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<td>Kamata, Masayoshi</td>
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NOTES ON THE COBORDISM GROUP $U^*(L^n(m))$

MASAYOSHI KAMATA

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1. Let $U^*(X)$ be the unitary cobordism group of a finite CW complex $X$. P.S. Landweber [4] and K. Shibata [6] determined the unitary cobordism group of the lens space $L^n(m)=S^{2n+1}/Z_m$. In this paper, we use the structure of the reduced unitary cobordism group of $L^n(m)$ to prove the following

**Theorem 1.** If positive integers $p$ and $q$ are relatively prime, there exists an isomorphism

$$\psi: \bar{U}^{ev}(L^n(p)) \oplus \bar{U}^{ev}(L^n(q)) \to \bar{U}^{ev}(L^n(pq)),$$

where $\bar{U}^{ev}(\cdot)=\sum_i \bar{U}^{2i}(\cdot)$.

Let $U_*(X)$ be the unitary bordism group of a space $X$. Denote by $BZ_m$ the classifying space of the group $Z_m$. Using the duality isomorphism $D: U_*(L^n(m)) \approx U^*(L^n(m))$ and the isomorphism $U_k(L^n(m)) \approx U_k(BZ_m)$ for $k<2n+1$ [3], we have $U_k(BZ_m) \approx \bar{U}^{2n+1-k}(L^n(m))$ for $k<2n+1$. Then, Theorem 1 implies the following

**Theorem 2.** If $p$ and $q$ are relatively prime, there exists an isomorphism

$$\psi_*: U_{od}(BZ_p) \oplus U_{od}(BZ_q) \to U_{od}(BZ_{pq}),$$

where $U_{od}(\cdot)=\sum_i U_{2i+1}(\cdot)$.

Using the spectral sequence [3], we obtain

$$U_{2i}(BZ_m) \approx U_{2k}.$$ 

For a prime $p$, $U_*(BZ_p)$ was determined in [1] and [3].

Denote by $\bar{K}(X)$ the reduced Grothendieck group of isomorphism classes of complex vector bundles over $X$. In [2], Conner and Floyd gave the isomorphism

$$\bar{K}(X) \approx \bar{U}^{ev}(X) \otimes U^*Z.$$ 

Therefore, Theorem 1 implies the following
Theorem 3. (N. Mahammed [5]) If $p$ and $q$ are relatively prime, there exists an isomorphism

$$K(L^*(p)) \oplus K(L^*(q)) \simeq K(L^*(pq)).$$

2. In this section we prove Theorem 1. Denote by $CP^n$ the $n$-dimensional complex projective space and by $\gamma$ the canonical complex line bundle over $CP^n$. Let $\pi: L^*(P) \to CP^n$ be the natural projection and put

$$x_p = \pi^*c_1(\gamma),$$

where $c_1(\gamma)$ is the first Chern class of $\gamma$ in the sense of Conner and Floyd [2].

Let $F(\ , )$ is the formal group law such that

$$F(c_1(\gamma), c_1(\gamma')) = c_1(\gamma \otimes \gamma')$$

for complex line bundles $\gamma$, $\gamma'$ over the same CW complex [7]. For a positive integer $m$, let $[m]_F(x) \in U^*[[x]]$ be a formal power series defined by the following formulas

$$[1]_F(x) = x$$
$$[k]_F(x) = F(x, [k-1]_F(x)).$$

In [6], K. Shibata gave the following

Theorem 2.1.

$$U^*(L^*(m)) \cong \Lambda_{\nu}(D[pt, i]) \oplus U^*[[x_m]]/(x_m^{m+1}, [m]_F(x_m)),$$

where $[pt, i] \in U^*_\partial(L^*(m))$ is the bordism class represented by an inclusion map of a point, $\Lambda_{\nu}$ ( ) is the exterior algebra over $U^*$ and $(x_m^{m+1}, [m]_F(x_m))$ denotes the ideal generated by $x_m^{m+1}$ and $[m]_F(x_m)$.

The same result can be obtained also by the method of P.S. Landweber [4] directly.

Considering the following short exact sequence

$$0 \to \bar{U}^*(L^*(m)) \to U^*(L^*(m)) \to U^* \to 0,$$

it follows from Theorem 2.1 that

$$\bar{U}^*(L^*(m)) \cong U^*[[x_m]]/(x_m^{m+1}, [m]_F(x_m)),$$

where $\bar{U}^*[[x_m]]$ is the kernel of the homomorphism

$$\varepsilon: U^*[[x_m]] \to U^*$$

defined by $\varepsilon(\sum a_k x_m^k) = a_\nu$.

