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INFINITESIMAL GENERATORS OF NONHOMOGENEOUS CONVOLUTION SEMIGROUPS ON LIE GROUPS

Dedicated to Professor Masatoshi Fukushima on the occasion of his 60th birthday

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1. Introduction

In 1956, Hunt [3] characterized all possible homogeneous convolution semigroups of probability distributions on a Lie group through the representations of their infinitesimal generators. Let $\{\mu_t\}_{t>0}$ be a convolution semigroup of probability distributions defined on a Lie group G of dimension d. It defines a semigroup of linear operators $\{T_t\}_{t>0}$ on $\mathscr{C}$ by setting $T_t f(\sigma)=\int f(\sigma\tau)\mu_t(d\tau)$, where $\mathscr{C}$ is the Banach space consisting of bounded continuous functions f on G (such that $\lim_{\sigma\to\infty} f(\sigma)$ exists if G is noncompact). Then the domain $\mathscr{D}(A)$ of its infinitesimal generator A contains $\mathscr{C}_2$ (a space consisting of $\mathscr{C}_2$-functions on G) and Af, $f\in\mathscr{C}_2$ is represented by

$$
(1.1)\qquad Af(\sigma)=\frac{1}{2}\sum_{i,j}a^{ij}X_iX_jf(\sigma)+\sum_i b^iX_if(\sigma) + \int_G \Big(f(\sigma\tau)-f(\sigma)-\sum_i x^i(\tau)X_if(\sigma)\Big)\nu(d\tau).
$$

Here $X_1,\cdots,X_d$ constitute a basis of the Lie algebra of G regarding them as left invariant first order differential operators (vector fields), $A=(a^{ij})$ is a symmetric nonnegative definite matrices, $b=(b^i)$ is a vector and ν is a measure on G such that $\nu(\{e\})=0$ and $\int\phi(\sigma)\nu(d\sigma)<\infty$, where e is the unit element of G. Further, $x^1,\cdots,x^d$, ϕ are $\mathscr{C}_2$-functions on G satisfying (3.1) and (3.2). Conversely the above operator determines a unique convolution semigroup.

In this paper we study nonhomogeneous convolution semigroups $\{\mu_{s,t}\}_{0<s<t<\infty}$ of probability distributions on a Lie group. In the first part (Sections 2–4), we characterize them by representing their infinitesimal generators $A(t), t>0$, similarly as (1.1), where the triple (A,b,ν) in the representation on $A(t)$ depends on t. We remark that a similar representation of the infinitesimal generator has been obtained by Maksimov [7] in the case where the underlying Lie group is compact. However, we are particularly interested in the nonhomogeneous convolution semigroup on a noncompact Lie group. Further, our Condition (***D***) needed for the representation theorem is milder than his.

In the second part of this paper (Sections 5–6), we study nonhomogeneous convolution semigroups having the self-similar property. A convolution semigroup $\{\mu_{s,t}\}$ is called self-similar with respect to $\{\gamma_r\}$, if $\gamma_r\mu_{s,t}=\mu_{rs,rt}$ holds for any $s<t$ and $r>0$, where $\{\gamma_r\}_{r>0}$ is a one parameter group of automorphisms of G called a dilation. Applying the representation theorem of the first part, we characterize all self-similar nonhomogeneous convolution semigroups through their infinitesimal generators. As a further application, we study selfdecomposable distributions on a Lie group.

Operator-stable distributions and operator-stable Lévy processes on Euclidean spaces or infinite dimensional vector spaces have been studied with details. See Jurek-Mason [4], Sato [10] and references therein. Recently the author [5] [6] studied stable distributions and stable (homogeneous) convolution semigroups on a simply connected nilpotent Lie group, which correspond to strictly operator-stable distributions and strictly operator-stable Lévy processes, respectively, on Euclidean space. The present self-similar nonhomogeneous convolution semigroup on a Lie group is a nonhomogeneous extension of the stable (homogeneous) convolution semigroup.

Our selfdecomposable distribution on a Lie group corresponds to an operator selfdecomposable distribution or a distribution of the class OL in Sato [10] on an Euclidean space. We shall imbed it into a nonhomogeneous self-similar convolution semigroup of distributions and then characterize the former through the infinitesimal generator of the latter. We remark that in the case of Euclidean space, the imbedding was done by Sato [10], where the latter is called a process of class L.

2. Nonhomogeneous convolution semigroups on a Lie group and their infinitesimal generators

Let G be a connected Lie group of dimension d. Elements of G are denoted by σ, τ, etc., and its unit element is denoted by e. Let $\mathscr{G}$ (or $\mathscr{G}'$) be its left invariant (or right invariant) Lie algebra. Elements of $\mathscr{G}$ (or $\mathscr{G}'$) are regarded as left invariant (or right invariant) first order differential operators (vector fields) and are denoted by X, Y, etc. (or X', Y', etc.) We fix its basis $\{X_1,\cdots,X_d\}$ (or $\{X'_1,\cdots,X'_d\}$).

Let μ be a distribution on G. For a bounded continuous function f on G, we set $\mu(f)=\int f d\mu$. For two distributions μ and ν on G, their *convolution* is a distribution on G defined by $\mu*\nu(A)=\int_G\mu(d\sigma)\nu(\sigma^{-1}A)$. A distribution μ is called *infinitely divisible in the generalized sense* if for any $\varepsilon>0$, there exist distributions $\nu_1,\cdots,\nu_n$ such that $\mu=\nu_1*\nu_2*\cdots\nu_n$ and $\nu_j(U_\varepsilon^c)<\varepsilon$ for any $j\leq n$, where U_ε is an ε-neighborhood of the unit e of G. In particular, if we can choose $\nu_1,\cdots,\nu_n$ as identical distributions, the distribution μ is called *infinitely divisible.* In the case where G is a Euclidean space, it is known that any infinitely divisible distribution in the generalized sense is infinitely divisible. However, the author does not know whether a similar fact is valid for distributions on Lie groups.

Let $\{\mu_{s,t}\}_{0\leq s<t<\infty}$ be a family of probability distributions on a Lie group G. It is called a *nonhomogeneous convolution semigroup* if it satisfies the following two properties.

(i) (Semigroup property) $\mu_{s,t}*\mu_{t,u}=\mu_{s,u}$ holds for any $0\leq s<t<u<\infty$.

(ii) (Continuity) For any $t_0>0$, $\lim_{h\to 0}\sup_{0\leq s<t<t_0,t-s<h}|\mu_{s,t}(f)-f(e)|=0$ holds for any bounded continuous function f.

Clearly each $\mu_{s,t}$ is an infinitely divisible distribution in the generalized sense.

A convolution semigroup $\{\mu_{s,t}\}$ is called *homogeneous* if for any $s<t$, $\mu_{s,t}$ depends only on $t-s$. We denote it by μ_{t-s}. Then one parameter family of distributions $\{\mu_t\}_{t>0}$ satisfies $\mu_s*\mu_t=\mu_{s+t}$ for any $s,t>0$ and $\mu_h\to\delta_e$ weakly as $h\to 0$, where δ_e is the unit measure concentrated at e. It is called a (*homogeneous*) *convolution semigroup.*

Let $\mathscr{C}=\mathscr{C}(G)$ be the set of all bounded continuous functions f on G (such that $\lim_{\sigma\to\infty}f(\sigma)$ exists if G is noncompact). It is a separable Banach space with the supremum norm $\|\ \|$.

We define

$$(2.1)\qquad P_{s,t}f(\sigma)=\int f(\sigma\tau)\mu_{s,t}(d\tau),\qquad f\in\mathscr{C}.$$

Then $\{P_{s,t}\}_{0\leq s<t<\infty}$ is a family of linear operators on $\mathscr{C}$ and satisfies $P_{s,t}P_{t,u}=P_{s,u}$ for all $s<t<u$. Further $\lim_{t\downarrow s}P_{s,t}f=f$ holds for all $f\in\mathscr{C}$.

We denote by $\mathscr{E}_2$ the totality of $f\in\mathscr{C}$ such that it is twice continuously differentiable. Let $\mathscr{C}_2$ (or $\mathscr{C}'_2$) be the totality of $f\in\mathscr{E}_2$ such that Xf and YZf (or $X'f$ and $Y'Z'f$) belong to $\mathscr{C}$ for any $X,Y,Z\in\mathscr{G}$ (or $X',Y',Z'\in\mathscr{G}'$). Set

$$(2.2)\qquad \|f\|'_2=\|f\|+\sum_{i=1}^{d}\|X'_if\|+\sum_{j,k=1}^{d}\|X'_jX'_kf\|.$$

Then $\mathscr{C}'_2$ is a Banach space with this norm. It holds $X'P_{s,t}f=P_{s,t}X'f$, etc., so that $P_{s,t}$ maps $\mathscr{C}'_2$ into itself and satisfies $\|P_{s,t}f\|'_2\leq\|f\|'_2$.

For the study of nonhomogeneous convolution semigroups, it is convenient to introduce the associated space-time homogeneous semigroups. We need some notations. Let $\tilde{G}=G\times[0,\infty)$ be the product manifold. Let $\tilde{\mathscr{C}}=\mathscr{C}(\tilde{G})$ be the set of all bounded continuous functions f on G such that $\lim_{\sigma\to\infty}f(\sigma,t)$ exists uniformly in $t\in[0,N]$ for any N if G is noncompact. Let $f=f(\sigma,t)\in\tilde{\mathscr{C}}$. When we fix the variable t and consider it as a function of σ, we denote it by $f_t(\sigma)$. Then $\tilde{\mathscr{C}}$ is a locally convex linear topological space with seminourms: $\|f\|_N^* = \sup_{0\le t\le N}\|f_t\|$. If a sequence $\{f_n\}$ of $\tilde{\mathscr{C}}$ converges to f of $\tilde{\mathscr{C}}$ with respect to these seminorms, the sequence is said to converge in the space $\tilde{\mathscr{C}}$.

Let $\tilde{\mathscr{E}}_{2,1}$be the set of all $f(\sigma,t)\in\tilde{\mathscr{C}}$ which are twice continuously differentiable with respect to σ and continuously differentiable with respect to t. Here and in the sequel, by the derivative at $t=0$, we mean the right derivative: $\lim_{h\downarrow 0}(f(\sigma,h)-f(\sigma,0))/h$. $\tilde{\mathscr{C}}_{2,1}$ is the subspace of $f\in\tilde{\mathscr{E}}_{2,1}$ such that Xf, YZf, $(\partial/\partial t)f$ belong to $\tilde{\mathscr{C}}$ for any $X,Y,Z\in\mathscr{G}$. The space $\tilde{\mathscr{C}}'_{2,1}$ is defined similarly. Obviously, the spaces $\mathscr{C}$, $\mathscr{C}_2$, etc., are imbedded in the spaces $\tilde{\mathscr{C}}$, $\tilde{\mathscr{C}}_{2,1}$, etc., respectively.

Now, define

$$\tilde{T}_r f(\sigma,t) = P_{t,t+r}(f_{t+r})(\sigma). \tag{2.3}$$

Then $\{\tilde{T}_r\}_{r>0}$ are continuous linear operators on $\tilde{\mathscr{C}}$ and satisfy $\tilde{T}_r\tilde{T}_s=\tilde{T}_{r+s}$ for $r,s>0$ and $\tilde{T}_r f\to f$ as $r\to 0$ in $\tilde{\mathscr{C}}$. Consider the Laplace transform of $\{\tilde{T}_r\}$. For $\lambda>0$, set

$$\tilde{R}_\lambda f(\sigma,s)\equiv\int_0^\infty e^{-\lambda r}\tilde{T}_r f(\sigma,s)dr=\int_0^\infty e^{-\lambda r}P_{s,s+r}(f_{s+r})(\sigma)dr.$$

Then $\tilde{R}_\lambda$ are continuous linear operators on $\tilde{\mathscr{C}}$ and satisfy the resolvent equation $\tilde{R}_\lambda f=\tilde{R}_\gamma f-(\lambda-\gamma)\tilde{R}_\lambda\tilde{R}_\gamma f$, for any $\lambda,\gamma>0$ and $\lambda\tilde{R}_\lambda f\to f$ in $\tilde{\mathscr{C}}$ as $\lambda\to\infty$. Therefore the map $\tilde{R}_\lambda:\tilde{\mathscr{C}}\to\tilde{\mathscr{C}}$ is one to one and the range $\{\tilde{R}_\lambda f: f\in\tilde{\mathscr{C}}\}$ is independent of λ, which we denote by $\tilde{\mathscr{D}}$. The infinitesimal generator of the semigroup $\{\tilde{T}_r\}$ is defined by

$$\tilde{A}f(\sigma)\equiv\lim_{h\to 0}\frac{\tilde{T}_h f(\sigma)-f(\sigma)}{h},$$

if the right hand side converges in the space $\tilde{\mathscr{C}}$. The domain of the operator $\tilde{A}$ is the set of all $f\in\tilde{\mathscr{C}}$ such that the above limit exists. It coincides with $\tilde{\mathscr{D}}$ and it holds $\tilde{A}f=(\lambda-\tilde{R}_\lambda^{-1})f$ for any $f\in\tilde{\mathscr{D}}$.

We introduce a differentiability condition for the resolvent $\tilde{R}_\lambda$ with respect to t.

Condition (***D***). For any $g\in\tilde{\mathscr{C}}'_{2,1}$, $\tilde{R}_\lambda g(\sigma,t)$ is continuously differentiable with respect to $t\in[0,\infty)$ and $(\partial/\partial t)\tilde{R}_\lambda g(\sigma,t)$ belongs to $\tilde{\mathscr{C}}$.

