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A STOCHASTIC RESOLUTION OF A COMPLEX MONGE-AMPERE EQUATION ON A NEGATIVELY CURVED KÄHLER MANIFOLD

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1. Introduction

The Dirichlet problem for the complex Monge-Ampère equation on a strongly pseudo-convex domain of $C^n$ was studied and solved by Bedford-Taylor [3]. The same problem for the Monge-Ampère equation on a negatively curved Kähler manifold has been recently proposed and solved by T. Asaba [2]. The main purpose of this paper is to solve the equation by using a method of the stochastic control presented by B. Gaveau [6].

Let $M$ be an $n$-dimensional simply connected Kähler manifold with metric $g$ whose sectional curvature $K$ satisfies

$$-b^2 \leq K \leq -a^2$$

on $M$ for some positive constants $b$ and $a$. $\omega_0$ denotes the associated Kähler form. We denote by $M(\infty)$ the Eberlein-O’Neill’s ideal boundary of $M$ and we always consider the cone topology on $\overline{M}=M \cup M(\infty)$ (see [4] for these notions). T. Asaba formulated the Monge-Ampère equation on $M$ in the following manner:

We write PSH($D$) for the family of locally bounded plurisubharmonic functions defined on a complex manifold $D$. When $u \in$ PSH($D$), the current $(dd^c u)^n = dd^c u \wedge \cdots \wedge dd^c u$ of type-(n, n) is defined as a positive Radon measure on $D$. Therefore, for given functions $f \in C(M)$ and $\varphi \in C(M(\infty))$, the complex Monge-Ampère equation

$$\left\{ \begin{array}{ll}
u \in \text{PSH}(M) \cap C(\overline{M}) \\
(dd^c u)^n = f \omega_0^n/n! & \text{on } M \\
u \mid_{M(\infty)} = \varphi
\end{array} \right.$$  (1)

can be considered. T. Asaba found a unique solution of (1) by imposing the following condition on $f$: there exist two positive constants $\mu_0$ and $C_0$ such that
Here and in the sequel \( r \) stands for the distance function from a fixed point of \( M \). Following a similar line to the proof performed by B. Gaveau [6], in which a stochastic proof of the existence of the Monge-Ampère equation on a strongly pseudo-convex domain of \( C^* \) was presented, we will prove not only the existence of the solution of (1) but also its uniqueness (§ 3, Thoerem B). Actually T. Asaba assumed condition (2) for a specific value of \( \mu_0 \). In what follows, we assume the condition (2) on \( f \) holding for some \( \mu_0 > 0 \) and \( C_0 > 0 \).

In accordance with the first part of B. Gaveau [6], a certain transience behavior of the sample path of the conformal martingales on \( M \) need to be studied. It was conjectured by H. Wu [13] that \( M \) is biholomorphic to a bounded domain of \( C^* \) (cf. Y.T. Siu [11] and R.E. Greene [7]). If this would be true, then the conformal martingales of the type considered by B. Gaveau [6] must hit the boundary of \( M \). In fact, we shall prove in Section 2 that the almost all sample paths of every non-degenerate conformal martingale converge to points of the ideal boundary \( M(\infty) \). We use the method of J.J. Prat [10], in which the sample paths’ property was proven for the Brownian motion on Riemannian manifolds with negative curvature bounded away from zero.

The basic estimates obtained in Section 2 will be further utilized after Section 3 in resolving the Monge-Ampère equation stochastically.

The author expresses his thanks to T. Asaba for private discussions.

2. Basic estimates for non-degenerate conformal martingales

We first define the notion of the conformal martingales on \( M \).

DEFINITION. Let \((\Omega, \mathcal{F}, P)\) be a probability space with a filtration \((\mathcal{F}_t)_{t\geq 0}\). An \( M \)-valued continuous stochastic process \((Z_t)_{0\leq t<\xi}\) defined up to a stopping time \( \xi > 0 \) is said to be a conformal martingale, if

1. there exists \( p \in M \) such that \( Z_0 = p \) a.s.
2. there exists a sequence of stopping times \( (T_n)_{n=1}^{\infty} \) such that \( T_n < \xi \), \( \lim T_n = \xi \) and \( (f(Z_t)_{t\geq 0}^\xi)_{t\geq 0} \) is a \( C \)-valued bounded \((\mathcal{F}_t)\)-martingale for every holomorphic function \( f \) on \( M \) (we need note that \( M \) is a Stein manifold and so \( M \) possessess enough holomorphic functions).