We define a homomorphism
\[ \psi: \bar{U}^e(L^*(p)) \oplus \bar{U}^e(L^*(q)) \to \bar{U}^e(L^*(pq)) \]

by \( \psi(P(x_p), Q(x_q)) = P([q]_F(x_{pq})) + Q([p]_F(x_{pq})) \), where \( P(x_p), Q(x_q) \) and \( P([q]_F(x_{pq})) + Q([p]_F(x_{pq})) \) are the classes of the formal power series \( P(x_p) \in \bar{U}^*[x_p], Q(x_q) \in \bar{U}^*[x_q] \) and \( P([q]_F(x_{pq})) + Q([p]_F(x_{pq})) \in \bar{U}^*[x_{pq}] \) respectively.

Using the associativity of the formal group law, we obtain

\[
[p]_F([q]_F(x)) = [p]_F([pq]_F(x)) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (3)
\]

From (2) and (3), it follows that the homomorphism \( \psi \) is well defined.

We define the multiplication in \( \bar{U}^e(L^*(p)) \oplus \bar{U}^e(L^*(q)) \) by

\[
(x, y) \cdot (x', y') = (xx', yy').
\]

We prove the following lemma, so that the homomorphism \( \psi \) is a ring homomorphism.

**Lemma 2.2.** If \( p \) and \( q \) are relatively prime, \( [p]_F(x_{pq}) \cdot [q]_F(x_{pq}) = 0 \) in \( \bar{U}^e(L^*(pq)) \).

**Proof.** We put

\[
I_{p,q} = (x_{pq}^{x_{pq}^2}, [pq]_F(x_{pq})).
\]

We show that \([p]_F(x_{pq}) \cdot [q]_F(x_{pq}) \in I_{p,q} \). From (3),

\[
p[q]_F(x) + \sum_{i=1}^{\infty} a_i ([q]_F(x))^i = [pq]_F(x),
\]

\[
q[p]_F(x) + \sum_{i=1}^{\infty} b_i ([p]_F(x))^i = [pq]_F(x),
\]

where \( x = x_{pq} \).

Since \( p \) and \( q \) are relatively prime, there exist integers \( a \) and \( b \) such that \( ap + bq = 1 \). Then, we have

\[
[p]_F(x) \cdot [q]_F(x)
\]

\[
= a[p]_F(x) \cdot ([pq]_F(x) - \sum_{i=1}^{\infty} a_i ([q]_F(x))^i)
\]

\[
+ b[q]_F(x) \cdot ([pq]_F(x) - \sum_{i=1}^{\infty} b_i ([p]_F(x))^i).
\]

We put

\[
X = [p]_F(x), \quad Y = [q]_F(x), \quad a'_i = aa_i \quad \text{and} \quad b' = bb_i.
\]

The equation (4) implies
Therefore,

\[ \{1 + \left( \sum_{i=1}^{\infty} a'_i Y^{i-1} + \sum_{i=2}^{\infty} b'_i X^{i-1} \right) \} = I \in I_{p,q}. \]

Therefore,

\[ XY = I(1 + A + A^2 + \cdots) \in I_{p,q}, \]

where \( A = -\left( \sum_{i=1}^{\infty} a'_i Y^{i-1} + \sum_{i=2}^{\infty} b'_i X^{i-1} \right) \).

q.e.d.

**Proposition 2.3.** If \( p \) and \( q \) are relatively prime, then \( \psi \) is epimorphic.

Proof. Since \( \psi \) is the ring homomorphism, we need only to prove the existence of the elements \( y \) and \( z \) which satisfy \( \psi(y, z) = x_{pq} \). We put

\[ [p]_p(x_{pq}) = \sum_{i=0}^{\infty} c_i x_{pq}^{i+1}, \quad c_0 = p \]

and

\[ [q]_p(x_{pq}) = \sum_{i=0}^{\infty} d_i x_{pq}^{i+1}, \quad d_0 = q. \]

We find series \( A = \sum_{i=0}^{\infty} a_i x_{pq}^i \) and \( B = \sum_{i=0}^{\infty} b_i x_{pq}^i \) which satisfy

\[ x_{pq} = A[p]_p(x_{pq}) + B[q]_p(x_{pq}), \]

that is, \( a_i \) and \( b_i \) satisfy the following

\[ pa_0 + qb_0 = 1, \quad (c_0 = p \text{ and } d_0 = q), \]

\[ a_1 c_0 + a_1 c_1 + b_1 d_0 + b_2 d_1 = 0 \]

\[ \sum_{i=0}^{k} a_{k-i} c_i + \sum_{i=0}^{k} b_{k-i} d_i \]

\[ = a_k c_0 + b_k d_0 + \sum_{i=1}^{\infty} (a_{k-i} c_i + b_{k-i} d_i) \]

\[ = 0. \]

Since \( p \) and \( q \) are relatively prime, there exist \( a_0 \) and \( b_0 \) which satisfy \( 1 = pa_0 + qb_0 \).

