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ON THE 2 BY 2 WEAKLY HYPERBOLIC SYSTEMS

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Abstract

We study the wellposedness in the Gevrey classes γ^s and in C^∞ of the Cauchy problem for 2 by 2 weakly hyperbolic systems. In this paper we shall give some conditions to the case that the characteristic roots oscillate rapidly and vanish at an infinite number of points.

1. Introduction

In this paper we shall consider the Cauchy problem, on $[0, T] \times \mathbb{R}^n_x$,

$$\begin{cases} \partial_t U - \displaystyle\sum_{j=1}^{n} A_j(t)\partial_{x_j} U + B(t)U = 0, \\ U(0, x) = U_0(x), \end{cases} \tag{1}$$

where

$$A_j \in AC([0, T]), \quad B \in L^1(0, T), \tag{2}$$

$AC([0, T])$ denoting the space of absolutely continuous functions.

Here, we restrict ourselves to the case when the $A_j(t)$'s are 2×2 matrices with real entries, whereas $B(t)$ is a complex 2×2 matrix. We write, for $(t, \xi) \in [0, T] \times \mathbb{R}^n$,

$$A(t, \xi) = \sum_{j=1}^{n} A_j(t)\xi_j = \begin{pmatrix} a(t, \xi) & b(t, \xi) \\ c(t, \xi) & d(t, \xi) \end{pmatrix}, \quad B(t) = \begin{pmatrix} e(t) & f(t) \\ g(t) & h(t) \end{pmatrix},$$

Finally, we assume that $A(t, \xi)$ is a hyperbolic matrix, which means that

$$\Delta(t, \xi) = (a - d)^2 + 4bc = (a - d)^2 + (b + c)^2 - (b - c)^2 \geq 0. \tag{3}$$

We shall denote by $\mathcal{C}^k([0, T])$, with $k = \nu + \alpha \in \mathbb{R}^+$, $\nu = [k] \in \mathbb{N}$, and $0 \leq \alpha < 1$, the space of $\mathcal{C}^\nu$ functions with ν-th derivative α-Hölder continuous (if $\alpha > 0$). Moreover, $\mathcal{C}^\infty = \mathcal{C}^\infty(\mathbb{R}^n)$ will be the space of infinitely differentiable functions, and $\gamma^s = \gamma^s(\mathbb{R}^n)$,

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$s \geq 1$, the space of Gevrey functions of order s, i.e., the functions satisfying

$$\sup_{x \in K} |D_x^\alpha \varphi(x)| \leq C_K \Lambda_K^{|\alpha|} \alpha!^s, \quad \text{for all} \quad K \Subset \mathbb{R}^n \text{ and } \alpha \in \mathbb{N}^n.$$

We say that the Cauchy problem (1) is well posed in γ^s (resp. in $\mathcal{C}^\infty$) if, for any $U_0 \in \gamma^s$ (resp. $U_0 \in \mathcal{C}^\infty$), there exists a unique solution $U(t,x)$ in $C^1([0,T];\gamma^s)$ (resp. in $C^1([0, T]; \mathcal{C}^\infty)$).

Concerning the second order equation $u_{tt} - t^k u_{xx} + t^l u_x = 0$, with $k, l \geq 0$, in case that $k - 2l - 2 > 0$ (resp. $k - 2l - 2 = 0$), Ivrii [10] proved the wellposedness in γ^s for $1 \leq s < (2k - 2l)/(k - 2l - 2)$ (resp. in $\mathcal{C}^\infty$). For hyperbolic equations of higher order, suitable Levi conditions on the lower order terms were proved to be sufficient for the wellposedness, by Kajitani, Wakabayashi and Yagdjian [12] and D'Ancona and Kinoshita [4]. The first goal of the present paper is to find analogous Levi conditions for 2 by 2 systems.

On the other hand, for the homogeneous equation $u_{tt} - c(t)u_{xx} = 0$, with $c(t) \geq 0$ belonging to $C^k([0,T])$, Colombini, Jannelli and Spagnolo [1] proved the wellposedness in γ^s for $1 \leq s < 1 + k/2$ (see also [3] and [14]). For equations of the more general type $u_{tt} - c(t)u_{xx} - d(t)u_{tx} = 0$ with $d(t)^2 + 4c(t) \geq 0$, where $c(t)$, $d(t)$ belong to $C^k([0, T])$, $k \geq 2$, Kinoshita and Spagnolo [9] proved the wellposedness in γ^s for $1 \leq s < 1 + k/2$, under the condition on the characteristic roots

$$\frac{\lambda_1^2 + \lambda_2^2}{(\lambda_1 - \lambda_2)^2} \leq M < \infty, \tag{4}$$

which is equivalent to each of the following ones on the coefficients:

$$\frac{|c(t)|}{d(t)^2 + 4c(t)} \leq M_1, \quad \frac{d(t)^2}{d(t)^2 + 4c(t)} \leq M_2, \tag{5}$$

where, M, M_1, M_2 are constants independent on t, ξ. A similar result holds true also for hyperbolic equations of higher order (see [9] and [4]).

Going back to the 2×2 systems, Nishitani [15] found a necessary and sufficient condition for the $\mathcal{C}^\infty$, wellposedness in case of analytic coefficients depending also on x. In [13] (see also [6]), this result was partially extended to systems with non-analytic, sufficiently smooth coefficients, by proving the γ^s wellposedness for $s < s(k)$, where k is the regularity of the coefficients. Here we shall prove a more precise result, by relating the degree of Gevrey wellposedness also with the order of vanishing of the discriminant of the system.

On the other hand, in our previous paper [7] the result of [1] was extended to $m \times m$ systems, $m = 2, 3$, with Hölder coefficients, i.e., with smoothness $0 \leq k \leq 1$. Thus, the second goal of the present paper is to study the case $k \geq 1$, and in particular to find a suitable generalization of (4) and (5) for 2×2 systems with $k \geq 2$.

We must mention that, from the point of view of wellposedness, the 2×2 systems obtained from a second order equation are not the best ones. Indeed, for some systems with a special structure, we expect stronger results; for instance the Cauchy problem for a symmetric system is always well posed in $\mathcal{C}^\infty$. This suggests that the wellposedness can be related to the another quantity, besides the difference of the roots.

To formulate our result, we associate to $A(t, \xi)$ the *traceless* matrix

$$A_0(t, \xi) = A(t, \xi) - \frac{\operatorname{Tr} A(t, \xi)}{2} I = \begin{pmatrix} (a-d)/2 & b \\ c & (d-a)/2 \end{pmatrix}. \tag{6}$$

Taking the matrix norm $\|X\|^2 = \operatorname{Tr}(XX^*) = \sum x_{ij}^2$, we have

$$\|A_0\|^2 = \frac{1}{2}\{(a-d)^2 + (b+c)^2 + (b-c)^2\} \geq \frac{1}{2}\Delta. \tag{7}$$

