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Osaka University
A STOCHASTIC RESOLUTION OF A COMPLEX
MONGE-AMPERE EQUATION ON A NEGATIVELY
CURVED KÄHLER MANIFOLD

HIROSHI KANEKO

(Received December 4, 1985)

1. Introduction

The Dirichlet problem for the complex Monge-Ampère equation on a
strongly pseudo-convex domain of $\mathbb{C}^n$ was studied and solved by Bedford-
Taylor [3]. The same problem for the Monge-Ampère equation on a negative-
ly curved Kähler manifold has been recently proposed and solved by T. Asaba
[2]. The main purpose of this paper is to solve the equation by using a method
of the stochastic control presented by B. Gaveau [6].

Let $M$ be an $n$-dimensional simply connected Kähler manifold with metric
$g$ whose sectional curvature $K$ satisfies

$$-b^2 \leq K \leq -a^2$$
on $M$ for some positive constants $b$ and $a$. $\omega_0$ denotes the associated Kahler
form. We denote by $M(\infty)$ the Eberlein-O’Neill’s ideal boundary of $M$ and
we always consider the cone topology on $\overline{M}=M \cup M(\infty)$ (see [4] for these
notions). T. Asaba formulated the Monge-Ampère equation on $M$ in the fol-
lowing manner:

We write $\text{PSH}(D)$ for the family of locally bounded plurisubharmonic
functions defined on a complex manifold $D$. When $u \in \text{PSH}(D)$, the current
$(dd^c u)^n=dd^c u \wedge \cdots \wedge dd^c u$ of type-$(n, n)$ is defined as a positive Radon measure
$n$-copies
on $D$. Therefore, for given functions $f \in C(M)$ and $\varphi \in C(M(\infty))$, the complex
Monge-Ampère equation

$$u \in \text{PSH}(M) \cap C(\overline{M})$$

$$\begin{cases}
(dd^c u)^n = f \omega_0^n/n! & \text{on } M \\
u |_{M(\infty)} = \varphi
\end{cases}$$

(1)
can be considered. T. Asaba found a unique solution of (1) by imposing the
following condition on $f$: there exist two positive constants $\mu_0$ and $C_0$ such that
Here and in the sequel $r$ stands for the distance function from a fixed point of $M$. Following a similar line to the proof performed by B. Gaveau [6], in which a stochastic proof of the existence of the Monge-Ampère equation on a strongly pseudo-convex domain of $C^*$ was presented, we will prove not only the existence of the solution of (1) but also its uniqueness (§ 3, Theorem B). Actually T. Asaba assumed condition (2) for a specific value of $\mu_0$. In what follows, we assume the condition (2) on $f$ holding for some $\mu_0 > 0$ and $C_0 > 0$.

In accordance with the first part of B. Gaveau [6], a certain transience behavior of the sample path of the conformal martingales on $M$ need to be studied. It was conjectured by H. Wu [13] that $M$ is biholomorphic to a bounded domain of $C^*$ (cf. Y.T. Siu [11] and R.E. Greene [7]). If this would be true, then the conformal martingales of the type considered by B. Gaveau [6] must hit the boundary of $M$. In fact, we shall prove in Section 2 that the almost all sample paths of every non-degenerate conformal martingale converge to points of the ideal boundary $M(\infty)$. We use the method of J.J. Prat [10], in which the sample paths’ property was proven for the Brownian motion on Riemannian manifolds with negative curvature bounded away from zero.

The basic estimates obtained in Section 2 will be further utilized after Section 3 in resolving the Monge-Ampère equation stochastically.

The author expresses his thanks to T. Asaba for private discussions.

2. Basic estimates for non-degenerate conformal martingales

We first define the notion of the conformal martingales on $M$.

**Definition.** Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{t \geq 0}$. An $M$-valued continuous stochastic process $(Z_t)_{0 \leq t < \zeta}$ defined up to a stopping time $\zeta > 0$ is said to be a conformal martingale, if

(i) there exists $p \in M$ such that $Z_0 = p$ a.s.

(ii) there exists a sequence of stopping times $(T_n)_{n=1}^\infty$ such that $T_n \leq \zeta$, $\lim_{n \to \infty} T_n = \zeta$, and $(f(Z_t \wedge T_n))_{t \geq 0}$ is a $C$-valued bounded $(\mathcal{F}_t)$-martingale for every holomorphic function $f$ on $M$ (we need note that $M$ is a Stein manifold and so $M$ possesses enough holomorphic functions).

