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## A STOCHASTIC RESOLUTION OF A COMPLEX MONGE-AMPÈRE EQUATION ON A NEGATIVELY CURVED KÄHLER MANIFOLD

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### 1. Introduction

The Dirichlet problem for the complex Monge-Ampère equation on a strongly pseudo-convex domain of  $\mathbf{C}^n$  was studied and solved by Bedford-Taylor [3]. The same problem for the Monge-Ampère equation on a negatively curved Kähler manifold has been recently proposed and solved by T. Asaba [2]. The main purpose of this paper is to solve the equation by using a method of the stochastic control presented by B. Gaveau [6].

Let  $M$  be an  $n$ -dimensional simply connected Kähler manifold with metric  $g$  whose sectional curvature  $K$  satisfies

$$-b^2 \leq K \leq -a^2$$

on  $M$  for some positive constants  $b$  and  $a$ .  $\omega_0$  denotes the associated Kähler form. We denote by  $M(\infty)$  the Eberlein-O'Neill's ideal boundary of  $M$  and we always consider the cone topology on  $\bar{M} = M \cup M(\infty)$  (see [4] for these notions). T. Asaba formulated the Monge-Ampère equation on  $M$  in the following manner:

We write  $\text{PSH}(D)$  for the family of locally bounded plurisubharmonic functions defined on a complex manifold  $D$ . When  $u \in \text{PSH}(D)$ , the current  $\underbrace{(dd^c u)^n}_{n\text{-copies}} = dd^c u \wedge \cdots \wedge dd^c u$  of type- $(n, n)$  is defined as a positive Radon measure

on  $D$ . Therefore, for given functions  $f \in C(M)$  and  $\varphi \in C(M(\infty))$ , the complex Monge-Ampère equation

$$(1) \quad \begin{cases} u \in \text{PSH}(M) \cap C(\bar{M}) \\ (dd^c u)^n = f \omega_0^n / n! & \text{on } M \\ u|_{M(\infty)} = \varphi \end{cases}$$

can be considered. T. Asaba found a unique solution of (1) by imposing the following condition on  $f$ : there exist two positive constants  $\mu_0$  and  $C_0$  such that

$$(2) \quad 0 \leq f \leq C_0 e^{-\mu_0 r}.$$

Here and in the sequel  $r$  stands for the distance function from a fixed point of  $M$ . Following a similar line to the proof performed by B. Gaveau [6], in which a stochastic proof of the existence of the Monge-Ampère equation on a strongly pseudo-convex domain of  $\mathbf{C}^n$  was presented, we will prove not only the existence of the solution of (1) but also its uniqueness (§ 3, Theorem B). Actually T. Asaba assumed condition (2) for a specific value of  $\mu_0$ . In what follows, we assume the condition (2) on  $f$  holding for some  $\mu_0 > 0$  and  $C_0 > 0$ .

In accordance with the first part of B. Gaveau [6], a certain transience behavior of the sample path of the conformal martingales on  $M$  need to be studied. It was conjectured by H. Wu [13] that  $M$  is biholomorphic to a bounded domain of  $\mathbf{C}^n$  (cf. Y.T. Siu [11] and R.E. Greene [7]). If this would be true, then the conformal martingales of the type considered by B. Gaveau [6] must hit the boundary of  $M$ . In fact, we shall prove in Section 2 that the almost all sample paths of every non-degenerate conformal martingale converge to points of the ideal boundary  $M(\infty)$ . We use the method of J.J. Prat [10], in which the sample paths' property was proven for the Brownian motion on Riemannian manifolds with negative curvature bounded away from zero.

The basic estimates obtained in Section 2 will be further utilized after Section 3 in resolving the Monge-Ampère equation stochastically.

The author expresses his thanks to T. Asaba for private discussions.

## 2. Basic estimates for non-degenerate conformal martingales

We first define the notion of the conformal martingales on  $M$ .

**DEFINITION.** Let  $(\Omega, \mathcal{F}, P)$  be a probability space with a filtration  $(\mathcal{F}_t)_{t \geq 0}$ . An  $M$ -valued continuous stochastic process  $(Z_t)_{0 \leq t < \zeta}$  defined up to a stopping time  $\zeta > 0$  is said to be a conformal martingale, if

- (i) there exists  $p \in M$  such that  $Z_0 = p$  a.s.
- (ii) there exists a sequence of stopping times  $(T_n)_{n=1}^\infty$  such that  $T_n < \zeta$ ,  $\lim T_n = \zeta$  and  $(f(Z_{t \wedge T_n}))_{t \geq 0}$  is a  $\mathbf{C}$ -valued bounded  $(\mathcal{F}_t)$ -martingale for every holomorphic function  $f$  on  $M$  (we need note that  $M$  is a Stein manifold and so  $M$  possess enough holomorphic functions).