Next, for $0 < \varepsilon < 1$, and $\xi \in \mathbb{R}^n$, we introduce the sets:

$$\Omega_\varepsilon = \Omega_\varepsilon(\xi) = \{t \in [0, T]\colon \sqrt{\Delta(t, \xi)} \leq \varepsilon|\xi|\},$$
$$\tilde{\Omega}_\varepsilon = \tilde{\Omega}_\varepsilon(\xi) = \{t \in [0, T]\colon \sqrt{2}\|A_0(t, \xi)\| \leq \varepsilon|\xi|\},$$

which depend only on $\xi/|\xi|$. The measure $\mu(\Omega_\varepsilon)$ is a measure of the defect of strict hyperbolicity. By (3) and (7) it follows that $\tilde{\Omega}_\varepsilon \subseteq \Omega_\varepsilon$. Then, denoting by $'$ the derivative in time, we define

$$\Gamma(t, \xi) = \frac{1}{4}\{(a-d)^2 + (b+c)^2\}, \tag{8}$$

$$\Theta(t, \xi) = \frac{1}{8}(b-c)\{(a-d)(b+c)' - (b+c)(a-d)'\}, \tag{9}$$

Note that by (3) and (7), it follows

$$\frac{1}{4}(b-c)^2 \leq \Gamma, \quad \frac{1}{4}\Delta \leq \Gamma \leq \frac{1}{2}\|A_0\|^2 \leq 2\Gamma, \tag{10}$$

and

$$|\Theta| \leq \frac{1}{\sqrt{2}}\Gamma\|A_0'\|. \tag{11}$$

By (11) and (2), it follows that $(\Theta\Gamma^{-1})(t,\xi)$ belongs to $L^1(0,T)$ for all ξ, with uniform norm as $|\xi| = 1$. We also note that

$$\operatorname{Tr}(A_0 B) = \frac{1}{2}\{(a-d)(h-e) - (b+c)(f+g) + (b-c)(f-g)\}. \tag{12}$$

The main result of this paper is the following:

Theorem 1.1. *Let* $A_j \in AC([0,T])$, $B \in L^1(0,T)$, *and assume* (3). *Moreover assume that, for some* $\alpha \geq 0$, $\beta > 0$, *and some* $M > 0$,

$$\mu(\Omega_\varepsilon) \leq M\varepsilon^\alpha, \tag{13}$$

$$\int_{[0,T]\setminus\Omega_\varepsilon} \frac{|\Theta/\Gamma + \mathrm{Tr}(A_0B)|}{\sqrt{\Delta}}\,dt + \int_{[0,T]\setminus\Omega_\varepsilon} \frac{|\sqrt{\Delta}'|}{\sqrt{\Delta}}\,dt + \int_{[0,T]\setminus\tilde{\Omega}_\varepsilon} \frac{\|A_0'\|}{\|A_0\|}\,dt \leq M\varepsilon^{-\beta} \tag{14}$$

for all $0 < \varepsilon < 1$ *and all* $|\xi| = 1$. *Then,* (1) *is well posed in* γ^s *for*

$$1 \leq s < 1 + \frac{\alpha+1}{\beta}. \tag{15}$$

Thus, in order to get larger $s \geq 1$ in (15), we must take larger $\alpha \geq 0$ and smaller $\beta > 0$.

Concerning the wellposedness in C^∞, we prove:

Theorem 1.2. *Let* $A_j \in AC([0,T])$, $B \in L^1(0,T)$, *and assume* (3). *Moreover assume that, for some* $M > 0$,

$$\int_{[0,T]\setminus\Omega_\varepsilon} \frac{|\Theta/\Gamma + \mathrm{Tr}(A_0B)|}{\sqrt{\Delta}}\,dt + \int_{[0,T]\setminus\Omega_\varepsilon} \frac{|\sqrt{\Delta}'|}{\sqrt{\Delta}}\,dt + \int_{[0,T]\setminus\tilde{\Omega}_\varepsilon} \frac{\|A_0'\|}{\|A_0\|}\,dt \leq M\log\varepsilon^{-1} \tag{16}$$

for all $0 < \varepsilon < 1$ *and all* $|\xi| = 1$. *Then,* (1) *is well posed in* C^∞.

REMARK 1.3. We can strengthen the assumptions (14) and (16) of Theorems 1.1 and 1.2, by replacing the first integral by

$$\int_{[0,T]\setminus\Omega_\varepsilon} \frac{|\Theta|/\Gamma + |\mathrm{Tr}(A_0B)|}{\sqrt{\Delta}}\,dt.$$

The meaning of (14) and (16) is the following:
(i) the conditions on $|\sqrt{\Delta}'|/\sqrt{\Delta}$ and $\|A_0'\|/\|A_0\|$ take care of the low regularity of the coefficients,
(ii) the condition on Θ/Γ is the analogous of (4) and (5) for a system, while the condition on $\mathrm{Tr}(A_0B)$ is a kind of Levi condition.

REMARK 1.4. The following are typical examples of "good" lower order terms.
(i) $B = \phi(t)I$, with $\phi(t)$ scalar function. Therefore: $\mathrm{Tr}(A_0B) = 0$.
(ii) $n = 1$, $B = A_1(t) \equiv A(t,\xi)\xi^{-1}$. Therefore: $\mathrm{Tr}(A_0B) = (1/2)\Delta\xi^{-1}$.
(iii) $n = 1$, $B = A_1'(t) \equiv A'(t,\xi)\xi^{-1}$. Therefore: $\mathrm{Tr}(A_0B) = (1/2)\Delta'\xi^{-1}$.
In all these cases (note that $\Delta'/\sqrt{\Delta} = 2(\sqrt{\Delta})' \in L^1(0,T)$), we have

$$\int_{[0,T]\setminus\Omega_\varepsilon} \frac{|\Theta/\Gamma + \mathrm{Tr}(A_0B)|}{\sqrt{\Delta}}\,dt \leq \int_{[0,T]\setminus\Omega_\varepsilon} \frac{|\Theta|/\Gamma}{\sqrt{\Delta}}\,dt + C$$

for some constant C. Hence, the presence of B does not affect the Gevrey, or $\mathcal{C}^\infty$, wellposedness ensured by Theorems 1.1 and 1.2.

EXAMPLE 1.5. Let $n = 1$, $b = \xi$, $c = t^k\xi$, $g = t^l$, $a = d = e = f = h = 0$. Then (1) is equivalent to the equation $u_{tt} - t^k u_{xx} + t^l u_x = 0$, and

$$\Delta = 4t^k\xi^2, \quad \Omega_\varepsilon = \left[0, \frac{1}{2}\varepsilon^{2/k}\right], \quad \Theta = 0, \quad \mathrm{Tr}(A_0 B) = t^l\xi.$$

Let $l \geq 0$ and $k - 2l - 2 > 0$ (resp. $k - 2l - 2 = 0$). Then, for $|\xi| = 1$,

$$\begin{aligned}\int_{[0,T]\setminus\Omega_\varepsilon} \left\{ \frac{|\mathrm{Tr}(A_0 B)|}{\sqrt{\Delta}} + \frac{|\sqrt{\Delta}'|}{\sqrt{\Delta}} \right\} dt &= \int_{(1/2)\varepsilon^{2/k}}^{T} \frac{t^l + kt^{k/2-1}}{2t^{k/2}}\, dt \\ &\leq C \int_{(1/2)\varepsilon^{2/k}}^{T} t^{l-k/2}\, dt \leq C'\varepsilon^{-(1-(2l+2)/k)} \\ &\qquad (\text{resp.} \leq C' \log \varepsilon^{-1}).\end{aligned}$$

On the other hand, the third term in (14) is estimated by $M\varepsilon^{-\beta}$ for all $\beta > 0$, since $\|A_0\| = |\xi|\sqrt{t^{2k} + 1}$. Thus, applying Theorem 1.1 with $\alpha = 2/k$ and $\beta = 1 - (2l + 2)/k$ (resp. Theorem 1.2), we get the γ^s wellposedness for $1 \leq s < (2k - 2l)/(k - 2l - 2)$ (resp. the C^∞ wellposedness). This coincides with the result of Ivrii [10].