Noting the trivialty of the bundle of unitary frames, we choose smooth vector fields \( X_1, \ldots, X_n \) of type-(1, 0) on \( M \) so that \( g(X_a, X_b) = \delta_{a,b} \) on \( M \). For a smooth function \( f \) defined on \( M \), we write \( Lf \) for the Levi-form of \( f \). The notion of conformal martingale is related to the Levi-form in the following way:

Proposition 1. For each conformal martingale \((Z_t)_{0\leq t<\xi}\) on \( M \), there is a non-negative hermitian matrix valued \((\mathcal{F}_t)\)-adapted process \((\sum_{a,b}(t))_{0\leq t<\xi}\) such that
it is increasing (in the sense that \( s \leq t \Rightarrow \sum_{a,b} \delta(s) \leq \sum_{a,b} \delta(t) \) as hermitian matrices a.s.) and that, for each real valued function \( f \in C^2(M) \)

\[
f(Z_t) - f(Z_0) = \sum_{a,b=1}^s \int_0^t Lf(X_a, X_b) d\sum_{a,b} \delta(s)
\]

is a local martingale.

Proof. Take countable local complex charts \((U_i; z^i, \ldots, z^i_t)_{i=1,2,\ldots}\) of \(M\) and closed sets \(V_i \subseteq U_i\) such that \(\{V_i\}_{i=1}^\infty\) covers \(M\). Since \(M\) is a Stein manifold, we may assume that \(z^i, \ldots, z^i_t\) are the restrictions to \(U_i\) of certain holomorphic functions on \(M\) for every \(i=1, 2, 3, \ldots\). Define a sequence of stopping times \(\sigma_s\) and random variables \(i_s\) successively as follows:

\[
\sigma_0 = 0
\]
\[
i_0 = \inf \{i; Z_0 \in V_i\}
\]
\[
\sigma_1 = \inf \{t > 0; Z_t \in U_{i_0}\}
\]
\[
\vdots
\]
\[
\sigma_k = \inf \{t > \sigma_{k-1}; Z_t \in U_{i_{k-1}}\}
\]
\[
i_k = \inf \{i; Z_{i_k} \in V_i\}
\]

By virtue of Ito’s formula, we obtain

\[
f(Z_t \wedge \sigma_k) - f(Z_t \wedge \sigma_{k-1}) = \sum_{a=1}^n \int_{t \wedge \sigma_k}^{t \wedge \sigma_{k-1}} \partial f/\partial z^a (Z_s) dz^a (Z_s)
\]

\[
+ \sum_{a,b=1}^s \int_{t \wedge \sigma_k}^{t \wedge \sigma_{k-1}} \partial f/\partial z^a (Z_s) dz^b (Z_s)
\]

\[
+ \sum_{a,b=1}^s \int_{t \wedge \sigma_k}^{t \wedge \sigma_{k-1}} \partial^2 f/\partial z^a \partial z^b (Z_s) d\langle z^a (Z_s), z^b (Z_s) \rangle,
\]

where \(z^a = z^a_{i_{k-1}}, \alpha = 1, 2, \ldots, n, k = 1, 2, 3, \ldots\). Define a hermitian matrix valued process \(\sigma(t)\) by \(\sum_{k=1}^n \sigma^a_k(t) (\partial/\partial z^a |_{Z_t}) = X_a |_{Z_t}, \alpha = 1, 2, \ldots, n\) and set

\[
\sum_{a,b} \delta(t) = \sum_{a,b} \int_0^t \sigma^a_k(s) \sigma^b_k(s) d\langle z^a (Z_s), z^b (Z_s) \rangle,
\]

then this can be well defined, independently of the choice of local coordinates, and further

\[
f(Z_t \wedge \sigma_k) - f(Z_t \wedge \sigma_{k-1}) = \sum_{a,b=1}^s \int_{t \wedge \sigma_k}^{t \wedge \sigma_{k-1}} Lf(X_a, X_b) z_z d\sum_{a,b} \delta(s)
\]

is a martingale. Since \(\lim_{k \to \infty} \sigma_k = T\), the proof is completed. q.e.d.

For our investigation, it is enough to consider exclusively conformal
martingales \((Z_t)_{0 \leq t \leq 	au}\) for which the following stopping times \(\tau_k\) \((k=0, 1, 2, 3, \cdots)\) are finite almost surely:

\[
\begin{align*}
\tau_0 &= 0 \\
\tau_1 &= \inf \{t > 0; \text{dist}(Z_t, Z_0) = 1\} \\
&\quad \vdots \\
\tau_{k+1} &= \inf \{t > \tau_k; \text{dist}(Z_t, Z_{\tau_k}) = 1\}
\end{align*}
\]

We call such property "admissible" and in what follows \(\tau_k\) means the above stopping time. Here, we present a basic estimate of the same type as in D. Sullivan [12].