Noting the triviality of the bundle of unitary frames, we choose smooth vector fields  $X_1, \dots, X_n$  of type- $(1, 0)$  on  $M$  so that  $g(X_\alpha, X_\beta) = \delta_{\alpha, \beta}$  on  $M$ . For a smooth function  $f$  defined on  $M$ , we write  $Lf$  for the Levi-form of  $f$ . The notion of conformal martingale is related to the Levi-form in the following way:

**Proposition 1.** *For each conformal martingale  $(Z_t)_{0 \leq t < \zeta}$  on  $M$ , there is a non-negative hermitian matrix valued  $(\mathcal{F}_t)$ -adapted process  $(\sum_{\alpha, \beta} (t))_{0 \leq t < \zeta}$  such that*

it is increasing (in the sense that  $s \leq t \Rightarrow \sum_{\alpha, \beta} (s) \leq \sum_{\alpha, \beta} (t)$  as hermitian matrices a.s.) and that, for each real valued function  $f \in C^2(M)$

$$f(Z_t) - f(Z_0) - \sum_{\alpha, \beta=1}^n \int_0^t Lf(X_\alpha, X_\beta)_{Z_s} d \sum_{\alpha, \beta}(s)$$

is a local martingale.

Proof. Take countable local complex charts  $(U_i; z_i^1, \dots, z_i^n)_{i=1,2,\dots}$  of  $M$  and closed sets  $V_i \subset U_i$  such that  $\{V_i\}_{i=1}^\infty$  covers  $M$ . Since  $M$  is a Stein manifold, we may assume that  $z_i^1, \dots, z_i^n$  are the restrictions to  $U_i$  of certain holomorphic functions on  $M$  for every  $i=1, 2, 3, \dots$ . Define a sequence of stopping times  $\sigma_k$  and random variables  $i_k$  successively as follows:

$$\begin{aligned} \sigma_0 &= 0 \\ i_0 &= \inf \{i; Z_0 \in V_i\} \\ \sigma_1 &= \inf \{t > 0; Z_t \notin U_{i_0}\} \\ &\dots \\ \sigma_k &= \inf \{t > \sigma_{k-1}; Z_t \notin U_{i_{k-1}}\} \\ i_k &= \inf \{i; Z_{\sigma_k} \in V_i\} \\ &\dots \end{aligned}$$

By virtue of Ito's formula, we obtain

$$\begin{aligned} f(Z_{t \wedge \sigma_k}) - f(Z_{t \wedge \sigma_{k-1}}) &= \sum_{\beta=1}^n \int_{t \wedge \sigma_{k-1}}^{t \wedge \sigma_k} \partial f / \partial z^\beta(Z_s) dz^\beta(Z_s) \\ &+ \sum_{\alpha=1}^n \int_{t \wedge \sigma_{k-1}}^{t \wedge \sigma_k} \partial f / \partial \bar{z}^\alpha(Z_s) d\bar{z}^\alpha(Z_s) \\ &+ \sum_{\alpha, \beta=1}^n \int_{t \wedge \sigma_{k-1}}^{t \wedge \sigma_k} \partial^2 f / \partial z^\alpha \partial \bar{z}^\beta(Z_s) d\langle z^\alpha(Z_s), \bar{z}^\beta(Z_s) \rangle, \end{aligned}$$

where  $z^\alpha = z_{i_{k-1}}^\alpha$ ,  $\alpha=1, 2, \dots, n$ ,  $k=1, 2, 3, \dots$ . Define a hermitian matrix valued process  $\sigma(t)$  by  $\sum_{\kappa=1}^n \sigma_\kappa^\alpha(t) (\partial / \partial z^\kappa |_{Z_t}) = \dot{X}_\alpha |_{Z_t}$ ,  $\alpha=1, 2, \dots, n$  and set

$$\sum_{\alpha, \beta=1}^n \sigma_{\alpha\beta}(t) = \sum_{\kappa, \lambda=1}^n \int_0^t \sigma_\kappa^\alpha(s) \sigma_\lambda^\beta(s) d\langle z^\kappa(Z_s), \bar{z}^\lambda(Z_s) \rangle,$$

then this can be well defined, independently of the choice of local coordinates, and further

$$f(Z_{t \wedge \sigma_k}) - f(Z_{t \wedge \sigma_{k-1}}) - \sum_{\alpha, \beta=1}^n \int_{t \wedge \sigma_{k-1}}^{t \wedge \sigma_k} Lf(X_\alpha, X_\beta)_{Z_s} d \sum_{\alpha, \beta}(s)$$

is a martingale. Since  $\lim_{k \rightarrow \infty} \sigma_k = \zeta$ , the proof is completed. q.e.d.

For our investigation, it is enough to consider exclusively conformal

martingales  $(Z_t)_{0 \leq t < \zeta}$  for which the following stopping times  $\tau_k$  ( $k=0, 1, 2, 3, \dots$ ) are finite almost surely:

$$\begin{aligned} \tau_0 &= 0 \\ \tau_1 &= \inf \{t > 0; \text{dist}(Z_t, Z_0) = 1\} \\ &\dots \\ \tau_{k+1} &= \inf \{t > \tau_k; \text{dist}(Z_t, Z_{\tau_k}) = 1\} \end{aligned}$$

We call such property ‘‘admissible’’ and in what follows  $\tau_k$  means the above stopping time. Here, we present a basic estimate of the same type as in D. Sullivan [12].

