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Watanabe, T.
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ADAMS OPERATIONS IN THE CONNECTIVE K-THEORY OF COMPACT LIE GROUPS

TAKASHI WATANABE

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1. Introduction

Let G be a compact, 1-connected, simple Lie group of rank 2 or 3. That is, G is one of the following:

$$SU(3), Sp(2), G_2, SU(4), Spin(7) \text{ and } Sp(3).$$

In [14], for these groups G , we have given a complete description of the Chern character ([7, §1])

$$ch: K^*(G) \rightarrow H^*(G; \mathbb{Q}).$$

Using this, one can easily compute the Adams operations ψ^r ([1]) on $K^*(G)$ for all $r \in \mathbb{Z}$ (see (2.5)).

Throughout this paper p will denote an odd prime. Let us introduce some spectra ([4, Part III]). Let $\mathbf{KZ}_{(p)}$ denote the ring spectrum representing complex K -theory localized at p . Let $\mathbf{kZ}_{(p)}$ be its (-1) -connected cover. So there is a map of ring spectra $\kappa: \mathbf{kZ}_{(p)} \rightarrow \mathbf{KZ}_{(p)}$ such that

$$\kappa_*: \pi_*(\mathbf{kZ}_{(p)}) = Z_{(p)}[u] \rightarrow \pi_*(\mathbf{KZ}_{(p)}) = Z_{(p)}[u, u^{-1}]$$

satisfies $\kappa_*(u) = u$ where $|u| = 2$. As is well known, there is a ring spectrum $\mathbf{g}(p)$ such that

$$\mathbf{kZ}_{(p)} \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} \mathbf{g}(p).$$

Here the injection $\iota: \mathbf{g}(p) \rightarrow \mathbf{kZ}_{(p)}$ is a map of ring spectra such that

$$\iota_*: \pi_*(\mathbf{g}(p)) = Z_{(p)}[v] \rightarrow \pi_*(\mathbf{kZ}_{(p)}) = Z_{(p)}[u]$$

satisfies $\iota_*(v) = u^{p-1}$ where $|v| = 2(p-1)$. For r prime to p there are maps of ring spectra

$$\psi^r: \mathbf{KZ}_{(p)} \rightarrow \mathbf{KZ}_{(p)},$$

$$\psi^r: \mathbf{kZ}_{(p)} \rightarrow \mathbf{kZ}_{(p)},$$

$$\psi^r: \mathbf{g}(p) \rightarrow \mathbf{g}(p)$$

which are called the stable Adams operations ([6], [5]). They commute with κ , ι and satisfy $\psi'(u)=ru$. Let

$$\theta_r: \mathbf{g}(p) \rightarrow \Sigma^{2(p-1)} \mathbf{g}(p)$$

be a unique map of spectra such that $(v \cdot) \theta_r \simeq \psi' - 1$ where $v \cdot: \Sigma^{2(p-1)} \mathbf{g}(p) \rightarrow \mathbf{g}(p)$ is multiplication by v . We denote by $\mathbf{j}(p; r)$ the fibre spectrum of θ_r . If r or r' generates the group of units of \mathbb{Z}/p^2 , then $\mathbf{j}(p; r) \simeq \mathbf{j}(p; r')$. In this case, we may write $\mathbf{j}(p)$ for $\mathbf{j}(p; r)$ and use a suitable r to discuss it. $\mathbf{j}(p)$ is known to be a ring spectrum (see [13]).

Let $\widetilde{\mathbf{j}(p)}_i(G)$ (resp. $\widetilde{\mathbf{j}(p)}^i(G)$) be the i -th reduced $\mathbf{j}(p)$ -homology (resp. cohomology) group of G . One of our targets is to compute the groups $\widetilde{\mathbf{j}(p)}_i(G)$ for all the above G and p . As will be mentioned in §3, the cases $(G, p) = (G_2, 3)$, $(Sp(3), 3)$ are most interesting. Then we obtain

Theorem 1.1. *For $i \leq 21$ and $G = G_2$, $Sp(3)$ the groups $\widetilde{\mathbf{j}(3)}_i(G)$ are listed in the following table:*

\backslash i G	0	1	2	3	4	5	6	7	8	9
G_2	0	0	0	$\mathbb{Z}_{(3)}$	0	0	$\mathbb{Z}/3$	0	0	0
$Sp(3)$	0	0	0	$\mathbb{Z}_{(3)}$	0	0	0	$\mathbb{Z}_{(3)}$	0	0
\backslash i G	10	11	12	13		14		15	16	
G_2	0	$\mathbb{Z}_{(3)}$	0	0		$\mathbb{Z}/3^3 \oplus \mathbb{Z}_{(3)}$		0	0	
$Sp(3)$	$\mathbb{Z}/3 \oplus \mathbb{Z}_{(3)}$	$\mathbb{Z}_{(3)}$	0	$\mathbb{Z}/3$		$\mathbb{Z}/3 \oplus \mathbb{Z}/3^3 \oplus \mathbb{Z}_{(3)}$		0	0	
\backslash i G	17		18		19		20		21	
G_2	$\mathbb{Z}/3$		$\mathbb{Z}/3^2$		0		0		$\mathbb{Z}/3$	
$Sp(3)$	$\mathbb{Z}/3$		$\mathbb{Z}/3 \oplus \mathbb{Z}/3^2 \oplus \mathbb{Z}_{(3)}$		0		0		$\mathbb{Z}/3 \oplus \mathbb{Z}/3^3 \oplus \mathbb{Z}_{(3)}$	

where \oplus indicates the direct sum of the groups.

Since G is parallelizable, the Poincaré duality isomorphism

$$E_i(G) \cong E^{n-i}(G)$$

holds for any spectrum \mathbf{E} , where $n = \dim G$ (see [4, Part III]). Therefore, to compute $\widetilde{\mathbf{j}(p)}_i(G)$ it suffices to compute $\widetilde{\mathbf{j}(p)}^{n-i}(G)$. Theorem 1.1 is a consequence of Theorem 4.6, in which the cup-product ring structure of $\widetilde{\mathbf{j}(p)}^*(G)$ is described for $(G, p) = (G_2, 3)$, $(Sp(3), 3)$.

