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| Title        | Jorgensen groups of parabolic type II (Countably infinite case)           |
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| Citation     | Osaka Journal of Mathematics. 41(3) P.491-P.506                           |
| Issue Date   | 2004-09   |
| Text Version | publisher   |
| URL          | <a href="https://doi.org/10.18910/5186">https://doi.org/10.18910/5186</a> |
| DOI          | 10.18910/5186   |
| rights       |   |
| Note         |   |

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## JØRGENSEN GROUPS OF PARABOLIC TYPE II (COUNTABLY INFINITE CASE)

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(Received February 21, 2003)

### 1. Introduction

**1.1.** It is one of the most important problems in the theory of Kleinian groups to decide whether or not a subgroup  $G$  of the Möbius transformation group is discrete. For this problem there are two important and useful theorems: One is Poincaré's polyhedron theorem, which gives a sufficient condition for  $G$  to be discrete. The other is Jørgensen's inequality, which gives a necessary condition for a two-generator Möbius transformation group  $\langle A, B \rangle$  to be discrete. Here we will consider extreme discrete groups (Jørgensen groups) for Jørgensen's inequality. This paper is the second part of a series of studies on Jørgensen groups (cf. Li-Oichi-Sato [4, 5]).

**1.2.** Let  $\text{Möb}$  denote the set of all linear fractional transformations (Möbius transformations)

$$A(z) = \frac{az + b}{cz + d}$$

of the extended complex plane  $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ , where  $a, b, c, d$  are complex numbers and the determinant  $ad - bc = 1$ . There is an isomorphism between  $\text{Möb}$  and  $PSL(2, \mathbf{C})$ . We always write elements of  $\text{Möb}$  as matrices with determinant 1 in this paper. We recall that  $\text{Möb}$  ( $= PSL(2, \mathbf{C})$ ) acts on the upper half space  $H^3$  of  $\mathbf{R}^3$  as the group of conformal isometries of hyperbolic 3-space.

In this paper we use a Kleinian group in the same meaning as a discrete group. Namely, a Kleinian group is a discrete subgroup of  $\text{Möb}$ . A Kleinian group  $G$  is of *the first kind* if the limit set  $\Lambda(G)$  of  $G$  is all of the extended complex plane  $\hat{\mathbf{C}}$  and it is of *the second kind* otherwise. A subgroup  $G$  of  $\text{Möb}$  is said to be *elementary* if there exists a finite  $G$ -orbit in  $\mathbf{R}^3$ .

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2000 *Mathematics Subject Classification* : Primary 30F40; Secondary 20H10, 32G15.

\*Partly supported by the Grants-in-Aid for Co-operative Research as well as Scientific Research, the Ministry of Education, Science, Sports and Culture, Japan.

**1.3.** The *trace*  $\text{tr}(A)$  of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (ad - bc = 1)$$

in  $SL(2, \mathbf{C})$  is defined by  $\text{tr}(A) = a + d$ . We remark that the trace of an element of Möb ( $= PSL(2, \mathbf{C})$ ) is not well-defined, but Jørgensen number (defined later) is still well-defined after choosing matrix representatives.

**1.4.** Let  $A^*$  and  $B^*$  be matrices in  $SL(2, \mathbf{C})$  representing the Möbius transformations  $A$  and  $B$ , respectively. As  $A^*$  and  $B^*$  are determined by  $A$  and  $B$  to within a factor of  $-1$ , we see that the commutator  $A^*B^*(A^*)^{-1}(B^*)^{-1}$  (resp.  $(A^*)^2$ ) are uniquely determined by  $A$  and  $B$  (resp.  $A$ ). Thus we may write  $\text{tr}(ABA^{-1}B^{-1}) = \text{tr}(A^*B^*(A^*)^{-1}(B^*)^{-1})$  and  $\text{tr}^2(A) = \text{tr}^2(A^*)$ .

In 1976 Jørgensen obtained the following important theorem, which gives a necessary condition for a non-elementary Möbius transformation group  $G = \langle A, B \rangle$  to be discrete.