When the coefficients of the system are sufficiently smooth, the terms $|\sqrt{\Delta}'|/\sqrt{\Delta}$ and $\|A_0'\|/\|A_0\|$ in (14) and (16) can be omitted, and from Theorems 1.1 and 1.2, we derive:

Corollary 1.6. *Let $A_j \in C^k([0, T])$ with $k \geq 2$, and $B \in L^1(0, T)$. Assume* (3). *Also assume that there is $M > 0$ such that, for all $|\xi| = 1$,*

$$\int_0^T \frac{|\Theta/\Gamma + \mathrm{Tr}(A_0 B)|}{\Delta^{1/2-1/k}}\, dt \leq M. \tag{17}$$

Then, (1) *is well posed in γ^s for $1 \leq s < 1 + k/2$.*

Corollary 1.7. *Instead of* (3), *assume that $\Delta(t, \xi) > 0$ for all $t > 0$. Also assume that the $A_j(t)$'s are analytic on $[0, T]$, and*

$$\frac{|\Theta/\Gamma + \mathrm{Tr}(A_0 B)|}{\sqrt{\Delta}} \leq \frac{M}{t}, \quad \forall t > 0, \tag{18}$$

with a uniform constant M for $|\xi| = 1$. Then, (1) *is well posed in $\mathcal{C}^\infty$.*

To prove Corollary 1.7, we use the inequality

$$\int_{\{\sqrt{\Delta}\geq\varepsilon\}} \frac{|\Delta'(t,\xi)|}{\Delta(t,\xi)}\,dt \leq C\log\varepsilon^{-1},$$

which is an easy consequence of the fact that the quadratic form

$$\Delta(t,\xi) \equiv \sum_{i,j}^{1,n} \delta_{ij}(t)\xi_i\xi_j \geq 0,$$

has analytic coefficients $\delta_{ij}(t)$.

Let us now prove Corollary 1.6. Under our assumptions, we do not get any information about the sets Ω_ε and $\tilde{\Omega}_\varepsilon$. Thus, in order to derive the wished result from Theorem 1.1, we are forced to take $\alpha = 0$ in condition (13). On the other side, we can take $\beta = 2/k$ in (14). Indeed we have, for $|\xi| = 1$. putting $\Phi = \Theta/\Gamma + \mathrm{Tr}(A_0 B)$,

$$\begin{aligned}\int_{[0,T]\setminus\Omega_\varepsilon} \frac{|\Phi| + |\sqrt{\Delta}'|}{\sqrt{\Delta}}\,dt &\leq \varepsilon^{-2/k}\int_{[0,T]\setminus\Omega_\varepsilon} \frac{|\Phi| + |\sqrt{\Delta}'|}{\Delta^{1/2-1/k}}\,dt \\ &= \varepsilon^{-2/k}\left\{\int_{[0,T]\setminus\Omega_\varepsilon} \frac{|\Phi|}{\Delta^{1/2-1/k}}\,dt + \int_{[0,T]\setminus\Omega_\varepsilon} \frac{|\Delta'|}{2\Delta^{1-1/k}}\,dt\right\} \\ &\leq M'\varepsilon^{-2/k}.\end{aligned}$$

The last inequality follows from (17), and from the assumption that $A \in C^k([0,T])$, whence $\Delta \in C^k([0,T])$, thanks to the following lemma:

Lemma 1.8 ([1]). *Let $f(t) \in C^k([0,T])$, $k \geq 1$, $f(t) \geq 0$. Then, $f(t)^{1/k} \in AC([0,T])$ and there exists $C = C(k,T) > 0$ such that*

$$\int_0^T \frac{|f'(t)|}{f(t)^{1-1/k}}\,dt \leq C\{\|f\|_{C^k([0,T])}\}^{1/k}. \tag{19}$$

If we drop the assumption of positivity by considering an arbitrary function $g(t) \in C^k([0,T])$with $k \geq 1$, we can apply (19) with $f = g^2$ to get the following estimate (see [16])

$$\int_0^T \frac{|g'(t)|}{|g(t)|^{1-2/k}}\,dt \leq C\{\|g\|_{C^k([0,T])}\}^{2/k}. \tag{20}$$

A similar estimate also holds to any matrix function $X(t) \in C^k$, $k \geq 1$. Indeed, by applying (20) to each entry of $X(t)$, we find:

$$\int_0^T \frac{\|X'(t)\|}{\|X(t)\|^{1-2/k}}\,dt \leq C\{\|X\|_{C^k([0,T])}\}^{2/k}. \tag{21}$$

Now, for ξ running in $\{|\xi| = 1\}$ the matrix function $A_0(t, \xi)$ belongs uniformly to $C^k([0, T])$, and $\|A_0\| \geq \varepsilon$ in $\tilde{\Omega}_\varepsilon$. Hence for $|\xi| = 1$ we get

$$\int_{[0,T]\setminus\tilde{\Omega}_\varepsilon} \frac{\|A_0'\|}{\|A_0\|}\, dt \leq \int_{[0,T]\setminus\tilde{\Omega}_\varepsilon} \frac{\|A_0'\|}{\|A_0\|^{1-2/k}\varepsilon^{2/k}}\, dt \leq C\varepsilon^{-2/k}.$$

This concludes the proof of Corollary 1.6.

REMARK 1.9. In the case $1 \leq k < 2$, we obtain the same conclusion of Corollary 1.6 without the assumption (17). Indeed, by applying Theorem 1.1 with $\alpha = 0$ and $\beta = 2/k$, we get, recalling that $\Theta/\Gamma \in L^1(0, T)$,

$$\int_{[0,T]\setminus\Omega_\varepsilon} \frac{|\Theta/\Gamma + \mathrm{Tr}(A_0 B)|}{\sqrt{\Delta}}\, dt \leq C\varepsilon^{-1} \leq C\varepsilon^{-2/k},$$

while

$$\int_{[0,T]\setminus\Omega_\varepsilon} \frac{|\sqrt{\Delta}'|}{\sqrt{\Delta}}\, dt = \int_{[0,T]\setminus\Omega_\varepsilon} \frac{|\Delta'|}{2\Delta}\, dt \leq \int_{[0,T]\setminus\Omega_\varepsilon} \frac{|\Delta'|}{2\Delta^{1-1/k}(\varepsilon^2)^{1/k}}\, dt \leq C\varepsilon^{-2/k}.$$

A similar estimate holds for $\|A_0'\|/\|A_0\|$, as proved above.

Also in the case $0 \leq k < 1$, the result of Corollary 1.6 holds true without the assumption (17): this was proved in our previous paper [7].

Summing up, we get the following:

Corollary 1.10. *Let $A_j \in C^k([0, T])$ with $0 \leq k < 2$, and $B \in L^1(0, T)$. Assume* (3). *Then,* (1) *is well posed in γ^s for $1 \leq s < 1 + k/2$.*

EXAMPLE 1.11. Let $n = 1$. If $a = d = 0$, $b = \xi$, $c = c(t)\xi$, $c(t) \geq 0$, and $B = 0$. Then, (1) is equivalent to the equation $u_{tt} - c(t)u_{xx} = 0$, and (17) is trivially fulfilled. Thus, if $c \in C^k([0, T])$, Corollary 1.6 ensures the wellposedness in γ^s for $s < 1 + k/2$, which is the result of [1].