**Proposition 2.** For any \(\mu \in (0, a)\), there exists a constant \(C_1 \in (0, 1)\) such that

\[
E[\exp(-\mu r(Z_{\tau_{k+1}}))] \leq C_1 E[\exp(-\mu r(Z_{\tau_k}))], \quad k = 0, 1, 2, 3, \cdots,
\]

for every admissible conformal martingale \((Z_t)_{0 \leq t \leq \tau}\).

**Proof.** A Jacobi field estimate—the Hessian comparison theorem presented in [8; Theorem A] implies

\[
L \exp(-\mu r) \leq (\mu(\mu-a)/2) \exp(\mu r)g \quad \text{in the sense [8].}
\]

By applying Proposition 1 to the function \(\exp(-\mu r)\), we then have

\[
E[\exp(-\mu r(Z_{\tau_{k+1}}))] = E[\exp(-\mu r(Z_{\tau_k}))]
+ E[\sum_{a, \beta=1}^{r-k+1} \exp(-\mu r(X_a, X_\beta) \delta_2 d \Sigma_{a, \beta}(s))]
\leq E[\exp(-\mu r(Z_{\tau_k}))]
+ (\mu(\mu-a)/2) E[\int_{\tau_k}^{\tau_{k+1}} \exp(-\mu r(Z_s)) d(\text{trace } \Sigma_{a, \beta}(s))],
\]

\(k = 0, 1, 2, \cdots\).

While, taking conditional expectation, we have

\[
E[\int_{\tau_k}^{\tau_{k+1}} \exp(-\mu r(Z_s)) d(\text{trace } \Sigma_{a, \beta}(s))]
= \int_M P(Z_{\tau_k} \in d\eta) E[\int_{\tau_k}^{\tau_{k+1}} \exp(-\mu r(Z_s)) d(\text{trace } \Sigma_{a, \beta}(s)) | Z_{\tau_k} = \eta]
\geq \int_M P(Z_{\tau_k} \in d\eta) \exp(-\mu(\mu+1)) E[\int_{\tau_k}^{\tau_{k+1}} d(\text{trace } \Sigma_{a, \beta}(s)) | Z_{\tau_k} = \eta],
\]

which is not less than \(\exp(-\mu)C_2^{-1} E[\exp(-\mu r(Z_{\tau_k}))]\) by virtue of Lemma 1 stated below. Hence we arrive at the desired estimate with \(C_1 = 1 + ((\mu(\mu-a)/2))C_2^{-1} \exp(-\mu)\). q.e.d.
In the above proof, we have used the next lemma, which also will be utilized in §4.

**Lemma 1.** There exists a positive constant $C_2$ depending only on $a$ and $b$ such that

$$C_2^{-1} \leq E\left[\int_{\tau_k}^{\tau_{k+1}} d(\text{trace } \sum_{\alpha, \beta}(s)) \mid \tau_k = \eta \right] \leq C_2$$

holds $P(\tau_k \in d\eta)$-a.s. $k = 0, 1, 2, 3, \ldots$, for every admissible conformal martingale $Z_t$.

**Proof.** For $f \in C_c^2(M)$, we know from Proposition 1 that

$$E[f(Z_{\tau_{k+1}}) - f(Z_{\tau_k}) - \sum_{\alpha, \beta=1}^s \int_{\tau_k}^{\tau_{k+1}} Lf(X_{\beta}, X_{\beta}) Z_s d \sum_{\alpha, \beta}(s) \mid \tau_k = \eta] = 0$$

$P(\tau_k \in d\eta)$-a.s., $k = 0, 1, 2, 3, \ldots$.