The remainder of this paper is organized as follows. In §2 we collect some results for later use. In §3 we describe the action of θ_r on $\mathbf{g}(p)^*(G)$.

In §4 we compute the rings $\widetilde{j(p)^*}(G)$.

2. Preliminaries

This section is devoted to describe the rings $K^*(G; Z_{(p)})$, $k^*(G; Z_{(p)})$, $g(p)^*(G)$ and the homomorphism $ch: K^*(G) \rightarrow H^*(G; Q)$.

Notice that G is assumed to be as in §1 and p is assumed to be an odd prime. According to Borel [9], G has no p -torsion and we have

Lemma 2.1. *There exist elements $x_{2m_i-1} \in H^{2m_i-1}(G; Z_{(p)})$, for $1 \leq i \leq l$ (where $l=2$ or 3), such that*

$$H^*(G; Z_{(p)}) = \Lambda(x_{2m_1-1}, x_{2m_2-1}, \dots, x_{2m_l-1})$$

where $2=m_1 \leq m_2 \leq \dots \leq m_l$ and Λ denotes an exterior algebra (over $Z_{(p)}$).

For this lemma and the values of m_i see [8].

We need the famous result of Hodgkin [11]:

Lemma 2.2. *Let $\{\rho_1, \dots, \rho_l\}$ be a system of ring generators of the complex representation ring $R(G)$. Then there exist elements $\beta(\rho_i) \in K^{-1}(G)$, for $1 \leq i \leq l$, such that*

$$K^*(G) = \Lambda(\beta(\rho_1), \dots, \beta(\rho_l)) \otimes Z[u, u^{-1}] .$$

Therefore

$$K^*(G; Z_{(p)}) = \Lambda(\beta(\rho_1), \dots, \beta(\rho_l)) \otimes Z_{(p)}[u, u^{-1}] .$$

The following proposition shows that

$$\begin{aligned} \kappa: k^*(G; Z_{(p)}) &\rightarrow K^*(G; Z_{(p)}) , \\ \iota: g(p)^*(G) &\rightarrow k^*(G; Z_{(p)}) \end{aligned}$$

are injective.

Proposition 2.3. *One can choose elements*

$$\xi_{2m_i-1} \in g(p)^{2m_i-1}(G) , \quad \text{for } 1 \leq i \leq l ,$$

such that

- (i) $g(p)^*(G) = \Lambda(\xi_{2m_1-1}, \dots, \xi_{2m_l-1}) \otimes Z_{(p)}[v]$.
- (ii) $k^*(G; Z_{(p)}) = \Lambda(\iota(\xi_{2m_1-1}), \dots, \iota(\xi_{2m_l-1})) \otimes Z_{(p)}[u]$.
- (iii) $K^*(G; Z_{(p)}) = \Lambda(\kappa\iota(\xi_{2m_1-1}), \dots, \kappa\iota(\xi_{2m_l-1})) \otimes Z_{(p)}[u, u^{-1}]$.
- (iv) The CW-filtration degree ([7, §2]) of ξ_{2m_i-1} is $2m_i-1$; or equivalently, $\kappa\iota(\xi_{2m_i-1})$ satisfies

$$ch(u^{m_i} \kappa\iota(\xi_{2m_i-1})) = cx_{2m_i-1} + \text{higher terms}$$

where c is a unit of $Z_{(p)}$.

Proof. By [7, §2.4] the Atiyah-Hirzebruch spectral sequence for $K^*(G; Z_{(p)})$ collapses. Therefore it follows from the naturality with respect to κ (resp. ι) that the Atiyah-Hirzebruch spectral sequence for $k^*(G; Z_{(p)})$ (resp. $g(p)^*(G)$) collapses. Thus Lemma 2.1 yields the result; in particular, for (iv) see [7, §2.5].

We quote from [14] the following

Lemma 2.4. *For our groups G , the Chern character*

$$ch: K^{-1}(G) = \tilde{K}(\Sigma G) \rightarrow H^*(\Sigma G; Q) \cong H^{*-1}(G; Q)$$

is given by:

(1) *If $G = SU(3)$, we have*

$$\begin{aligned} ch\beta(\lambda_1) &= -x_3 + \frac{1}{2}x_5, \\ ch\beta(\lambda_2) &= -x_3 - \frac{1}{2}x_5 \end{aligned}$$

(where $\{\lambda_1, \lambda_2\}$ generates $R(SU(3))$).

(2) *If $G = Sp(2)$, we have*

$$\begin{aligned} ch\beta(\lambda_1) &= x_3 - \frac{1}{6}x_7, \\ ch\beta(\lambda_2) &= 2x_3 + \frac{2}{3}x_7. \end{aligned}$$

(3) *If $G = G_2$, we have*

$$\begin{aligned} ch\beta(\rho_1) &= 2x_3 + \frac{1}{60}x_{11}, \\ ch\beta(\Lambda^2\rho_1) &= 10x_3 - \frac{5}{12}x_{11}. \end{aligned}$$

(4) *If $G = SU(4)$, we have*

$$\begin{aligned} ch\beta(\lambda_1) &= -x_3 + \frac{1}{2}x_5 - \frac{1}{6}x_7, \\ ch\beta(\lambda_2) &= -2x_3 + \frac{2}{3}x_7, \\ ch\beta(\lambda_3) &= -x_3 - \frac{1}{2}x_5 - \frac{1}{6}x_7. \end{aligned}$$