EXAMPLE 1.12. Let $n = 1$. If $a = 0$, $b = \xi$, $c = c(t)\xi$, $d = d(t)\xi$, and $B = 0$, (1) is equivalent to the equation $u_{tt} - d(t)u_{tx} - c(t)u_{xx} = 0$. Now, if $\Delta = d^2 + 4c \geq 0$, and $c, d \in C^k([0, T])$ with $k \geq 2$, the assumption (17) in Corollary 1.6 is a consequence of the condition (5). Indeed, for $|\xi| = 1$, such a condition implies

$$\begin{aligned}\int_{[0,T]\setminus\Omega_\varepsilon} \frac{|\Theta|/\Gamma}{\Delta^{1/2-1/k}}\, dt &\leq C_1 \int_{[0,T]\setminus\Omega_\varepsilon} \frac{|c'| + |d'|}{\{d^2 + 4c\}^{1/2-1/k}}\, dt \\ &\leq C_2 \left\{ \int_0^T \frac{|c'|}{|c|^{1/2-1/k}}\, dt + \int_0^T \frac{|d'|}{|d|^{1-2/k}}\, dt \right\} \leq M.\end{aligned}$$

Here we have applied the inequality (20) to the function $d(t)$, and then to the function $c(t)$ but with k replaced by $h = 4k/(k+2)$, so that $1/2 - 1/k = 1 - 2/h$. Note that $k \geq 2$ implies $k \geq h$, hence $c \in C^h$ since $c \in C^k$. Thus, Corollary 1.6 gives the γ^s wellposedness for $s < 1 + k/2$, which coincides with the result of [9].

Before stating the next Corollary, we consider the following conditions, where the constants M are uniform for $|\xi| = 1$.

$$|a - d| \leq M\sqrt{\Delta}, \tag{22}$$

$$|b + c| \leq M\sqrt{\Delta}, \tag{23}$$

$$|b - c| \leq M\sqrt{\Delta}, \tag{24}$$

noting that, by the identity $(a-d)^2 + (b+c)^2 = \Delta + (b-c)^2$, it follows

$$(22) \text{ and } (23) \iff (24). \tag{25}$$

Then, we have:

Corollary 1.13. i) *Assume* (24). *Then, if* $A_j \in C^k([0, T])$, *the Cauchy problem* (1), *for any* $B \in L^1(0, T)$, *is well posed in* γ^s *for* $s < 1+k/2$, *while, if the* $A_j(t)$*'s are analytic*, (1) *is well posed in* $\mathcal{C}^\infty$.
ii) *Assume either* (22), *or* (23). *Let* $A_j \in C^k([0, T])$ (*resp.* A_j *analytic*). *Then*, (1) *is well posed in* γ^s *for* $s < 1 + k/2$ (*resp. in* $\mathcal{C}^\infty$), *provided* $B \in L^1(0, T)$ *satisfy the uniform estimate, for* $|\xi| = 1$,

$$\int_0^T \frac{|\operatorname{Tr}(A_0 B)|}{\Delta^{1/2-1/k}}\, dt \leq M, \quad (\textit{resp.}\ |\operatorname{Tr}(A_0 B)| \leq M\sqrt{\Delta}/t). \tag{26}$$

Proof. i) Since $2\|A_0\|^2 = \Delta + 2(b-c)^2$ (see (7)), by (24) it follows

$$\Delta \leq 2\|A_0\|^2 \leq (1 + 2M^2)\Delta.$$

Now $|\Theta|/\Gamma \leq C_1\|A_0'\|$ (see (11)), while, for $B_0(t) = B(t) - \{\operatorname{Tr} B(t)\}I$,

$$|\operatorname{Tr}(A_0 B)| = |\operatorname{Tr}(A_0 B_0)| \leq C_2\|A_0\|\,\|B_0\|,$$

hence we get, since $B_0(t) \in L^1(0, T)$,

$$\frac{|\Theta/\Gamma + \operatorname{Tr}(A_0 B)|}{\sqrt{\Delta}} \leq C\frac{\|A_0'\|}{\|A_0\|} + \psi(t), \quad \text{with} \quad \psi \in L^1(0, T).$$

Proceeding as in the proof of Corollary 1.7, we reach the conclusion.

ii) Writing (9) in the form

$$\Theta = \frac{1}{8}(b-c)(a-d)(b+c)\{(b+c)'(b+c)^{-1} - (a-d)'(a-d)^{-1}\},$$

and noting that $|b \pm c| \le 2\sqrt{\Gamma}$, we see that each one among (22) and (23) implies

$$\frac{|\Theta|/\Gamma}{\sqrt{\Delta}} \le \frac{1}{2} M \left\{ \frac{|(b+c)'|}{|b+c|} + \frac{|(a-d)'|}{|a-d|} \right\}.$$

Thus, applying (20) to the $\mathcal{C}^k$ functions $b+c$, $a-d$, and using (26), we reach the conclusion. The analytic case can be handled in a similar way. □

REMARK 1.14. By (12) we derive that, under the assumption (22), the condition (26) is fulfilled, in particular, if one has

$$|f(t) + g(t)| + |f(t) - g(t)| \le C\sqrt{\Delta}.$$

REMARK 1.15. It is easily seen that (24) is equivalent to say that the matrix $A(t,\xi)$ is *uniformly symmetrizable* (but, in general, not smoothly). Thus, Corollary 1.13 provides another proof of Theorem 1.3 of Colombini and Nishitani [2]. In the case $b = c$, $A(t,\xi)$ is symmetric. On the other hand, recalling that $\Delta = (a-d)^2 + 4bc$, we get a special case of (22) by assuming $bc \ge 0$. In such a case, $A(t,\xi)$ is a *pseudo-symmetric* matrix in the sense of [5].

NOTATION 1.16. In the following we shall write, for the sake of brevity,

$$\alpha = \frac{a+d}{2}, \quad \beta = \frac{b+c}{2}, \quad \gamma = \frac{c-b}{2}, \quad \delta = \frac{a-d}{2}. \tag{27}$$

Moreover we put, accordingly with [13] and [15],

$$D_1 = \gamma\delta' - \delta\gamma', \quad D_2 = \beta\gamma' - \gamma\beta', \quad D^\sharp = D_1 + iD_2. \tag{28}$$

Therefore, the quantities (3), (7), (8), and (9) take the form

$$\Delta = 4(\beta^2 + \delta^2 - \gamma^2), \quad A_0^2 = 2(\beta^2 + \gamma^2 + \delta^2), \quad \Gamma = \beta^2 + \delta^2, \tag{29}$$

$$\Theta = \gamma(\beta\delta' - \delta\beta') = \beta D_1 - \delta D_2, \tag{30}$$

and by Schwarz' inequality it follows

$$|\Theta| \le \sqrt{\Gamma}|D^\sharp|. \tag{31}$$

REMARK 1.17. Let $B(t) \equiv 0$. By (31), we derive that the condition (17) of Corollary 1.6 (resp. the condition (18) of Corollary 1.7) will be fulfilled if, for $|\xi| = 1$,

$$\int_0^T \frac{|D^\sharp|/\sqrt{\Gamma}}{\Delta^{1/2-1/k}}\, dt \le M, \quad \left(\text{resp. } \frac{|D^\sharp|/\sqrt{\Gamma}}{\sqrt{\Delta}} \le \frac{M}{t}\right). \tag{32}$$

Hence, in the case when $A \in \mathcal{C}^\infty$, this yields a new proof of the γ^∞ wellposedness proved in [13] (see also [6]). In case of analytic coefficients, we obtain the $\mathcal{C}^\infty$ wellposedness proved by Nishitani in [15] (where also the case of x-depending coefficients was considered).