Assuming the above condition, $\tilde{R}_\lambda f$belongs $\tilde{\mathscr{C}}'_{2,1}$ if $f\in\tilde{\mathscr{C}}'_{2,1}$. Indeed, we have

$$X'\tilde{R}_\lambda f(\sigma,s)=\tilde{R}_\lambda X'f(\sigma,s),\qquad Y'Z'\tilde{R}_\lambda f(\sigma,s)=\tilde{R}_\lambda Y'Z'f(\sigma,s).$$

Further, we have $\lambda X'\tilde{R}_\lambda f\to X'f$, $\lambda Y'Z'\tilde{R}_\lambda f\to Y'Z'f$. Therefore $\|\lambda(\tilde{R}_\lambda f)_t-f_t\|'_2\to 0$ uniformly in $t\in[0,N]$ as $\lambda\to\infty$ for any $N>0$, if $f\in\tilde{\mathscr{C}}'_{2,1}$. However, $\{\lambda\tilde{R}_\lambda f\}$ may not converge to f with respect to the strong topology of $\tilde{\mathscr{C}}'_{2,1}$, since $(\partial/\partial t)\lambda\tilde{R}_\lambda f$ may not converge to $(\partial/\partial t)f$.

We set

$$\tilde{\mathscr{D}}_{2,1}=\{\tilde{R}_\lambda g : g\in\tilde{\mathscr{C}}'_{2,1}\}. \tag{2.4}$$

It does not depend on λ because of the resolvent equation of $\tilde{R}_\lambda$.

The *infinitesimal generator* $\{A(t)\}_{t\geq 0}$ *of* $\{P_{s,t}\}$ *or the convolution semigroup* $\{\mu_{s,t}\}$ is defined by

$$A(t)f(\sigma)\equiv\lim_{h\to 0}\frac{P_{t,t+h}f(\sigma)-f(\sigma)}{h}, \tag{2.5}$$

if the right hand side converges in the spae $\tilde{\mathscr{C}}$ as a function of (σ,t). Its domain $\mathscr{D}(\{A(t)\})$ is the set of all $f\in\mathscr{C}$ such that the above limit exists.

The following proposition describes a relation between the infinitesimal generator $\{A(t)\}$ of the convolution semigroup $\{\mu_{s,t}\}$ and the infinitesimal generator $\tilde{A}$ of the associated space-time semigroup $\{\tilde{T}_r\}$.

Proposition 2.1. *Assume Condition* (***D***). *Then for any* $f\in\tilde{\mathscr{D}}_{2,1}$, *the limit*

$$A(t)f_t(\sigma)\equiv\lim_{h\to 0}\frac{1}{h}(P_{t,t+h}f_t(\sigma)-f_t(\sigma)) \tag{2.6}$$

exists in the space $\tilde{\mathscr{C}}$ *as a function of* (σ,t). *Further, we have*

$$\tilde{A}f(\sigma,t)=A(t)f_t(\sigma)+\frac{\partial f}{\partial t}(\sigma,t). \tag{2.7}$$

Proof. Let $f=\tilde{R}_\lambda g$. Note that $\tilde{T}_h f(\sigma,t)=P_{t,t+h}(f_{t+h})(\sigma)$. Then,

$$\frac{1}{h}(P_{t,t+h}f_t-f_t)(\sigma)=\frac{1}{h}(\tilde{T}_h f(\sigma,t)-f(\sigma,t))-\frac{1}{h}P_{t,t+h}(f_{t+h}-f_t)(\sigma)$$
$$\to\tilde{A}f(\sigma,t)-\frac{\partial f}{\partial t}(\sigma,t)$$

in the space $\tilde{\mathscr{C}}$. The proof is complete.

3. Representation of the infinitesimal generators

We shall represent the infinitesimal generator of the convolution semigroup. The following theorem has been proved by Hunt [3] in the case where the convolution semigroup is homogeneous. Maksimov [7] proved, in the case where the convolution semigroup is nonhomogeneous but the underlying Lie group is compact, a similar result under the differentiability condition of $P_{a,t}$ with respect to $a\leq t\leq b$, which is stronger than our Condition (***D***).

In the following theorem, we shall fix $x^1,\cdots,x^d$, ϕ of $\mathscr{C}_2$ satisfying

(3.1) $$x^i(e)=0,\ i=1,\cdots,d, \qquad X_ix^j(e)=\delta_{ij},\ i,j=1,\cdots,d,$$

(3.2) $$\phi(e)=0,\quad \phi(\sigma)>0\ (\sigma\neq e),\quad \lim_{\sigma\to\infty}\phi(\sigma)>0,\quad \text{if } G \text{ is noncompact,}$$

$$\phi\cong\sum(x^i)^2 \text{ near } e.$$

Theorem 3.1. *Assume Condition (D). Then* $\mathscr{C}_2\subset\mathscr{D}(\{A(t)\})$. *Further, for any* $f\in\mathscr{C}_2$, $A(t)f(\sigma)$ *are represented by integro-differential operators* $L(t)f$:

(3.3) $$L(t)f(\sigma)=\frac{1}{2}\sum_{i,j}a^{ij}(t)X_iX_jf(\sigma)+\sum_i b^i(t)X_if(\sigma)$$
$$+\int_G\Big(f(\sigma\tau)-f(\sigma)-\sum_i x^i(\tau)X_if(\sigma)\Big)\nu_t(d\tau),$$

where $A(t)=(a^{ij}(t))$, $b(t)=(b^i(t))$ *and* ν_t *satisfy the following properties.*
(a) $A(t)=(a^{ij}(t))$, $t\geq 0$, *are symmetric nonnegative definite matrices continuous in* t,
(b) $b(t)=(b^i(t))$, $t\geq 0$, *are continuous functions of* t,
(c) ν_t, $t\geq 0$, *are positive measures on* G *such that* $\nu_t(\{e\})=0$ *and the integrals* $\nu_t(\phi)\equiv\int\phi(\tau)\nu_t(d\tau)$ *are finite for any* $t\geq 0$ *and* $\nu_t(\phi f)$ *is continuous in* $t\geq 0$ *for any* $f\in\mathscr{C}$.
The matrices $A(t)$ *and the measures* ν_t *are uniquely determined from the nonhomogeneous convolution semigroup* $\{\mu_{s,t}\}$, *but the vectors* $b(t)$ *may depend on the choice of the functions* $x^1,\cdots,x^d$.

The proof of the theorem will be given after three lemmas. Our argument in these lemmas is close to Hunt [3].

Lemma 3.2. *For* $f\in\mathscr{C}$, *set*

(3.4) $$A(t)_hf(\sigma)=\frac{1}{h}(P_{t,t+h}f(\sigma)-f(\sigma)).$$

If $f\in\mathscr{C}'_2$, $A(t)_hf(e)$ *converges uniformly in* $t\in[0,N]$ *as* $h\to 0$ *for any* $N>0$.

Proof. Let $t_0\geq 0$. We show the uniform convergence of $\{A(t)_hf(e)\}_{h>0}$ with respect to t as $h\to 0$ on a certain neighborhood of t_0. For each t, there exists $\psi^{(t)}$ of $\tilde{\mathscr{D}}_{2,1}$ such that

$$\psi^{(t)}(e,t)=0,\qquad X_i\psi^{(t)}(e,t)=0,\qquad X_i^2\psi^{(t)}(e,t)=2$$

and $\psi^{(t)}(\tau,t)>0$ if $\tau\neq e$, $\lim_{\tau\to\infty}\psi^{(t)}(\tau,t)>0$ if G is noncompact. Indeed, on a certain neighborhood $U(t_0)$ of t_0, the family of functions $\{\psi^{(t)};t\in U(t_0)\}$ can be chosen in the form $\psi^{(t)}=\Sigma_k^n c_k(t)g^k$, where $g^k\in\tilde{\mathscr{D}}_{2,1}$ and $c_k(t)$ are continuous functions of

t. Then, since $A(t)_h g_t^k(e)$ converges uniformly in $t\in U(t_0)$ as $h\to 0$ by Proposition 2.1, the family of functions $\{A(t)_h\psi_t^{(t)}(e),\ t\in U(t_0),\ h>0\}$ is uniformly bounded.

Let $0<\delta<1$. In the sequel, we shall choose, on a certain neighborhood $V(t_0)(\subset U(t_0))$ of t_0, a family of functions $\{g^{(t)}\in\tilde{\mathscr{D}}_{2,1}: t\in V(t_0)\}$ satisfying

$$|f-f(e)-g_t^{(t)}+g_t^{(t)}(e)|\leq\delta\psi_t^{(t)},\qquad \forall t\in V(t_0). \tag{3.5}$$

Since $\psi^{(t)}(\tau,t)\cong\Sigma_i x^i(\tau)^2$ holds near e, there exists an ε neighborhood $U_\varepsilon=\{\tau:\Sigma_i x^i(\tau)^2<\varepsilon^2\}$ of e $(0<\varepsilon<1)$ and $V(t_0)$ such that for any $t\in V(t_0)$,

$$\begin{aligned}\psi^{(t)}(\tau,t)&\geq\frac{1}{2}\varepsilon^2,\qquad \text{if } \tau\in U_\varepsilon^c,\\ &\geq\frac{1}{2}\sum_i x^i(\tau)^2,\qquad \text{if } \tau\in U_\varepsilon.\end{aligned}$$

Set $c=\delta\varepsilon^2/4$. Choose $\lambda_0>0$ such that $g_1\equiv\lambda_0\tilde{R}_{\lambda_0}f$ satisfies $\|f-(g_1)_t\|_2'\leq c$ for any $t\in V(t_0)$. Then we can choose $g_2^{(t)}\in\tilde{\mathscr{D}}_{2,1}$ satisfying the following (i)-(iii).

(i) $\|(g_2^{(t)})_t\|_2'<c$ and $\|(\partial g_2^{(t)}/\partial t\|\leq 1$ for $t\in V(t_0)$.

(ii) $g^{(t)}\equiv g_1+g_2^{(t)}$ satisfies

$$g_t^{(t)}(e)=f(e),\qquad Xg_t^{(t)}(e)=Xf(e),\qquad YZg_t^{(t)}(e)=YZf(e),\qquad \forall X,Y,Z\in\mathscr{G}.$$

(iii) The functions $g^{(t)}$ are represented in the form $g^{(t)}=\Sigma_l^k c_l'(t)g^l$, where $g^l\in\tilde{\mathscr{D}}_{2,1}$ and $c_l'(t)$ are continuous in $t\in V(t_0)$.

We shall prove that $g^{(t)}$ satisfies (3.5). If $\tau\in U_\varepsilon^c$, it holds

$$\begin{aligned}&|f(\tau)-f(e)-g_t^{(t)}(\tau)+g_t^{(t)}(e)|\\ &\quad\leq\|f-g_t^{(t)}\|_2'\leq\|f-(g_1)_t\|_2'+\|(g_2^{(t)})_t\|_2'\leq 2c=\delta\varepsilon^2/2\leq\delta\psi_t^{(t)}(\tau)\end{aligned}$$

for any $t\in V(t_0)$. Further, if $\tau\in U_\varepsilon$ and $t\in V(t_0)$, by the mean value theorem, the left hand side of the above is dominated by

$$\begin{aligned}&\frac{1}{2}|\sum_{i,j}x^i(\tau)x^j(\tau)X_i'X_j'(f-g_t^{(t)})(\tau*)|\\ &\quad\leq\|f-g_t^{(t)}\|_2'\sum_i x^i(\tau)^2\leq\frac{1}{2}\delta\varepsilon^2\sum_i x^i(\tau)^2\leq\delta\psi_t^{(t)}(\tau),\end{aligned}$$

where $\tau*\in U_\varepsilon$.

Now integrate both sides of (3.5) by the measure $h^{-1}\mu_{t,t+h}$. Then we obtain the inequality:

$$|A(t)_h f(e)-A(t)_h g_t^{(t)}(e)|\leq\delta A(t)_h\psi_t^{(t)}(e),$$

since $\psi_t^{(t)}(e)=0$. We have further,

$$P_{t,t+h}\psi_{t+h}^{(t)}(e)=\tilde{T}_h\psi^{(t)}(e,t)=\int_0^h \tilde{T}_r\tilde{A}\psi^{(t)}f(e,t)dr$$
$$=\int_t^{t+h} P_{t,u}(A(u)\psi_u^{(t)}+\frac{\partial\psi^{(t)}}{\partial u})(e)du,$$

by Proposition 2.1. Therefore,

$$A(t)_h\psi_t^{(t)}(e)=\frac{1}{h}\int_t^{t+h} P_{t,u}(A(u)\psi_u^{(t)}+\frac{\partial\psi^{(t)}}{\partial u})(e)du-P_{t,t+h}\left(\frac{1}{h}(\psi_{t+h}^{(t)}-\psi_t^{(t)})\right)(e),$$

which converges uniformly in $t\in V(t_0)$ as $h\to 0$, since $\psi^{(t)}=\Sigma c_k(t)g^k$ and $c_k(t)$ are continuous in t. This implies

$$|A(t)_hf(e)-A(t)_hg_t^{(t)}(e)|\leq O(\delta),\qquad \forall t\in V(t_0),$$

i.e., $A(t)_hf(e)$ and $A(t)_hg_t^{(t)}(e)$ differ by $O(\delta)$ (uniformly in h). Further, $A(t)_hg_t^{(t)}(e)$ converges uniformly on $V(t_0)$ as $h\to 0$ by property (iii). Since this is valid for any $\delta>0$, $A(t)_hf(e)$ should also converge. The proof is complete.