**Proposition 2.** *For any  $\mu \in (0, a)$ , there exists a constant  $C_1 \in (0, 1)$  such that*

$$E[\exp(-\mu r(Z_{\tau_{k+1}}))] \leq C_1 E[\exp(-\mu r(Z_{\tau_k}))], \quad k = 0, 1, 2, 3, \dots,$$

for every admissible conformal martingale  $(Z_t)_{0 \leq t < \zeta}$ .

Proof. A Jacobi field estimate—the Hessian comparison theorem presented in [8; Theorem A] implies

$$L \exp(-\mu r) \leq (\mu(\mu - a)/2) \exp(\mu r)g \quad \text{in the sense [8].}$$

By applying Proposition 1 to the function  $\exp(-\mu r)$ , we then have

$$\begin{aligned} E[\exp(-\mu r(Z_{\tau_{k+1}}))] &= E[\exp(-\mu r(Z_{\tau_k}))] \\ &+ E\left[\sum_{\alpha, \beta=1}^n \int_{\tau_k}^{\tau_{k+1}} L \exp(-\mu r)(X_\alpha, X_\beta)_{Z_s} d\Sigma_{\alpha, \beta}(s)\right] \\ &\leq E[\exp(-\mu r(Z_{\tau_k}))] \\ &+ (\mu(\mu - a)/2) E\left[\int_{\tau_k}^{\tau_{k+1}} \exp(-\mu r(Z_s)) d(\text{trace } \Sigma_{\alpha, \beta}(s))\right], \\ &k = 0, 1, 2, \dots \end{aligned}$$

While, taking conditional expectation, we have

$$\begin{aligned} &E\left[\int_{\tau_k}^{\tau_{k+1}} \exp(-\mu r(Z_s)) d(\text{trace } \Sigma_{\alpha, \beta}(s))\right] \\ &= \int_M P(Z_{\tau_k} \in d\eta) E\left[\int_{\tau_k}^{\tau_{k+1}} \exp(-\mu r(Z_s)) d(\text{trace } \Sigma_{\alpha, \beta}(s)) \mid Z_{\tau_k} = \eta\right] \\ &\geq \int_M P(Z_{\tau_k} \in d\eta) \exp(-\mu(r(\eta) + 1)) E\left[\int_{\tau_k}^{\tau_{k+1}} d(\text{trace } \Sigma_{\alpha, \beta}(s)) \mid Z_{\tau_k} = \eta\right], \end{aligned}$$

which is not less than  $\exp(-\mu)C_2^{-1} E[\exp(-\mu r(Z_{\tau_k}))]$  by virtue of Lemma 1 stated below. Hence we arrive at the desired estimate with  $C_1 = 1 + ((\mu(\mu - a)/2))C_2^{-1} \exp(-\mu)$ . q.e.d.

In the above proof, we have used the next lemma, which also will be utilized in § 4.

**Lemma 1.** *There exists a positive constant  $C_2$  depending only on  $a$  and  $b$  such that*

$$C_2^{-1} \leq E \left[ \int_{\tau_k}^{\tau_{k+1}} d(\text{trace } \Sigma_{\alpha, \beta}(s)) \mid Z_{\tau_k} = \eta \right] \leq C_2$$

holds  $P(Z_{\tau_k} \in d\eta)$ -a.s.  $k=0, 1, 2, 3, \dots$ , for every admissible conformal martingale  $Z_t$ .

Proof. For  $f \in C_b^2(M)$ , we know from Proposition 1 that

$$E[f(Z_{\tau_{k+1}}) - f(Z_{\tau_k}) - \sum_{\alpha, \beta=1}^n \int_{\tau_k}^{\tau_{k+1}} Lf(X_\alpha, X_\beta)_{Z_s} d\Sigma_{\alpha, \beta}(s) \mid Z_{\tau_k} = \eta] = 0$$

$$P(Z_{\tau_k} \in d\eta)\text{-a.s.}, k = 0, 1, 2, 3, \dots$$

Taking a countably dense subset of  $C_b^2(M)$  and by the approximation procedure we know that the exceptional  $\eta$ -set in the above statement can be taken independently of  $f \in C_b^2(M)$ . Choose  $f = f^{(n)}(p) \in C_b^2(M)$  which coincides with  $\text{dist}(p, \eta)^2$  on a neighborhood of  $\{p; \text{dist}(p, \eta) \leq 1\}$ . Then it turns out that

$$1 = E \left[ \sum_{\alpha, \beta=1}^n \int_{\tau_k}^{\tau_{k+1}} Lf(X_\alpha, X_\beta)_{Z_s} d\Sigma_{\alpha, \beta}(s) \mid Z_{\tau_k} = \eta \right] \quad P(Z_{\tau_k} \in d\eta)\text{-a.s.}$$