REMARK 1.18. Besides $D^\sharp$ (see (28)), let us introduce the complex function $a^\sharp = \beta + i\delta$. When $A(t)$ is analytic, the following necessary and sufficient condition for the $\mathcal{C}^\infty$ wellposedness of (1) was given in [15]:

$$|D^\sharp + a^\sharp \operatorname{Tr}(A_0 B)| + |D^\sharp + a^\sharp \overline{\operatorname{Tr}(A_0 B)}| \leq M |a^\sharp| \frac{\sqrt{\Delta}}{t},$$

or equivalently, since $(a^\sharp)^{-1} = \overline{a^\sharp}\Gamma^{-1}$,

$$\frac{|D^\sharp \overline{a^\sharp}\Gamma^{-1} + \operatorname{Tr}(A_0 B)|}{\sqrt{\Delta}} + \frac{|D^\sharp \overline{a^\sharp}\Gamma^{-1} + \overline{\operatorname{Tr}(A_0 B)}|}{\sqrt{\Delta}} \leq \frac{M}{t}. \tag{33}$$

Now, a simple computation gives

$$\begin{aligned} D^\sharp \overline{a^\sharp} &= (D_1\beta + D_2\delta) + i(D_2\beta - D_1\delta) \\ &= \{\gamma(\beta\delta' - \delta\beta')\} + i\{(\beta^2 + \delta^2)\gamma' - \gamma(\beta\beta' + \delta\delta')\}, \\ &= \Theta + i\left\{\frac{1}{4}\Delta\gamma' - \frac{1}{8}\gamma\Delta'\right\}. \end{aligned} \tag{34}$$

In view of (33), we must estimate the complex function $D^\sharp \overline{a^\sharp}\Gamma^{-1}(\sqrt{\Delta})^{-1}$. The imaginary part is easily estimated. Indeed, since $|\gamma'| \leq C\|A_0'\|$, $\sqrt{\Gamma} \geq (1/2)\sqrt{\Delta}$, $\sqrt{\Gamma} \geq (1/2)\|A_0\|$, and $|\gamma| \leq \sqrt{\Gamma}$ (see (10)), we get

$$\frac{\Delta|\gamma'|}{\Gamma\sqrt{\Delta}} \leq \frac{\sqrt{\Delta}\cdot C\|A_0'\|}{\Gamma} \leq \frac{\sqrt{\Delta}\cdot C\|A_0'\|}{(1/4)\sqrt{\Delta}\|A_0\|} = 4C\frac{\|A_0'\|}{\|A_0\|},$$

$$\frac{|\gamma\Delta'|}{\Gamma\sqrt{\Delta}} \leq \frac{\sqrt{\Gamma}|\Delta'|}{\Gamma\sqrt{\Delta}} \leq 2\frac{|\Delta'|}{\Delta} = 4\frac{|\sqrt{\Delta}'|}{\sqrt{\Delta}}.$$

Hence, by (34) it follows

$$\frac{|\Im(D^\sharp\overline{a^\sharp})|}{\Gamma\sqrt{\Delta}} \leq C\left\{\frac{|\sqrt{\Delta}'|}{\sqrt{\Delta}} + \frac{\|A_0'\|}{\|A_0\|}\right\}.$$

Now, in the analytic case, the left hand side of last inequality is majorized, up to a constant factor, by $1/t$, hence in (33) we have only to take care of $\Re(D^\sharp\overline{a^\sharp}) \equiv \Theta$. In conclusion, (33) is equivalent to

$$\frac{|\Theta/\Gamma + \operatorname{Tr}(A_0 B)|}{\sqrt{\Delta}} \leq \frac{M}{t}, \tag{35}$$

which is just our condition (18).

A similar argument applies to the non-analytic coefficients (see [6]).

2. The energy estimate

It is well known that the Cauchy problem (1) is well posed in the class of real analytic functions. Therefore, in the following we shall assume $s > 1$. By Fourier transform with respect to x, the system (1) turns into

$$\begin{cases} \hat{U}' = iA(t,\xi)\hat{U} - B(t)\hat{U}, \\ \hat{U}(0,\xi) = \hat{U}_0(\xi). \end{cases} \tag{36}$$

Fixed a non-increasing, smooth function $\varphi(r) \geq 0$ for $r \geq 0$, such that $\varphi \equiv 1$ for $r \leq 1$, $\varphi \equiv 0$ for $r \geq 2$, and $|\varphi'| \leq 2$, we define

$$\omega_\varepsilon(t,\xi) = \varepsilon|\xi|\varphi(\varepsilon^{-1}|\xi|^{-1}\sqrt{\Delta(t,\xi)}),$$
$$\tilde{\omega}_\varepsilon(t,\xi) = \varepsilon|\xi|\varphi(\varepsilon^{-1}|\xi|^{-1}\sqrt{2}\|A_0(t,\xi)\|).$$

Then we have

$$\omega_\varepsilon(t,\xi) = \begin{cases} \varepsilon|\xi|, & \text{for } \quad t \in \Omega_\varepsilon(\xi), \\ 0, & \text{for } \quad t \notin \Omega_{2\varepsilon}(\xi), \end{cases} \tag{37}$$

and the same for $\tilde{\omega}_\varepsilon$, with $\tilde{\Omega}_\varepsilon$ in place of Ω_ε. Moreover we have

$$|\omega_\varepsilon'| \leq 2|\sqrt{\Delta}'|.$$

Finally, recalling that $\sqrt{\Delta} \leq \sqrt{2}\|A_0\|$, it holds

$$\tilde{\omega}_\varepsilon(t,\xi) \leq \omega_\varepsilon(t,\xi). \tag{38}$$

The basic tool in our proof is the energy density

$$E(t,\xi) = \frac{|A_0(t,\xi)\hat{U}|^2 + \{(1/4)\Delta(t,\xi) + (1/2)\omega_\varepsilon(t,\xi)^2\}|\hat{U}|^2}{\Delta_\varepsilon(t,\xi)} \tag{39}$$

where we put

$$\Delta_\varepsilon = \Delta + \omega_\varepsilon^2. \tag{40}$$

We note that

$$E(t,\xi) \geq \frac{1}{4}|\hat{U}|^2 + |A_0\hat{U}|^2\Delta_\varepsilon^{-1} \geq \frac{|A_0\hat{U}|\,|\hat{U}|}{\sqrt{\Delta_\varepsilon}} \geq \frac{|(A_0\hat{U},\hat{U})|}{\sqrt{\Delta_\varepsilon}}. \tag{41}$$

If A is symmetric, i.e., if $b = c$, we see that $|A_0\hat{U}|^2 = (1/4)\Delta|\hat{U}|^2$. Hence, $E(t,\xi)$ equals $(1/2)|\hat{U}|^2$, the classical energy for symmetric systems.