Lemma 3.3. *Let* $\bar{x}^1,\cdots,\bar{x}^d$, $\bar{\phi}$ *be the functions of* $\mathscr{C}'_2$ *satisfying* (3.1) *and* (3.2). *Then, for any* $t\geq 0$, $\bar{A}(t)f(e)\equiv\lim_{h\to 0}A(t)_hf(e), f\in\mathscr{C}'_2$ *is represented by* $\bar{L}(t)f(e)$, *where* $\bar{L}(t)$ *are integro-differential operators represented by*

(3.6)
$$\bar{L}(t)f(\sigma)=\frac{1}{2}\sum_{i,j}\bar{a}^{ij}(t)X_iX_jf(\sigma)+\sum_i\bar{b}^i(t)X_if(\sigma)$$
$$+\int_{\bar{G}}(f(\sigma\tau)-f(\sigma)-\sum_i\bar{x}^i(\tau)X_if(\sigma))\bar{v}_t(d\tau).$$

Here $(\bar{a}^{ij}(t))$ *is a symmetric nonnegative definite matrix,* $(\bar{b}^i(t))$ *is a vector.* $\bar{v}_t$ *is a measure on* $\bar{G}$ *such that* $\bar{v}_t(\{e\})=0$ *and* $\bar{v}_t(\bar{\phi}_t)<\infty$, *where* $\bar{G}=G$ *if* G *is compact and* $\bar{G}=G\cup\{\infty\}$ *(a one point compactification of* G*) if* G *is noncompact.*

Proof. Consider the family of positive measures

$$\bar{F}_{t,h}(E)\equiv\frac{1}{h}\int_E\bar{\phi}(\tau)\mu_{t,t+h}(d\tau).$$

Then $\bar{F}_{t,h}(G)$ is equal to $A(t)_h\bar{\phi}(e)$, which converges to a finite value as $h\to 0$ because $\bar{\phi}\in\mathscr{C}'_2$. Therefore, for each $t\geq 0$, $\{\bar{F}_{t,h};h>0\}$ is a family of a uniformly bounded measures on $\bar{G}$. Furthemore, since $\int f(\tau)\bar{F}_{t,h}(d\tau)=A(t)_h(f\bar{\phi})(e)$, it converges as $h\to 0$ for any $f\in\mathscr{C}'_2$. Therefore the family of the measures $\{\bar{F}_{t,h}\}$ converges weakly as

$h \to 0$. We denote the limit measure by $\bar{F}_t$.

Now for a given $f \in \mathscr{C}'_2$, set

$$c = f(e), \quad c_i = X_i f(e), \quad c_{ij} = X_i X_j (f - \sum_k c_k \bar{x}^k)(e).$$

Then the function $g \equiv f - c - \Sigma c_i \bar{x}^i - \Sigma c_{ij} \bar{x}^i \bar{x}^j$ is of the form $\psi \bar{\phi}$ with $\psi \in \mathscr{C}$ vanishing at e. Therefore we have,

$$\begin{aligned}\bar{A}(t)f(e) &= \bar{A}(t)(c + \sum c_i \bar{x}^i + \sum c_{ij} \bar{x}^i \bar{x}^j)(e) + \lim_{h \to 0} \frac{1}{h} \int \psi \bar{\phi}_t \mu_{t,t+h}(d\tau) \\ &= \sum \bar{A}(t)(\bar{x}^i)(e) c_i + \sum \bar{A}(t)(\bar{x}^i \bar{x}^j)(e) c_{ij} + \int_{\bar{G}} \psi(\tau) \bar{F}_t(d\tau).\end{aligned}$$

Set

$$\bar{b}^i(t) = \bar{A}(t)(\bar{x}^i)(e), \qquad \bar{a}^{ij}(t) = \bar{A}(t)(\bar{x}^i \bar{x}^j)(e),$$

and define measures $\bar{v}_t, t \geq 0$, on $\bar{G}$ by

$$\bar{v}_t(E) = \int_E \bar{\phi}(\tau)^{-1} \bar{F}_t(d\tau) \quad \text{if } E \subset G - \{e\}, \quad \bar{v}_t(\{e\}) = 0.$$

Then $\bar{A}(t)f(e)$ is represented by (3.6) at $\sigma = e$.

REMARK. By the definition, $\bar{a}^{ij}(t)$, $\bar{b}^i(t)$ and $\bar{v}_t(f \bar{\phi}_t)$ $(f \in \mathscr{C})$ are continuous with respect to the parameter t.

Lemma 3.4. *Let $f \in \mathscr{C}'_2$. If it is of compact supports, $A(t)_h f(\sigma)$ converges in the space $\tilde{\mathscr{C}}$ as functions of (σ, t). Let $\bar{A}(t)f(\sigma)$ be its limit. Then it is represented by $\bar{L}(t)f(\sigma)$ of* (3.6) *for any $\sigma \in G$.*

Proof. Apply the result of Lemma 3.3 to the function $f \circ L_\sigma$, where L_σ is the left translation by σ. Since $A(t)_h f(\sigma) = A(t)_h (f \circ L_\sigma)(e)$, it converges to $\bar{L}(t)(f \circ L_\sigma)(e)$ by Lemma 3.3. Further, $\bar{L}(t)(f \circ L_\sigma)(e) = \bar{L}(t)f(\sigma)$ holds since we have $X(f \circ L_\sigma)(e) = Xf(\sigma)$, etc. Therefore for each $\sigma \in G, A(t)_h f(\sigma)$ converges to $\bar{L}(t)f(\sigma)$ as $h \to 0$.

We want to show that if the support of f is compact, $A(t)_h f(\sigma)$ converges to $\bar{L}(t)f(\sigma)$ boundedly. Let t_0, δ, ε be as in the proof of Lemma 3.2. For each $(t, \sigma) \in \tilde{G}$, we can choose $g_2^{(t,\sigma)} \in \tilde{\mathscr{D}}_{2,1}$, satisfying the following (i′)-(iii′).

(i′) $\|(g_2^{(t,\sigma)})_t\|'_2 < c$ and $\|(\partial g_2^{(t,\sigma)} / \partial t\| \leq 1$ for $t \in V(t_0)$, where $c = \delta \varepsilon^2 / 4$.

(ii′) $g^{(t,\sigma)} \equiv g_1 + g_2^{(t,\sigma)}$ satisfies

$$g_t^{(t,\sigma)}(\sigma) = f(\sigma), \quad X g_t^{(t,\sigma)}(\sigma) = Xf(\sigma), \quad YZ g_t^{(t,\sigma)}(\sigma) = YZf(\sigma), \quad \forall X, Y, Z \in \mathscr{G}.$$

(iii′) There exist a finite open covering $\{U_1,\cdots,U_n\}$ of $K\equiv\operatorname{supp}(f)$ and a neighborhood $V(t_0)$ of t_0 such that for each $(t,\sigma)\in V(t_0)\times U_i$, $g^{(t,\sigma)}$ is written as $\Sigma_l^k c_l(t,\sigma)g^l$, where $g^l\in\tilde{\mathscr{D}}_{2,1}$ and $c_l(t,\sigma)$ are continuous in $(t,\sigma)\in V(t_0)\times U_i$. For $(t,\sigma)\in V(t_0)\times K^c$, $g^{(t,\sigma)}$ is written as $\Sigma_l^m c_l'(g')^l$, where $(g')^l\in\tilde{\mathscr{D}}_{2,1}$.
Then it holds

$$|f\circ L_\sigma-f\circ L_\sigma(e)-g_t^{(t,\sigma)}+g_t^{(t,\sigma)}(e)|\leq\delta\bar{\psi}_t^{(t)},\qquad\forall t\in V(t_0),\tag{3.7}$$

as in the proof of Lemma 3.2. Further, we have

$$\sup_{t\in V(t_0)}\sup_{\sigma}\|A(t)g_t^{(t,\sigma)}\|=M<\infty,\tag{3.8}$$

by property (iii′). Integrate both sides of (3.7) by the measure $h^{-1}\mu_{t,t+h}$. Then we obtain

$$|A(t)_h(f\circ L_\sigma)(e)-A(t)_h g_t^{(t,\sigma)}(e)|=O(\delta),$$

similarly as in the proof of Lemma 3.2. Note that $g^{(t,\sigma)}$ belongs to $\tilde{\mathscr{D}}_{2,1}$. Then,

$$A(t)_h g_t^{(t,\sigma)}(e)=\frac{1}{h}\int_t^{t+h}P_{t,u}\left(A(u)g_u^{(t,\sigma)}+\frac{\partial g^{(t,\sigma)}}{\partial u}\right)(e)du-P_{t,t+h}\left(\frac{1}{h}(g_{t+h}^{(t,\sigma)}-g_t^{(t,\sigma)})\right)(e).$$

Therefore,

$$|A(t)_h g_t^{(t,\sigma)}(e)|\leq\sup_{t\leq u\leq t+h}\|A(u)g_u^{(t,\sigma)}\|+2\|\frac{\partial g^{(t,\sigma)}}{\partial u}\|\leq M+2+\|\frac{\partial g_1}{\partial u}\|<\infty,$$

holds for any $\sigma\in G$ and $t+h\in V(t_0)$. Consequently, $A(t)_h f(\sigma)=A(t)_h(f\circ L_\sigma)(e)$ converges to $\bar{L}(t)f(\sigma)$ boundedly as $h\to 0$.

Now note

$$(P_{s,t+h}f-P_{s,t}f)/h=P_{s,t}A(t)_h f.$$

Let $h\to 0$. Then we obtain $\frac{\partial}{\partial t}P_{s,t}f=P_{s,t}\bar{L}(t)f$, since $A(t)_h f$ converges to $\bar{L}(t)f$ boundedly.

Integrating the last equality with respect to t, we obtain

$$P_{s,t}f=f+\int_s^t P_{s,u}\bar{L}(u)f\,du.\tag{3.9}$$

Then we have

$$A(t)_h f=\frac{1}{h}\int_t^{t+h}P_{t,u}\bar{L}(u)f\,du.$$

Since $\bar{L}(u)f\in\mathscr{C}$ and $\|\bar{L}(u)f-\bar{L}(t)f\|\to 0$ as $u\to t$, $\|\bar{A}(t)_h f-\bar{L}(t)f\|\to 0$ holds uniformly in $t\in[0,N]$ as $h\to 0$ for any $N>0$. The proof is complete.

Proof of Theorem 3.1. We have seen in the proof of Lemma 3.4 and its proof that if $f\in\mathscr{C}'_2\cap\mathscr{C}_2$ is of compact support, then (3.9) holds. Now for any $f\in\mathscr{C}_2$, there exists a sequence $\{f_n\}$ of $\mathscr{C}_2\cap\mathscr{C}'_2$ of compact supports, such that $f_n\to f$, $Xf_n\to Xf$ and $YZf_n\to YZf$ hold uniformly on compact sets for any $X,Y,Z\in\mathscr{G}$ and further the convergences are bounded convergences. Then $\bar{L}(t)f_n\to\bar{L}(t)f$ boundedly. Therefore, equality (3.9) is valid for any $f\in\mathscr{C}_2$. Then $A(t)_h f(\sigma)$ converges to $\bar{L}(t)f(\sigma)$ in the space $\tilde{\mathscr{C}}$ as functions of (σ,t). This proves $\mathscr{C}_2\subset\mathscr{D}(\{A(t)\})$ and $A(t)f=\bar{L}(t)f$.

We will prove that $\bar{\nu}_t(\infty)=0$ in the case where G is noncompact. Let $\{f_n\}$ be a sequence of $\mathscr{C}_2$ such that $f_n(\infty)=1$ for all n, and $f_n(\sigma)\to 0$, $Xf_n(\sigma)\to 0$, $YZf_n(\sigma)\to 0$ boundedly for any $\sigma\in G$. Then $\bar{L}(t)f_n(\sigma)$ converge to a constant function $h=\bar{\nu}_t(\infty)$ boundedly. Since (3.9) is valid for any f_n, we have $\int_s^t P_{s,u}h\,du=0$, proving $h=0$.

Now let $x^1,\cdots,x^d$, ϕ be functions of $\mathscr{C}_2$ satisfying (3.1) and (3.2). These functions belong to $\mathscr{D}(\{A(t)\})$. Then we can apply Lemma 3.3 with these functions. Then we get $A(t)f=L(t)f$, where the coefficients of $L(t)$ is determined by $b^i(t)=A(t)(x^i)(e)$, $a^{ij}(t)=A(t)(x^ix^j)(e)$, etc.

The uniqueness of $a^{ij}(t)$ is obvious, since $a^{ij}(t)=A(t)(x^ix^j)(e)$ holds, where $x^1,\cdots,x^d$ are any functions satisfying (3.1). It is clearly symmetric. Further, for any complex numbers $z^1,\cdots,z^d$, we have

$$\sum_{i,j}a^{ij}(t)z^i\bar{z}^j=A(t)\left(|\sum_{i=1}^d x^iz^i|^2\right)(e)=\lim_{h\to 0}\frac{1}{h}P_{t,t+h}\left(|\sum_{i=1}^d x^iz^i|^2\right)(e)\geq 0.$$

Therefore $(a^{ij}(t))$ is nonnegative definite. Next, let $f\in\mathscr{C}_2$ be a function such that $f(e)=X_if(e)=X_iX_jf(e)=0$ for any i, j. Then, $A(t)f(e)=\int f(\tau)\nu_t(d\tau)$. Therefore, the uniqueness of the Lévy measure ν_t follows.

In applications, it is sometimes convenient to extend the domain of the infinitesimal generators of the convolution semigroups. We first introduce some notations. Let $\mathscr{B}$ be the set of all bounded continuous functions on G and let $\mathscr{B}_2$ be the set of all twice continuously differentiable functions on G such that Xf, XYf belong to $\mathscr{B}$ for any $X,Y,Z\in\mathscr{G}$. Let $\tilde{\mathscr{B}}$ be the set of all bounded continuous functions on $\tilde{G}$ and let $\tilde{\mathscr{B}}_{2,1}$ be the set of all functions $f(\sigma,t)\in\tilde{\mathscr{B}}$ which is twice continuously differentiable with respect to σ and continuously differentiable with respect to t such that Xf, XYf, $(\partial/\partial t)f$ belong to $\tilde{\mathscr{B}}$ for any $X,Y,Z\in\mathscr{G}$. Then it holds $\mathscr{C}\subset\mathscr{B}$, etc.

The semigroups $\{P_{s,t}\}$ and $\{\tilde{T}_r\}$ associated with a convolution semigroup $\{\mu_{s,t}\}$ can be extended to the spaces $\mathscr{B}$ and $\tilde{\mathscr{B}}$, respectively. For $f\in\mathscr{B}_2$, we define integro-differential operators $\{L(t)\}$ by (3.3).