**Theorem A** (Jørgensen [1]). *Suppose that the Möbius transformations  $A$  and  $B$  generate a non-elementary discrete group. Then*

$$J(A, B) := |\text{tr}^2(A) - 4| + |\text{tr}(ABA^{-1}B^{-1}) - 2| \geq 1.$$

*The lower bound 1 is best possible.*

**1.5.**

**DEFINITION 1.** Let  $A$  and  $B$  be Möbius transformations. The *Jørgensen number*  $J(A, B)$  for the ordered pair  $(A, B)$  is defined by

$$J(A, B) := |\text{tr}^2(A) - 4| + |\text{tr}(ABA^{-1}B^{-1}) - 2|.$$

**DEFINITION 2.** A subgroup  $G$  of Möb is called a *Jørgensen group* if  $G$  satisfies the following four conditions:

- (1)  $G$  is a two-generator group.
- (2)  $G$  is a discrete group.
- (3)  $G$  is a non-elementary group.
- (4) There exist generators  $A$  and  $B$  of  $G$  such that  $J(A, B) = 1$ .

**1.6.** Jørgensen and Kiikka showed the following.

**Theorem B** (Jørgensen-Kiikka [2]). *Let  $\langle A, B \rangle$  be a non-elementary discrete group with  $J(A, B) = 1$ . Then  $A$  is elliptic of order at least seven or  $A$  is parabolic.*

If  $\langle A, B \rangle$  is a Jørgensen group such that  $A$  is parabolic and  $J(A, B) = 1$ , then we call it a *Jørgensen group of parabolic type*. There are infinitely many Jørgensen groups of parabolic type (Jørgensen-Lascurain-Pignataro [3], Sato [8]).

Now it gives rise to the following problem.

**Problem 1.** Find all Jørgensen groups of parabolic type.

**1.7.** Let  $\langle A, B \rangle$  be a marked two-generator group such that  $A$  is parabolic. Then we can normalize  $A$  and  $B$  as follows:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_{\sigma,\mu} = \begin{pmatrix} \mu\sigma & \mu^2\sigma - 1/\sigma \\ \sigma & \mu\sigma \end{pmatrix},$$

where  $\sigma \in \mathbf{C} \setminus \{0\}$  and  $\mu \in \mathbf{C}$ . See [4] for this normalization. We can easily see that  $J(A, B_{\sigma,\mu}) = |\sigma|^2$ .

We denote by  $G_{\sigma,\mu}$  the marked group generated by  $A$  and  $B_{\sigma,\mu}$ :  $G_{\sigma,\mu} = \langle A, B_{\sigma,\mu} \rangle$ . We say that  $(\sigma, \mu) \in \mathbf{C} \setminus \{0\} \times \mathbf{C}$  is the point representing a marked group  $G_{\sigma,\mu}$  and that  $G_{\sigma,\mu}$  is the marked group corresponding to a point  $(\sigma, \mu)$ .

**1.8.** In [8], Sato considered the case of  $\mu = ik$  ( $k \in \mathbf{R}$ ). Namely, he considered marked two-generator group  $G_{\sigma,ik} = \langle A, B_{\sigma,ik} \rangle$  generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_{\sigma,ik} = \begin{pmatrix} ik\sigma & -k^2\sigma - 1/\sigma \\ \sigma & ik\sigma \end{pmatrix},$$

where  $\sigma \in \mathbf{C} \setminus \{0\}$  and  $k \in \mathbf{R}$ .

Now we have the following conjecture.

**Conjecture.** For any Jørgensen group  $G$  of parabolic type there exists a marked group  $G_{\sigma,ik}$  ( $\sigma \in \mathbf{C} \setminus \{0\}, k \in \mathbf{R}$ ) such that  $G_{\sigma,ik}$  is conjugate to  $G$ .

If this conjecture is true, then it is sufficient to consider the case of  $\mu = ik$  in order to find all Jørgensen groups of parabolic type. In this paper we only consider the case of  $\mu = ik$ .

**1.9.** Let  $C$  be the following cylinder:

$$C = \{(\sigma, ik) \mid |\sigma| = 1, k \in \mathbf{R}\}.$$

**Theorem C** (Sato [8]). *If a marked two-generator group  $G_{\sigma,ik} = \langle A, B_{\sigma,ik} \rangle$  ( $\sigma \in \mathbf{C} \setminus \{0\}, k \in \mathbf{R}$ ) is a Jørgensen group, then the point  $(\sigma, ik)$  representing  $G_{\sigma,ik}$  lies on the cylinder  $C$ .*

If  $(\sigma, ik)$  is a point on the cylinder  $C$ , then we can set  $\sigma = -ie^{i\theta}$  ( $0 \leq \theta \leq 2\pi$ ). If a point  $(-ie^{i\theta}, ik)$  on the cylinder  $C$  represents a Jørgensen group, then we say that the group is a *Jørgensen group of parabolic type*  $(\theta, k)$ .