Noting the triviality of the bundle of unitary frames, we choose smooth vector fields $X_1, \ldots, X_n$ of type-(1, 0) on $M$ so that $g(X_a, X_b) = \delta_{a, b}$ on $M$. For a smooth function $f$ defined on $M$, we write $Lf$ for the Levi-form of $f$. The notion of conformal martingale is related to the Levi-form in the following way:

**Proposition 1.** For each conformal martingale $(Z_t)_{0 \leq t < \zeta}$ on $M$, there is a non-negative hermitian matrix valued $(\mathcal{F}_t)$-adapted process $(\sum_{a, b} \delta_{a, b}(t))_{0 \leq t < \zeta}$ such that
it is increasing (in the sense that \( s \leq t \Rightarrow \sum_{a, \beta} (s) \leq \sum_{a, \beta} (t) \) as hermitian matrices a.s.) and that, for each real valued function \( f \in C^2(M) \)

\[
f(Z_t) - f(Z_0) - \sum_{a, \beta=1}^n \int_0^t Lf(X_a, X_\beta)_{Z_s} d\sum_{a, \beta}(s)
\]

is a local martingale.

Proof. Take countable local complex charts \((U_i; z^i_1, \ldots, z^i_r)_{i=1,2,\ldots}\) of \(M\) and closed sets \(V_i \subset U_i\) such that \(\{V_i\}_{i=1}^\infty\) covers \(M\). Since \(M\) is a Stein manifold, we may assume that \(z^i_1, \ldots, z^i_r\) are the restrictions to \(U_i\) of certain holomorphic functions on \(M\) for every \(i=1,2,3,\ldots\). Define a sequence of stopping times \(\sigma_k\) and random variables \(i_k\) successively as follows:

\[
\begin{align*}
\sigma_0 &= 0 \\
i_0 &= \inf \{i; Z_0 \in V_i\} \\
\sigma_1 &= \inf \{t > 0; Z_t \in U_{i_0}\} \\
\vdots \\
\sigma_k &= \inf \{t > \sigma_{k-1}; Z_t \in U_{i_{k-1}}\} \\
i_k &= \inf \{i; Z_{\sigma_k} \in V_i\}
\end{align*}
\]

By virtue of Ito's formula, we obtain

\[
f(Z_{t \wedge \sigma_k}) - f(Z_{t \wedge \sigma_{k-1}}) = \sum_{\beta=1}^n \int_{t \wedge \sigma_{k-1}}^{t \wedge \sigma_k} \partial f/\partial z^\alpha(Z_s) dz^\alpha(Z_s) \\
+ \sum_{a, \beta=1}^n \int_{t \wedge \sigma_{k-1}}^{t \wedge \sigma_k} \partial f/\partial z^\alpha(Z_s) dz^\beta(Z_s) \\
+ \sum_{a, \beta=1}^n \int_{t \wedge \sigma_{k-1}}^{t \wedge \sigma_k} \partial^2 f/\partial z^\alpha \partial z^\beta(Z_s) d\langle z^\alpha(Z_s), z^\beta(Z_s) \rangle,
\]

where \(z^\alpha = z^\alpha_{i_{k-1}}, \alpha = 1, 2, \ldots, n, k = 1, 2, 3, \ldots\). Define a hermitian matrix valued process \(\sigma(t)\) by \(\sum_{k=1}^n \sigma_k(t) (\partial / \partial z^\alpha |_{Z_t}) = X_a |_{Z_t}, \alpha = 1, 2, \ldots, n\) and set

\[
\sum_{a, \beta}(t) = \sum_{k=1}^n \int_0^{t \wedge \sigma_k} \sigma_k^\alpha(s) \sigma_k^\beta(s) d\langle z^\alpha(Z_s), z^\beta(Z_s) \rangle,
\]

then this can be well defined, independently of the choice of local coordinates, and further

\[
f(Z_{t \wedge \sigma_k}) - f(Z_{t \wedge \sigma_{k-1}}) - \sum_{a, \beta=1}^n \int_{t \wedge \sigma_{k-1}}^{t \wedge \sigma_k} Lf(X_a, X_\beta)_{Z_s} d\sum_{a, \beta}(s)
\]

is a martingale. Since \(\lim \sigma_k = \zeta\), the proof is completed. q.e.d.

For our investigation, it is enough to consider exclusively conformal
martingales \((Z_t)_{0 \leq t < \xi}\) for which the following stopping times \(\tau_k (k=0, 1, 2, 3, \ldots)\) are finite almost surely:

\[
\begin{align*}
\tau_0 &= 0 \\
\tau_1 &= \inf \{ t > 0; \text{dist}(Z_t, Z_0) = 1 \} \\
\vdots \\
\tau_{k+1} &= \inf \{ t > \tau_k; \text{dist}(Z_t, Z_{\tau_k}) = 1 \}
\end{align*}
\]

We call such property "admissible" and in what follows \(\tau_k\) means the above stopping time. Here, we present a basic estimate of the same type as in D. Sullivan [12].