Corollary 3.5. *Let $f\in\mathscr{B}_2$. For any $s\geq 0$, $P_{s,t}f$ is continuously differentiable with respect to $t\in(s,\infty)$, and satisfies*

$$\frac{\partial P_{s,t}f}{\partial t}=P_{s,t}L(t)f, \quad \forall t>s. \tag{3.10}$$

4. Problems related to the infinitesimal generators

4.1. Stochastic differential equation. Existence of the nonhomogeneous convolution semigroup

Let $\{\mu_{s,t}\}_{0\leq s<t<\infty}$ be a nonhomogeneous convolution semigroup on a Lie group G. Then on a certain probability space $(\Omega,\mathscr{F},P)$, we can define a stochastic process $\{\varphi_t;t\geq 0\}$ with values in G satisfying the following equality:

$$\begin{aligned}P(\varphi_0=e,\ \varphi_{t_1}\in A_1,\ \varphi_{t_2}\in A_2,\ \cdots,\ \varphi_{t_n}\in A_n)\\ =\int\cdots\int_{A_1\times\cdots\times A_n}\mu_{0,t_1}(d\sigma_1)\mu_{t_1,t_2}(\sigma_1^{-1}d\sigma_2)\cdots\mu_{t_{n-1},t_n}(\sigma_{n-1}^{-1}d\sigma_n)\end{aligned} \tag{4.1}$$

for any $0<t_1<\cdots<t_n<\infty$ and Borel sets $A_1,\cdots,A_n$ of G. The stochastic process $\{\varphi_t,t\geq 0\}$ has independent increments, i.e., G-valued random variables $\varphi_{t_{i-1}}^{-1}\varphi_{t_i}$, $i=1,\cdots,n$ are independent for any $0=t_0<t_1<\cdots<t_n<\infty$. Indeed the equality (4.1) implies

$$P(\varphi_{t_{i-1}}^{-1}\varphi_{t_i}\in B_i,\ i=1,\cdots,n)=\prod_{i=1}^{n}\mu_{t_{i-1},t_i}(B_i) \tag{4.2}$$

for any Borel sets B_i, $i=1,\cdots,n$. The process $\{\varphi_t,t\geq 0\}$ is called a *process with independent increments* on the Lie group G associated with the nonhomogeneous convolution semigroup $\{\mu_{s,t}\}$. Conversely let $\{\varphi_t,t\geq 0\}$ be a stochastic process with values in G continuous in probability such that it has independent increments and $\varphi_0=e$. The process $\{\varphi_t,t\geq 0\}$ is called a *process with independent increments*. Define $\mu_{s,t}(B)=P(\varphi_s^{-1}\varphi_t\in B)$. Then $\{\mu_{s,t}\}$ is a nonhomogeneous convolution semigroup. It is said to be *associated with the process with independent increments* $\{\varphi_t,t\geq 0\}$.

The process with independent increments $\{\varphi_t,t\geq 0\}$ has a modification such that it is right continuous with the left hand limit with respect to time t, provided that the associated nonhomogeneous convolution semigroup $\{\mu_{s,t}\}$ satisfies Condition (***D***). Indeed, if $f\in\mathscr{C}_2$, $f(\varphi_t)-f(e)-\int_0^t L(s)f(\varphi_s)ds$ is a martingale with mean 0 because of the equality (3.10). Then the stochastic process $f(\varphi_t),t\geq 0$, has a modification which is right continuous with left hand limits. Since this is valid for any $f\in\mathscr{C}_2$, the existence of such a modification for the process $\varphi_t,t\geq 0$, follows.

Now suppose we are given integro-differential operators $\{L(t)\}$ represented by

(3.3), where the coefficients $a^{ij}(t)$, $b^i(t)$, ν_t satisfies condtions (a)-(c) of Theorem 3.1. We show the existence of the convolution semigroup whose infinitesimal generator is represented by the integro-differential operators $\{L(t)\}$, constructing a process with independent increments by solving a stochastic differential equation on the Lie group.

Let $(A(t),b(t),\nu_t)$ be an arbitrary triple satisfying conditions (a)-(c) of Theorem 3.1. Then there exists a nonhomogeneous Brownian motion $B_t=(B_t^1,\cdots,B_t^d)$ such that the mean of B_t-B_s is $\int_s^t b(r)dr$ and the covariance is $\int_s^t(a^{ij}(r))dr$ and a nonhomogeneous Poisson random measure $N((s,t]\times E)$ on G with intensity measure $dtd\nu_t(\tau)$ independent of B_t.

We consider a stochastic differential equation on the Lie group G driven by B_t and $N((s,t]\times E)$:

$$
\begin{aligned}
(4.3) \qquad f(\varphi_t)=f(\sigma)&+\sum_i\int_s^t X_if(\varphi_{u-})\circ dB_u^i \\
&+\int_s^{t+}\int_G (f(\varphi_{u-}\tau)-f(\varphi_{u-}))\tilde{N}(dud\tau) \\
&+\int_s^{t+}\int_G (f(\varphi_{u-}\tau)-f(\varphi_{u-})-\sum_i X_if(\varphi_{u-})x^i(\tau))\nu_u(d\tau)du,
\end{aligned}
$$

where $\tilde{N}((s,t]\times E)=N((s,t]\times E)-\int_s^t\nu_u(E)du$ and f is a test function of $\mathscr{B}_2$. The integral $\int\cdots\circ dB_u$ is the Stratonovich integral.

Theorem 4.1. *For any $s\geq 0$ and $\sigma\in G$, there exists a unique solution of the stochastic differential equation* (4.3) *driven by B_t and $N((s,t]\times E)$. Denote the solution by $\varphi_{s,t}(\sigma)$ and set $\varphi_{s,t}=\varphi_{s,t}(e)$. Then it has the following properties*:

(1) *$\sigma\varphi_{s,t}(\tau)=\varphi_{s,t}(\sigma\tau)$ holds a.s. for any σ, τ and $s<t$.*

(2) *$\varphi_{s,t}\varphi_{t,u}=\varphi_{s,u}$ holds a.s. for any $s<t<u$.*

(3) *$\varphi_t\equiv\varphi_{0,t}$ is a process with independent increments.*

The proof can be carried out similarly as in Applebaum-Kunita [1]. We omit the details of the proof. The above theorem implies:

Theorem 4.2. *Suppose we are given a family of triple $(A(t),b(t),\nu_t)$, $t\geq 0$, satisfying conditions* (a)-(c) *of* Theorem 3.1. *Then there exists a unique nonhomogeneous convolution semigroup $\{\mu_{s,t}\}$ such that the domain $\mathscr{D}(\{A(t)\})$ of the infinitesimal generator $\{A(t)\}$ includes $\mathscr{C}_2$ and $A(t)f$, $f\in\mathscr{C}_2$ is represented by $L(t)f$ of* (3.3).

4.2. An extension of the convolution semigroup

We have so far considered the convolution semigroup $\{\mu_{s,t}\}$ defined for $0\leq s<t<\infty$. However, in some applications, we encounter a convolution semigroup $\{\mu_{s,t}\}$ defined only for $0<s<t<\infty$, i.e., $\mu_{0,t}$ is not defined. Thus the limit $\mu_{0,t}\equiv\lim_{s\to 0}\mu_{s,t}$ may or may not exists. Even so, we can obtain the representation of the infinitesimal generator under a condition similar to Condition (***D***).

Let $\{\mu_{s,t}\}_{0<s<t<\infty}$ be a convolution semigroup on a Lie group G. Then for any $s>0$, there exists a stochastic process $\varphi_{s,t}, t\geq s$ with values in G with independent increments such that $\varphi_{s,s}=e$ and $\mu_{t,u}(A)=P(\varphi_{s,t}^{-1}\varphi_{s,u}\in A)$ holds for all $s<t<u$. The semigroup of linear operators $\{P_{s,t}\}_{0<s<t<\infty}$, its infinitesimal generator $\{A(t)\}_{t>0}$ and the domain $\mathscr{D}(\{A(t)\})$ are defined in the same way as in Section 2.

Denote $G\times(0,\infty)$ by $\tilde{G}^o$. Let $\tilde{\mathscr{C}}^o$ be the set of all continuous functions on $\tilde{G}^o$ such that $\sup_{t>1/N}\|f_t\|<\infty$ and $\lim_{\sigma\to\infty}f(\sigma,t)$ exists uniformly in $t\in[1/N,N]$ for any $N>0$. It is a locally convex linear topological space with seminorms $\|f\|_N^{**}=\sup_{1/N\leq t\leq N}\|f_t\|$. The resolvents $\tilde{R}_\lambda$ are defined on the space $\tilde{\mathscr{C}}^o$. Let $\tilde{\mathscr{C}}_{2,1}^{o,\prime}$ be the set of all $f\in\tilde{\mathscr{C}}^o$, which is twice continuously differentiable with respect to σ and continuously differentiable with respect to $t>0$ and $X'f$, $Y'Z'f$, $(\partial/\partial t)f$ belong to $\tilde{\mathscr{C}}^o$ for any $X',Y',Z'\in\mathscr{G}$, where X', etc., are right invariant vector fields.

Let $\tilde{\mathscr{C}}_{2,1}^d$ be a dense subspace of $\tilde{\mathscr{C}}_{2,1}^{o,\prime}$. We introduce a differentiability condition of the resolovent $\tilde{R}_\lambda$, which is slightly weaker than Condition (***D***) introduced in Section 2.

Condition ($\boldsymbol{D}_0$). For any $g\in\tilde{\mathscr{C}}_{2,1}^d$, $\tilde{R}_\lambda g(\sigma,t)$ is continuously differentiable with respect to $t\in(0,\infty)$ in the space $\tilde{\mathscr{C}}^o$.

Now if we restrict the time set of the convolution semigroup $\{\mu_{s,t}\}$ to $[\varepsilon,\infty)$, where $\varepsilon>0$, then we can apply Theorem 3.1 and its Corollary for this convolution semigroup. Therefore the assertions of the theorems are valid if "$t\geq 0$" is replaced by "$t>0$" in the corresponding statement.

We are interested in the case where $\mu_{0,t}\equiv\lim_{s\downarrow 0}\mu_{s,t}$ exists. If the limit exists, the extended family of distributins $\{\mu_{s,t}\}_{0\leq s<t<\infty}$ becomes a convolution semigroup in the sense of Section 2. The existence of $\mu_{0,t}$ is equivalent to the convergence (in probability) of $\varphi_{s,t}$ as $s\to 0$. Further, if the limit exists, $\varphi_t\equiv\lim_{s\to 0}\varphi_{s,t}$ becomes a process with independent increments associated with the extended semigroup.

Theorem 4.3. *Assume Condition* ($\boldsymbol{D}_0$) *for the convolution semigroup* $\{\mu_{s,t}\}_{0<s<t<\infty}$. *Then* $\mathscr{C}_2\subset\mathscr{D}(\{A(t)\})$. *Further,* $A(t)f$, $f\in\mathscr{C}_2$ *is represented by* $\boldsymbol{L}(t)f$ *of* (3.3).

Furthermore, if $\mu_{0,t}$ *exists, the triple* $(A(t),b(t),\nu_t)_{t>0}$ *satisfies the following integrability condition*:

(d) *The triple* $(A(t),b(t),\nu_t)$ *is integrable on the interval* (0,1), *i.e.*, (4.4) $$\int_0^1 |A(t)|dt<\infty, \quad \int_0^1 |b(t)|dt<\infty,$$

(4.5) $$\int_0^1 \nu_t(\phi)dt<\infty,$$

where ϕ is a function of $\mathscr{C}_2$ satisfying (3.2).

Conversely, suppose that we are given a triple $(A(t),b(t),\nu_t)_{t>0}$ satisfying (a)-(c) *of* Theorem 3.1 *and* (d). *Then there exists a unique nonhomogeneous convolution semigroup $\{\mu_{s,t}\}_{0<s<t<\infty}$ such that the domain $\mathscr{D}(\{A(t)\})$ of the infinitesimal generator $\{A(t)\}$ includes $\mathscr{C}_2$ and $A(t)f$, $f\in\mathscr{C}_2$, is represented by $L(t)f$ of* (3.3). *Further, $\mu_{0,t}$ exists for any $t>0$.*

In order to prove the theorem, we shall apply the orthogonal representation theory of the Lie group.

Let $g:G\to O(n)$ be a C^∞-homomorphism, where $O(n)$ is the linear Lie group of orthogonal $n\times n$-matrices. We call g an *orthogonal representation of G of degree n.* For an orthogonal representation g of degree n, we define an $n\times n$-matrix $\mu(g)$ by $\mu(g)=(\mu(g_{ij}))$. Then we have

Lemma 4.4. (1) *It holds $\mu*\nu(g)=\mu(g)\nu(g)$ for any distributions μ, ν on G and orthogonal representation g.*

(2) *If μ is an infinitely divisible distibution on G in the generalized sense, the matrix $\mu(g)$ is invertible for any orthogonal representation g.*

Proof. For any distributions μ, ν and any orthogonal representation g, we have

$$\mu*\nu(g)=\iint\mu(d\sigma)\nu(\sigma^{-1}d\tau)g(\tau)=\iint\mu(d\sigma)\nu(d\tau)g(\sigma\tau)$$
$$=\iint\mu(d\sigma)\nu(d\tau)g(\sigma)g(\tau)=\mu(g)\nu(g),$$

proving the first assertion.

We prove the second assertion. Since μ is infinitely divisible in the generalized sense, for any $\varepsilon>0$, there exists $\mu_1,\cdots,\mu_n$ such that $\mu=\mu_1*\cdots*\mu_n$ and $\mu_j(U_\varepsilon^c)<\varepsilon$ for any $j\le n$, where U_ε is an ε-neighborhood of the unit e of G. Then it holds $\mu(g)=\mu_1(g)\cdots\mu_n(g)$. Since $g(e)$ is the identity and $g(\sigma)$ is continuous in σ, $\mu_j(g)$ are invertible for any $j\le n$ for a sufficiently small ε. Therefore $\mu(g)$ is also invertible. The proof is complete.

We shall modify the Peter-Weyl theory concerning the completeness of representations of a compact Lie group, so that it can be applied to a noncompact

Lie group. Set

(4.6) $\Delta_n=\{g_{ij}(\sigma): g(\sigma)=(g_{ij}(\sigma))$ are orthogonal representations of degree $n\}$

and define $\Delta=\cup_n\Delta_n$. It is a system of bounded C^∞-functions.