Now it gives rise to the following problem.

**Problem 2.** Find all Jørgensen groups of parabolic type  $(\theta, k)$ .

We divide Jørgensen groups of this type into three parts as follows:

Part 1.  $|k| \leq \sqrt{3}/2$ ,  $0 \leq \theta \leq 2\pi$  (finite case).

Part 2.  $\sqrt{3}/2 < |k| \leq 1$ ,  $0 \leq \theta \leq 2\pi$  (countably infinite case).

Part 3.  $1 < |k|$ ,  $0 \leq \theta \leq 2\pi$  (uncountably infinite case).

By some lemmas in [8], it suffices to consider the case of  $0 \leq \theta \leq \pi/2$  and  $k \geq 0$  in order to find Jørgensen groups of parabolic type  $(\theta, k)$ .

In the previous paper [4] we found all Jørgensen groups in the case where  $0 \leq \theta \leq \pi/2$  and  $0 \leq k \leq \sqrt{3}/2$ , that is, we obtain the following theorem.

**Theorem D** (finite case) (Li-Oichi-Sato [4]). (i) *There are sixteen Jørgensen groups in  $D = \{(\theta, k) \in \mathbf{R} \mid 0 \leq \theta \leq \pi/2, 0 \leq k \leq \sqrt{3}/2\}$ .*

(ii) *Nine of them are Kleinian groups of the first kind and seven groups are of the second kind.*

Furthermore, in [5] we found all Jørgensen groups in the case where  $0 \leq \theta \leq \pi/2$  and  $k > 1$  (uncountably infinite case). Therefore we found all Jørgensen groups of parabolic type  $(\theta, k)$ , that is, Problem 2 is completely solved.

Thanks are due to the referees for their careful reading and valuable suggestions.

## 2. Main theorem

In this section we will state that we find all Jørgensen groups in Part 2.

For simplicity we write  $B_{\theta, k}$  for  $B_{-ie^{i\theta}, ik}$ . Let  $A$  and  $B_{\theta, k}$  ( $k \in \mathbf{R}, 0 \leq \theta \leq \pi/2$ ) be the following matrices:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{\theta, k} = \begin{pmatrix} ke^{i\theta} & ie^{-i\theta}(k^2e^{2i\theta} - 1) \\ -ie^{i\theta} & ke^{i\theta} \end{pmatrix}.$$

We obtain the following theorem.

**Main Theorem.** *The group  $G_{\theta, k}$  with  $0 \leq \theta \leq \pi/2$  and  $\sqrt{3}/2 < k \leq 1$  is a Jørgensen group (discrete) if and only if one of the following conditions holds.*

(a)  $\theta = 0$  and  $k = 1$ . In this case,  $G_{\theta, k}$  is a Kleinian group of the second kind, and  $\Omega(G_{\theta, k})/G_{\theta, k}$  is a union of two Riemann surfaces with signature  $(0; 2, 3, \infty)$ .

(b)  $\theta = 0$  and  $k = \cos(\pi/n)$  ( $n = 7, 8, \dots$ ). In this case,  $G_{\theta, k}$  is a Kleinian group of

the second kind, and  $\Omega(G_{\theta,k})/G_{\theta,k}$  is a union of two Riemann surfaces with signatures  $(0; 2, 3, n)$  and  $(0; 2, 3, \infty)$ .

(c)  $\theta = \pi/4$  and  $k = 1$ . In this case,  $G_{\theta,k}$  is a Kleinian group of the first kind, and the volume  $V(G_{\pi/4,1})$  of the 3-orbifold for  $G_{\pi/4,1}$  is

$$V(G_{\pi/4,1}) = 8 \left\{ 2L\left(\frac{\pi}{4}\right) - L\left(\frac{\pi}{12}\right) - L\left(\frac{5\pi}{12}\right) \right\},$$

where  $L(\theta)$  is the Lobachevskiĭ function:

$$L(\theta) = - \int_0^\theta \log |2 \sin u| du.$$

(d) (Sato-Yamada [9])  $\theta = \pi/2$  and  $k = 1$ . In this case,  $G_{\theta,k}$  is a Kleinian group of the second kind, and  $\Omega(G_{\theta,k})/G_{\theta,k}$  is a Riemann surface with signature  $(0; 3, 3, \infty)$ .