(5) *If $G = Spin(7)$, we have*

$$\begin{aligned} ch\beta(\lambda'_1) &= 2x_3 - \frac{2}{3}x_7 + \frac{1}{60}x_{11}, \\ ch\beta(\lambda'_2) &= 10x_3 + \frac{2}{3}x_7 - \frac{5}{12}x_{11}, \\ ch\beta(\Delta_7) &= 2x_3 + \frac{1}{3}x_7 + \frac{1}{60}x_{11}. \end{aligned}$$

(6) If $G = Sp(3)$, we have

$$\begin{aligned} ch\beta(\lambda_1) &= x_3 - \frac{1}{6}x_7 + \frac{1}{120}x_{11}, \\ ch\beta(\lambda_2) &= 4x_3 + \frac{1}{3}x_7 - \frac{13}{60}x_{11}, \\ ch\beta(\lambda_3) &= 6x_3 + x_7 + \frac{11}{20}x_{11}. \end{aligned}$$

An application of this result is a quick calculation of the operation ψ' on $K^*(G)$. For example, in $K^{-1}(SU(3))$ we have

$$\begin{aligned} (2.5) \quad \psi'(\beta(\lambda_1)) &= \frac{r^2(r+1)}{2}\beta(\lambda_1) + \frac{r^2(-r+1)}{2}\beta(\lambda_2), \\ \psi'(\beta(\lambda_2)) &= \frac{r^2(-r+1)}{2}\beta(\lambda_1) + \frac{r^2(r+1)}{2}\beta(\lambda_2) \end{aligned}$$

(cf. the proof of Proposition 3.3).

3. The operation θ_r on $g(p)^*(G)$

In this section we first recall the facts we need about the p -localization of G . With this as a background, we shall describe the action of θ_r on $g(p)^*(G)$.

Let $B_n(p)$, for $n \geq 1$, be the S^{2n+1} -bundle over $S^{2n+2p-1}$ such that

$$H^*(B_n(p); \mathbb{Z}/p) = \Lambda(x_{2n+1} \mathcal{P}^1 x_{2n+1}),$$

It has a cell structure:

$$(3.1) \quad B_n(p) \simeq S^{2n+1} \cup e^{2n+1+2(p-1)} \cup e^{4n+2+2(p-1)}.$$

Then G is called p -regular if and only if it is homotopy equivalent to a product of spheres when localized at p , and G is called quasi p -regular if and only if it is homotopy equivalent to a product of spaces $B_n(p)$ and spheres when localized at p .

The following result is due to Mimura and Toda [12].

Lemma 3.2. *We have*

- (1) $SU(3) \xrightarrow[p]{} S^3 \times S^5$ for $p \geq 3$.
- (2) $Sp(2) \xrightarrow[p]{} S^3 \times S^7$ for $p \geq 5$;
 $Sp(2) \xrightarrow[3]{} B_1(3)$.
- (3) $G_2 \xrightarrow[p]{} S^3 \times S^{11}$ for $p \geq 7$;
 $G_2 \xrightarrow[5]{} B_1(5)$.

$$(4) \quad SU(4) \xrightarrow{p} S^3 \times S^5 \times S^7 \quad \text{for } p \geq 5;$$

$$SU(4) \xrightarrow{5} B_1(3) \times S^5.$$

$$(5) \quad Spin(7) \xrightarrow{p} S^3 \times S^7 \times S^{11} \quad \text{for } p \geq 7;$$

$$Spin(7) \xrightarrow{5} B_1(5) \times S^7.$$

$$(6) \quad Sp(3) \xrightarrow{p} S^3 \times S^7 \times S^{11} \quad \text{for } p \geq 7;$$

$$Sp(3) \xrightarrow{5} B_1(5) \times S^7.$$

We first consider the cases in which G is p -regular.

Proposition 3.3. *In the following cases there are elements $\xi_{2m_i-1} \in g(p)^{2m_i-1}(G)$, for $1 \leq i \leq l$, as in Proposition 2.3, which satisfy:*

$$(1) \quad G = SU(3), \quad p \geq 3.$$

$$(a) \quad u^2\kappa\iota(\xi_3) = -\frac{1}{2}\beta(\lambda_1) - \frac{1}{2}\beta(\lambda_2) \xrightarrow{ch} x_3$$

$$u^3\kappa\iota(\xi_5) = \beta(\lambda_1) - \beta(\lambda_2) \xrightarrow{ch} x_5.$$

$$(b) \quad \theta_r(\xi_3) = 0, \quad \theta_r(\xi_5) = 0.$$

$$(2) \quad G = Sp(2), \quad p \geq 5.$$

$$(a) \quad u^2\kappa\iota(\xi_3) = \frac{2}{3}\beta(\lambda_1) + \frac{1}{6}\beta(\lambda_2) \xrightarrow{ch} x_3$$

$$u^4\kappa\iota(\xi_7) = -2\beta(\lambda_1) + \beta(\lambda_2) \xrightarrow{ch} x_7.$$

$$(b) \quad \theta_r(\xi_3) = 0, \quad \theta_r(\xi_7) = 0.$$

$$(3) \quad G = G_2, \quad p \geq 7.$$

$$(a) \quad u^2\kappa\iota(\xi_3) = \frac{5}{6}\beta(\rho_1) + \frac{1}{30}\beta(\Lambda^2\rho_1) \xrightarrow{ch} 2x_3$$

$$u^6\kappa\iota(\xi_{11}) = 5\beta(\rho_1) - \beta(\Lambda^2\rho_1) \xrightarrow{ch} \frac{1}{2}x_{11}.$$

$$(b) \quad \theta_r(\xi_3) = 0, \quad \theta_r(\xi_{11}) = 0.$$

$$(4) \quad G = SU(4), \quad p \geq 5.$$

$$(a) \quad u^2\kappa\iota(\xi_3) = -\frac{1}{3}\beta(\lambda_1) - \frac{1}{6}\beta(\lambda_2) - \frac{1}{3}\beta(\lambda_3) \xrightarrow{ch} x_3$$