Lemma 4.5 (cf. Pontryagin [9]). *The system Δ is locally uniformly complete in $\mathscr{B}$, i.e., any element of $\mathscr{B}$ can be approximated uniformly on each compact subset of G by a sequence of linear sums of elements of Δ.*

Proof. We will follow the argument of [9]. Let $\boldsymbol{K}$ be the set of all real C^∞-functions $k(\sigma)$ with compact supports such that $k(\sigma)=k(\sigma^{-1})$ holds for all $\sigma\in G$. For $k\in\boldsymbol{K}$, we consider an integral equation

$$\varphi(\sigma)=\lambda\int k(\sigma^{-1}\tau)\varphi(\tau)d\tau, \tag{4.7}$$

on the real $\boldsymbol{L}_2(d\tau)$ space, where $d\tau$ is a (left) Haar measure. Any nontrivial solution $\varphi\neq 0$ of (4.7) belonging to $\boldsymbol{L}_2(d\tau)$ is called an *eigen function of the kernel* k and λ is called an *eigen value of the kernel* k.

Let $\Delta'(k)$ be the set of all eigen functions of the above integral equation with all possible eigen values. Define $\Delta'=\cup_{k\in\boldsymbol{K}}\Delta'(k)$. Then Δ' is a locally uniformly complete system in $\mathscr{B}$. See [9], Section 29.

For a given $k\in\boldsymbol{K}$, let $\varphi_1,\cdots,\varphi_n$ be a complete system of orthonormal eigen functions with an eigen value λ. These are C^∞-functions since k is a C^∞-function with a compact support. Let α be any fixed element of G. Then, since $d\tau$ is a Haar measure, $\varphi_i(\alpha\sigma)$, $i=1,\cdots,n$ are also eigen functions with the same eigen value λ. Then there exists a matrix $g(\alpha)$ such that

$$\varphi_i(\alpha\sigma)=\sum_j g_{ij}(\alpha)\varphi_j(\sigma), \qquad i=1,\cdots,n. \tag{4.8}$$

Noting that $\{\varphi_1,\cdots,\varphi_n\}$ are orthonormal, it is easy to verify that the matrix $g(\alpha)$ is orthogonal. It is a C^∞-function of α and satisfies $g(\alpha\beta)=g(\alpha)g(\beta)$ for all $\alpha,\beta\in G$. Therefore g is an orthogonal representation of G, i.e., $g_{ij}\in\Delta$.

Now, setting $\sigma=e$ in (4.8), we have $\varphi_i(\alpha)=\Sigma_j g_{ij}(\alpha)\varphi_j(e)$ for all $\alpha\in G$. This shows that each φ_i is a linear sum of elements of Δ. Therefore, any $f\in\mathscr{B}$ can be approximated uniformly on each compact sebset of G by a sequence of linear sums of elements of Δ. The proof is complete.

Proof of Theorem 4.3. Suppose first that $\mu_{0,t}$ exists. Consider the $n\times n$-matrix function $\boldsymbol{L}(r)g(\sigma)\equiv(\boldsymbol{L}(r)g_{ij}(\sigma))$. Note that g_{ij} belongs to $\mathscr{B}_2$. A direct computation yields $\boldsymbol{L}(r)g(\sigma)=g(\sigma)\boldsymbol{L}(r)g(e)$. Then we have $(d/dt)\mu_{s,t}(g)=\mu_{s,t}(g)\boldsymbol{L}(t)g(e)$ by Corollary

3.5. Then we obtain $\mu_{s,t}(g)^{-1}(d/dt)\mu_{s,t}(g)=L(t)g(e)$. Let $s\to 0$. Then $\mu_{0,t}(g)^{-1}(d/dt)\mu_{0,t}(g)=L(t)g(e)$. The left hand side is integrable near 0 since $\mu_{0,t}(g)^{-1}\to I$ as $t\to 0$. Therefore $L(t)g(e)$ is integrable on (0,1).

We take a component of $g=(g^{kl})$, say g^{kl} and denote it by f. We show that

$$\sum_{i,j}\left(\int_0^t a^{ij}(u)du\right)X_i f(\sigma)X_j f(\sigma)+\int_0^t\int_G (f(\sigma\tau)-f(\sigma))^2\nu_u(d\tau)du<\infty. \tag{4.9}$$

Let $\varphi_{s,t}$, $t\geq s$, be a process with independent increments such that $\varphi_{s,s}=e$ and the law of $\varphi_{s,t}$ is $\mu_{s,t}$. We may assume that it is a solution of the stochastic differential equation (4.3), where B_t is a Brownian motion with mean 0 and $\mathrm{Cov}(B_t-B_s)=\int_s^t A(u)du$ and $N((s,t]\times E)$ is a Poisson random measure with intensity measure $\nu_t dt$. Equation (4.3) is written as

$$f(\varphi_{s,t})=\delta_{kl}+(k,l)\text{-component of }\int_s^t g(\varphi_{s,u})L(u)g(e)du+M_{s,t}(f), \tag{4.10}$$

where $M_{s,t}(f)$ is a martingale, whose bracket process is given by

$$\langle M_{s,t}(f)\rangle=\sum_{i,j}\int_s^t a^{ij}(u)X_i f(\varphi_{s,u})X_j f(\varphi_{s,u})du \tag{4.11}$$
$$+\int_s^t\int_G (f(\varphi_{s,u}\tau)-f(\varphi_{s,u}))^2\nu_u(d\tau)du.$$

Since $\varphi_{s,t}\to\varphi_{0,t}$ in probability, $\int_s^t g(\varphi_{s,u})L(u)g(e)du\to\int_0^t g(\varphi_{0,u})L(u)g(e)du$. Therefore, $\lim_{s\downarrow 0}M_{s,t}(f)$ exists boundedly. This proves $\lim_{s\downarrow 0}\langle M_{s,t}(f)\rangle<\infty$ a.s. Then we obtain (4.9).

The above argument implies

$$\int_0^t |a^{ij}(u)|du<\infty, \tag{4.12}$$

$$\int_0^t\int_G (f(\sigma\tau)-f(\sigma))^2\nu_u(d\tau)du<\infty. \tag{4.13}$$

Then we get $\int_0^1|b^i(u)|du<\infty$, since $\int_0^t|L(u)g|du<\infty$. We can repeat a similar argument for $f\in\mathscr{B}_2$ instead of g. Then we obtain (4.13) for any $f\in\mathscr{B}_2$. Then we have $\int_0^t(\int\phi(\tau)\nu_u(d\tau))du<\infty$. We have thus obtained (4.4) and (4.5).

Conversely suppose we are given a triple $(A(t),b(t),\nu_t)_{t>0}$ satisfying (a)-(d). Then there exists a Brownian motion B_t starting from 0 at time 0 with mean $\int_0^t b(u)du$ and covariance $\int_0^t A(u)du$, and a Poisson random measure $N((s,t]\times E)$ with the intensity measure $\nu_t dt$ independent of B_t. Then the stochastic differential equation (4.3) has a unique solution $\varphi_{s,t}(\sigma)$ for any $0\leq s<t<\infty$. It defines a convolution semigroup $\{\mu_{s,t}\}_{0\leq s<t<\infty}$ by setting $\mu_{s,t}(E)=P(\varphi_{s,t}\in E)$, where $\varphi_{s,t}=\varphi_{s,t}(e)$. It admits the required property. The proof is complete.

4.3. The case where the Lie group is simply connected and nilpotent

We shall obtain another representation of the infinitesimal generator in the case where the Lie group G is simply connected and nilpotent. An important fact on a simply connected, nilpotent Lie group is that any $\sigma\in G$ is represented uniquely as $\sigma=\exp(\Sigma_i x_i X_i)$, where $X_1,\cdots,X_d$ is a fixed basis of $\mathcal{G}$ and $x=(x_1,\cdots,x_d)\in \boldsymbol{R}^d$. Further, the exponential map $\exp:\mathcal{G}\rightarrow G$ is a diffeomorphism. See Hochschild [2].

We shall restrict our attension to the convolution semigroup satisfying Condition $(\boldsymbol{D}_0)$.

Theorem 4.6. *Let G be a simply connected and nilpotent Lie group. Let $\{\mu_{s,t}\}_{0<s<t<\infty}$ be a nonhomogeneours convolution semigroup on G satisfying Condition $(\boldsymbol{D}_0)$. Let $\{A(t)\}_{t>0}$ be its infinitesimal generators. Set $\xi(x)=\exp\Sigma_i x_i X_i$. Then for $f\in\mathscr{C}_2$, $A(t)f$ is represented by the integro-differential operator $\boldsymbol{L}(t)f$, where*

$$
\begin{aligned}
(4.14)\qquad \boldsymbol{L}(t)f(\sigma)=&\frac{1}{2}\sum_{i,j}a^{ij}(t)X_iX_jf(\sigma)+\sum_i b^i(t)X_if(\sigma)\\
&+\int_{\boldsymbol{R}^d}\{f(\sigma\xi(x))-f(\sigma)-\sum_i\frac{x_i}{1+|x|^2}X_if(\sigma)\}\nu_t(dx),
\end{aligned}
$$

where $A(t)=(a^{ij}(t))$ and $b(t)=(b^i(t))$ satisfy (a) and (b) of Theorem 3.1, *respectively (replacing "$t\geq 0$" by "$t>0$") and*
(c') ν_t, $t>0$, are Lévy measures on $\boldsymbol{R}^d$, i.e., they satisfy

$$
(4.15)\qquad \nu_t(\{0\})=0,\qquad \int_{\boldsymbol{R}^d}\frac{|x|^2}{1+|x|^2}\nu_t(dx)<\infty
$$

for any $t\geq 0$. Further, $\int_{\boldsymbol{R}^d}(|x|^2/(1+|x|^2))f(x)\nu_t(dx)$ is continuous in $t>0$ for any bounded continuous function f on $\boldsymbol{R}^d$.

Further, if $\mu_{0,t}=\lim_{s\downarrow 0}\mu_{s,t}$ exists, the following (d') is satisfied.
(d') The triple $(A(t),b(t),\nu_t)$ is integrable on the interval $(0,1)$, i.e., $A(t)$, $b(t)$ satisfies (4.4) and ν_t satisfies

$$
(4.16)\qquad \int_0^1\left(\int_{\boldsymbol{R}^d}\frac{|x|^2}{1+|x|^2}\nu_t(dx)\right)dt<\infty.
$$

Conversely, suppose that we are given triple $(A(t),b(t),\nu_t)_{t>0}$ satisfying (a), (b), (c') and (d'). Then there exists a unique nonhomogeneous convolution semigroup $\{\mu_{s,t}\}_{0<s<t<\infty}$ such that the domain $\mathscr{D}(\{A(t)\})$ of the infinitesimal generator $\{A(t)\}_{t>0}$ includes $\mathscr{C}_2$ and $A(t)f$, $f\in\mathscr{C}_2$, is represented by $\boldsymbol{L}(t)f$ of (4.14). Further, $\mu_{0,t}$

exists.

Proof. We shall apply Theorem 3.1 and Theorem 4.3. Let $x=(x_1,\cdots,x_d)$ be the global canonical coordinate of G, i.e., for $\sigma=\exp(\Sigma x_i X_i)$, we set $x_i(\sigma)=x_i$. Define the functions $x^1,\cdots,x^d$, ϕ of $\mathscr{C}_2$ by $x^i(\sigma)=x_i(\sigma)/(1+|x(\sigma)|^2)$ and $\phi(\sigma)=|x(\sigma)|^2/(1+|x(\sigma)|^2)$. We define a measure $\xi^{-1}\nu_t$ on $\boldsymbol{R}^d$ by $\xi^{-1}\nu_t(E)=\nu_t(\xi(E))$ for any Borel sets E of $\boldsymbol{R}^d$. For simplicity of the notation, we denote the measure $\xi^{-1}\nu_t$ by ν_t. Then (3.3) is written as (4.14). Properties (c′) and (d′) follow from properties (c) and (d) of Theorems 3.1 and 4.3, immediately.

The triple $(A(t),b(t),\nu_t)$ of Theorem 4.6 determines the operators $\boldsymbol{L}(t)$, so that it determines and characterizes the convolution semigroup $\{\mu_{s,t}\}$. It is called the *characteristics* of the convolution semigroup $\{\mu_{s,t}\}$.

5. Infinitesimal generators of self-similar nonhomogeneous convolution semigroups

As an application of Theorem 4.6, we shall determine the infinitesimal generators of all self-similar nonhomogeneous convolution semigroups. It will turn out that the infinitesimal generator $A(t)$ can not be defined at $t=0$ in many cases.

Let β be an automorphism of the Lie group G, i.e., it is a diffeomorphism of G and satisfies $\beta(\tau\sigma)=\beta(\tau)\beta(\sigma)$ for any $\tau,\sigma\in G$. For a distribution μ on G, we denote by $\beta\mu$ the distribution such that $\beta\mu(A)=\mu(\beta^{-1}(A))$ holds for any Borel set A of G. Then it holds $(\beta\mu)(f)=\mu(f\circ\beta)$, where $f\circ\beta$ is the composition of the function f and the map β. Further, the equality $\beta(\mu*\nu)=(\beta\mu)*(\beta\nu)$ holds for any distributions μ, ν.

Let $\{\mu_{s,t}\}$ be a nonhomogeneous convolution semigroup of distributions on G. Set $\mu_{s,t}^{(\beta)}=\beta\mu_{s,t}$. Then $\{\mu_{s,t}^{(\beta)}\}$ is a nonhomogeneous convolution semigroup. Indeed, we have $\beta\mu_{s,t}*\beta\mu_{t,u}=\beta(\mu_{s,t}*\mu_{t,u})=\beta\mu_{s,u}$ for any $s<t<u$ and $\beta\mu_{s,t}(f)=\mu_{s,t}(f\circ\beta)\to f(e)$ as $|t-s|\to 0$ for any $f\in\mathscr{B}$. We discuss the infinitesimal generator of $\{\mu_{s,t}^{(\beta)}\}$ in connection with that of $\{\mu_{s,t}\}$.