Proposition 2.1. *For every smooth solution $\hat{U}(t,\xi)$ of* (36), *it holds*

$$E'(t) \le C\left\{\frac{|\Theta/\Gamma + \operatorname{Tr}(A_0B)| + |\sqrt{\Delta}'|}{\sqrt{\Delta+\omega_\varepsilon^2}} + \frac{\|A_0'\|}{\|A_0\| + \tilde{\omega}_\varepsilon} + \omega_\varepsilon + \|B(t)\|\right\}E(t). \tag{42}$$

Proof. We differentiate (39) with respect to t. Since

$$\begin{aligned}
\partial_t(1/\Delta_\varepsilon) &= -\Delta_\varepsilon'\Delta_\varepsilon^{-2} = -\{\Delta' + 2\omega_\varepsilon'\omega_\varepsilon\}\Delta_\varepsilon^{-2},\\
\partial_t\left|A_0\hat{U}\right|^2 &= 2\Re\{(A_0'\hat{U},\, A_0\hat{U}) + i(A_0A\hat{U},\, A_0\hat{U}) - (A_0B\hat{U},\, A_0\hat{U})\},\\
\partial_t|U|^2 &= 2\Re\{i(A\hat{U},\, \hat{U}) - (B\hat{U},\, \hat{U})\},
\end{aligned}$$

we find the equality

$$E' = \{\Psi_1(t,\xi) + \Psi_2(t,\xi) + \Psi_3(t,\xi)\}\Delta_\varepsilon(t,\xi)^{-1} \tag{43}$$

where

$$\Psi_1 = -\{\Delta' + 2\omega_\varepsilon'\omega_\varepsilon\}E + \left\{\frac{1}{4}\Delta' + \omega_\varepsilon'\omega_\varepsilon\right\}|\hat{U}|^2, \tag{44}$$

$$\Psi_2 = 2\Re\left\{i(A_0A\hat{U},\, A_0\hat{U}) + \left\{\frac{1}{4}\Delta + \frac{1}{2}\omega_\varepsilon^2\right\}((iA - B)\hat{U},\, \hat{U})\right\}, \tag{45}$$

$$\Psi_3 = 2\Re\{(A_0'\hat{U},\, A_0\hat{U}) - (A_0B\hat{U},\, A_0\hat{U})\}. \tag{46}$$

We estimate the terms in the right hand side of (43):

Estimate of $\boldsymbol{\Psi_1\Delta_\varepsilon^{-1}}$. We recall the inequalities $\omega_\varepsilon \le \sqrt{\Delta_\varepsilon}$, $|\omega_\varepsilon'| \le 2|\sqrt{\Delta}'|$, and the identity $\Delta' = 2\sqrt{\Delta}'\sqrt{\Delta}$. Then, by (41) it follows

$$\frac{|\Psi_1|}{\Delta_\varepsilon} \le C\frac{|\sqrt{\Delta}'|}{\sqrt{\Delta_\varepsilon}}E. \tag{47}$$

Estimate of $\boldsymbol{\Psi_2\Delta_\varepsilon^{-1}}$. From the definition (6), it follows the identity $A_0^2 = (1/4)\Delta I$. Therefore

$$i(A_0A\hat{U},\, A_0\hat{U}) = i(A_0(A_0+\alpha I)\hat{U},\, A_0\hat{U}) = i\left\{\frac{1}{4}\Delta(\hat{U},\, A_0\hat{U}) + \alpha|A_0\hat{U}|^2\right\},$$

where $\alpha = (1/2)\operatorname{Tr} A$ is real. On the other side,

$$\left(\frac{1}{4}\Delta + \frac{1}{2}\omega_\varepsilon^2\right)(iA\hat{U},\, \hat{U}) = i\left\{\left(\frac{1}{4}\Delta + \frac{1}{2}\omega_\varepsilon^2\right)[(A_0\hat{U},\, \hat{U}) + \alpha|\hat{U}|^2]\right\}.$$

Thus, noting that $(\hat{U},\, A_0\hat{U}) + (A_0\hat{U},\, \hat{U}) \in \mathbb{R}$, we get the equality

$$\Psi_2 = \frac{1}{2}\omega_\varepsilon^2 2\Re\{i(A_0\hat{U},\, \hat{U})\} - \left\{\frac{1}{4}\Delta + \frac{1}{2}\omega_\varepsilon^2\right\}2\Re(B\hat{U},\, \hat{U}),$$

from which, using again (41) and noting that $\omega_\varepsilon \le \sqrt{\Delta_\varepsilon}$, we derive

$$\frac{|\Psi_2|}{\Delta_\varepsilon} \le C(\omega_\varepsilon + \|B(t)\|)E. \tag{48}$$

Estimate of $\boldsymbol{\Psi_3\Delta_\varepsilon^{-1}}$. Drawing inspiration from the ring of *quaternions*, we consider the following base of the space of 2×2 real matrices:

$$\mathbf{e_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{e_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{e_3} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{e_4} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

noting the relations

$$\mathbf{e_2^2} = \mathbf{e_1}, \quad \mathbf{e_3^2} = -\mathbf{e_1},\ \mathbf{e_4^2} = \mathbf{e_1},$$
$$\mathbf{e_3e_4} = -\mathbf{e_4e_3} = \mathbf{e_2}, \quad \mathbf{e_2e_4} = -\mathbf{e_4e_2} = \mathbf{e_3}, \quad \mathbf{e_2e_3} = -\mathbf{e_3e_2} = \mathbf{e_4}.$$

Thus, our matrix A takes the form:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha+\delta & \beta-\gamma \\ \beta+\gamma & \alpha-\delta \end{pmatrix} = \alpha\mathbf{e_1} + \beta\mathbf{e_2} + \gamma\mathbf{e_3} + \delta\mathbf{e_4},$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ are defined in (27), while

$$A_0 = \beta\mathbf{e_2} + \gamma\mathbf{e_3} + \delta\mathbf{4_4} = \gamma\mathbf{e_3} + K\mathbf{e_2}. \tag{49}$$

where

$$K = \beta\mathbf{e_1} - \delta\mathbf{e_3}. \tag{50}$$

The ring $\mathcal{L}\{\mathbf{e_1}, \mathbf{e_3}\}$ generated by $\{\mathbf{e_1}, \mathbf{e_3}\}$ can be identified with the complex field $\mathbb{C}$, via the isomorphism $x + iy \mapsto x\mathbf{e_1} + y\mathbf{e_3}$. In particular:

$$KK^* = (\beta\mathbf{e_1} - \delta\mathbf{e_3})(\beta\mathbf{e_1} + \delta\mathbf{e_3}) = (\beta^2+\delta^2)\mathbf{e_1} = \Gamma\mathbf{e_1}. \tag{51}$$

To estimate of $(A_0'\hat{U}, A_0\hat{U})$, we put A_0' in the form $A_0' = P + QA_0$, for suitable $P, Q \in \mathcal{L}\{\mathbf{e_1}, \mathbf{e_3}\}$. More precisely, restricting ourselves to the non-singular set $\{\Delta(t,\xi) \neq 0\}$, we derive from (49) the equality

$$\begin{aligned} A_0' &= \gamma'\mathbf{e_3} + K'\mathbf{e_2} = \gamma'\mathbf{e_3} + K'K^{-1}(K\mathbf{e_2}) = \gamma'\mathbf{e_3} + K'K^{-1}(A_0 - \gamma\mathbf{e_3}) \\ &= (\gamma'K - \gamma K')K^{-1}\mathbf{e_3} + K'K^{-1}A_0. \end{aligned}$$

But $K^{-1} = \Gamma^{-1}K^*$ by (51), hence we obtain

$$A_0' = \Gamma^{-1}H + \Gamma^{-1}K'K^*A_0, \tag{52}$$

where we put

$$H = (\gamma' K - \gamma K')K^*\mathbf{e_3}. \tag{53}$$

We observe that, up to a multiplicative constant, the functions γ, $\|K\|$, $\|K^*\|$ and $\|A_0\|$ are majorized by $\sqrt{\Gamma}$, while $|\gamma'|$ and $\|K'\|$ are majorized by $\|A_0'\|$. Consequently, $\|H\|\Gamma^{-1}$ and $\|K'K^*A_0\|\Gamma^{-1}$ are majorized by $\|A_0'\|$, and hence belong to $L^1(0, T)$ for all ξ. Thus, the identity (52) is a.e. true on the whole interval $[0, T]$.