$$u^3\kappa\iota(\xi_5) = \beta(\lambda_1) - \beta(\lambda_3) \xrightarrow{ch} x_5$$

$$u^4\kappa\iota(\xi_7) = -\beta(\lambda_1) + \beta(\lambda_2) - \beta(\lambda_3) \xrightarrow{ch} x_7.$$

$$(b) \quad \theta_r(\xi_3) = 0, \quad \theta_r(\xi_5) = 0, \quad \theta_r(\xi_7) = 0.$$

$$(5) \quad G = Spin(7), \quad p \geq 7.$$

$$\begin{aligned}
 (a) \quad u^2\kappa\iota(\xi_3) &= \frac{3}{10}\beta(\lambda'_1) + \frac{1}{30}\beta(\lambda'_2) + \frac{8}{15}\beta(\Delta_7) & 2x_3 \\
 u^4\kappa\iota(\xi_7) &= -\beta(\lambda'_1) & + \beta(\Delta_7) \xrightarrow{ch} & x_7 \\
 u^6\kappa\iota(\xi_{11}) &= \beta(\lambda'_1) - \beta(\lambda'_2) + 4\beta(\Delta_7) & \frac{1}{2}x_{11} .
 \end{aligned}$$

$$(b) \quad \theta_r(\xi_3) = 0, \quad \theta_r(\xi_7) = 0, \quad \theta_r(\xi_{11}) = 0.$$

$$(6) \quad G = Sp(3), \quad p \geq 7.$$

$$\begin{aligned}
 (a) \quad u^2\kappa\iota(\xi_3) &= \frac{2}{5}\beta(\lambda_1) + \frac{1}{10}\beta(\lambda_2) + \frac{1}{30}\beta(\lambda_3) & x_3 \\
 u^4\kappa\iota(\xi_7) &= -\frac{7}{2}\beta(\lambda_1) + \frac{1}{2}\beta(\lambda_2) + \frac{1}{4}\beta(\lambda_3) \xrightarrow{ch} & x_7 \\
 u^6\kappa\iota(\xi_{11}) &= \beta(\lambda_1) - 2\beta(\lambda_2) + \beta(\lambda_3) & x_{11} .
 \end{aligned}$$

$$(b) \quad \theta_r(\xi_3) = 0, \quad \theta_r(\xi_7) = 0, \quad \theta_r(\xi_{11}) = 0.$$

Proof. We show (1) only, because the others can be shown quite similarly. Since $\{\beta(\lambda_1), \beta(\lambda_2)\}$ forms a Z -basis for $K^{-1}(SU(3))$ by Lemma 2.2 (and [14, §2]), it is easy to see that $\{-\frac{1}{2}\beta(\lambda_1) - \frac{1}{2}\beta(\lambda_2), \beta(\lambda_1) - \beta(\lambda_2)\}$ forms a $Z_{(p)}$ -basis for $K^{-1}(SU(3); Z_{(p)})$; their images under ch are as required by Lemma 2.4. On the other hand, by Proposition 2.3 $\{u^2\kappa\iota(\xi_3), u^3\kappa\iota(\xi_5)\}$ is a $Z_{(p)}$ -basis for $K^{-1}(SU(3); Z_{(p)})$. These (together with (b)) permit us to conclude that there exist $\xi_i \in g(p)^i(SU(3))$, $i=3, 5$, satisfying (a).

To prove (b) we compute $\psi^r(u^2\kappa\iota(\xi_3))$ and $\psi^r(u^3\kappa\iota(\xi_5))$ in $\tilde{K}(\Sigma SU(3))$. By use of the formula $ch^q\psi^r = r^q ch^q$ [1, Theorem 5.1 (vi)] where ch^q is the composition

$$\tilde{K}(\Sigma G) \xrightarrow{ch} \tilde{H}^*(\Sigma G; Q) \xrightarrow{\pi_{2q}} H^{2q}(\Sigma G; Q)$$

(where π_{2q} is the projection to the $2q$ -dimensional component), we have

$$ch\psi^r(u^2\kappa\iota(\xi_3)) = r^2 x_3 = ch(r^2 u^2 \kappa\iota(\xi_3)).$$

Since $ch: \tilde{K}(\Sigma G) \rightarrow \tilde{H}^*(\Sigma G; Q)$ is injective, it follows that

$$\psi^r(u^2\kappa\iota(\xi_3)) = r^2 u^2 \kappa\iota(\xi_3).$$

Since $\psi^r(u^2) = r^2 u^2$, it follows that

$$\psi^r(\kappa\iota(\xi_3)) = \kappa\iota(\xi_3).$$

Since ψ^r commutes with κ , ι and κ , ι are injective, it follows that

$$\psi^r(\xi_3) = \xi_3.$$

Similarly we have $\psi^r(\xi_5) = \xi_5$. So (b) follows by the definition of θ_r .

In view of Lemma 3.2, all statements in Proposition 3.3 except (a) are clear. But, if one wants to discuss a homomorphism $f^*: g(p)^*(G') \rightarrow g(p)^*(G)$ which is induced by a homomorphism of compact Lie groups $f: G \rightarrow G'$, it seems to us that (a) is necessary.