Taking a countably dense subset of $C_c^2(M)$ and by the approximation procedure we know that the exceptional $\eta$-set in the above statement can be taken independently of $f \in C_c^2(M)$. Choose $f = f^{(\eta)}(p) \in C_c^2(M)$ which coincides with dist$(p, \eta)$ on a neighborhood of $\{p; \text{dist}(p, \eta) \leq 1\}$. Then it turns out that

$$1 = E\left[\sum_{\alpha, \beta=1}^s \int_{\tau_k}^{\tau_{k+1}} Lf(X_{\alpha}, X_{\beta}) Z_s d \sum_{\alpha, \beta}(s) \mid \tau_k = \eta \right] \quad P(\tau_k \in d\eta)$-a.s.$$

Again by the Hessian comparison theorem, we find that there exists a constant $C_2$ depending only on the curvature bounds $a$ and $b$ such that

$$C_2 g \leq Lf^{(\eta)} \leq C_2^{-1} g \quad \text{on } \{p; \text{dist}(p, \eta) \leq 1\},$$

so we have

$$C_2^{-1} \leq E\left[\int_{\tau_k}^{\tau_{k+1}} d(\text{trace } \sum_{\alpha, \beta}(s)) \mid \tau_k = \eta \right] \leq C_2$$

$P(\tau_k \in d\eta)$-a.s. q.e.d.

The next theorem is an immediate consequence of Proposition 2 combined with the geometrical method employed by D. Sullivan [12] and J.J. Prat [10].

**Theorem A.** For every admissible conformal martingale $(Z_t)_{0 \leq t \leq \xi}$, the following are true:

(i) The limit $\lim_{t \uparrow \xi} Z_t$ exists in $M(\infty)$ a.s.

(ii) For any $\xi \in M(\infty)$, $\varepsilon > 0$ and neighborhood $V \subset M(\infty)$ of $\xi$, there exists a neighborhood $U \subset M$ of $\xi$ relative to the cone topology such that

$$P(\lim_{t \uparrow \xi} Z_t \in V) \geq 1 - \varepsilon,$$

whenever $Z_t$ strats from a point of $U$. $U$ does not depend on the choice of $(Z_t)_{0 \leq t \leq \xi}$. 
3. The stochastic solution of the Monge-Ampère equation—the statement of the main theorem

Let $K_p$ be the family of all admissible conformal martingales $Z=(Z_t)_{t \leq t \leq \tau(Z)}$ on $M$ such that $Z$ starts from $p \in M$ and the associate process $(\Sigma_{a,b}(t))_{0 \leq t \leq \tau(Z)}$ in Proposition 1 possesses a density $(A_{a,b}(t))_{0 \leq t \leq \tau(Z)}$ with respect to the Lebesgue measure $dt$ with $\det A_{a,b}(t) \geq 1$ for $t \geq 0$ a.s. For $Z \in K_p$, set

$$w(p, Z) = E[-C(n) \int_0^{\tau(Z)} f^{1/n}(Z_t) dt + \varphi(Z_{\tau(Z)})],$$

where $C(n)=n/8(n!)^{1/n}$. By virtue of Lemma 2 in the next section, we know that, if $Z=(Z_t)$ is the conformal diffusion generated by the Kahler metric $g$ on $M$, then $w(p, Z)$ is exactly the solution of the Dirichlet problem with boundary condition on the sphere at infinity:

$$\begin{cases} \Delta_g u/2 = C(n)f^{1/n} \\ u|_{M(\infty)} = \varphi \end{cases}$$

for the Laplace-Beltrami operator $\Delta_g$ related to $g$. Now, we can describe the solution of the Monge-Ampère equation (1), using the above stochastic notations.

**Theorem B.** The function

$$u(p) = \inf_{Z \in K_p} w(p, Z), \quad p \in M$$

is the unique solution of the Monge-Ampère equation (1).

In the following sections, we shall prove this theorem. The proof will be performed by the stochastic control method due to B. Gaveau [6].

4. Continuity of the stochastic solution

In this section, we shall prove the continuity of the function $u$ defined by (3).

**Proposition 3.** $u$ can be extended to a continuous function on $\bar{M}$ and $u|_{M(\infty)} = \varphi$.

We have to prepare several lemmas for the proof.

**Lemma 2.** For each $Z \in K_p$, there exist positive constants $v$ and $C_2$ depending only on the constants $\mu_0$, $C_0$ in (2) and the curvature bounds such that

