Watanabe, T.
Osaka J. Math.
23 (1986), 617–632

ADAMS OPERATIONS IN THE CONNECTIVE K-THEORY OF COMPACT LIE GROUPS

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(Received May 25, 1985)

1. Introduction

Let G be a compact, 1-connected, simple Lie group of rank 2 or 3. That is, G is one of the following:

$$SU(3), Sp(2), G_2, SU(4), Spin(7) \quad \text{and} \quad Sp(3).$$

In [14], for these groups G, we have given a complete description of the Chern character ([7, §1])

$$ch: K^*(G) \to H^*(G; Q).$$

Using this, one can easily compute the Adams operations ψ^r ([1]) on $K^*(G)$ for all $r \in Z$ (see (2.5)).

Throughout this paper p will denote an odd prime. Let us introduce some spectra ([4, Part III]). Let $\boldsymbol{KZ}_{(p)}$ denote the ring spectrum representing complex K-theory localized at p. Let $\boldsymbol{kZ}_{(p)}$ be its (-1)-connected cover. So there is a map of ring spectra $\kappa: \boldsymbol{kZ}_{(p)} \to \boldsymbol{KZ}_{(p)}$ such that

$$\kappa_*: \pi_*(\boldsymbol{kZ}_{(p)}) = Z_{(p)}[u] \to \pi_*(\boldsymbol{KZ}_{(p)}) = Z_{(p)}[u, u^{-1}]$$

satisfies $\kappa_*(u) = u$ where $|u| = 2$. As is well known, there is a ring spectrum $\boldsymbol{g}(p)$ such that

$$\boldsymbol{kZ}_{(p)} \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} \boldsymbol{g}(p).$$

Here the injection $\iota: \boldsymbol{g}(p) \to \boldsymbol{kZ}_{(p)}$ is a map of ring spectra such that

$$\iota_*: \pi_*(\boldsymbol{g}(p)) = Z_{(p)}[v] \to \pi_*(\boldsymbol{kZ}_{(p)}) = Z_{(p)}[u]$$

satisfies $\iota_*(v) = u^{p-1}$ where $|v| = 2(p-1)$. For r prime to p there are maps of ring spectra

$$\begin{aligned}\psi^r &: \boldsymbol{KZ}_{(p)} \to \boldsymbol{KZ}_{(p)}, \\ \psi^r &: \boldsymbol{kZ}_{(p)} \to \boldsymbol{kZ}_{(p)}, \\ \psi^r &: \boldsymbol{g}(p) \to \boldsymbol{g}(p)\end{aligned}$$

which are called the stable Adams operations ([6], [5]). They commute with κ, ι and satisfy $\psi^r(u)=ru$. Let

$$\theta_r\colon \boldsymbol{g}(p) \to \Sigma^{2(p-1)}\boldsymbol{g}(p)$$

be a unique map of spectra such that $(v\cdot)\theta_r\simeq\psi^r-1$ where $v\cdot\colon \Sigma^{2(p-1)}\boldsymbol{g}(p)\to\boldsymbol{g}(p)$ is multiplication by v. We denote by $\boldsymbol{j}(p;r)$ the fibre spectrum of θ_r. If r or r' generates the group of units of Z/p^2, then $\boldsymbol{j}(p;r)\simeq\boldsymbol{j}(p;r')$. In this case, we may write $\boldsymbol{j}(p)$ for $\boldsymbol{j}(p;r)$ and use a suitable r to discuss it. $\boldsymbol{j}(p)$ is known to be a ring spectrum (see [13]).

Let $\widetilde{j(p)}_i(G)$ (resp. $\widetilde{j(p)}^i(G)$) be the i-th reduced $\boldsymbol{j}(p)$-homology (resp. cohomology) group of G. One of our targets is to compute the groups $\widetilde{j(p)}_i(G)$ for all the above G and p. As will be mentioned in §3, the cases $(G,p)=(G_2,3)$, $(Sp(3),3)$ are most interesting. Then we obtain

Theorem 1.1. *For $i\leq 21$ and $G=G_2$, $Sp(3)$ the groups $\widetilde{j(3)}_i(G)$ are listed in the following table*:

$G \backslash i$	0	1	2	3	4	5	6	7	8	9
G_2	0	0	0	$Z_{(3)}$	0	0	$Z/3$	0	0	0
$Sp(3)$	0	0	0	$Z_{(3)}$	0	0	0	$Z_{(3)}$	0	0

$G \backslash i$	10	11	12	13	14	15	16
G_2	0	$Z_{(3)}$	0	0	$Z/3^3\oplus Z_{(3)}$	0	0
$Sp(3)$	$Z/3\oplus Z_{(3)}$	$Z_{(3)}$	0	$Z/3$	$Z/3\oplus Z/3^3\oplus Z_{(3)}$	0	0

$G \backslash i$	17	18	19	20	21
G_2	$Z/3$	$Z/3^2$	0	0	$Z/3$
$Sp(3)$	$Z/3$	$Z/3\oplus Z/3^3\oplus Z_{(3)}$	0	0	$Z/3\oplus Z/3^3\oplus Z_{(3)}$

where $\oplus$ indicates the direct sum of the groups.

Since G is parallelizable, the Poincaré duality isomorphism

$$E_i(G) \cong E^{n-i}(G)$$

holds for any spectrum $\boldsymbol{E}$, where $n=\dim G$ (see [4, Part III]). Therefore, to compute $\widetilde{j(p)}_i(G)$ it suffices to compute $\widetilde{j(p)}^{n-i}(G)$. Theorem 1.1 is a consequence of Theorem 4.6, in which the cup-product ring structure of $\widetilde{j(p)}^*(G)$ is described for $(G,p)=(G_2,3)$, $(Sp(3),3)$.

