of genus-1 simplified broken Lefschetz fibrations under the assumption on the number of Lefschetz singularities. Behrens [5] also classified 4-manifolds admitting genus-1 simple wrinkled fibrations. Since we can change genus-1 simple wrinkled fibrations

into genus-1 simplified broken Lefschetz fibrations by application of unsink deformations, we can regard the classification problem of genus-1 simple wrinkled fibrations as a part of the classification of genus-1 simplified broken Lefschetz fibrations. In this thesis, we gave the complete classification of diffeomorphism types of total spaces of genus-1 simplified broken Lefschetz fibrations (see Theorem 3.0.1). This result includes all the results above and gives the affirmative answer to Conjecture 5.3 in [14] and the negative answer to Problem 24 in [3].

We next study hyperelliptic simplified broken Lefschetz fibrations (and more generally, hyperelliptic directed broken Lefschetz fibrations) in Chapter 4. In smooth category, a hyperelliptic Lefschetz fibration is defined as a Lefschetz fibration such that all the vanishing cycles of the fibration are preserved by the hyperelliptic involution of the standard surface Σ_g for a suitable choice of identification of a reference fiber with the surface Σ_g. Siebert and Tian [29] and Fuller [11] proved that the total space of a hyperelliptic Lefschetz fibration admits an involution which preserves any fiber of the Lefschetz fibration. In this thesis, we define hyperelliptic simplified broken Lefschetz fibrations in the same way as in the case of Lefschetz fibrations and generalize Siebert, Tian and Fuller's result on hyperelliptic Lefschetz fibrations to that on hyperelliptic simplified broken Lefschetz fibrations. We prove that the total space of a hyperelliptic simplified broken Lefschetz fibration admits an involution preserving any fiber of the fibration provided that the genus of the fibration is greater than or equal to 3 (see (i) of Theorem 4.3.1). By using the involution, we also prove that the rational homology class of the total space of a hyperelliptic simplified broken Lefschetz fibration of genus $g \geq 3$ represented by a regular fiber is not trivial (see (ii) of Theorem 4.3.1). The second statement implies that we cannot drop the assumption on a genus from the first statement. Note that we can generalize our results above to those on hyperelliptic directed broken Lefschetz fibrations (see Theorem 4.3.8).

Matsumoto defined the local signature of a Lefschetz singular fiber and proved that the signature of the total space of a Lefschetz fibration of genus-1 [23] and genus-2 [24] is equal to the sum of the local signatures of the Lefschetz singular fibers of the Lefschetz fibration. Endo [9] generalized Matsumoto's signature formulae to a formula for the signatures of hyperelliptic Lefschetz fibrations. In this thesis, we further generalize Matsumoto and Endo's signature formulae to a formula for the signatures of hyperelliptic directed broken Lefschetz fibrations as follows: we define a certain rational valued homomorphism $h_{g,c}$ on the subgroup of the hyperelliptic mapping class group which consists of mapping classes preserving c, where $c \subset \Sigma_g$ is a simple closed curve preserved by the hyperelliptic involution. We prove that the signature of the total space of a hyperelliptic directed broken Lefschetz fibration is equal to the sum of the local signatures of the Lefschetz singular fibers of the fibration and the values of the monodromies along the images of folds of the fibration under the homomorphisms $h_{g,d_1}, \ldots, h_{g,d_m}$, where $d_1, \ldots, d_m$ are vanishing cycles of folds (see Theorem 4.5.1).

Chapter 2

Broken Lefschetz fibrations and vanishing cycles

In this chapter, we give the definition of broken Lefschetz fibrations and summarize the results on monodromies and vanishing cycles of broken Lefschetz fibrations we need in this thesis.

2.1 Broken Lefschetz fibrations

Let X and B be connected, oriented, compact, smooth manifolds of dimension 4 and 2, respectively, and $f : X \to B$ a smooth map. Assume that f satisfies the condition $f^{-1}(\partial B) = \partial X$. A critical point $p \in X$ of f is called an *indefinite fold singularity* if there exist real coordinates (t, x, y, z) of X around p and (s, w) of B around $f(p)$ such that f is locally described as follows:

$$(t, x, y, z) \mapsto (s, w) = (t, x^2 + y^2 - z^2).$$

In this paper, we will refer to this singularity as a *fold* for simplicity. A critical point $p \in X$ of f is called a *Lefschetz singularity* if there exist complex coordinates (z, w) of X around p compatible with the orientation of X and ξ of B around $f(p)$ compatible with the orientation of B such that f is locally described as follows:

$$(z, w) \mapsto \xi = zw.$$

Definition 2.1.1. A map f is called a *broken Lefschetz fibration* if f satisfies the following conditions:

- the set $\mathrm{Crit}(f) \subset X$ of critical points of f consists of folds and Lefschetz singularities;

- the restriction $f|_{Z_f}$ is a generic immersion, where $Z_f \subset X$ is the set of folds;

- the restriction $f|_{\mathcal{C}_f}$ is injective, where $\mathcal{C}_f \subset X$ is the set of Lefschetz singularities of f.

A broken Lefschetz fibration f is called a *Lefschetz fibration* if the set of critical points of f consists of Lefschetz singularities. All the regular fiber of a Lefschetz fibration are diffeomorphic to some genus-g surface Σ_g. A Lefschetz fibration f has *genus-g* if the genus of a regular

fiber of f is equal to g. We will refer to broken Lefschetz fibrations and Lefschetz fibrations as *BLFs* and *LFs*, respectively.

Definition 2.1.2. Two BLFs $f_1 : X_1 \to B_1$ and $f_2 : X_2 \to B_2$ are said to be *equivalent* if there exist diffeomorphisms $\Theta : X_1 \to X_2$ and $\theta : B_1 \to B_2$ which make the following diagram commute:

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\ \Theta\ } & X_2 \\
{\scriptstyle f_1}\Big\downarrow & & \Big\downarrow{\scriptstyle f_2} \\
B_1 & \xrightarrow{\ \theta\ } & B_2.
\end{array}
$$

Let $f : X \to S^2$ be a BLF over the 2-sphere. We assume the following conditions:

(a) the restriction of f to the set of singularities is injective;

(b) the image $f(Z_f)$ is a disjoint union of embedded circles parallel to the equator of S^2.

We put $f(Z_f) = Z_1 \amalg \cdots \amalg Z_m$, where Z_i is a connected component of $f(Z_f)$. We take an embedding $\gamma : [0,1] \to S^2$ so that γ satisfies the following properties:

1. the image $\gamma([0,1])$ is contained in the complement of $f(\mathcal{C}_f)$;

2. γ starts at the south pole $p_s \in S^2$ and connects the south pole to the north pole $p_n \in S^2$;

3. γ intersects each component of $f(Z_f)$ at a single point transversely.

We put $\{q_i\} = Z_i \cap \alpha([0,1])$ and $\gamma(t_i) = q_i$. We assume that $q_1, \ldots, q_m$ appear in this order when we go along γ from p_s to p_n (see Figure 2.1.1). The preimage $f^{-1}(\gamma([0,1]))$ is
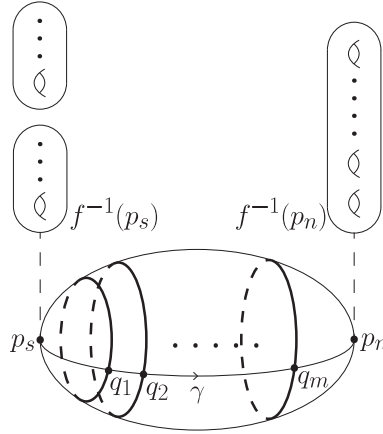
Figure 2.1.1: A path γ. The bold circles describe $f(Z_f)$.

a 3-manifold which is a cobordism between $f^{-1}(p_s)$ and $f^{-1}(p_n)$. By the definition of folds, $f^{-1}(\gamma([0, t_i + \varepsilon]))$ is obtained from $f^{-1}(\gamma([0, t_i - \varepsilon]))$ by either 1 or 2-handle attachment for each $i = 1, \ldots, m$. Thus, we obtain a handle decomposition of the cobordism $f^{-1}(\gamma([0,1]))$.

Definition 2.1.3. A BLF f is said to be *directed* if it satisfies the following conditions:

1. f satisfies the conditions (a) and (b) above;

2. all the handles of the handle decomposition of $f^{-1}(\gamma([0,1]))$ have index-1;

3. all Lefschetz singularities of f are in the preimage of the component of $S^2 \backslash (Z_1 \amalg \cdots \amalg Z_m)$ which contains the point p_n.

In this paper, we will refer to a directed broken Lefschetz fibration as a DBLF.

Definition 2.1.4. A DBLF $f : X \to S^2$ is called a *simplified broken Lefschetz fibration* if the set of folds of f is connected and that all the fibers of f are connected. In this paper, we will refer to a simplified broken Lefschetz fibration as an SBLF.

Let f be an SBLF. The set Z_f is either the empty set or an embedded circle in X. If Z_f is empty, then f is an LF over S^2. If Z_f is not empty, the image $f(Z_f)$ is an embedded circle in S^2. Thus, $S^2 \setminus \mathrm{Int}\, \nu f(Z_f)$ consists of two 2-disks D_1 and D_2, where $\nu f(Z_f)$ is a regular neighborhood of $f(Z_f)$. Furthermore, the genus of a regular fiber of the fibration $\mathrm{res} f : f^{-1}(D_1) \to D_1$ is higher than the genus of a regular fiber of the fibration $\mathrm{res} f : f^{-1}(D_2) \to D_2$ by 1. we call $f^{-1}(D_1)$ (resp. $f^{-1}(D_2)$) the *higher side* (resp. *lower side*) of f and $f^{-1}(\nu f(Z_f))$ the *round cobordism* of f. By the definition, all the Lefschetz singularities of f are in the higher side of f. We call the genus of a regular fiber in the higher side the *genus* of f.

2.2 Monodromy representations of Lefschetz fibrations

Let $f : X \to B$ be a genus-g LF. We fix a regular value $p_0 \in B$ of f and an orientation-preserving diffeomorphism $\psi_0 : f^{-1}(p_0) \to \Sigma_g$. For a loop $\gamma : (I, \partial I) \to (B \setminus f(\mathcal{C}_f), p_0)$, the pull-back $\gamma^* f = \{(t, x) \in I \times M \mid \gamma(t) = f(x)\}$ is isomorphic to the trivial Σ_g-bundle. We take a trivialization $\Psi : \gamma^* f \to I \times \Sigma_g$ so that the restriction $\Psi|_{\{0\} \times \Sigma_g}$ coincides with ψ_0. We put $\Psi(t, x) = (t, \psi_t(x))$. We denote by $[\psi_1 \circ \psi_0^{-1}]$ the isotopy class of the map $\psi_1 \circ \psi_0^{-1}$ and by $\mathcal{M}_g$ the mapping class group of Σ_g, that is, the set of isotopy classes of orientation-preserving diffeomorphisms. We define the map $\rho_f : \pi_1(B \setminus f(\mathcal{C}_f), p_0) \to \mathcal{M}_g$ as follows:

$$\rho_f([\gamma]) = [\psi_1 \circ \psi_0^{-1}].$$

This map is well-defined and called a *monodromy representation* of f. The readers should refer to [13] for details of monodromy representations of LFs.

Remark 2.2.1. In order to make monodromy representations of LFs homomorphisms, we define a group structure of the mapping class group $\mathcal{M}_g$ using the *opposite* multiplication to the composition as maps, that is, we define the multiplication $[f] \cdot [g]$ as $[g \circ f]$ for elements $[f], [g] \in \mathcal{M}_g$.

Definition 2.2.2. Let B_i $(i = 1, 2)$ be a connected surface (possibly with boundary or punctures), q_i a point in B_i and $\rho_i : \pi_1(B_i, q_i) \to \mathcal{M}_g$ a representation. The representations ρ_1 and ρ_2 are said to be *equivalent* if there exist an element $\varphi \in \mathcal{M}_g$ and an orientation

preserving diffeomorphism $h : (B_1, q_1) \to (B_2, q_2)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\pi_1(B_1, q_1) & \xrightarrow{\rho_1} & \mathcal{M}_g \\
{\scriptstyle h_*}\downarrow & & \downarrow{\scriptstyle \mathrm{conj}(\varphi)} \\
\pi_1(B_2, q_2) & \xrightarrow{\rho_2} & \mathcal{M}_g,
\end{array}
$$

where $\mathrm{conj}(\varphi)$ is the inner automorphism of $\mathcal{M}_g$ by the element φ.

Note that a monodromy representation of an LF depends on various choices in construction. However, the equivalent class of a monodromy representation of an LF does not depend on the choices.

Theorem 2.2.3 ([20], [24])**.** *Let $f_i : X_i \to B_i$ $(i = 1, 2)$ be an LF with genus-g. We assume that f_1 satisfies one of the following conditions:*

- *the genus g of f_1 is greater than 1;*

- *the base space B_1 of f_1 has non-empty boundary;*

- *the set $\mathrm{Crit}\, f_1$ of critical points of f_1 is not empty.*

Then f_1 and f_2 are equivalent if and only if the corresponding monodromy representations ρ_{f_1} and ρ_{f_2} are equivalent.

2.3   Surgery homomorphisms

For a simple closed curve $c \subset \Sigma_g$, we define the subgroup $\mathcal{M}_g(c)$ of the mapping class group $\mathcal{M}_g$ as follows:

$$
\mathcal{M}_g(c) = \{[T] \in \mathcal{M}_g | T(c) = c\}.
$$

For an element $\varphi \in \mathcal{M}_g(c)$, we take a mapping class $\Phi_c(\varphi)$ in the following way; we first take a representative $T \in \varphi$ so that T preserves the curve c setwise. The restriction $T|_{\Sigma_g \backslash c}$ is also a diffeomorphism. We can extend $T|_{\Sigma_g \backslash c}$ to a self-diffeomorphism of S_c, where S_c is obtained by applying surgery to Σ_g along c. We denote by $\Phi_c(\varphi)$ the isotopy class of this extended diffeomorphism. The topology of S_c is determined easily as follows:

$$
S_c \cong \begin{cases} \Sigma_{g-1} & (\text{if } c \text{ is type I}), \\ \Sigma_h \amalg \Sigma_{g-h} & (\text{if } c \text{ is type II}_h), \end{cases}
$$

where c is said to be *type* I if c is non-separating, and *type* II$_h$ if c is separating curve which bounds a genus-h surface (see Figure 2.3.1). Thus, we obtain:

$$
\mathrm{Mod}(S_c) \cong \begin{cases} \mathcal{M}_{g-1} & (\text{if } c \text{ is type I}), \\ \mathcal{M}_h \times \mathcal{M}_{g-h} & (\text{if } c \text{ is type II}_h \text{ and } 2h \neq g), \\ (\mathcal{M}_h \times \mathcal{M}_h) \rtimes \mathbb{Z}/2\mathbb{Z} & (\text{if } c \text{ is type II}_h \text{ and } 2h = g), \end{cases}
$$

where $\mathrm{Mod}(S_c)$ is the mapping class group of S_c, and in the last case, a generator of $\mathbb{Z}/2\mathbb{Z}$ is represented by a map which exchanges the components. We can prove the following lemma by the argument quite similar to that in [5].
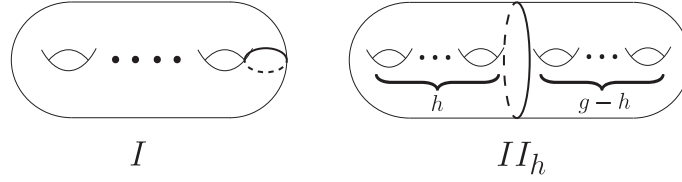
Figure 2.3.1: type I and type II_h

Lemma 2.3.1. *The map $\Phi_c : \mathcal{M}_g(c) \ni \varphi \mapsto \Phi_c(\varphi) \in \mathrm{Mod}(S_c)$ is well-defined.*

We call the homomorphism $\Phi_c : \mathcal{M}_g(c) \to \mathrm{Mod}(S_c)$ a *surgery homomorphism* along c. To see the relation between surgery homomorphisms and monodromies, we consider a BLF $f : X \to S^1 \times I$ over the annulus. We assume the following conditions:

- the set $\mathrm{Crit}(f)$ consists of one component of folds;

- the restriction $f|_{\mathrm{Crit}(f)}$ is injective;

- the image $f(\mathrm{Crit}(f))$ is equal to $S^1 \times \{\frac{1}{2}\}$.

We take a point $p_0 \in S^1$. By the assumptions, we can obtain a Morse function $\mathrm{res}\, f : f^{-1}(\{p_0\} \times I) \to I$ with one critical point. Suppose that the index of the critical point of this Morse function in index 1. We denote by $d \subset f^{-1}(p_0, 1)$ a attaching circle of the 2-handle. We fix an identification of $f^{-1}(p_0, 1)$ with a disjoint union $\Sigma_{g_1} \amalg \cdots \amalg \Sigma_{g_k}$. Suppose that d is contained in Σ_{g_1}. The fiber $f^{-1}(p_0, 0)$ consists of k components if d is non-separating, and $k+1$ components if d is separating. The fiber $f^{-1}(p_0, 0)$ can be identified with $S_d \amalg \Sigma_{g_2} \amalg \cdots \amalg \Sigma_{g_k}$ using the fixed identification of $f^{-1}(p_0, 1)$. We take a simple loop $\gamma_i \subset S^1 \times I \setminus f(\mathrm{Crit}(f))$ based at (p_0, i) $(i = 0, 1)$ which is parallel to the boundary. We denote by ψ_i the monodromy along γ_i.

Theorem 2.3.2 (cf. [1] and [2])**.** *The element ψ_1 preserves the curve d and is mapped to ψ_0 by the following homomorphisms:*

$$\Phi_d \times \mathrm{id}_{\mathcal{M}_{g_2}} \times \cdots \times \mathrm{id}_{\mathcal{M}_{g_k}} : \mathcal{M}_{g_1}(d) \times \mathcal{M} \to \mathrm{Mod}(S_c) \times \mathcal{M},$$

where we put $\mathcal{M} = \mathcal{M}_{g_2} \times \cdots \times \mathcal{M}_{g_k}$.

2.4 Vanishing cycles of directed broken Lefschetz fibrations

For a genus-g DBLF $f : X \to S^2$, we put $f(\mathcal{C}_f) = \{p_1, \ldots, p_l\}$. We take a path $\gamma \subset S^2$ as in Section 2.1. By the definition of DBLFs, we can obtain the preimage $f^{-1}(\gamma([0, t_i - \varepsilon]))$ from the preimage $f^{-1}(\gamma([0, t_i + \varepsilon]))$ by attaching a 2-handle. We call an attaching circle $d_i \subset f^{-1}(\gamma(t_i + \varepsilon))$ of the 2-handle a *vanishing cycle* of folds.

We take paths $\gamma_1, \ldots, \gamma_l$ so that γ_i connects p_n to p_i and that the paths $\gamma, \gamma_1, \ldots, \gamma_l$ are mutually disjoint except on the point p_0. Suppose that the indices of the paths are given

so that $\gamma, \gamma_1, \ldots, \gamma_l$ appear in this order when we go around p_0 counterclockwise. As in the case of folds, the paths $\gamma_1, \ldots, \gamma_l$ determine *vanishing cycles* of Lefschetz singularities (the reader refer to [13], for example, for details of vanishing cycles of Lefschetz singularities). We denote by $c_i \subset f^{-1}(p_0)$ the vanishing cycle determined by γ_i. We identify the fiber $f^{-1}(p_0)$ with Σ_g and regard the vanishing cycles as simple closed curves in Σ_g. Denote by $D_h \subset S^2$ the connected component of $S^2 \setminus \nu f(Z_f)$ which contains the point p_n. Since the restriction $f|_{f^{-1}(D_h)}$ is an LF, we can take a monodromy representation $\rho_f : \pi_1(D_h \setminus f(\mathcal{C}_f), p_0) \to \mathcal{M}_g$.

Theorem 2.4.1 ([20], [24])**.** *For each reference path γ_i, denote by a_i a loop in $D^2 \setminus f(\mathcal{C}_f)$ based at p_0 by connecting p_0 with a small counterclockwise circle around p_i using γ_i. Then the monodromy $\rho_f(a_i)$ is equal to the right-handed Dehn twist t_{c_i}.*

Since the product $a_1 \cdot \cdots \cdot a_l \in \pi_1(D_h \setminus f(\mathcal{C}_f), p_0)$ is represented by a loop parallel to the boundary of D_h, we immediately obtain:

Corollary 2.4.2. *The element $t_{c_1} \cdot \cdots \cdot t_{c_l}$ is the monodromy of f along a loop parallel to the boundary ∂D_h. Moreover, this element is contained in the kernel of the composition $\Phi_{d_1} \circ \cdots \circ \Phi_{d_m}$.*

Proof. The first statement is obvious, while the second statement holds since the lowest genus side of f does not contain any singularities. $\qquad\square$

2.5   Hurwitz cycle systems of simplified broken Lefschetz fibrations

Let $f : X \to S^2$ be a genus-g SBLF with non-empty folds. As explained in the previous section, we can obtain vanishing cycles $c = d_1, c_1, \ldots, c_l$ of folds and Lefschetz singularities of f. We call a system of reference paths $\gamma, \gamma_1, \ldots, \gamma_l$ which give vanishing cycles $c, c_1, \ldots, c_l$ a *Hurwitz path system* and a sequence $W_f = (c; c_1, \ldots, c_l)$ a *Hurwitz cycle system* of f. There are two types of modifications of Hurwitz cycle systems. The first one, which we will refer to as an *elementary transformation*, is as follows:

$$(c; c_1, \ldots, c_i, c_{i+1}, \ldots, c_n) \longrightarrow (c; c_1, \ldots, c_{i+1}, t_{c_{i+1}}(c_i), \ldots, c_n).$$

It is easy to see that this modification can be realized by replacing a Hurwitz path system as described in the left side of Figure 2.5.1. The second modification, *simultaneous action by $h \in \mathcal{M}_g$*, is as follows.

$$(c; c_1, \ldots, c_n) \longrightarrow (h(c); h(c_1), \ldots, h(c_n)).$$

This modification corresponds to substitution of an identification of the reference fiber with Σ_g. Two sequences $(c; c_1, \ldots, c_l)$ and $(d; d_1, \ldots, d_l)$ of simple closed curves in Σ_g are said to be *Hurwitz equivalent* if one can be obtained from the other by successive application of simultaneous actions, elementary transformations and their inverse. Note that for a given SBLF f, any sequence W which is Hurwitz equivalent to W_f can be realized as a Hurwitz cycle system of f by replacing reference paths and an identification $f^{-1}(p_0) \cong \Sigma_g$.
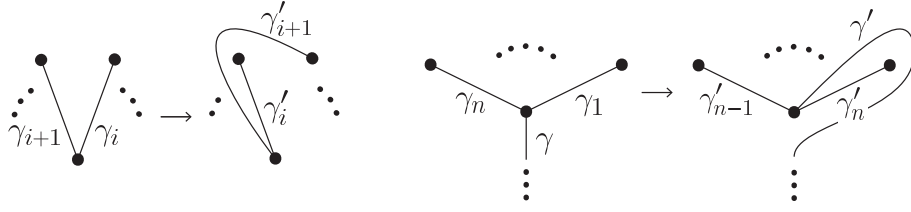
Figure 2.5.1: Left: modification of a Hurwitz path system corresponding to an elementary transformation. Right: another modification of a path system.

Remark 2.5.1. There is another modification of a Hurwitz cycle system which is described as follows:

$$(c; c_1, \ldots, c_n) \longrightarrow (t_{c_1}(c); c_2, \ldots, c_n, c_1).$$

It is easy to verify that this modification is induced by the modification of a Hurwitz path system described in the right side of Figure 2.5.1. Furthermore, this modification can be realized by simultaneous action by t_{c_1}, followed by successive application of inverse transformations of elementary transformations. This modification will play a key role in the proof of the theorem below.

Theorem 2.5.2. *Let $f_i : X_i \to S^2$ be an SBLF with genus $g \geq 3$ ($i = 1, 2$). The fibrations f_1 and f_2 are isomorphic if and only if the corresponding Hurwitz cycle systems W_{f_1} and W_{f_2} are equivalent.*

Remark 2.5.3. This theorem would not hold if the assumption on genera of fibrations are dropped. Indeed, there exist infinitely many SBLFs with small genera which are mutually not isomorphic but have the same Hurwitz cycle systems.

Proof of Theorem 2.5.2. We first prove the only if part. Suppose that f_1 and f_2 are isomorphic, and we fix diffeomorphisms $\Phi : X_1 \to X_2$ and $\varphi : S^2 \to S^2$ satisfying the condition in the definition. We take reference paths $\gamma, \gamma_1, \ldots, \gamma_n$ of the fibration f_1 as explained above. We denote by W_{f_1} the corresponding Hurwitz cycle system of f_1 derived from these paths, together with an identification $\phi : f_1^{-1}(y_0) \to \Sigma_g$. We can use the paths $\varphi(\gamma), \varphi(\gamma_1), \ldots, \varphi(\gamma_n)$ and a diffeomorphism $\phi \circ \Phi^{-1} : f_2^{-1}(\varphi(y_0)) \to \Sigma_g$ to obtain a Hurwitz cycle system W_{f_2} of the fibration f_2. It is easy to verify that W_{f_1} is equal to W_{f_2}. Thus, all we need to prove is a Hurwitz cycle system of f_1 derived from different reference paths $\gamma', \gamma'_1, \ldots, \gamma'_n$ is Hurwitz equivalent to W_{f_1}. By the argument similar to that in the solution of Exercise 8.2.7(c) in [13], we can prove that the system $\gamma', \gamma'_1, \ldots, \gamma'_n$ can be changed into the system $\gamma, \gamma_1, \ldots, \gamma_n$ up to isotopy by successive application of the two moves in Figure 2.5.1. This completes the proof of the only if part.

We next prove the if part. By the assumption, we can take reference paths of f_1 and f_2, and identifications of reference fibers with the surface Σ_g so that the corresponding Hurwitz cycle systems W_{f_1} and W_{f_2} coincide. We decompose X_i into the three parts $X_i^{(r)}, X_i^{(h)}$ and $X_i^{(l)}$, that is, the preimage of a regular neighborhood of $f(Z_f)$, the highest side and the lowest side of f_i. The restriction $f_i|_{X_i^{(h)}}$ is an LF. By Theorem 2.2.3, the fibrations $f_1|_{X_1^{(h)}}$ and $f_2|_{X_2^{(h)}}$ are isomorphic. In particular, we can take a fiber-preserving diffeomorphism $\Phi_h : X_1^{(h)} \to$

$X_2^{(h)}$. Since there are no singularities on the boundary of the highest side, we can take an identification of the boundary $\partial X_1^{(h)}$ with the mapping torus $T(\theta) = I \times \Sigma_g / (1, x) \sim (0, \theta(x))$, where $\theta : \Sigma_g \to \Sigma_g$ is a diffeomorphism. This identification, together with a diffeomorphism Φ_h, gives an identification of $\partial X_2^{(h)}$ with $T(\theta)$. We denote by $c \subset \Sigma_g$ the vanishing cycle of folds in W_{f_1} (note that this cycle coincides with that in W_{f_2}). By Corollary 2.4.2, the isotopy class of θ is contained in $\mathcal{M}_g(c)$.

we can assume that θ preserves a regular neighborhood $\nu(c)$. For each $i = 1, 2$, we take an identification of $f_i(X_i^{(r)})$ with the annulus $I \times D^1 / (1, t) \sim (0, t)$ so that the restriction $f_i|_{\partial X_i^{(h)}} : T(\theta) \to I \times \{1\} / \sim$ becomes the projection, and that the image of indefinite folds is equal to the circle $I \times \{0\} / \sim$. The following lemma can be proved easily:

Lemma 2.5.4. *We denote by $Z_i \subset X_i$ the set of indefinite folds of f_i. There exist a sufficiently small number $\varepsilon > 0$ and a diffeomorphism $\Psi_i : I \times D^1_\varepsilon \times D^2_\varepsilon / (1, x, y_1, y_2) \sim (0, \pm x, y_1, \pm y_2) \to \nu(Z_i)$, where D^d_ε is the d-dimensional ball with radius ε, which make the following diagram commute:*

$$
\begin{array}{ccc}
I \times D^1_\varepsilon \times D^2_\varepsilon / \sim & \xrightarrow{\ \Psi_i\ } & \nu(Z_i) \\
{\scriptstyle \pi} \downarrow & \swarrow {\scriptstyle f_i} & \\
I \times [-\varepsilon, \varepsilon] / \sim & &
\end{array}
\quad,
$$

where π is defined as $\pi(t, x, y_1, y_2) = (t, -x^2 + {y_1}^2 + {y_2}^2)$.

For a positive number $s \leq 2$, we define a path $\gamma_{t,s} : [0, s] \to I \times D^1 / \sim$ as $\gamma_t(x) = (t, 1 - x)$. A connected component of the set $\mathrm{Sub}(S^1, \Sigma_g)$ of circles in Σ_g is simply connected if $g \geq 2$ (see Theorem 2.7.H of [17] for example). Thus, we can take a horizontal distribution $\mathcal{H}_i$ of $f_i|_{X_i \setminus Z_i}$ so that it satisfies the following conditions:

1. in the image of Ψ_i, $\mathcal{H}_i$ is equal to the horizontal distribution derived from the product metric of $I \times D^1_\varepsilon \times D^2_\varepsilon / \sim$,

2. the parallel transport $PT^{\mathcal{H}_i}_{\gamma_{t, 1 - \frac{\varepsilon}{2}}}$ of $\mathcal{H}_i$ along $\gamma_{t, 1 - \frac{\varepsilon}{2}}$ maps $\{t\} \times \nu(c)$ to the following set:

$$
\pi^{-1}\left(\frac{\varepsilon}{2}\right) \cap \left\{ (t, x, y_1, y_2) \in I \times D^1_\varepsilon \times D^2_\varepsilon / \sim \ \big| \ |x| \leq \frac{\varepsilon}{2} \right\},
$$

3. the parallel transport $PT^{\mathcal{H}_1}_{\gamma_{t, 1 - \frac{\varepsilon}{2}}}$ is equal to the parallel transport $PT^{\mathcal{H}_2}_{\gamma_{t, 1 - \frac{\varepsilon}{2}}}$ under the identifications.

Using the distributions, we can define a diffeomorphism $\Phi_r : X_1^{(r)} \to X_2^{(r)}$ as follows:

$$
\Phi_r(w) = \begin{cases} (t, x, y) \in \nu(Z_2) & (w = (t, x, y) \in \nu(Z_1) \cap I \times D^1_{\frac{2\varepsilon}{3}} \times D^2_{\frac{2\varepsilon}{3}} / \sim), \\ PT^{\mathcal{H}_2}_{\gamma_{t,s}}(z) \in X_2^{(r)} & (w = PT^{\mathcal{H}_1}_{\gamma_{t,s}}(z) \in X_1^{(r)}, z \in I \times (\Sigma_g \setminus \nu(c)) / \sim). \end{cases}
$$

It is easy to see that this map is fiber-preserving. In particular, the restriction $\Phi_r : \partial X_1^{(l)} \to \partial X_2^{(l)}$ is a fiber-preserving diffeomorphism. Since the connected component of the group $\mathrm{Diff}^+(\Sigma_{g-1})$ is contractible if $g \geq 3$, this restriction can be extended to a fiber-preserving diffeomorphism $\Phi_l : X_l^{(1)} \to X_l^{(2)}$. Combing the three diffeomorphisms Φ_h, Φ_r and Φ_l, we can obtain the desired map Φ. This completes the proof of Theorem 2.5.2. $\qquad \square$

2.6 Kirby diagrams of broken Lefschetz fibrations

In this section, we will give a quick review of a handle decomposition of a total space of a directed broken Lefschetz fibration which reflects its fibration structure. The readers should refer to [2] for details of them.