(e) (Sato-Yamada [9])  $\theta = \pi/2$  and  $k = \cos(\pi/n)$  ( $n = 7, 8, \dots$ ). In this case,  $G_{\theta,k}$  is a Kleinian group of the second kind, and  $\Omega(G_{\theta,k})/G_{\theta,k}$  is a Riemann surface with signature  $(0; 3, 3, n)$ .

The proof of the main theorem is given in Sections 4–9.

**Corollary.** *There are countably infinite Jørgensen groups on the region  $\{(\theta, k) \mid 0 \leq \theta \leq \pi/2, \sqrt{3}/2 < k \leq 1\}$ .*

### 3. Poincaré’s polyhedron theorem

In this section we will state Poincaré’s polyhedron theorem following Maskit [6, p. 73]. The theorem gives a sufficient condition for a subgroup of the Möbius transformation group to be discrete.

Let  $\mathbf{P}$  be a polyhedron in the upper half space  $H^3$ . We assume that for each side  $s$  of  $\mathbf{P}$ , there is a side  $s'$ , not necessarily distinct from  $s$ , and there is an element  $g_s \in \text{Möb}$ , satisfying the following conditions:

- (i)  $g_s(s) = s'$
- (ii)  $g_{s'} = g_s^{-1}$ .

Then the isometries  $g_s$  are called the *side pairing transformations*.

Let  $G$  be the group generated by the side pairing transformations. If there is a side  $s$ , with  $s' = s$ , then condition (i) implies that  $g_s^2 = 1$ . If this occurs, the relation  $g_s^2 = 1$ , is called a *reflection relation*.

The side pairing transformations induce an equivalence relation on  $\bar{\mathbf{P}}$  (the closure of  $\mathbf{P}$ ). Let  $\mathbf{P}^*$  be the space of equivalence classes so that projection  $p: \bar{\mathbf{P}} \rightarrow \mathbf{P}^*$  is continuous and open. We assume that for every point  $z \in \mathbf{P}^*$ ,  $p^{-1}(z)$  is a finite set.

Next we will define a cycle of edges of  $\mathbf{P}$  and a cycle transformation. Let  $e_1$  be an edge. It lies on the boundary of two sides of  $\mathbf{P}$ . Let  $s_1$  be one side of them. Then there is a side  $s'_1$  and there is a side pairing transformation  $g_1$  with  $g_1(s_1) = s'_1$ . Set

$e_2 = g_1(e_1)$ . The edge  $e_2$  lies on the boundary of two sides, one of them is  $s'_1$ . Let  $s_2$  be the other side. Then there are a side  $s'_2$  and a side pairing transformation  $g_2$  with  $g_2(s_2) = s'_2$ . Continuing this manner, we generate a sequence  $\{e_n\}$  of edges and a sequence  $\{g_n\}$  of side pairing transformations.

Since each point of  $e_1$  is equivalent to at most finitely many other points of  $\bar{\mathbf{P}}$ , the sequence of edges is periodic. Let  $m$  be the least period. The cyclically ordered sequence of edges  $\{e_1, e_2, \dots, e_m\}$  is called a *cycle of edges* and  $m$  is the *period* of the cycle. Two cycles are *equivalent* if they both contain the same set of edges. Each edge lies in exactly one equivalence class of cycle. We have  $g_m \cdots g_1(e_1) = e_1$ , and call  $h = g_m \cdots g_1$  the *cycle transformation* at  $e_1$ . The relation of the form  $h^l = 1$  is called the *cycle relation*. The following theorem is well-known.

**Theorem E** (Poincaré’s Polyhedron Theorem (Maskit [6, p. 73])). *Let  $\mathbf{P}$  be a polyhedron with side pairing transformations satisfying the following conditions (1) through (6). Then,  $G$ , the group generated by the side pairing transformations, is discrete and  $\mathbf{P}$  is a fundamental polyhedron for  $G$ , and the reflection relations and cycle relations form a complete set of relations for  $G$ :*