Let $d\beta$ be the differential of the automorphism β. Then $d\beta$ defines an automorphism of the Lie algabre $\mathscr{G}$, i.e., $d\beta$ is an invertible linear map of $\mathscr{G}$ onto itself satisfying the relation $d\beta[X,Y]=[d\beta X,d\beta Y]$ for all $X,Y\in\mathscr{G}$, where $[,]$ is the Lie bracket. It holds $(d\beta X)f=X(f\circ\beta)\circ\beta^{-1}$ for any $X\in\mathscr{G}$ and $f\in\mathscr{C}_2$. Now if we fix a basis $\{X_1,\cdots,X_d\}$ of the Lie algebra $\mathscr{G}$, $d\beta$ induces naturally an invertible linear tranformation of $\boldsymbol{R}^d$, which we denote by the same notation $d\beta$.

For a measure ν on $\boldsymbol{R}^d$, we denote by $d\beta\nu$ the measure such that $(d\beta\nu)(A)=\nu(d\beta^{-1}(A))$ holds for any Borel set A.

Lemma 5.1. *Let $\{\mu_{s,t}\}$ be a nonhomogeneous convolution semigroup on a simply connected nilpotent Lie group G satisfying Condition $(\boldsymbol{D}_0)$. Let β be an automorphism* *of G. Then $\{\mu_{s,t}^{(\beta)} \equiv \beta\mu_{s,t}\}$ is a nonhomogeneous convolution semigroup satisfying Condition ($\boldsymbol{D}_0$). Let $\{A^{(\beta)}(t)\}$ be its infinitesimal generator. Then it is represented by*

$$
\begin{aligned}
(5.1)\qquad \boldsymbol{L}^{(\beta)}(t)f(\sigma) = & \sum_{i,j} a^{ij}(t)d\beta X_i d\beta X_j f(\sigma) \\
& + \int_{\boldsymbol{R}^d} \{f(\sigma\xi(x)) - f(\sigma) - \sum_i \frac{x_i}{1+|x|^2} X_i f(\sigma)\}(d\beta\nu_t)(dx) \\
& + \sum_i b^i(t) d\beta X_i f(\sigma) \\
& + \sum_i \left(\int_{\boldsymbol{R}^d} \{\frac{x^i}{1+|x|^2} - \frac{x^i}{1+|(d\beta)^{-1}x|^2}\}(d\beta\nu_t)(dx) \right) X_i f(\sigma),
\end{aligned}
$$

where $((a^{ij}(t)),(b^i(t)),\nu_t)$ is the characteristics of the convolution semigroup $\{\mu_{s,t}\}$.

Proof. Let $\{P_{s,t}^{(\beta)}\}$ be the semigroup of linear operators associated with $\{\mu_{s,t}^{(\beta)}\}$. Then it satisfies $P_{s,t}^{(\beta)}f = P_{s,t}(f\circ\beta)\circ\beta^{-1}$, where $\{P_{s,t}\}$ is the semigroup of $\{\mu_{s,t}\}$. Then, the corresponding resolvents $\tilde{R}_\lambda^{(\beta)}$ and $\tilde{R}_\lambda$ are related by $\tilde{R}_\lambda^{(\beta)}f = \tilde{R}_\lambda(f\circ\beta)\circ\beta^{-1}$. Therefore $\tilde{R}_\lambda^{(\beta)}$ satisfies Condition ($\boldsymbol{D}_0$) if $\tilde{R}_\lambda$ satisfies it. Let $\{A^{(\beta)}(t)\}$ be its infinitesimal generator. It satisfies $A^{(\beta)}(t)f = A(t)(f\circ\beta)\circ\beta^{-1}$ for any $f\in\mathscr{C}_2$. Since $A(t)f$ is represented by $\boldsymbol{L}(t)f$ of (4.14), $A^{(\beta)}(t)f$ is represented as $\boldsymbol{L}^{(\beta)}(t)f$ of (5.1). The proof is complete.

Let $\{\gamma_r\}_{r>0}$ be a one parameter group of automorphisms of the Lie group G, i.e., (i) for each $r>0$, γ_r is an automorphism of G, (ii) $\gamma_r\gamma_s = \gamma_{rs}$ holds for any $r,s>0$, (iii) γ_r is continuous in $r\in(0,\infty)$. It is called a *dilation* if it satisfies (iv) $\gamma_r(\sigma)\to e$ uniformly on compact sets as $r\to 0$. A dilation can not be defined on an arbitrary Lie group. Indeed, if a dilation exists on the Lie group G, the Lie group is necessarily simply connected and nilpotent.

Given a dilation $\{\gamma_r\}$, we set $\delta_t = \gamma_{e^t}$, $-\infty < t < \infty$. Then $\{\delta_t\}$ is a one parameter group of diffeomorphisms of G. Let $\boldsymbol{Y}$ be its infinitesimal generator (a complete C^∞-vector field):

$$
(5.2)\qquad \boldsymbol{Y}f(\sigma) = \lim_{t\to 0}\frac{f(\delta_t(\sigma)) - f(\sigma)}{t}.
$$

Then, it holds

$$
(5.3)\qquad \frac{\partial}{\partial t} f\circ\gamma_t = \frac{1}{\mathrm{t}}\boldsymbol{Y}(f\circ\gamma_t).
$$

Note that $\boldsymbol{Y}$ is not necessarily an element of $\mathscr{G}$.

A nonhomogeneous convolution semigroup $\{\mu_{s,t}\}_{0\leq s<t<\infty}$ is called *self-similar with respect to a dilation* $\{\gamma_r\}_{r>0}$ if $\gamma_r\mu_{s,t}=\mu_{rs,rt}$ holds for any $r>0$ and $0\leq s<t<\infty$. Let $\{\varphi_t, t\geq 0\}$ be a G-valued stochastic process with independent increments associated with $\{\mu_{s,t}\}$. Then it is self-similar with respect to $\{\gamma_r\}$, if and only if the law of the process $\{\varphi_t^{(\gamma_r)}\equiv\gamma_r(\varphi_t), t\geq 0\}$ is equal to the law of the process $\{\varphi_t^{(r)}\equiv\varphi_{rt}, t\geq 0,\}$ for any $r>0$.

In this section we will characterize all self-similar nonhomogeneous convolution semigroups through the representation of the infinitesimal generators. We first show that any self-similar convolution semigroup satisfies Condition $(\boldsymbol{D}_0)$ of Section 4 so that the infinitesimal generator admits the representation (4.14).

Lemma 5.2. *Let $\{\mu_{s,t}\}$ be a nonhomogeneous convolution semigroup on a simply connected nilpotent Lie group G. Suppose that it is self-similar with respect to a dilation $\{\gamma_r\}$. Then its resolvent $\tilde{R}_\lambda$ satisfies Condition $(\boldsymbol{D}_0)$.*

Proof. The associated semigroup $\{P_{s,t}\}$ satisfies $P_{s,t}(f\circ\gamma_r)\circ\gamma_r^{-1}(\sigma)=P_{rs,rt}f(\sigma)$. Therefore,

$$\begin{aligned}\tilde{R}_\lambda f(\sigma,t)&=\int_0^\infty e^{-\lambda s}P_{t,t+s}(f_{t+s})(\sigma)ds\\&=\int_1^\infty e^{-\lambda t(r-1)}P_{t,tr}(f_{tr})(\sigma)tdr\\&=\int_1^\infty e^{-\lambda t(r-1)}tP_{1,r}(f_{tr}\circ\gamma_t)\circ\gamma_t^{-1}(\sigma)dr.\end{aligned}$$

Set $\tilde{\mathscr{C}}^d_{2,1}=\{f\in\tilde{\mathscr{C}}^{o,\prime}_{2,1}: Yf\in\tilde{\mathscr{B}}\}$. It is a dense subset of $\tilde{\mathscr{C}}^{o,\prime}_{2,1}$. For $f\in\tilde{\mathscr{C}}^d_{2,1}$, the above is continuously differentiable with respect to $t>0$. The proof is complete.

We shall study the infinitesimal generator of a self-similar nonhomogeneous convolution semigroup. Let $d\gamma_r$ be the differential of the automorphism γ_r. Then $\{d\gamma_r\}_{r>0}$ is a one parameter group of automorphisms of $\mathscr{G}$. It satisfies $d\gamma_r X\to 0$ as $r\to 0$ for any $X\in\mathscr{G}$. The linear map $d\gamma_r$ is represented by $d\gamma_r=\exp(\log r)Q$, where Q is a linear map of $\mathscr{G}$ such that all of its eigen values have positive real parts. Further it satisfies $Q[X,Y]=[QX,Y]+[X,QY]$ for all $X,Y\in\mathscr{G}$. The map $d\gamma_r$ is often written as r^Q and the linear map Q is called the *exponent* of the dilation $\{\gamma_r\}_{r>0}$. The adjoint (transpose) of Q is denoted by Q'.

Theorem 5.3. *Let $\{\mu_{s,t}\}_{0\leq s<t<\infty}$ be a nonhomogeneous convolution semigroup on a simply connected nilpotent Lie group G. Suppose that it is self-similar with respect to a dilation $\{\gamma_r\}$ with exponent Q. Then its infinitesimal generator $\{A(t)\}_{t>0}$ admits the representation $\{\boldsymbol{L}(t)\}_{t>0}$ of* (4.14). *Further, the triple $(A(t),b(t),\nu_t)$*

satisfies the following equalities:

(5.4) $$A(t)=t^{-1}t^{Q}A(1)t^{Q'}, \quad \forall t>0.$$

(5.5) $$\nu_t=t^{-1}(t^{Q}\nu_1), \quad \forall t>0,$$

where ν_1 *is a Lévy measure satisfying*

(5.6) $$\int_{\boldsymbol{R}^d}\log(1+|x|^2)\nu_1(dx)<\infty.$$

(5.7) $$b(t)=t^{Q-1}b(1)+\int_{\boldsymbol{R}^d}\{\frac{x}{1+|x|^2}-\frac{x}{1+|t^{-Q}x|^2}\}t^{-1}(t^{Q}\nu_1)(dx), \quad \forall t>0.$$

Conversely suppose that we are given an arbitrary triple $(A(1),b(1),\nu_1)$ *of a symmetric nonnegative definite matrix* $A(1)$, *a vector* $b(1)$ *and a Lévy measure* ν_1 *satisfying* (5.6). *Let* $\{\gamma_r\}$ *be an arbitrary dilation and let* Q *be its exponent. Then there exists a unique convolution semigroup* $\{\mu_{s,t}\}_{0\le s<t<\infty}$ *whose infinitesimal generator* $\{A(t)\}_{t>0}$ *admits the representation* $\{\boldsymbol{L}(t)\}$ *of* (4.14) *with characteristics* (5.4), (5.5), (5.7). *It is self-similar with respect to* $\{\gamma_r\}$.

Proof. Let $\{A^{(r)}(t)\}$ be the infinitesimal generator of $\{\mu_{s,t}^{(r)}\}$, where $\mu_{s,t}^{(r)}\equiv\mu_{rs,rt}$. Then it holds $A^{(r)}(t)=rA(rt)$, where $\{A(t)\}$ is the infinitesimal generator of $\{\mu_{s,t}\}$. Next let $\{A^{(\gamma_r)}(t)\}$ be the infinitesimal generator of the nonhomogeneous convolution semigroup $\{\mu_{s,t}^{(\gamma_r)}\}$. Since $\mu_{s,t}^{(\gamma_r)}=\mu_{s,t}^{(r)}$ holds for any $s<t$ and $r>0$, we have $A^{(\gamma_r)}(t)f=rA(rt)f$, $f\in\mathscr{C}_2$ for any $r>0$ and $t>0$. This implies, in particular, $A(t)f=t^{-1}A^{(\gamma_t)}(1)f$, $f\in\mathscr{C}_2$. Now $A^{(\gamma_t)}(1)f$ is represented by $\boldsymbol{L}^{(\gamma_t)}(1)f$ of (5.1). Since $d\gamma_t=t^{Q}$ holds, we get equalities (5.4), (5.5) and (5.7) by comparing the coefficients of operators $\boldsymbol{L}(t)$ of (4.14) and $t^{-1}\boldsymbol{L}^{(\gamma_t)}(1)$.

We shall prove the integrability condition (5.6) by making use of the integrability condition (d') of Theorem 4.6. Setting $t=e^{-u}$ in (4.16), the left hand side of (4.16) equals

(5.8) $$\begin{aligned}\int_0^1 t^{-1}\{\int_{\boldsymbol{R}^d}\frac{|x|^2}{1+|x|^2}(t^{Q}\nu_1)(dx)\}dt&=\int_0^\infty\{\int_{\boldsymbol{R}^d}\frac{|x|^2}{1+|x|^2}(e^{-uQ}\nu_1)(dx)\}du\\&=\int_{\boldsymbol{R}^d}\{\int_0^\infty\frac{|e^{-uQ}x|^2}{1+|e^{-uQ}x|^2}du\}\nu_1(dx).\end{aligned}$$

Since,

$$c_1\log(1+|x|^2)\le\int_0^\infty\frac{|e^{-uQ}x|^2}{1+|e^{-uQ}x|^2}du\le c_2\log(1+|x|^2)$$

holds with some positive constants c_1, c_2 (Urbanik [13]), the integral (5.8) is finite if and only if integrability (5.6) holds.