Definition 2.6.1. Let X be a smooth 4-manifold and we put

$$R_i^{\pm} = I \times D^i \times D^{3-i} / ((1, x_1, x_2, x_3) \sim (0, \pm x_1, x_2, \pm x_3))(i = 1, 2).$$

Let $\psi : I \times \partial D^i \times D^{3-i} / \sim \to \partial M$ be an embedding. We call $M \bigcup_{\psi} R_i^{\pm}$ a 4-manifold obtained by attaching a *round i-handle* and R_i^+(resp. R_i^-) *(4-dimensional) untwisted* (resp. *twisted*) *round i-handle.*

Remark 2.6.2. Both untwisted and twisted round handles are diffeomorphic to $S^1 \times D^3$, but these round handles have distinct attaching regions. The attaching region of an untwisted round i-handle is the trivial $S^{i-1} \times D^{3-i}$-bundle over S^1, while that of a twisted one is a non-trivial $S^{i-1} \times D^{3-i}$-bundle over S^1.

By the definition of round handles, we can regard 4-dimensional round i handle attachment as S^1-family of 3-dimensional i-handle attachment. We call an attaching sphere of a i-handle in this family an *attaching sphere* of a round handle.

Lemma 2.6.3 ([2])**.** *For $i \in \{1, 2\}$, round i-handle attachment is given by i-handle attachment followed by $(i + 1)$-handle attachment whose attaching sphere goes over the belt sphere of the i-handle geometrically twice, algebraically zero times if the round handle is untwisted and twice if the round handle is twisted.*

Proof. The handle $R_i^{\pm}$ can be decomposed into two parts $[0, \frac{1}{2}] \times D^i \times D^{3-i}$ and $[\frac{1}{2}, 1] \times D^i \times D^{3-i}$. Attachment of a round i-handle is equivalent to attachment of the former part followed by attachment of the latter part. It is easy to see that the former (resp. the latter) attachment can be regarded as 2-handle (resp. 3-handle) attachment. □

Let $f : X \to S^2$ be a DBLF of genus-g. We use the same notations $Z_1, \ldots, Z_m \subset S^2$ as in section 2.4 and denote by $\widetilde{Z_i} \subset X$ the connected component of folds of f on Z_i. We can decompose S^2 into two disks D_h, D_l and m annuli $\nu Z_1, \ldots, \nu Z_m$. Since Lemma 2.5.4 also works for each component $\widetilde{Z_i}$, we immediately obtain:

Lemma 2.6.4 ([2])**.** *The manifold $f^{-1}(D_h \amalg \nu Z_1 \amalg \cdots \amalg \nu Z_i)$ can be obtained from $f^{-1}(D_h \amalg \nu Z_1 \amalg \cdots \amalg \nu Z_{i-1})$ by attaching a round 2-handle. Moreover, an attaching circle the round 2-handle is along a vanishing cycle of $\widetilde{Z_i}$. When we regard this round 2-handle attachment as 2-handle attachment followed by 3-handle attachment, the 2-handle is attached along a vanishing cycle of $\widetilde{Z_i}$ whose framing is along the fiber.*

Using Lemma 2.6.4, we can obtain a Kirby diagram of the total space of a DBLF. Several examples of Kirby diagrams obtained in this way can be found in [2], for example. The procedure of handle decomposition above also implies the following corollary, whose proof is left to the reader.

Corollary 2.6.5 (cf. [2]). *Let $c_1, \ldots, c_l \subset \Sigma_g$ and $d_i \subset (\cdots((\Sigma_g)_{c_1})_{c_2} \cdots)_{c_{i-1}}$ $(1 \geq i \geq m)$ be simple closed curves. Suppose that the element $t_{c_1} \cdot \ldots \cdot t_{c_l}$ is contained in the kernel of the map $\Phi_{d_m} \circ \cdots \circ \Phi_{d_1}$. Then, these exists a genus-g DBLF $f : X \to S^2$ such that vanishing cycles of f obtained as in section 2.4 coincides with $c_1, \ldots, c_l, d_1 \ldots, d_m$.*

Chapter 3

Classification of genus-1 simplified broken Lefschetz fibrations

In this chapter, we classify total spaces of genus-1 simplified broken Lefschetz fibrations. We prove the following theorem:

Theorem 3.0.1. *The following* 4*-manifolds admits a relatively minimal genus-1 SBLF with non-empty folds and $l \geq 0$ Lefschetz singularities:*

- $\#k\mathbb{CP}^2\#(l-k)\overline{\mathbb{CP}^2}$ *($0 \geq k \geq l-1$);*

- $\#\frac{l}{2}S^2 \times S^2$. *Note that this manifold appears only if l is even;*

- $S^1 \times S^3 \# S \# l\overline{\mathbb{CP}^2}$, *where S is an S^2-bundle over S^2;*

- $L\#l\overline{\mathbb{CP}^2}$, *where L is either of the manifolds L_n or L'_n ($n \geq 2$), which is defined by Pao [27].*

Conversely, every 4*-manifold which admits a relatively minimal genus-1 SBLF with non-empty folds is diffeomorphic to one of the manifolds above.*

3.1   The mapping class group of the torus

We take elements $\alpha, \beta \subset H_1(T^2; \mathbb{Z})$ so that the algebraic intersection $\alpha \cdot \beta$ is 1. We define a homomorphism $\Psi : \mathcal{M}_1 \to SL(2, \mathbb{Z})$ so that a pair $(T_*(\alpha), T_*(\beta))$ is equal to $(\alpha, \beta) \cdot {}^t\Psi([T])$ for every diffeomorphism $T : T^2 \to T^2$. It is known that Ψ is an isomorphism. In the rest of this section, we identify the group $\mathcal{M}_1$ with $SL(2, \mathbb{Z})$ via this isomorphism.

Since a primitive element in $H_1(T^2; \mathbb{Z})$ uniquely determines the isotopy class of an oriented loop in T^2, we represent the isotopy class of a simple closed curve by its homology class (after giving some orientation). With this understood, the Dehn twist along a primitive

element $\gamma = p\alpha + q\beta \in H_1(T^2; \mathbb{Z})$ makes sense and the corresponding linear representation is determined by the Picard-Lefschetz formula as follows:

$$t_\gamma = \begin{pmatrix} 1 - pq & -q^2 \\ p^2 & 1 + pq \end{pmatrix}.$$

In particular, we obtain $t_\alpha = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $t_\beta = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Denote these matrices by X_1 and X_2, respectively. The group $SL(2, \mathbb{Z})$ has the following finite presentation (see [22], for example):

$$SL(2, \mathbb{Z}) = < X_1, X_2 | (X_1 X_2)^6, X_1 X_2 X_1 X_2^{-1} X_1^{-1} X_2^{-1} > .$$

3.2 Chart descriptions of monodromy representations

Toward classification of total spaces of genus-1 SBLFs, we first prove that any genus-1 SBLF has a Hurwitz cycle system which we can easily deal with. The goal of this section is to prove the following theorem:

Theorem 3.2.1. *Let $f : X \to S^2$ be a relatively minimal genus-1 SBLF with non-empty folds. A Hurwitz cycle system W_f is Hurwitz equivalent to the following sequence:*

$$(\alpha; S_r T(n_1, \ldots, n_s)),$$

where $T(n_1, \ldots, n_s) = (\beta + n_1\alpha, \ldots, \beta + n_s\alpha)$ and $S_r = (\alpha, \ldots, \alpha)$ (r α's are contained in this sequence).

To prove this, we will introduce a graphical description of Hurwitz cycle systems.

Definition 3.2.2. A finite graph Γ in D^2 (possibly being empty or having *hoops* that are closed edges without vertices) is called a *chart* if Γ satisfies the following conditions:

(1) the degree of each vertex is equal to either 1, 6 or 12;

(2) each vertex in ∂D^2 has degree-1;

(3) each edge in Γ is labeled 1 or 2 and is oriented;

(4) the edge from a degree-1 vertex in $\text{Int}(D^2)$ is oriented toward the vertex;

(5) the six edges from a degree-6 vertex are labeled alternately with 1 and 2. Moreover, three consecutive edges are oriented toward the vertex and the other edges are oriented away from it;

(6) the twelve edges from a degree-12 vertex are labeled alternately with 1 and 2 and all the edges are oriented in the same way, oriented toward or away from the vertex (see Figure 3.2.1);

(7) an interior of each edge is contained in $\text{int}D^2$;

(8) let $\{v_1, \ldots, v_n\}$ be the set of vertices in ∂D^2 and assume that the indices are given so that $v_1, \ldots, v_n$ appear in this order when we go along ∂D^2 counterclockwise. Denote by i_k the label of the edge e_i from v_i. We put $\varepsilon_k = +1$ if e_i is oriented toward v_i and put $\varepsilon_k = +1$ otherwise. For some k, the sequence $((i_k, \varepsilon_k), \ldots, (i_n, \varepsilon_n), (i_1, \varepsilon_1), \ldots, (i_{k-1}, \varepsilon_{k-1}))$ can be divided into the following subsequences:

(a) $((1, \varepsilon))$;

(b) $((i, \varepsilon), (j, \varepsilon), (i, \varepsilon), (j, \varepsilon), (i, \varepsilon), (j, \varepsilon))$, where $(\{i, j\} = \{1, 2\})$ and ε is either $+1$ or -1.

We call such a sequence a *boundary sequence* of Γ and two subsequences above the *unit subsequences*.

degree 1
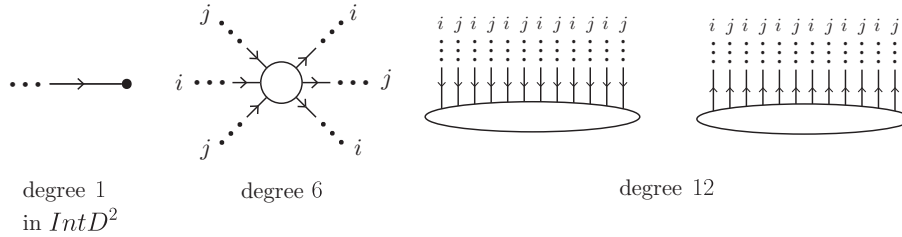in $Int D^2$

degree 6

degree 12

Figure 3.2.1: vertices of a chart, where $\{i, j\} = \{1, 2\}$.

An example of a chart is illustrated in Figure 3.2.2. The corresponding sequence mentioned in the condition (8) of the definition is as follows:

$$((1, -1), (1, -1), (1, -1), (1, -1), (2, 1), (1, 1), (2, 1), (1, 1), (2, 1), (1, 1)).$$

This sequence satisfies the condition (8).

For a chart Γ, we denote by $V(\Gamma)$ the set of all the vertices of Γ, and by S_Γ the subset of $V(\Gamma)$ consisting of the degree-1 vertices in $\text{Int}(D^2)$. Let v be a vertex of Γ. An edge e from v is called an *incoming edge* of v if e is oriented toward v and an *outgoing edge* of v if e is oriented away from v. A degree-1 or 12 vertex of a chart is *positive* (resp. *negative*) if all the edges from the vertex is outgoing edge (resp. incoming edge) of the vertex. Note that each degree-1 vertex in $\text{Int}(D^2)$ is negative by the definition of charts. Among the six edges from a degree-6 vertex v, three consecutive edges are incoming edges of v and the other edges are outgoing edges. We call the middle edge in the three incoming or outgoing edges a *middle edge* and another edge a *non-middle edge*. An edge in a chart is called a (d_1, d_2)-*edge* if its end points are a degree-d_1 vertex and a degree-d_2 vertex, where $d_1, d_2 \in \{1, 6, 12\}$ and $d_1 \leq d_2$. An edge in a chart is called a (∂, d)-*edge* if one of its end points is in ∂D^2 and the other is degree-d vertex, where $d \in \{1, 6, 12\}$, and we call an edge whose two end points are in ∂D^2 a (∂, ∂)-*edge*. A $(\partial, *)$-edge is called a *boundary edge*, where $* \in \{1, 6, 12, \partial\}$. Let $((i_1, \varepsilon_1), \ldots, (i_n, \varepsilon_n))$ be a boundary sequence of Γ. A union of six vertices and edges which correspond to a subsequence $((i, \varepsilon), (j, \varepsilon), (i, \varepsilon), (j, \varepsilon), (i, \varepsilon), (j, \varepsilon))$ is called a *boundary comb* of Γ.
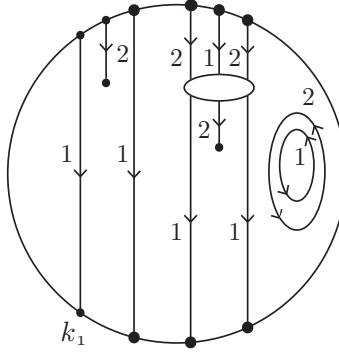
Figure 3.2.2: An example of a chart.

Let Γ be a chart in D^2. A path $\eta : [0,1] \to D^2$ is said to be *in general position* with respect to Γ if $\eta([0,1])$ is away from any vertices of Γ and intersects edges of Γ transversely. We put $\eta([0,1]) \cap \Gamma = \{p_1, \ldots, p_n\}$. Assume that $p_1, \ldots, p_n$ appear in this order when we go along η from $\eta(0)$ to $\eta(1)$. We denote by $k_i \in \{1, 2\}$ a label of the edge of Γ which goes through p_i and by ε_i the sign of the intersection between η and Γ at p_i. We call a word $w_\Gamma(\eta) = X_{k_1}^{\varepsilon_1} \cdot \ldots \cdot X_{k_n}^{\varepsilon_n}$ the *intersection word* of η with respect to Γ. We regard this word as an element of $SL(2, \mathbb{Z})$ by identifying the letters X_1, X_2 with the matrices defined in section 3.1.

Definition 3.2.3. Let Γ be a chart in D^2. We take a point $p_0 \in D^2 \setminus V(\Gamma)$. We define a homomorphism

$$\rho_\Gamma : \pi_1(D^2 \setminus S_\Gamma, p_0) \to SL(2, \mathbb{Z})$$

in the following way: for an element $\xi \in \pi_1(D^2 \setminus S_\Gamma, p_0)$, we choose a representative path $\eta : [0,1] \to D^2 \setminus S_\Gamma$ of ξ which is in general position with respect to Γ. Then we put $\rho_\Gamma(\xi) = w_\Gamma(\eta)$. We call the homomorphism ρ_Γ a *monodromy representation associated with* Γ. We can prove ρ_Γ is well-defined by the same way as in the proof of Lemma 12 of [18].

Since the monodromy along the boundary of the highest side of a genus-1 SBLF preserves a vanishing cycle of folds, we can assume that the monodromy is equal to $\pm X_1^m$ for some integer m after changing an identification of the reference fiber with T^2. Thus, the following lemma can be proved in the same way as in the proof of Theorem 15 of [18].

Lemma 3.2.4. *For any genus-1 SBLF $f : X \to S^2$, there exists a chart Γ in D^2 such that the monodromy representation of the highest side of f is equal to the monodromy representation associated with Γ up to equivalence.*

We next introduce several moves of charts which do not change the associated monodromy representations. The following lemma can be obtained in a similar way to the proof of Lemma 16 in [18].

Lemma 3.2.5. *Let Γ_1 and Γ_2 be charts and $E \subset D^2$ a 2-disk. We assume that E contains no degree-1 vertices of Γ_1 and Γ_2 in $\mathrm{Int}(D^2)$, that Γ_1 coincides with Γ_2 outside of E and that the complement $D^2 \setminus E$ is path connected. Then, the monodromy representation associated with Γ_1 is equal to that associated with Γ_2.*

Definition 3.2.6. Suppose that two charts Γ_1 and Γ_2 are in the situation of Lemma 3.2.5. We say that Γ_1 is obtained from Γ_2 by a *CI-move* in E. A typical CI-move is described in Figure 3.2.3, which is called a *channel change*.
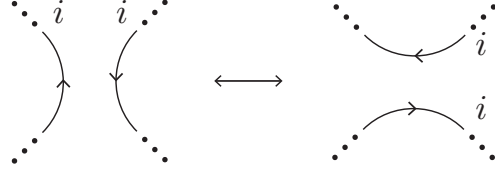

Figure 3.2.3: a channel change

Lemma 3.2.7. *Let Γ_1 and Γ_2 be charts and $E \subset D^2$ a 2-disk. We assume that Γ_1 is different from Γ_2 in E as described in Figure 3.2.4, that Γ_1 coincides with Γ_2 outside of E and that the complement $D^2 \setminus E$ is path connected. Then, the monodromy representation associated with Γ_1 is equal to that associated with Γ_2 up to equivalence.*
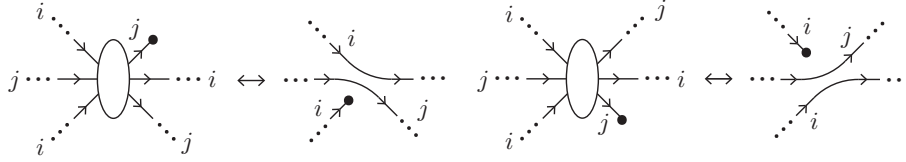

Figure 3.2.4: CII-moves

We omit a proof of Lemma 3.2.7 since it is quite similar to that of Lemma 18 of [18].

Definition 3.2.8. Suppose that two charts Γ_1 and Γ_2 are in the situation of Lemma 3.2.7. We say that Γ_1 is obtained from Γ_2 by a *CII-move* in E.

By a *C-move*, we mean a CI-move, a CII-move or an isotopic deformation in D^2. Two charts are said to be *C-move equivalent* if one can be obtained from the other by successive application of C-moves. Note that monodromy representations associated with C-move equivalent charts are equivalent by Lemma 3.2.5 and Lemma 3.2.7.

Lemma 3.2.9. *Let Γ be a chart. By successive application of C-moves, we can change Γ into a chart which has no degree-12 vertices.*

Proof. We choose a decomposition of the boundary sequence of Γ into the unit subsequences. Let v_1 and v_2 be consecutive vertices in ∂D^2 which are not contained in the same boundary comb. We denote by S the connected component of $\partial D^2 \setminus (\partial D^2 \cap \Gamma)$ between v_1 and v_2. We can move all the degree-12 vertices in Γ into a region of $\partial D^2 \setminus \Gamma$ containing S by using CI-moves illustrated in Figure 12 of [18]. By the CI-move illustrated in Figure 3.2.5, we can eliminate all the degree-12 vertices in Γ. This completes the proof of Lemma 3.2.9.   □
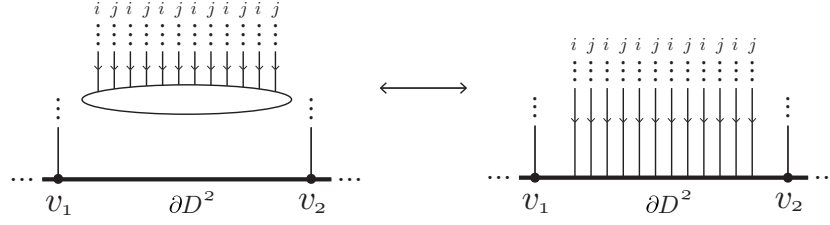
Figure 3.2.5: CI-move used to get rid of degree-12 vertices. The bold lines represent ∂D^2.

Lemma 3.2.10. *Let Γ be a chart. By successive application of C-moves, we can change Γ into a chart Γ' such that each $(1,6)$-edge e in Γ' satisfies the following conditions:*

  (i) e is a middle edge;

  (ii) the label of e is 2;

  (iii) let K be the connected component of $D^2 \setminus \Gamma'$ whose closure contains e. The set $K \cap \partial D^2$ is not empty.

The idea of the proof of Lemma 3.2.10 is similar to that of the proof of Lemma 22 in [18]. However, the two proofs are slightly different because of the difference of the definition of charts. Thus, we give the full proof below.

Proof. Let $n(\Gamma)$ be the sum of the number of degree-6 vertices and the number of $(1,6)$-edges in Γ. The proof proceeds by induction on $n(\Gamma)$.

If $n(\Gamma) = 0$, the conclusion of Lemma 3.2.10 holds since Γ has no $(1,6)$-edges. We assume that $n(\Gamma) > 0$ and there exists a $(1,6)$-edge which does not satisfy at least one of the conditions *(i)*, *(ii)* or *(iii)* of Lemma 3.2.10.

Case.1: Suppose that Γ has a non-middle $(1,6)$-edge. Let v be a degree-6 vertex which is an end point of a $(1,6)$-edge. We can apply a CII-move around v and eliminate this vertex. The number $n(\Gamma)$ decreases and the conclusion holds by the induction hypothesis.

Case.2: Suppose that Γ has a middle $(1,6)$-edge e whose label is 1. Let v_0 and v_1 the end points of e whose degrees are 1 and 6, respectively. We denote by K the connected component of $D^2 \setminus \Gamma$ whose closure contains v_0, v_1 and e. We take a sequence $f_1, \ldots, f_m$ of edges of Γ with signs as in the proof of Lemma 22 of [18]. For each f_i, we take a letter $w(f_i) = X_k^\varepsilon$, where k is equal to the label of the edge f_i and ε is equal to the sign of f_i. Note that both f_1 and f_m are equal to e and the sign of f_1 is negative, while the sign of f_m is positive, since the vertex v_0 is negative.

Case.2.1: There exists a consecutive pair f_i and f_{i+1} such that two edges share a vertex and

$$(w(f_i), w(f_{i+1})) = (X_1^{-1}, X_2^{-1}).$$

Case.2.2: There exists a consecutive pair f_i and f_{i+1} such that two edges share a vertex and

$$(w(f_i), w(f_{i+1})) = (X_2, X_1).$$

Case.2.3: The set $K \cap \partial D^2$ is empty.

If one of the above three cases occurs, then the conclusion holds by the same argument as that in Lemma 22 of [18].

Case.2.4: Suppose that $K \cap \partial D^2$ is not empty. Then one of the edges $f_1, \ldots, f_m$ is a boundary edge. By Cases 2.1 and 2.2, we can assume that $(w(f_i), w(f_{i+1}))$ is not equal to either of the subsequences (X_1^{-1}, X_2^{-1}) and (X_2, X_1) if f_i and f_{i+1} share a vertex. Let f_k be a boundary edge with the smallest index. By the assumption above, $w(f_k)$ is equal to either X_1^{-1} or X_2. If $w(f_k) = X_1^{-1}$, we can decrease the number of $(1,6)$-edges by applying C-moves illustrated in Figure 3.2.6. Thus, the conclusion holds by the induction hypothesis. Suppose
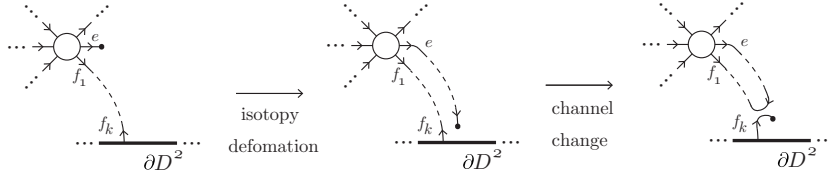

Figure 3.2.6: The bold line in the figure describes ∂D^2.

that $w(f_k) = X_2$. One of $f_{k+1}, \ldots, f_m$ is a boundary edge but not a $(\partial, 1)$-edge. Let f_l be such an edge with the smallest index.

Case.2.4.1: Suppose that $w(f_l) = X_1$. Then we can decrease the number of $(1,6)$-edges by applying C-moves illustrated in Figure3.2.7 and the conclusion holds by the induction hypothesis.
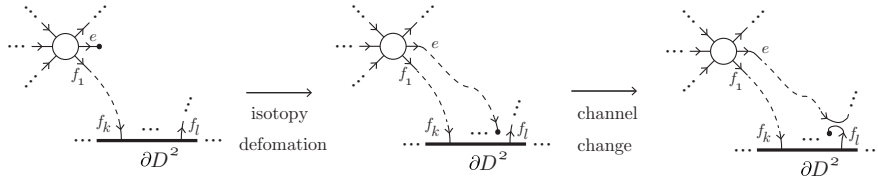
Figure 3.2.7:

Case. 2.4.2: Suppose that $w(f_l) = X_2$. We fix a decomposition of the boundary sequence of Γ into the unit subsequences. It is easy to see that a boundary comb which contains f_k is distinct from a boundary comb which contains f_l. Thus, we can apply C-moves as shown in Figure 3.2.8 and the conclusion holds by induction.

Case.2.4.3: Suppose that $w(f_l) = X_2^{-1}$. If both f_k and f_l were contained in a same boundary comb, there would be at least one $(\partial, 1)$-edge between f_k and f_l. However, all the degree-1 edges are negative. This contradiction says that a boundary comb that contains f_k is distinct from a boundary comb that contains f_l. Thus, we can apply C-moves similar to that we used in Case.2.4.2 and the conclusion holds by induction hypothesis.

Case.2.4.4: Suppose that $w(f_l) = X_1^{-1}$. If each $f_{l+1}, \ldots, f_m$ were not a boundary edge, then $(w(f_l), \ldots, w(f_m))$ would be equal to $(X_1^{-1}, X_2, X_1^{-1}, X_2, \ldots)$. This contradicts $w(f_m) = X_1$. Thus, at least one of $f_{l+1}, \ldots, f_m$ is a boundary edge. Let $f_{k'}$ be such an edge with the smallest index. The word $w(f_{k'})$ is equal to either X_1^{-1} or X_2. If $w(f_{k'}) = X_1^{-1}$, the conclusion holds

Figure 3.2.8: We first apply CI-move between the two boundary comb which contain f_k and f_l, respectively, and we obtain a new 1-labeled (∂, ∂)-edge. Then we move v_0 near this edge by isotopy deformation and apply a channel change.
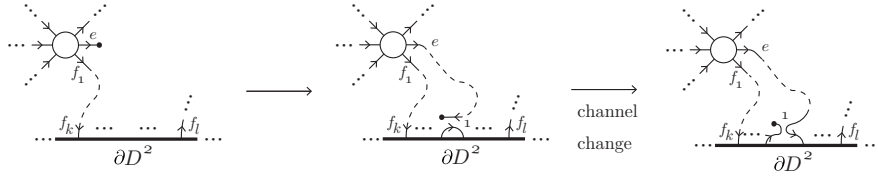
by the above argument. If $w(f_{k'}) = X_2$, one of four cases above occurs for $f_{k'}$. When one of the former three cases occurs, the conclusion holds by the same argument. When Case.2.4.4 occurs for $f_{k'}$, we can take $f_{k''}$ again as we take $f_{k'}$. We can repeat the above argument and the conclusion holds since m is finite.

Case.3: Suppose that Γ has a middle $(1,6)$-edge whose label is 2 which does not satisfy the condition *(iii)* in Lemma 3.2.10. We define K as we defined in Case.2. By the assumption, $K \cap \partial D^2$ is empty. Thus, we can prove the conclusion by the same argument as that in Cases 2.1, 2.2 and 2.3.

Combining the conclusions of Cases.1, 2 and 3, we complete the proof of Lemma 3.2.10. □

Proof of Theorem 3.2.1. By Lemma 3.2.4, we can take a chart Γ such that the associated monodromy representation ρ_Γ is equal to the monodromy representation of the higher side of f up to inner automorphisms of $SL(2, \mathbb{Z})$. We first remove all the degree-12 vertices in Γ by applying Lemma 3.2.9. By applying Lemma 3.2.10, we change the chart Γ into a chart such that all the $(1,6)$-edges satisfy conditions *(i)*, *(ii)* and *(iii)* in Lemma 3.2.10. In the process of the proof of Lemma 3.2.10, the number of degree-12 vertices does not increase. Thus, the chart obtained by the above process has no degree-12 vertices. Let $\{v_1, \ldots, v_m\}$ be the set of degree-1 vertices of Γ in ∂D^2. We choose the indices of v_i so that $v_1, \ldots, v_m$ appear in this order when we go along ∂D^2 counterclockwise. We further assume that v_1 and v_m are not contained in the same boundary comb. We denote by e_i a boundary edge whose end point is v_i. We put $S_\Gamma = \{p_1, \ldots, p_n\}$. Let K_i be a connected component of $D^2 \setminus \Gamma$ whose closure contains p_i. By the assumption on Γ, each p_i is an end point of either $(1,6)$-edge or $(\partial, 1)$-edge. For each p_i which is an end point of $(1,6)$-edge, we choose a connected component E_i of $K_i \cap \partial D^2$. We denote the two points of ∂E_i by v_{k_i} and v_{k_i+1}, where $k_i \in \{1, \ldots, m\}$ and $v_{m+1} = v_1$. Let V be a sufficiently small collar neighborhood of ∂D^2 in D^2 and p_0 a point in $V \cap K$, where K is a connected component of $D^2 \setminus \Gamma$ whose closure contains a connected component of $\partial D^2 \setminus \{v_1, \ldots, v_m\}$ between v_m and v_1. We take embedded paths A_i $(i = 1, \ldots, n)$ in D^2 starting from p_0 as follows:

(a) if $i \neq j$, then $A_i \cap A_j = \{p_0\}$;

(b) if p_i is an end point of a $(\partial, 1)$-edge e_j, then A_i starts from p_0, travels in V across the edges $e_1, \ldots, e_{j-1}$, goes into K_i and ends at p_i;

(c) if p_i is an end point of a $(1,6)$-edge, then A_i starts from p_0, travels in V across the edges $e_1, \ldots, e_{k_i - 1}$, goes into K_i and ends at p_i.

For example, the paths $A_1, \ldots, A_n$ are as shown in Figure3.2.9 for the charts described in Figure3.2.2. Let a_i be an element of $\pi_1(D^2 \setminus S_\Gamma, p_0)$ which is represented by a curve obtained
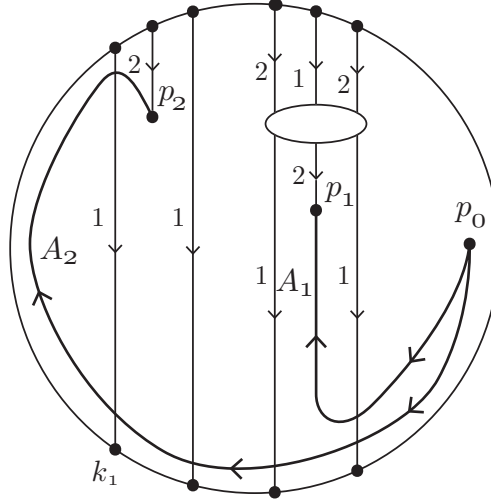

Figure 3.2.9: Examples of paths $A_1, \ldots, A_n$ determined by the condition (a) and the constructions (b) and (c).

by connecting counterclockwise circle around p_i to the base point p_0 by using A_i. It is sufficient to prove that each $\rho_\Gamma(a_i)$ is equal to either X_1 or $X_1^{-n} X_2 X_1^n$, where n is an integer.

Case.1: Suppose that p_i is an end point of $(1,6)$-edge and e_{k_i} is not contained in a boundary comb. Then the intersection word of A_i is equal to X_1^n. Thus, $\rho_\Gamma(a_i)$ is equal to X_1 if the label of the $(1,6)$-edge is 1 and $X_1^n X_2 X_1^{-n}$ if the label of the $(1,6)$-edge is 2.