Again by the Hessian comparison theorem, we find that there exists a constant  $C_2$  depending only on the curvature bounds  $a$  and  $b$  such that

$$C_2 g \leq Lf^{(n)} \leq C_2^{-1} g \quad \text{on } \{p; \text{dist}(p, \eta) \leq 1\},$$

so we have

$$C_2^{-1} \leq E \left[ \int_{\tau_k}^{\tau_{k+1}} d(\text{trace } \Sigma_{\alpha, \beta}(s)) \mid Z_{\tau_k} = \eta \right] \leq C_2$$

$$P(Z_{\tau_k} \in d\eta)\text{-a.s.} \qquad \qquad \qquad \text{q.e.d.}$$

The next theorem is an immediate consequence of Proposition 2 combined with the geometrical method employed by D. Sullivan [12] and J.J. Prat [10].

**Theorem A.** *For every admissible conformal martingale  $(Z_t)_{0 \leq t < \xi}$ , the following are true :*

(i) *The limit  $\lim_{t \uparrow \xi} Z_t$  exists in  $M(\infty)$  a.s.*

(ii) *For any  $\xi \in M(\infty)$ ,  $\varepsilon > 0$  and neighborhood  $V \subset M(\infty)$  of  $\xi$ , there exists a neighborhood  $U \subset \bar{M}$  of  $\xi$  relative to the cone topology such that*

$$P(\lim_{t \uparrow \xi} Z_t \in V) \geq 1 - \varepsilon,$$

whenever  $Z_t$  starts from a point of  $U$ .  $U$  does not depend on the choice of  $(Z_t)_{0 \leq t < \xi}$ .

**3. The stochastic solution of the Monge-Ampère equation—the statement of the main theorem**

Let  $K_p$  be the family of all admissible conformal martingales  $Z=(Z_t)_{0 \leq t < \zeta(Z)}$  on  $M$  such that  $Z$  starts from  $p \in M$  and the associate process  $(\sum_{\alpha, \beta}(t))_{0 \leq t < \zeta(Z)}$  in Proposition 1 possesses a density  $(A_{\alpha, \beta}(t))_{0 \leq t < \zeta(Z)}$  with respect to the Lebesgue measure  $dt$  with  $\det A_{\alpha, \beta}(t) \geq 1$  for  $t \geq 0$  a.s. For  $Z \in K_p$ , set

$$w(p, Z) = E[-C(n) \int_0^{\zeta(Z)} f^{1/n}(Z_t) dt + \varphi(Z_{\zeta(Z)})],$$

where  $C(n) = n/8(n!)^{1/n}$ . By virtue of Lemma 2 in the next section, we know that, if  $Z=(Z_t)$  is the conformal diffusion generated by the Kahler metric  $g$  on  $M$ , then  $w(p, Z)$  is exactly the solution of the Dirichlet problem with boundary condition on the sphere at infinity:

$$\begin{cases} \Delta_g u/2 = C(n)f^{1/n} \\ u|_{M(\infty)} = \varphi \end{cases}$$

for the Laplace-Beltrami operator  $\Delta_g$  related to  $g$ . Now, we can describe the solution of the Monge-Ampère equation (1), using the above stochastic notations.

**Theorem B.** *The function*

$$(3) \quad u(p) = \inf_{Z \in K_p} w(p, Z), \quad p \in M$$

*is the unique solution of the Monge-Ampère equation (1).*

In the following sections, we shall prove this theorem. The proof will be performed by the stochastic control method due to B. Gaveau [6].

**4. Continuity of the stochastic solution**

In this section, we shall prove the continuity of the function  $u$  defined by (3).

**Proposition 3.**  *$u$  can be extended to a continuous function on  $\bar{M}$  and  $u|_{M(\infty)} = \varphi$ .*

We have to prepare several lemmas for the proof.

**Lemma 2.** *For each  $Z \in K_p$ , there exist positive constants  $\nu$  and  $C_3$  depending only on the constants  $\mu_0, C_0$  in (2) and the curvature bounds such that*

$$E \left[ \int_0^{\zeta(Z)} f(Z_t)^{1/n} dt \right] \leq C_2 \exp(-\nu r(p)).$$

Proof. By the assumption (2) imposed on  $f$ , for  $\nu \leq \mu_0$ , we know

$$\begin{aligned} E \left[ \int_0^{\zeta(Z)} f(Z_t)^{1/n} dt \right] &\leq C_0 E \left[ \int_0^{\zeta(Z)} \exp(-\nu r(Z_t)/n) dt \right] \\ &\leq C_0 \sum_{k=0}^{\infty} E \left[ \int_{\tau_k}^{\tau_{k+1}} \exp(-\nu r(Z_t)/n) dt \right], \end{aligned}$$