$$E \left[ \int_0^{\tau(Z)} f(Z_t)^{1/n} dt \right] \leq C_2 \exp(-v\varphi(p)).$$
Proof. By the assumption (2) imposed on \( f \), for \( \nu \leq \mu_0 \), we know
\[
E \left[ \int_0^{\tau_0} f(Z_t) v(t) dt \right] 
\leq C_0 E \left[ \int_0^{\tau_0} \exp \left( -vr(Z_t) \right) dt \right] 
\leq C_0 \sum_{k=0}^{\infty} E \left[ \int_{\tau_k}^{\tau_{k+1}} \exp \left( -vr(Z_t) \right) dt \right],
\]
where \( \tau_0 = 0, \tau_1 = \inf \{ t > 0 ; \text{dist}(Z_t, Z_0) = 1 \} \) and \( \tau_{k+1} = \inf \{ t > \tau_k ; \text{dist}(Z_t, Z_\tau) = 1 \} \). We may assume that \( \nu \) is so small that \( \nu/n \) is less than \( a \). Because
\[
E \left[ \int_{\tau_k}^{\tau_{k+1}} \exp \left( -vr(Z_t) \right) dt \right] \leq E \left[ \int_{\tau_k}^{\tau_{k+1}} \exp \left( -vr(Z_t) \right) dt \right] \left( \text{trace } \Sigma \right),
\]
we have
\[
E \left[ \int_{\tau_k}^{\tau_{k+1}} \exp \left( -vr(Z_t) \right) dt \right] \leq \exp \left( a \right) C_2 E \left[ \exp \left( -vr(Z_t) \right) \right],
\]
in view of the proof of Proposition 2. Further by virtue of the basic estimate (Proposition 2) we know
\[
\sum_{k=0}^{\infty} E \left[ \exp \left( -vr(Z_t) \right) \right] \leq \left( 1 - C_1 \right)^{-1} \exp \left( -vr(p) \right).
\]
The desired inequality holds for \( C_2 = \exp \left( a \right) C_0 C_2 \left( 1 - C_1 \right) \). q.e.d.

Combining this with the result on the weak convergence of the hitting distribution in Theorem A (ii), we know that for arbitrary \( \xi \in M(\infty) \) and any \( \varepsilon > 0 \), there exists a neighborhood \( U \) of \( \xi \) such that
\[
(4) \quad p \in U \Rightarrow |w(p, Z) - \varphi(\xi)| < \varepsilon,
\]
when \( Z \in K_p \). Furthermore, we can show the following lemma.

**Lemma 3.** For any \( \varepsilon > 0 \), there exist a positive large constant \( R \) and a small constant \( \gamma_0 \) such that, if
\[
p \in D_R = \{ \eta \in M ; r(\eta) \leq R \}
\]
and \( \text{dist}(p, q) < \gamma_0 \), then
\[
|w(p, Z) - w(q, Z')| < \varepsilon,
\]
for any \( Z \in K_p \) and \( Z' \in K_q \).

Proof. For any \( \varepsilon > 0 \), there exist some points \( \xi_1, \cdots, \xi_n \in M(\infty) \) and open sets \( U_i \supseteq \xi_i \) such that
\[
p \in U_i \text{ and } Z \in K_p,
\]
\[
\Rightarrow |w(p, Z) - \varphi(\xi_i)| < \varepsilon/2.
\]
for all $i=1, 2, \ldots, n$ and $M(\infty) \subset \bigcup_{i=1}^{n} U_i$. Take a closed neighborhood $U_i \subset U_i$ of $\xi_i$ so that $M(\infty) \subset \bigcup_{i=1}^{n} U_i$. Then, there exists $R>0$ satisfying $M \setminus D_R \subset \bigcup_{i=1}^{n} U_i$. Therefore for sufficiently small, $\gamma_0$ we know that

$$\text{dist}(p, q) < \gamma_0, p \in D_R$$

$$\Rightarrow |w(p, Z) - w(q, Z')|$$

$$\leq |w(p, Z) - \varphi(\xi_i)| + |\varphi(\xi_i) - w(q, Z')|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

whenever $Z \in K_p$ and $Z' \in K_q$, by choosing $i$ so that $p \in U_i$.

Because the holomorphic tangent bundle is holomorphically trivial, there exists a frame of holomorphic vector fields $Y_1, \ldots, Y_n$. Let $\Phi_\delta(p) = \text{Exp}(\text{Re}\sum_{i=1}^{n} z_i^i Y_i)(p)$, for $p \in M$ and $z = (z^1, \ldots, z^n)$ in $\mathbb{C}^n$. This transformation on $M$ was considered in T. Asaba [2] and proven to enjoy the next property:

For any $R>0$, there exists $\Delta_\delta = \{z \in \mathbb{C}^n; \sum_{i=1}^{n} |z_i^i|^2 < \delta\}$ such that $\Phi_\delta(p)$ is a smooth mapping from $\Delta_\delta \times D_R$ to $M$ satisfying the following properties (i), (ii) and (iii).

(i) For each $z \in \Delta_\delta$, $\Phi_\delta$ gives a biholomorphic mapping from the domain $D_R$ to $\Phi_\delta(D_R)$.

(ii) $\Phi_\delta$ is the identity transformation on $D_R$.