- (1) *For each side  $s$  of  $\mathbf{P}$ , there is a side  $s'$  and there is an element  $g_s \in G$  satisfying  $g_s(s) = s'$  and  $g_{s'} = g_s^{-1}$ .*
- (2)  *$g_s(\mathbf{P}) \cap \mathbf{P} = \emptyset$ .*
- (3) *For every point  $z \in \mathbf{P}^*$ ,  $p^{-1}(z)$  is a finite set.*
- (4) *Let  $e$  be an edge and let  $h$  be the cycle transformation at  $e$ . Then for each edge  $e$ , there is a positive integer  $t$  such that  $h^t = 1$ .*
- (5) *Let  $\{e_1, e_2, \dots, e_m\}$  be any cycle of edges of  $\mathbf{P}$  and let  $\alpha(e_k)$  ( $k = 1, 2, \dots, m$ ) be the angle measured from inside  $\mathbf{P}$  at the edge  $e_k$ . Let  $q$  be the smallest positive integer such that  $h^q = 1$ , where  $h$  is the cycle transformation at  $e_1$ . Then the equality*

$$\sum_{k=1}^m \alpha(e_k) = \frac{2\pi}{q}$$

*holds.*

- (6)  *$\mathbf{P}^*$  is complete.*

**4. Proof of the case  $\theta = 0$**

In this section we will prove the main theorem in the case of  $\theta = 0$ . For  $\theta = 0$  we have

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_k := B_{0,k} = \begin{pmatrix} k & i(k^2 - 1) \\ -i & k \end{pmatrix} \text{ (} k \in \mathbf{R}\text{)}.$$

We set  $G_k = \langle A, B_k \rangle$ .

**4.1.** We set  $C_k = AB_kA^{-1}B_k^{-1}$  and  $D_k = B_k^{-1}C_k$ . Then we have

$$C_k = \begin{pmatrix} -ik & -(k^2 - 1) + ik \\ -1 & 1 + ik \end{pmatrix} \text{ and } D_k = \begin{pmatrix} -i & i \\ 0 & i \end{pmatrix}.$$

Furthermore we have

$$C_kA = \begin{pmatrix} -ik & -(k^2 - 1) \\ -1 & ik \end{pmatrix} \text{ and } D_kA = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

We can easily see the following lemmas and so omit the proofs.

**Lemma 4.1.** *Let  $A, B_k, C_k, D_k, C_kA$  and  $D_kA$  be the Möbius transformations represented by the above matrices. Then the following hold.*

- (i)  $A$  is a parabolic transformation with a fixed point  $\infty$ .
- (ii) (1)  $B_k$  is a parabolic transformation with a fixed point  $0$  if  $k = 1$ .  
 (2)  $B_k$  is an elliptic transformation of order  $n$  (integer  $n \geq 7$ ) if  $0 < k < 1$ , where  $k = \cos(\pi/n)$ .
- (iii)  $C_k$  is an elliptic transformation of order three with fixed points  $(1+i(2k \pm \sqrt{3}))/2$ .
- (iv)  $D_k$  is an elliptic transformation of order two with fixed points  $1/2$  and  $\infty$ .
- (v)  $C_kA$  is an elliptic transformation of order two with fixed points  $(k \pm 1)i$ .
- (vi)  $D_kA$  is an elliptic transformation of order two with fixed points  $0$  and  $\infty$ .

**Lemma 4.2.** *Let  $A, B_k, C_k, D_k, C_kA$  and  $D_kA$  be the Möbius transformations represented by the above matrices. Let  $G_k$  be the group generated by  $A$  and  $B_k$ , and  $G_k^*$  be the group generated by  $A, C_k$  and  $D_k$ :  $G_k = \langle A, B_k \rangle$  and  $G_k^* = \langle A, C_k, D_k \rangle$ . Then  $G_k = G_k^*$ .*

**4.2.** We will prove (a) and (b) of the main theorem. The part is proved parallel to 6.2 in [4]. Let  $\theta = 0$  and  $k = 1$  or  $k = \cos \pi/n$  ( $n = 7, 8, \dots$ ). Let  $G_k^*$  be the group generated by  $A, C_k$  and  $D_k$ , that is,  $G_k^* = \langle A, C_k, D_k \rangle$ . We denote by  $H^3$  and  $\bar{H}^3$  the upper half space and its closure, respectively:  $H^3 = \{(z, t) \in \mathbf{C} \times \mathbf{R} \mid t > 0\}$  and  $\bar{H}^3 = \{(z, t) \in \mathbf{C} \times \mathbf{R} \mid t \geq 0\}$ . We will define some sides  $F_j$  ( $j = 1, 2, \dots, 6$ ) in  $\bar{H}^3$  as follows:

- $F_1 = \{(z, t) \in \bar{H}^3 \mid x = 0, y \geq 0, t \geq 0, |z - ik|^2 + t^2 \geq 1\}$ ,
- $F_2 = \{(z, t) \in \bar{H}^3 \mid x = 1, y \geq 0, t \geq 0, |z - (1 + ik)|^2 + t^2 \geq 1\}$ ,
- $F_3 = \{(z, t) \in \bar{H}^3 \mid 0 \leq x \leq 1/2, y = 0, t \geq 0, |z - ik|^2 + t^2 \geq 1\}$ ,
- $F_4 = \{(z, t) \in \bar{H}^3 \mid 1/2 \leq x \leq 1, y = 0, t \geq 0, |z - (1 + ik)|^2 + t^2 \geq 1\}$ ,
- $F_5 = \{(z, t) \in \bar{H}^3 \mid 0 \leq x \leq 1/2, y \geq 0, |z - ik|^2 + t^2 = 1, t \geq 0\}$ ,
- $F_6 = \{(z, t) \in \bar{H}^3 \mid 1/2 \leq x \leq 1, y \geq 0, |z - (1 + ik)|^2 + t^2 = 1, t \geq 0\}$ .

Let  $\mathbf{P}$  be the polyhedron in  $H^3$  bounded by the above six sides  $F_j$  ( $j = 1, 2, \dots, 6$ ) and the complex plane  $\mathbf{C}$ . Then there are three side pairing transformations  $A, D_k$  and  $C_k$  of  $\mathbf{P}$  as follows:  $F_2 = A(F_1)$ ,  $F_4 = D_k(F_3)$ ,  $F_5 = C_k(F_6)$  (see



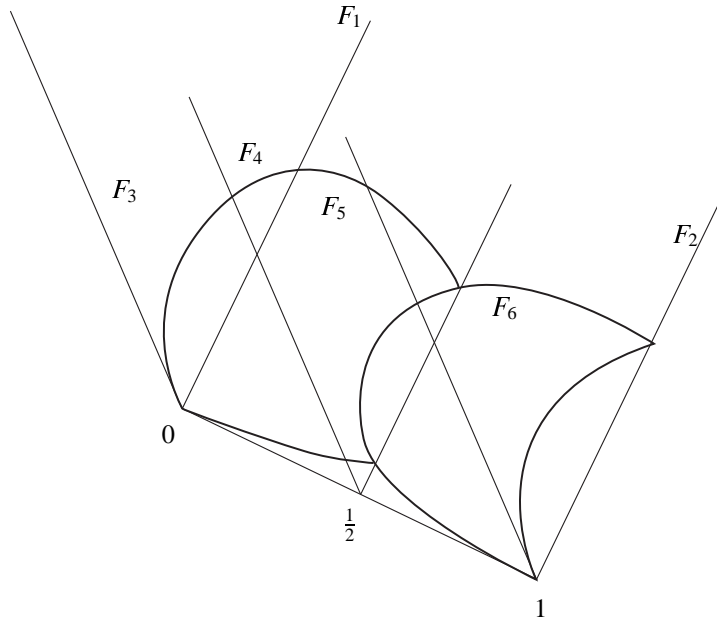


Fig. 4.1. ( $k = 1$ )

Fig. 4.1 and Fig. 4.2).

Next we denote the edges  $e_j$  ( $j = 1, 2, \dots, 6$ ) of  $\mathbf{P}$  as follows:

$$e_1 = F_1 \cap F_3; e_2 = F_2 \cap F_4; e_3 = F_3 \cap F_4; e_4 = F_1 \cap F_5; e_5 = F_2 \cap F_6; e_6 = F_5 \cap F_6.$$