where  $\tau_0=0, \tau_1=\inf \{t>0; \text{dist}(Z_t, Z_0)=1\}, \dots, \tau_{k+1}=\inf \{t>\tau_k; \text{dist}(Z_t, Z_{\tau_k})=1\}, \dots$ . We may assume that  $\nu$  is so small that  $\nu/n$  is less than  $a$ . Because  $E \left[ \int_{\tau_k}^{\tau_{k+1}} \exp(-\nu r(Z_t)/n) dt \right] \leq E \left[ \int_{\tau_k}^{\tau_{k+1}} \exp(-\nu r(Z_t)/n) d(\text{trace } \Sigma_{\alpha, \beta}(t)) \right]$ , we have  $E \left[ \int_{\tau_k}^{\tau_{k+1}} \exp(-\nu r(Z_t)/n) dt \right] \leq \exp(a) C_2 E[\exp(-\nu r(Z_{\tau_k})/n)]$ , in view of the proof of Proposition 2. Further by virtue of the basic estimate (Proposition 2) we know

$$\sum_{k=0}^{\infty} E[\exp(-\nu r(Z_{\tau_k})/n)] \leq (1 - C_1)^{-1} \exp(-\nu r(p)/n).$$

The desired inequality holds for  $C_3 = \exp(a) C_0 C_2 (1 - C_1)$ . q.e.d.

Combining this with the result on the weak convergence of the hitting distribution in Theorem A (ii), we know that for arbitrary  $\xi \in M(\infty)$  and any  $\varepsilon > 0$ , there exists a neighborhood  $U$  of  $\xi$  such that

$$(4) \quad p \in U \Rightarrow |w(p, Z) - \varphi(\xi)| < \varepsilon,$$

when  $Z \in K_p$ . Furthermore, we can show the following lemma.

**Lemma 3.** *For any  $\varepsilon > 0$ , there exist a positive large constant  $R$  and a small constant  $\gamma_0$  such that, if*

$$p \notin D_R = \{\eta \in M; r(\eta) < R\}$$

and  $\text{dist}(p, q) < \gamma_0$ , then

$$|w(p, Z) - w(q, Z')| < \varepsilon,$$

for any  $Z \in K_p$  and  $Z' \in K_q$ .

Proof. For any  $\varepsilon > 0$ , there exist some points  $\xi_1, \dots, \xi_n \in M(\infty)$  and open sets  $U_i \ni \xi_i$  such that

$$\begin{aligned} p \in U_i \text{ and } Z \in K_p \\ \Rightarrow |w(p, Z) - \varphi(\xi_i)| < \varepsilon/2 \end{aligned}$$



for all  $i=1, 2, \dots, n$  and  $M(\infty) \subset \bigcup_{i=1}^n U_i$ . Take a closed neighborhood  $U'_i \subset U_i$  of  $\xi_i$  so that  $M(\infty) \subset \bigcup_{i=1}^n U'_i$ . Then, there exists  $R>0$  satisfying  $M \setminus D_R \subset \bigcup_{i=1}^n U'_i$ . Therefore for sufficiently small,  $\gamma_0$  we know that

$$\begin{aligned} \text{dist}(p, q) &< \gamma_0, p \notin D_R \\ \Rightarrow |w(p, Z) - w(q, Z')| \\ &\leq |w(p, Z) - \varphi(\xi_i)| + |\varphi(\xi_i) - w(q, Z')| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon, \end{aligned}$$

whenever  $Z \in K_p$  and  $Z' \in K_q$ , by choosing  $i$  so that  $p \in U'_i$ . q.e.d.

Because the holomorphic tangent bundle is holomorphically trivial, there exists a frame of holomorphic vector fields  $Y_1, \dots, Y_n$ . Let  $\Phi_z(p) = \text{Exp}(\text{Re} \sum_{i=1}^n z^i Y_i)(p)$ , for  $p \in M$  and  $z = (z^1, \dots, z^n)$  in  $\mathbf{C}^n$ . This transformation on  $M$  was considered in T. Asaba [2] and proven to enjoy the next property:

For any  $R>0$ , there exists  $\Delta_\delta = \{z \in \mathbf{C}^n; \sum_{i=1}^n |z^i|^2 < \delta\}$  such that  $\Phi_z(p)$  is a smooth mapping from  $\Delta_\delta \times D_R$  to  $M$  satisfying the following properties (i), (ii) and (iii).

- (i) For each  $z \in \Delta_\delta$ ,  $\Phi_z$  gives a biholomorphic mapping from the domain  $D_R$  to  $\Phi_z(D_R)$ .
- (ii)  $\Phi_0$  is the identity transformation on  $D_R$ .
- (iii) For  $p \in D_R$ ,  $\Phi_*(p)$  defines a diffeomorphism from  $\Delta_\delta$  to some neighborhood of  $p$ .

Using this transformation  $\Phi$ , we can prove the continuity of the stochastic solution  $u$ .