(iii) For $p \in D_R$, $\Phi_\delta(p)$ defines a diffeomorphism from $\Delta_\delta$ to some neighborhood of $p$.

Using this transformation $\Phi$, we can prove the continuity of the stochastic solution $u$.

**Lemma 4.** For any $\varepsilon>0$ and $R>0$, there exists $\varepsilon>0$ such that for each $Z \in K_p$ and $q$ enjoying $p \in D_R$ and $\text{dist}(p, q) < \gamma$, we can always find $Z' \in K_q$ so that

$$|w(p, Z) - w(q, Z')| < \varepsilon.$$
where $\tau = \inf \{ t > 0; Z_t \in D_\delta \}$.

We next perform the time change by letting $\tilde{Z}^{\Phi_t(p)}(Z_t) = Z^{\Phi_t(p)}_{t_\tau}$, up to the explosion time of $\tilde{Z}^{\Phi_t(p)}(Z_t) = (\tilde{Z}^{\Phi_t(p)}_{t_\tau})_{t \geq 0}$, where $\tau = \inf \{ t > 0; \int_0^t (\det A_{\Phi_t(u)}(u))^{1/n} du \leq t \}$, $(A_{\Phi_t(u)}(u))_{t \geq 0}$ being the density of the increasing process associated with $Z^{\Phi_t(p)}(Z_t) = (Z^{\Phi_t(p)}_{t_\tau})_{t \geq 0}$ according to Proposition 1.

On the other hand, taking conditional expectation, we have

$$w(\eta, W) = E[-C(n) \int_0^\tau f^{1/n}(Z_t) dt + \varphi(Z_{\zeta(Z)}^-) | Z_\tau = \eta] P(Z_\tau \in d\eta).$$

If we set $W_t = Z_{t+\tau}$ and let

$$w(\eta, W) = E[-C(n) \int_0^\tau f^{1/n}(W_t) dt + \varphi(Z_{\zeta(Z)}^-) | Z_\tau = \eta]$$

for $W = (W_t)_{0 \leq t < \zeta(Z)}$, then

$$w(p, W) = E[-C(n) \int_0^\tau f^{1/n}(Z_t) dt + \int_{D_R} w(\eta, W) P(Z_\tau \in d\eta).$$

Similarly, letting $\sigma$ be the first exit time from $\Phi_t(D_R)$ of $\tilde{Z}^{\Phi_t(p)}$, we set $W^{\Phi_t(p)}(Z_t) = (W^{\Phi_t(p)}_{t_\sigma})_{t \geq 0}$, 0 $\leq t < \zeta(Z)^{-\sigma}$ and then, for $W^{\Phi_t(p)} = (W^{\Phi_t(p)}_{t_\sigma})_{t \geq 0}$,

$$w(\eta, W^{\Phi_t(p)}) = E[-C(n) \int_0^{\zeta(W^{\Phi_t(p)})} f^{1/n}(W^{\Phi_t(p)}) dt + \varphi(W^{\Phi_t(p)}_{\zeta(W^{\Phi_t(p)})}^-) | Z^{\Phi_t(p)}_\sigma = \eta].$$

Then

$$w(\Phi_t(p), Z^{\Phi_t(p)}) = E[-C(n) \int_0^{\sigma} f^{1/n}(\tilde{Z}^{\Phi_t(p)}) dt + \int_{\Phi_t(D_R)} w(\eta', W^{\Phi_t(p)}) P(Z^{\Phi_t(p)}_\sigma \in d\eta').$$

Therefore, after all we have that

$$w(p, Z) - w(\Phi_t(p), Z^{\Phi_t(p)}) = E[-C(n) (\int_0^\tau f^{1/n}(Z_t) dt - \int_0^{\sigma} f^{1/n}(\tilde{Z}^{\Phi_t(p)}_{t_\tau}) dt)] + \int_{D_R} \{w(\eta, W) - w(\Phi_t(\eta), W^{\Phi_t(p)})\} P(Z_\tau \in d\eta).$$

From Lemma 2, there exists $\delta > 0$ such that the absolute value of the second term of the right hand side is less than $\varepsilon/2$ for every $\varepsilon \in \Delta_\delta$. While the continuity of $f^{1/n}$ shows that the first term of the right hand side is less than $\varepsilon/2$ in
the absolute value, whenever \( z \in \Delta_{\delta} \).