**Proposition 2.** For any \(\mu \in (0, a)\), there exists a constant \(C_1 \in (0, 1)\) such that

\[
E[\exp(-\mu r(Z_{\tau_{k+1}}))] \leq C_1 E[\exp(-\mu r(Z_{\tau_k}))], \quad k = 0, 1, 2, 3, \ldots,
\]

for every admissible conformal martingale \((Z_t)_{0 \leq t < \xi}\).

Proof. A Jacobi field estimate—the Hessian comparison theorem presented in [8; Theorem A] implies

\[
L \exp(-\mu r) \leq (\mu(\mu - a)/2) \exp(\mu r) \text{ in the sense [8].}
\]

By applying Proposition 1 to the function \(\exp(-\mu r)\), we then have

\[
E[\exp(-\mu r(Z_{\tau_{k+1}}))] = E[\exp(-\mu r(Z_{\tau_k}))]
+ E[\sum_{a, \beta = 1}^{k+1} \int_{\tau_k}^{\tau_{k+1}} L \exp(-\mu r(X_a, X_\beta) d \Sigma_{a, \beta}(s))]
\leq E[\exp(-\mu r(Z_{\tau_k}))]
+ (\mu(\mu - a)/2) E[\int_{\tau_k}^{\tau_{k+1}} \exp(-\mu r(Z_s))d(\text{trace } \Sigma_{a, \beta}(s))],
\]

\(k = 0, 1, 2, \ldots\).

While, taking conditional expectation, we have

\[
E[\int_{\tau_k}^{\tau_{k+1}} \exp(-\mu r(Z_s))d(\text{trace } \Sigma_{a, \beta}(s))]
= \int_M P(Z_{\tau_k} \in d\eta)E[\int_{\tau_k}^{\tau_{k+1}} \exp(-\mu r(Z_s))d(\text{trace } \Sigma_{a, \beta}(s))|Z_{\tau_k} = \eta]
\geq \int_M P(Z_{\tau_k} \in d\eta) \exp(-\mu(r(\eta) + 1))E[\int_{\tau_k}^{\tau_{k+1}} d(\text{trace } \Sigma_{a, \beta}(s))|Z_{\tau_k} = \eta],
\]

which is not less than \(\exp(-\mu)C_2^{-1} E[\exp(-\mu r(Z_{\tau_k}))]\) by virtue of Lemma 1 stated below. Hence we arrive at the desired estimate with \(C_1 = 1 + ((\mu(\mu - a)/2))C_2^{-1} \exp(-\mu)\). q.e.d.
In the above proof, we have used the next lemma, which also will be utilized in § 4.

**Lemma 1.** There exists a positive constant $C_2$ depending only on $a$ and $b$ such that

$$C_2^{-1} \leq E \left[ \int_{\tau_k}^{\tau_{k+1}} d(\text{trace } \sum_{a, \beta} s) \mid Z_{\tau_k} = \eta \right] \leq C_2$$

holds $P(Z_{\tau_k} \in d\eta)$-a.s. $k = 0, 1, 2, 3, \ldots$, for every admissible conformal martingale $Z_t$.

**Proof.** For $f \in C^2_b(M)$, we know from Proposition 1 that

$$E[f(Z_{\tau_{k+1}}) - f(Z_{\tau_k}) - \sum_{a, \beta} \int_{\tau_k}^{\tau_{k+1}} Lf(X_a, X_{\beta}) Z_s d \sum_{a, \beta}(s) \mid Z_{\tau_k} = \eta] = 0$$

$$P(Z_{\tau_k} \in d\eta)$$-a.s., $k = 0, 1, 2, 3, \ldots$.

Taking a countably dense subset of $C^2_b(M)$ and by the approximation procedure we know that the exceptional $\eta$-set in the above statement can be taken independently of $f \in C^2_b(M)$. Choose $f = f^{(a)}(p) \in C^2_b(M)$ which coincides with $\text{dist}(p, \eta)$ on a neighborhood of $\{p; \text{dist}(p, \eta) \leq 1\}$. Then it turns out that

$$1 = E \left[ \sum_{a, \beta} \int_{\tau_k}^{\tau_{k+1}} Lf(X_a, X_{\beta}) Z_s d \sum_{a, \beta}(s) \mid Z_{\tau_k} = \eta \right] \quad P(Z_{\tau_k} \in d\eta)$$-a.s.

