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## AN UPPER BOUND FOR LOEWY LENGTHS OF PROJECTIVE MODULES IN $p$ -SOLVABLE GROUPS

WOLFGANG WILLEMS

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**1. Introduction.** Let  $p$  be a prime,  $F$  a field of characteristic  $p$  and let  $G$  be a finite group. With every  $FG$ -module  $M$  there is attached a non-negative integer  $L(M)$  called the Loewy length of  $M$ . If  $J(FG)$  denotes the Jacobson radical of  $FG$ , then  $L(M)$  is the smallest integer  $n$  such that  $MJ(FG)^n$  is the zero module. The Loewy length of a module is of some interest since knowledge of its value or at least of good bounds for it can be very useful determining of the structure of the module.

In a significant paper [5], Jennings solved the structure problem for the group algebra of a  $p$ -group. Unfortunately, little is known about the algebra structure of an arbitrary group algebra, resp. of a  $p$ -block. However, in case of  $p$ -solvable groups, there are many papers concerning bounds for the Loewy length of a group algebra, resp. of a  $p$ -block ([6], [7], [8], [12], [14], [15], [16], [17]).

Recently Ninomiya found a lower bound for the Loewy length of a projective indecomposable module depending on the order of a vertex of its head ([13]). The aim of this note is to determine an upper bound.

Throughout this paper all groups in questions are finite and  $p$ -solvable all modules finitely generated  $FG$ -modules where  $F$  denotes a field of characteristic  $p$ . The notation used in the following is consistent with that in the books of Feit [2] and Huppert/Blackburn [4].

### 2. Results

**Theorem 1.** *If  $P$  is a projective indecomposable module, then*  
 $L(P) \leq \max\{|V| \mid V \text{ is a vertex of a composition factor of } P\}$ .

**Theorem 2.** *Equality holds in Theorem 1 if and only if the defect group of the block to which  $P$  belongs is cyclic.*

**Corollary** (Koshitani, Okuyama, Tsushima). *Let  $B$  be a  $p$ -block with defect group  $D$ . Then  $L(B) \leq |D|$  and equality holds if and only if  $D$  is cyclic.*

According to all the examples we know, it seems reasonable to ask the

**Question.** *Let  $M$  be an irreducible module with vertex  $V$ . Is it true that  $L(P_G(M)) \leq |V|$  where  $P_G(M)$  denotes the projective cover of  $M$ ?*

To see that the problem considered here differs from the analogue for  $p$ -blocks and defect groups, let us mention the following remarkable example ([10], [11]).

Let  $p$  be the prime 3 and let  $G$  denote the semidirect product of  $SL(2,3)$  with the standard module. Then  $G$  possesses an irreducible module  $M$  with

$$L(P_G(M)) = |vx(M)|$$

where the vertex  $vx(M)$  of  $M$  is elementary abelian of order 9.

### 3. Proofs

In what follows we may always assume that  $F$  is algebraically closed. The reduction to such a field is routine since the radical of a group algebra, vertices and defect groups are well behaved by field extensions.

Proof of Theorem 1.

We argue by induction on  $(|G|_p, |G|)$  where  $|G|_p$  denotes the  $p$ -part of the order  $|G|$  of  $G$ .

Write  $P = P_G(M)$  for some irreducible module  $M$ .

(1) First assume that  $O_p(G) \neq \langle 1 \rangle$ .

Let  $E$  be a normal abelian  $p$ -subgroup of  $G$ . Put  $J = J(FE)$  and let  $G$  act by conjugation on the powers  $J^i$  of  $J$ . By a result of Alperin, Collins and Sibley [1], we have with  $\bar{G} = G/E$  an  $FG$ -isomorphism

$$P_{\bar{G}}(M) \otimes J^i/J^{i+1} \cong P_G(M)J^i/P_G(M)J^{i+1}.$$

Since  $E \subseteq C_G(J^i/J^{i+1})$ , the left hand side is an  $F\bar{G}$ -module, hence a projective  $F\bar{G}$ -module. As  $FG$ -modules, all composition factors of

$$X := P_{\bar{G}}(M) \otimes J^i/J^{i+1}$$

are composition factors of  $P_G(M)$ .