Conversely suppose that the triple $(A(1), b(1), \nu_1)$ satisfies the conditions mentioned in the theorem. Define the triple $(A(t), b(t), \nu_t)$ by (5.4), (5.5) and (5.7). Then it satisfies (a), (b), (c'), (d') of Theorems 3.1 and 4.6 because of (5.6). Therefore there exists a nonhomogeneous convolution semigroup $\{\mu_{s,t}\}$ with characteristics $(A(t), b(t), \nu_t)$. We will show that it is self-similar with respect to $\{\gamma_r\}$. It is enough to prove $\mu_{s,t}^{(\gamma_r)} = \mu_{s,t}^{(r)}$ for all $r>0$ and $s<t$. It is easy to verify that the triple $(A(t), b(t), \nu_t)$ satisfies

$$A(rt) = r^{-1} r^{Q} A(t) r^{Q'}, \qquad \nu_{rt} = r^{-1}(r^{Q}\nu_t),$$

$$b(rt) = r^{Q-1} b(t) + \int_{\mathbf{R}^d} \left\{ \frac{x}{1+|x|^2} - \frac{x}{1+|r^{-Q}x|^2} \right\} r^{-1}(r^{Q}\nu_t)(dx),$$

for any $t>0$ and $r>0$. Therefore the infinitesimal generator $\{A^{(\gamma_r)}(t)\}$ of $\{\mu_{s,t}^{(\gamma_r)}\}$ satisfies $A^{(\gamma_r)}(t) = A^{(r)}(t) = rA(rt)$ for any $r>0$ and $t>0$, where $\{A^{(r)}(t)\}$ and $\{A(t)\}$ are infinitesimal generators of $\{\mu_{s,t}^{(r)}\}$ and $\{\mu_{s,t}\}$, respectively. This implies $\mu_{s,t}^{(\gamma_r)} = \mu_{s,t}^{(r)}$ for any $s<t$ and $r>0$. The proof is complete.

Theorem 5.3 tells us that an arbitrary integro-differential operator $\boldsymbol{L}(1)$ (or the characteristics $(A(1), b(1), \nu_1)$) satisfying the integrability condition (5.6) and an arbitrary dilation $\{\gamma_r\}$ give us a unique nonhomogeneous $\{\gamma_r\}$-self-similar convolution semigroup. However, if we restrict our attention to homogeneous ones, it is much more restrictive.

In Kunita [6], a homogeneous convolution semigroup $\{\mu_t\}$ is called *stable with respect to a dilation* $\{\gamma_r\}$, if $\gamma_r\mu_t = \mu_{rt}$ holds for all $t>0$ and $r>0$. Therefore for a homogeneous convolution semigroup, the self-similar property and the stable property are identical. The following Corollary indicates how strongly the characteristics and the dilation are related for the stable convolution semigroups.

Corollary 5.4 (cf. Kunita [6]). *Let* $\{\mu_t\}$ *be a homogeneous convolution semigroup on a simply connected nilpotent Lie group G. It is self-similar (stable) with respect to a dilation* $\{\gamma_r\}$ *with exponent Q, if and only if its characteristics* (A, b, ν) *satisfies the following equalities:*

$$QA + AQ' = A, \tag{5.9}$$

$$\nu(E) = \int_S \lambda(d\theta) \int_0^\infty \chi_E(r^{Q}\theta) r^{-2} dr, \tag{5.10}$$

where $S = \{\theta \in \mathbf{R}^d; |\theta| = 1,\ |r^{Q}\theta| > 1$ *for all* $r>1\}$ *and* λ *is a bounded measure on S, and b satisfies*

(5.11) $$(Q-I)b=\int_{R^d}\frac{2\langle Qx,x\rangle}{(1+|x|^2)^2}x\nu(dx).$$

Proof. Theorem 5.3 tells us that the homogeneous convolution semigroup is self-similar with respect to r^Q if and only if its characteristics (A,b,ν) satisfies

$$A=t^{-1}t^QAt^{Q'},\qquad \nu=t^{-1}(t^Q\nu),$$

$$b=t^{Q-1}b+\int_{R^d}\{\frac{x}{1+|x|^2}-\frac{x}{1+|t^{-Q}x|^2}\}\nu(dx),$$

for all $t>0$. These three equalities are equivalent to (5.9), (5.10) and (5.11), respectively. See the proof of Theorem 2.1 in [6].

REMARK. The equality (5.11) indicates the following two cases:
a) If 1 is not an eigen value of Q, the vector b is determined by ν and Q. It is given by the following b_1:

(5.12) $$b_1=\int_{R^d}\frac{2\langle Qx,x\rangle}{(1+|x|^2)^2}(Q-I)^{-1}x\nu(dx).$$

b) If 1 is an eigen value of Q, the measure ν satisfies an additional equality:

(5.13) $$\int_{R^d}\frac{2\langle Qx,x\rangle}{(1+|x|^2)^2}T_{W_1}x\nu(dx)\in\tilde{W}_1,$$

where W_1 is the invariant subspace of R^d generated by eigen vectors associated with eigen value 1, T_{W_1} is the projector to the space W_1 and $\tilde{W}_1=\{(Q-I)x;x\in W_1\}$. Further the vector b is given by b_1+b_0, where b_1 is the vector of (5.13) and b_0 is an element of $\hat{W}_1=\{x:Qx=x\}$.

Finally, we give some examples of stable, and nonstable self-similar Brownian motions on a Heisenberg group. Let G be a Heisenberg group. It is diffeomorphic to R^3 and is a simply connected nilpotent Lie group of step 2. There exists a basis $\{X_1,X_2,X_3\}$ of the Lie algebra $\mathcal{G}$ of G satisfying $[X_1,X_2]=X_3$ and $[X_1,X_3]=[X_2,X_3]=0$. Consider a differential operator:

(5.14) $$L^{(c)}=c_1X_1^2+c_2X_2^2+c_3X_3^2,$$

where $c=(c_1,c_2,c_3)$ are nonnegative constants. Then there exists a unique homogeneous convolution semigroup $\{\mu_t^{(c)}\}$ with the infinitesimal generator $L^{(c)}$. Let Q be a 3×3 diagonal matrix with diagonal elements α_1, α_2 and α_3 where α_1, α_2, α_3 are positive constants. Then for any $r>0$, r^Q defines an automorphism of $\mathcal{G}$ if and only if $\alpha_3=\alpha_1+\alpha_2$. If the equality holds, there exists an automorphism

γ_r of G such that $d\gamma_r = r^Q$. Then $\{\gamma_r\}_{r>0}$ is a dilation with the exponent Q. The homogeneous convolution semigroup $\{\mu_t^{(c)}\}$ is self-similar with respect to the dilation, if and only if $c_3=0$ and the exponent satisfies $\alpha_1=\alpha_2=1/2$ and $\alpha_3=1$. See Kunita [6].

However, if c is a time dependent function $c=c(t)$, then the situation is completely different. Denote the right hand side of (5.14) by $L^{(c)}(t)$. Then there exists a unique nonhomogeneous convolution semigroup $\{\mu_{s,t}^{(c)}\}$ with the infinitesimal generator $L^{(c)}(t)$. Further, it is self-similar with respect to the dilation with exponent Q such that $\alpha_3=\alpha_1+\alpha_2$, if and only if $L^{(c)}(t)$ is represented by

$$L^{(c)}(t)=(c_1 t^{2\alpha_1-1})X_1^2+(c_2 t^{2\alpha_2-1})X_2^2+(c_3 t^{2(\alpha_1+\alpha_2)-1})X_3^2, \tag{5.15}$$

where c_1, c_2, c_3 are arbitrary nonnegative constants.

6. Selfdecomposable distributions and the associated self-similar nonhomogeneous convolution semigroups

6.1. Selfdecomposable distributions

Let β be an automorphism of the Lie group G. A distribution μ on G is called *β-decomposable* if there exists a distribution μ_β such that $\mu=\beta\mu*\mu_\beta$. Let $\{\gamma_r\}$ be a dilation on G. A distribution μ is called $\{\gamma_r\}$-selfdecomposable if it is γ_r-decomposable for any $0<r<1$. If $\{\mu_{s,t}\}$ is self-similar with respect to $\{\gamma_r\}$, $\mu_{0,t}$ is $\{\gamma_r\}$-selfdecomposable for any $t>0$.

The following theorem is a generalization of Sato [10], where a Q-selfdecomposable distribution on $\boldsymbol{R}^d$ is imbedded into a nonhomogeneous convolution semigroup on $\boldsymbol{R}^d$.

Theorem 6.1. *Let μ be a distribution on a simply connected nilpotent Lie group G equipped with a dilation $\{\gamma_r\}$. Suppose that μ is infinitely divisible in the generalized sense and $\{\gamma_r\}$-selfdecomposable. Then there exists a unique nonhomogeneous convolution semigroup $\{\mu_{s,t}\}$ on G, self-similar with respect to $\{\gamma_r\}$ such that $\mu_{0,1}=\mu$.*

For the proof of the theorem, we need a lemma.

Lemma 6.2. (1) *Let μ and ν be infinitely divisible distributions in the generalized sense. Suppose that there exists a distribution ξ such that $\mu=\nu*\xi$. Then ξ is uniquely determined.*

(2) *Let μ be infinitely divisible in the generalized sense. Let $\{\nu_n\}$ be a sequence of infinitely divisible distributions in the generalized sense converging weakly to an infinitely divisible distribution ν in the generalized sense. Suppose there exists a sequence of distributions $\{\xi_n\}$ such that $\mu=\nu_n*\xi_n$. Then the sequence $\{\xi_n\}$ converges weakly to a distribution ξ such that $\mu=\nu*\xi$.*

Proof. Let g be an orthogonal representation of G. Then we have the equality $\mu(g)=\nu(g)\xi(g)$. Since $\nu(g)$ is invertible, we obtain $\xi(g)=\nu(g)^{-1}\mu(g)$. Let Δ be the system of bounded C^{∞}-functions on G introduced by (4.6). Since Δ is locally uniformly complete in $\mathscr{B}$ by Lemma 4.5, ξ is uniquely determined by ν and μ. This proves the first assertion of the lemma.

We shall prove the second assertion. Since $\mu(g)=\nu_n(g)\xi_n(g)$ and $\{\nu_n(g)\}$ converges to the invertible matrix $\nu(g)$, $\{\xi_n(g)\}$ converges for any orthogonal representation g. We shall prove that the sequence $\xi_n, n=1,2,\cdots$ converges weakly to a distribution ξ. Let $\bar{G}=G\cup\infty$ be a one point compactification of G. It is a semigroup by setting $\sigma\infty=\infty\sigma=\infty$ for any $\sigma\in\bar{G}$. All distributions $\mu,\nu,\xi_n,n=1,2,\cdots$ can be regarded as distributions on $\bar{G}$ by setting $\mu(\{\infty\})=0$, etc. Then there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ converging weakly to a distribution $\bar{\xi}$ on the space $\bar{G}$, since $\bar{G}$ is a compact space. It satisfies $\mu=\nu*\bar{\xi}$ i.e., for any continuous function $\bar{f}$ on $\bar{G}$, we have

$$\mu(\bar{f})=\nu*\xi(\bar{f})+\bar{f}(\infty)\bar{\xi}(\{\infty\}),$$

where ξ is a measure on G such that $\xi=\bar{\xi}$ on G. We shall prove $\bar{\xi}(\{\infty\})=0$. Let $\chi_{\{\infty\}}$ be the indicator function of the set $\{\infty\}$ and $\bar{f}_n$ be a sequence of continuous functions over $\bar{G}$ such that $\bar{f}_n\downarrow\chi_{\{\infty\}}$. Then $\mu(\bar{f}_n)=\nu*\xi(\bar{f}_n)+\bar{f}_n(\infty)\bar{\xi}(\{\infty\})$. Let n tend to infinity. Both of $\mu(\bar{f}_n)$ and $\nu*\xi(\bar{f}_n)$ converges to 0. Since $\lim\bar{f}_n(\infty)=1$, we obtain $\bar{\xi}(\{\infty\})=0$. We have thus proved the equality $\mu=\nu*\xi$. Since ξ is unique by (1), the sequence $\{\xi_n\}$ converges weakly to ξ on the space G. The proof is complete.

Proof of Theorem 6.1. Set $\mu_{0,t}=\gamma_t\mu$. If $0<t<1$, we have $\mu=\mu_{0,t}*\mu_t$. We set $\mu_{0,1}=\mu$ and $\mu_{t,1}=\mu_t$. Then it holds $\mu_{0,1}=\mu_{0,t}*\mu_{t,1}$. For $0\le s<t<\infty$, we set $\mu_{s,t}=\gamma_t\mu_{s/t,1}$. Then we have $\mu_{s,1}=\mu_{s,t}*\mu_{t,1}$ if $s<t<1$. Indeed, since

$$\mu_{0,s}*\mu_{s,t}*\mu_{t,1}=\gamma_t\mu_{0,s/t}*\gamma_t\mu_{s/t,1}*\mu_{t,1}=\gamma_t(\mu_{0,s/t}*\mu_{s/t,1})*\mu_{t,1}=\gamma_t\mu*\mu_{t,1}=\mu,$$

we have $\mu_{0,s}*(\mu_{s,t}*\mu_{t,1})=\mu_{0,s}*\mu_{s,1}$. This proves $\mu_{s,t}*\mu_{t,1}=\mu_{s,1}$ in view of Lemma 6.2 (1). Next for any $0\le s<t<u<\infty$, we have

$$\mu_{s,t}*\mu_{t,u}=\gamma_u(\mu_{s/u,t/u}*\mu_{t/u,1})=\gamma_u\mu_{s/u,1}=\mu_{s,u}.$$

Therefore $\{\mu_{s,t}\}$ is a nonhomogeneous convolution semigroup.

We will prove the continuity of the semigroup $\{\mu_{s,t}\}$. Note the equality $\mu_{0,t}=\mu_{0,s}*\mu_{s,t}$. Since $\mu_{0,s}=\gamma_{s/t}\mu_{0,t}$, $\mu_{0,s}$ converges weakly to $\mu_{0,t}$ as $s\to t$. Therefore $\mu_{s,t}$ converges weakly to δ_ε as $s\to t$ by Lemma 6.2 (2). Furthermore, noting $\mu_{s,t}=\gamma_t\mu_{s/t,1}$, we obtain $\sup_{s,t<\varepsilon}|\mu_{s,t}(f)-f(e)|\to 0$ as $\varepsilon\to 0$, and for any $\varepsilon>0$,

$$\sup_{s,t>\varepsilon,|s-t|<h}|\mu_{s,t}(f)-f(e)|\to 0 \quad \text{as } h\to 0.$$

These two convergences imply the continuity of the semigroup.