Case.2: Suppose that p_i is an end point of $(\partial, 1)$-edge and the edge is not contained in a boundary comb. Then the intersection word of A_i is equal to X_1^n and the conclusion holds.

Case.3: Suppose that p_i is an end point of $(1,6)$-edge and e_{k_i} is contained in a boundary comb. Let e_l and e_{l+6} be two edges at the end of the boundary comb which contains e_{k_i}. Then one of 24 cases illustrated in Figure3.2.10 occurs.

The intersection word of a path which starts from p_0, travels in V across the edges $e_1, \ldots, e_{l-1}$, ends near the boundary comb is equal to X_1^n, where n is an integer. Since the label of the $(1,6)$-edge which contains p_i as an end point is 2, $\rho_\Gamma(a_i)$ is calculated as follows:

$$
\rho_\Gamma(a_i) = \begin{cases}
X_1^{n+1} X_2 X_1^{-n-1} & \text{(if (1), (2), (22) or (23) occurs)}, \\
X_1^{n-1} X_2 X_1^{-n+1} & \text{(if (5), (6), (10), (11), (13) or (14) occurs)}, \\
X_1^n X_2 X_1^n & \text{(if (7), (12), (17), (18), (19) or (24) occurs)}, \\
X_1 & \text{(otherwise)}.
\end{cases}
$$

For each case, the conclusion holds.
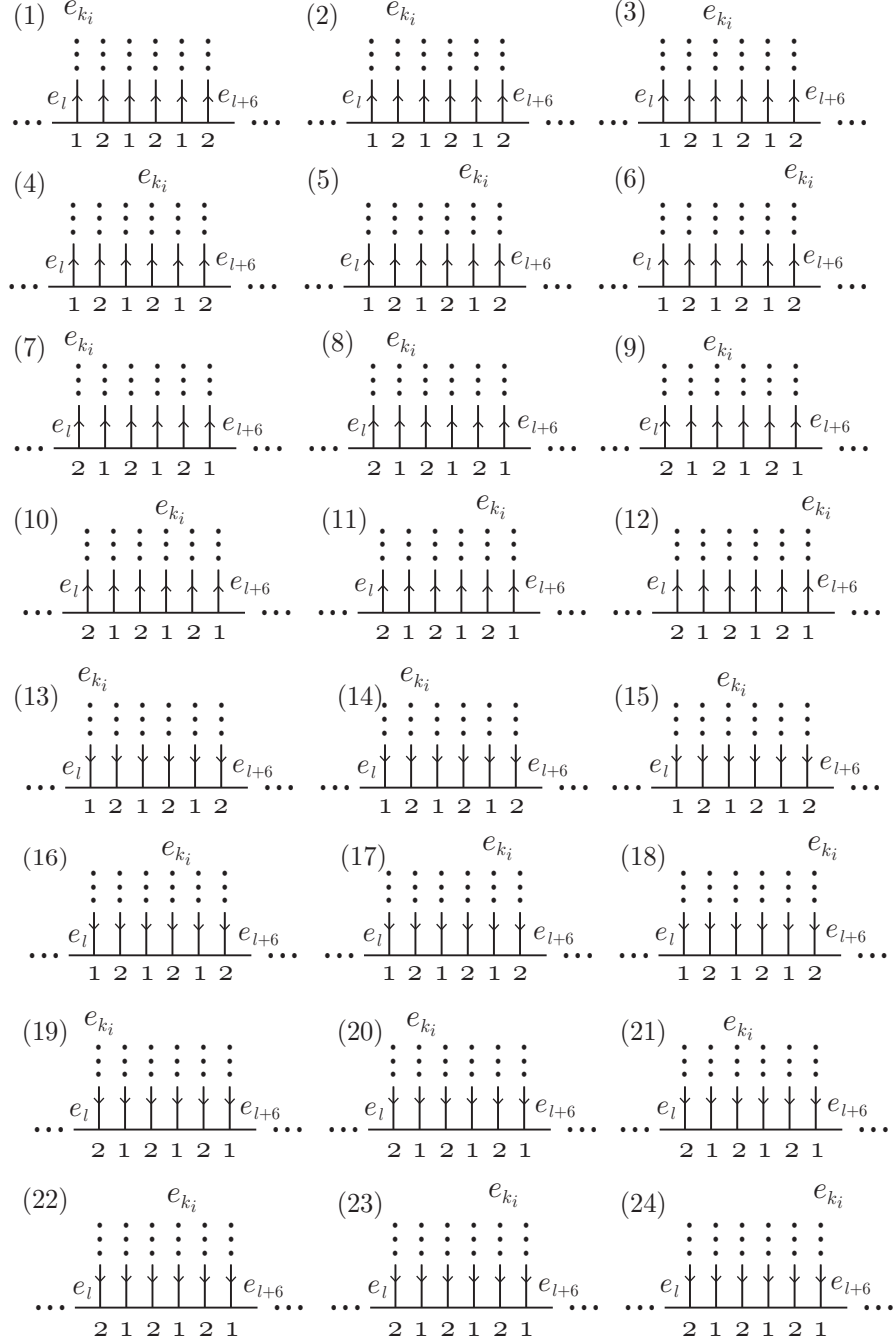
Figure 3.2.10: 24 cases about e_{k_i} and the boundary comb containing e_{k_i}.

Case.4: Suppose that p_i is an end point of $(\partial, 1)$-edge e_j and e_j is contained in a boundary comb. Let e_l and e_{l+6} be two edges at the end of the boundary comb which contains e_j. Since the degree-1 vertex p_i is negative, one of 12 cases illustrated in Figure3.2.11 occurs. We assume that the intersection word of a path which starts from p_0, travels in V across the edges $e_1, \ldots, e_{l-1}$, ends near the boundary comb is equal to X_1^n, where n is an integer. By using the relation $X_1 X_2 X_1 X_2^{-1} X_1^{-1} X_2^{-1} = (X_1 X_2)^6 = E$, $\rho_\Gamma(a_i)$ is calculated as follows:

$$\rho_\Gamma(a_i) = \begin{cases} X_1 & \text{(if (1), (4), (9) or (12) occurs)}, \\ X_1^{n+1} X_2 X_1^{-n-1} & \text{(if (2) or (5) occurs)}, \\ X_1^n X_2 X_1^{-n} & \text{(if (3), (6), (7) or (10) occurs)}, \\ X_1^{n-1} X_2 X_1^{-n+1} & \text{(otherwise)}. \end{cases}$$
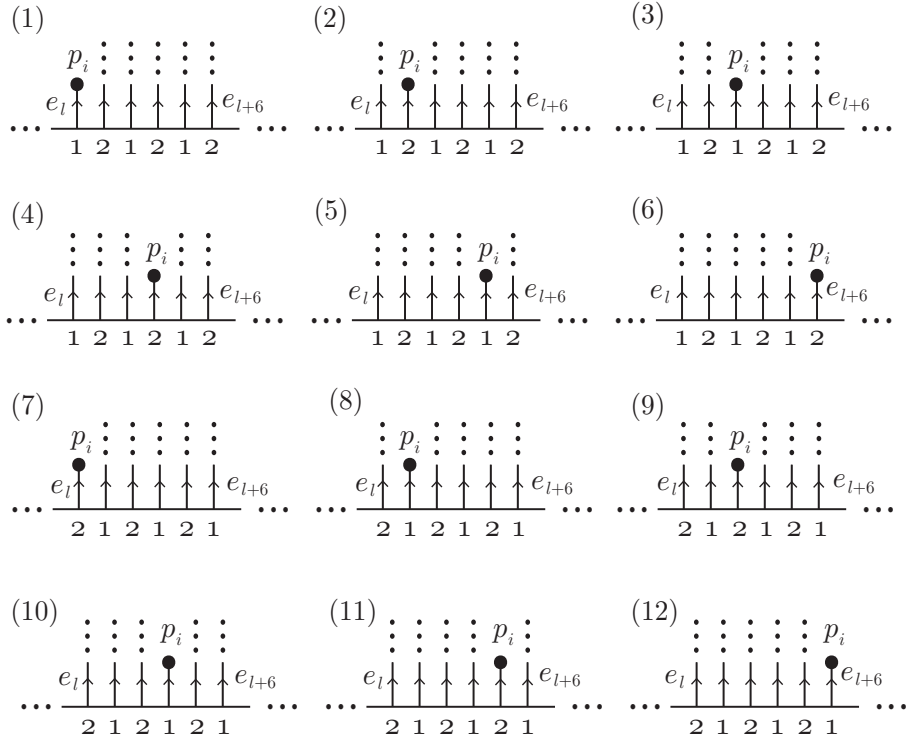
For each cases, the conclusion holds.

Figure 3.2.11: 12 cases about e_j and the boundary comb containing e_j.

Combining the conclusions we obtain in Cases.1, 2, 3 and 4, we complete the proof of Theorem 3.2.1. $\qquad\square$

3.3   Examples of genus-1 simplified broken Lefschetz fibrations

For a sequence $W = (c; c_1, \ldots, c_n)$ of simple closed curves in T^2, we denote by $w(W)$ a product $t_{c_1} \cdot \cdots \cdot t_{c_n}$ of Dehn twists. By Corollary 2.6.5, if a sequence $W = (c; c_1, \ldots, c_n)$ satisfies the condition $w(W) \in \mathrm{Ker}(\Phi_c)$, there exists a genus-1 SBLF $f : X \to S^2$ such that a Hurwitz cycle system of f coincides with the sequence W. Furthermore, the condition $w(W) \in \mathrm{Ker}(\Phi_c)$ is equivalent to the condition $t_{c_1} \cdot \cdots \cdot t_{c_n} \in \mathcal{M}_1(c)$ (or the condition $t_c w(W) t_c^{-1} = w(W)$) since the mapping class group of S^2 is trivial. In this section, we give some examples of sequences of simple closed curves in T^2 which satisfy the condition above and determine what 4-manifolds may admit genus-1 SBLFs whose Hurwitz cycle systems coincide with these examples.

Proposition 3.3.1. *We define sequences S_r and T_s of simple closed curves as follows:*

$$S_r = (\alpha; \alpha, \ldots, \alpha) \quad (r + 1 \ \alpha\text{'s are contained}),$$
$$T_s = (\alpha; \beta + n_1 \alpha, \ldots, \beta + n_s \alpha),$$

where $n_1 = 2s - 3$, $n_s = -2s + 3$ and $n_i = 2s - 6 + 4(i - 1)$ $(i = 2, \ldots, s - 1)$. Then $t_\alpha w(S_r) t_\alpha^{-1} = w(S_r)$ and $t_\alpha w(T_s) t_\alpha^{-1} = w(T_s)$. In particular, these sequences are Hurwitz systems of some genus-1 SBLF.

Proof. It is obvious that $w(S_r)$ is equal to X_1^r and in particular the statement for $w(S_r)$ holds. We prove $w(T_s) = (-1)^{s+1} X_1^{-5s+6}$ by induction on s. Since $(X_1 X_2)^3 = -E$ and $X_1 X_2 X_1 = X_2 X_1 X_2$, we obtain:

$$X_2 X_1{}^2 X_2 X_1{}^2 = X_2 X_1 (X_2 X_1 X_2) X_1$$
$$= -E.$$

Thus, $w(T_2)$ and $w(T_3)$ are computed as follows:

$$\begin{aligned}
w(T_2) &= (X_1^{-1} X_2 X_1)(X_1 X_2 X_1^{-1}) \\
&= X_1^{-1} (X_2 X_1{}^2 X_2) X_1^{-1} \\
&= X_1^{-1} (-X_1^{-2}) X_1^{-1} \\
&= -X_1^{-4}. \\
w(T_3) &= (X_1^{-3} X_2 X_1{}^3) X_2 (X_1{}^3 X_2 X_1^{-3}) \\
&= X_1^{-3} X_2 X_1 (X_1{}^2 X_2 X_1{}^2) X_1 X_2 X_1^{-3} \\
&= X_1^{-3} X_2 X_1 (-X_2^{-1}) X_1 X_2 X_1^{-3} \\
&= -X_1^{-3} (X_2 X_1 X_2^{-1}) X_1 X_2 X_1^{-3} \\
&= -X_1^{-3} (X_1^{-1} X_2 X_1) X_1 X_2 X_1^{-3} \\
&= -X_1^{-4} (X_2 X_1{}^2 X_2) X_1^{-3} \\
&= X_1^{-9}.
\end{aligned}$$

By the definition of T_s, $w(T_s)$ is represented by $w(T_{s-2})$ as follows:

$$w(T_s) = (X_1^{-2s+3} X_2 X_1^{2s-3})(X_1^{-2s+6} X_2 X_1^{2s-6})(X_1^{-2s+7} X_2^{-1} X_1^{2s-7}) w(T_{s-2})$$

$$(X_1^{2s-7}X_2^{-1}X_1^{-2s+7})(X_1^{2s-6}X_2X_1^{-2s+6})(X_1^{2s-3}X_2X_1^{-2s+3})$$
$$=X_1^{-2s+3}X_2X_1{}^2(X_1X_2X_1)X_2^{-1}X_1^{2s-7}w(T_{s-2})$$
$$\quad X_1^{2s-7}X_2^{-1}(X_1X_2X_1)X_1{}^2X_2X_1^{-2s+3}$$
$$=X_1^{-2s+3}(X_2X_1{}^2X_2)X_1^{2s-6}w(T_{s-2})X_1^{2s-6}(X_2X_1{}^2X_2)X_1^{-2s+3}$$
$$=X_1^{-2s+3}(-X_1^{-2})X_1^{2s-6}w(T_{s-2})X_1^{2s-6}(-X_1^{-2})X_1^{-2s+3}$$
$$=X_1^{-5}w(T_{s-2})X_1^{-5}.$$

Thus the conclusion holds by the induction hypothesis. This completes the proof of Proposition 3.3.1. $\qquad\square$

Theorem 3.3.2. *Let $f : X \to S^2$ be a genus-1 SBLF. Suppose that W_f is equal to S_r. Then X is diffeomorphic to one of the following 4-manifolds:*

(1) $\#r\overline{\mathbb{CP}^2}$;

(2) $L\#r\overline{\mathbb{CP}^2}$;

(3) $S^1 \times S^3 \# S \# r\overline{\mathbb{CP}^2}$,

where S is either of the manifolds $S^2 \times S^2$ and $S^2 \tilde{\times} S^2$ and L is either of the manifolds L_n and L'_n (for some $n \geq 2$).

Before proving Theorem 3.3.2, we review the definition and some properties of L_n and L'_n. For more details, see [27]. Let N_0 and N_1 be 4-manifolds diffeomorphic to $D^2 \times T^2$. The boundaries of N_0 and N_1 are $\partial D^2 \times T^2$. Let (t, x, y) be a coordinate of $\mathbb{R}^3$. We identify $\partial D^2 \times T^2$ with $\mathbb{R}^3/\mathbb{Z}^3$. The group $GL(3, \mathbb{Z})$ naturally acts on $\mathbb{R}^3$ and this action descends to an action on the lattice $\mathbb{Z}^3$. Thus, $GL(3, \mathbb{Z})$ acts on $\partial D^2 \times T^2$. For an element A of $GL(3, \mathbb{Z})$, we denote by f_A a self-diffeomorphism of $\partial D^2 \times T^2$ defined as follows:

$$f_A([t, x, y]) = [(t, x, y)^t A].$$

We define elements A_n and A'_n of $GL(3, \mathbb{Z})$ as follows:

$$A_n = \begin{pmatrix} 0 & 1 & 1 \\ 0 & n & n-1 \\ 1 & n & 0 \end{pmatrix} , \ A'_n = \begin{pmatrix} 0 & 1 & 1 \\ 0 & n & n-1 \\ 1 & n-1 & 0 \end{pmatrix}.$$

We denote by S^1_x and S^1_y circles with coordinates x and y, respectively, and take an embedded ball D^3 in $D^2 \times S^1_y$. The manifold $D^3 \times S^1_x$ is contained in $(D^2 \times S^1_y) \times S^1_x = D^2 \times T^2 = N_0$. We define L_n and L'_n as follows:

$$L_n = S^2 \times D^2 \cup_{\mathrm{id}} (N_0 \setminus (\mathrm{Int}\, D^3 \times S^1)) \cup_{f_{A_n}} N_1,$$
$$L'_n = S^2 \times D^2 \cup_{\mathrm{id}} (N_0 \setminus (\mathrm{Int}\, D^3 \times S^1)) \cup_{f_{A'_n}} N_1,$$

where we identify S^{m-1} with ∂D^m.

Remark 3.3.3. The original definitions of L_n and L'_n are different from the definition given above. However, both two definitions are equivalent (c.f. Lemma V.7 in [27]). Note that

Auroux, Donaldson and Katzarkov also mentioned these manifolds in Example 1 of section 8.2 of [1], although they did not state that the manifolds they gave in [1] are the manifolds L_n and L'_n. Indeed, N_1 (resp. $N_0 \setminus (\text{int} B^3 \times S^1)$, $D^2 \times S^2$) in our paper corresponds to X_- (resp. W, X_+) in [1].

We next take handle decompositions of L_n and L'_n. Since N_1 is $D^2 \times T^2$, we can draw a Kirby diagram of N_1 as in the left side of Figure 3.3.1. The coordinate (t, x, y) is also described as in Figure 3.3.1. The manifold $B^3 \times S^1$ has a handle decomposition consisting of

Figure 3.3.1: The left diagram is a Kirby diagram of N_1, while the right one is a diagram of $(N_0 \setminus (\text{int} B^3 \times S^1)) \cup_{f_{A_n}} N_1$, which is also a diagram of $(N_0 \setminus (\text{int} B^3 \times S^1)) \cup_{f_{A'_n}} N_1$. t represents the coordinate of ∂D^2, while x and y represent the coordinates of T^2.
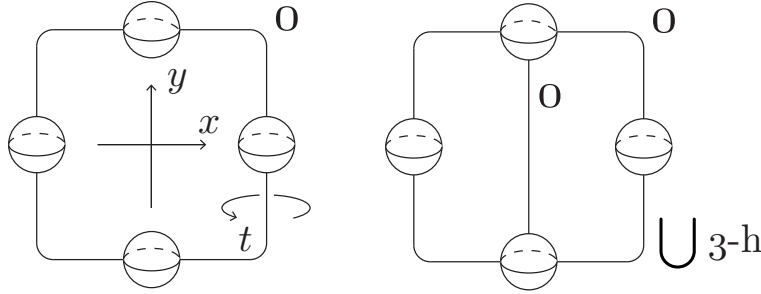
a 0-handle and a 1-handle. Thus, we can decompose $N_0 \setminus (\text{int} B^3 \times S^1)$ as follows:

$$N_0 \setminus (\text{int} B^3 \times S^1) = \partial N_0 \times I \cup (\text{2-handle}) \cup (\text{3-handle}).$$
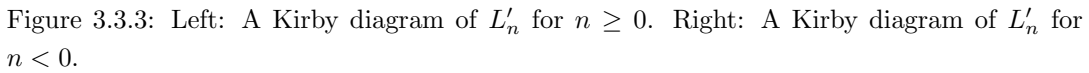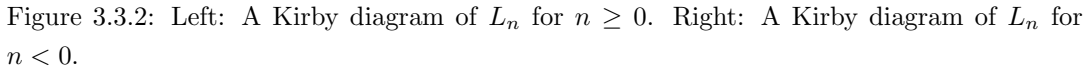
Let $C_1 \subset \partial N_0$ be an attaching circle of the 2-handle. By the construction of the decomposition, we obtain:

$$C_1 = \{[t, 0, 0] \in \partial N_0 | t \in [0, 1]\}.$$

Since $f_{A_n}([t, 0, 0]) = f_{A'_n}([t, 0, 0]) = [0, 0, t]$, an attaching circle of the 2-handle is in a regular fiber and along y-axis in the diagram of N_1. Since $f_{A_n}([t, 0, \varepsilon]) = f_{A'_n}([t, 0, \varepsilon]) = [\varepsilon, (n-1)\varepsilon, t]$ for sufficiently small $\varepsilon > 0$, the framing of the 2-handle is along a regular fiber. Thus, we can draw Kirby diagrams of $(N_0 \setminus (\text{int} B^3 \times S^1)) \cup_{f_{A_n}} N_1$ and $(N_0 \setminus (\text{int} B^3 \times S^1)) \cup_{f_{A'_n}} N_1$ as shown in the right side of Figure 3.3.1. We can decompose $D^2 \times S^2$ as follows:

$$D^2 \times S^2 = \partial D^2 \times S^2 \times I \cup (\text{2-handle}) \cup (\text{4-handle}).$$

Let $C_2 \subset \partial N_0$ be the image under h of the attaching circle of the 2-handle of $D^2 \times S^2$. After moving C_2 by isotopy in N_0, we obtain:

$$C_2 = \{[0, t, \delta] \in \partial N_0 | t \in [0, 1]\},$$

where $\delta > 0$ is sufficiently small. The framing of the 2-handle is $\{[0, t, \delta'] \in \partial N_0 | t \in [0, 1]\}$, where $\delta' > \delta$ is sufficiently small. Since $f_{A_n}([0, t, \delta]) = [t + \delta, nt + (n-1)\delta, nt]$, we can describe the attaching circle of the 2-handle of $D^2 \times S^2$ contained in L_n and the knot representing the framing of the 2-handle in the diagram described in Figure 3.3.1. Eventually, we can draw a

Figure 3.3.2: Left: A Kirby diagram of L_n for $n \geq 0$. Right: A Kirby diagram of L_n for $n < 0$.

Figure 3.3.3: Left: A Kirby diagram of L'_n for $n \geq 0$. Right: A Kirby diagram of L'_n for $n < 0$.

Kirby diagram of L_n as shown in Figure 3.3.2. Similarly, we can draw a Kirby diagram of L'_n as shown in Figure 3.3.3. By the diagrams of L_n and L'_n described in Figure 3.3.2 and Figure 3.3.3, both L_n and L'_n admit genus-1 SBLFs without Lefschetz singularities. We can easily prove by Kirby calculus that L_{-n} (resp. L'_{-n}) is diffeomorphic to L_n (resp. L'_n).

Proof of Theorem 3.3.2. The higher side of f is obtained by attaching r 2-handles to a trivial T^2 bundle over D^2. Each attaching circle of the 2-handle is in a regular fiber and isotopic to a simple closed curve α. Since $w(W_f) = X_1{}^r$, a 2-handle of a round 2-handle is attached along α in a regular fiber of the boundary of the higher side. We obtain X by attaching a 2-handle and a 4-handle to the 4-manifold obtained by successive handle attachment to $D^2 \times T^2$. If the attaching circle of the 2-handle of $D^2 \times S^2$ goes through the 1-handle that the 2-handle of the round handle goes through, we can slide the 2-handle of $D^2 \times S^2$ to the 2-handle of the round handle so that the 2-handle of $D^2 \times S^2$ does not go through the 1-handle. Thus, a Kirby diagram of X is one of the diagrams in Figure 3.3.4, where n and l are integers. It is obvious that the two 4-manifolds illustrated in Figure 3.3.4 are diffeomorphic to each other. We denote by $X_{n,l}$ the 4-manifold illustrated in Figure 3.3.4. Note that l framed knot in Figure 3.3.4 represents a 2-handle of $D^2 \times S^2$ and the attachment of the lower side depends only on the parity of l. In particular, $X_{n,l}$ and $X_{n,l'}$ are diffeomorphic if $l \equiv l'$ (mod 2).
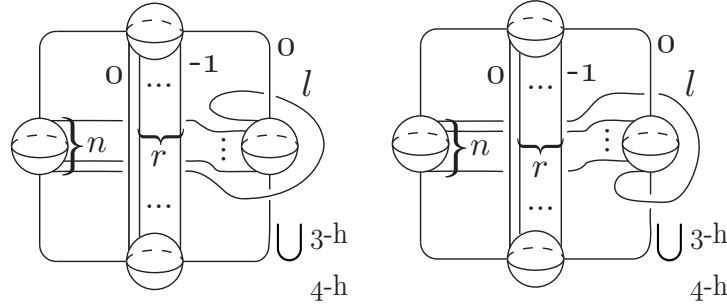
Figure 3.3.4: A Kirby diagram of a genus-1 SBLF with Hurwitz cycle system S_r. Framings of r 2-handles parallel to the 2-handle of the round 2-handle are all -1.

We change a Kirby diagram of $X_{n,l}$ as shown in Figure 3.3.5. We first slide r 2-handles representing Lefschetz singularities to the 2-handle of the round 2-handle. We next slide the 2-handle of $D^2 \times T^2$ to the 2-handle of the round 2-handle and move this 2-handle so that the attaching circle of the 2-handle does not go through 1-handles.
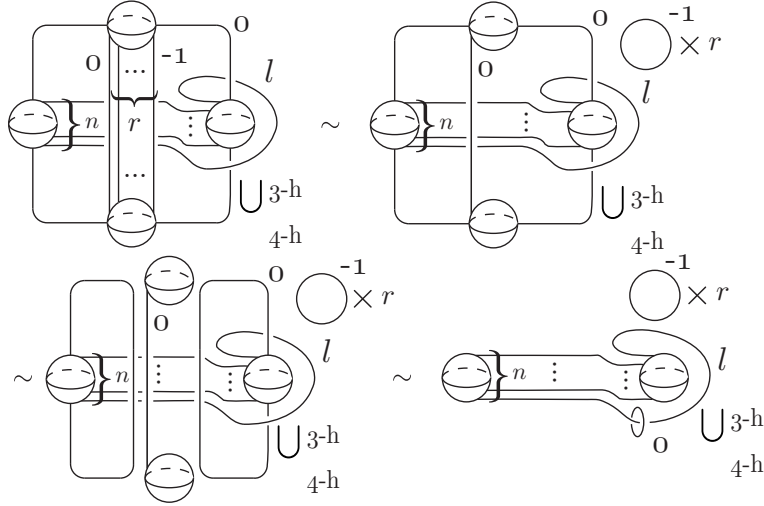

Figure 3.3.5: A Kirby diagram of $X_{n,l}$.

The diagram of $X_{0,l}$ consists of r 2-handles with (-1)-framing, a 1-handle and the Hopf link such that the two components have 0 and l-framing, respectively. Thus, we obtain:

$$X_{0,l} = S^1 \times S^3 \# S \# r\overline{\mathbb{CP}^2},$$

where S is equal to $S^2 \times S^2$ if l is even and $S^2 \widetilde{\times} S^2$ if l is odd.

The diagram of $X_{1,l}$ has two canceling pairs of handles. By canceling these pairs, we can change the diagram of $X_{1,l}$ into the diagram which has only r 2-handles with (-1)-framing. Thus, we obtain:

$$X_{1,l} = \# r\overline{\mathbb{CP}^2}.$$

We can change the diagrams of L_n and L'_n illustrated in Figure 3.3.2 and 3.3.3 as shown in Figure 3.3.6. The upper three diagrams in Figure 3.3.6 describe L_n, where $n \geq 0$. We first
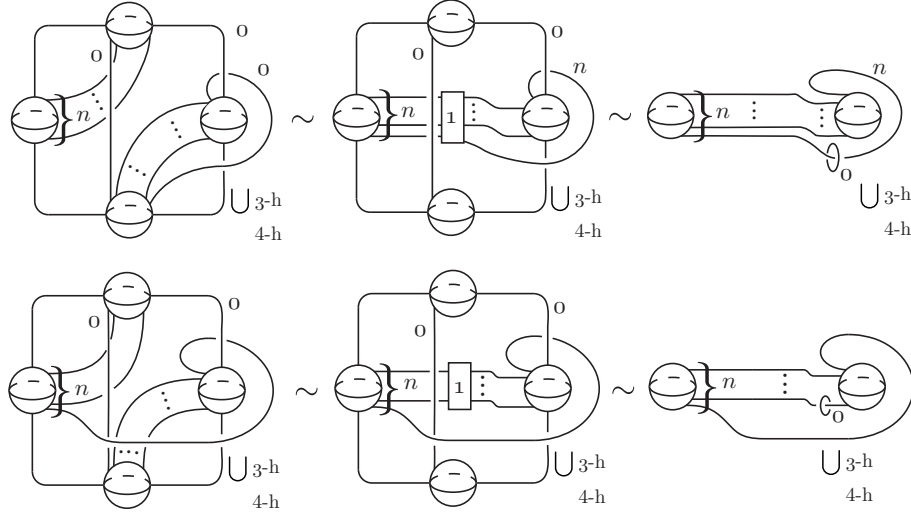

Figure 3.3.6: The upper three diagrams describe L_n, while the lower three diagrams describe L'_n.

slide 2-handle of $D^2 \times S^2$ to the 2-handle of N_0. Then we slide the 2-handle of $D^2 \times T^2$ to the 2-handle of N_0 and eliminate a canceling pair. The lower three diagrams in Figure 3.3.6 describe L'_n, where $n \geq 0$. We can apply the same move as above to the far left diagram and we obtain the far right diagram. Eventually, we obtain:

$$X_{n,l} = L \# r \overline{\mathbb{CP}^2},$$

where L is either L_n or L'_n. This completes the proof of Theorem 3.3.2. $\qquad \square$

Theorem 3.3.4. *Let $f : X \to S^2$ be a genus-1 SBLF. Suppose that $W_f = T_s$. Then X is diffeomorphic to $S \# (s-2)\mathbb{CP}^2$, where S is either of the manifolds $S^2 \times S^2$ and $S^2 \tilde{\times} S^2$.*

Proof. We prove the statement by induction on s. Let X_s be a total space of genus-1 SBLF f with $W_f = T_s$. We first look at the manifold X_2. We can draw a Kirby diagram of X_2 as shown in Figure 3.3.7. We slide the 2-handles representing Lefschetz singularities and the 2-handle of $D^2 \times T^2$ to the 2-handle of the round handle. Then we eliminate the obvious canceling pair and slide the (-2)-framed knot and the l-framed knot to the 0-framed knot. We can change the l-framed knot and the 0-framed meridian of this into the Hopf link by using the 0-framed meridian. We can obtain the last diagram of Figure 3.3.7 by canceling two pairs of handles. Eventually, we can prove that X_2 is diffeomorphic to $S^2 \times S^2$ if m is even or $S^2 \tilde{\times} S^2$ if m is odd.