Before considering the cases in which G is quasi p -regular, we describe $g(p)^*(B_1(p))$ and the θ_r -action on it. Since θ_r detects \mathcal{P}^1 (see [13, Lemma 1.1]), it follows from the Atiyah-Hirzebruch spectral sequence argument using (3.1) that

(3.4) *There exist $\xi_i \in g(p)^i(B_1(p))$, for $i=3, 2p+1$, such that*

- (i) $g(p)^*(B_1(p)) = \Lambda(\xi_3, \xi_{2p+1}) \otimes Z_{(p)}[v]$.
- (ii) *The operation θ_r is given by*

$$\theta_r(\xi_3) = \xi_{2p+1}, \quad \theta_r(\xi_{2p+1}) = 0.$$

Proposition 3.5. *In the following cases there are elements $\xi_{2m_i-1} \in g(p)^{2m_i-1}(G)$, for $1 \leq i \leq l$, as in Proposition 2.3, which satisfy:*

(1) $G=Sp(2)$, $p=3$.

$$\begin{aligned} (a) \quad u^2\kappa\iota(\xi_3) &= \frac{1}{2}\beta(\lambda_2) \xrightarrow{ch} x_3 + \frac{1}{3}x_7 \\ u^4\kappa\iota(\xi_7) &= -2\beta(\lambda_1) + \beta(\lambda_2) \longrightarrow x_7. \\ (b) \quad \theta_2(\xi_3) &= \xi_7, \quad \theta_2(\xi_7) = 0. \end{aligned}$$

(2) $G=G_2$, $p=5$.

$$\begin{aligned} (a) \quad u^2\kappa\iota(\xi_3) &= \beta(\rho_1) & 2x_3 + \frac{1}{60}x_{11} \\ u^6\kappa\iota(\xi_{11}) &= 5\beta(\rho_1) - \beta(\Lambda^2\rho_1) \xrightarrow{ch} & \frac{1}{2}x_{11}. \\ (b) \quad \theta_2(\xi_3) &= \frac{1}{2}\xi_{11}, \quad \theta_2(\xi_{11}) = 0. \end{aligned}$$

(3) $G=SU(4)$, $p=3$.

$$\begin{aligned} (a) \quad u^2\kappa\iota(\xi_3) &= -\frac{1}{2}\beta(\lambda_1) & -\frac{1}{2}\beta(\lambda_3) & x_3 + \frac{1}{6}x_7 \\ u^3\kappa\iota(\xi_5) &= \beta(\lambda_1) & -\beta(\lambda_3) \xrightarrow{ch} & x_5 \\ u^4\kappa\iota(\xi_7) &= -\beta(\lambda_1) + \beta(\lambda_2) - \beta(\lambda_3) & & x_7. \\ (b) \quad \theta_2(\xi_3) &= \frac{1}{2}\xi_7, \quad \theta_2(\xi_5) = 0, \quad \theta_2(\xi_7) = 0. \end{aligned}$$

(4) $G=Spin(7)$, $p=5$.

$$\begin{aligned} (a) \quad u^2\kappa\iota(\xi_3) &= \frac{1}{3}\beta(\lambda'_1) & +\frac{2}{3}\beta(\Delta_7) & 2x_3 + \frac{1}{60}x_{11} \\ u^4\kappa\iota(\xi_7) &= -\beta(\lambda'_1) & +\beta(\Delta_7) \xrightarrow{ch} & x_7 \\ u^6\kappa\iota(\xi_{11}) &= \beta(\lambda'_1) - \beta(\lambda'_2) + 4\beta(\Delta_7) & & \frac{1}{2}x_{11}. \end{aligned}$$

$$(b) \quad \theta_2(\xi_3) = \frac{1}{2}\xi_{11}, \quad \theta_2(\xi_7) = 0, \quad \theta_2(\xi_{11}) = 0.$$

(5) $G=Sp(3), \quad p=5.$

$$(a) \quad u^2\kappa\iota(\xi_3) = \frac{5}{12}\beta(\lambda_1) + \frac{1}{12}\beta(\lambda_2) + \frac{1}{24}\beta(\lambda_3) \quad x_3 + \frac{1}{120}x_{11}$$

$$u^4\kappa\iota(\xi_7) = -\frac{7}{2}\beta(\lambda_1) + \frac{1}{2}\beta(\lambda_2) + \frac{1}{4}\beta(\lambda_3) \xrightarrow{ch} x_7$$

$$u^6\kappa\iota(\xi_{11}) = 2\beta(\lambda_1) - 2\beta(\lambda_2) + \beta(\lambda_3) \quad x_{11}.$$

$$(b) \quad \theta_2(\xi_3) = \frac{1}{8}\xi_{11}, \quad \theta_2(\xi_7) = 0, \quad \theta_2(\xi_{11}) = 0.$$

Proof. We prove (1) only; the proof for the others is similar. First, (a) follows from Proposition 2.3 and Lemma 2.4 as in the proof of Proposition 3.3. To prove (b) we compute $\psi^2(u^2\kappa\iota(\xi_3))$. In $\tilde{K}(\Sigma Sp(2))$ we have

$$\begin{aligned} ch\psi^2(u^2\kappa\iota(\xi_3)) &= 2^2x_3 + \frac{2^4}{3}x_7 \\ &= 2^2(x_3 + \frac{1}{3}x_7) + 2^2x_7 \\ &= 2^2chu^2\kappa\iota(\xi_3) + 2^2chu^4\kappa\iota(\xi_7). \end{aligned}$$

Therefore

$$\psi^2(u^2\kappa\iota(\xi_3)) = 2^2u^2\kappa\iota(\xi_3) + 2^2u^4\kappa\iota(\xi_7).$$

Since $\iota(v)=u^2$ (where $p=3$), it follows that

$$\psi^2(\xi_3) = \xi_3 + v\xi_7.$$

Similarly we have

$$\psi^2(\xi_{11}) = \xi_{11}.$$

These imply the result.

There remain the cases in which G is neither p -regular nor quasi p -regular.