The remainder of this paper is organized as follows. In §2 we collect some results for later use. In §3 we describe the action of θ_r on $g(p)^*(G)$.

In §4 we compute the rings $\widetilde{j(p)}^*(G)$.

2. Preliminaries

This section is devoted to describe the rings $K^*(G; Z_{(p)})$, $k^*(G; Z_{(p)})$, $g(p)^*(G)$ and the homomorphism $ch: K^*(G) \to H^*(G; Q)$.

Notice that G is assumed to be as in §1 and p is assumed to be an odd prime. According to Borel [9], G has no p-torsion and we have

Lemma 2.1. *There exist elements* $x_{2m_i-1} \in H^{2m_i-1}(G; Z_{(p)})$, *for* $1 \leq i \leq l$ (*where* $l=2$ *or* 3), *such that*

$$H^*(G; Z_{(p)}) = \Lambda(x_{2m_1-1}, x_{2m_2-1}, \cdots, x_{2m_l-1})$$

where $2=m_1 \leq m_2 \leq \cdots \leq m_l$ *and* Λ *denotes an exterior algebra* (*over* $Z_{(p)}$).

For this lemma and the values of m_i see [8].
We need the famous result of Hodgkin [11]:

Lemma 2.2. *Let* $\{\rho_1, \cdots, \rho_l\}$ *be a system of ring generators of the complex representation ring* $R(G)$. *Then there exist elements* $\beta(\rho_i) \in K^{-1}(G)$, *for* $1 \leq i \leq l$, *such that*

$$K^*(G) = \Lambda(\beta(\rho_1), \cdots, \beta(\rho_l)) \otimes Z[u, u^{-1}] .$$

Therefore

$$K^*(G; Z_{(p)}) = \Lambda(\beta(\rho_1), \cdots, \beta(\rho_l)) \otimes Z_{(p)}[u, u^{-1}] .$$

The following proposition shows that

$$\kappa: k^*(G; Z_{(p)}) \to K^*(G; Z_{(p)}) ,$$
$$\iota : g(p)^*(G) \to k^*(G; Z_{(p)})$$

are injective.

Proposition 2.3. *One can choose elements*

$$\xi_{2m_i-1} \in g(p)^{2m_i-1}(G) , \qquad \text{for} \quad 1 \leq i \leq l ,$$

such that

(i) $g(p)^*(G) = \Lambda(\xi_{2m_1-1}, \cdots, \xi_{2m_l-1}) \otimes Z_{(p)}[v]$.

(ii) $k^*(G; Z_{(p)}) = \Lambda(\iota(\xi_{2m_1-1}), \cdots, \iota(\xi_{2m_l-1})) \otimes Z_{(p)}[u]$.

(iii) $K^*(G; Z_{(p)}) = \Lambda(\kappa\iota(\xi_{2m_1-1}), \cdots, \kappa\iota(\xi_{2m_l-1})) \otimes Z_{(p)}[u, u^{-1}]$.

(iv) *The CW-filtration degree* ([7, §2]) *of* ξ_{2m_i-1} *is* $2m_i-1$; *or equivalently,* $\kappa\iota(\xi_{2m_i-1})$ *satisfies*

$$ch(u^{m_i}\kappa\iota(\xi_{2m_i-1})) = cx_{2m_i-1} + \textit{higher terms}$$

where c *is a unit of* $Z_{(p)}$.

Proof. By [7, §2.4] the Atiyah-Hirzebruch spectral sequence for $K^*(G; Z_{(p)})$ collapses. Therefore it follows from the naturality with respect to κ (resp. ι) that the Atiyah-Hirzebruch spectral sequence for $k^*(G; Z_{(p)})$ (resp. $g(p)^*(G)$) collapses. Thus Lemma 2.1 yields the result; in particular, for (iv) see [7, §2.5].

We quote from [14] the following

Lemma 2.4. *For our groups G, the Chern character*

$$ch\colon K^{-1}(G) = \tilde{K}(\Sigma G) \to \tilde{H}^*(\Sigma G; Q) \cong \tilde{H}^{*-1}(G; Q)$$

is given by:

(1) *If* $G = SU(3)$, *we have*

$$ch\beta(\lambda_1) = -x_3 + \frac{1}{2}x_5,$$

$$ch\beta(\lambda_2) = -x_3 - \frac{1}{2}x_5$$

(*where* $\{\lambda_1, \lambda_2\}$ *generates* $R(SU(3))$).

(2) *If* $G = Sp(2)$, *we have*

$$ch\beta(\lambda_1) = x_3 - \frac{1}{6}x_7,$$

$$ch\beta(\lambda_2) = 2x_3 + \frac{2}{3}x_7.$$

(3) *If* $G = G_2$, *we have*

$$ch\beta(\rho_1) = 2x_3 + \frac{1}{60}x_{11},$$

$$ch\beta(\Lambda^2\rho_1) = 10x_3 - \frac{5}{12}x_{11}.$$

(4) *If* $G = SU(4)$, *we have*

$$ch\beta(\lambda_1) = -x_3 + \frac{1}{2}x_5 - \frac{1}{6}x_7,$$

$$ch\beta(\lambda_2) = -2x_3 + \frac{2}{3}x_7,$$

$$ch\beta(\lambda_3) = -x_3 - \frac{1}{2}x_5 - \frac{1}{6}x_7.$$

(5) *If* $G = Spin(7)$, *we have*

$$ch\beta(\lambda_1') = 2x_3 - \frac{2}{3}x_7 + \frac{1}{60}x_{11},$$

$$ch\beta(\lambda_2') = 10x_3 + \frac{2}{3}x_7 - \frac{5}{12}x_{11},$$

$$ch\beta(\Delta_7) = 2x_3 + \frac{1}{3}x_7 + \frac{1}{60}x_{11}.$$

(6) *If* $G=Sp(3)$, *we have*

$$ch\beta(\lambda_1)= x_3-\frac{1}{6}x_7+\frac{1}{120}x_{11}\,,$$

$$ch\beta(\lambda_2)= 4x_3+\frac{1}{3}x_7-\frac{13}{60}x_{11}\,,$$

$$ch\beta(\lambda_3)= 6x_3+x_7+\frac{11}{20}x_{11}\,.$$

An application of this result is a quick calculation of the operation ψ^r on $K^*(G)$. For example, in $K^{-1}(SU(3))$ we have