Suppose that s is greater than or equal to 3. f has the configuration of reference paths and the corresponding vanishing cycles as described in the far left of Figure 3.3.8. Since two vanishing cycles of folds determined by the dashed arcs in the far left of Figure 3.3.8 intersect at a single point transversely, we can apply fold merge to f as described in the figure. We
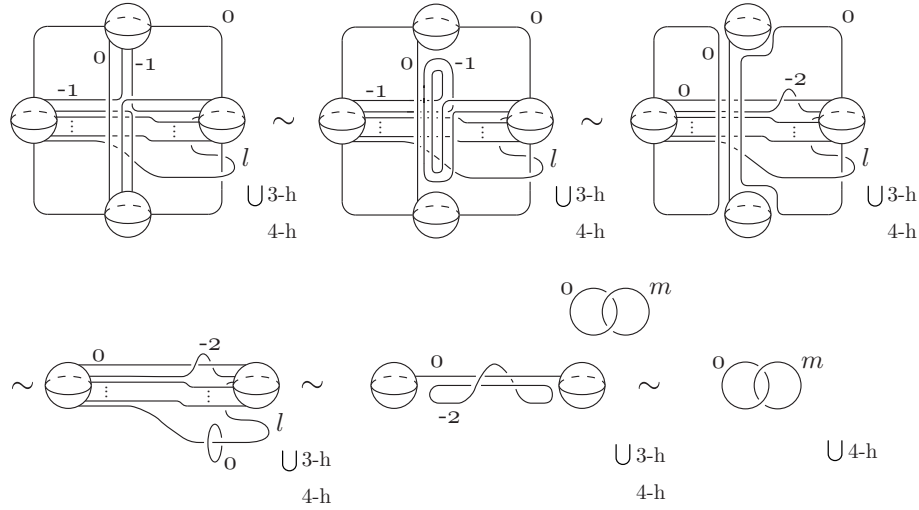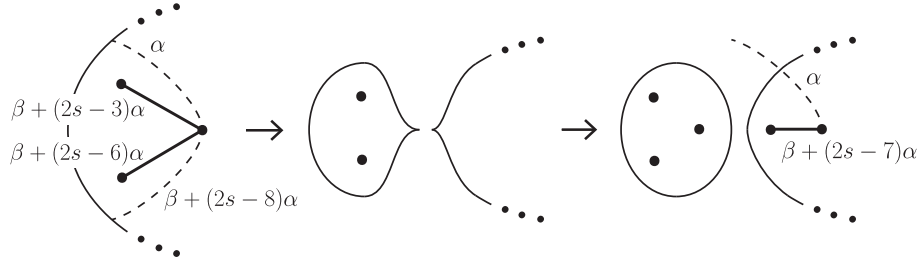
Figure 3.3.7: A diagram of a SBLF whose Hurwitz system is T_2.

Figure 3.3.8: Left: configuration of reference paths. Right: configuration of the image of singularities after applying several homotopies. Four dots in this picture are the image of Lefschetz singularities.

can further apply unsink to f and the configuration of the image of singularities of f are changed as in Figure 3.3.8. It is easy to verify that we can apply wrinkle on the preimage of a disk which contains the circular image of folds and the three Lefschetz critical values in the circle. By application of wrinkle, the folds and three cusps are changed into an achiral Lefschetz singularity with null-homotopic vanishing cycle (the reader should refer to [21] or [6] for details of several homotopies in the above argument). Eventually, we can obtain a genus-1 SBLF $f' : X' \to S^2$ so that X is diffeomorphic to $X' \# \mathbb{CP}^2$. Furthermore, a Hurwitz cycle system of f' is as follows:

$$(\alpha; \beta + (2s-7)\alpha, \beta + (2s-10)\alpha, \beta + (2s-14)\alpha, \ldots, \beta + (-2s+6)\alpha, \beta + (-2s+3)\alpha).$$

This sequence is Hurwitz equivalent to the sequence T_{s-1}. Thus, the conclusion holds by induction hypothesis and this completes the proof of Theorem 3.3.4.   □

We can obtain the following corollary by the same argument as in the proof of Theorem 3.3.4:

Corollary 3.3.5. *Let $f : X \to S^2$ be a genus-1 SBLF with non-empty folds. Suppose that W_f is Hurwitz equivalent to the following sequence:*

$$(\alpha; \beta, \beta - 3\alpha, c_1, \ldots, c_l),$$

where c_i is an element in $H_1(T^2; \mathbb{Z})$. Then there exists a genus-1 SBLF $f' : X' \to S^2$ which satisfies the following conditions;

1. *$W_{f'}$ is Hurwitz equivalent to $(\alpha; \beta - 4\alpha, c_1, \ldots, c_l)$,*

2. *X is diffeomorphic to $X' \# \mathbb{CP}^2$.*

3.4 Further properties of Hurwitz cycle systems

We define matrices A and B in $SL(2, \mathbb{Z})$ as follows:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}.$$

Both X_1 and X_2 are represented by A and B as follows:

$$X_1 = ABA, \ X_2 = BA^2.$$

Since X_1 and X_2 generate the group $SL(2, \mathbb{Z})$, the matrices A and B also generate $SL(2, \mathbb{Z})$. Let a, b, x_1 and x_2 be elements of $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{\pm E\}$ represented by A, B, X_1 and X_2, respectively, where we denote by E the unit matrix. Then $PSL(2, \mathbb{Z})$ has the following finite presentation (see [22]):

$$PSL(2, \mathbb{Z}) = < a, b | a^3, b^2 > .$$

In particular, $PSL(2, \mathbb{Z})$ is isomorphic to the free product $\mathbb{Z}/3 * \mathbb{Z}/2$. The sequence $(w_1, \ldots, w_n)$ of elements of $PSL(2, \mathbb{Z})$ is called *reduced* if the set $\{w_i, w_{i+1}\}$ is equal to either of the sets $\{a, b\}$ and $\{a^2, b\}$ for each $i \in \{1, \ldots, n-1\}$.

Lemma 3.4.1 ([22]). *For every element g of $PSL(2, \mathbb{Z})$, there exists a reduced sequence $(w_1, \ldots, w_n)$ of $PSL(2, \mathbb{Z})$ such that $g = w_1 \cdot \cdots \cdot w_n$. Moreover, such a sequence is unique.*

Let $W = (\alpha; c_1, \ldots, c_l)$ be a Hurwitz cycle system of some genus-1 SBLF. By Theorem 3.2.1, we can assume that W is equal to $(\alpha; S_r T(n_1, \ldots, n_s))$. The element $t_{\beta + n_i \alpha}$ is equal to $X_1^{-n_i} X_2 X_1^{n_i}$. The product $w(W) = t_{c_1} \cdot \cdots \cdot t_{c_l}$ is represented by X_1 and X_2 as follows:

$$w(W) = X_1^r X_1^{-n_1} X_2 X_1^{n_1 - n_2} X_2 \cdots X_1^{n_{s-1} - n_s} X_2 X_1^{n_s}.$$

By Corollary 2.4.2, $w(W)$ preserves the curve α. In particular, $w(W)$ is equal to $t_\alpha^i (t_\alpha t_\beta)^{3j}$ for some integers $i, j \in \mathbb{Z}$. Since $(t_\alpha t_\beta)^{3j}$ is equal to $(-E)^j$, we can obtain the following relation in $PSL(2, \mathbb{Z})$:

$$x_2 x_1^{n_1 - n_2} x_2 \cdots x_2 x_1^{n_{s-1} - n_s} x_2 = x_1^k,$$

where k is some integer.

Lemma 3.4.2. *Suppose that each $n_i - n_{i+1}$ is not equal to either 1, 2 or 3. Then $x_2 x_1^{n_1-n_2} \cdot \cdots \cdot x_1^{n_{s-1}-n_s} x_2$ is equal to bS or $a^2 b a^2 b S$, where $S = w_1 \cdot \cdots \cdot w_k$ and $(w_1, \ldots, w_k)$ is a reduced sequence such that $w_1 = a$ or a^2.*

Proof. We prove this statement by induction on s.

We first look at the case $s = 2$. $x_2 x_1^{n_1-n_2} x_2$ is calculated as follows:

$$x_2 x_1^{n_1-n_2} x_2 = \begin{cases} ba^2 \cdot a(ba^2)^{n_1-n_2-1} ba \cdot ba^2 & (\text{if } n_1 - n_2 \geq 4), \\ ba^2 \cdot a^2(ba)^{-n_1+n_2-1} ba^2 \cdot ba^2 & (\text{if } n_1 - n_2 \leq 0), \end{cases}$$

$$= \begin{cases} a^2 ba^2 (ba^2)^{n_1-n_2-3} baba^2 & (\text{if } n_1 - n_2 \geq 4), \\ (ba)^{-n_1+n_2} ba^2 ba^2 & (\text{if } n_1 - n_2 \leq 0). \end{cases}$$

Thus, the statement holds.

We then look at the general case. By the induction hypothesis, we obtain:

$$x_2 x_1^{n_2-n_3} \cdot \cdots \cdot x_1^{n_{s-1}-n_s} x_2 = bS \text{ or } a^2 ba^2 bS,$$

where S is the product of a reduced sequence starting from a or a^2. We denote by S' the words above. We can calculate $x_2 x_1^{n_1-n_2}$ as follows:

$$x_2 x_1^{n_1-n_2} = \begin{cases} ba^2 \cdot a(ba^2)^{n_1-n_2-1} ba & (\text{if } n_1 - n_2 \geq 4), \\ ba^2 \cdot a^2(ba)^{-n_1+n_2-1} ba^2 & (\text{if } n_1 - n_2 \leq 0), \end{cases}$$

$$= \begin{cases} a^2 ba^2 (ba^2)^{n_1-n_2-4} ba^2 ba & (\text{if } n_1 - n_2 \geq 4), \\ (ba)^{-n_1+n_2} ba^2 & (\text{if } n_1 - n_2 \leq 0), \end{cases}$$

Thus, we obtain:

$$x_2 x_1^{n_1-n_2} \cdot \cdots \cdot x_1^{n_{s-1}-n_s} x_2$$

$$= \begin{cases} a^2 ba^2 (ba^2)^{n_1-n_2-4} ba^2 babS & (\text{if } n_1 - n_2 \geq 4 \text{ and } S' = bS), \\ a^2 ba^2 (ba^2)^{n_1-n_2-4} ba^2 baa^2 ba^2 bS & (\text{if } n_1 - n_2 \geq 4 \text{ and } S' = a^2 ba^2 bS), \\ (ba)^{-n_1+n_2} ba^2 bS & (\text{if } n_1 - n_2 \leq 0 \text{ and } S' = bS), \\ (ba)^{-n_1+n_2} ba^2 a^2 ba^2 bS & (\text{if } n_1 - n_2 \leq 0 \text{ and } S' = a^2 ba^2 bS), \end{cases}$$

$$= \begin{cases} a^2 ba^2 (ba^2)^{n_1-n_2-4} ba^2 babS & (\text{if } n_1 - n_2 \geq 4 \text{ and } S' = bS), \\ a^2 ba^2 (ba^2)^{n_1-n_2-4} babS & (\text{if } n_1 - n_2 \geq 4 \text{ and } S' = a^2 ba^2 bS), \\ (ba)^{-n_1+n_2} ba^2 bS & (\text{if } n_1 - n_2 \leq 0 \text{ and } S' = bS), \\ (ba)^{-n_1+n_2} baba^2 bS & (\text{if } n_1 - n_2 \leq 0 \text{ and } S' = a^2 ba^2 bS). \end{cases}$$

This completes the proof of Lemma 3.4.2. $\qquad\qquad\square$

Corollary 3.4.3. *Assume s is greater than 1. There exists an integer $i \in \{1, \ldots, s-1\}$ such that $n_i - n_{i+1}$ is equal to either 1, 2 or 3.*

Proof. Suppose that $n_i - n_{i+1}$ is not equal to either 1, 2 or 3 for any $i \in \{1, \ldots, s-1\}$. By Lemma 3.4.2, $w(W)$ is not trivial. The element x_1^k is equal to $a(ba^2)^{k-1} ba$ if $k \geq 1$ or $a^2(ba)^{-k+1}$. This contradicts the condition $w(W) = x_1^k$ for some k. $\qquad\square$

3.5   Classification

Proof of Theorem 3.0.1. We first prove that each 4-manifold in the statement of Theorem 3.0.1 admits a genus-1 SBLF. We proved that the manifolds S, $\#r\overline{\mathbb{CP}^2}$, $S^1 \times S^3 \# S \# r\overline{\mathbb{CP}^2}$ and $L \# r\overline{\mathbb{CP}^2}$ admit genus-1 SBLFs in the proof of Theorem 3.3.2. By Corollary 2.6.5, there exists a genus-1 SBLF $f : X(l,k) \to S^2$ with Hurwitz cycle system $(\alpha; S_{l-k-1}T_{k+1})$ for any $l \geq 3$ and $k \in [1, l-1]$. It is easy to verify that $X(l,k)$ is diffeomorphic to $\#k\mathbb{CP}^2 \#(l-k)\overline{\mathbb{CP}^2}$. Thus, it is sufficient to construct a genus-1 SBLF on $\#kS^2 \times S^2$. A diagram in Figure 3.5.1 represents the total space of a genus-1 SBLF $f_k : X_k \to S^2$ whose Hurwitz cycle system is equal to $(\alpha; kT_2)$. It is easy to prove that the manifold M_k is diffeomorphic to $\#kS^2 \times S^2$.
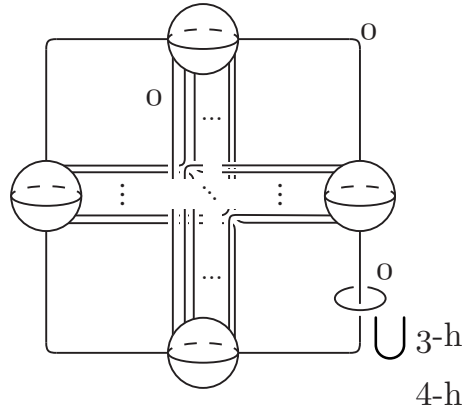

Figure 3.5.1: A Kirby diagram of X_k. All the 2-handles derived from Lefschetz singularities have (-1)-framing.

The details are left to the readers.

We next prove that the total space of a genus-1 SBLF f is diffeomorphic to one of the manifolds in the statement of Theorem 3.0.1. Denote by l the number of Lefschetz singularities of f. We will prove the statement by induction on l. The statement for the case $l = 0$ has already been proved in the proof of Theorem 3.3.2. We assume $l > 0$ and W_f is equal to $(\alpha; S_r T(n_1, \ldots, n_s))$.

We first consider the case X is not simply connected. In this case, it is easy to verify that $s = 0$ and therefore, there exists a genus-1 SBLF $f : X' \to S^2$ without Lefschetz singularities such that X is diffeomorphic to $X' \# r\overline{\mathbb{CP}^2}$. Thus, we can deduce the conclusion from induction hypothesis.

We next consider the case X is simply connected. If r is not equal to 0, we can reduce the number of Lefschetz singularities by blowing down X and the conclusion holds by induction hypothesis. Assume that r is equal to 0 and $s = l$. By Lemma 3.4.2, there exists a number $i \in \{1, \ldots, l-1\}$ such that $n_i - n_{i+1}$ is equal to either 1, 2 or 3. If $n_i - n_{i+1}$ is equal to 1, then W_f is Hurwitz equivalent to the following sequence:

$$(\alpha; S_1 T(n_1 + 1, \ldots, n_{i-1} + 1, n_i, n_{i+2} \ldots, n_l))$$

since the sequence $T(n_i, n_{i+1})$ is Hurwitz equivalent to $S_1 T(n_i)$. Thus, the conclusion holds by induction hypothesis. If $n_i - n_{i+1}$ is equal to 2, then W_f is Hurwitz equivalent to the

following sequence:

$$(\alpha; T(n_i, n_{i+1})T(n_1 - 4, \ldots, n_{i-1} - 4, n_{i+2} \ldots, n_l))$$

since the composition $t_{\beta+n_i\alpha} t_{\beta+n_{i+1}\alpha}$ is equal to $(t_\alpha t_\beta)^3 t_\alpha^{-4}$. By the argument similar to that in the proof of Lemma 6.13 in [16], we can prove that there exists a genus-1 SBLF $f' : X' \to S^2$ such that X is diffeomorphic to $X' \# S$, where S is an S^2-bundle over S^2. Thus, the conclusion holds by induction hypothesis. If $n_i - n_{i+1}$ is equal to 3, then W_f is Hurwitz equivalent to $(\alpha; T(0, -3)W)$, where W is some sequence which consists of $l-2$ simple closed curves. We can apply Corollary 3.3.5 to this fibration and the conclusion holds by the induction hypothesis. This completes the proof of Theorem 3.0.1. $\qquad\square$

Chapter 4

Hyperelliptic broken Lefschetz fibrations

4.1 Preliminaries

4.1.1 Hyperelliptic mapping class groups

Let Σ_g be a closed oriented surface of genus $g \geq 1$. Denote by $\iota : \Sigma_g \to \Sigma_g$ an involution described in Figure 4.1.1. Let $C(\iota)$ denote the centralizer of ι in the diffeomorphism
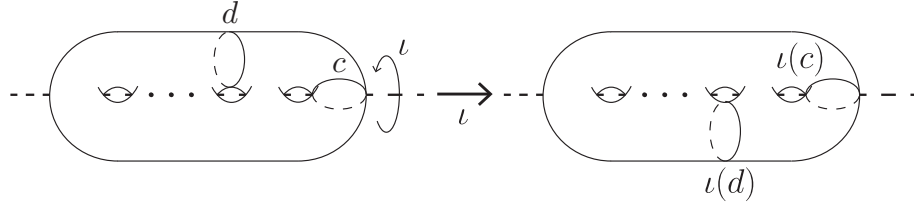

Figure 4.1.1: the hyperelliptic involution on the surface Σ_g.

group $\mathrm{Diff}_+ \Sigma_g$, and endow $C(\iota) \subset \mathrm{Diff}_+ \Sigma_g$ with the relative topology. The inclusion homomorphism $C(\iota) \to \mathrm{Diff}_+ \Sigma_g$ induces a natural homomorphism $\pi_0 C(\iota) \to \mathcal{M}_g$ between their path-connected components.

Theorem 4.1.1 (Birman-Hilden [7])**.** *When $g \geq 2$, the homomorphism*

$$\pi_0 C(\iota) \to \mathcal{M}_g$$

is injective.

Denote the image of this homomorphism by $\mathcal{H}_g$ for $g \geq 1$. This group is called the *hyperelliptic mapping class group*. In fact, they showed the above result in more general settings later. See [7] for details.

In the following, we review some properties of the hyperelliptic mapping class group. Let S be a 2-disk or a 2-sphere. For a positive integer n and distinct points $\{p_i\}_{i=1}^n$ in $\mathrm{Int}\, S$,

Denote by $\mathrm{Diff}_+(S, \partial S, \{p_1, p_2, \cdots, p_n\})$ the group defined by

$$\mathrm{Diff}_+(S, \partial S, \{p_1, p_2, \cdots, p_n\})$$
$$= \{T \in \mathrm{Diff}_+ S \mid T|_{\partial S} = \mathrm{id}_{\partial S}, T(\{p_1, p_2, \cdots, p_n\}) = \{p_1, p_2, \cdots, p_n\}\}.$$

Denote by $\mathcal{M}_0^n$ or $\mathcal{M}_{0,1}^n$ its mapping class group when $S = S^2$ or $S = D^2$, respectively. Let D_i be a disk in $\mathrm{Int}\, S$ which includes p_i and p_{i+1} but is disjoint from all p_j for $j \neq i, i+1$, and denote by $\nu(\partial D_i)$ a neighborhood of the boundary ∂D_i in D_i. Choose a diffeomorphism $T_i \in \mathrm{Diff}_+(S, \partial S, \{p_1, p_2, \cdots, p_n\})$ such that $T_i|_{D_i}$ interchanges the points p_i and p_{i+1}, $T_i|_{X-\mathrm{Int}\, D_i}$ is the identity map, and T_i^2 is isotopic to the Dehn twist along ∂D_i (see Birman-Hilden p.87-88 for details). The mapping class group $\mathcal{M}_0^n$ and $\mathcal{M}_{0,1}^n$ is generated by $\{\sigma_i\}_{i=1}^{n-1}$, where σ_i is the mapping class represented by the diffeomorphism T_i.

Identifying the quotient space $\Sigma_g / \langle \iota_g \rangle$ with S^2, let $p_1, \ldots, p_{2g+2} \subset S^2$ be the branched points of the quotient map $\Sigma_g \to \Sigma_g / \langle \iota_g \rangle$. By the definition, any diffeomorphism T in $C(\iota_g)$ satisfies $T\iota_g(x) = \iota_g T(x)$ for $x \in \Sigma_g$. Hence, there exists a unique diffeomorphism $\bar{T} \in \mathrm{Diff}_+ S^2$ such that the diagram

$$
\begin{array}{ccc}
\Sigma_g & \xrightarrow{\,T\,} & \Sigma_g \\
p \downarrow & & \downarrow p \\
S^2 & \xrightarrow{\,\bar{T}\,} & S^2
\end{array}
$$

commutes. Moreover, it satisfies $\bar{T}(\{p_1, p_2, \cdots, p_{2g+2}\}) = \{p_1, p_2, \cdots, p_{2g+2}\} \subset S^2$. By the above diagram, we can define

$$\mathcal{P}_g : \mathcal{H}_g^s \to \mathcal{M}_0^{2g+2}$$

by $\mathcal{P}_g([T]) = [\bar{T}]$.

Theorem 4.1.2 (Birman-Hilden [7]). *Let $g \geq 1$. the sequence*

$$1 \longrightarrow \langle \iota_g \rangle \longrightarrow \mathcal{H}_g^s \xrightarrow{\mathcal{P}_g} \mathcal{M}_0^{2g+2} \longrightarrow 1$$

is exact.

They showed the homomorphism $\mathcal{P}_g : \mathcal{H}_g^s \to \mathcal{M}_0^{2g+2}$ maps the Dehn twist t_{c_i} to σ_i in [7]. Furthermore, they proved:

Proposition 4.1.3. *Let $g \geq 1$. The group $\mathcal{H}_g^s$ is generated by $\{t_{c_1}, \cdots, t_{c_{2g+1}}\}$, where c_i is a simple closed curve in Σ_g described in Figure 4.1.2.*
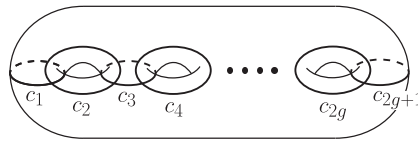

Figure 4.1.2: simple closed curves $c_1, \cdots, c_{2g+1}$

4.1.2 Hyperelliptic fibrations

A Lefschetz fibration is said to be *hyperelliptic* if we can take an identification of the fiber of a base point with the closed oriented surface so that the image of the monodromy representation of the fibration is contained in the hyperelliptic mapping class group. Thus, it is natural to generalize this definition to directed (and especially simplified) BLFs as follows: Let $f : M \to S^2$ be a DBLF. We use the same notations as those we use in Sections 2.1 and 2.4. We put $r_i = \alpha\left(\frac{t_i + t_{i+1}}{2}\right)$ $(i = 1, \ldots, m-1)$ and $r_m = p_n$. We can regard the vanishing cycle d_i we took in Section 2.4 a simple closed curve in $f^{-1}(r_i)$. Once we fix an identification of $f^{-1}(r_m)$ with $\Sigma_{g_1} \amalg \cdots \amalg \Sigma_{g_k}$, we obtain an involution ι_i on $f^{-1}(r_i)$ induced by the hyperelliptic involution on $f^{-1}(r_m)$ since we can identify $f^{-1}(r_{i-1}) \setminus \{$two points$\}$ with $f^{-1}(r_i) \setminus d_i$ by using α. f is said to be *hyperelliptic* if it satisfies the following conditions for a suitable identification of $f^{-1}(r_m)$ with $\Sigma_{g_1} \amalg \cdots \amalg \Sigma_{g_k}$:

- the image of the monodromy representation of the Lefschetz fibration res $f : f^{-1}(D_h) \to D_h$ is contained in the group $\mathcal{H}_g$;

- d_i is preserved by the involution ι_i up to isotopy.

In this paper, we will call a hyperelliptic SBLF HSBLF for short.

Remark 4.1.4. Every SBLF whose genus is less than or equal to 2 is hyperelliptic since $\mathcal{H}_g = \mathcal{M}_g$ and all simple closed curves in Σ_g are preserved by ι if $g \le 2$.

4.1.3 Meyer's signature cocycle and the local signature

It is known that, for an HLF $f : X \to \Sigma$ over a closed oriented surface Σ, the signature Sign X is described as the sum of invariants of the singular fiber germs in X. We review this invariant.

Let φ, ψ be elements in the mapping class group $\mathcal{M}_g$. We denote by $E_{\varphi,\psi}$ a Σ_g-bundle over a pair of pants $S^2 - \amalg_{i=1}^3 \text{Int } D^2$ whose monodromies along α and β in Figure 4.1.3 are φ and ψ, respectively.
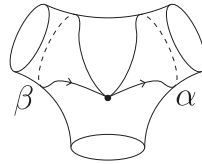
Figure 4.1.3: paths α and β

Theorem 4.1.5 (Meyer [25])**.** *Define a 2-cochain $\tau_g : \mathcal{M}_g \times \mathcal{M}_g \to \mathbb{Z}$ of the mapping class group by $\tau_g(\varphi, \psi) = -\text{Sign } E_{\varphi,\psi}$. Then, τ_g is a 2-cocycle, and the order of its homology class is as follows.*

1. The order of $[\tau_1] \in H^2(\mathcal{M}_1; \mathbb{Z})$ is 3,

2. The order of $[\tau_2] \in H^2(\mathcal{M}_2; \mathbb{Z})$ is 5,

3. *When* $g \geq 3$, $[\tau_g] \neq 0 \in H^2(\mathcal{M}_g; \mathbb{Q})$.

Proposition 4.1.6 (Endo [9]). *If we restrict τ_g to $\mathcal{H}_g$, the order of $[\tau_g] \in H^2(\mathcal{H}_g; \mathbb{Z})$ is $2g+1$.*

Since τ_g represents a trivial homology class in $H^2(\mathcal{H}_g; \mathbb{Q})$, there exists a cobounding function $\phi_g : \mathcal{H}_g \to \mathbb{Q}$ of it. Furthermore, since $H_1(\mathcal{H}_g; \mathbb{Q})$ is trivial, this cobounding function ϕ_g is unique.

Lemma 4.1.7 (Endo [9]). *Let $f : X \to \Sigma$ be a Σ_g-bundle over a compact oriented surface Σ. Assume that the image of the monodromy representation $\pi_1(\Sigma, p_0) \to \mathcal{M}_g$ is in $\mathcal{H}_g$ if we choose a suitable identification $f^{-1}(p_0) \cong \Sigma_g$. Let $\{\partial_j\}_{j=1}^n$ denote the boundary components of Σ, and give orientations coming from Σ. Then, we have*

$$\mathrm{Sign}\, X = -\sum_{j=1}^{l} \phi(\psi_j),$$

where $\psi_j \in \mathcal{H}_g$ is the monodromy along ∂_j.

Using this function, he generalized the local signature of LFs of genus 1 [23] and of genus 2 [24] constructed by Matsumoto. Let $f : X \to \Sigma$ be an HLF of genus g over a closed oriented surface Σ, and $p_1, \cdots, p_l$ the image of the set of Lefschetz singularities under f. For the Lefschetz singular fiber $f^{-1}(p_j)$, define a rational number $\sigma_{\mathrm{loc}}(f^{-1}(p_j))$ by

$$\sigma_{\mathrm{loc}}(f^{-1}(p_j)) = -\phi_g(\varphi_j) + \mathrm{Sign}(f^{-1}\nu(p_j)),$$

where $\varphi_j \in \mathcal{H}_g$ is the monodromy along $\partial\nu(p_j)$. He computed the values for Lefschetz singular fibers as follows:

Lemma 4.1.8 (Endo [9]). *We call a Lefschetz singular fiber is type* I *or type* II_h *if the vanishing cycle is type* I *or type* II_h, *respectively. Then we have*

$$\sigma_{\mathrm{loc}}(\mathrm{I}) = -\frac{g+1}{2g+1}, \quad \sigma_{\mathrm{loc}}(\mathrm{II}_h) = \frac{4h(g-h)}{2g+1} - 1.$$

Furthermore, Endo gave the following formula for signatures of HLFs.

Theorem 4.1.9 (Endo [9]). *Let $f : X \to \Sigma$ be an HLF as above. Then we have*

$$\mathrm{Sign}\, X = \sum_{i=1}^{l} \sigma_{\mathrm{loc}}(f^{-1}(p_j)).$$

4.2   A subgroup $\mathcal{H}_g(c)$

Let c be an essential simple closed curve in the surface Σ_g which is preserved by the involution $\iota \in \mathrm{Diff}_+ \Sigma_g$ as a set. Let $\mathcal{H}_g(c)$ denote the subgroup of the hyperelliptic mapping class group defined by $\mathcal{H}_g(c) := \mathcal{H}_g \cap \mathcal{M}_g(c)$. As introduced in Theorem 4.1.1, the hyperelliptic mapping class group $\mathcal{H}_g$ is isomorphic to the group consisting of the path-connected components of $C(\iota)$. Hence, the group $\mathcal{H}_g(c)$ consists of the mapping classes which can be represented by both of elements in the centralizer $C(\iota)$ and elements in $\mathrm{Diff}_+(\Sigma_g, c)$. Let $\mathcal{H}_g^s(c)$ denote the subgroup of $\pi_0 C(\iota)$ defined by $\mathcal{H}_g^s(c) := \{[T] \in \pi_0 C(\iota) \,|\, T(c) = c\}$. In this section, we will prove the following lemma.

Lemma 4.2.1. *Assume that g is greater than or equal to 1. The natural map $\pi_0 C(\iota) \to \mathcal{H}_g$ in Theorem 4.1.1 restricts to a surjective map between the groups $\mathcal{H}_g^s(c)$ and $\mathcal{H}_g(c)$. Furthermore, this restriction is an isomorphism if $g \geq 2$.*

To prove the lemma, It is enough to show that the homomorphism maps $\mathcal{H}_g^s(c)$ onto $\mathcal{H}_g(c)$. Let $[T]$ be a mapping class in $\mathcal{H}_g(c)$. We can choose a representative $T : \Sigma_g \to \Sigma_g$ in the centralizer $C(\iota)$. Since it is isotopic to some diffeomorphism on Σ_g which preserves the curve c, the curve $T(c)$ is isotopic to c.

We call an isotopy $L_0 : \Sigma_g \times [0,1] \to \Sigma_g$ is symmetric if $L_0(*, t) \in C(\iota)$ for any $t \in [0,1]$. In the following, we will construct a symmetric isotopy $L : \Sigma_g \times [0,1] \to \Sigma_g$ satisfying

$$L(*, 0) = T, \text{ and } L(c, 1) = c \subset \Sigma_g.$$

It indicates that $L(*, 1)$ represents an element in $\mathcal{H}_g^s(c)$, and $[L(*, 1)] = [T] \in \pi_0 C(\iota)$. Hence, we see that the homomorphism $\mathcal{H}_g^s(c) \to \mathcal{H}_g(c)$ is surjective.

To construct the symmetric isotopy $L : \Sigma_g \times [0,1] \to \Sigma_g$, we need a proposition, so called the bigon criterion.

Proposition 4.2.2 (Farb-Margalit Proposition 1.7 [10])**.** *Let S be a compact surface. The geometric intersection number of two transverse simple closed curves in S is minimal if and only if they do not form a bigon.*

We may assume that the curves c and $T(c)$ are transverse by changing the diffeomorphism T in terms of some symmetric isotopy. Since c and $T(c)$ are isotopic, the minimal intersection number of them is 0. Hence, there exist bigons such that each of their boundaries is the union of an arc of c and an arc of $T(c)$. Choose an innermost bigon Δ among them.

Let α be the arc $c \cap \partial \Delta$ and β the arc $T(c) \cap \partial \Delta$, respectively. Since Δ is a bigon, the endpoints of them coincide. Denote them by $\{x_1, x_2\} \subset \partial \Delta$.

Lemma 4.2.3.
$$\text{Int } \Delta \cap (T(c) \cup c) = \emptyset$$

Proof. If the set $\text{Int } \Delta \cap c$ is non-empty, there exists an arc of c in Δ which forms an bigon with the arc β. Since the bigon Δ is innermost, it is a contradiction. We can also show that $\text{Int } \Delta \cap T(c) = \emptyset$ in the same way. $\qquad\square$

Note that the bigon $\iota(\Delta)$ is also innermost. By Lemma 4.2.3, we have $\Delta \cap \iota(\Delta) = \partial \Delta \cap \partial \iota(\Delta)$.