Let  $\theta_j = \alpha(e_j)$  ( $j = 1, 2, \dots, 6$ ) be the angles measured from inside  $\mathbf{P}$  at  $e_j$ . Then it is easily seen that  $\theta_1 = \theta_2 = \pi/2$ ,  $\theta_3 = \pi$ ,  $\theta_4 = \theta_5 = \pi/2$ ,  $\theta_6 = 2\pi/3$ . Then  $\theta_1 + \theta_2 = 2\pi/2$ ,  $\theta_3 = 2\pi/2$ ,  $\theta_4 + \theta_5 = 2\pi/2$ ,  $\theta_6 = 2\pi/3$ . Since  $e_2 = A(e_1)$  and  $e_1 = D_k(e_2)$  (resp.  $e_5 = A(e_4)$  and  $e_4 = C_k(e_5)$ ) we have that  $D_k A$  (resp.  $C_k A$ ) are the cycle transformations at  $e_1$  (resp.  $e_4$ ) and that both of the cycle transformations are elliptic of order two by Lemma 4.1. Since  $e_3 = D_k(e_3)$  (resp.  $e_6 = C_k(e_6)$ ), we have  $D_k$  (resp.  $C_k$ ) are the cycle transformations at  $e_3$  (resp.  $e_6$ ) and that both of the cycle transformations are elliptic of order two (resp. of order three) by Lemma 4.1.

By the above we see that the conditions (1) through (5) in Theorem E are satisfied. The completeness condition (6) in Theorem E is easily shown by using Ratcliffe [7, Theorem 11.1.2]. Thus  $G_k^*$  is a Kleinian group of the second kind. By Lemma 4.2,  $G_k$  is a Kleinian group of the second kind and so a Jørgensen group.

We define some half lines, segments and circular arcs  $s_j$  ( $j = 1, 2, \dots, 8$ ) on the complex plane  $\mathbf{C}$  as follows:

- $s_1 = \{z \in \mathbf{C} \mid x = 0, y \geq k + 1\}$ ,
- $s_2 = \{z \in \mathbf{C} \mid x = 1, y \geq k + 1\}$ ,
- $s_3 = \{z \in \mathbf{C} \cap F_5 \mid 0 \leq x \leq 1/2, k + \sqrt{3}/2 \leq y \leq k + 1\}$ ,
- $s_4 = \{z \in \mathbf{C} \cap F_6 \mid 1/2 \leq x \leq 1, k + \sqrt{3}/2 \leq y \leq k + 1\}$ ,

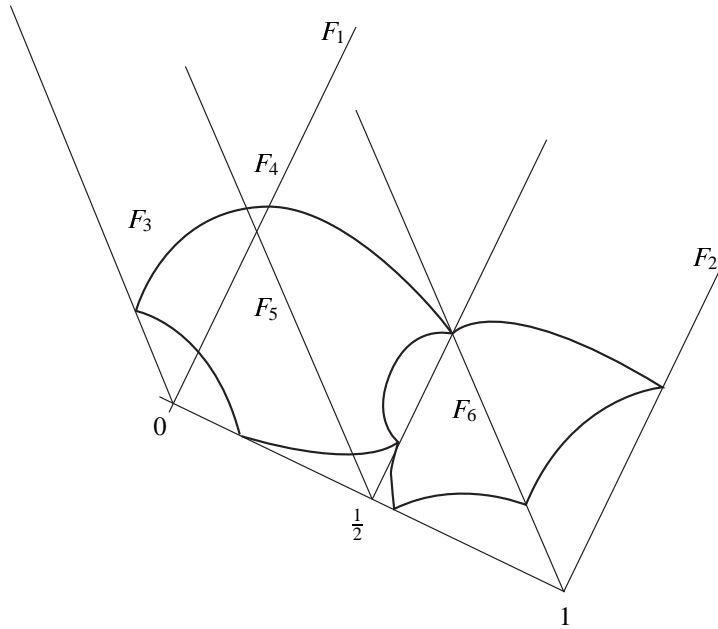


Fig. 4.2.  $(\frac{\sqrt{3}}{2} < k < 1)$

$$\begin{aligned}
 s_5 &= \{z \in \mathbf{C} \cap F_5 \mid \sqrt{1-k^2} \leq x \leq 1/2, 0 \leq y \leq k - \sqrt{3}/2\}, \\
 s_6 &= \{z \in \mathbf{C} \cap F_6 \mid 1/2 \leq x \leq 1 - \sqrt{1-k^2}, 0 \leq y \leq k - \sqrt{3}/2\}, \\
 s_7 &= \{z \in \mathbf{C} \mid \sqrt{1-k^2} \leq x \leq 1/2, y = 0\}, \\
 s_8 &= \{z \in \mathbf{C} \mid 1/2 \leq x \leq 1 - \sqrt{1-k^2}, y = 0\}.
 \end{aligned}$$