Proposition 3.6. *In the following cases there are elements $\xi_{2m_i-1} \in g(p)^{2m_i-1}(G)$, for $1 \leq i \leq l$, as in Proposition 2.3, which satisfy:*

(1) $G=G_2, \quad p=3.$

$$(a) \quad u^2\kappa\iota(\xi_3) = \beta(\rho_1) \xrightarrow{ch} 2x_3 + \frac{1}{60}x_{11}$$

$$u^6\kappa\iota(\xi_{11}) = 5\beta(\rho_1) - \beta(\Lambda^2\rho_1) \xrightarrow{ch} \frac{1}{2}x_{11}.$$

$$(b) \quad \theta_2(\xi_3) = \frac{1}{2}v\xi_{11}, \quad \theta_2(\xi_{11}) = 0.$$

(2) $G=Spin(7)$, $p=3$.

$$\begin{aligned}
 (a) \quad u^2\kappa\iota(\xi_3) &= \beta(\lambda'_1) & 2x_3 - \frac{2}{3}x_7 + \frac{1}{60}x_{11} \\
 u^4\kappa\iota(\xi_7) &= -\beta(\lambda'_1) & +\beta(\Delta_7) \xrightarrow{ch} & x_7 \\
 u^6\kappa\iota(\xi_{11}) &= \beta(\lambda'_1) - \beta(\lambda'_2) + 4\beta(\Delta_7) & & \frac{1}{2}x_{11}.
 \end{aligned}$$

$$(b) \quad \theta_2(\xi_3) = -2\xi_7 + \frac{1}{2}v\xi_{11}, \quad \theta_2(\xi_7) = 0, \quad \theta_2(\xi_{11}) = 0.$$

(3) $G=Sp(3)$, $p=3$.

$$\begin{aligned}
 (a) \quad u^2\kappa\iota(\xi_3) &= \beta(\lambda_1) & x_3 - \frac{1}{6}x_7 + \frac{1}{120}x_{11} \\
 u^4\kappa\iota(\xi_7) &= -4\beta(\lambda_1) + \beta(\lambda_2) & \xrightarrow{ch} & x_7 - \frac{1}{4}x_{11} \\
 u^6\kappa\iota(\xi_{11}) &= 2\beta(\lambda_1) - 2\beta(\lambda_2) + \beta(\lambda_3) & & x_{11}.
 \end{aligned}$$

$$(b) \quad \theta_2(\xi_3) = -\frac{1}{2}\xi_7, \quad \theta_2(\xi_7) = -\frac{3}{4}\xi_{11}, \quad \theta_2(\xi_{11}) = 0.$$

This proposition follows from the calculation similar to that in the proof of Proposition 3.3. We omit the details of the proof.

It is known [10] that

$$Spin(7) \xrightarrow{\sim} Sp(3).$$

Therefore $j(3)^*(Spin(7)) \cong j(3)^*(Sp(3))$. Henceforth we exclude to consider the former.

4. The $j(p)$ -cohomology of G

In Lemma 4.2 we present formulas on the multiplicative structure of $\widetilde{j(p)}^*(X)$ (where X satisfies a certain condition). In the rest of this section we compute $\widetilde{j(p)}^*(G)$ for all pairs (G, p) . Finally we comment on $\widetilde{j(p)}_*(G)$.

Throughout this section, the letters X and Y will stand for finite connected CW -complexes.

Consider the fibration sequence

$$\Sigma^{2p-3}g(p) \xrightarrow{\delta} j(p) \xrightarrow{\eta} g(p) \xrightarrow{\theta} \Sigma^{2p-2}g(p).$$

It leads to a short exact sequence

$$\begin{aligned}
 (4.1) \quad 0 \rightarrow \text{Coker } (\theta: \widetilde{g(p)}^{i-1}(X) \rightarrow \widetilde{g(p)}^{i+2p-3}(X)) \xrightarrow{\delta} \\
 \widetilde{j(p)}^i(X) \xrightarrow{\eta} \text{Ker } (\theta: \widetilde{g(p)}^i(X) \rightarrow \widetilde{g(p)}^{i+2p-2}(X)) \rightarrow 0
 \end{aligned}$$

for any $i \in \mathbb{Z}$. In this situation we shall use the following notation. For any $x \in \widetilde{g(p)}^*(X)$ we write \tilde{x} for $\delta(x) \in \widetilde{j(p)}^{*-2p+3}(X)$; therefore, if $x \in \text{Im } (\theta)$, we have $\tilde{x}=0$. Suppose now that $x \in \text{Ker } (\theta)$. Then we denote by \tilde{x} an element such that $\eta(\tilde{x})=x$; it is unique if $\widetilde{g(p)}^*(X)$ is (p) -torsion free. This condition is satisfied for $X=G$ by Proposition 2.3.

Lemma 4.2. *Suppose that $\widetilde{g(p)}^*(X)$ is torsion free. Then, with the above notations, for any $x, y \in \widetilde{g(p)}^*(X)$, the following formulas hold in $\widetilde{j(p)}^*(X)$:*

- (i) $\tilde{x} \cup \tilde{y} = \widetilde{x \cup y}$.
- (ii) $\tilde{x} \cup \bar{y} = \widetilde{x \cup y}$.
- (iii) $\tilde{x} \cup \tilde{y} = \widetilde{x \cup y}$.
- (iv) $\tilde{x} \cup \bar{y} = 0$.

Proof. Parts (i), (ii) and (iii) are proved by using the same technique as in [13, §4]; we refer to it for the details. In this proof we will use the facts which are shown there, without specific reference.

It remains to prove part (iv). Since η is a map of ring spectra and $\eta \delta \simeq 0$, we have

$$\eta(\tilde{x} \cup \bar{y}) = \eta(\delta(x) \cup \delta(y)) = \eta \delta(x) \cup \eta \delta(y) = 0 \cup 0 = 0.$$