$$\psi^r(\beta(\lambda_1))=\frac{r^2(r+1)}{2}\beta(\lambda_1)+\frac{r^2(-r+1)}{2}\beta(\lambda_2)\,, \tag{2.5}$$

$$\psi^r(\beta(\lambda_2))=\frac{r^2(-r+1)}{2}\beta(\lambda_1)+\frac{r^2(r+1)}{2}\beta(\lambda_2)$$

(cf. the proof of Proposition 3.3).

3. The operation θ_r on $g(p)^*(G)$

In this section we first recall the facts we need about the p-localization of G. With this as a background, we shall describe the action of θ_r on $g(p)^*(G)$.

Let $B_n(p)$, for $n\geq 1$, be the S^{2n+1}-bundle over $S^{2n+2p-1}$ such that

$$H^*(B_n(p);\, Z/p)=\Lambda(x_{2n+1}\ \mathcal{P}^1x_{2n+1})\,,$$

It has a cell structure:

$$B_n(p)\simeq S^{2n+1}\cup e^{2n+1+2(p-1)}\cup e^{4n+2+2(p-1)}\,. \tag{3.1}$$

Then G is called p-regular if and only if it is homotopy equivalent to a product of spheres when localized at p, and G is called quasi p-regular if and only if it is homotopy equivalent to a product of spaces $B_n(p)$ and spheres when localized at p.

The following result is due to Mimura and Toda [12].

Lemma 3.2. *We have*

(1) $SU(3)\underset{p}{\simeq}S^3\times S^5$ *for* $p\geq 3$.

(2) $Sp(2)\underset{p}{\simeq}S^3\times S^7$ *for* $p\geq 5$;

$Sp(2)\underset{3}{\simeq}B_1(3)$.

(3) $G_2\underset{p}{\simeq}S^3\times S^{11}$ *for* $p\geq 7$;

$G_2\underset{5}{\simeq}B_1(5)$.

(4) $SU(4) \underset{p}{\simeq} S^3 \times S^5 \times S^7$ *for* $p \geq 5$;

$SU(4) \underset{3}{\simeq} B_1(3) \times S^5$.

(5) $Spin(7) \underset{p}{\simeq} S^3 \times S^7 \times S^{11}$ *for* $p \geq 7$;

$Spin(7) \underset{5}{\simeq} B_1(5) \times S^7$.

(6) $Sp(3) \underset{p}{\simeq} S^3 \times S^7 \times S^{11}$ *for* $p \geq 7$;

$Sp(3) \underset{5}{\simeq} B_1(5) \times S^7$.

We first consider the cases in which G is p-regular.

Proposition 3.3. *In the following cases there are elements* $\xi_{2m_i-1} \in g(p)^{2m_i-1}(G)$, *for* $1 \leq i \leq l$, *as in Proposition* 2.3, *which satisfy*:

(1) $G = SU(3)$, $p \geq 3$.

(a)
$$\begin{array}{lll} u^2\kappa\iota(\xi_3) = -\frac{1}{2}\beta(\lambda_1) - \frac{1}{2}\beta(\lambda_2) & \xrightarrow{ch} & x_3 \\ u^3\kappa\iota(\xi_5) = \beta(\lambda_1) - \beta(\lambda_2) & & x_5\,. \end{array}$$

(b) $\theta_r(\xi_3) = 0$, $\theta_r(\xi_5) = 0$.

(2) $G = Sp(2)$, $p \geq 5$.

(a)
$$\begin{array}{lll} u^2\kappa\iota(\xi_3) = \frac{2}{3}\beta(\lambda_1) + \frac{1}{6}\beta(\lambda_2) & \xrightarrow{ch} & x_3 \\ u^4\kappa\iota(\xi_7) = -2\beta(\lambda_1) + \beta(\lambda_2) & & x_7\,. \end{array}$$

(b) $\theta_r(\xi_3) = 0$, $\theta_r(\xi_7) = 0$.

(3) $G = G_2$, $p \geq 7$.

(a)
$$\begin{array}{lll} u^2\kappa\iota(\xi_3) = \frac{5}{6}\beta(\rho_1) + \frac{1}{30}\beta(\Lambda^2\rho_1) & \xrightarrow{ch} & 2x_3 \\ u^6\kappa\iota(\xi_{11}) = 5\beta(\rho_1) - \beta(\Lambda^2\rho_1) & & \frac{1}{2}x_{11}\,. \end{array}$$

(b) $\theta_r(\xi_3) = 0$, $\theta_r(\xi_{11}) = 0$.

(4) $G = SU(4)$, $p \geq 5$.

(a)
$$\begin{array}{lll} u^2\kappa\iota(\xi_3) = -\frac{1}{3}\beta(\lambda_1) - \frac{1}{6}\beta(\lambda_2) - \frac{1}{3}\beta(\lambda_3) & & x_3 \\ u^3\kappa\iota(\xi_5) = \beta(\lambda_1) - \beta(\lambda_3) & \xrightarrow{ch} & x_5 \\ u^4\kappa\iota(\xi_7) = -\beta(\lambda_1) + \beta(\lambda_2) - \beta(\lambda_3) & & x_7\,. \end{array}$$

(b) $\theta_r(\xi_3) = 0$, $\theta_r(\xi_5) = 0$, $\theta_r(\xi_7) = 0$.