Going back to (46), we note that for any 2×2 matrix X it holds $X + X^{\mathrm{co}} = \{\mathrm{Tr}\, X\}I$, where X^{co} is the *cofactor matrix*. Thus, putting

$$\tau = \mathrm{Tr}(A_0B),$$

and noting that $A^{\mathrm{co}} = -A$, we can write

$$A_0B = \tau I - B^{\mathrm{co}}A_0^{\mathrm{co}} = \tau I + B^{\mathrm{co}}A_0. \tag{54}$$

Introducing (52) and (54) in (46), we obtain

$$\Psi_3 = 2\Re\{(\{\Gamma^{-1}H - \tau I\}\hat{U}, A_0\hat{U}) + \Gamma^{-1}(K'K^*A_0\hat{U}, A_0\hat{U}) - (B^{\mathrm{co}}A_0\hat{U}, A_0\hat{U})\}.$$

Now, by (53) and (50) we easily derive, recalling (30), that

$$\begin{aligned} H &= \{\gamma(\delta\beta' - \beta\delta')\}\mathbf{e_1} + \{(\beta^2 + \delta^2)\gamma' - \gamma(\beta\beta' + \delta\delta')\}\mathbf{e_3} \\ &= -\Theta\mathbf{e_1} + \left\{\frac{1}{4}\Delta\gamma' - \frac{1}{8}\Delta'\gamma\right\}\mathbf{e_3}, \end{aligned}$$

hence, noting that $\mathbf{e_1}$, $\mathbf{e_3}$ are matrices with norm $\sqrt{2}$, and that $|B^{\mathrm{co}}A_0\hat{U}| \le \|B^{\mathrm{co}}\||A_0U|$, we get

$$\begin{aligned} |\Psi_3(t,\xi)| &\le 2\left(\left|\left\{(\Theta\Gamma^{-1} + \tau)\mathbf{e_1} + \left(\frac{1}{4}\Delta\gamma' - \frac{1}{8}\gamma\Delta'\right)\Gamma^{-1}\mathbf{e_3}\right\}\hat{U}, A_0\hat{U}\right)\right| \\ &\quad + 2\Gamma^{-1}\|K'K^*\||A_0\hat{U}|^2 + 2|B^{\mathrm{co}}A_0\hat{U}||A_0\hat{U}| \\ &\le 2\sqrt{2}\left\{|\Theta\Gamma^{-1} + \tau| + \left|\frac{1}{4}\Delta\gamma' - \frac{1}{8}\gamma\Delta'\right|\Gamma^{-1}\right\}|\hat{U}||A_0\hat{U}| \\ &\quad + 2\Gamma^{-1}\|K'K^*\||A_0\hat{U}|^2 + 2\|B^{\mathrm{co}}(t)\||A_0\hat{U}|^2. \end{aligned}$$

In order to estimate $\Psi_3\Delta_\varepsilon^{-1}$ we first compute, recalling (41),

$$\begin{aligned}\left\{\left|\frac{1}{4}\Delta\gamma' - \frac{1}{8}\gamma\Delta'\right|\Gamma^{-1}|\hat{U}||A_0\hat{U}|\right\}\Delta_\varepsilon^{-1} &\le \{\Delta|\gamma'| + |\gamma||\Delta'|\}\Gamma^{-1}\sqrt{\Delta_\varepsilon^{-1}}E \\ &\le C_1\left\{\Delta\|A_0'\|\Gamma^{-1}\sqrt{\Delta_\varepsilon^{-1}} + |\Delta'|\sqrt{(\Gamma\Delta_\varepsilon)^{-1}}\right\}E \\ &\le C_2\left\{\|A_0'\|(\|A_0\| + \tilde{\omega}_\varepsilon)^{-1} + |\sqrt{\Delta}'|\sqrt{\Delta_\varepsilon^{-1}}\right\}E.\end{aligned}$$

Indeed, from the inequalities $\Delta \le 4\Gamma$, $\|A_0\|^2 \le 4\Gamma$, it follows (by (38)):

$$\frac{\Delta}{\Gamma\sqrt{\Delta_\varepsilon}} = \frac{\Delta}{\Gamma\sqrt{\Delta + \omega_\varepsilon^2}} \le \frac{4\Gamma}{\Gamma\sqrt{4\Gamma + \omega_\varepsilon^2}} \le \frac{4}{\sqrt{\|A_0\|^2 + \omega_\varepsilon^2}} \le \frac{4\sqrt{2}}{\|A_0\| + \tilde{\omega}_\varepsilon},$$

and $|\Delta'| = |2\sqrt{\Delta}'\sqrt{\Delta}| \le 4|\sqrt{\Delta}'|\sqrt{\Gamma}$.

Next we estimate the term $\{\Gamma^{-1}\|K'K^*\||A_0\hat{U}|^2\}\Delta_\varepsilon^{-1}$. By (41) it follows

$$|A_0\hat{U}|^2 \le \min\{\Delta_\varepsilon E, \|A_0\||A_0\hat{U}||\hat{U}|\} \le \min\{\Delta_\varepsilon, \|A_0\|\sqrt{\Delta_\varepsilon}\}E,$$

while $\|K'\| \le C\|A_0'\|$, $\|K^*\| \le C\sqrt{\Gamma}$, $\|A_0\| \le 2\sqrt{\Gamma}$. Hence we find

$$\begin{aligned}\{\Gamma^{-1}\|K'K^*\||A_0\hat{U}|^2\}\Delta_\varepsilon^{-1} &\le C\Gamma^{-1}\|A_0'\|\sqrt{\Gamma}\min\{\Delta_\varepsilon, \|A_0\|\sqrt{\Delta_\varepsilon}\}E\Delta_\varepsilon^{-1} \\ &= C\|A_0'\|\min\{\sqrt{\Gamma^{-1}}, \|A_0\|\sqrt{(\Gamma\Delta_\varepsilon)^{-1}}\}E \\ &\le C\|A_0'\|\min\left\{\sqrt{\Gamma^{-1}}, 2\sqrt{\Delta_\varepsilon^{-1}}\right\}E.\end{aligned}$$

But $\min\{1/x, 1/y\} \le 2/(x+y)$, thus we conclude that

$$\{\Gamma^{-1}\|K'K^*\||A_0\hat{U}|^2\}\Delta_\varepsilon^{-1} \le C\{\|A_0'\|(\|A_0\| + \tilde{\omega}_\varepsilon)^{-1}\}E.$$

Finally, we have

$$\|B^{\mathrm{co}}\||A_0\hat{U}|^2 \le \|B\|\Delta_\varepsilon E.$$

Summing up, we have proved the estimate

$$\frac{|\Psi_3|}{\Delta_\varepsilon} \le C\left\{\frac{|\Theta/\Gamma + \mathrm{Tr}(A_0B)| + |\sqrt{\Delta}'|}{\sqrt{\Delta + \omega_\varepsilon^2}} + \frac{\|A_0'\|}{\|A_0\| + \tilde{\omega}_\varepsilon} + \|B\|\right\}E. \tag{55}$$

Inserting (44), (45) and (46) into (43), we get (42). □

3. Proof of Theorem 1.1

Since $B(t) \in L^1(0, T)$, from (42) it follows

$$E(t, \xi) \le C E(0, \xi) \exp\left\{ \int_0^T \left[\frac{J}{\sqrt{\Delta + \omega_\varepsilon^2}} + \frac{\|A_0'\|}{\|A_0\| + \tilde{\omega}_\varepsilon} + \omega_\varepsilon + 1 \right] dt \right\} \tag{56}$$

where we put

$$J(t, \xi) = |\Theta/\Gamma + \mathrm{Tr}(A_0 B)| + |\sqrt{\Delta}'|. \tag{57}$$