Lemma 4.2.4.
$$\partial \Delta \cap \partial \iota(\Delta) \subset \{x_1, x_2\}.$$

Proof. Since $\partial \alpha = \partial \beta = \alpha \cap \beta = \{x_1, x_2\}$, it suffices to show that $\text{Int } \alpha \cap \partial \iota(\Delta) = \text{Int } \beta \cap \partial \iota(\Delta) = \emptyset$. Since $\alpha \cap T(c) = \{x_1, x_2\}$, we have $\text{Int } \alpha \cap \iota(\beta) = \emptyset$. Next, we will show that $\text{Int } \alpha \cap \text{Int } \iota(\alpha) = \emptyset$. We assume $\text{Int } \alpha \cap \text{Int } \iota(\alpha) \neq \emptyset$. Since c is simple and contains α and $\iota(\alpha)$, α and $\iota(\alpha)$ must coincide. In particular, we have $\partial \alpha = \partial \iota(\alpha)$. So $\beta \cup \iota(\beta)$ forms a simple closed curve, and this curve is null-homotopic because both of the arcs β and $\iota(\beta)$ are homotopic to $\alpha = \iota(\alpha)$ relative to their boundaries. Since $T(c)$ is simple and contains β and $\iota(\beta)$, $T(c)$ and $\beta \cup \iota(\beta)$ must coincide. This contradicts that $T(c)$ is essential. In the same way, we can show that $\text{Int } \beta \cap \partial \iota(\Delta) = \emptyset$. $\qquad\square$

Let Σ_g^ι denote the fixed point set of the involution ι on Σ_g.

Lemma 4.2.5. *If c is non-separating, the set $c \cap \Sigma_g^\iota$ consists of 2 points, and*

$$c \cap \Sigma_g^\iota = T(c) \cap \Sigma_g^\iota.$$

If c is separating,

$$c \cap \Sigma_g^\iota = T(c) \cap \Sigma_g^\iota = \emptyset.$$

Proof. Endow the curves c and $T(c)$ with arbitrary orientations.

First, consider the case when c is a non-separating simple closed curve. In this case, the curve $T(c)$ is also non-separating. They represent nontrivial homology classes in $H_1(\Sigma_g; \mathbf{Z})$. Since the involution ι acts on $H_1(\Sigma_g; \mathbf{Z})$ by -1, it changes the orientations of c and $T(c)$. Hence, both of the sets $c \cap \Sigma_g^\iota$ and $T(c) \cap \Sigma_g^\iota$ consist of 2 points.

We will show that $T(c) \cap \Sigma_g^\iota = c \cap \Sigma_g^\iota$. Since c and $T(c)$ are isotopic, the Dehn twists t_c and $t_{T(c)}$ represent the same element in $\mathcal{H}_g$. The mapping classes $\Psi([t_c])$ and $\Psi([t_{T(c)}])$ in $\mathcal{M}_0^{2g+2}$ permute the branched points $p(c \cap \Sigma_g^\iota)$ and $p(T(c) \cap \Sigma_g^\iota)$, respectively. Hence, the sets $p(c \cap \Sigma_g^\iota)$ and $p(T(c) \cap \Sigma_g^\iota)$ coincide. It shows that $c \cap \Sigma_g^\iota = T(c) \cap \Sigma_g^\iota$.

Next, let c be a separating simple closed curve. Since ι preserves the orientations of the subsurfaces bounded by c or $T(c)$, it also preserves the orientation of c and $T(c)$. In general, if an involution acts on a circle preserving its orientation, it does not have a fixed point. Hence, we have $c \cap \Sigma_g^\iota = T(c) \cap \Sigma_g^\iota = \emptyset$. $\qquad\square$

Proof of Lemma 4.2.1. Let c be a non-separating curve. By Lemma 4.2.5, the geometric intersection number of c and $T(c)$ is at least 2. Hence, there is an innermost bigon Δ. By Lemma 4.2.4, the cardinality $\sharp(\Delta \cap \iota(\Delta))$ is equal to 0, 1, or 2 as shown in Figure 4.2.1.
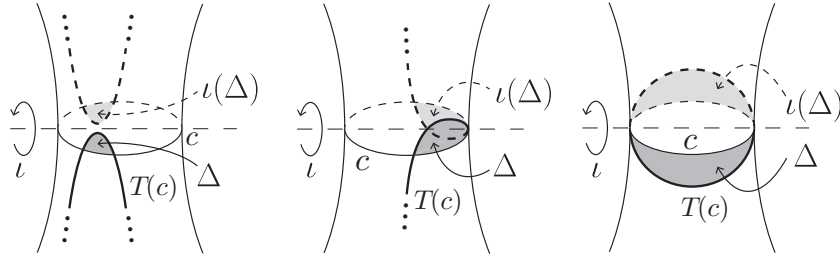

Figure 4.2.1: Left: $\sharp(\Delta \cap \iota(\Delta)) = 0$. Center: $\sharp(\Delta \cap \iota(\Delta)) = 1$. Right: $\sharp(\Delta \cap \iota(\Delta)) = 2$. The bold curves describe the curves $T(c)$.

Firstly, assume that $\sharp(\Delta \cap \iota(\Delta)) = 0$. In this case, there is a symmetric isotopy $L_1 : \Sigma_g \times [0, 1] \to \Sigma_g$ such that $L_1(*, 0)$ is the identity, and $L_1(*, 1)$ collapses the bigon Δ as in Figure 4.2.2. Therefore, we can decrease the geometric intersection number of c and $T(c)$ by 4 by replacing the diffeomorphism T by $L_1(*, 1)T$.

Secondly, assume that $\sharp(\Delta \cap \iota(\Delta)) = 1$. In this case, we also have a symmetric isotopy $L_2 : \Sigma_g \times [0, 1] \to \Sigma_g$ which decreases the geometric intersection number by 2 as in Figure 4.2.3. Note that $\Delta \cap \iota(\Delta)$ is a branched point, and $L_2(*, t)$ fixes it for any $t \in [0, 1]$.
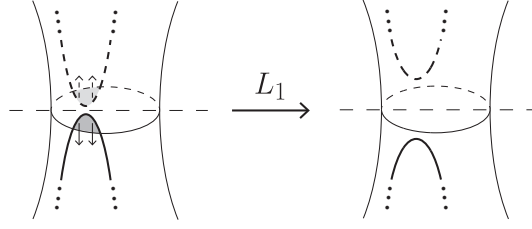
Figure 4.2.2: An isotopy L_1.
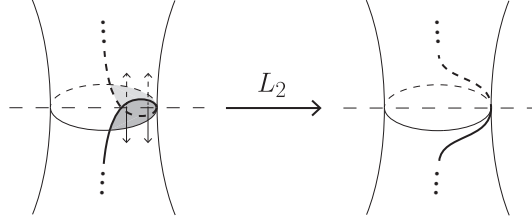
Figure 4.2.3: An isotopy L_2.

After replacing the diffeomorphism T in these two cases, the branch points $\{x_1, x_2\}$ remains in $c \cap T(c)$. Hence, if we repeat to replace T, the case when $\sharp(\Delta \cap \iota(\Delta))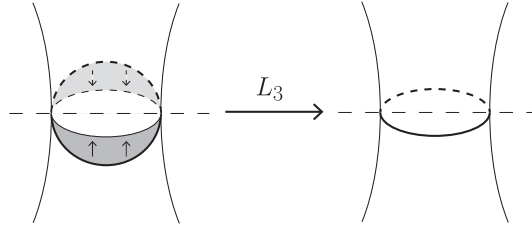 = 2$ will definitely occur. In this case, there is a symmetric isotopy $L_3 : \Sigma_g \times [0, 1] \to \Sigma_g$ such that

$$L_3(*, 0) \text{ is the identity map,}$$
$$L_3(\beta, 1) = \alpha,$$
$$L_3(\iota(\beta), 1) = \iota(\alpha),$$

as in Figure 4.2.4. It indicates that $L_3(*, 1)T$ preserves the curve c. By combining these isotopies, we have obtained a desired symmetric isotopy.

Figure 4.2.4: An isotopy L_3.

Next, let c be a separating curve. If the geometric intersection number of c and $T(c)$ is 0, the curves c and $T(c)$ bound an annulus A. Since ι acts on A without fixed points, $A/\langle \iota \rangle$ is also an annulus. Hence, we can make a symmetric isotopy which moves $T(c)$ to c.

Assume that the geometric intersection number is not 0. Since we have $c \cap \Sigma_g^\iota = T(c) \cap \Sigma_g^\iota = \emptyset$, the cardinality $\sharp(\Delta \cap \iota(\Delta)) \neq 1$. By Lemma 4.2.4, we have $\sharp(\Delta \cap \iota(\Delta)) = 0$ or 2. By the same argument as in the case when c is non-separating, we can collapse the bigons Δ and

$\iota(\Delta)$. $\square$

4.3 An involution on hyperelliptic broken Lefschetz fibrations

In this section, we prove the following theorem:

Theorem 4.3.1. *Let $f : X \to S^2$ be a genus-g hyperelliptic simplified broken Lefschetz fibration. We assume that g is greater than or equal to 3.*

(i) *Let s be the number of Lefschetz singularities of f whose vanishing cycles are separating. Then there exists an involution*

$$\omega : M \to M$$

such that the fixed point set of ω is the union of (possibly unorientable) surfaces and s isolated points. Moreover, ω can be extended to an involution

$$\overline{\omega} : X \# s\overline{\mathbb{CP}^2} \to X \# s\overline{\mathbb{CP}^2}$$

so that $X \# s\overline{\mathbb{CP}^2}/\overline{\omega}$ is diffeomorphic to $S \# 2s\overline{\mathbb{CP}^2}$, where S is S^2-bundle over S^2, and that the quotient map

$$/\overline{\omega} : X \# s\overline{\mathbb{CP}^2} \to X \# s\overline{\mathbb{CP}^2}/\overline{\omega} \cong S \# 2s\overline{\mathbb{CP}^2}$$

is a double branched covering.

(ii) *A regular fiber F of the fibration f represents a non-trivial rational homology class of X, that is, $[F] \neq 0$ in $H_2(X; \mathbb{Q})$.*

Proof of (i) in Theorem 4.3.1. Let $f : X \to S^2$ be genus-$g \geq 3$ HSBLF, $c_i \subset \Sigma_g$ $(i = 1, \dots, l)$ a vanishing cycle of a Lefschetz singularity of f and $c \subset \Sigma_g$ a vanishing cycle of indefinite folds of f. We assume that $c_1, \dots, c_n$ and c are preserved by the involution $\iota : \Sigma_g \to \Sigma_g$. By the argument in Section 2.6, we can decompose X as follows:

$$X = D^2 \times \Sigma_g \cup (h_1^2 \amalg \cdots \amalg h_n^2) \cup R^2 \cup D^2 \times \Sigma_{g-1},$$

where $h_i^2 = D^2 \times D^2$ is the 2-handle attached along the simple closed curve $\{p_i\} \times c_i \in \partial D^2 \times \Sigma_g$ and R^2 is a round 2-handle. We first prove existence of an involution ω by using the above decomposition.

Step 1: We define an involution ω_1 on $D^2 \times \Sigma_g$ as follows:

$$\omega_1 = \mathrm{id} \times \iota : \quad \begin{array}{ccc} D^2 \times \Sigma_g & \longrightarrow & D^2 \times \Sigma_g \\ \cup & & \cup \\ (z, x) & \longmapsto & (z, \iota(x)). \end{array}$$

In the following steps, we will define an involution on each component in the above decomposition of X which is compatible with the involution ω_1.

Step 2: We next define an involution $\omega_{2,i}$ on the 2-handle h_i^2 attached along $\{q_i\} \times c_i \subset \partial D^2 \times \Sigma_g$. We will abuse the notation by denoting the attaching circle $\{q_i\} \times c_i$ by c_i.

We take a tubular neighborhood νc_i in $\{q_i\} \times \Sigma_g$ and an identification

$$\nu c_i \cong S^1 \times [-1, 1]$$

so that c_i corresponds to the circle $S^1 \times \{0\}$ under the identification. We assume that the standard orientation of $S^1 \times [-1, 1]$ coincides with that of $\{q_i\} \times \Sigma_g$. We take a sufficiently small neighborhood I_{q_i} of q_i in ∂D^2 as follows:

$$I_{q_i} = \{q_i \cdot \exp(\sqrt{-1}\theta) \in \partial D^2 | \theta \in [-\varepsilon_1, \varepsilon_1]\},$$

where $\varepsilon_1 > 0$ is a sufficiently small number. We further identify the neighborhood I_{q_i} with the unit interval $[-1, 1]$ by using the following map:

$$
\begin{array}{ccc}
[-1, 1] & \xrightarrow{\sim} & I_{q_i} \\
\cup & & \cup \\
s & \longmapsto & q_i \cdot \exp(\sqrt{-1}\varepsilon_1 s).
\end{array}
$$

We regard $I_{q_i} \times [-1, 1]$ as the subset of $\mathbb{C}$ by the following embedding:

$$
\begin{array}{ccc}
I_{q_i} \times [-1, 1] & \hookrightarrow & \{z \in \mathbb{C} | |\operatorname{Re} z| \leq 1, |\operatorname{Im} z| \leq 1\} \\
\cup & & \cup \\
(s, t) & \longmapsto & s + t\sqrt{-1}.
\end{array}
$$

We put $J = \{z \in \mathbb{C} | |\operatorname{Re} z| \leq 1, |\operatorname{Im} z| \leq 1\}$. The orientation of $\partial D^2 \times \Sigma_g$ is opposite to the natural orientation of $J \times S^1$. Thus, the attaching map of the 2-handle h_i^2 is described as follows:

$$
\begin{array}{ccc}
\varphi_i : \quad \partial D^2 \times D^2 & \longrightarrow & J \times S^1 \subset \partial D^2 \times \Sigma_g \\
\cup & & \cup \\
(w_1, w_2) & \longmapsto & (\varepsilon_2 w_2 w_1, w_1),
\end{array}
$$

where $\varepsilon_2 > 0$ is a sufficiently small number. Note that the map φ_i is orientation-preserving if we give the natural orientation of $\partial D^2 \times D^2$.

Case 2.1: If c_i is non-separating, we can take a tubular neighborhood $\nu c_i \cong S^1 \times [-1, 1]$ so that the involution ω_1 acts on νc_i as follows:

$$
\begin{array}{ccc}
\omega_1|_{\nu c_i} : \quad S^1 \times [-1, 1] & \longrightarrow & S^1 \times [-1, 1] \\
\cup & & \cup \\
(z, t) & \longmapsto & (\bar{z}, -t).
\end{array}
$$

Since the involution $\omega_1 : D^2 \times \Sigma_g \to D^2 \times \Sigma_g$ preserves the first component, ω_1 acts on $I_{q_i} \times \nu c_i \cong J \times S^1$ as follows:

$$
\begin{array}{ccc}
\omega_1|_{J \times S^1} : \quad J \times S^1 & \longrightarrow & J \times S^1 \\
\cup & & \cup \\
(z_1, z_2) & \longmapsto & (\overline{z_1}, \overline{z_2}).
\end{array}
$$

We define an involution $\omega_{2,i}$ on the 2-handle h_i^2 as follows:

$$\omega_{2,i}: \begin{array}{ccc} D^2 \times D^2 & \longrightarrow & D^2 \times D^2 \\ \cup & & \cup \\ (w_1, w_2) & \longmapsto & (\overline{w_1}, \overline{w_2}). \end{array}$$

Then the following diagram commutes:

$$\begin{array}{ccc} \partial D^2 \times D^2 & \xrightarrow{\omega_{2,i}} & \partial D^2 \times D^2 \\ \varphi_i \downarrow & & \downarrow \varphi_i \\ J \times S^1 & \xrightarrow{\omega_1} & J \times S^1. \end{array}$$

Thus, we can define an involution $\omega_1 \cup \omega_{2,i}$ on the manifold $D^2 \times \Sigma_g \cup_{\varphi_i} h_i^2$.

Case 2.2: If c_i is separating, we can take a tubular neighborhood $\nu c_i \cong S^1 \times [-1, 1]$ so that the involution ω_1 acts on νc_i as follows:

$$\omega_1|_{\nu c_i}: \begin{array}{ccc} S^1 \times [-1, 1] & \longrightarrow & S^1 \times [-1, 1] \\ \cup & & \cup \\ (z, t) & \longmapsto & (-z, t). \end{array}$$

Then ω_1 acts on $I_{q_i} \times \nu c_i \cong J \times S^1$ as follows:

$$\omega_1|_{J \times S^1}: \begin{array}{ccc} J \times S^1 & \longrightarrow & J \times S^1 \\ \cup & & \cup \\ (z_1, z_2) & \longmapsto & (z_1, -z_2). \end{array}$$

We define an involution $\omega_{2,i}$ on the 2-handle h_i^2 as follows:

$$\omega_{2,i}: \begin{array}{ccc} D^2 \times D^2 & \longrightarrow & D^2 \times D^2 \\ \cup & & \cup \\ (w_1, w_2) & \longmapsto & (-w_1, -w_2). \end{array}$$

Then the following diagram commutes:

$$\begin{array}{ccc} \partial D^2 \times D^2 & \xrightarrow{\omega_{2,i}} & \partial D^2 \times D^2 \\ \varphi_i \downarrow & & \downarrow \varphi_i \\ J \times S^1 & \xrightarrow{\omega_1} & J \times S^1. \end{array}$$

Thus, we can define an involution $\omega_1 \cup \omega_{2,i}$ on the manifold $D^2 \times \Sigma_g \cup_{\varphi_i} h_i^2$.

Combining Case 2.1 and Case 2.2, we can define the involution $\tilde{\omega}_2$ on the 4-manifold $X_h = D^2 \times \Sigma_g \cup (h_1^2 \amalg \cdots \amalg h_n^2)$ as follows:

$$\tilde{\omega}_2 = \omega_1 \cup (\omega_{2,1} \cup \cdots \cup \omega_{2,n}).$$

Before giving an involution on the round 2-handle, we look at the Σ_g-bundle structure of ∂X_h. The projection $\pi_h : \partial X_h \to \partial D^2$ of this bundle is described as follows:

- for an element $(z, x) \in \partial D^2 \times \Sigma_g \setminus (\amalg \operatorname{Int} \varphi_i(\partial D^2 \times D^2))$, π_h is defined as follows:

$$\pi_h(z, x) = z,$$

- for an element $(w_1, w_2) \in D^2 \times \partial D^2 \subset \partial h_i^2$, π_h is defined as follows:

$$\pi_h(w_1, w_2) = q_i \cdot \exp(\sqrt{-1}\varepsilon_1\varepsilon_2(\operatorname{Re} w_1 \operatorname{Re} w_2 - \operatorname{Im} w_1 \operatorname{Im} w_2).$$

Indeed, the map π_h is well-defined. To see this, we only need to verify the following equation:

$$q_i \cdot \exp(\sqrt{-1}\varepsilon_1\varepsilon_2(\operatorname{Re} w_1 \operatorname{Re} w_2 - \operatorname{Im} w_1 \operatorname{Im} w_2)) = p_1 \circ \varphi_i(w_1, w_2),$$

where $(w_1, w_2) \in D^2 \times \partial D^2 \subset \partial h_i^2$ and $p_1 : J \times S^1 \to I_{q_i}$ is the projection. $p_1 \circ \varphi_i(w_1, w_2)$ is calculated as follows:

$$\begin{aligned}
p_1 \circ \varphi_i(w_1, w_2) &= p_1(\varepsilon_2 w_2 w_1, w_1) \\
&= q_i \cdot \exp(\sqrt{-1}\varepsilon_1 \operatorname{Re}(\varepsilon_2 w_2 w_1)) \\
&= q_i \cdot \exp(\sqrt{-1}\varepsilon_1\varepsilon_2(\operatorname{Re} w_1 \operatorname{Re} w_2 - \operatorname{Im} w_1 \operatorname{Im} w_2))
\end{aligned}$$

This implies that the definition of π_k above makes sense.

Lemma 4.3.2. *The involution $\tilde{\omega}_2$ preserves the fibers of π_h. Moreover, there exists a lift V of the vector field $\dfrac{d}{d\theta}\exp(\sqrt{-1}\theta)$ by the map π_h which is compatible with the involution $\tilde{\omega}_2$, that is,*

$$\tilde{\omega}_{2*}(V) = V.$$

Proof of Lemma 4.3.2. It is easy to verify that $\tilde{\omega}_2$ preserves the fibers of π_h by direct calculation. The details of this are left to the readers.

To prove existence of a lift V, we construct V explicitly. We define a vector field V_1 on $\partial D^2 \times \Sigma_g \setminus (\amalg\varphi_i(\partial D^2 \times D^2))$ as follows:

$$V_1(\exp(\sqrt{-1}\theta_0), x) = \left.\frac{d}{d\theta}\exp(\sqrt{-1}\theta)\right|_{\theta=\theta_0} \in T_{(\exp(\sqrt{-1}\theta_0),x)}(\partial D^2 \times \Sigma_g),$$

for a point $(\exp(\sqrt{-1}\theta_0), x) \in \partial D^2 \times \Sigma_g \setminus (\amalg \operatorname{Int} \varphi_i(\partial D^2 \times D^2))$. The vector field V_1 is described in $J \times S^1$ as follows:

$$V_1(s + t\sqrt{-1}, z) = \frac{1}{\varepsilon_1}\left.\frac{\partial}{\partial s}\right|_s \in T_{(s+t\sqrt{-1},z)}(J \times S^1).$$

We also define a vector field V_2 on $D^2 \times \partial D^2 \subset \partial h_i^2$ as follows:

$$V_2(w_1, w_2) = \frac{\varrho(|w_1|^2)}{\varepsilon_1\varepsilon_2 |w_1|^2}\left(x_1\frac{\partial}{\partial x_2} - y_1\frac{\partial}{\partial y_2}\right) + \frac{1 - \varrho(|w_1|^2)}{\varepsilon_1\varepsilon_2}\left(x_2\frac{\partial}{\partial x_1} - y_2\frac{\partial}{\partial y_1}\right),$$

where $w_i = x_i + y_i\sqrt{-1}$ and $\varrho : [0, 1] \to [0, 1]$ is a monotone increasing smooth function which satisfies the following conditions:

- $\varrho(t) = 0$ for $t \in \left[0, \frac{1}{3}\right]$;

- $\varrho(t) = 1$ for $t \in \left[\frac{2}{3}, 1\right]$.

For $(w_1, w_2) \in \partial D^2 \times \partial D^2$, $d\varphi_i(V_2(w_1, w_2))$ is calculated as follows:

$$
\begin{aligned}
&d\varphi_i(V_2(w_1, w_2)) \\
=&d\varphi_i\left(\frac{1}{\varepsilon_1\varepsilon_2}\left(x_1\frac{\partial}{\partial x_2} - y_1\frac{\partial}{\partial y_2}\right)\right) && (\because |w_1| = 1) \\
=&\frac{1}{\varepsilon_1\varepsilon_2}x_1 d\varphi_i\left(\frac{\partial}{\partial x_2}\right) - \frac{1}{\varepsilon_1\varepsilon_2}y_1 d\varphi_i\left(\frac{\partial}{\partial y_2}\right) \\
=&\frac{1}{\varepsilon_1}x_1\left(x_1\frac{\partial}{\partial s} + y_1\frac{\partial}{\partial t}\right) - \frac{1}{\varepsilon_1}y_1\left(-y_1\frac{\partial}{\partial s} + x_1\frac{\partial}{\partial t}\right) \\
=&\frac{1}{\varepsilon_1}({x_1}^2 + {y_1}^2)\frac{\partial}{\partial s} \\
=&V_1(\varphi_i(w_1, w_2)).
\end{aligned}
$$

Hence, we can define a vector field $V = V_1 \cup V_2$ on the manifold ∂X_h. Moreover, it can be shown that V_1 and V_2 is a lift of the vector field $\frac{d}{d\theta}\exp(\sqrt{-1}\theta)$ by the map π_h. Thus, the vector field V is a lift of $\frac{d}{d\theta}\exp(\sqrt{-1}\theta)$. We can show that the vector field V is compatible with the involution $\tilde{\omega}_2$ by direct calculation. This completes the proof of Lemma 4.3.2. $\quad\square$

We choose a base point $q_0 \in \partial D^2 \setminus (\mathrm{III}_{q_i})$ and define a map $\Theta_V : f^{-1}(q_0) \to f^{-1}(q_0)$ as follows:

$$\Theta_V(x) = c_{V,x}(2\pi),$$

where $c_{V,x}$ is the integral curve of the vector field V constructed in Lemma 4.3.2 which satisfies $c_{V,x}(0) = x$. We identify $f^{-1}(q_0)$ with the surface Σ_g via the projection onto the second component. Then the map Θ_V is contained in the centralizer $C(\iota) \subset \mathrm{Diff}_+ \Sigma_g$ since the vector field V is compatible with $\tilde{\omega}_2$. The isotopy class represented by Θ_V is the monodromy of the boundary of X_h. In particular, this class is contained in the group $\mathcal{H}_g(c)$. By Lemma 4.2.1, there exists an isotopy $H_t : \Sigma_g \to \Sigma_g$ satisfying the following conditions:

- $H_0 = \Theta_V$;

- H_1 preserves the curve c as a set;

- for each level t, H_t is in the centralizer $C(\iota)$.

We obtain the following isomorphism of Σ_g-bundles:

$$\partial X_h \cong [0, 1] \times \Sigma_g / ((1, x) \sim (0, H_1(x))).$$

We identify the above Σ_g-bundles via the isomorphism. Under this identification, the involution $\tilde{\omega}_2$ acts on ∂X_h as follows:

$$\tilde{\omega}_2(t, x) = (t, \iota(x)),$$

where (t, x) is an element in $[0, 1] \times \Sigma_g / ((1, x) \sim (0, H_1(x))) \cong \partial X_h$.

Step 3: In this step, we define an involution ω_3 on the round 2-handle R^2. Since c is non-separating and c is preserved by ι, c contains two fixed points of the involution ι. We denote

these points by v_1 and v_2. We can take a tubular neighborhood $\nu c \cong S^1 \times [-1,1]$ in Σ_g so that the involution ι acts on νc as follows:

$$\iota(z,t) = (\bar{z}, -t).$$

By perturbing the map H_1, we can assume that H_1 preserves the neighborhood νc. Since the genus of the fibration f is not equal to 1, the attaching region of the round 2-handle R^2 is $[0,1] \times \nu c / ((1,x) \sim (0, H_1(x)))$.

Case 3.1: If H_1 preserves the orientation of c and two points v_1 and v_2, then the round handle R^2 is untwisted and the restriction $H_1|_{\nu c}$ is described as follows:

$$H_1(z,t) = (z,t),$$

where $(z,t) \in S^1 \times [-1,1] \cong \nu c$. Moreover, the attaching map of the round handle is described as follows:

$$\begin{array}{ccc}
\varphi: & [0,1] \times \partial D^2 \times D^1 / \sim & \longrightarrow & [0,1] \times S^1 \times [-1,1] / \sim \\
& \cup & & \cup \\
& (s,z,t) & \longmapsto & (s,z,t),
\end{array}$$

where $[0,1] \times \partial D^2 \times D^1$ is the attaching region of R^2 and $[0,1] \times S^1 \times [-1,1] \cong [0,1] \times \nu c$ is the subset of ∂X_h. We define an involution ω_3 on the round handle as follows:

$$\begin{array}{ccc}
\omega_3: & [0,1] \times D^2 \times D^1 / \sim & \longrightarrow & [0,1] \times D^2 \times D^1 / \sim \\
& \cup & & \cup \\
& (s,z,t) & \longmapsto & (s,\bar{z},-t),
\end{array}$$

Then the following diagram commutes:

$$\begin{array}{ccc}
[0,1] \times \partial D^2 \times D^1 & \xrightarrow{\omega_3} & [0,1] \times \partial D^2 \times D^1 \\
\varphi \downarrow & & \downarrow \varphi \\
[0,1] \times S^1 \times [-1,1] & \xrightarrow{\tilde{\omega}_2} & [0,1] \times S^1 \times [-1,1].
\end{array}$$

Therefore, we obtain an involution $\tilde{\omega}_3 = \tilde{\omega}_2 \cup \omega_3$ on $X_h \cup X_r = X_h \cup R^2$.

Case 3.2: If H_1 preserves the orientation of c but does not preserve two points v_1 and v_2, then the round handle R^2 is untwisted and the restriction $H_1|_{\nu c}$ is described as follows:

$$H_1(z,t) = (-z,t),$$

The attaching map of the round handle is described as follows:

$$\begin{array}{ccc}
\varphi: & [0,1] \times \partial D^2 \times D^1 / \sim & \longrightarrow & [0,1] \times S^1 \times [-1,1] / \sim \\
& \cup & & \cup \\
& (s,z,t) & \longmapsto & (s, \exp(\pi\sqrt{-1}s)z, t).
\end{array}$$

We define an involution ω_3 on the round handle as follows:

$$\begin{array}{ccc}
\omega_3: & [0,1] \times D^2 \times D^1 / \sim & \longrightarrow & [0,1] \times D^2 \times D^1 / \sim \\
& \cup & & \cup \\
& (s,z,t) & \longmapsto & (s, \exp(-2\pi\sqrt{-1}s)\bar{z}, -t),
\end{array}$$

Then we can define an involution $\tilde{\omega}_3 = \tilde{\omega}_2 \cup \omega_3$ on $X_h \cup X_r = X_h \cup R^2$ by the same reason as in Case 3.1.

Case 3.3: If H_1 does not preserve the orientation of c but preserves two points v_1 and v_2, then the round handle R^2 is twisted and the restriction $H_1|_{\nu c}$ is described as follows:

$$H_1(z,t) = (\bar{z}, -t),$$

where $(z,t) \in S^1 \times [-1,1] \cong \nu c$. Moreover, the attaching map of the round handle is described as follows:

$$
\begin{array}{ccc}
\varphi : & [0,1] \times \partial D^2 \times D^1/\sim & \longrightarrow & [0,1] \times S^1 \times [-1,1]/\sim \\
& \cup & & \cup \\
& (s,z,t) & \longmapsto & (s,z,t).
\end{array}
$$

We define an involution ω_3 on the round handle as follows:

$$
\begin{array}{ccc}
\omega_3 : & [0,1] \times D^2 \times D^1/\sim & \longrightarrow & [0,1] \times D^2 \times D^1/\sim \\
& \cup & & \cup \\
& (s,z,t) & \longmapsto & (s,\bar{z},-t),
\end{array}
$$

Then we can define an involution $\tilde{\omega}_3 = \tilde{\omega}_2 \cup \omega_3$ on $X_h \cup X_r = X_h \cup R^2$.

Case 3.4: If H_1 preserves neither the orientation of c nor two points v_1 and v_2, then the round handle R^2 is twisted and the restriction $H_1|_{\nu c}$ is described as follows:

$$H_1(z,t) = (-\bar{z}, -t),$$

where $(z,t) \in S^1 \times [-1,1] \cong \nu c$. Moreover, the attaching map of the round handle is described as follows:

$$
\begin{array}{ccc}
\varphi : & [0,1] \times \partial D^2 \times D^1/\sim & \longrightarrow & [0,1] \times S^1 \times [-1,1]/\sim \\
& \cup & & \cup \\
& (s,z,t) & \longmapsto & (s,\exp(\pi\sqrt{-1}s)z,t).
\end{array}
$$

We define an involution ω_3 on the round handle as follows:

$$
\begin{array}{ccc}
\omega_3 : & [0,1] \times D^2 \times D^1/\sim & \longrightarrow & [0,1] \times D^2 \times D^1/\sim \\
& \cup & & \cup \\
& (s,z,t) & \longmapsto & (s,\exp(-2\pi\sqrt{-1}s)\bar{z},-t),
\end{array}
$$

Then we can define an involution $\tilde{\omega}_3 = \tilde{\omega}_2 \cup \omega_3$ on $X_h \cup X_r = X_h \cup R^2$.