(5) $G = Spin(7)$, $p \geq 7$.

(a)
$$\begin{array}{llr} u^2\kappa\iota(\xi_3)=\frac{3}{10}\beta(\lambda_1')+\frac{1}{30}\beta(\lambda_2')+\frac{8}{15}\beta(\Delta_7) & & 2x_3 \\ u^4\kappa\iota(\xi_7)=-\beta(\lambda_1') \qquad\qquad + \ \beta(\Delta_7) & \xrightarrow{ch} & x_7 \\ u^6\kappa\iota(\xi_{11})= \ \beta(\lambda_1') - \beta(\lambda_2') + 4\beta(\Delta_7) & & \frac{1}{2}x_{11}\,. \end{array}$$

(b) $\theta_r(\xi_3)=0\,,\quad \theta_r(\xi_7)=0\,,\quad \theta_r(\xi_{11})=0\,.$

(6) $G=Sp(3),\ p\geq 7.$

(a)
$$\begin{array}{llr} u^2\kappa\iota(\xi_3)= \ \frac{2}{5}\beta(\lambda_1)+\frac{1}{10}\beta(\lambda_2)+\frac{1}{30}\beta(\lambda_3) & & x_3 \\ u^4\kappa\iota(\xi_7)=-\frac{7}{2}\beta(\lambda_1)+\frac{1}{2}\beta(\lambda_2)+\frac{1}{4}\beta(\lambda_3) & \xrightarrow{ch} & x_7 \\ u^6\kappa\iota(\xi_{11})= \ \beta(\lambda_1) - 2\beta(\lambda_2) + \beta(\lambda_3) & & x_{11}\,. \end{array}$$

(b) $\theta_r(\xi_3)=0\,,\quad \theta_r(\xi_7)=0\,,\quad \theta_r(\xi_{11})=0\,.$

Proof. We show (1) only, because the others can be shown quite similarly. Since $\{\beta(\lambda_1), \beta(\lambda_2)\}$ forms a Z-basis for $K^{-1}(SU(3))$ by Lemma 2.2 (and [14, §2]), it is easy to see that $\{-\frac{1}{2}\beta(\lambda_1)-\frac{1}{2}\beta(\lambda_2),\ \beta(\lambda_1)-\beta(\lambda_2)\}$ forms a $Z_{(p)}$-basis for $K^{-1}(SU(3); Z_{(p)})$; their images under ch are as required by Lemma 2.4. On the other hand, by Proposition 2.3 $\{u^2\kappa\iota(\xi_3), u^3\kappa\iota(\xi_5)\}$ is a $Z_{(p)}$-basis for $K^{-1}(SU(3); Z_{(p)})$. These (together with (b)) permit us to conclude that there exist $\xi_i\in g(p)^i(SU(3))$, $i=3, 5$, satisfying (a).

To prove (b) we compute $\psi^r(u^2\kappa\iota(\xi_3))$ and $\psi^r(u^3\kappa\iota(\xi_5))$ in $\tilde{K}(\Sigma SU(3))$. By use of the formula $ch^q\psi^r=r^q ch^q$ [1, Theorem 5.1 (vi)] where ch^q is the composition

$$\tilde{K}(\Sigma G)\xrightarrow{ch}\tilde{H}^*(\Sigma G; Q)\xrightarrow{\pi_{2q}}\tilde{H}^{2q}(\Sigma G; Q)$$

(where π_{2q} is the projection to the $2q$-dimensional component), we have

$$ch\psi^r(u^2\kappa\iota(\xi_3))=r^2x_3=ch(r^2u^2\kappa\iota(\xi_3))\,.$$

Since $ch\colon \tilde{K}(\Sigma G)\to\tilde{H}^*(\Sigma G; Q)$ is injective, it follows that

$$\psi^r(u^2\kappa\iota(\xi_3))=r^2u^2\kappa\iota(\xi_3)\,.$$

Since $\psi^r(u^2)=r^2u^2$, it follows that

$$\psi^r(\kappa\iota(\xi_3))=\kappa\iota(\xi_3)\,.$$

Since ψ^r commutes with κ, ι and κ, ι are injective, it follows that

$$\psi^r(\xi_3)=\xi_3\,.$$

Similarly we have $\psi^r(\xi_5)=\xi_5$. So (b) follows by the definition of θ_r.

In view of Lemma 3.2, all statements in Proposition 3.3 except (a) are clear. But, if one wants to discuss a homomorphism $f^*: g(p)^*(G')\to g(p)^*(G)$ which is induced by a homomorphism of compact Lie groups $f\colon G\to G'$, it seems to us that (a) is necessary.

Before considering the cases in which G is quasi p-regular, we describe $g(p)^*(B_1(p))$ and the θ_r-action on it. Since θ_r detects $\mathcal{P}^1$ (see [13, Lemma 1.1]), it follows from the Atiyah-Hirzebruch spectral sequence argument using (3.1) that

(3.4) *There exist* $\xi_i\in g(p)^i(B_1(p))$, *for* $i=3,\ 2p+1$, *such that*

(i) $g(p)^*(B_1(p)) = \Lambda(\xi_3, \xi_{2p+1})\otimes Z_{(p)}[v]$.

(ii) *The operation* θ_r *is given by*

$$\theta_r(\xi_3) = \xi_{2p+1},\ \theta_r(\xi_{2p+1}) = 0\,.$$

Proposition 3.5. *In the following cases there are elements* $\xi_{2m_i-1}\in g(p)^{2m_i-1}(G)$, *for* $1\le i\le l$, *as in Proposition* 2.3, *which satisfy*:

(1) $G=Sp(2),\ p=3$.