Again by the Hessian comparison theorem, we find that there exists a constant $C_2$ depending only on the curvature bounds $a$ and $b$ such that

$$C_2 g \leq Lf^{(a)} \leq C_2^{-1} g \quad \text{on } \{p; \text{dist}(p, \eta) \leq 1\},$$

so we have

$$C_2^{-1} \leq E \left[ \int_{\tau_k}^{\tau_{k+1}} d(\text{trace } \sum_{a, \beta} s) \mid Z_{\tau_k} = \eta \right] \leq C_2$$

$$P(Z_{\tau_k} \in d\eta)$$-a.s. q.e.d.

The next theorem is an immediate consequence of Proposition 2 combined with the geometrical method employed by D. Sullivan [12] and J.J. Prat [10].

**Theorem A.** For every admissible conformal martingale $(Z_t)_{0 \leq t < \xi}$, the following are true:

(i) The limit $\lim_{t \uparrow \xi} Z_t$ exists in $M(\infty)$ a.s.

(ii) For any $\xi \in M(\infty)$, $\varepsilon > 0$ and neighborhood $V \subset M(\infty)$ of $\xi$, there exists a neighborhood $U \subset M$ of $\xi$ relative to the cone topology such that

$$P(\lim_{t \uparrow \xi} Z_t \in V) \geq 1 - \varepsilon,$$

whenever $Z_t$ strats from a point of $U$. $U$ does not depend on the choice of $(Z_t)_{0 \leq t < \xi}$. 
3. The stochastic solution of the Monge-Ampère equation— the statement of the main theorem

Let \( K_p \) be the family of all admissible conformal martingales \( Z = (Z_t)_{t \leq t(z)} \) on \( M \) such that \( Z \) starts from \( p \in M \) and the associate process \( (\Sigma_{a,b}(t))_{t \leq t(z)} \) in Proposition 1 possesses a density \( (A_{a,b}(t))_{t \leq t(z)} \) with respect to the Lebesgue measure \( dt \) with \( \det A_{a,b}(t) \geq 1 \) for \( t \geq 0 \) a.s. For \( Z \in K_p \), set

\[
\omega(p, Z) = E[-C(n) \int_0^{t(z)} f^{1/n}(Z_t) dt + \varphi(Z_t(z))],
\]

where \( C(n) = n/B(n!)/n \). By virtue of Lemma 2 in the next section, we know that, if \( Z = (Z_t) \) is the conformal diffusion generated by the Kahler metric \( g \) on \( M \), then \( \omega(p, Z) \) is exactly the solution of the Dirichlet problem with boundary condition on the sphere at infinity:

\[
\begin{align*}
\Delta g u/2 & = C(n)f^{1/n} \\
u|_{M(\infty)} & = \varphi
\end{align*}
\]

for the Laplace-Beltrami operator \( \Delta g \) related to \( g \). Now, we can describe the solution of the Monge-Ampère equation (1), using the above stochastic notations.

**Theorem B.** The function

\[
(3) \quad u(p) = \inf_{Z \in K_p} \omega(p, Z), \quad p \in M
\]

is the unique solution of the Monge-Ampère equation (1).

In the following sections, we shall prove this theorem. The proof will be performed by the stochastic control method due to B. Gaveau [6].

4. Continuity of the stochastic solution

In this section, we shall prove the continuity of the function \( u \) defined by (3).

**Proposition 3.** \( u \) can be extended to a continuous function on \( \bar{M} \) and \( u|_{M(\infty)} = \varphi \).

We have to prepare several lemmas for the proof.

**Lemma 2.** For each \( Z \in K_p \), there exist positive constants \( v \) and \( C_2 \) depending only on the constants \( \mu_0, C_0 \) in (2) and the curvature bounds such that