**Lemma 4.** *For any  $\varepsilon>0$  and  $R>0$ , there exists  $\gamma>0$  such that for each  $Z \in K_p$  and  $q$  enjoying  $p \in D_R$  and  $\text{dist}(p, q) < \gamma$ , we can always find  $Z' \in K_q$  so that*

$$|w(p, Z) - w(q, Z')| < \varepsilon.$$

Proof. To begin, replace  $R$  by a sufficiently large one and choose  $\gamma_0$  so that the implication in Lemma 3 holds for  $\varepsilon/2$  instead of  $\varepsilon$ . Fix  $Z \in K_p$ . We then consider the holomorphic local transformation  $\Phi$  and the Kähler diffusion  $B_t(\eta)$  on  $M$  starting from  $\eta \in M$ , independent of  $Z$  and measurable in  $t, z$  and  $\omega$ . Let

$$(5) \quad Z_t^{\Phi_z(p)} = \begin{cases} \Phi_z(Z_t), & t \leq \tau \\ B_{t-\tau}(\Phi_z(Z_\tau)), & t > \tau, \end{cases}$$

where  $\tau = \inf \{t > 0; Z_t \notin D_R\}$ .

We next perform the time change by letting  $\hat{Z}_t^{\Phi_z(\rho)} = Z_{\tau_t}^{\Phi_z(\rho)}$ , up to the explosion time of  $\hat{Z}^{\Phi_z(\rho)} = (\hat{Z}_t^{\Phi_z(\rho)})_{t \geq 0}$ , where  $\tau_t = \inf \{s > 0; \int_0^s (\det A_{\alpha, \beta}(u))^{1/n} du \geq t\}$ ,  $(A_{\alpha, \beta}(t))_{t \geq 0}$  being the density of the increasing process associated with  $Z^{\Phi_z(\rho)} = (Z_t^{\Phi_z(\rho)})_{t \geq 0}$  according to Proposition 1.

On the other hand, taking conditional expectation, we have

$$w(p, Z) = W[-C(n) \int_0^\tau f^{1/n}(Z_t) dt] + \int_{\partial D_R} E[C(n) \int_\tau^{\xi(Z)} f^{1/n}(Z_t) dt + \varphi(Z_{\xi(Z)}) | Z_\tau = \eta] P(Z_\tau \in d\eta).$$

If we set  $W_t = Z_{t+\tau}$  and let

$$w(\eta, W) = E[-C(n) \int_0^{\xi(Z)-\tau} f^{1/n}(W_t) dt + \varphi(Z_{\xi(Z)}) | Z_\tau = \eta]$$

for  $W = (W_t)_{0 \leq t < \xi(Z)-\tau}$ , then

$$w(p, W) = E[-C(n) \int_0^\tau f^{1/n}(Z_t) dt] + \int_{\partial D_R} w(\eta, W) P(Z_\tau \in d\eta).$$

Similarly, letting  $\sigma$  be the first exit time from  $\Phi_z(D_R)$  of  $\hat{Z}^{\Phi_z(\rho)}$ , we set  $W_t^{\Phi_z(\rho)} = \hat{Z}_{t+\sigma}^{\Phi_z(\rho)}$ ,  $0 \leq t < \xi(\hat{Z}^{\Phi_z(\rho)}) - \sigma$  and then, for  $W^{\Phi_z(\rho)} = (W_t^{\Phi_z(\rho)})_{t \geq 0}$ ,

$$w(\eta, W^{\Phi_z(\rho)}) = E[-C(n) \int_0^{\xi(W^{\Phi_z(\rho)})} f^{1/n}(W_t^{\Phi_z(\rho)}) dt + \varphi(W_{\xi(W^{\Phi_z(\rho)})}^{\Phi_z(\rho)}) | \hat{Z}_\sigma^{\Phi_z(\rho)} = \eta].$$

Then

$$w(\Phi_z(p), \hat{Z}^{\Phi_z(\rho)}) = E[-C(n) \int_0^\sigma f^{1/n}(\hat{Z}_t^{\Phi_z(\rho)}) dt] + \int_{\partial \Phi_z(D_R)} w(\eta', W^{\Phi_z(\rho)}) P(\hat{Z}_\sigma^{\Phi_z(\rho)} \in d\eta')$$

Therefore, after all we have that

$$w(p, Z) - w(\Phi_z(p), Z^{\Phi_z(\rho)}) = E[-C(n) (\int_0^\tau f^{1/n}(Z_t) dt - \int_0^\sigma f^{1/n}(\hat{Z}_t^{\Phi_z(\rho)}) dt)] + \int_{\partial D_R} \{w(\eta, W) - w(\Phi_z(\eta), W^{\Phi_z(\rho)})\} P(Z_\tau \in d\eta).$$

From Lemma 2, there exists  $\delta > 0$  such that the absolute value of the second term of the right hand side is less than  $\varepsilon/2$  for every  $z \in \Delta_\delta$ . While the continuity of  $f^{1/n}$  shows that the first term of the right hand side is less than  $\varepsilon/2$  in

the absolute value, whenever  $z \in \Delta_\delta$ .

Because, for sufficiently small  $\gamma$ , the  $\gamma$ -neighborhood of each  $p \in D_R$  is contained in the image of  $\Delta_\delta$  by the mapping  $\Phi_z(p)$ , for  $q = \Phi_z(p)$ ,  $Z' = \hat{Z}^{\Phi_z(p)}$  is the required conformal martingale in our lemma. q.e.d.