We will next prove the uniqueness of the convolution semigroup. Let $\{\hat{\mu}_{s,t}\}$ be another convolution semigroup satisfying $\gamma_r\hat{\mu}_{s,t}=\hat{\mu}_{rs,rt}$ and $\hat{\mu}_{0,1}=\mu$. Then we have $\hat{\mu}_{0,s}=\gamma_s\mu=\mu_{0,s}$ for any $s>0$. Since $\hat{\mu}_{0,s}*\hat{\mu}_{s,t}=\gamma_t\mu=\mu_{0,s}*\mu_{s,t}$ and $\hat{\mu}_{0,s}=\mu_{0,s}$ hold, we have $\hat{\mu}_{s,t}=\mu_{s,t}$ for any $s<t$ by Lemma 6.2 (1). The proof is complete.

We do not discuss the existence of the density function of the selfdecomposable distribution. But we derive a partial differential equation satisfied by the density function. Let Y be the infinitesimal generator ($=$a complete C^∞ vector field) of $\delta_t=\gamma_{e^t}$, $-\infty<t<\infty$, defined by (5.2). Then it holds

$$Yf(\sigma)=\lim_{t\to1}\frac{f(\gamma_t(\sigma))-f(\sigma)}{t-1}. \tag{6.1}$$

It is represented by

$$Yf(\sigma)=\sum_{j,k}Q_{jk}x_j(\sigma)X_kf, \tag{6.2}$$

where $\sigma=\exp\Sigma x_j(\sigma)X_j$.

Theorem 6.3. *Suppose that $\mu_{0,t}$ has a density function $g_t(\sigma)$ with respect to the Haar measure $d\sigma$. Then it satisfies*

$$\left(L(t)-\frac{1}{t}Y\right)^*g_t=0, \tag{6.3}$$

where $(L(t)-\frac{1}{t}Y)^*$ is the formal adjoint represented as

$$\begin{aligned}\left(L(t)-\frac{1}{t}Y\right)^*f(\sigma)=&\frac{1}{2}\sum_{i,j}a^{ij}(t)X_iX_jf(\sigma)-\sum_i b^i(t)X_if(\sigma)\\&+\frac{1}{t}Yf(\sigma)+\frac{1}{t}(\mathrm{tr}Q)f(\sigma)\\&+\int_{R^d}\left(f(\sigma\xi(-x))-f(\sigma)+\sum_j\frac{x_j}{1+|x|^2}X_jf(\sigma)\right)\nu_t(dx).\end{aligned} \tag{6.4}$$

Proof. Let $f\in\mathscr{C}_2$. Then $P_{0,t}f$ satisfies $\frac{d}{dt}P_{0,t}f=P_{0,t}L(t)f$. Since $P_{0,t+h}f=P_{0,t}(f\circ\gamma_{1+h/t})$, we have

$$\frac{d}{dt}P_{0,t}f=\lim_{h\to0}\frac{P_{0,t}(f\circ\gamma_{1+h/t})-P_{0,t}f}{h}=\frac{1}{t}P_{0,t}Yf.$$

Therefore we have

$$\mu_{0,t}((L(t)-\frac{1}{t}Y)f)=0,$$

for any $f\in\mathscr{C}_2$. Then the density function g_t satisfies (6.3). The proof is complete.

6.2. The case of Euclidean space. Operator-self-similar convolution semigroups and operator-selfdecomposable distributions

We shall consider the case where $G=\boldsymbol{R}^d$. Let $\{\mu_{s,t}\}$ be a nonhomogeneous convolution semigroup on $\boldsymbol{R}^d$ satisfying Condition $(\boldsymbol{D}_0)$. Then $\mu_{s,t}$ are infinitely divisible distributions for all $0<s<t$. We shall compute their characteristic functions. Set $f_z(x)=\exp i\langle z,x\rangle$. Then $\mu_{s,t}(f_z)(0)\equiv\hat{\mu}_{s,t}(z)$ coincides with the characteristic function (Fourier transform) of the distribution $\mu_{s,t}$. Further we have $L(t)f_z(x)=\Phi(t,z)f_z(x)$, where

$$\Phi(t,z)=-\frac{1}{2}\sum_{i,j}a^{ij}(t)z_iz_j+i\sum_i b^i(t)z_i+\int(\exp i\langle z,x\rangle-1-\frac{i\langle z,x\rangle}{1+|x|^2})\nu_t(dx).$$

Therefore, if $0<s<t$ we have from (3.10), $\hat{\mu}_{s,t}(z)-1=\int_s^t\Phi(r,z)\hat{\mu}_{s,r}(z)dr$. Differentiating both sides with respect to t, we obtain $\frac{d}{dt}\hat{\mu}_{s,t}(z)=\Phi(t,z)\hat{\mu}_{s,t}(z)$. Integrating the differential equation, we arrive at Lévy-Khinchine's formula:

$$\hat{\mu}_{s,t}(z)=\exp\int_s^t\Phi(r,z)dr \tag{6.5}$$

$$=\exp\left(-\frac{1}{2}\langle z,(\int_s^t A(r)dr)z\rangle+i\langle\int_s^t b(r)dr,z\rangle+\int(\exp i\langle z,x\rangle-1-\frac{i\langle z,x\rangle}{1+|x|^2})N_{s,t}(dx)\right),$$

where $N_{s,t}=\int_s^t\nu_r dr$. Consequently, the *generating elements* of the infinitely divisible distribution $\mu_{s,t}$ is given by

$$\left(\int_s^t A(r)dr,\int_s^t b(r)dr,\int_s^t \nu_r dr\right). \tag{6.6}$$

Conversely, suppose we are given a triple $(A(t),b(t),\nu_t)$ which satisfies (a), (b), (c′) and (d′) of Theorem 3.1 and Theorem 4.6. Then there exists a unique convolution semigroup $\{\mu_{s,t}\}$ on $\boldsymbol{R}^d$ such that the characteristic functions of $\{\mu_{s,t}\}$ are given by the Lévy-Khinchine's formula (6.5).

An automorphism β of $\boldsymbol{R}^d$ is nothing but an invertible linear transformation. Then the dilation $\{\gamma_t\}_{r>0}$ is represented by $\gamma_r=\exp(\log rQ)\equiv r^Q,r>0,$ where Q is a linear transformation such that the real parts of its eigen values are all positive. A distribution μ on $\boldsymbol{R}^d$ is $\{r^Q\}$-selfdecomposable if and only if μ is $\exp(-tQ)$-selfdecomposable in Jurek-Mason [4] or Q-selfdecomposable in Sato [10].

We give a simple proof of Urbanik's characterization of selfdecomposable distribution by means of the generating elements. A merit of our proof is that we get more informations for each term in the representations of the generating elements.

Let Q be a given $d\times d$-matrix such that the real parts of eigen values of Q are all positive.

Theorem 6.4 (Urbanik [13]). *Let μ be a distribution on $\boldsymbol{R}^d$ with the generating elements (A,b,ν). It is selfdecomposable with respect to $\{r^Q\}$ if and only if A and ν satisfy the following properties* (i) *and* (ii):
(i) *Thr matrix A is represented by*

$$A=\int_0^\infty e^{-uQ}A(1)e^{-uQ'}du, \tag{6.7}$$

with a symmetric nonnegative definite matrix $A(1)$.
(ii) *The Lévy measure ν is represented by*

$$\nu=\int_0^\infty (e^{-uQ}\nu_1)du, \tag{6.8}$$

with a Lévy measure ν_1 satisfying (5.6).

Proof. Let μ be a distribution, selfdecomposable with respect to $\{r^Q\}$. There exists a unique nonhomogeneous convolution semigroup $\{\mu_{s,t}\}$ on $\boldsymbol{R}^d$, self-similar with respect to $\{r^Q\}$ such that $\mu_{0,1}=\mu$. Then in view of Theorem 5.3, the generating elements of the infinitely divisible distribution $\mu_{s,t}$ are given by

$$\int_s^t \frac{1}{r}r^Q A(1)r^{Q'}dr,\quad \int_s^t \frac{1}{r}(r^Q\nu_1)dr, \tag{6.9}$$

$$\int_s^t r^{Q-1}b(1)dr+\int_s^t \frac{1}{r}\left(\int\left(\frac{x}{1+|r^{-Q}x|^2}-\frac{x}{1+|x|^2}\right)(r^Q\nu_1)(dx)\right)dr.$$

Consider the case $s=0$ and $t=1$. Setting $u=-\log r$ in (6.9), we obtain (6.7), (6.8) and

$$b=\left(\int_0^\infty e^{-uQ}du\right)b(1)+\int_0^\infty\left(\int\left(\frac{x}{1+|e^{-uQ}x|^2}-\frac{x}{1+|x|^2}\right)(e^{-uQ}\nu_1)(dx)\right)du. \tag{6.10}$$

Conversely suppose that the generating elements (A,b,ν) of the distribution μ satisfies (i) and (ii). Define $b(1)$ by the relation (6.10). Next define the triple

$(A(t),b(t),\nu_t)$ by (5.4), (5.5) and (5.7) using the triple $(A(1),b(1),\nu_1)$. Then it satisfies the integrability condition (d') of Theorem 4.6, because of (5.6). Therefore there exists a nonhomogeneous convolution semigroup $\{\mu_{s,t}\}$ with characteristics $(A(t),\nu_t,b(t))$, which is self-similar with respect to $\{r^Q\}$ by Theorem 5.3. Then the distribution $\mu_{0,1}$ is selfdecomposable with respect to $\{r^Q\}$, whose generating elements are given by (6.7), (6.8) and (6.10). Therefore $\mu=\mu_{0,1}$, proving that μ is selfdecomposable with respect to $\{r^Q\}$. The proof is complete.

Here is another representation of the Lévy measure ν in Theorem 6.4. The following is due to Yamazato, Wolfe, Jurek and others. We refer to Yamazato [14] and Sato-Yamazato [11].

Theorem 6.5. *Let μ be a distribution on $\boldsymbol{R}^d$ with the generating elements (A,ν,b). It is selfdecomposable with respect to $\{r^Q\}$ if and only if A and ν satisfy the following properties* (i') *and* (ii'):

(i') *$A(1)\equiv QA+AQ'$ is a nonnegative definite matrix, where Q' is the transpose of Q.*
(ii') *The Lévy measure ν is represented by*

$$\nu(E)=\int_S \lambda(d\theta)\int_0^\infty \chi_E(r^Q\theta)k_\theta(r)r^{-1}dr, \tag{6.11}$$

where λ is a bounded measure on $S=\{\theta;|\theta|=1,\ |r^Q\theta|>1$ for all $r>0\}$ and $k_\theta(r)$ is nonincreasing in $r\in(0,\infty)$, measurable in $\theta\in S$ and satisfies

$$\int_S \lambda(d\theta)\int_0^\infty \frac{|r^Q\theta|^2}{1+|r^Q\theta|^2}k_\theta(r)r^{-1}dr<\infty. \tag{6.12}$$

Proof. It is sufficient to prove that properties (i') and (ii') are equivalent to the properties (i) and (ii) of the previous theorem, respectively. Suppose (i) is satisfied. Then,

$$\begin{aligned} QA+AQ' &= -\int_0^\infty \{\frac{d}{dt}(e^{-tQ})A(1)e^{tQ'}+e^{-tQ}A(1)\frac{d}{dt}(e^{-tQ'})\}dt \\ &= -e^{-tQ}A(1)e^{-tQ'}|_0^\infty=A(1). \end{aligned}$$

Conversely suppose that the matrix $A(1)\equiv QA+AQ'$ is nonnegative definte. Define $\tilde{A}=\int_0^\infty e^{-tQ}A(1)e^{-tQ'}dt$. It satisfies $A(1)=Q\tilde{A}+\tilde{A}Q'$. Then we must have $A=\tilde{A}$, proving the equivalence of (i) and (i').

Suppose next that (ii) of Theorem 6.4 is satisfied. Let us remark that every $x\in\boldsymbol{R}^d-\{0\}$ is represented by $x=s^Q\theta$, where $s\in(0,\infty)$ and $\theta\in S$. Define a measure λ on S by $\lambda(F)=\nu_1(\{s^Q\theta;s>1,\ \theta\in F\})$. It is a bounded measure. There exists a family of (conditional) measures $\{\mu_\theta:\theta\in S\}$ on $(0,\infty)$, measurable with respect to θ such that

$$\nu_1(E)=\int_S \lambda(d\theta)\int_0^\infty \chi_E(s^Q\theta)\mu_\theta(ds) \tag{6.13}$$

holds for all Borel sets E. Then we have

$$\begin{aligned}\int_0^\infty (e^{-tQ}\nu_1)(E)dt&=\int_{R^d}\nu_1(dx)\int_0^\infty \chi_E(e^{-tQ}x)dt \\ &=\int_S \lambda(d\theta)\int_0^\infty\int_0^\infty \mu_\theta(ds)dt\chi_E((se^{-t})^Q\theta) \\ &=\int_S \lambda(d\theta)\int_0^\infty r^{-1}dr\int_r^\infty \mu_\theta(ds)\chi_E(r^Q\theta) \\ &=\int_S \lambda(d\theta)\int_0^\infty \chi_E(r^Q\theta)\mu_\theta([r,\infty))r^{-1}dr.\end{aligned} \tag{6.14}$$

(We set $se^{-t}=r$ in the above computation.) Therefore, setting $k_\theta(r)=\mu_\theta([r,\infty))$, we obtain the representation (6.11). Further, we have by (6.13),

$$\begin{aligned}\int_{R^d}\log(1+|x|^2)\nu_1(dx)&=-\int_S \lambda(d\theta)\int_0^\infty \log(1+|r^Q\theta|^2)dk_\theta(r) \\ &=\int_S \lambda(d\theta)\int_0^\infty \frac{\langle Qr^{Q-1}\theta,r^Q\theta\rangle}{1+|r^Q\theta|^2}k_\theta(r)dr.\end{aligned} \tag{6.15}$$

The last equality follows from integration by parts. See Lemma 2.2 in [11]. Therefore (5.6) holds if and only if the last member of the above is finite, or equivalently, (6.12) is satisfied

Conversely suppose that (ii′) is satisfied . Set

$$\nu_1(E)=-\int_S \lambda(d\theta)\int_0^\infty \chi_E(r^Q\theta)dk_\theta(r).$$

It is a Lévy measure satisfying (5.6). Then a computation similar to (6.14) implies the equality (6.8). The proof is complete.

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