(a)
$$\begin{array}{lll} u^2\kappa\iota(\xi_3) = \frac{1}{2}\beta(\lambda_2) & \overset{ch}{} & x_3+\frac{1}{3}x_7 \\ u^4\kappa\iota(\xi_7) = -2\beta(\lambda_1)+\beta(\lambda_2) & \longrightarrow & x_7\,. \end{array}$$

(b) $\theta_2(\xi_3)=\xi_7,\quad \theta_2(\xi_7)=0\,.$

(2) $G=G_2,\ p=5$.

(a)
$$\begin{array}{lll} u^2\kappa\iota(\xi_3) = \beta(\rho_1) & & 2x_3+\frac{1}{60}x_{11} \\ & \xrightarrow{ch} & \\ u^6\kappa\iota(\xi_{11}) = 5\beta(\rho_1)-\beta(\Lambda^2\rho_1) & & \frac{1}{2}x_{11}\,. \end{array}$$

(b) $\theta_2(\xi_3) = \frac{1}{2}\xi_{11},\quad \theta_2(\xi_{11}) = 0\,.$

(3) $G=SU(4),\ p=3$.

(a)
$$\begin{array}{lll} u^2\kappa\iota(\xi_3) = -\frac{1}{2}\beta(\lambda_1) \qquad -\frac{1}{2}\beta(\lambda_3) & & x_3+\frac{1}{6}x_7 \\ u^3\kappa\iota(\xi_5) = \qquad \beta(\lambda_1) \qquad - \beta(\lambda_3) & \xrightarrow{ch} & x_5 \\ u^4\kappa\iota(\xi_7) = \quad -\beta(\lambda_1) + \beta(\lambda_2)-\beta(\lambda_3) & & x_7\,. \end{array}$$

(b) $\theta_2(\xi_3) = \frac{1}{2}\xi_7,\quad \theta_2(\xi_5) = 0\,,\quad \theta_2(\xi_7) = 0\,.$

(4) $G=Spin\,(7),\ p=5$.

(a)
$$\begin{array}{lll} u^2\kappa\iota(\xi_3) = \frac{1}{3}\beta(\lambda_1') \qquad +\frac{2}{3}\beta(\Delta_7) & & 2x_3+\frac{1}{60}x_{11} \\ u^4\kappa\iota(\xi_7) = -\beta(\lambda_1') \qquad + \beta(\Delta_7) & \xrightarrow{ch} & x_7 \\ u^6\kappa\iota(\xi_{11}) = \quad \beta(\lambda_1')-\beta(\lambda_2')+4\beta(\Delta_7) & & \frac{1}{2}x_{11}\,. \end{array}$$

(b) $\theta_2(\xi_3)=\frac{1}{2}\xi_{11},\quad \theta_2(\xi_7)=0,\quad \theta_2(\xi_{11})=0.$

(5) $G=Sp(3),\quad p=5.$

(a)
$$\begin{array}{rlcl} u^2\kappa\iota(\xi_3) = & \frac{5}{12}\beta(\lambda_1)+\frac{1}{12}\beta(\lambda_2)+\frac{1}{24}\beta(\lambda_3) & & x_3+\frac{1}{120}x_{11} \\ u^4\kappa\iota(\xi_7) = & -\frac{7}{2}\beta(\lambda_1)+\frac{1}{2}\beta(\lambda_2)+\frac{1}{4}\beta(\lambda_3) & \xrightarrow{ch} & x_7 \\ u^6\kappa\iota(\xi_{11}) = & 2\beta(\lambda_1)-2\beta(\lambda_2)+\beta(\lambda_3) & & x_{11}\,. \end{array}$$

(b) $\theta_2(\xi_3)=\frac{1}{8}\xi_{11},\quad \theta_2(\xi_7)=0,\quad \theta_2(\xi_{11})=0\,.$

Proof. We prove (1) only; the proof for the others is similar. First, (a) follows from Proposition 2.3 and Lemma 2.4 as in the proof of Proposition 3.3. To prove (b) we compute $\psi^2(u^2\kappa\iota(\xi_3))$. In $\tilde{K}(\Sigma Sp(2))$ we have

$$\begin{aligned} ch\,\psi^2(u^2\kappa\iota(\xi_3)) &= 2^2x_3+\frac{2^4}{3}x_7 \\ &= 2^2(x_3+\frac{1}{3}x_7)+2^2x_7 \\ &= 2^2 ch u^2\kappa\iota(\xi_3)+2^2 ch u^4\kappa\iota(\xi_7)\,. \end{aligned}$$

Therefore

$$\psi^2(u^2\kappa\iota(\xi_3)) = 2^2u^2\kappa\iota(\xi_3)+2^2u^4\kappa\iota(\xi_7)\,.$$

Since $\iota(v)=u^2$(where $p=3$), it follows that

$$\psi^2(\xi_3)=\xi_3+v\xi_7\,.$$

Similarly we have

$$\psi^2(\xi_{11})=\xi_{11}\,.$$

These imply the result.

There remain the cases in which G is neither p-regular nor quasi p-regular.

Proposition 3.6. *In the following cases there are elements $\xi_{2m_i-1}\in g(p)^{2m_i-1}(G)$, for $1\leq i\leq l$, as in Proposition 2.3, which satisfy:*

(1) $G=G_2,\quad p=3.$

(a)
$$\begin{array}{rlcl} u^2\kappa\iota(\xi_3) = & \beta(\rho_1) & \xrightarrow{ch} & 2x_3+\frac{1}{60}x_{11} \\ u^6\kappa\iota(\xi_{11}) = & 5\beta(\rho_1)-\beta(\Lambda^2\rho_1) & & \frac{1}{2}x_{11}\,. \end{array}$$

(b) $\theta_2(\xi_3)=\frac{1}{2}v\xi_{11},\quad \theta_2(\xi_{11})=0.$

(2) $G=Spin(7)$, $p=3$.