To conclude the proof of Theorem 1.1, we prove that the assumption (14) allow us to estimate the growth of the the integral in (56), as $\varepsilon \to 0$. Thus, we prove the following lemma:

Lemma 3.1. *Let $J(t) \equiv J(t, \xi) \ge 0$ be a function in $L^1(0, T)$, homogeneous in ξ of degree 1, with L^1 norm bounded as $|\xi| = 1$. Assume that, for some $\beta > 0$ and some $C > 0$, one has, for all $|\xi| = 1$ and $\varepsilon \le 1$,*

$$\int_{[0,T]\setminus\Omega_\varepsilon} \frac{J(t)}{\sqrt{\Delta(t)}}\, dt \le C\varepsilon^{-\beta}. \tag{58}$$

Then, there exists $C' > 0$ such that

$$\int_0^T \frac{J(t)}{\sqrt{\Delta(t) + \omega_\varepsilon^2(t)}}\, dt \le C'\varepsilon^{-\beta}. \tag{59}$$

Proof. i) Let $\beta \ge 1$. By the definitions of Ω_ε and ω_ε, we derive, for all $\xi = 1$,

$$\Delta(t) + \omega_\varepsilon^2(t) \begin{cases} = \Delta(t) + \varepsilon^2 \ge \varepsilon^2, & \text{for} \quad t \in \Omega_\varepsilon(\xi), \\ \ge \varepsilon^2 + \omega_\varepsilon^2 \ge \varepsilon^2, & \text{for} \quad t \notin \Omega_\varepsilon(\xi). \end{cases}$$

Hence

$$\int_0^T \frac{J}{\sqrt{\Delta + \omega_\varepsilon^2}}\, dt \le \int_0^T \frac{J}{\varepsilon}\, dt \le C\varepsilon^{-1} \le C\varepsilon^{-\beta}.$$

ii) Let $0 < \beta < 1$. We split the domain of integration in (60) as $[0, T] = \Omega_\varepsilon \cup ([0, T] \setminus \Omega_\varepsilon)$. By (58), it follows

$$\int_{[0,T]\setminus\Omega_\varepsilon} \frac{J}{\sqrt{\Delta + \omega_\varepsilon^2}}\, dt \le \int_{[0,T]\setminus\Omega_\varepsilon} \frac{J}{\sqrt{\Delta}}\, dt \le C\varepsilon^{-\beta}.$$

On the other hand, writing

$$\Omega_\varepsilon = \bigcup_{j=0}^{\infty}\{\Omega_j \setminus \Omega_{j+1}\}, \quad \text{where} \quad \Omega_j := \Omega_{\varepsilon 2^{-j}},$$

and recalling that, for $|\xi| = 1$, $\omega_\varepsilon(t) \equiv \varepsilon|\xi| \equiv \varepsilon$ on Ω_ε, we have

$$\int_{\Omega_\varepsilon} \frac{J}{\sqrt{\Delta + \omega_\varepsilon^2}}\,dt = \sum_{j=0}^{\infty} \int_{\Omega_j\setminus\Omega_{j+1}} \frac{J}{\sqrt{\Delta + \varepsilon^2}}\,dt.$$

Now,

$$\int_{\Omega_j\setminus\Omega_{j+1}} \frac{J}{\sqrt{\Delta + \varepsilon^2}}\,dt \le \sup_{\Omega_j}\left\{\frac{\sqrt{\Delta}}{\sqrt{\Delta + \varepsilon^2}}\right\} \cdot \int_{\Omega_j\setminus\Omega_{j+1}} \frac{J}{\sqrt{\Delta}}\,dt.$$

On Ω_j it holds $\sqrt{\Delta} \le \varepsilon 2^{-j}$, hence $\sqrt{\Delta}/(\sqrt{\Delta} + \varepsilon) \le \varepsilon 2^{-j}/\varepsilon = 2^{-j}$; while by (58) it follows

$$\int_{\Omega_j\setminus\Omega_{j+1}} \frac{J}{\sqrt{\Delta}}\,dt \le \int_{[0,T]\setminus\Omega_{j+1}} \frac{J}{\sqrt{\Delta}}\,dt \le C\{\varepsilon 2^{-(j+1)}\}^{-\beta} = C\varepsilon^{-\beta}2^{\beta(j+1)}.$$

Thus,

$$\int_{\Omega_j\setminus\Omega_{j+1}} \frac{J}{\sqrt{\Delta + \varepsilon^2}}\,dt \le 2^{-j} \cdot C\varepsilon^{-\beta}2^{\beta(j+1)} = C2^{\beta}\varepsilon^{-\beta}2^{-j(1-\beta)}.$$

In conclusion we obtain, since $\beta < 1$,

$$\int_{\Omega_\varepsilon} \frac{J}{\sqrt{\Delta + \omega_\varepsilon^2}}\,dt \le \sum_{j=0}^{\infty} C2^{\beta}\varepsilon^{-\beta}2^{-j(1-\beta)} = C'\varepsilon^{-\beta}\sum_{j=0}^{\infty} 2^{-j(1-\beta)} \le C'\varepsilon^{-\beta}.$$

which concludes the proof of Lemma 3.1. □

Let us go back to (56). Proceeding as in Lemma 3.1, we prove that (14) implies

$$\int_0^T \frac{\|A_0'\|}{\|A_0\| + \tilde{\omega}_\varepsilon}\,dt \le C\varepsilon^{-\beta}. \tag{60}$$

Finally, recalling (37), we have

$$\int_0^T \omega_\varepsilon\,dt = \varepsilon|\xi|\mu(\Omega_{2\varepsilon}(\xi)) \le \varepsilon|\xi| \cdot C\varepsilon^{\alpha} = C\varepsilon^{\alpha+1}|\xi|. \tag{61}$$

In conclusion, if we introduce in (56) the estimates (59), (60) and (61), we obtain

$$E(t,\xi) \le CE(0,\xi)\exp\{C\varepsilon^{-\beta} + C\varepsilon^{\alpha+1}|\xi|\}.$$

We can now conclude the proof of the theorem. It is sufficient to give an estimate for $|\xi| \ge 1$, since for $|\xi| \le 1$ we have directly the estimate $|\hat{U}(t,\xi)| \le C|\hat{U}_0(\xi)|$ from the ordinary differential system (36), with ξ as a parameter. Thus, for $|\xi| \ge 1$ we choose

$$\varepsilon = |\xi|^{-1/(\alpha+\beta+1)}$$

and this leads to the final energy inequality

$$E(t,\xi) \le CE(0,\xi)\exp\{|\xi|^{\beta/(\alpha+\beta+1)}\}.$$

Therefore, by a standard argument, we obtain the wellposedness in γ^s for $s < 1 = (\alpha+1)/\beta$. This concludes the proof of Theorem 1.1. $\square$

4. Proof of Theorem 1.2

Theorem 1.2 can be proved quite similarly to Theorem 1.1. The only relevant difference is that, in place of Lemma 3.1, we must use the following lemma (which can be proved in a similar way):

Lemma 4.1. *Assume that, for all* $|\xi| = 1$ *and* $\varepsilon \le 1$,

$$\int_{[0,T]\setminus\Omega_\varepsilon} \frac{J}{\sqrt{\Delta}}\,dt \le C\log\varepsilon^{-1}.$$

Then

$$\int_0^T \frac{J}{\sqrt{\Delta+\omega_\varepsilon^2}}\,dt \le C'\log\varepsilon^{-1}.$$

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