Because, for sufficiently small \( \gamma \), the \( \gamma \)-neighborhood of each \( p \in D_R \) is contained in the image of \( \Delta_{\delta} \) by the mapping \( \Phi_*(p) \), for \( q = \Phi_*(p) \), \( Z_\tau^w(q) \) is the required conformal martingale in our lemma.

Proof of Proposition 3. The last inequality in Lemma 4 implies \( w(p, Z) \leq u(q) - \varepsilon \). Taking the infimum over \( Z \in K_p \), we can conclude that \( u(p) \leq u(q) - \varepsilon \), whenever \( p, q \in D_R \) and \( \text{dist}(p, q) < \gamma \). Exchanging the role of \( p \) and \( q \), we see that \( u \) is a continuous function on \( M \). Recalling the estimate (4) noted after Lemma 2, we know that \( \lim_{p \to \xi} u(p) = \varphi(\xi) \) for each \( \xi \in M(\infty) \). This completes the proof.

q.e.d.

5. The Bellman principle

The purpose of this section is to establish the Bellman principle in order to localize the stochastic expression of the function \( u \) defined by (3).

Proposition 4. For every bounded domain \( D \) of \( M \) and \( p \in D \), we obtain

\[
u(p) = \inf_{z \in K_p} E[-C(n) \int_{0}^{\tau_p(z)} f^{\mu}(Z_i)dt + u(Z_{\tau_p(z)})],
\]

where \( \tau_p(Z) = \inf\{t > 0; Z_t \not\in D\} \).

Proof. Fix \( \varepsilon > 0 \) and take \( R \) so that \( D_R \supseteq \overline{D} \). For each \( q \in \partial D \) there exist \( \delta > 0 \) and \( Z \in K_q \) such that, for \( z \in \Delta_{\delta} \),

\[|w(\Phi_*(p), Z_\tau^w(q)) - u(q)| > \varepsilon,
\]

where \( Z_\tau^w(q) \) is the conformal martingale defined by (5). Therefore, we can select several points \( q_1, \ldots, q_n \in \partial D \) and their neighborhoods \( \Delta(q_1), \ldots, \Delta(q_n) \) so that \( \partial D \subseteq \bigcup_{i=1}^{n} \Delta(q_i) \) (disjoint union), the image of \( \Phi_*(q_i) \) contains \( \Delta(q_i) \) and

\[|w(\Phi_*(q_i), Z_\tau^w(q_i)) - u(q_i)| < \varepsilon,
\]

whenever \( Z_\tau^w(q_i) \) is in \( \Delta(q_i) \), \( i = 1, 2, \ldots, n \).

For each \( Z \in K_p \), we set

\[Z^*_t = \begin{cases} Z_t, & \text{if } t \leq \tau_p(Z) \\ \Phi_*(q_i), & \text{if } t > \tau_p(Z), \quad Z_{\tau_p(Z)} \in \Delta(q_i) \quad \text{and} \\ Z_{t - \tau_p(Z)} \end{cases},
\]

where we take \( Z_\tau^w(q_i) \) and \( Z \) to be independent. Then \( Z^* = (Z^*_t) \in K_p \). By the same method of B. Gaveau [6; pp. 400–403], we can prove that
Since $\varepsilon > 0$ is arbitrary, the proof is completed. q.e.d.

6. Proof of the main theorem

Finally, we shall finish the proof of the main theorem by showing the next two propositions.

Proposition 5. $u$ is a plurisubharmonic function and $(dd^c u)^n = \omega^n / n!$ on $M$.

Proposition 6. If $u_0$ is a solution of (1), then

$$u_0(p) = \inf_{\beta \in K_x} E[-\varphi(Z)]$$

In particular, (1) has a unique solution.

Proof of Proposition 5. Let $p$ be an arbitrary point of $M$. Choose a complex local coordinate system $(D, z^1, \ldots, z^n)$ around $p$ such that $\psi = (z^1, \ldots, z^n)$ defines a biholomorphic mapping from $D$ to the complex unit ball $B = \{ |z^1|, \ldots, z^n| < 1 \}$ for some local coordinates $z^1, \ldots, z^n$. For the push forward function $U(z) = (\psi \circ u)(z) = u(\psi^{-1}(z))$

$$U(z) = \inf_{\beta \in K_x} E[-\varphi(Z)]$$

where $g_{ij} = g(\partial / \partial z^i, \partial / \partial z^j)$ and $K_x$ is the family of all $C^n$-valued conformal martingales $Z$ which start from $z \in B$ such that $a_{ij}(t) = d\langle z'(Z_t), z'(Z_t) \rangle / dt$ satisfy $\det(a_{ij}(t)) \geq 1$, $t \geq 0$ a.s.