(a)
$$\begin{array}{lll} u^2\kappa\iota(\xi_3)=\beta(\lambda_1') & & 2x_3-\frac{2}{3}x_7+\frac{1}{60}x_{11} \\ u^4\kappa\iota(\xi_7)=-\beta(\lambda_1') \qquad +\beta(\Delta_7) & \xrightarrow{ch} & x_7 \\ u^6\kappa\iota(\xi_{11})=\beta(\lambda_1')-\beta(\lambda_2')+4\beta(\Delta_7) & & \frac{1}{2}x_{11}\,. \end{array}$$

(b) $\theta_2(\xi_3)=-2\xi_7+\frac{1}{2}v\xi_{11}\,,\quad \theta_2(\xi_7)=0\,,\quad \theta_2(\xi_{11})=0\,.$

(3) $G=Sp(3)$, $p=3$.

(a)
$$\begin{array}{lll} u^2\kappa\iota(\xi_3)=\beta(\lambda_1) & & x_3-\frac{1}{6}x_7+\frac{1}{120}x_{11} \\ u^4\kappa\iota(\xi_7)=-4\beta(\lambda_1)+\beta(\lambda_2) & \xrightarrow{ch} & x_7-\frac{1}{4}x_{11} \\ u^6\kappa\iota(\xi_{11})=2\beta(\lambda_1)-2\beta(\lambda_2)+\beta(\lambda_3) & & x_{11}\,. \end{array}$$

(b) $\theta_2(\xi_3)=-\frac{1}{2}\xi_7,\quad \theta_2(\xi_7)=-\frac{3}{4}\xi_{11},\quad \theta_2(\xi_{11})=0\,.$

This proposition follows from the calculation similar to that in the proof of Proposition 3.3. We omit the details of the proof.

It is known [10] that

$$Spin\,(7)\underset{p}{\simeq}Sp\,(3)\,.$$

Therefore $j(3)^*(Spin(7))\cong j(3)^*(Sp(3))$. Henceforth we exclude to consider the former.

4. The $j(p)$-cohomology of G

In Lemma 4.2 we present formulas on the multiplicative structure of $\widetilde{j(p)}^*(X)$ (where X satisfies a certain condition). In the rest of this section we compute $\widetilde{j(p)}^*(G)$ for all pairs (G,p). Finally we comment on $\widetilde{j(p)}_*(G)$.

Throughout this section, the letters X and Y will stand for finite connected CW-complexes.

Consider the fibration sequence

$$\Sigma^{2p-3}g(p)\xrightarrow{\delta}j(p)\xrightarrow{\eta}g(p)\xrightarrow{\theta}\Sigma^{2p-2}g(p)\,.$$

It leads to a short exact sequence

$$\begin{aligned}(4.1)\quad 0\to \operatorname{Coker}\,(\theta\colon \widetilde{g(p)}^{i-1}(X)\to\widetilde{g(p)}^{i+2p-3}(X))\xrightarrow{\delta}\\ \widetilde{j(p)}^{i}(X)\xrightarrow{\eta}\operatorname{Ker}\,(\theta\colon \widetilde{g(p)}^{i}(X)\to\widetilde{g(p)}^{i+2p-2}(X))\to 0\end{aligned}$$

for any $i \in Z$. In this situation we shall use the following notation. For any $x \in \widetilde{g(p)}^*(X)$ we write $\bar{x}$ for $\delta(x) \in \widetilde{j(p)}^{*-2p+3}(X)$; therefore, if $x \in \mathrm{Im}(\theta)$, we have $\bar{x}=0$. Suppose now that $x \in \mathrm{Ker}(\theta)$. Then we denote by $\tilde{x}$ an element such that $\eta(\tilde{x})=x$; it is unique if $\widetilde{g(p)}^*(X)$ is (p-)torsion free. This condition is satisfied for $X=G$ by Proposition 2.3.

Lemma 4.2. *Suppose that $\widetilde{g(p)}^*(X)$ is torsion free. Then, with the above notations, for any $x, y \in \widetilde{g(p)}^*(X)$, the following formulas hold in $\widetilde{j(p)}^*(X)$:*

(i) $\tilde{x} \cup \tilde{y} = \widetilde{x \cup y}$.
(ii) $\tilde{x} \cup \bar{y} = \overline{x \cup y}$.
(iii) $\bar{x} \cup \tilde{y} = \overline{x \cup y}$.
(iv) $\bar{x} \cup \bar{y} = 0$.

Proof. Parts (i), (ii) and (iii) are proved by using the same technique as in [13, §4]; we refer to it for the details. In this proof we will use the facts which are shown there, without specific reference.

It remains to prove part (iv). Since η is a map of ring spectra and $\eta\delta \simeq 0$, we have

$$\eta(\bar{x} \cup \bar{y}) = \eta(\delta(x) \cup \delta(y)) = \eta\delta(x) \cup \eta\delta(y) = 0 \cup 0 = 0.$$

Hence there exists a $z \in g(p)^*(X)$ such that $\bar{x} \cup \bar{y} = \bar{z}$. This equality implies that, in the following diagram, the outer square is commutative:

$$\begin{tikzcd}
X \arrow[r, "d"] \arrow[ddrrr, bend right, "z"'] & X \wedge X \arrow[rr, "\bar{x} \wedge \bar{y}"] \arrow[d, "x \wedge y"] \arrow[dr, phantom, "\mathrm{I}"] & & \Sigma^m j \wedge \Sigma^n j \arrow[r, "\mu_j"] & \Sigma^{m+n} j \\
& \Sigma^{m+2p-3} g \wedge \Sigma^{n+2p-3} g \arrow[rr, "1 \wedge \delta"] & & \Sigma^{m+2p-3} g \wedge \Sigma^n j \arrow[u, "\delta \wedge 1"] \arrow[d, "1 \wedge \eta"] \arrow[r, phantom, "\mathrm{II}"] & {} \\
& & & \Sigma^{m+2p-3} g \wedge \Sigma^n g \arrow[r, "\mu_g"] & \Sigma^{m+n+2p-3} g \arrow[uu, "\delta"']
\end{tikzcd}$$