Hence by the inductive hypothesis we get

$$\begin{aligned} L(X) &\leq \max\{|V| \mid V \text{ is a vertex of a composition factor of } X\} \\ &\leq \max\{|\bar{V}| \mid \bar{V} \text{ is a vertex of a composition factor of } P_{\bar{G}}(M)\} \end{aligned}$$

and therefore

$$\begin{aligned} L(P_G(M)) &\leq \max\{|\bar{V}| \mid \bar{V} \text{ is a vertex of a composition factor of } P_{\bar{G}}(M)\} \cdot L(FE) \\ &\leq \max\{|\bar{V}| \mid \bar{V} \text{ is a vertex of a composition factor of } P_{\bar{G}}(M)\} \cdot |E| \\ &= \max\{|V| \mid V \text{ is a vertex of a composition factor of } P_G(M)\}. \end{aligned}$$

Thus it remains to deal with the case

$$H: = O_{p'}(G) \neq \langle 1 \rangle.$$

Let  $N$  be an irreducible constituent of  $M_H$  and let  $I$  denote the inertial group of  $N$  in  $G$ , i.e.

$$I = I_G(N) = \{g \mid g \in G, N \otimes g \cong N\}.$$

(2) Next we consider the case  $I < G$ .

By Clifford's theory, there exists an irreducible  $FI$ -module  $X$  such that  $M \cong X^G$  and  $X_H \cong eN$  for some  $e \in N$ . According to Proposition 2.7 in [18] we have

$$P_G(M) \cong P_I(X)^G.$$

We claim that for all composition factors  $Y$  of  $P_I(X)$  the induced modules  $Y^G$  are irreducible. Since  $vx(Y) \cong vx(Y^G)$  we are done by the inductive hypothesis. Thus assume that  $Y$  is a composition factor of  $P_I(X)$  and  $Z$  an irreducible submodule of  $Y^G$ . Obviously,  $Y_H \cong fN$  for some  $f \in N$ .

If  $\{g_1=1, g_2, \dots, g_r\}$  denotes a right transversal of  $I$  in  $G$ , then

$$Z_H \subseteq (Y^G)_H = Y_H \otimes g_1 \oplus \dots \oplus Y_H \otimes g_r$$

where the  $Y_H \otimes g_i$  are precisely the homogeneous components of  $(Y^G)_H$ .

Now choose a homogeneous component of  $Z_H$ , say  $W$ . We may assume that  $W \subseteq Y_H \otimes g_1$ , otherwise we consider a suitable  $G$ -conjugate of  $W$ . Since  $I$  acts irreducibly on  $Y$ , we get  $W_H = Y_H \otimes g_1$  and therefore  $Z = Y^G$ .

(3) Finally, let  $I=G$ .

Now, Fong's reduction theorem ([2], Chap. X) asserts that there exists a finite group  $\tilde{G}$  and a short exact sequence

$$\langle 1 \rangle \rightarrow Z \rightarrow \tilde{G} \rightarrow G \xrightarrow{f} \langle 1 \rangle$$

with  $Z$  a cyclic  $p'$ -group in the center of  $\tilde{G}$ .

Furthermore,

- (i)  $\tilde{G}$  contains a normal subgroup  $\tilde{H} \cong H$  with  $Z\tilde{H} = Z \times \tilde{H} = f^{-1}(H)$ .
- (ii) There is an  $F\tilde{G}$ -module  $\tilde{N}$  on which  $\tilde{H}$  acts irreducibly and an  $F\tilde{G}$ -module  $\tilde{M}$  with  $\tilde{H} \subseteq \ker(\tilde{M})$  such that  $M$  considered as an  $F\tilde{G}$ -module is isomorphic to  $\tilde{M} \otimes \tilde{N}$ .
- (iii)  $P_{\tilde{G}}(M) \cong P_{\tilde{G}}(\tilde{M}) \otimes \tilde{N}$ .