Eventually, we obtain the involution $\tilde{\omega}_3$ on $X_h \cup X_r$ in any cases. We next look at Σ_{g-1}-bundle structure of $\partial(X_h \cup X_r)$. The projection $\pi_r : \partial(X_h \cup X_r) \to [0,1]/\{0,1\}$ of this bundle is described as follows:

$$\pi_r(s,x) = s \quad \big((s,x) \in \big([0,1] \times \Sigma_g/(1,x) \sim (0,H_1(x))\big) \setminus ([0,1] \times \nu c/\sim)\big);$$
$$\pi_r(s,z,t) = s \quad \big((s,z,t) \in [0,1] \times D^2 \times \partial D^1\big).$$

Indeed, it is easy to show that π_r is well-defined.

Lemma 4.3.3. *The involution $\tilde{\omega}_3$ preserves the fibers of π_r. Moreover, there exists a lift $\tilde{V}$ of the vector field $\dfrac{d}{ds}$ on $[0,1]/\{0,1\}$ by the map π_r which is compatible with the involution $\tilde{\omega}_3$.*

Proof of Lemma 4.3.3. It is obvious that the involution $\tilde{\omega}_3$ preserves the fibers of π_r. We construct $\tilde{V}$ as we do in Lemma 4.3.2. We define a vector field $\tilde{V}_1$ on $([0,1] \times \Sigma_g / \sim) \setminus ([0,1] \times \nu c / \sim)$ as follows:

$$\tilde{V}_1(s, x) = \frac{d}{ds}.$$

We first consider the case H_1 preserves the points v_1 and v_2. In this case, we define a vector field $\tilde{V}_2$ on the round handle R^2 as follows:

$$\tilde{V}_2(s, z, t) = \frac{d}{ds},$$

where $(s, z, t) \in [0,1] \times D^2 \times \partial D^1 \subset \partial R^2$. It is easy to verify the equality $d\varphi \left(\frac{d}{ds} \right) = \frac{d}{ds}$. Hence, we can define vector field $\tilde{V} = \tilde{V}_1 \cup \tilde{V}_2$ on $\partial(X_h \cup X_r)$. It is obvious that $\tilde{V}$ is a lift of the vector field $\frac{d}{ds}$ on $[0,1]/\{0,1\}$ by π_r and is compatible with the involution $\tilde{\omega}_3$.

We next consider the case H_1 does not preserve the points v_1 and v_2. In this case, we define a vector field $\tilde{V}_2$ on R^2 as follows:

$$\tilde{V}_2(s, x + y\sqrt{-1}, t) = \frac{d}{ds} + \pi y \frac{\partial}{\partial x} - \pi x \frac{\partial}{\partial y},$$

where $(s, x + y\sqrt{-1}, t) \in [0,1] \times D^2 \times \partial D^1 \subset \partial R^2$. The differential $d\varphi(\tilde{V}_2(s, x + \sqrt{-1}y, t))$ is calculated as follows:

$$
\begin{aligned}
&d\varphi(\tilde{V}_2(s, x + \sqrt{-1}y, t)) \\
=&d\varphi \left(\frac{d}{ds} + \pi y \frac{\partial}{\partial x} - \pi x \frac{\partial}{\partial y} \right) \\
=& \left(\frac{d}{ds} + \pi(-x \sin \pi s - y \cos \pi s) \frac{d}{dx} + \pi(x \cos \pi s - y \sin \pi s) \frac{d}{dy} \right) \\
&+ \pi y \left(\cos \pi s \frac{d}{dx} + \sin \pi s \frac{d}{dy} \right) - \pi x \left(-\sin \pi s \frac{d}{dx} + \cos \pi s \frac{d}{dy} \right) \\
=& \frac{d}{ds} \\
=& \tilde{V}_1(\varphi(s, x + \sqrt{-1}y, t)).
\end{aligned}
$$

Hence, we can define a vector field $\tilde{V} = \tilde{V}_1 \cup \tilde{V}_2$ on $\partial(X_h \cup X_r)$. It is obvious that $\tilde{V}$ is a lift of the vector field $\frac{d}{ds}$ on $[0,1]/\{0,1\}$ by π_r. To verify that $\tilde{V}$ is compatible with the involution $\tilde{\omega}_3$, we need to prove that the following equation holds for any points $x \in \partial(X_h \cup X_r)$:

$$d\tilde{\omega}_3(\tilde{V}(x)) = \tilde{V}(\tilde{\omega}_3(x)).$$

If x is contained in $[0,1] \times \Sigma_g / \sim \setminus ([0,1] \times \nu c / \sim)$, the above equation can be proved easily. If $x = (s, x + \sqrt{-1}y, t) \in [0,1] \times D^2 \times \partial D^1 \subset \partial R^2$, then $d\tilde{\omega}_3(\tilde{V}(x))$ is calculated as follows:

$$
\begin{aligned}
&d\tilde{\omega}_3(\tilde{V}(x)) \\
=&d\tilde{\omega}_3(\frac{d}{ds} + \pi y \frac{\partial}{\partial x} - \pi x \frac{\partial}{\partial y}) \\
=& \left(\frac{d}{ds} + 2\pi(-x \sin 2\pi s - y \cos 2\pi s) \frac{\partial}{\partial x} + 2\pi(-x \cos 2\pi s + y \sin 2\pi s) \frac{\partial}{\partial y} \right)
\end{aligned}
$$

$$+ \pi y \left(\cos 2\pi s \frac{\partial}{\partial x} - \sin 2\pi s \frac{\partial}{\partial y} \right) - \pi x \left(-\sin 2\pi s \frac{\partial}{\partial x} - \cos 2\pi s \frac{\partial}{\partial y} \right)$$

$$= \frac{d}{ds} + \pi(-x \sin 2\pi s - y \cos 2\pi s) \frac{\partial}{\partial x} + \pi(-x \cos 2\pi s + y \sin 2\pi s) \frac{\partial}{\partial y}$$

$$= \tilde{V}(\tilde{\omega}_3(x)).$$

Thus, $\tilde{V}$ is compatible with the involution $\tilde{\omega}_3$. This completes the proof of Lemma 4.3.3. $\square$

We define the map $\Theta_{\tilde{V}} : \pi_r^{-1}(0) \to \pi_r^{-1}(0)$ as follows:

$$
\begin{array}{ccc}
\Theta_{\tilde{V}} : & \pi_r^{-1}(0) & \longrightarrow & \pi_r^{-1}(0) \\
 & \cup\!\!| & & \cup\!\!| \\
 & x & \longmapsto & c_{\tilde{V},x}(1),
\end{array}
$$

where $c_{\tilde{V},x}$ is the integral curve of $\tilde{V}$ starting at x. We identify the fiber $\pi_r^{-1}(0)$ with the surface Σ_{g-1}. The map $\Theta_{\tilde{V}}$ is contained in the centralizer $C(\iota)$ since $\tilde{V}$ is compatible with $\tilde{\omega}_3$. Moreover, $\Theta_{\tilde{V}}$ is isotopic to the identity map. By Theorem 4.1.1, we can take an isotopy $\tilde{H}_t : \Sigma_{g-1} \to \Sigma_{g-1}$ which satisfies the following conditions:

- $\tilde{H}_0 = \Theta_{\tilde{V}}$;

- $\tilde{H}_1$ is the identity map;

- $\tilde{H}_t$ is contained in the centralizer $C(\iota)$.

Note that such an isotopy may not be taken if the condition $g \geq 3$ is dropped. Indeed, the map $\pi_0 C(\iota) \to \mathcal{M}_1$ induced by the inclusion is not injective.

By using the isotopy $\tilde{H}_t$, we obtain the following isomorphism of Σ_{g-1}-bundle:

$$\partial(X_h \cup X_r) \cong [0,1] \times \Sigma_{g-1}/(1,x) \sim (0,x).$$

The involution $\tilde{\omega}_3$ acts on $[0,1] \times \Sigma_{g-1}/(1,x) \sim (0,x)$ via the above isomorphism as follows:

$$\tilde{\omega}_3(s,x) = (s, \iota(x)).$$

Step 4: We define an involution ω_4 on $D^2 \times \Sigma_{g-1}$ as follows:

$$\omega_4(z,x) = (z, \iota(x)),$$

where $(z,x) \in D^2 \times \Sigma_{g-1}$. Let $\Phi : [0,1] \times \Sigma_{g-1}/ \sim \to \partial D^2 \times \Sigma_{g-1}$ be the attaching map of the lower side. Since the genus of the fibration f is greater than 2, we can assume that Φ is given by $\Phi(s,x) = (\exp(2\pi\sqrt{-1}s), x)$. In particular, the following diagram commutes:

$$
\begin{array}{ccc}
[0,1] \times \Sigma_{g-1}/ \sim & \xrightarrow{\tilde{\omega}_3} & [0,1] \times \Sigma_{g-1}/ \sim \\
\Phi \downarrow & & \downarrow \Phi \\
\partial D^2 \times \Sigma_{g-1} & \xrightarrow{\omega_4} & \partial D^2 \times \Sigma_{g-1}.
\end{array}
$$

Hence, we obtain an involution $\omega = \tilde{\omega}_3 \cup \omega_4$ on X.

We next look at the fixed point set of ω. The involution ω is equal to $\mathrm{id} \times \iota$ on $D^2 \times \Sigma_g$. Thus, we obtain:

$$X^\omega \cup D^2 \times \Sigma_g = D^2 \times \{v_1, \dots, v_{2g+2}\},$$

where $v_1, \dots, v_{2g+2} \in \Sigma_g$ are the fixed points of ι. Note that $X^\omega \cup D^2 \times \Sigma_g$ has the natural orientation derived from the orientation of D^2.

The involution ω acts on the 2-handle $h_i^2 = D^2 \times D^2$ as follows:

$$\omega(w_1, w_2) = \begin{cases} (\overline{w_1}, \overline{w_2}) & (c_i\text{:non-separating}), \\ (-w_1, -w_2) & (c_i\text{:separating}), \end{cases}$$

where $(w_1, w_2) \in D^2 \times D^2$. Thus, the fixed point set $h_i^{2^\omega}$ is equal to $(D^2 \cap \mathbb{R}) \times (D^2 \cap \mathbb{R})$ if c_i is non-separating, and is equal to $\{(0,0)\}$ if c_i is separating. Furthermore, if c_i is non-separating, we can give an orientation to $(D^2 \cap \mathbb{R}) \times (D^2 \cap \mathbb{R})$ which is compatible with the orientation of $D^2 \times \{v_1, \dots, v_{2g+2}\}$. Hence, the fixed point set $X_h{}^\omega$ is the union of the oriented surfaces and the s points, where s is the number of Lefschetz singularities of f whose vanishing cycle is separating.

The involution ω acts on the round 2-handle R^2 in the following way:

- if H_1 preserves the two points v_1 and v_2, then $\omega(s, z, t)$ is equal to $(s, \overline{z}, -t)$ for $(s, z, t) \in R^2 = [0,1] \times D^2 \times D^1 / \sim$;

- if H_1 does not preserve the two points v_1 and v_2, then $\omega(s, z, t)$ is equal to $(s, \exp(-2\pi\sqrt{-1}s)\overline{z}, -t)$ for $(s, z, t) \in R^2 = [0,1] \times D^2 \times D^1 / \sim$.

The fixed point set R^{2^ω} is equal to $[0,1] \times (D^2 \cap \mathbb{R}) \times \{0\} / \sim$ if H_1 preserves the two points v_1 and v_2, and $R_2{}^\omega$ is equal to the following set otherwise:

$$\{(s, z, 0) \in R^2 \mid z = r\exp(-\pi\sqrt{-1}s), r \in [-1, 1]\}.$$

In particular, the fixed point set R^{2^ω} is equal to the annulus or the Möbius band. As explained in the previous paragraph, we can give an orientation of the 2-dimensional part of $X_h{}^\omega$ in the canonical way. It is easy to see that any orientation of R^{2^ω} is not compatible with this canonical orientation of $X_h{}^\omega$. In particular, even if R^{2^ω} is the annulus, the 2-dimensional part of the fixed point set $(X_h \cup X_r)^\omega$ may not be orientable. Indeed, this part is orientable if and only if R^{2^ω} is the annulus, and there is a connected component in $X_h{}^\omega$ whose boundary contains only one component of ∂R^{2^ω}.

The involution ω is equal to $\mathrm{id} \times \iota$ on $D^2 \times \Sigma_{g-1}$. Thus, the fixed point set $(D^2 \times \Sigma_{g-1})^\omega$ is equal to $D^2 \times \{\tilde{v}_1, \dots, \tilde{v}_{2g}\}$, where $\{\tilde{v}_1, \dots, \tilde{v}_{2g}\}$ is the set of the fixed points of ι. Eventually, X^ω is the union of the closed surfaces and the s points. The 2-dimensional part of X^ω is orientable if and only if that of $(X_h \cup X_r)^\omega$ is orientable. This completes the proof of the statement in Theorem 4.3.1 on the fixed point set of ω.

We next extend the involution ω to the manifold $X \# s\overline{\mathbb{CP}^2}$. We assume that the curves $c_{k_1}, \dots, c_{k_s}$ are separating. We construct the manifold $X \# s\overline{\mathbb{CP}^2}$ by blowing up X s times at $(0,0) \in h_{k_i}^2$ $(i = 1, \dots, s)$. We can obtain a natural decomposition of $X \# s\overline{\mathbb{CP}^2}$ as follows:

$$D^2 \times \Sigma_g \cup (h_1^2 \amalg \overset{\hat{k}_1, \dots, \hat{k}_s}{\dots\dots\dots} \amalg h_n^2) \cup (\tilde{h}_{k_1} \amalg \cdots \amalg \tilde{h}_{k_s}) \cup R^2 \cup D^2 \times \Sigma_{g-1},$$

where $\tilde{h}_{k_i} = \{((w_1, w_2), [l_1 : l_2]) \in D^2 \times D^2 \times \mathbb{CP}^1 \mid w_1 l_2 - w_2 l_1 = 0\} \cong h_{k_i} \# \overline{\mathbb{CP}^2}$. We define an involution $\overline{\omega}$ on $X \# s\overline{\mathbb{CP}^2}$ as follows:

$$\overline{\omega}(x) = \omega(x) \quad (x \in X \# s\overline{\mathbb{CP}^2} \setminus (\tilde{h}_{k_1} \amalg \cdots \amalg \tilde{h}_{k_s})),$$
$$\overline{\omega}((w_1, w_2), [l_1 : l_2]) = ((-w_1, -w_2), [l_1 : l_2]) \quad (((w_1, w_2), [l_1 : l_2]) \in \tilde{h}_{k_i}).$$

It is obvious that $\overline{\omega}$ is an extension of ω. The fixed point set of $\overline{\omega}$ is the union of the 2-dimensional part of X^ω and s 2-spheres.

We next prove that $X \# s\overline{\mathbb{CP}^2}/\overline{\omega}$ is diffeomorphic to $S \# 2s\overline{\mathbb{CP}^2}$, where S is an S^2-bundle over S^2. Since Σ_g/ι is diffeomorphic to S^2, it is easy to see that $D^2 \times \Sigma_g/\overline{\omega}$ is diffeomorphic to $D^2 \times S^2$. Thus, the manifold $X \# s\overline{\mathbb{CP}^2}$ is obtained by attaching $h_j/\overline{\omega}$ $(j \neq k_1, \ldots, k_s)$, $\tilde{h}_{k_i}/\overline{\omega}$, $R^2/\overline{\omega}$ and $D^2 \times \Sigma_{g-1}/\overline{\omega} \cong D^2 \times S^2$ to $D^2 \times S^2$.

Lemma 4.3.4. *Suppose that c_i is non-separating. Then,*

$$(D^2 \times \Sigma_g \cup_{\varphi_i} h_i^2)/\overline{\omega} \cong D^2 \times S^2.$$

Proof of Lemma 4.3.4. If we identify $h_i^2 = D^2 \times D^2$ with D^4, then $\overline{\omega}$ is equal to the covering transformation of the double covering $D^4 \to D^4$ branched at the unknotted 2-disk in D^4. In particular, we obtain $h_i^2/\overline{\omega}$ is diffeomorphic to D^4. Moreover, the attaching region of h_i^2 corresponds to the 3-disk in ∂D^4 under the diffeomorphism. Denote by $\overline{\varphi_i} : h_i^2/\overline{\omega} \to \partial D^2 \times \Sigma_g/\overline{\omega}$ the embedding induced by φ_i. We obtain:

$$\begin{aligned}
(D^2 \times \Sigma_g \cup_{\varphi_i} h_i^2)/\overline{\omega} &\cong (D^2 \times \Sigma_g/\overline{\omega}) \cup_{\overline{\varphi_i}} h_i^2/\overline{\omega} \\
&\cong D^2 \times S^2 \natural D^4 \\
&\cong D^2 \times S^2.
\end{aligned}$$

This completes the proof of Lemma 4.3.4. $\square$

Lemma 4.3.5. *For each $i \in \{1, \ldots, s\}$, $(D^2 \times \Sigma_g \cup_{\varphi_i} \tilde{h}_{k_i}^2)/\overline{\omega} \cong D^2 \times S^2 \# 2\overline{\mathbb{CP}^2}$.*

Proof of Lemma 4.3.5. By eliminating the corner of $D^2 \times D^2$, we identify $\tilde{h}_{k_i}^2$ with the following manifold:

$$H = \{((w_1, w_2), [l_1 : l_2]) \in D^4 \times \mathbb{CP}^1 \mid w_1 l_2 - w_2 l_1 = 0\}.$$

Under this identification, the attaching region of $\tilde{h}_{k_i}^2$ corresponds to the tubular neighborhood of the circle $\{((w_1, 0), [1 : 0]) \in \partial H \mid |w_1| = 1\}$ in ∂H. Let $p_2 : H \to \mathbb{CP}^1$ be the projection onto the second component. The map p_2 is the D^2-bundle over the 2-sphere with Euler number -1. We define $D_1, D_2 \subset \mathbb{CP}^1$, and local trivializations ψ_1 and ψ_2 of p_2 as follows:

$$D_1 = \{[l_1 : l_2] \in \mathbb{CP}^1 \mid |l_1| \geq |l_2|\},$$
$$D_2 = \{[l_1 : l_2] \in \mathbb{CP}^1 \mid |l_2| \geq |l_1|\},$$

$$\psi_1 : \quad D^2 \times D^2 \quad \longrightarrow \quad p_2^{-1}(D_1)$$
$$\cup \qquad\qquad\qquad\qquad \cup$$
$$(w_1, w_2) \quad \longmapsto \quad \left(\frac{w_2}{\sqrt{1 + |w_1|^2}}(1, w_1), [1 : w_1]\right),$$

$$\psi_2: \quad D^2 \times D^2 \quad \longrightarrow \quad p_2^{-1}(D_2)$$
$$\cup \qquad\qquad\qquad\qquad \cup$$
$$(w_1, w_2) \quad \longmapsto \quad \left(\frac{w_2}{\sqrt{1 + |w_1|^2}}(w_1, 1), [w_1, 1] \right).$$

Denote $p_2^{-1}(D_1)$ and $p_2^{-1}(D_2)$ by H_1 and H_2, respectively. We identify H_1 and H_2 with $D^2 \times D^2$ by the above trivializations. The manifold H can be identified with $D^2 \times D^2 \cup_\Psi D^2 \times D^2$, where $\Psi = \psi_1^{-1} \circ \psi_2 : (w_1, w_2) \longmapsto (\frac{1}{w_1}, w_1 w_2)$. Under the identification, the attaching region of H corresponds to $\partial D^2 \times D^2 \subset \partial H_1$.

We define $\tilde{H} = \tilde{H}_1 \cup_{\tilde{\Psi}} \tilde{H}_2$, where $\tilde{H}_i = D^2 \times D^2$ $(i = 1, 2)$ and $\tilde{\Psi} : \partial D^2 \times D^2 \to \partial D^2 \times D^2$ is a diffeomorphism defined as follows:

$$\tilde{\Psi}(w_1, w_2) = (\frac{1}{w_1}, w_1{}^2 w_2).$$

We can define $\mathcal{P} : H \to \tilde{H}$ as follows:

$$\mathcal{P}(w_1, w_2) = \begin{cases} (w_1, w_2{}^2) \in \tilde{H}_1 & ((w_1, w_2) \in H_1), \\ (w_1, w_2{}^2) \in \tilde{H}_2 & ((w_1, w_2) \in H_2). \end{cases}$$

The map $\mathcal{P}$ is a double branched covering branched at the 0-section of $\tilde{H}$ as a D^2-bundle. Moreover, $\tilde{\omega}$ is the non-trivial covering transformation of $\mathcal{P}$. Thus, we obtain $H/\tilde{\omega}$ is diffeomorphic to $\tilde{H}$.

Since the attaching region of H is mapped by $\mathcal{P}$ to $D^2 \times \partial D^2 \subset \partial \tilde{H}_1$, we can regard $\tilde{H}_1$ and $\tilde{H}_2$ as 2-handles. Thus, $(D^2 \times \Sigma_g \cup_{\varphi_i} \tilde{h}_{k_i}^2)/\overline{\omega}$ is obtained by attaching the 2-handles $\tilde{H}_1$ and $\tilde{H}_2$ to $D^2 \times S^2$. To prove the statement, we look at the attaching maps of $\tilde{H}_1$ and $\tilde{H}_2$.

We take an identification $\nu c_{k_i} \cong J \times S^1$ as we take in Step 2 of the construction of ω. The attaching map φ_{k_i} of the 2-handle $h_{k_i}^2$ satisfies $\varphi_{k_i}(w_1, w_2) = (\varepsilon_2 w_2 w_1, w_1)$. Since the manifold H is obtained by eliminating the corner of $\tilde{h}_{k_i}^2$, the attaching map of H_1 is described as follows:

$$\Phi: \quad \partial H_1 \supset \quad D^2 \times \partial D^2 \quad \longrightarrow \quad J \times S^1$$
$$\cup \qquad\qquad\qquad\qquad \cup$$
$$(w_1, w_2) \quad \longmapsto \quad (\varepsilon_2 w_2{}^2 w_1, w_2).$$

For an element $(z_1, z_2) \in J \times S^1$, the image $\overline{\omega}(z_1, z_2)$ is equal to $(z_1, -z_2)$. Thus, the manifold $J \times S^1 / \overline{\omega}$ is diffeomorphic to $J \times S^1$ and the quotient map $/\overline{\omega} : J \times S^1 \to J \times S^1 / \overline{\omega} \cong J \times S^1$ satisfies the equality $/\overline{\omega}(z_1, z_2) = (z_1, z_2{}^2)$. The attaching map $\tilde{\Phi} : D^2 \times \partial D^2 \to J \times S^1$ of $\tilde{H}_1$ satisfies the equality $\tilde{\Phi}(w_1, w_2) = (\varepsilon_2 w_2 w_1, w_2)$. It is easy to see that the attaching circle of $\tilde{H}_1$ is equal to the circle $c_{k_i}/\overline{\omega}$. Moreover, the framing of $\tilde{\Phi}$ is -1 relative to the framing along $\{*\} \times S^2 \subset \partial D^2 \times S^2$.

By the definition of $\tilde{\Psi}$, the attaching circle of $\tilde{H}_2$ is equal to the belt circle of $\tilde{H}_1$, which is isotopic to the meridian of the attaching circle of $\tilde{H}_1$. In particular, there exists the natural framing of the attaching circle of $\tilde{H}_2$ which is represented by the meridian of the attaching circle of $\tilde{H}_1$ parallel to the attaching circle of $\tilde{H}_2$. Since the Euler number of $\tilde{H}$ as a D^2-bundle is equal to -2, the framing of the attaching map $\tilde{\Psi}$ is equal to -2 relative to the natural framing. Therefore, we can draw a Kirby diagram of $(D^2 \times \Sigma_g \cup_{\varphi_i} \tilde{h}_{k_i}^2)/\overline{\omega}$ as shown

in Figure 4.3.1. It is obvious that this manifold is diffeomorphic to $D^2 \times S^2 \# 2\overline{\mathbb{CP}^2}$ and this completes the proof of Lemma 4.3.5. $\qquad\square$
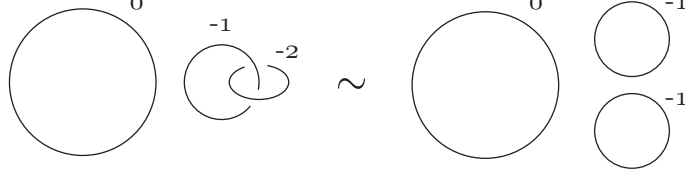

Figure 4.3.1: the (-1)-framed knot describes $\tilde{H}_1$, while the (-2)-framed knot describes $\tilde{H}_2$.

By applying the arguments in Lemma 4.3.4 and 4.3.5 successively, we can prove that $X_h \# s\overline{\mathbb{CP}^2}/\varpi$ is diffeomorphic to $D^2 \times S^2 \# 2s\overline{\mathbb{CP}^2}$.

Lemma 4.3.6. $((X_h \cup X_r) \# s\overline{\mathbb{CP}^2})/\varpi \cong D^2 \times S^2 \# 2s\overline{\mathbb{CP}^2}$.

Proof of Lemma 4.3.6. We can decompose R^2 into two components as follows:

$$R^2 = \left[0, \frac{1}{2}\right] \times D^2 \times D^1 \cup \left[\frac{1}{2}, 1\right] \times D^2 \times D^1.$$

Denote $\left[0, \frac{1}{2}\right] \times D^2 \times D^1$ and $\left[\frac{1}{2}, 1\right] \times D^2 \times D^1$ by R_1 and R_2, respectively. It is easy to see that R_i/ϖ is diffeomorphic to D^4 and R_i is the double covering of $D^4 \cong R_i/\varpi$ branched at the unknotted 2-disk.

The attaching region of R_1 is equal to $[0, \frac{1}{2}] \times \partial D^2 \times D^1$. The quotient $[0, \frac{1}{2}] \times \partial D^2 \times D^1/\varpi$ is a 3-ball in $\partial D^4 \cong \partial R_1$. Thus, we obtain:

$$(X_h \cup R_1)/\varpi \cong X_h/\varpi \cup R_1/\varpi$$
$$\cong D^2 \times S^2 \# 2s\overline{\mathbb{CP}^2} \natural D^4$$
$$\cong D^2 \times S^2 \# 2s\overline{\mathbb{CP}^2}.$$

The attaching region of R_2 is equal to $\left[\frac{1}{2}, 1\right] \times \partial D^2 \times D^1 \cup \{\frac{1}{2}, 1\} \times D^2 \times D^1$. The quotient $\left[\frac{1}{2}, 1\right] \times \partial D^2 \times D^1/\varpi$ is a 3-ball D_0 in $\partial D^4 \cong \partial R_2$, while $\{\frac{1}{2}, 1\} \times D^2 \times D^1/\varpi$ is a disjoint union of two 3-balls $D_1 \amalg D_2$ in ∂D^4. Both of the intersections $D_0 \cap D_1$ and $D_0 \cap D_2$ are 2-disks in ∂D_0. Eventually, the attaching region of R_2 is a 3-ball in ∂D^4. Thus, we can prove $(X_h \cup R_1 \cup R_2)/\varpi$ is diffeomorphic to $D^2 \times S^2 \# 2s\overline{\mathbb{CP}^2}$. This completes the proof of Lemma 4.3.6. $\qquad\square$

It is easy to see that $D^2 \times \Sigma_{g-1}/\varpi$ is diffeomorphic to $D^2 \times S^2$, and is attached to $(X_h \cup X_r)/\varpi$ so that the following diagram commutes:

$$(X_h \cup X_r)/\varpi \supset \quad S^1 \times S^2 \quad \longrightarrow \quad \partial D^2 \times S^2 \quad \subset D^2 \times \Sigma_{g-1}/\varpi$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$S^1 \quad \longrightarrow \quad \partial D^2,$$

where the upper horizontal arrow in the diagram represents the attaching map, the lower horizontal arrow represents the identity map, and vertical arrows represent the projection onto the first component (in other word, the attaching map is a bundle map as a S^2-bundle over S^1). In particular, we obtain:

$$X \# s\overline{\mathbb{CP}^2}/\overline{\omega} \cong S \# 2s\overline{\mathbb{CP}^2}.$$

It is obvious that the quotient map $/\overline{\omega} : X \# s\overline{\mathbb{CP}^2} \to S \# 2s\overline{\mathbb{CP}^2}$ is a double branched covering. This completes the proof of the statement (i) in Theorem 4.3.1. $\square$

Proof of (ii) in Theorem 4.3.1. Let $F_h \subset X$ be a regular fiber in the higher side of f. It is easy to see that F_h represents the same rational homology class of X as that represented by F. Let $\omega : X \to X$ be the involution constructed in the proof of (i) in Theorem 4.3.1. If f has no indefinite fold singularities, then the 2-dimensional part of the fixed point set X^ω of the involution ω is an orientable surface and the algebraic intersection number between this part and F_h is equal to $2g + 2$, especially is non-zero. Thus, the statement (ii) in Theorem 4.3.1 holds.

Suppose that f has indefinite fold singularities. We first prove that F_h represents a non-trivial rational homology class of $X_h \cup X_r$. To prove this, we construct an element $\mathcal{S}$ in the group $H_2(X_h \cup X_r, \partial(X_h \cup X_r); \mathbb{Q})$ such that $[F_h] \cdot \mathcal{S} \neq 0$. Let $\tilde{S}$ be the intersection between the 2-dimensional part of X^ω and X_h, which is the union of compact oriented surfaces. We use the notations H_1, c, v_1, v_2 and R^2 as we used in the proof of (i) in Theorem 4.3.1.

Case 1: If the map H_1 preserves the orientation of c and two points v_1 and v_2, then R^2 is untwisted and $\tilde{S} \cap R^2 = \{(s, \pm 1, 0) \in R^2 \mid s \in [0, 1]\}$ is a disjoint union of two circles. We define four annuli A_1, A_2, A_3 and A_4 as follows:

$$\begin{aligned}
A_1 &= \{(s, t, 0) \in R^2 \mid s \in [0, 1], t \in [0, 1]\}, \\
A_2 &= \{(s, t, 0) \in R^2 \mid s \in [0, 1], t \in [-1, 0]\}, \\
A_3 &= \{(s, 0, t) \in R^2 \mid s \in [0, 1], t \in [0, 1]\}, \\
A_4 &= \{(s, 0, t) \in R^2 \mid s \in [0, 1], t \in [-1, 0]\}.
\end{aligned}$$

The union $S = \tilde{S} \cup A_1 \cup A_2 \cup A_3 \cup A_4$ represents the homology class of the pair $(X_h \cup X_r, \partial(X_h \cup X_r))$ after giving suitable orientations to the annuli A_1, A_2, A_3 and A_4. We denote this class by $\mathcal{S}$. It is easy to verify that the intersection number $\mathcal{S} \cdot [F_h]$ is equal to $2g + 2$, especially is non-zero.