where d is the diagonal map; $x \in g(p)^{m+2p-3}(X)$, $y \in g(p)^{n+2p-3}(X)$; $\boldsymbol{g}=\boldsymbol{g}(p)$, $\boldsymbol{j}=\boldsymbol{j}(p)$; μ_g and μ_j are multiplications in $\boldsymbol{g}(p)$ and $\boldsymbol{j}(p)$ respectively. The commutativity of square I is obvious and that of square II was shown in [13, Lemma 4.4]. Thus we have

$$\begin{aligned}
\bar{z} &= \bar{x} \cup \bar{y} = \mu_j(\bar{x} \wedge \bar{y})d \\
&= \mu_j(\delta \wedge 1)(1 \wedge \delta)(x \wedge y)d \\
&= \delta\mu_g(1 \wedge \eta)(1 \wedge \delta)(x \wedge y)d \\
&= 0.
\end{aligned}$$

By virtue of this lemma, if one computes $\widetilde{j(p)}^*(X)$ by using (4.1), then its ring structure is automatically known.

We now record some basic data for $\boldsymbol{j}(p)$. Since $\psi^r(v)=r^{p-1}v$, the coefficient ring of $\boldsymbol{j}(p)$ is given by

$$\pi_*(\boldsymbol{j}(p)) = Z_{(p)}\{\tilde{1}\} \oplus \bigoplus_{i\geq 1} Z/p^{1+\nu_p(i)}\{\overline{v^{i-1}}\} \tag{4.3}$$

where the formula

$$\nu_p(r^{i(p-1)}-1) = 1+\nu_p(i)$$

([2, Lemma (2.12)]) is essential. We also have the Cartan formula for θ_r: for any $x, y \in g(p)^*(X)$,

$$\theta_r(x\cup y) = \theta_r(x)\cup y + x\cup\theta_r(y) + v\cdot\theta_r(x)\cup\theta_r(y) \tag{4.4}$$

(cf. [13, Lemma 4.1]).

Let us enter into a computation of $\widetilde{j(p)}^*(G)$. As is well known, the cofibration

$$X\vee Y\rightarrow X\times Y\rightarrow X\wedge Y$$

leads to a split short exact sequence

$$0\rightarrow \widetilde{j(p)}^i(X\wedge Y)\rightarrow \widetilde{j(p)}^i(X\times Y)\rightarrow \widetilde{j(p)}^i(X)\oplus\widetilde{j(p)}^i(Y)\rightarrow 0$$

for any $i\in Z$. Therefore by Lemma 3.2, in order to compute $\widetilde{j(p)}^*(G)$ when G is p-regular or quasi p-regular, it suffices to determine $\widetilde{j(p)}^*(B_1(p))$. From (3.4) we deduce

Proposition 4.5. *The ring $\widetilde{j(p)}^*(B_1(p))$ is given by:*

$$\begin{aligned}\widetilde{j(p)}^*(B_1(p)) = {} & \widetilde{j(p)}^*(S^0)\{\widetilde{\xi_3\xi_{2p+1}}\}\oplus Z_{(p)}\{\widetilde{\xi_{2p+1}}\}\\ & \oplus Z_{(p)}\{\widetilde{(r^{p-1}-1)\xi_3 - v\xi_{2p+1}}\}\\ & \oplus \bigoplus_{i\geq 1} Z/p^{2+\nu_p(i)+\nu_p(i+1)}\{\overline{v^{i-1}\xi_3}\}\end{aligned}$$

where the relations

$$\begin{aligned}\overline{\xi_{2p+1}} &= 0\,,\\ \overline{v^i\xi_{2p+1}} &= (r^{-i(p-1)}-1)\overline{v^{i-1}\xi_3} \qquad (\text{for } i\geq 1)\end{aligned}$$

hold.

Proof. By using (4.4), in $g(p)^*(B_1(p))$ we have

$$\begin{aligned}\theta_r(v^i\xi_3\xi_{2p+1}) &= (r^{i(p-1)}-1)v^{i-1}\xi_3\xi_{2p+1}\,,\\ \theta_r(v^i\xi_{2p+1}) &= (r^{i(p-1)}-1)v^{i-1}\xi_{2p+1}\,,\\ \theta_r(v^i\xi_3) &= (r^{i(p-1)}-1)v^{i-1}\xi_3 + r^{i(p-1)}v^i\xi_{2p+1}\,.\end{aligned}$$

So the kernel and cokernel of θ_r are easily calculated and the result follows.

In this way, if G is p-regular or quasi p-regular, the ring $j(p)^*(G)$ can be described. For the remaining cases, from parts (1) and (3) of Proposition 3.6 we deduce

Theorem 4.6. *With the notation as in Lemma* 4.2, *the ring* $\widetilde{j(3)}^*(G)$ *for* $G=G_2$, $Sp(3)$ *is given by*:

(1) $G=G_2$.

i	$\widetilde{j(3)}^i(G_2)$
14	$Z_{(3)}\{\widetilde{\xi_3\xi_{11}}\}$
13	0
12	0
11	$Z/3\{\overline{\xi_3\xi_{11}}\}\oplus Z_{(3)}\{\widetilde{\xi_{11}}\}$
10	0
9	0
8	$Z/3\{\overline{\xi_{11}}\}$
7	$Z/3\{\overline{v\xi_3\xi_{11}}\}$
6	0
5	0
4	0
3	$Z/3^2\{\overline{v^2\xi_3\xi_{11}}\}\oplus Z_{(3)}\{\widetilde{3\xi_3-\frac{1}{10}v^2\xi_{11}}\}$
2	0
1	0
0	$Z/3^3\{\overline{\xi_3}\}$
-1	$Z/3\{\overline{v^3\xi_3\xi_{11}}\}$
-2	0
-3	0
-4	$Z/3^2\{\overline{v\xi_3}\}$
-5	$Z/3\{\overline{v^4\xi_3\xi_{11}}\}$
-6	0
-7	0