\[
E\left[ \int_0^{t(z)} f(Z_t)^{1/n} dt \right] \leq C_2 \exp(-vr(p)).
\]
Proof. By the assumption (2) imposed on $f$, for $\nu \leq \mu_0$, we know
\[
E \left[ \int_0^{\tau_0} f(Z_t) \nu dt \right] 
\leq C_0 E \left[ \int_0^{\tau_0} \exp \left( -\nu r(Z_t) \right) dt \right] 
\leq C_0 \sum_{k=0}^{\tau_{k+1}} E \left[ \int_{\tau_k}^{\tau_{k+1}} \exp \left( -\nu r(Z_t) \right) dt \right],
\]
where $\tau_0 = 0$, $\tau_1 = \inf \{ t > 0; \text{dist}(Z_t, Z_0) = 1 \}$, $\tau_2 = \inf \{ t > 0; \text{dist}(Z_t, Z_{\tau_1}) = 1 \}$, $\ldots$. We may assume that $\nu$ is so small that $\nu/n$ is less than $a$. Because
\[
E \left[ \int_{\tau_k}^{\tau_{k+1}} \exp \left( -\nu r(Z_t) \right) dt \right] \leq E \left[ \int_{\tau_k}^{\tau_{k+1}} \exp \left( -\nu r(Z_t) \right) d(\text{trace} \Sigma_a, \beta(t)) \right],
\]
we have
\[
E \left[ \int_{\tau_k}^{\tau_{k+1}} \exp \left( -\nu r(Z_t) \right) dt \right] \leq \exp(a) C_2 E \left[ \exp \left( -\nu r(Z_{\tau_k}) \right) \right],
\]
in view of the proof of Proposition 2. Further by virtue of the basic estimate (Proposition 2) we know
\[
\sum_{k=0}^{\tau_{_0}} E \left[ \exp \left( -\nu r(Z_{\tau_k}) \right) \right] \leq (1-C_1)^{-1} \exp \left( -\nu T \right).
\]
The desired inequality holds for $C_3 = \exp (a) C_0 C_2 (1-C_1)$. q.e.d.

Combining this with the result on the weak convergence of the hitting distribution in Theorem A (ii), we know that for arbitrary $\xi \in M(\infty)$ and any $\varepsilon > 0$, there exists a neighborhood $U$ of $\xi$ such that
\[
(4) \quad p \in U \Rightarrow |w(p, Z) - \varphi(\xi)| < \varepsilon,
\]
when $Z \in K_p$. Furthermore, we can show the following lemma.

**Lemma 3.** For any $\varepsilon > 0$, there exist a positive large constant $R$ and a small constant $\gamma_0$ such that, if
\[
p \in D_R = \{ \eta \in M; r(\eta) < R \}
\]
and $\text{dist}(p, q) < \gamma_0$, then
\[
|w(p, Z) - w(q, Z')| < \varepsilon,
\]
for any $Z \in K_p$ and $Z' \in K_q$.

Proof. For any $\varepsilon > 0$, there exist some points $\xi_1, \ldots, \xi_n \in M(\infty)$ and open sets $U_i \ni \xi_i$ such that
\[
p \in U_i \text{ and } Z \in K_p
\Rightarrow |w(p, Z) - \varphi(\xi_i)| < \varepsilon/2
for all $i=1, 2, \cdots, n$ and $M(\infty) \subset \bigcup_{i=1}^{n} U_i$. Take a closed neighborhood $U'_i \subset U_i$ of $\xi_i$ so that $M(\infty) \subset \bigcup_{i=1}^{n} U'_i$. Then, there exists $R>0$ satisfying $M \setminus D_R \subset \bigcup_{i=1}^{n} U'_i$.

Therefore for sufficiently small, $\gamma_0$ we know that

$$\text{dist}(p, q) < \gamma_0, \quad p \in D_R$$

$$\Rightarrow |w(p, Z) - w(q, Z')|$$

$$\leq |w(p, Z) - \varphi(\xi)| + |\varphi(\xi) - w(q, Z')|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

whenever $Z \in K_p$ and $Z' \in K_q$, by choosing $i$ so that $p \in U'_i$. q.e.d.

Because the holomorphic tangent bundle is holomorphically trivial, there exists a frame of holomorphic vector fields $Y_1, \cdots, Y_n$. Let $\Phi_i(p) = \text{Exp}(\text{Re} \sum_{i=1}^{n} \varphi_i Y_i)(p)$, for $p \in M$ and $z = (z^1, \cdots, z^n)$ in $C^n$. This transformation on $M$ was considered in T. Asaba [2] and proven to enjoy the next property:

For any $R>0$, there exists $\Delta_\delta = \{z \in C^n; \sum_{i=1}^{n} |z^i|^2 < \delta\}$ such that $\Phi_i(p)$ is a smooth mapping from $\Delta_\delta \times D_R$ to $M$ satisfying the following properties (i), (ii) and (iii).

(i) For each $z \in \Delta_\delta$, $\Phi_i$ gives a biholomorphic mapping from the domain $D_R$ to $\Phi_i(D_R)$.

(ii) $\Phi_0$ is the identity transformation on $D_R$.

(iii) For $p \in D_R$, $\Phi_i(p)$ defines a diffeomorphism from $\Delta_\delta$ to some neighborhood of $p$.

Using this transformation $\Phi$, we can prove the continuity of the stochastic solution $u$.