Hence there exists a $z \in \widetilde{g(p)}^*(X)$ such that $\tilde{x} \cup \bar{y} = \bar{z}$. This equality implies that, in the following diagram, the outer square is commutative:

$$\begin{array}{ccccccc}
 X & \xrightarrow{d} & X \wedge X & \xrightarrow{\tilde{x} \wedge \bar{y}} & \Sigma^m j \wedge \Sigma^n j & \xrightarrow{\mu_j} & \Sigma^{m+n} j \\
 & & \downarrow x \wedge y & \text{I} & \uparrow \delta \wedge 1 & & \uparrow \delta \\
 & & \Sigma^{m+2p-3} g \wedge \Sigma^{n+2p-3} g & \xrightarrow{1 \wedge \delta} & \Sigma^{m+2p-3} g \wedge \Sigma^n j & \xrightarrow{\text{II}} & \Sigma^{m+n+2p-3} g \\
 & & z & & \downarrow 1 \wedge \eta & & \uparrow \delta \\
 & & & & \Sigma^{m+2p-3} g \wedge \Sigma^n g & \xrightarrow{\mu_g} & \Sigma^{m+n+2p-3} g
 \end{array}$$

where d is the diagonal map; $x \in \widetilde{g(p)}^{m+2p-3}(X)$, $y \in \widetilde{g(p)}^{n+2p-3}(X)$; $g = g(p)$, $j = j(p)$; μ_g and μ_j are multiplications in $g(p)$ and $j(p)$ respectively. The commutativity of square I is obvious and that of square II was shown in [13, Lemma 4.4]. Thus we have

$$\begin{aligned}
 \bar{z} &= \tilde{x} \cup \bar{y} = \mu_j(\tilde{x} \wedge \bar{y})d \\
 &= \mu_j(\delta \wedge 1)(1 \wedge \delta)(x \wedge y)d \\
 &= \delta \mu_g(1 \wedge \eta)(1 \wedge \delta)(x \wedge y)d \\
 &= 0.
 \end{aligned}$$

By virtue of this lemma, if one computes $\widetilde{j(p)}^*(X)$ by using (4.1), then its ring structure is automatically known.

We now record some basic data for $j(p)$. Since $\psi^r(v) = r^{p-1}v$, the coefficient ring of $j(p)$ is given by

$$(4.3) \quad \pi_*(j(p)) = Z_{(p)}\{\tilde{1}\} \oplus \bigoplus_{i \geq 1} Z/p^{1+\nu_p(i)}\{\overline{v^{i-1}}\}$$

where the formula

$$\nu_p(r^{i(p-1)} - 1) = 1 + \nu_p(i)$$

([2, Lemma (2.12)]) is essential. We also have the Cartan formula for θ_r : for any $x, y \in g(p)^*(X)$,

$$(4.4) \quad \theta_r(x \cup y) = \theta_r(x) \cup y + x \cup \theta_r(y) + v \cdot \theta_r(x) \cup \theta_r(y)$$

(cf. [13, Lemma 4.1]).

Let us enter into a computation of $\widetilde{j(p)}^*(G)$. As is well known, the cofibration

$$X \vee Y \rightarrow X \times Y \rightarrow X \wedge Y$$

leads to a split short exact sequence

$$0 \rightarrow \widetilde{j(p)}^i(X \wedge Y) \rightarrow \widetilde{j(p)}^i(X \times Y) \rightarrow \widetilde{j(p)}^i(X) \oplus \widetilde{j(p)}^i(Y) \rightarrow 0$$

for any $i \in \mathbb{Z}$. Therefore by Lemma 3.2, in order to compute $\widetilde{j(p)}^*(G)$ when G is p -regular or quasi p -regular, it suffices to determine $\widetilde{j(p)}^*(B_1(p))$. From (3.4) we deduce

Proposition 4.5. *The ring $\widetilde{j(p)}^*(B_1(p))$ is given by:*

$$\begin{aligned} \widetilde{j(p)}^*(B_1(p)) &= \widetilde{j(p)}^*(S^0)\{\widetilde{\xi_3\xi_{2p+1}}\} \oplus Z_{(p)}\{\widetilde{\xi_{2p+1}}\} \\ &\quad \oplus Z_{(p)}\{(r^{p-1}-1)\widetilde{\xi_3} - v\widetilde{\xi_{2p+1}}\} \\ &\quad \oplus \bigoplus_{i \geq 1} Z/p^{2+\nu_p(i)+\nu_p(i+1)}\{\overline{v^{i-1}\xi_3}\} \end{aligned}$$

where the relations

$$\begin{aligned} \overline{\xi_{2p+1}} &= 0, \\ \overline{v^i\xi_{2p+1}} &= (r^{-i(p-1)} - 1)\overline{v^{i-1}\xi_3} \quad (\text{for } i \geq 1) \end{aligned}$$

hold.

Proof. By using (4.4), in $g(p)^*(B_1(p))$ we have

$$\begin{aligned} \theta_r(v^i\xi_3\xi_{2p+1}) &= (r^{i(p-1)} - 1)v^{i-1}\xi_3\xi_{2p+1}, \\ \theta_r(v^i\xi_{2p+1}) &= (r^{i(p-1)} - 1)v^{i-1}\xi_{2p+1}, \\ \theta_r(v^i\xi_3) &= (r^{i(p-1)} - 1)v^{i-1}\xi_3 + r^{i(p-1)}v^i\xi_{2p+1}. \end{aligned}$$

So the kernel and cokernel of θ , are easily calculated and the result follows.

In this way, if G is p -regular or quasi p -regular, the ring $j(p)^*(G)$ can be described. For the remaining cases, from parts (1) and (3) of Proposition 3.6 we deduce

Theorem 4.6. *With the notation as in Lemma 4.2, the ring $\widetilde{j(3)^*(G)}$ for $G=G_2, Sp(3)$ is given by:*

$$(1) \quad G=G_2.$$

i	$\widetilde{j(3)^i(G_2)}$
14	$Z_{(3)} \widetilde{\{\xi_3 \xi_{11}\}}$
13	0
12	0
11	$Z/3 \widetilde{\{\xi_3 \xi_{11}\}} \oplus Z_{(3)} \widetilde{\{\xi_{11}\}}$
10	0
9	0
8	$Z/3 \widetilde{\{\xi_{11}\}}$
7	$Z/3 \widetilde{\{v \xi_3 \xi_{11}\}}$
6	0
5	0
4	0
3	$Z/3^2 \widetilde{\{v^2 \xi_3 \xi_{11}\}} \oplus Z_{(3)} \widetilde{\{3 \xi_3 - \frac{1}{10} v^2 \xi_{11}\}}$
2	0
1	0
0	$Z/3^3 \widetilde{\{\xi_3\}}$
-1	$Z/3 \widetilde{\{v^3 \xi_3 \xi_{11}\}}$
-2	0
-3	0
-4	$Z/3^2 \widetilde{\{v \xi_3\}}$
-5	$Z/3 \widetilde{\{v^4 \xi_3 \xi_{11}\}}$
-6	0
-7	0