Case 2: If the map H_1 preserves the orientation of c but does not preserve two points v_1 and v_2, then R^2 is untwisted and $\tilde{S} \cap R^2 = \{(s, \pm \exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0, 1]\}$ is a circle. We define three annuli A_5, A_6 and A_7 as follows:

$$\begin{aligned}
A_5 &= \{(s, t\exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0, 1], t \in [0, 1]\} \\
&\quad \cup \{(s, -t\exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0, 1], t \in [0, 1]\}, \\
A_6 &= \{(s, 0, t) \in R^2 \mid s \in [0, 1], t \in [0, 1]\}, \\
A_7 &= \{(s, 0, t) \in R^2 \mid s \in [0, 1], t \in [-1, 0]\}.
\end{aligned}$$

The union $S = \tilde{S} \cup A_5 \cup A_6 \cup A_7$ represents the homology class of the pair $(X_h \cup X_r, \partial(X_h \cup X_r))$ after giving suitable orientations to the annuli A_5, A_6 and A_7. We denote this class by $\mathcal{S}$. It is easy to verify that the intersection number $\mathcal{S} \cdot [F_h]$ is equal to $2g + 2$, especially is non-zero.

Case 3: If the map H_1 does not preserve the orientation of c but preserves two points v_1 and v_2, then R^2 is twisted and $\tilde{S} \cap R^2 = \{(s, \pm 1, 0) \in R^2 \mid s \in [0, 1]\}$ is a disjoint union of two circles. We define three annuli A_8, A_9 and A_{10} as follows:

$$
\begin{aligned}
A_8 =& \{(s, t, 0) \in R^2 \mid s \in [0, 1], t \in [0, 1]\}, \\
A_9 =& \{(s, t, 0) \in R^2 \mid s \in [0, 1], t \in [-1, 0]\}, \\
A_{10} =& \{(s, 0, t) \in R^2 \mid s \in [0, 1], t \in [0, 1]\} \\
& \cup \{(s, 0, t) \in R^2 \mid s \in [0, 1], t \in [-1, 0]\}.
\end{aligned}
$$

The union $S = \tilde{S} \cup A_8 \cup A_9 \cup A_{10}$ represents the homology class of the pair $(X_h \cup X_r, \partial(X_h \cup X_r))$ after giving suitable orientations to the annuli A_8, A_9 and A_{10}. We denote this class by $\mathcal{S}$. It is easy to verify that the intersection number $\mathcal{S} \cdot [F_h]$ is equal to $2g + 2$, especially is non-zero.

Case 4: If the map H_1 preserves neither the orientation of c nor two points v_1 and v_2, then R^2 is twisted and $\tilde{S} \cap R^2 = \{(s, \pm \exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0, 1]\}$ is a circle. We define two annuli A_{11} and A_{12} as follows:

$$
\begin{aligned}
A_{11} =& \{(s, t\exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0, 1], t \in [0, 1]\} \\
& \cup \{(s, -t\exp(-\pi\sqrt{-1}s), 0) \in R^2 \mid s \in [0, 1], t \in [0, 1]\}, \\
A_{12} =& \{(s, 0, t) \in R^2 \mid s \in [0, 1], t \in [0, 1]\}, \\
& \cup \{(s, 0, t) \in R^2 \mid s \in [0, 1], t \in [-1, 0]\}.
\end{aligned}
$$

The union $S = \tilde{S} \cup A_{11} \cup A_{12}$ represents the homology class of the pair $(X_h \cup X_r, \partial(X_h \cup X_r))$ after giving suitable orientations to the annuli A_{11} and A_{12}. We denote this class by $\mathcal{S}$. It is easy to verify that the intersection number $\mathcal{S} \cdot [F_h]$ is equal to $2g + 2$, especially is non-zero.

Eventually, we can construct the element $\mathcal{S}$ satisfying the desired condition in any cases. Thus, we complete to prove $[F_h]$ is not trivial in $H_2(X_h \cup X_r; \mathbb{Q})$.

We are now ready to prove the statement (ii) in Theorem 4.3.1. There exists the following exact sequence which is the part of the Meyer-Vietoris exact sequence:

$$
H_2(S^1 \times \Sigma_{g-1}; \mathbb{Q}) \xrightarrow{i_1 \oplus i_2} H_2(X_h \cup X_r; \mathbb{Q}) \oplus H_2(D^2 \times \Sigma_{g-1}; \mathbb{Q}) \xrightarrow{j_1 - j_2} H_2(M; \mathbb{Q}).
$$

Suppose that $(j_1 - j_2)([F_h], 0) = [F_h] = 0$. There exists an element $\mu \in H_2(S^1 \times \Sigma_{g-1}; \mathbb{Q})$ which satisfies the equality $(i_1 \oplus i_2)(\mu) = ([F_h], 0)$. By a Künneth formula, we obtain the following isomorphism:

$$
H_2(S^1 \times \Sigma_{g-1}; \mathbb{Q}) \cong H_2(\Sigma_{g-1}; \mathbb{Q}) \oplus \left(H_1(\Sigma_{g-1}; \mathbb{Q}) \otimes H_1(S^1; \mathbb{Q}) \right).
$$

Since the map $i_2 : H_2(S^1 \times \Sigma_{g-1}; \mathbb{Q}) \to H_2(D^2 \times \Sigma_{g-1}; \mathbb{Q}) \cong H_2(\Sigma_{g-1}; \mathbb{Q})$ is regarded as the projection onto the first component via the above isomorphism, The element μ is contained in $H_1(\Sigma_{g-1}; \mathbb{Q}) \otimes H_1(S^1; \mathbb{Q})$. The involution ω acts on the component $H_2(\Sigma_{g-1}; \mathbb{Q})$ trivially and on the component $H_1(\Sigma_{g-1}; \mathbb{Q}) \otimes H_1(S^1; \mathbb{Q})$ by multiplying -1. Thus, we obtain:

$$
\omega_*(\mu) = -\mu.
$$

The composition $i_1 \circ \omega_*$ is equal to $\omega_* \circ i_1$ since i_1 is induced by the inclusion map. Thus, we obtain:

$$
\begin{aligned}
[F_h] &= \omega_*([F_h]) \\
&= \omega_* \circ i_1(\mu) \\
&= i_1 \circ \omega_*(\mu) \\
&= i_1 \circ (-\mu) = -[F_h].
\end{aligned}
$$

This means that $2[F_h] = 0$ in $H_2(X_h \cup X_r; \mathbb{Q})$. This contradicts $[F_h] \neq 0$. Therefore, we obtain $[F_h] \neq 0$ in $H_2(M; \mathbb{Q})$ and this completes the proof of the statement. $\qquad\square$

Remark 4.3.7. By the argument similar to that in the proof of Theorem 4.3.1, we can generalize Theorem 4.3.1 to DBLFs as follows:

Theorem 4.3.8. *Let $f : X \to S^2$ be an HDBLF. Suppose that the genus of every connected component of fiber of f is greater than or equal to 2.*

(i) Let s_1 be the number of Lefschetz singularities of f whose vanishing cycles are separating. We define s_2 as follows:

$$
s_2 = max\{s \in \mathbb{N} \mid f^{-1}(x) \text{ has } s \text{ components. } x \in S^2\}.
$$

Then, there exists an involution

$$
\omega : X \to X
$$

such that the fixed point set of ω is the union of (possibly non-orientable) surfaces and s_1 isolated points. Moreover, the involution ω can be extended to an involution

$$
\overline{\omega} : X \# s_1 \overline{\mathbb{CP}^2} \to X \# s_1 \overline{\mathbb{CP}^2}
$$

such that $X \# s_1 \overline{\mathbb{CP}^2} / \overline{\omega}$ is diffeomorphic to $\# s_2 S \# 2s_1 \overline{\mathbb{CP}^2}$, where S is S^2-bundle over S^2, and the quotient map

$$
/\overline{\omega} : X \# s_1 \overline{\mathbb{CP}^2} \to X \# s_1 \overline{\mathbb{CP}^2} / \overline{\omega} \cong \# s_2 S \# 2s_1 \overline{\mathbb{CP}^2}
$$

is the double branched covering.

(ii) Let $F \in X$ be a regular fiber of f. Then F represents a non-trivial rational homology class of X.

We leave the details of the proof of Theorem 4.3.8 to the readers.

4.4   A generating set of $\mathcal{H}_g(c)$

In this section, we investigate the abelianization and a generating set of the group $\mathcal{H}_g(c)$. In the last paragraphs of Subsection 4.4.1 and Subsection 4.4.2, we will prove the following proposition:

Proposition 4.4.1. *Assume that g is greater than or equal to 1.*

1. *Let c be a non-separating simple closed curve of type* I *in Figure 2.3.1. The group $\mathcal{H}_g(c)$ is generated by*

$$\{t_{c_1}, \cdots, t_{c_{2g-1}}, t_{c_{2g+1}}, \iota_g\}.$$

2. *Let $1 \leq h \leq g - 1$, and c a separating simple closed curve of type* II_h *in Figure 2.3.1. The group $\mathcal{H}_g(c^{\mathrm{ori}})$ is generated by*

$$\{t_{c_1}, t_{c_2}, \cdots, t_{c_{2h}}, t_{c_{2h+2}}, t_{c_{2h+3}}, \cdots, t_{c_{2g+1}}\}.$$

4.4.1   When c is non-separating

First, consider the case when c is type I. For simplicity, we choose c as in Figure 2.3.1. Let $\gamma \in \Sigma_g / \langle \iota_g \rangle$ be the projection of the curve c by $p : \Sigma_g \to \Sigma_g / \langle \iota_g \rangle$. Identifying $\Sigma_g / \langle \iota_g \rangle$ with S^2, define a group $\mathcal{M}_0^{2g}(\gamma)$ by

$$\mathcal{M}_0^{2g}(\gamma) = \{[T] \in \mathcal{M}_0^{2g+2} \,|\, T(\gamma) = \gamma\}.$$

For a diffeomorphism $T \in C(\iota_g)$, we have a diffeomorphism

$$\bar{T} \in \mathrm{Diff}_+(S^2, p_1, p_2, \cdots, p_{2g+1}, p_{2g+2})$$

defined by $pT = \bar{T}p$ as in Section 4.1.1. Moreover, if $T \in C(\iota_g)$ preserves c setwise, $\bar{T}$ also preserves the path γ setwise. Hence, the image $\mathcal{P}_g(\mathcal{H}_g^s(c))$ is contained in $\mathcal{M}_0^{2g}(\gamma)$. Conversely, if $\bar{T} \in \mathrm{Diff}_+(S^2, p_1, p_2, \cdots, p_{2g+1}, p_{2g+2})$ preserves the path γ setwise, there is a diffeomorphism $T \in C(\iota_g)$ such that $T(c) = c$ and $pT = \bar{T}p$. Thus, we have $\mathcal{P}_g(\mathcal{H}_g^s(c)) = \mathcal{M}_0^{2g}(\gamma)$. Consider the exact sequence obtained by restricting the homomorphism $\mathcal{P}_g : \mathcal{H}_g^s \to \mathcal{M}_0^{2g+2}$ in Theorem 4.1.2 to $\mathcal{H}_g^s(c)$.

Lemma 4.4.2. *For $g \geq 1$, the exact sequence*

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathcal{H}_g^s(c) \xrightarrow{\mathcal{P}_g} \mathcal{M}_0^{2g}(\gamma) \longrightarrow 1$$

splits. In particular, we have $\mathcal{H}_g^s(c) \cong \mathbb{Z}/2\mathbb{Z} \times \mathcal{M}_0^{2g}(\gamma)$.

Proof. Define a map $\lambda : \mathcal{H}_g^s(c) \to \mathbb{Z}/2\mathbb{Z}$ by $\lambda(\varphi) = 0$ if $\varphi_*[c] = [c] \in H_1(\Sigma_g; \mathbb{Z})$, and $\lambda(\varphi) = 1$ if $\varphi_*[c] = -[c] \in H_1(\Sigma_g; \mathbb{Z})$. Then, λ is a homomorphism, and satisfies $\lambda([\iota_g]) = 1 \in \mathbb{Z}/2\mathbb{Z}$. Thus, it induces a splitting of the exact sequence. $\qquad\square$

Let $s : \partial D^2 \to \partial D^2$ denote the half-rotation of the circle. Let $\mathcal{M}_{0,\mathrm{half}}^{2g}$ denote the group which consists of the path-connected components of $\{T \in \mathrm{Diff}_+(D^2, p_1, p_2, \cdots, p_{2g}) \,|\, T|_{\partial D^2} = s \text{ or } \mathrm{id}_{\partial D^2}\}$.

Lemma 4.4.3. *Assume that g is greater than or equal to 1. The group $\mathcal{M}_0^{2g}(\gamma)$ is isomorphic to $\mathcal{M}_{0,\mathrm{half}}^{2g}$.*

Proof. Let $\mathcal{M}_0^{2g}(\gamma^{\mathrm{ori}})$ be a subgroup of $\mathcal{M}_0^{2g}(\gamma)$ consists of mapping classes which preserve the orientation of the path γ. First, we prove the isomorphism

$$\mathcal{M}_0^{2g}(\gamma^{\mathrm{ori}}) \cong \mathcal{M}_{0,1}^{2g}.$$

Let $\mathrm{Diff}_+(S^2, \{p_1, \cdots, p_{2g+2}\}, [\gamma])$ be the group which consists of orientation-preserving diffeomorphisms $T : S^2 \to S^2$ satisfying the following conditions:

- $T(\{p_1, \cdots, p_{2g+2}\}) = \{p_1, \cdots, p_{2g+2}\}$;

- there exists a closed neighborhood $\nu(\gamma)$ of γ where $T|_{\nu(\gamma)}$ is the identity map.

Let T be a representative of a mapping class in $\mathcal{M}_0^{2g}(\gamma^{\mathrm{ori}})$. Using the isotopy extension theorem, we can change T into a diffeomorphism satisfying the conditions above by some isotopy. Moreover, we can also prove that

$$\mathcal{M}^{2g}(\gamma^{\mathrm{ori}}) \cong \pi_0 \operatorname{Diff}_+(S^2, \{p_1, \cdots, p_{2g+2}\}, [\gamma]),$$

using the isotopy extension theorem. Similarly, we denote by

$$\operatorname{Diff}_+(S^2 - \operatorname{Int} D^2, p_1, \cdots, p_{2g}, [\partial D^2])$$

a group of orientation-preserving diffeomorphisms $T : S^2 - \operatorname{Int} D^2 \to S^2 - \operatorname{Int} D^2$ such that there exists a closed neighborhood $\nu(\partial D^2)$ on which $T|_{\nu(\partial D^2)}$ is the identity map. We can also show that

$$\mathcal{M}_{0,1}^{2g} \cong \pi_0 \operatorname{Diff}_+(S^2 - \operatorname{Int} D^2, p_1, \cdots, p_{2g}, [\partial D^2]).$$

Separate the circle ∂D^2 into two arcs $\alpha : [0,1] \to \partial D^2$ and $\beta : [0,1] \to \partial D^2$ such that $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. If we identify $\alpha(t)$ and $\beta(t)$ in $S^2 - \operatorname{Int} D^2$, the quotient space is diffeomorphic to S^2. Choose an identification L of the $(2g+3)$-tuples

$$(S^2 - \operatorname{Int} D^2/(\alpha(t) \sim \beta(t)), p_1, \cdots, p_{2g}, \alpha(0), \alpha(1))$$
$$\cong (S^2, p_1, \cdots, p_{2g}, p_{2g+1}, p_{2g+2}).$$

Since a diffeomorphism $T \in \operatorname{Diff}_+(S^2 - \operatorname{Int} D^2)$ satisfying $T|_{\nu(\partial D^2)} = \operatorname{id}_{\nu(\partial D^2)}$ induces a diffeomorphism $\bar{T}$ of $S^2 - \operatorname{Int} D^2/(\alpha(t) \sim \beta(t))$, we have the isomorphism $\mathcal{M}_{0,1}^{2g} \cong \mathcal{M}^{2g}(\gamma^{\mathrm{ori}})$ defined by $[T] \mapsto [L\bar{T}L^{-1}]$.

Next, we prove $\mathcal{M}_0^{2g}(\gamma) \cong \mathcal{M}_{0,\mathrm{half}}^{2g}$. Choose a diffeomorphism $r \in \operatorname{Diff}_+(S^2 - \operatorname{Int} D^2)$ such that $r\alpha(t) = \beta(1-t)$ and $r(\{p_1, \cdots, p_{2g}\}) = \{p_1, \cdots, p_{2g}\}$. It induces a diffeomorphism $\bar{r} \in \operatorname{Diff}_+ S^2$ such that $\bar{r}(\{p_1, \cdots, p_{2g}\}) = \{p_1, \cdots, p_{2g}\}$, $\bar{r}(p_{2g+1}) = p_{2g+2}$, and $\bar{r}(p_{2g+2}) = p_{2g+1}$. Consider the group consisting of diffeomorphisms T of S^2 such that $T(\{p_1, \cdots, p_{2g+2}\}) = \{p_1, \cdots, p_{2g+2}\}$, and $T|_{\nu(\gamma)}$ is equal to $\bar{r}|_{\nu(\gamma)}$ or $\operatorname{id}_{\nu(\gamma)}$ for some closed neighborhood $\nu(\gamma)$ instead of $\operatorname{Diff}_+(S^2, \{p_1, \cdots, p_{2g+2}\}, [\gamma])$. In the same way, consider the group consisting of diffeomorphisms T of $S^2 - \operatorname{Int} D^2$ such that $T(\{p_1, \cdots, p_{2g}\}) = \{p_1, \cdots, p_{2g}\}$, and $T|_{\nu(\partial D^2)}$ is equal to $r|_{\nu(\partial D^2)}$ or $\operatorname{id}_{\nu(\partial D^2)}$ instead of the group

$$\operatorname{Diff}_+(S^2 - \operatorname{Int} D^2, p_1, \cdots, p_{2g}, [\partial D^2]).$$

Then, we have the isomorphism between their path-connected components, similarly. Thus, we have $\mathcal{M}^{2g}(\gamma) \cong \mathcal{M}_{0,\mathrm{half}}^{2g}$. $\qquad\square$

We can define a homomorphism $\mathcal{M}_{0,\mathrm{half}}^{2g} \to \langle s \rangle$ by mapping $[T]$ to $T|_{\partial D^2}$, where $\langle s \rangle$ is the cyclic group of order 2 generated by s. Then, the kernel is the subgroup $\mathcal{M}_{0,1}^{2g}$.

Lemma 4.4.4. *For $g \geq 1$, the exact sequence*

$$1 \longrightarrow \mathcal{M}_{0,1}^{2g} \longrightarrow \mathcal{M}_{0,\mathrm{half}}^{2g} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

splits.

Proof. We may assume $p_1, \cdots, p_{2g}$ are arranged in the disk as in Figure 4.4.1. Consider an involution $\mu \in \mathrm{Diff}_+(D^2, p_1, \cdots, p_{2g})$ which rotates the disk 180 degrees and interchanges the points p_i and p_{g+i} for $i = 1, \cdots, g$. Define a homomorphism $j : \mathbb{Z}/2\mathbb{Z} \to \mathcal{M}_{0,\mathrm{half}}^{2g}$ by $j(1) = \mu$.
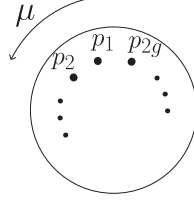

Figure 4.4.1: $p_1, \cdots, p_{2g}$ in D^2

This induces the splitting of the above exact sequence. $\qquad\qquad\square$

Lemma 4.4.5. *Let $g \geq 1$, and c a non-separating simple closed curve such that $\iota_g(c) = c$. Then, we have*

$$H_1(\mathcal{H}_g^s(c); \mathbb{Z}) = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2.$$

Proof. By Lemma 4.4.2 and Lemma 4.4.4, we have

$$H_1(\mathcal{H}_g^s(c); \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus H_1(\mathcal{M}_0^{2g}(\gamma); \mathbb{Z}), \ \ H_1(\mathcal{M}_{0,\mathrm{half}}^{2g}; \mathbb{Z}) \cong H_1(\mathcal{M}_{0,1}^{2g}; \mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}.$$

We showed $\mathcal{M}_0^{2g}(\gamma) \cong \mathcal{M}_{0,\mathrm{half}}^{2g}$ in Lemma 4.4.3, and it is known that $H_1(\mathcal{M}_{0,1}^{2g}; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ (see, for example, Section 9.1.3 and 9.2 of [10]). Hence, we have $H_1(\mathcal{H}_g^s(c); \mathbb{Z}) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$. $\qquad\square$

Consider the case when $g = 1$. As is well-known, the group $\mathcal{H}_1$ coincides with $\mathcal{M}_1$. Hence, $\mathcal{H}_1(c)$ also coincides with $\mathcal{M}_1(c)$. If $c = c_3$ in Figure 4.1.2, the group $\mathcal{M}_1(c)$ is described as

$$\mathcal{M}_1(c) = \left\{ \begin{pmatrix} \epsilon & n \\ 0 & \epsilon \end{pmatrix} \in \mathrm{SL}(2; \mathbb{Z}) \ \middle| \ \epsilon \in \{\pm 1\}, n \in \mathbb{Z} \right\}.$$

By mapping $[T] \in \mathcal{M}_1(c)$ to $\epsilon \in \mathbb{Z}/2\mathbb{Z}$, we have a split exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{M}_1(c) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$

Thus, we have $H_1(\mathcal{H}_1(c); \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Combining Lemma 4.4.5, Lemma 4.2.1, and the case when $g = 1$ as above, we have:

Lemma 4.4.6. *Let c be a non-separating simple closed curve such that $\iota_g(c) = c$. Then, we have*

$$H_1(\mathcal{H}_g(c); \mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2 & \text{when } g \geq 2, \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{when } g = 1. \end{cases}$$

Proof of Proposition 4.4.1 (i). Let $\sigma \in \mathcal{M}_{0,\mathrm{half}}^{2g}$ denote the half twist along ∂D^2. By the exact sequence in Lemma 4.4.4, the group $\mathcal{M}_{0,\mathrm{half}}^{2g}$ is generated by $\{\sigma_1, \cdots, \sigma_{2g-1}, \sigma\}$. By Theorem 2 of [7], we have $\mathcal{P}_g(t_{c_i}) = \sigma_i$ for $i = 1, \cdots, 2g$ and $\mathcal{P}_g(t_{c_{2g+1}}) = \sigma$. By the exact sequence in Lemma 4.4.2, the group $\mathcal{H}_g(c)$ is generated by t_{c_i} for $i = 1, 2, \cdots, 2g - 1, 2g + 1$ and ι_g. $\qquad\square$

4.4.2 When c is separating

Next, consider the case when c is type II_h. For simplicity, we choose c as in Figure 2.3.1.

As we will see in Section 4.5.1, when the vanishing cycle of Z_i in the hyperelliptic directed BLF is separating, the image of the monodromy representation along $\partial_0 A_i$ is contained in $\mathcal{H}_g(c^{\mathrm{ori}})$. Hence, we only consider the group $\mathcal{H}_g(c^{\mathrm{ori}})$ in this section instead of $\mathcal{H}_g(c)$. Of course, if $g \neq 2h$, we have $\mathcal{H}_g(c) = \mathcal{H}_g(c^{\mathrm{ori}})$ since any diffeomorphism of Σ_g which preserves c setwise acts trivially on $\pi_0(\Sigma_g - c)$.

First, consider the case when $h = 0, g$. For any diffeomorphism T of Σ_g, we can change T so that it preserves c setwise by some isotopy. Thus, we have $\mathcal{H}_g(c^{\mathrm{ori}}) = \mathcal{H}_g$.

In the following, we only consider the case $1 \leq h \leq g-1$. Choose a disk D in $\Sigma_g - \bigcup_{i=1}^{2g} c_i$ so that $\iota_g(D) = D$, where c_i is the simple closed curve in Figure 4.1.2. Denote by $\Sigma_{g,1}$ the subsurface $\Sigma_g - \mathrm{Int}\, D$, and by $\iota_{g,1}$ the restriction of ι_g to $\Sigma_{g,1}$. The mapping class group $\mathcal{M}_{g,1}$ of $\Sigma_{g,1}$ is defined by $\mathcal{M}_{g,1} = \pi_0 \mathrm{Diff}_+(\Sigma_{g,1}, \partial\Sigma_{g,1})$, where $\mathrm{Diff}_+(\Sigma_{g,1}, \partial\Sigma_{g,1})$ is the diffeomorphism group of $\Sigma_{g,1}$ with C^∞ topology which fixes the boundary pointwise.

We identify the subsurfaces of Σ_g bounded by c with $\Sigma_{h,1}$ and $\Sigma_{g-h,1}$ so that $\iota_g|_{\Sigma_{h,1}} = \iota_{h,1}$ and $\iota_g|_{\Sigma_{g-h,1}} = \iota_{g-h,1}$. For $T_1 \in \mathrm{Diff}_+(\Sigma_{h,1}, \partial\Sigma_{h,1})$ and $T_2 \in \mathrm{Diff}_+(\Sigma_{g-h,1}, \partial\Sigma_{g-h,1})$, the diffeomorphism $T_1 \cup T_2 \in \mathrm{Diff}_+\Sigma_g$ preserves the curve c. Hence, we can define a map

$$\Psi : \mathcal{M}_{h,1} \times \mathcal{M}_{g-h,1} \to \mathcal{M}_g(c^{\mathrm{ori}})$$

by $\Psi([T_1], [T_2]) = [T_1 \cup T_2]$. This is a well-defined homomorphism.

Define a subgroup $\mathcal{H}_{g,1}$ of $\mathcal{M}_{g,1}$ by $\mathcal{H}_{g,1} = \{[T] \in \mathcal{M}_{g,1} \,|\, \iota_{g,1} T \iota_{g,1}^{-1} = T\}$. Apparently, the image $\Psi(\mathcal{H}_{h,1} \times \mathcal{H}_{g-h,1})$ is contained in the subgroup $\mathcal{H}_g(c^{\mathrm{ori}}) \subset \mathcal{M}_g(c^{\mathrm{ori}})$.

Lemma 4.4.7. *Let $g \geq 2$. When $1 \leq h \leq g-1$, the sequence*

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{H}_{h,1} \times \mathcal{H}_{g-h,1} \xrightarrow{\Psi} \mathcal{H}_g(c^{\mathrm{ori}}) \longrightarrow 1$$

is exact.

Proof. By Theorem 3.18 in [10], we have

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{M}_{h,1} \times \mathcal{M}_{g-h,1} \xrightarrow{\Psi} \mathcal{M}_g(c^{\mathrm{ori}}) \longrightarrow 1.$$

The kernel of Ψ is generated by $(t_{\partial\Sigma_1}, t_{\partial\Sigma_2}^{-1})$, and it is contained in $\mathcal{H}_{h,1} \times \mathcal{H}_{g-h,1}$. Thus, we only need to prove $\Psi(\mathcal{H}_{h,1} \times \mathcal{H}_{g-h,1}) = \mathcal{H}_g(c^{\mathrm{ori}})$.

Let φ be a mapping class in $\mathcal{H}_g(c)$. By Lemma 4.2.1, we can choose a representative $T \in \mathrm{Diff}_+ \Sigma_g$ of φ satisfying $T\iota_g = \iota_g T$ and $T(c) = c$. Using some isotopy, we may assume $T|_c$ is the identity map. Then, $T|_{\Sigma_{h,1}}$ and $T|_{\Sigma_{g-h,1}}$ represent mapping classes in $\mathcal{H}_{h,1}$ and $\mathcal{H}_{g-h,1}$, respectively. Since $\Psi([T|_{\Sigma_{h,1}}], [T|_{\Sigma_{g-h,1}}]) = [T]$, we obtain $\Psi(\mathcal{H}_{h,1} \times \mathcal{H}_{g-h,1}) = \mathcal{H}_g(c^{\mathrm{ori}})$. $\square$

We define a group $C(\iota_{g,1})$ as follows:

$$C(\iota_{g,1}) = \{T \in \mathrm{Diff}_+(\Sigma_{g,1}, \partial\Sigma_{g,1}) \,|\, \iota_{g,1} T \iota_{g,1}^{-1} = T\}.$$

We have the homomorphism $\mathcal{P}_{g,1} : \pi_0(C(\iota_{g,1})) \to \mathcal{M}_{0,1}^{2g+1}$ defined by $[T] \mapsto [\bar{T}]$ in the same way as $\mathcal{P}_g : \mathcal{H}_g^s \to \mathcal{M}_0^{2g+2}$ in Subsection 4.1.1. Since any isotopy of $\mathrm{Diff}_+(D^2, \partial D^2, \{p_1, \cdots, p_{2g+1}\})$

can be lifted to an isotopy of $C(\iota_{g,1})$, $\mathrm{Ker}(\mathcal{P}_{g,1})$ is represented by the deck transformation $\iota_{g,1}$ or $\mathrm{id}_{\Sigma_{g,1}}$. Since $C(\iota_{g,1})$ does not contain $\iota_{g,1}$, the kernel of the homomorphism $\mathcal{P}_{g,1}$ is trivial. Furthermore, $\mathcal{P}_{g,1} : \pi_0 C(\iota_{g,1}) \to \mathcal{M}_{0,1}^{2g+1}$ is an isomorphism since $\mathcal{M}_{0,1}^{2g+1}$ is generated by $\{\sigma_i\}_{i=1}^{2g}$ and $\mathcal{P}_{g,1}(t_{c_i}) = \sigma_i$ for $i = 1, \cdots, 2g$.

Lemma 4.4.8. *For $g \geq 1$, the natural homomorphism $\pi_0 C(\iota_{g,1}) \to \mathcal{H}_{g,1}$ is an isomorphism.*

Proof. By the definition of $\mathcal{H}_{g,1}$, the natural homomorphism $\pi_0(C(\iota_{g,1})) \to \mathcal{H}_{g,1}$ is surjective. Hence, it suffices to show the injectivity.

Embed $\Sigma_{g,1}$ in Σ_{g+1} so that $\iota_{g+1}|_{\Sigma_{g,1}} = \iota_{g,1}$. For a diffeomorphism T of $\Sigma_{g,1}$, we can extend T to a diffeomorphism $\tilde{T}$ of Σ_{g+1} by the identity map on $\Sigma_{g+1} \setminus \Sigma_{g,1}$. Thus, we have homomorphisms $\pi_0(C(\iota_{g,1})) \to \pi_0(C(\iota_{g+1}))$ and $\mathcal{H}_{g,1} \to \mathcal{H}_{g+1}$ defined by $[T] \mapsto [\tilde{T}]$. By gluing a disk with three marked points to D^2, we can also define a homomorphism $\mathcal{M}_{0,1}^{2g+1} \to \mathcal{M}_0^{2g+4}$ in the same way. By Theorem 3.18 in [10], the latter homomorphism is injective.

If we consider $(\Sigma_{g+1} \setminus \mathrm{Int}\,\Sigma_{g,1})/\langle \iota_{g+1} \rangle$ as a disk with three marked points, we have a commutative diagram

$$
\begin{array}{ccccc}
\mathcal{M}_{0,1}^{2g+1} & \xleftarrow{\;\mathcal{P}_{g,1}\;} & \pi_0 C(\iota_{g,1}) & \longrightarrow & \mathcal{H}_{g,1} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M}_0^{2g+4} & \xleftarrow{\;\mathcal{P}_{g+1}\;} & \pi_0 C(\iota_{g+1}) & \longrightarrow & \mathcal{H}_{g+1}.
\end{array}
$$

The left side shows that $\pi_0 C(\iota_{g,1}) \to \pi_0 C(\iota_{g+1})$ is injective. By Theorem 4.1.1, the right side shows that $\pi_0 C(\iota_{g,1}) \to \mathcal{H}_{g,1}$ is also injective. $\qquad\square$

Lemma 4.4.9. *Let $g \geq 2$ and $1 \leq h \leq g-1$. Let c be a separating simple closed curve which bounds subsurfaces of genus h and $g-h$ and satisfies $\iota_g(c) = c$. Then, we have*

$$
H_1(\mathcal{H}_g(c^{\mathrm{ori}}); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z},
$$

where $d = \gcd(4h(2h+1), 4(g-h)(2g-2h+1))$.