**Lemma 4.** For any $\varepsilon>0$ and $R>0$, there exists $\gamma>0$ such that for each $Z \in K_p$ and $q$ enjoying $p \in D_R$ and $\text{dist}(p, q) < \gamma$, we can always find $Z' \in K_q$ so that

$$|w(p, Z) - w(q, Z')| < \varepsilon.$$

**Proof.** To begin, replace $R$ by a sufficiently large one and choose $\gamma_0$ so that the implication in Lemma 3 holds for $\varepsilon/2$ instead of $\varepsilon$. Fix $Z \in K_p$. We then consider the holomorphic local transformation $\Phi$ and the Kahler diffusion $B_t(\eta)$ on $M$ starting from $\eta \in M$, independent of $Z$ and measurable in $t, z$ and $\omega$.

Let

$$Z_{t}(\eta) = \begin{cases} \Phi_i(Z_i), & t \leq \tau \\ B_{t-\tau}(\Phi_i(Z_i)), & t > \tau, \end{cases}$$

(5)
where $\tau = \inf \{ t > 0 ; Z_t \in D_\varepsilon \}$.

We next perform the time change by letting $\hat{Z}^{\Phi_t(z)} = Z^{\Phi_t(z)}_{\tau_t}$, up to the explosion time of $\hat{Z}^{\Phi_t(z)} = (\hat{Z}^{\Phi_t(z)}_t)_{t \geq 0}$, where $\tau_t = \inf \{ s > 0 ; \int_0^s (\det A_{\Phi_t(z)}(u))^{1/n} du \geq t \}$. $\left( A_{\Phi_t(z)}(t) \right)_{t \geq 0}$ being the density of the increasing process associated with $Z^{\Phi_t(z)} = (Z^{\Phi_t(z)}_t)_{t \geq 0}$ according to Proposition 1.

On the other hand, taking conditional expectation, we have

$$w(p, Z) = W[-C(n) \int_0^{\tau} f^{1/n}(Z_t) dt] + \int_{\partial B} E[C(n) \int_t^{\tau} f^{1/n}(Z_t) dt + \varphi(Z_t) \mid Z_t = \eta] P(Z_t \in d\eta).$$

If we set $W_t = Z_{t+\tau}$ and let

$$w(\eta, W) = E[-C(n) \int_0^{\tau} f^{1/n}(W_t) dt + \varphi(Z_{\tau}) \mid Z_{\tau} = \eta]$$

for $W = (W_t)_{0 \leq t < \tau}$, then

$$w(p, W) = E[-C(n) \int_0^{\tau} f^{1/n}(Z_t) dt] + \int_{\partial B} w(\eta, W) P(Z_t \in d\eta).$$

Similarly, letting $\sigma$ be the first exit time from $D_B$ of $\hat{Z}^{\Phi_t(z)}$, we set $W^{\Phi_t(z)} = \hat{Z}^{\Phi_t(z)}_{t \sigma}$, $0 \leq t < \tau(\hat{Z}^{\Phi_t(z)}) - \sigma$ and then, for $W^{\Phi_t(z)} = (W^{\Phi_t(z)}_t)_{t \geq 0}$,

$$w(\eta, W^{\Phi_t(z)}) = E[-C(n) \int_0^{\tau} f^{1/n}(W^{\Phi_t(z)}_t) dt + \varphi(W^{\Phi_t(z)}_{\tau(\hat{Z}^{\Phi_t(z)})}) \mid \hat{Z}^{\Phi_t(z)}_{\tau(\hat{Z}^{\Phi_t(z)})} = \eta].$$

Then

$$w\left( \Phi_t(z), \hat{Z}^{\Phi_t(z)} \right) = E[-C(n) \int_0^{\sigma} f^{1/n}(\hat{Z}^{\Phi_t(z)}_t) dt] + \int_{\partial B} w(\eta', W^{\Phi_t(z)}) P(\hat{Z}^{\Phi_t(z)} \in d\eta').$$

Therefore, after all we have that

$$w(p, Z) - w\left( \Phi_t(z), Z^{\Phi_t(z)} \right) = E[-C(n) \left( \int_0^{\tau} f^{1/n}(Z_t) dt - \int_0^{\sigma} f^{1/n}(\hat{Z}^{\Phi_t(z)}_t) dt \right)] + \int_{\partial B} \{ w(\eta, W) - w(\Phi_t(z), W^{\Phi_t(z)}) \} P(Z_t \in d\eta).$$

From Lemma 2, there exists $\delta > 0$ such that the absolute value of the second term of the right hand side is less than $\varepsilon/2$ for every $z \in \Delta_\delta$. While the continuity of $f^{1/n}$ shows that the first term of the right hand side is less than $\varepsilon/2$ in
the abo absolute value, whenever $z \in \Delta_b$.