**Proof of Proposition 3.** The last inequality in Lemma 4 implies  $w(p, Z) \geq u(q) - \varepsilon$ . Taking the infimum over  $Z \in K_p$ , we can conclude that  $u(p) \geq u(q) - \varepsilon$ , whenever  $p, q \in D_R$  and  $\text{dist}(p, q) < \gamma$ . Exchanging the role of  $p$  and  $q$ , we see that  $u$  is a continuous function on  $M$ . Recalling the estimate (4) noted after Lemma 2, we know that  $\lim_{p \rightarrow \xi} u(p) = \varphi(\xi)$  for each  $\xi \in M(\infty)$ . This completes the proof. q.e.d.

### 5. The Bellman principle

The purpose of this section is to establish the Bellman principle in order to localize the stochastic expression of the function  $u$  defined by (3).

**Proposition 4.** *For every bounded domain  $D$  of  $M$  and  $p \in D$ , we obtain*

$$u(p) = \inf_{Z \in K_p} E \left[ -C(n) \int_0^{\tau_D(Z)} f^{1/n}(Z_t) dt + u(Z_{\tau_D(Z)}) \right],$$

where  $\tau_D(Z) = \inf\{t > 0; Z_t \notin D\}$ .

**Proof.** Fix  $\varepsilon > 0$  and take  $R$  so that  $D_R \supset \bar{D}$ . For each  $q \in \partial D$  there exist  $\delta > 0$  and  $Z \in K_q$  such that, for  $z \in \Delta_\delta$ ,

$$|w(\Phi_z(q), \hat{Z}^{\Phi_z(q)}) - u(q)| > \varepsilon,$$

where  $Z^{\Phi_z(q)}$  is the conformal martingale defined by (5). Therefore, we can select several points  $q_1, \dots, q_n \in \partial D$  and their neighborhoods  $\Delta(q_1), \dots, \Delta(q_n)$  so that  $\partial D \subset \bigcup_{i=1}^n \Delta(q_i)$  (disjoint union), the image of  $\Phi_z(q_i)$  contains  $\Delta(q_i)$  and

$$|w(\Phi_z(q_i), \hat{Z}^{\Phi_z(q_i)}) - u(q_i)| < \varepsilon,$$

whenever  $Z^{\Phi_z(q_i)}$  is in  $\Delta(q_i)$ ,  $i = 1, 2, \dots, n$ .

For each  $Z \in K_p$ , we set

$$Z_i^* = \begin{cases} Z_t, & \text{if } t \leq \tau_D(Z) \\ \hat{Z}_{t-\tau_D(Z)}^{\Phi_z(q_i)} & \text{if } t > \tau_D(Z), Z_{\tau_D(Z)} \in \Delta(q_i) \text{ and} \\ & \Phi_z(q_i) = Z_{\tau_D(Z)}, i = 1, 2, \dots, n, \end{cases}$$

where we take  $Z^{\Phi_z(q_i)}$  and  $Z$  to be independent. Then  $Z^* = (Z_i^*) \in K_p$ . By the same method of B. Gaveau [6; pp. 400-403], we can prove that

$$\begin{aligned}
 u(p) - \varepsilon &\leq E\left[-C(n) \int_0^{\tau_D} f^{1/n}(Z_t) dt + u(Z_{\tau_D})\right] \\
 &\leq E\left[-C(n) \int_0^{\xi(Z)} f^{1/n}(Z_t) dt + \varphi(Z_{\xi(Z)})\right].
 \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary, the proof is completed. q.e.d.

**6. Proof of the main theorem**

Finally, we shall finish the proof of the main theorem by showing the next two propositions.

**Proposition 5.**  *$u$  is a plurisubharmonic function and  $(dd^c u)^n = f \omega_0^n/n!$  on  $M$ .*

**Proposition 6.** *If  $u_0$  is a solution of (1), then*

$$u_0(p) = \inf_{z \in K_p} E\left[-C(n) \int_0^{\xi} f^{1/n}(Z_t) dt + \varphi(Z_{\xi})\right].$$

*In particular, (1) has a unique solution.*

**Proof of Proposition 5.** Let  $p$  be an arbitrary point of  $M$ . Choose a complex local coordinate system  $(D, z^1, \dots, z^n)$  around  $p$  such that  $\psi = (z^1, \dots, z^n)$  defines a biholomorphic mapping from  $D$  to the complex unit ball  $B = \{(z^1, \dots, z^n) \in \mathbf{C}^n; \sum_{i=1}^n |z^i|^2 < 1\}$ . For the push forward function  $U(z) = (\psi_* u)(z) = u(\psi^{-1}(z))$ ,

$$U(z) = \inf_{z \in K_z} E\left[-C(n) \int_0^{\tau_{B(z)}} (\psi_*(f \det(g)))_{i\bar{j}}^{-1/n}(Z_t) dt + U(Z_{\tau_{B(z)}})\right],$$

where  $g_{i\bar{j}} = g(\partial/\partial z^i, \partial/\partial \bar{z}^j)$  and  $K_z$  is the family of all  $\mathbf{C}^n$ -valued conformal martingales  $Z$  which start from  $z \in B$  such that  $a_{i\bar{j}}(t) = d\langle z^i(Z_t), z^j(Z_t) \rangle / dt$  satisfy  $\det(a_{i\bar{j}}(t)) \geq 1, t \geq 0$  a.s.