Proof. Since $\mathcal{H}_{h,1} \cong \mathcal{M}_{0,1}^{2h+1}$, we have $H_1(\mathcal{H}_{h,1}; \mathbb{Z}) \cong \mathbb{Z}$. By the chain relation (see, for example, Proposition 4.12 in [10]), the mapping class $(t_{c_1} \cdots t_{c_{2h}})^{4h+2} \in \mathcal{H}_{h,1}$ coincides with the Dehn twist $t_{\partial \Sigma_{h,1}}$ along the boundary. In the same way, we have $(t_{c_1} \cdots t_{c_{2(g-h)}})^{4(g-h)+2} = t_{\partial \Sigma_{g-h,1}} \in \mathcal{H}_{g-h,1}$.

The kernel of the homomorphism $\mathcal{H}_{h,1} \times \mathcal{H}_{g-h,1} \to \mathcal{H}_g(c^{\mathrm{ori}})$ is the cyclic group generated by $(t_{\partial \Sigma_{h,1}}, t_{\partial \Sigma_{g-h,1}}^{-1})$. Hence, we have

$$
\begin{aligned}
& H_1(\mathcal{H}_g(c^{\mathrm{ori}}); \mathbb{Z}) \\
\cong\; & (\mathbb{Z} \oplus \mathbb{Z})/\langle (4h(2h+1), -4(g-h)(2g-2h+1)) \rangle \\
\cong\; & \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}.
\end{aligned}
$$

$\qquad\square$

Proof of Proposition 4.4.1 (ii). As explained in the paragraph before Lemma 4.4.8, $\mathcal{H}_{g,1}$ is generated by $t_{c_1}, \cdots, t_{c_{2g}}$. Thus, $\mathcal{H}_g(c)$ is generated by the following set by Lemma 4.4.7:

$$
\{t_{c_i} \mid i = 1, 2, \ldots, 2h, 2h+2, 2h+3, \ldots, 2g+2\}.
$$

This completes the proof of Proposition 4.4.1 (ii). $\qquad\square$

4.5   Localization of the signature

In this section, we generalize Theorem 4.1.9 for a signature formula for HDBLFs. To give the statement of the generalization precisely, we first introduce a homomorphism

$$h_{g,c} : \mathcal{H}_g(c) \to \mathbb{Q},$$

which we will define in Subsection 4.5.3. We will also calculate the value of generators of $\mathcal{H}_g(c)$ given in Proposition 4.4.1 (see Proposition 4.5.6).

   Let $f : X \to S^2$ be an HDBLF. We use the same notations as those we use in Sections 2.1 and 2.4. The boundary $\partial \nu Z_i$ has two components. We denote by $\partial_h \nu Z_i$ the component of $\partial \nu Z_i$ whose preimage contains vanishing cycles of folds (the right side of Z_i in Figure 2.1.1). There is a unique component of $f^{-1}(\partial_h \nu Z_i)$ which contains vanishing cycles of folds. Let g_i be the genus of a fiber in this component and we identify the fiber with Σ_{g_i}. We can regard d_i with a simple closed curve in Σ_{g_i}. Denote by φ_i the restriction of a monodromy of f along $\partial_h \nu Z_i$ to the component Σ_g. By the definition of HDBLFs, this element in contained in the group $\mathcal{H}_{g_i}(d_i)$. The purpose of this section is to prove the following theorem:

Theorem 4.5.1. *Let $f : X \to S^2$ be an HDBLF as above. Then, we have*

$$\text{Sign}\, X = \sum_{i=1}^{m} h_{g_i, d_i}(\varphi_i) + \sum_{j=1}^{l} \sigma_{\text{loc}}(f^{-1}(p_j)).$$

4.5.1   Signatures of round cobordisms

We use the same notation as in Sections 2.1 and 2.4. . Let $f : X \to S^2$ be a DBLF. Lemma 2.6.4 implies that the manifold $f^{-1}(\nu Z_i)$ can be obtained by attaching a round 2-handle to a surface bundle over an annulus. Moreover, when the vanishing cycle d_i is a separating curve, the monodromy φ is in $\mathcal{M}_{g_i}(d_i^{\text{ori}})$. This is because, if φ changes the orientation of d_i, the monodromy along $\partial_h \nu Z_i$ permutes the component of the fiber. Inductively, the monodromy along $\partial_h \nu Z_1$ permutes the component of the fiber. However, since $f^{-1}(D_l)$ is a trivial surface bundle over a disk, the image of this monodromy under the map Φ_{d_1} must be trivial.

   For a mapping class $\varphi \in \mathcal{M}_g(c)$ represented by $T \in \text{Diff}_+ \Sigma_g$ satisfying $T(c) = c$, define a mapping torus V_φ by $V_\varphi = \Sigma_g \times [0,1]/((0, T(x)) \sim (1, x))$. We can identify $f^{-1}(\partial_h \nu Z_i)$ with V_φ for some $\varphi \in \mathcal{M}_{g_i}(d_i)$. Let $R = I \times D^2 \times D^1/\sim$ be a round 2-handle which is untwisted if φ preserves the orientation of d_i and is twisted otherwise. Choose an embedding

$$j : I \times \partial D^2 \times D^1/\sim \to V_\varphi$$

such that $j(0, \partial D^2, 0) = c \times \{0\} \subset V_\varphi$ and $p_1 \circ j(t, x, s) = t$, By Lemma 2.6.4, we can obtain the following diffeomorphism:

$$f^{-1}(\nu Z_i) \cong (V_\varphi \times [0,1]) \cup_j R.$$

Note that the isotopy class of the attaching map $j : I \times \partial D^2 \times D^1/\sim \to V_\varphi \times \{0\}$ is unique if the genus g is greater than or equal to 2. Eventually, we obtain:

Lemma 4.5.2.

$$\text{Sign}\, f^{-1}(\nu Z_i) = \text{Sign}((V_\varphi \times [0,1]) \cup_j R).$$

4.5.2 Wall's non-additivity formula

In [28], Sato defined a class function $m : \mathcal{M}_{g,2} \to \mathbb{QP}^1$. We review this function, and calculate the signature of the compact 4-manifold $(V_\varphi \times [0,1]) \cup_j R$ in Section 4.5.1.

For a mapping class $\varphi = [T] \in \mathcal{M}_{g,2}$, let $V'_\varphi = \Sigma_{g,2} \times [0,1]/(0, T(x)) \sim (1, x)$ be its mapping torus. Choose points x_1 and x_2 in each boundary component of $\Sigma_{g,2}$, and define a continuous map by $l_i : S^1 \to V'_\varphi$ by $l_i(t) = (t, x_i)$ for $i = 1, 2$. Let ∂_1 and ∂_2 be the two boundary components of $\Sigma_{g,2}$. Denote by e_1, e_2, e_3, and e_4 the homology classes $[l_1]$, $[l_2]$, $[\partial_1 \times \{0\}]$, and $[\partial_2 \times \{0\}]$, respectively. Then, for some $p, q \in \mathbb{Q}$, the set $\{e_1 + e_2, p(e_3 - e_4) + q e_1\}$ forms a basis of $\mathrm{Ker}(H_1(\partial V'_\varphi; \mathbb{Q}) \to H_1(V'_\varphi; \mathbb{Q}))$. The element $[p : q] \in \mathbb{QP}^1$ is unique, and we can define a function $m : \mathcal{M}_{g,2} \to \mathbb{QP}^1$ by $m(\varphi) = [p : q]$. Since it satisfies $m(\varphi t_{\partial_1} t_{\partial_2}^{-1}) = m(\varphi)$, it induces the class function on $\mathcal{M}_g(c^{\mathrm{ori}})$. For simplicity, we also denote it by $m : \mathcal{M}_g(c^{\mathrm{ori}}) \to \mathbb{QP}^1$.

Define a map $s : \mathcal{M}_g(c) \to \mathbb{Z}$ by $s(\varphi) = \mathrm{Sign}((V_\varphi \times [0,1]) \cup R)$. We can write the signature $s(\varphi)$ with the function $m : \mathcal{M}_g(c^{\mathrm{ori}}) \to \mathbb{QP}^1$ as follows:

Lemma 4.5.3. *Let $\varphi \in \mathcal{M}_g(c)$. Then, we have*

$$s(\varphi) = \begin{cases} \mathrm{sign}(m(\varphi)), & \textit{if } c : \textit{non-separating}, \varphi \textit{ preserves the orientation of } c, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. We apply Wall's nonadditivity Formula to the pasting of the round 2-handle. First, we review his formula. Let X_-, X_0, and X_+ be compact 3-manifolds, and let Y_- and Y_+ be compact 4-manifolds such that

$$\partial X_- = \partial X_+ = \partial X_+ = Z, \quad \partial Y_- = X_- \cup X_0, \quad \partial Y_+ = X_+ \cup X_0.$$

We denote by Y and X the compact 4-manifold $Y = Y_- \cup Y_+$ and the space $X = X_- \cup X_0 \cup X_+$, respectively. Suppose that Y is oriented inducing orientations of Y_- and Y_+. Orient the other manifolds so that

$$\partial_*[Y_-] = [X_0] - [X_-],$$
$$\partial_*[Y_+] = [X_+] - [X_0],$$
$$\partial_*[X_-] = \partial_*[X_+] = \partial_*[X_0] = [Z].$$

We define vector spaces V, A, B, and C as follows:

$$V = H_1(Z; \mathbb{Q}),$$
$$A = \mathrm{Ker}(H_1(Z; \mathbb{Q}) \to H_1(X_-; \mathbb{Q})),$$
$$B = \mathrm{Ker}(H_1(Z; \mathbb{Q}) \to H_1(X_0; \mathbb{Q})),$$
$$C = \mathrm{Ker}(H_1(Z; \mathbb{Q}) \to H_1(X_+; \mathbb{Q})).$$

On the vector space $W = B \cap (C + A)/((B \cap C) + (B \cap A))$, Wall defined a symmetric bilinear map $\Psi : W \times W \to \mathbb{Q}$ as follows. Let $I : H_1(Z; \mathbb{Q}) \times H_1(Z; \mathbb{Q}) \to \mathbb{Q}$ denote the intersection form, and $b, b' \in B \cap (C + A)$. Since $b' \in B \cap (C + A)$, there exist $c' \in C$ and $a' \in A$ such that $a' + b' + c' = 0$. Then, define a map $\Psi : W \times W \to \mathbb{Q}$ by $\Psi'([b], [b']) = I(b, c')$. He showed that this map is well-defined and symmetric. Denote by $\mathrm{Sign}(V; B, C, A)$ the signature of this symmetric bilinear form. His signature formula is:

Theorem 4.5.4 (Wall [30])**.**

$$\text{Sign}\, Y = \text{Sign}\, Y_- + \text{Sign}\, Y_+ - \text{Sign}(V; B, C, A).$$

Next, we apply his formula to our settings. We should let Y_- and Y_+ denote the manifolds

$$Y_- = R = I \times D^2 \times D^1/\sim \ \text{ and } Y_+ = V_\varphi \times [0,1],$$

respectively. The rest of the manifolds are

$$\partial Y_- = (I \times \partial D^2 \times D^1/\sim) \cup (I \times D^2 \times \partial D^1/\sim),$$
$$\partial Y_+ = (V_\varphi \times \{1\}) \amalg (V_\varphi \times \{0\}),$$
$$X_0 = I \times \partial D^2 \times D^1/\sim, \quad X_- = I \times D^2 \times \partial D^1/\sim,$$
$$X_+ = (V_\varphi \times \{1\}) \amalg (V_\varphi \times \{0\} - \text{Int}\, j(X_0)), \quad Z = I \times \partial D^2 \times \partial D^1/\sim.$$

Consider the case when $T|_{\nu(c)} = id$. Choose a point x in ∂D^2. Define continuous maps $f_i : S^1 \to S^1 \times \partial D^2 \times \partial D^1$ by $f_i(t) = (t, x, (-1)^i)$ for $i = 1, 2$. The set consisting of the homology classes $e_1 = [\partial D^2 \times \{-1\}]$, $e_2 = [\partial D^2 \times \{1\}]$, $e_3 = [f_1]$, and $e_4 = [f_2]$ in $H_1(Z; \mathbb{Q})$ forms a basis.

When c is separating, we have $A = C = \mathbb{Q}e_1 \oplus \mathbb{Q}e_2$. Hence, we obtain $W = (B \cap (C + A))/((B \cap C) + (B \cap A)) = 0$. When c is non-separating, $\text{Sign}(V_\varphi \times [0,1] \cup R)$ is calculated in Lemma 3.4 of [28].

Consider the case when T does not preserve the orientation of c. In this case, the curve c is non-separating. Define a continuous map $f : S^1 \to I \times \partial D^2 \times \partial D^1/\sim$ by

$$f(t) = \begin{cases} (2t, x, -1,) & \text{when } 0 \le t \le \dfrac{1}{2}, \\ (2t-1, x, 1,) & \text{when } \dfrac{1}{2} \le t \le 1. \end{cases}$$

The set of homology classes consisting of $e_1 = [\partial D^2 \times \{-1\}]$ and $e_2 = [f]$ in $H_1(Z; \mathbb{Q})$ forms a basis. In this case, $A = B = \mathbb{Q}e_1$. Hence, we have $W = 0$. □

4.5.3 The homomorphism $h_{g,c}$

Let c be a simple closed curve in Σ_g. Since the neighborhood $\nu(c)$ of c is diffeomorphic to $\partial D^2 \times [-1, 1]$, we obtain a manifold $L(c) = \Sigma_g \times [0,1] \cup_{\nu(c)} (D^2 \times [-1, 1])$ by gluing $D^2 \times [-1, 1]$ along $\nu(c)$. This is diffeomorphic to a fiber of the projection $(V_\varphi \times [0,1]) \cup R \to S^1$. We denote $\tilde{V}_\varphi = (V_\varphi \times [0,1]) \cup R$ in the following.

Let φ and ψ be mapping classes in $\mathcal{M}_g(c)$. For example, by gluing the $L(c)$-bundles $\tilde{V}_\varphi \times [0,1]$ and $\tilde{V}_\psi \times [0,1]$ on an annulus, we obtain a $L(c)$-bundle over $S^2 - \amalg_{i=1}^3 \text{Int}\, D^2$ whose fiberwise boundary is $E_{\varphi,\psi} \amalg -E_{\Phi(\varphi),\Phi(\psi)}$ and the whole boundary is

$$\left(E_{\varphi,\psi} \amalg -E_{\Phi(\varphi),\Phi(\psi)}\right) \cup_{\partial E_{\varphi,\psi} \amalg -\partial E_{\Phi(\varphi),\Phi(\psi)}} \left(-\tilde{V}_\varphi \amalg -\tilde{V}_\psi \amalg -\tilde{V}_{(\varphi\psi)^{-1}}\right).$$

Hence, we have

$$\text{Sign}\, E_{\varphi,\psi} - \text{Sign}\, E_{\Phi(\varphi),\Phi(\psi)} - \text{Sign}\, \tilde{V}_\varphi - \text{Sign}\, \tilde{V}_\psi - \text{Sign}\, \tilde{V}_{(\varphi\psi)^{-1}} = 0.$$

If we rewrite it by Meyer's signature cocycle and the function $s : \mathcal{M}_g(c) \to \mathbb{Z}$, we have

$$-\tau_g(\varphi, \psi) + \Phi^* \tau_{g-1}(\varphi, \psi) - \delta s(\varphi, \psi) = 0 \in C^2(\mathcal{M}_g(c); \mathbb{Z})(c\text{:type I}),$$

$$-\tau_g(\varphi, \psi) + \Phi^*(\tau_h \times \tau_{g-h})(\varphi, \psi) - \delta s(\varphi, \psi) = 0 \in C^2(\mathcal{M}_g(c^{\mathrm{ori}}); \mathbb{Z})(c\text{:type II}_h).$$

If we restrict the Meyer cocycles to $\mathcal{H}_g$, we have $\tau_g = \delta\phi_g \in C^2(\mathcal{H}_g; \mathbb{Q})$, and $\tau_{g-1} = \delta\phi_{g-1} \in C^2(\mathcal{H}_{g-1}; \mathbb{Q})$. Thus, we have proved:

Lemma 4.5.5. *When c is type I, define a function $h_{g,c} : \mathcal{H}_g(c) \to \mathbb{Q}$ by*

$$h_{g,c}(\varphi) = s(\varphi) + \phi_g(\varphi) - \Phi^*\phi_{g-1}(\varphi).$$

When c is type II$_h$, define $h_{g,c} : \mathcal{H}_g(c^{\mathrm{ori}}) \to \mathbb{Q}$ by

$$h_{g,c}(\varphi) = s(\varphi) + \phi_g(\varphi) - \Phi^*(\phi_h \times \phi_{g-h})(\varphi).$$

Then, both of these maps are homomorphisms.

The values of generators of $\mathcal{H}_g(c)$ given in Proposition 4.4.1 under the map $h_{g,c}$ is calculated as follows:

Proposition 4.5.6. *Suppose that the genus g is greater than or equal to 1.*

1. *Let c be a non-separating simple closed curve of type I in Figure 2.3.1. The values of the homomorphism $h_{g,c} : \mathcal{H}_g(c) \to \mathbb{Q}$ are*

$$h_{g,c}(\iota_g) = 0,$$

$$h_{g,c}(t_{c_i}) = -\frac{1}{4g^2 - 1} \quad for\ i = 1 \cdots, 2g - 1,$$

$$h_{g,c}(t_{c_{2g+1}}) = -\frac{g}{2g+1}.$$

2. *Let $0 \leq h \leq g$, and c a separating simple closed curve of type II$_h$ in Figure 2.3.1. When $1 \leq h \leq g - 1$, the values of the homomorphism $h_{g,c} : \mathcal{H}_g(c^{\mathrm{ori}}) \to \mathbb{Q}$ are*

$$h_{g,c}(t_{c_i}) = \frac{g+1}{2g+1} - \frac{h+1}{2h+1} \quad for\ i = 1, \cdots, 2h,$$

$$h_{g,c}(t_{c_i}) = \frac{g+1}{2g+1} - \frac{g-h+1}{2(g-h)+1} \quad for\ i = 2h+2, \cdots, 2g.$$

When $h = 0, g$, the homomorphism $h_{g,c}$ is the zero map.

Proof. First, consider the case when the vanishing cycle c is type I in Figure 2.3.1. Since $h_{g,c}$ is a homomorphism, we have $h_{g,c}(\iota_g) = 0$. The mapping classes t_{c_i} for $i = 1, 2, \cdots, 2g - 1$ are mutually conjugate in $\mathcal{H}_g(c)$. Therefore, we have $h_{g,c}(t_{c_1}) = \cdots = h_{g,c}(t_{c_{2g-1}})$. By the chain relation, we have $(t_{c_1} \cdots t_{c_{2g-1}})^{2g} = t_{c_{2g+1}}^2$. Thus, we obtain $h_{g,c}(t_{c_{2g+1}}) = g(2g-1)h_{g,c}(t_{c_1})$. Hence, it suffices to show that $h_{g,c}(\sigma_{2g+1}) = -g/(2g+1)$.

In Lemma 3.3 of [9], Endo showed that $\phi_g(t_{c_{2g+1}}) = (g+1)/(2g+1)$. Since $\Phi(t_{c_{2g+1}}) = 1 \in \mathcal{M}_{g-1}$, we have $\Phi^*\phi_{g-1}(t_{c_{2g+1}}) = 0$. By Lemma 4.5.3, we have

$$s(t_{c_{2g+1}}) = \mathrm{sign}\, m(t_{c_{2g+1}}) = \mathrm{sign}([1 : -1]) = -1.$$

Thus, we obtain $h_{g,c}(t_{c_{2g+1}}) = -g/(2g+1)$.

Next, consider the case when the vanishing cycle c is type II$_h$ in Figure 2.3.1. When $1 \leq h \leq g - 1$, this follows from Lemma 3.3 of [9] since $s(t_{c_i}) = 0$. When $h = 0, g$, $h_{g,c}$ is the zero map since $H^1(\mathcal{H}_g(c); \mathbf{Q}) = H^1(\mathcal{H}_g; \mathbf{Q}) = 0$. $\qquad\square$

4.5.4   A local signature formula

In this subsection, we prove Theorem 4.5.1.

Proof of Theorem 4.5.1. We prepare the hyperelliptic mapping class group of the non-connected surface $f^{-1}(r_i)$, where the monodromy of it along $\partial_h \nu Z_i$ lies. Identify $f^{-1}(r_i)$ with some standard surface $S_i = \Sigma_{n_i(1)} \amalg \cdots \amalg \Sigma_{n_i(k_i)}$, where $n_i(1), \cdots, n_i(k_i)$ are non-negative integers. We may assume that the action on $f^{-1}(r_i)$ induced by ι_g in Subsection 4.1.2 coincides with $\iota_{n_i(1)} \amalg \cdots \amalg \iota_{n_i(k_i)}$, and the vanishing cycle d_i lies in $\Sigma_{n_i(1)}$ ($n_i(1) = g_i$). Define groups $\mathcal{H}_{S_i}$ and $\mathcal{H}_{S_i}(d_i)$ by $\mathcal{H}_{S_i} = \mathcal{H}_{n_i(1)} \times \cdots \mathcal{H}_{n_i(k_i)}$ and $\mathcal{H}_{S_i}(d_i) = \mathcal{H}_{n_i(1)}(d_i) \times \mathcal{H}_{n_i(2)} \times \cdots \mathcal{H}_{n_i(k_i)}$, respectively. By the definition HDBLFs, the monodromy $\tilde{\varphi}_m$ along $\partial_h \nu Z_m$ is contained in $\mathcal{H}_{S_m}(d_m)$. As stated in Section 4.5.1, the monodromy $\tilde{\varphi}_{m-1}$ of $f^{-1}(r_{m-1})$ along $\partial_h \nu Z_{m-1}$ is the image of $\tilde{\varphi}_m \in \mathcal{H}_{S_m}(d_m)$ under $\Phi_{d_m} : \mathcal{H}_{S_m}(d_m) \to \mathcal{H}_{S_{m-1}}$. By Theorem 2.3.2, it is contained in $\mathcal{H}_{S_{m-1}}(d_{m-1})$. Define a natural homomorphism $\Phi_{S_i} : \mathcal{H}_{S_i}(d_i) \to \mathcal{H}_{S_{i-1}}$ by $\Phi_{S_i}(x_1, x_2, \cdots, x_{k_i}) = (\Phi_{d_i}(x_1), x_2, \cdots, x_{k_i})$, for $i = 1, \cdots, m$. Inductively, the monodromy $\tilde{\varphi}_i$ along $\partial_h \nu Z_i$ is contained in $\mathcal{H}_{S_i}(d_i)$, and $\tilde{\varphi}_{i-1} = \Phi_{S_i}(\tilde{\varphi}_i)$.

By the Novikov additivity, we have

$$\operatorname{Sign} X = \sum_{i=1}^{m} \operatorname{Sign} f^{-1}(\nu Z_i) + \operatorname{Sign} f^{-1}(D_l) + \operatorname{Sign} f^{-1}(D_h)$$

$$= \sum_{i=1}^{m} \operatorname{Sign} f^{-1}(\nu Z_i) + \operatorname{Sign} f^{-1}\left(D_h - \coprod_{j=1}^{l} \operatorname{Int} \nu(p_j)\right)$$

$$+ \sum_{j=1}^{l} \operatorname{Sign} f^{-1}(\nu(p_j)).$$

Define the Meyer function $\phi_{S_i} : \mathcal{H}_{S_i} \to \mathbb{Q}$ by $\phi_{S_i}(x_1, \cdots, x_{k_i}) = \sum_{j=1}^{k_i} \phi_{S_i}(x_j)$. We take a loop a_j around the image $p_j \in D_h$ as in Theorem 2.4.1 Let $\psi_j \in \mathcal{H}_g$ denote the monodromy along the loop a_j.

We denote by M_i the component of $f^{-1}(\partial_h \nu Z_i)$ which contains vanishing cycles of $\widetilde{Z}_i$. By Lemma 4.1.7 and Lemma 4.5.3, we have

$$\operatorname{Sign} X = \sum_{i=1}^{m} s(\varphi_i) + \left(-\phi_g(\tilde{\varphi}_m^{-1}) - \sum_{j=1}^{l} \phi_g(\psi_j)\right) + \sum_{j=1}^{l} \operatorname{Sign} f^{-1}(\nu(p_j)),$$

where φ_i is the monodromy of M_i along $\partial_h \nu Z_i$. Since $f^{-1}(D_l)$ is a trivial bundle, we have $\tilde{\varphi}_0 = 1 \in \mathcal{H}_{S_0}$. Since $\Phi_{S_i}(\tilde{\varphi}_i) = \tilde{\varphi}_{i-1} \in \mathcal{H}_{S_{i-1}}(d_{i-1})$, we have

$$\sum_{i=1}^{m} (\phi_{S_i}(\tilde{\varphi}_i) - \Phi_{S_i}^* \phi_{S_{i-1}}(\tilde{\varphi}_i)) = \phi_g(\tilde{\varphi}_m).$$

Since the Meyer function has the property $\phi_g(\varphi^{-1}) = -\phi_g(\varphi)$ (see [9]) for any $\varphi \in \mathcal{H}_g$, we obtain

$$\operatorname{Sign} X = \sum_{i=1}^{m} \left(s(\varphi_i) + \phi_{S_i}(\tilde{\varphi}_i) - \Phi_{S_i}^* \phi_{S_{i-1}}(\tilde{\varphi}_i)\right)$$

$$+ \sum_{j=1}^{l} (-\phi_g(\psi_j) + \operatorname{Sign} f^{-1}(\nu(p_j))).$$

By the definition of Φ_{S_i}, we have

$$\phi_{S_i}(x_1, \cdots, x_{k_i}) - \Phi^*_{S_{i-1}} \phi_{S_{i-1}}(x_1, \cdots, x_{k_{i-1}})$$

$$= \begin{cases} \phi_{g_i}(x_1) - \Phi^*_{d_i} \phi_{g_i}(x_1), & (d_i\text{:nonseparating}), \\ \phi_{g_i}(x_1) - \Phi^*_{d_i}(\phi_h \times \phi_{g_i - h})(x_1), & (d_i \text{ bounds subsurfaces of genus } h). \end{cases}$$

Thus, we have

$$\operatorname{Sign} X = \sum_{i=1}^{m} h_{g_i, d_i}(\varphi_i) + \sum_{j=1}^{l} \sigma_{\mathrm{loc}}(f^{-1}(p_j)).$$

This completes the proof of Theorem 4.5.1. $\square$

4.5.5   Examples of calculation of signatures

Let $c_1, \ldots, c_{2g+1} \subset \Sigma_g$ be simple closed curves described in Figure 4.1.2.

Example 4.5.7. As shown in the proof of Theorem 1.4 in [15], there exists an SBLF $f_{g,n}$: $X_{g,n} \to S^2$ which has the following Hurwitz cycle system:

$$(c_{2g+1}; (c_{2g}, \ldots, c_2, c_1, c_1, c_2, \ldots, c_{2g})^{2n}).$$

By the definition of $f_{g,n}$, it is hyperelliptic. We denote by $p_1, \ldots, p_{8gn} \in S^2$ the critical values of $f_{g,n}$. By using the formula in Theorem 4.5.1, the signature of $X_{g,n}$ can be calculated as follows:

$$\begin{aligned} \operatorname{Sign} X_{g,n} &= \sum_{i=1}^{8gn} \sigma_{\mathrm{loc}}(f_{g,n}^{-1}(p_i)) + h((t_{c_{2g}} \cdots t_{c_2} t_{c_1}^2 t_{c_2} \cdots t_{c_{2g}})^{2n}) \\ &= 8gn \cdot \frac{-g-1}{2g+1} + h(t_{c_{2g+1}}^{-4n}) \\ &= \frac{-8g^2 n - 8gn}{2g+1} + (-4n) \cdot \frac{-g}{2g+1} \\ &= -4gn. \end{aligned}$$

It is easy to see that $X_{g,n}$ is simply connected and that the Euler characteristic of $X_{g,n}$ is $8gn - 4g + 6$. As shown in [15], $X_{g,n}$ is spin if and only if both of the integers g and n are even. Thus, by Freedman's theorem, $X_{g,n}$ is homeomorphic to $\#\frac{gn}{4}E(2)\#(\frac{5gn}{4} - 2g + 2)S^2 \times S^2$ if both g and n are even and $\#(2gn - 2g + 2)\mathbb{CP}^2\#(6gn - 2g + 2)\overline{\mathbb{CP}^2}$ otherwise.

Example 4.5.8. As shown in the proof of Theorem 1.4 in [15], there exists an SBLF $\tilde{f}_{g,n}$: $\tilde{X}_{g,n} \to S^2$ which has the following Hurwitz cycle system:

$$(c_{2g+1}; (c_{2g}, \ldots, c_2, c_1, c_1, c_2, \ldots, c_{2g})^{2n}, (c_1, \ldots, c_{2g-2})^{2(2g-1)n}).$$

By the definition of $\tilde{f}_{g,n}$, it is hyperelliptic. We denote by $\tilde{p}_1, \ldots, \tilde{p}_{8g^2 n - 4gn + 4n} \in S^2$ the critical values of $\tilde{f}_{g,n}$. By using the formula in Theorem 4.5.1, the signature of $\tilde{X}_{g,n}$ can be

calculated as follows:

$$
\begin{aligned}
\operatorname{Sign} \tilde{X}_{g,n} =\ & \sum_{i=1}^{8g^2n-4gn+4n} \sigma_{\text{loc}}(\tilde{f}_{g,n}^{-1}(\tilde{p}_i)) \\
& + h((t_{c_{2g}} \cdots t_{c_2} t_{c_1}{}^2 t_{c_2} \cdots t_{c_{2g}})^{2n} \cdot (t_{c_1} \cdots t_{c_{2g-2}})^{2(2g-1)n}) \\
=\ & (8g^2n - 4gn + 4n) \cdot \frac{-g-1}{2g+1} + 2n \cdot h(t_{c_{2g+1}}^{-2} \cdot \iota_g) \\
& + 2(2g-1)n \cdot h(t_{c_1} \cdots t_{c_{2g-2}}) \\
=\ & \frac{-8g^3n + 4g^2n - 4gn - 8g^2n + 4gn - 4n}{2g+1} - 4n \cdot \frac{-g}{2g+1} \\
& + 2(2g-1)n(2g-2) \cdot \frac{-1}{4g^2-1} \\
=\ & -4g^2n.
\end{aligned}
$$

It is easy to see that $\tilde{X}_{g,n}$ is simply connected, and that the Euler characteristic of $\tilde{X}_{g,n}$ is $8g^2n - 4gn + 4n - 4g + 6$. As shown in [15], $\tilde{X}_{g,n}$ is spin if and only if g is even. Thus, we can easily determine the homeomorphism type of $\tilde{X}_{g,n}$ as in Example 4.5.7.

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