Because, for sufficiently small $\gamma$, the $\gamma$-neighborhood of each $p \in D_R$ is contained in the image of $\Delta_b$ by the mapping $\Phi_s(p)$, for $q = \Phi_s(p)$, $Z^{\Phi_s(p)}$ is the required conformal martingale in our lemma.

Proof of Proposition 3. The last inequality in Lemma 4 implies $\omega(p, Z) \leq u(q) - \varepsilon$. Taking the infimum over $Z \in K_p$, we can conclude that $u(p) \leq u(q) - \varepsilon$, whenever $p, q \in D_R$ and $\text{dist}(p, q) < \gamma$. Exchanging the role of $p$ and $q$, we see that $u$ is a continuous function on $M$. Recalling the estimate (4) noted after Lemma 2, we know that $\lim_{\tau(x, t)^*} u(p) = p(x)$ for each $\xi \in M(\infty)$. This completes the proof.

q.e.d.

5. The Bellman principle

The purpose of this section is to establish the Bellman principle in order to localize the stochastic expression of the function $u$ defined by (3).

Proposition 4. For every bounded domain $D$ of $M$ and $p \in D$, we obtain

$$u(p) = \inf_{Z \in K_p} E[-C(n) \int_0^{\tau_D(Z)} f u'(Z_t) dt + u(Z_{\tau_D(Z)})],$$

where $\tau_D(Z) = \inf\{t > 0; Z_t \in D\}$.

Proof. Fix $\varepsilon > 0$ and take $R$ so that $D_R \supset \bar{D}$. For each $q \in \partial D$ there exist $\delta > 0$ and $Z \in K_q$ such that, for $z \in \Delta_b$,

$$|\omega(\Phi_s(q), Z^{\Phi_s(q)}) - u(q)| > \varepsilon,$$

where $Z^{\Phi_s(q)}$ is the conformal martingale defined by (5). Therefore, we can select several points $q_1, \ldots, q_n \in \partial D$ and their neighborhoods $\Delta(q_i), \ldots, \Delta(q_n)$ so that $\partial D \subset \bigcup_{i=1}^n \Delta(q_i)$ (disjoint union), the image of $\Phi_s(q_i)$ contains $\Delta(q_i)$ and

$$|\omega(\Phi_s(q_i), Z^{\Phi_s(q_i)}) - u(q_i)| < \varepsilon,$$

whenever $Z^{\Phi_s(q_i)}$ is in $\Delta(q_i), i = 1, 2, \ldots, n$.

For each $Z \in K_p$, we set

$$Z_t^* = \begin{cases} Z_t, & \text{if } t \leq \tau_D(Z) \\ Z_t^{\Phi_s(q_i)}, & \text{if } t > \tau_D(Z), Z_{\tau_D(Z)} \in \Delta(q_i) \text{ and} \\ \Phi_s(q_i) = Z_{\tau_D(Z)}, & i = 1, 2, \ldots, n, \end{cases}$$

where we take $Z^{\Phi_s(q_i)}$ and $Z$ to be independent. Then $Z^* = (Z_t^*) \in K_p$. By the same method of B. Gaveau [6; pp. 400–403], we can prove that
Since $\varepsilon > 0$ is arbitrary, the proof is completed.

q.e.d.

6. Proof of the main theorem

Finally, we shall finish the proof of the main theorem by showing the next two propositions.

Proposition 5. $u$ is a plurisubharmonic function and $(dd^c u)^n = f \omega^n/n!$ on $M$.

Proposition 6. If $u_0$ is a solution of (1), then

$$u_0(p) = \inf_{z \in K_p} E[-C(n) \int_0^T f^{1/n}(Z_t) dt + \varphi(Z_t)].$$

In particular, (1) has a unique solution.

Proof of Proposition 5. Let $p$ be an arbitrary point of $M$. Choose a complex local coordinate system $(D, z^1, \ldots, z^n)$ around $p$ such that $\varphi = (z^1, \ldots, z^n)$ defines a biholomorphic mapping from $D$ to the complex unit ball $B = \{ |z^1|^2 + \cdots + |z^n|^2 < 1 \}$. For the push forward function $U(z) = (\psi u)(z) = u(\psi^{-1}(z))$,

$$U(z) = \inf_{z \in K_z} E[-C(n) \int_0^T (\psi^*(f \det(g)))^{1/n}(Z_t) dt + U(Z_{\tau_p(z)})],$$

where $g_{ij} = g(\partial/\partial z^i, \partial/\partial z^j)$ and $K_z$ is the family of all $C^\infty$-valued conformal martingales $Z$ which start from $z \in B$ such that $a_{i\overline{j}}(t) = d\langle z^i(Z_t), z^j(Z_t) \rangle / dt$ satisfy $\det(a_{i\overline{j}}(t)) \geq 1$, $t \geq 0$ a.s.