Consider the following Monge-Ampère equation

$$(6) \quad \begin{cases} v \in PSH(B) \cap C(\bar{B}) \\ (dd^c v)^n = \psi_*(f \det(g_{i\bar{j}})) dV \\ v|_{\partial B} = U|_{\partial B}, \end{cases}$$

where  $dV$  stands for the Lebesgue measure on  $\mathbf{C}^n$ . Because of the strongly pseudo-convexity of  $B$ , we see that Theorem 4 and Remark of B. Gaveau [6; pp. 402–403] ensure the following stochastic description of the solution  $v_0$  of (6):

$$\begin{aligned}
 v_0(z) &= \inf_{z \in K_z} E\left[-C(n) \int_0^{\tau_{B(z)}} (\psi_*(f \det(g_{i\bar{j}})))^{1/n}(Z_t) dt \right. \\
 &\quad \left. + U(Z_{\tau_{B(z)}})\right], \quad z \in B.
 \end{aligned}$$

Hence, we know that  $v_0=U$  on  $B$  and  $u(p)=\psi_*v_0(p)\in PSH(D)$  and that  $(dd^c u)^n=f\omega_0^n/n!$  on  $D$ . q.e.d.

Proof of Proposition 6. To begin, take the countable family of charts  $(U_i; z_i^1, \dots, z_i^n)_{i=1}^\infty$  appeared in the proof of Proposition 1, we may assume that each  $\psi_i=(z_i^1, \dots, z_i^n)$  gives a biholomorphic mapping between  $U_i$  and the unit ball  $B\subset C^n$ . By virtue of Theorem 4 of B. Gaveau [6], for any  $\varepsilon>0$ , there exists a  $Z^{(1)}\in K_p$  such that

$$E[-C(n)\int_0^{\sigma_1} f^{1/n}(Z_t)dt+u_0(Z_{\sigma_1}^{(1)})]\leq u_0(p)+\varepsilon/2,$$

where  $\sigma_1$  is the stopping time for  $Z^{(1)}$  defined in the proof of Proposition 1. For each  $q\in\partial U_i$  there exists  $\delta>0$  and  $Z\in K_q$  such that

$$w(\Phi_z(q), \hat{Z}^{\Phi_z(q)})<u_0(q)+\varepsilon/2^2,$$

whenever  $z\in\Delta_\delta$ . Using the same argument as in the proof of Proposition 4, we can construct  $Z^{(2)}\in K_p$  which satisfies

$$Z_{i\wedge\sigma_1}^{(1)}=Z_{i\wedge\sigma_1}^{(2)}$$

and

$$E[-C(n)\int_0^{\sigma_2} f^{1/n}(Z_t^{(2)})dt+u_0(Z_{\sigma_2}^{(2)})]\leq u_0(p)+\varepsilon/2+\varepsilon/2^2,$$

where  $\sigma_2$  is defined for  $Z^{(2)}$  in the same way as above. Repeating this procedure, we obtain a sequence  $(Z^{(k)})_{k=1}^\infty\subset K_p$  so that  $Z_{i\wedge\sigma_{k-1}}^{(k-1)}=Z_{i\wedge\sigma_{k-1}}^{(k)}$ ,  $t\geq 0$ . a.s. and that

$$E[-C(n)\int_0^{\sigma_k} f^{1/n}(Z_t^{(k)})dt+u_0(Z_{\sigma_k}^{(k)})]\leq u_0(p)+\sum_{i=1}^k \varepsilon/2^i,$$

where  $\sigma_k$  is defined for  $Z^{(k)}$  as above.

Define  $Z_t=Z_t^{(k)}$ , if  $t<\sigma_k$ . Then we can easily check that  $Z=(Z_t)\in K_p$  and that  $\lim_{k\rightarrow\infty}\sigma_k=\zeta(Z)$ . Hence, we know

$$E[-C(n)\int_0^\zeta f^{1/n}(Z_t)dt+\varphi(Z_\zeta)]\leq u_0(p)+\varepsilon.$$

Letting  $\varepsilon\rightarrow 0$ , we can conclude that

$$u_0(p)\geq \inf_{Z\in K_p} E[-C(n)\int_0^\zeta f^{1/n}(Z_t)dt+\varphi(Z_\zeta)].$$

On the other hand, we can inductively obtain, for each  $Z\in K_p$ ,

$$u_0(p)\leq E[-C(n)\int_0^{\sigma_k} f^{1/n}(Z_t)dt+u_0(Z_{\sigma_k})], \quad k=1, 2, 3, \dots,$$

and so we have the opposite inequality, by letting  $k\rightarrow\infty$ . q.e.d.

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