Since  $\tilde{H} \subseteq O_{p'}(\text{Ker}(\tilde{M}))$ , we have  $\tilde{H} \subseteq \text{Ker}(\tilde{Y})$  for all composition factors  $\tilde{Y}$  of  $P_{\tilde{G}}(\tilde{M})$  (see [18], 3.1). Thus by ([4], Chap. VII, 9.12), the tensor product  $\tilde{Y} \otimes \tilde{N}$  is irreducible. Because of ([4], Chap. VII, 14.1 and 14.2), there are isomorphisms

$$P_{\bar{c}}(M) \cong P_{\bar{c}/Z}(M) \cong P_G(M) \quad \text{and} \quad P_{\bar{c}}(\tilde{M}) \cong P_{\bar{c}/\tilde{H}}(\tilde{M}).$$

In particular, each composition factor  $Y$  of  $P_G(M)$  is of the form  $Y \cong \tilde{Y} \otimes \tilde{N}$  for some composition factor  $\tilde{Y}$  of  $P_{\bar{c}}(\tilde{M})$ . Since  $\dim N$  is prime to  $p$ , we get  $|vx(Y)| = |vx(\tilde{Y})|$ , by ([3], 2.1). Now,  $|\tilde{G}/\tilde{H}|_p = |G|_p$  and  $O_p(\tilde{G}/\tilde{H}) \neq \langle 1 \rangle$ . Apply part (1) of the proof to  $P_{\bar{c}/\tilde{H}}(\tilde{M}) \cong P_{\bar{c}}(\tilde{M})$  and the proof is complete.

Proof of Theorem 2.

Assume first that the defect group  $D$  of the  $p$ -block  $B$  to which  $P$  belongs is cyclic. In this case it's well-known that

$$B \cong \text{Mat}(n, FU)$$

where  $D \triangleleft U \leq \text{holomorph}(D)$  and  $U/D$  is a cyclic  $p'$ -group. From this we deduce quite easily that all the projective indecomposable modules in  $B$  are uniserial of length  $|D|$ . Since  $D$  is a vertex for all irreducible modules in  $B$ , the assertion follows.

For the other direction in Theorem 2 assume that  $P = P_G(M)$  for some irreducible module  $M$  and

$$\begin{aligned} L(P_G(M)) &= \max\{|V| \mid V \text{ is a vertex of a composition factor of } P_G(M)\} \\ &= |V_0|. \end{aligned}$$

We claim by induction on  $(|G|_p, |G|)$  that  $V_0$  must be cyclic. Then it's well-known that  $V_0$  coincides up to  $G$ -conjugation with the defect group  $D$ . To do this assume first that  $O_{p'}(G)$  is contained in the center  $Z(G)$  of  $G$ . In this case

$$O_{p',p}(G) = O_{p'}(G) \times O_p(G)$$

and  $G/O_{p',p}(G)$  acts faithfully on  $O_p(G)$ . Let  $E$  be an abelian normal  $p$ -subgroup of  $G$  and put  $\bar{G} = G/E$ .

Similar to part (1) of the proof of Theorem 1 we get

$$L(P_{\bar{c}}(M)) = |V_0| \quad \text{and} \quad E \text{ has to be cyclic.}$$

By the inductive hypothesis,  $V_0$  is cyclic. Now, if  $G$  has at least two minimal normal  $p$ -subgroups, then  $V_0$  is abelian.

In particular,  $O_p(G)$  is abelian and therefore cyclic. Since  $G/O_{p',p}(G)$  acts faithfully on  $O_p(G)$ ,  $O_p(G)$  is a Sylow  $p$ -subgroup of  $G$ . Hence  $G$  has only one minimal normal  $p$ -subgroup  $E$ . This implies that  $E$  is contained in the center of  $O_p(G)$ . Since  $O_p(G)/E$  is cyclic,  $O_p(G)$  must be abelian, hence cyclic and therefore a Sylow  $p$ -subgroup of  $G$ . The proof can now be finished by the same line as we did in (2) and (3) of the proof of Theorem 1.

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Johannes Gutenberg-Universität  
 Fachbereich 17 Mathematik  
 Postfach 3980 6500 Mainz  
 F.R.G.