Consider the following Monge-Ampère equation

$$
\begin{cases}
  \psi \in PSH(B) \cap C(\overline{B}) \\
  (dd^c \psi)^n = \psi^*(f \det(g_{i\overline{j}}))dV \\
  \psi |_{\partial B} = U |_{\partial B},
\end{cases}
$$

where $dV$ stands for the Lebesgue measure on $C^\infty$. Because of the strongly pseudo-convexity of $B$, we see that Theorem 4 and Remark of B. Gaveau [6; pp. 402–403] ensure the following stochastic description of the solution $v_0$ of (6):

$$v_0(z) = \inf_{z \in K_z} E[-C(n) \int_0^T (\psi^*(f \det(g_{i\overline{j}})))^1 (Z_t) dt + U(Z_{\tau_p(z)})], \quad z \in B.$$
Hence, we know that $v_0 = U$ on $B$ and $u(p) = \psi_k v_0(p) \in \text{PSH}(D)$ and that $(dd^c)^n = f \omega_0^n/n!$ on $D$.

Proof of Proposition 6. To begin, take the countable family of charts $(U_i; z_1^i, \ldots, z_l^i)_{r=1}^\infty$ appeared in the proof of Proposition 1, we may assume that each $\psi_i = (z_1^i, \ldots, z_l^i)$ gives a biholomorphic mapping between $U_i$ and the unit ball $B \subset \mathbb{C}^l$. By virtue of Theorem 4 of B. Gaveau [6], for any $\varepsilon > 0$, there exists a $Z(\varepsilon) \in K_p$ such that

$$E[-C(n) \int_0^\sigma f^{1/2}(Z_i) dt + u_0(Z(\varepsilon))] \leq u_0(p) + \varepsilon/2,$$

where $\sigma$ is the stopping time for $Z(t)$ defined in the proof of Proposition 1. For each $q \in \partial U_i$ there exists $\delta > 0$ and $Z \in K_q$ such that $w(\Phi(q), \tilde{Z}(\varepsilon)) < u_0(q) + \varepsilon/2$, whenever $z \in \Delta_\delta$. Using the same argument as in the proof of Proposition 4, we can construct $Z(\varepsilon) \in K_p$ which satisfies

$$Z_{i/\varepsilon} = Z_{i/\varepsilon},$$

and

$$E[-C(n) \int_0^\sigma f^{1/n}(Z_i)^2 dt + u_0(Z_{i/\varepsilon})] \leq u_0(p) + \varepsilon/2 + \varepsilon/2^2,$$

where $\sigma_2$ is defined for $Z(\varepsilon)$ in the same way as above. Repeating this procedure, we obtain a sequence $(Z(h))_{h=1}^\infty \subset K_p$ so that $Z_{i/\varepsilon_{h-1}} = Z_{i/\varepsilon_{h-1}}$, $t \geq 0$, a.s. and that

$$E[-C(n) \int_0^\sigma f^{1/n}(Z_i)^2 dt + u_0(Z_{h/\varepsilon})] \leq u_0(p) + \sum_{i=1}^h \varepsilon/2^i,$$

where $\sigma_h$ is defined for $Z(h)$ as above.

Define $Z_t = Z(t)$, if $t < \sigma_h$. Then we can easily check that $Z = (Z_t) \in K_p$ and that $\lim_{h \to \infty} \sigma_h = \xi(Z)$. Hence, we know

$$E[-C(n) \int_0^\xi f^{1/n}(Z_i)^2 dt + \varphi(Z_i)] \leq u_0(p) + \varepsilon.$$

Letting $\varepsilon \to 0$, we can conclude that

$$u_0(p) \geq \inf_{z \in K_p} E[-C(n) \int_0^\xi f^{1/n}(Z_i)^2 dt + \varphi(Z_i)].$$

On the other hand, we can inductively obtain, for each $Z \in K_p$,

$$u_0(p) \leq E[-C(n) \int_0^\sigma f^{1/n}(Z_i)^2 dt + u_0(Z_{s_k})], \quad k = 1, 2, 3, \ldots,$$

and so we have the opposite inequality, by letting $k \to \infty$. q.e.d.
References


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