Consider the following Monge-Ampère equation

$$\begin{cases}
  v \in PSH(B) \cap C(\bar{B}) \\
  (dd^c v)^n = \rho^*(f \det(g_{ij}))/dV \\
  v\,|_{\partial B} = U\,|_{\partial B},
\end{cases}$$

where $dV$ stands for the Lebesgue measure on $C^n$. Because of the strongly pseudo-convexity of $B$, we see that Theorem 4 and Remark of B. Gaveau [6; pp. 402-403] ensure the following stochastic description of the solution $v_0$ of (6):

$$v_0(z) = \inf_{\beta \in K_x} E[-\varphi(Z)]$$

where $\varphi(Z)$ is a plurisubharmonic function and $(dd^c \varphi)^n = \omega^n / n!$ on $M$. Since $\varphi > 0$ is arbitrary, the proof is completed. q.e.d.
Hence, we know that \( v_0 = U \) on \( B \) and \( u(p) = \psi_k v_0(p) \in PSH(D) \) and that 
\[
(dd^c u)^n = f \omega_0^n / n 
\] on \( D \). q.e.d.

Proof of Proposition 6. To begin, take the countable family of charts 
\( (U_i; z^i_1, \ldots, z^i_r) \) \( r = 1, \ldots, r \) appeared in the proof of Proposition 1, we may assume that 
each \( \psi_i = (z^i_1, \ldots, z^i_r) \) gives a biholomorphic mapping between \( U_i \) and the unit ball \( B \subset \mathbb{C}^r \). By virtue of Theorem 4 of B. Gaveau [6], for any \( \varepsilon > 0 \), there exists a \( Z^{(1)}(r) \in K \) such that 
\[
E[-C(n) \int_0^\sigma f^{1/n}(Z_i) dt + u_0(Z^{(1)}(\sigma))] \leq u_0(p) + \varepsilon / 2,
\]
where \( \sigma \) is the stopping time for \( Z^{(1)} \) defined in the proof of Proposition 1. For each \( q \in \partial U_i \) there exists \( \delta > 0 \) and \( Z \in K_q \) such that 
\[
\varrho(\Phi_i(q), Z^{(1)}(\delta)) < u_0(q) + \varepsilon / 2,
\]
whenever \( z \in \Delta_k \). Using the same argument as in the proof of Proposition 4, we can construct \( Z^{(2)}(r) \in K \) which satisfies 
\[
Z^{(1)}(r) = Z^{(2)}(r) \wedge \sigma
\]
and 
\[
E[-C(n) \int_0^\sigma f^{1/n}(Z^{(2)}(i)) dt + u_0(Z^{(2)}(\sigma))] \leq u_0(p) + \varepsilon / 2 + \varepsilon / 2^2,
\]
where \( \sigma \) is defined for \( Z^{(2)} \) in the same way as above. Repeating this procedure, we obtain a sequence \( (Z^{(k)}) \) \( k = 1, 2, 3, \ldots \) \( \in K \) so that \( Z^{(k)} \wedge \sigma_{k-1} = Z^{(k)}(\sigma) \), \( t \geq 0 \) a.s. and that 
\[
E[-C(n) \int_0^\sigma f^{1/n}(Z^{(k)}(i)) dt + u_0(Z^{(k)}(\sigma))] \leq u_0(p) + \sum_{i=1}^k \varepsilon / 2^i,
\]
where \( \sigma_k \) is defined for \( Z^{(k)} \) as above.

Define \( Z = Z^{(k)}(i) \), if \( t < \sigma_k \). Then we can easily check that \( Z \in K \) and that \( \lim_{k \to \infty} \sigma_k = \varepsilon (Z) \). Hence, we know 
\[
E[-C(n) \int_0^\varepsilon f^{1/n}(Z_i) dt + \varphi(Z_i)] \leq u_0(p) + \varepsilon.
\]
Letting \( \varepsilon \to 0 \), we can conclude that 
\[
u_0(p) \geq \inf_{x \in K} E[-C(n) \int_0^\varepsilon f^{1/n}(Z_i) dt + \varphi(Z_i)] .
\]
On the other hand, we can inductively obtain, for each \( Z \in K \), 
\[
u_0(p) \leq E[-C(n) \int_0^\sigma f^{1/n}(Z_i) dt + u_0(Z_{\sigma})] , \quad k = 1, 2, 3, \ldots ,
\]
and so we have the opposite inequality, by letting \( k \to \infty \). q.e.d.
References


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