Suppose that \( a_j \) and \( b_j \) are determined for \( j < k \). Put

\[ a_k = -a_0 \sum_{i=1}^{k} (a_{k-i} c_i + b_{k-i} d_i) \]

and

\[ b_k = -b_0 \sum_{i=1}^{k} (a_{k-i} c_i + b_{k-i} d_i), \]

then \( a_k \) and \( b_k \) satisfy the above relation. Therefore,

\[ x_{pq} = \sum_{k=0}^{\infty} P_{k,1} x_{pq}^k. \]
where

\[ P_{k,1} = a_k[p]_p(x_{pq}) + b_k[q]_p(x_{pq}). \]

Suppose that

\[ x_{pq} = \sum_{k=0}^{\infty} P_{k,m} x_{pq}^k, \]

where \( P_{k,m} \) is a polynomial of \([p]_p(x_{pq})\) and \([q]_p(x_{pq})\) with the coefficients in \( U^* \), and for \( k \geq 1 \)

\[ P_{k,m} = x_{pq}^m Q_{k,m}, \quad Q_{k,m} \in U^*[[x_{pq}]]. \]

Then, we have

\[ x_{pq} = P_{0,m} + \sum_{k=1}^{\infty} P_{k,m} \left( \sum_{j=0}^{\infty} P_{j,m} x_{pq}^j \right)^k \]

\[ = P_{0,m} + \sum_{k=1}^{\infty} P_{k,m} \{ P_{0,m} + \sum_{j=1}^{\infty} P_{j,m} x_{pq}^j \}^k. \]

Put

\[ P_{0,m} + \sum_{k=1}^{\infty} P_{k,m} \{ P_{0,m} + \sum_{j=1}^{\infty} P_{j,m} x_{pq}^j \}^k = \sum_{k=0}^{\infty} P_{k,m+1} x_{pq}^k. \]

Then, we have

\[ P_{0,m+1} = P_{0,m} + \sum_{k=1}^{\infty} P_{k,m}(P_{0,m})^k \]

and since \( P_{j,m} = x_{pq}^m Q_{j,m} \) for \( j \geq 1 \), there exists \( Q_{j,m+1} \in U^*[[x_{pq}]] \) such that

\[ P_{j,m+1} = x_{pq}^m Q_{j,m+1}, \quad j \geq 1. \]

By induction, we have

\[ x_{pq} = P_{0,n} + \sum_{k=1}^{\infty} P_{k,n} x_{pq}^k, \]

and for \( k \geq 1 \)

\[ P_{k,n} = x_{pq}^m Q_{k,n}, \quad Q_{k,n} \in U^*[[x_{pq}]]. \]

Therefore,

\[ x_{pq} - P_{0,n} \in I_{p,q} = \{ x_{pq}^{n+1}, \ [p]_p(x_{pq}) \}. \]

Put

\[ P_{n} = P([p]_p(x_{pq})) + Q([q]_p(x_{pq}))+[p]_p(x_{pq}) \cdot Q([q]_p(x_{pq})) \cdot R, \]

where \( R \in U^*[[x_{pq}]] \).

From Lemma 2.2,

\[ x_{pq} - P([p]_p(x_{pq})) - Q([q]_p(x_{pq})) \in I_{p,q}. \]

Therefore, we obtain
Proposition 2.4. The order of the group $\bar{U}^{2s}(L^n(m))$ is $m^t$, $t = \sum_{i=-s+1}^{s} \tau_i$, where $\tau_i$ is the number of partitions of $i$ for $i \geq 0$ and $\tau_i = 0$ for $i < 0$.

Proof. Consider the spectral sequence $E^{p,a}_r$ associated with $\bar{U}^{2s}(L^n(m))$. There is a filtration

$$
\bar{U}^{2s}(L^n(m)) = J^{0,2s} \supset J^{1,2s-1} \supset \cdots \supset J^{n,2s-2s-1} = 0
$$

with $J^{p,q}/J^{p+1,q-1} = H^q(L^n(m); U^p)$. Then, for $1 \leq s+i \leq n$,

the order of $J^{2s+2s-2s+i/2,J^{2s+2s-2s-2s-1+i/2}} = \begin{cases} m^i & \text{if } i \geq 0, \\ 1 & \text{otherwise.} \end{cases}$

Therefore, the order of $\bar{U}^{2s}(L^n(m))$ is $m^t$, $t = \sum_{i=-s+1}^{s} \tau_i$. q.e.d.

From the Proposition 2.4, we have the following

Corollary 2.5. The order of $\bar{U}^{2s}(L^n(p)) \oplus \bar{U}^{2s}(L^n(q))$ is equal to that of $\bar{U}^{2s}(L^n(pq))$.

Proposition 2.3 and Corollary 2.5 prove Theorem 1.

OSAKA CITY UNIVERSITY

References