(2) $G=Sp(3)$.

i	$\widetilde{j(3)}^i(Sp(3))$
21	$Z_{(3)}\{\widetilde{\xi_3\xi_7\xi_{11}}\}$
20	0
19	0
18	$Z/3\{\overline{\xi_3\xi_7\xi_{11}}\}\oplus Z_{(3)}\{\widetilde{\xi_7\xi_{11}}\}$
17	0
16	0
15	0
14	$Z/3\{\overline{v\xi_3\xi_7\xi_{11}}\}\oplus Z_{(3)}\{\widetilde{3\xi_3\xi_{11}+\frac{1}{2}v\xi_7\xi_{11}}\}$
13	0
12	0
11	$Z/3\{\overline{\xi_3\xi_{11}}\}\oplus Z_{(3)}\{\widetilde{\xi_{11}}\}$
10	$Z/3^2\{\overline{v^2\xi_3\xi_7\xi_{11}}\}\oplus Z_{(3)}\{\widetilde{3\xi_3\xi_7+\frac{3}{4}v\xi_3\xi_{11}+\frac{1}{40}v^2\xi_7\xi_{11}}\}$
9	0
8	$Z/3\{\overline{\xi_{11}}\}$
7	$Z/3\{\overline{\xi_3\xi_7-\frac{1}{16}v\xi_3\xi_{11}}\}\oplus Z/3^3\{\overline{v\xi_3\xi_{11}}\}\oplus Z_{(3)}\{\widetilde{\xi_7+\frac{1}{4}v\xi_{11}}\}$
6	$Z/3\{\overline{v^3\xi_3\xi_7\xi_{11}}\}$
5	0
4	$Z/3\{\overline{v\xi_{11}}\}$
3	$Z/3\{\overline{v\xi_3\xi_7-\frac{4}{5}v^2\xi_3\xi_{11}}\}\oplus Z/3^3\{\overline{v^2\xi_3\xi_{11}}\}\oplus Z_{(3)}\{\widetilde{3\xi_3+\frac{1}{2}v\xi_7+\frac{1}{10}v^2\xi_{11}}\}$
2	$Z/3\{\overline{v^4\xi_3\xi_7\xi_{11}}\}$
1	0
0	$Z/3\{\overline{3\xi_3-\frac{8}{5}v^2\xi_{11}}\}\oplus Z/3^3\{\overline{\xi_3}\}$

Proof. For (1) we have

$$\begin{aligned}
\theta_2(v^i\xi_3\xi_{11}) &= (2^{2i}-1)v^{i-1}\xi_3\xi_{11}\,,\\
\theta_2(v^i\xi_{11}) &= (2^{2i}-1)v^{i-1}\xi_{11}\,,\\
\theta_2(v^i\xi_3) &= (2^{2i}-1)v^{i-1}\xi_3+2^{2i-1}v^{i+1}\xi_{11}\,.
\end{aligned}$$

For (2) we have

$$\theta_2(v^i\xi_3\xi_7\xi_{11}) = (2^{2i}-1)v^{i-1}\xi_3\xi_7\xi_{11}\,,$$
$$\theta_2(v^i\xi_7\xi_{11}) = (2^{2i}-1)v^{i-1}\xi_7\xi_{11}\,,$$
$$\theta_2(v^i\xi_3\xi_{11}) = (2^{2i}-1)v^{i-1}\xi_3\xi_{11}-2^{2i-1}v^i\xi_7\xi_{11}\,,$$
$$\theta_2(v^i\xi_{11}) = (2^{2i}-1)v^{i-1}\xi_{11}\,,$$
$$\theta_2(v^i\xi_3\xi_7) = (2^{2i}-1)v^{i-1}\xi_3\xi_7-2^{2i-2}3v^i\xi_3\xi_{11}+2^{2i-3}3v^{i+1}\xi_7\xi_{11}\,,$$
$$\theta_2(v^i\xi_7) = (2^{2i}-1)v^{i-1}\xi_7-2^{2i-2}3v^i\xi_{11}\,,$$
$$\theta_2(v^i\xi_3) = (2^{2i}-1)v^{i-1}\xi_3-2^{2i-1}v^i\xi_7\,.$$

So the result follows from elementary calculations of the kernel and cokernel of θ_2.

Proof of Theorem 1.1.

By using the Poincaré duality isomorphism

$$\begin{aligned} j(p)_i(G) &= \widetilde{j(p)}_i(G)\oplus\widetilde{j(p)}_i(S^0) \\ &\cong \widetilde{j(p)}^{n-i}(G)\oplus\widetilde{j(p)}^{n-i}(S^0) = j(p)^{n-i}(G) \end{aligned}$$

where $n=\dim G$, Theorem 1.1 follows from Theorem 4.6 and (4.3).

Finally we talk about the Pontrjagin ring structure of $\widetilde{j(p)}_*(G)$. Since in Lemma 2.2 each $\beta(\rho_i)$ is primitive (see [11]), the ring structure of $K_*(G)$ can be determined. Furthermore, the ψ^r-action on $K_*(G)$ can be determined by using the formula

$$\psi^r(a\cap\alpha) = \psi^r(a)\cap\psi^r(\alpha)$$

where $a\in K^*(G)$, $\alpha\in K_*(G)$ and $\cap$ denotes the cap product. Therefore the ring structure of $\widetilde{j(p)}_*(G)$ will be obtained by using the homology instead of the cohomology and taking the same course as in this paper.

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