(2) $G = Sp(3)$.

i	$\widetilde{j(3)^i(Sp(3))}$
21	$Z_{(3)}\{\overline{\xi_3\xi_7\xi_{11}}\}$
20	0
19	0
18	$Z/3\{\overline{\xi_3\xi_7\xi_{11}}\} \oplus Z_{(3)}\{\overline{\xi_7\xi_{11}}\}$
17	0
16	0
15	0
14	$Z/3\{\overline{v\xi_3\xi_7\xi_{11}}\} \oplus Z_{(3)}\{3\xi_3\xi_{11} + \frac{1}{2}v\xi_7\xi_{11}\}$
13	0
12	0
11	$Z/3\{\overline{\xi_3\xi_{11}}\} \oplus Z_{(3)}\{\overline{\xi_{11}}\}$
10	$Z/3^2\{\overline{v^2\xi_3\xi_7\xi_{11}}\} \oplus Z_{(3)}\{3\xi_3\xi_{11} + \frac{3}{4}v\xi_3\xi_{11} + \frac{1}{40}v^2\xi_7\xi_{11}\}$
9	0
8	$Z/3\{\overline{\xi_{11}}\}$
7	$Z/3\{\overline{\xi_3\xi_7 - \frac{1}{16}v\xi_3\xi_{11}}\} \oplus Z/3^3\{\overline{v\xi_3\xi_{11}}\} \oplus Z_{(3)}\{\xi_7 + \frac{1}{4}v\xi_{11}\}$
6	$Z/3\{\overline{v^3\xi_3\xi_7\xi_{11}}\}$
5	0
4	$Z/3\{\overline{v\xi_{11}}\}$
3	$Z/3\{v\xi_3\xi_7 - \frac{4}{5}v^2\xi_3\xi_{11}\} \oplus Z/3^3\{\overline{v^2\xi_3\xi_{11}}\} \oplus Z_{(3)}\{3\xi_3 + \frac{1}{2}v\xi_7 + \frac{1}{10}v^2\xi_{11}\}$
2	$Z/3\{\overline{v^4\xi_3\xi_7\xi_{11}}\}$
1	0
0	$Z/3\{3\xi_3 - \frac{8}{5}v^2\xi_{11}\} \oplus Z/3^3\{\overline{\xi_3}\}$

Proof. For (1) we have

$$\begin{aligned}\theta_2(v^i\xi_3\xi_{11}) &= (2^{2i}-1)v^{i-1}\xi_3\xi_{11}, \\ \theta_2(v^i\xi_{11}) &= (2^{2i}-1)v^{i-1}\xi_{11}, \\ \theta_2(v^i\xi_3) &= (2^{2i}-1)v^{i-1}\xi_3 + 2^{2i-1}v^{i+1}\xi_{11}.\end{aligned}$$

For (2) we have

$$\begin{aligned}
 \theta_2(v^i\xi_3\xi_7\xi_{11}) &= (2^{2i}-1)v^{i-1}\xi_3\xi_7\xi_{11}, \\
 \theta_2(v^i\xi_7\xi_{11}) &= (2^{2i}-1)v^{i-1}\xi_7\xi_{11}, \\
 \theta_2(v^i\xi_3\xi_{11}) &= (2^{2i}-1)v^{i-1}\xi_3\xi_{11}-2^{2i-1}v^i\xi_7\xi_{11}, \\
 \theta_2(v^i\xi_{11}) &= (2^{2i}-1)v^{i-1}\xi_{11}, \\
 \theta_2(v^i\xi_3\xi_7) &= (2^{2i}-1)v^{i-1}\xi_3\xi_7-2^{2i-2}3v^i\xi_3\xi_{11}+2^{2i-3}3v^{i+1}\xi_7\xi_{11}, \\
 \theta_2(v^i\xi_7) &= (2^{2i}-1)v^{i-1}\xi_7-2^{2i-2}3v^i\xi_{11}, \\
 \theta_2(v^i\xi_3) &= (2^{2i}-1)v^{i-1}\xi_3-2^{2i-1}v^i\xi_7.
 \end{aligned}$$

So the result follows from elementary calculations of the kernel and cokernel of θ_2 .

Proof of Theorem 1.1.

By using the Poincaré duality isomorphism

$$\begin{aligned}
 j(p)_i(G) &= \widetilde{j(p)}_i(G) \oplus \widetilde{j(p)}_i(S^0) \\
 &\cong \widetilde{j(p)}^{n-i}(G) \oplus \widetilde{j(p)}^{n-i}(S^0) = j(p)^{n-i}(G)
 \end{aligned}$$

where $n = \dim G$, Theorem 1.1 follows from Theorem 4.6 and (4.3).

Finally we talk about the Pontrjagin ring structure of $\widetilde{j(p)}_*(G)$. Since in Lemma 2.2 each $\beta(\rho_i)$ is primitive (see [11]), the ring structure of $K_*(G)$ can be determined. Furthermore, the ψ' -action on $K_*(G)$ can be determined by using the formula

$$\psi'(a \cap \alpha) = \psi'(a) \cap \psi'(\alpha)$$

where $a \in K^*(G)$, $\alpha \in K_*(G)$ and \cap denotes the cap product. Therefore the ring structure of $\widetilde{j(p)}_*(G)$ will be obtained by using the homology instead of the cohomology and taking the same course as in this paper.

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Department of Mathematics
Osaka Women's University
Daisen-cho, Sakai
Osaka 590, Japan