We denote by  $S_1$  (resp.  $S_2$ ) the polygons bounded by the curves  $s_1, s_2, s_3$  and  $s_4$  (resp.  $s_5, s_6, s_7$  and  $s_8$ ), and we denote by  $S_3$  (resp.  $S_4$ ) the mirror images of  $S_1$  (resp.  $S_2$ ) with respect to the real axis. Let  $p_j$  ( $j = 1, 2, \dots, 7$ ) be the vertices of the polygons  $S_1$  and  $S_2$  as follows:  $p_1 = s_1 \cap s_3, p_2 = s_2 \cap s_4, p_3 = s_3 \cap s_4, p_4 = s_5 \cap s_6, p_5 = s_5 \cap s_7, p_6 = s_6 \cap s_8$  and  $p_7 = s_7 \cap s_8$ . Since  $p_2 = A(p_1), p_1 = C_k(p_2), p_3 = C_k(p_3)$  (resp.  $p_4 = C_k(p_4), p_5 = C_k(p_6), p_6 = D_k(p_5)$  and  $p_7 = D_k(p_7)$ );  $s_2 = A(s_1), s_4 = C_k(s_3), s_5 = C_k(s_6), s_8 = D_k(s_8)$  and  $S_3 = D_k(S_1)$  (resp.  $S_4 = D_k(S_2)$ ) we have by Lemma 4.1 that  $\Omega(G_k)/G_k$  is a union of two Riemann surfaces with signature  $(0; 2, 3, \infty)$  for  $k = 1$  and that  $\Omega(G_k)/G_k$  is a union of two Riemann surfaces with signatures  $(0; 2, 3, n)$  and  $(0; 2, 3, \infty)$  for  $k = \cos(\pi/n)$  ( $n = 7, 8, \dots$ ).  $\square$

**4.3.** We will prove that  $G_{\theta,k}$  is not discrete in the case of  $\theta = 0$  and  $k$  with  $\cos(\pi/2m) < k < \cos(\pi/(2m+2))$  and  $k \neq \cos(\pi/(2m+1))$  ( $m = 3, 4, \dots$ ). We set

$$T_k = \left( \frac{1}{2\sqrt{1-k^2}} \right)^{1/2} \begin{pmatrix} 1 & -\sqrt{1-k^2} \\ 1 & \sqrt{1-k^2} \end{pmatrix} \quad (0 < k < 1).$$

Then we have

$$A^* := T_k A T_k^{-1} = \frac{1}{2\sqrt{1-k^2}} \begin{pmatrix} 2\sqrt{1-k^2} - 1 & 1 \\ -1 & 2\sqrt{1-k^2} + 1 \end{pmatrix} \quad (0 < k < 1),$$

$$B_k^* := T_k B_k T_k^{-1} = \begin{pmatrix} k + i\sqrt{1-k^2} & 0 \\ 0 & k - i\sqrt{1-k^2} \end{pmatrix} \quad (0 < k < 1).$$

Thus

$$(B_k^*)^n = \begin{pmatrix} (k + i\sqrt{1-k^2})^n & 0 \\ 0 & (k - i\sqrt{1-k^2})^n \end{pmatrix} \quad (0 < k < 1).$$

We set  $\cos \theta = k$ . Then  $e^{i\theta} = k + i\sqrt{1-k^2}$  and  $e^{-i\theta} = k - i\sqrt{1-k^2}$ .

Then we have

$$J(A, B_k^n) = J(A^*, (B_k^*)^n) = \left| \frac{-2 + 2 \cos 2n\theta}{4(\sin \theta)^2} \right|^2.$$

Thus  $0 \leq J(A, B_k^n) < 1$  if and only if  $\cos(\pi/2m) < k < \cos(\pi/(2m+2))$  ( $n = 2m+1, m = 3, 4, \dots$ ). We note that if  $k = \cos(\pi/(2m+1))$  ( $m = 3, 4, \dots$ ), then  $G_k = \langle A, B_k^{2m+1} \rangle$  is an elementary group. For the other cases  $G_k$  is a non-elementary group. By Theorem A, we can see that  $G_k$  is not a Kleinian group and not a Jørgensen group for  $k$  with  $\cos(\pi/2m) < k < \cos(\pi/(2m+2))$  and  $k \neq \cos(\pi/(2m+1))$  ( $m = 3, 4, \dots$ ). □

### 5. Proof of the case $\theta = \pi/4$

In this section we will prove the main theorem in the case of  $\theta = \pi/4$ . For  $\theta = \pi/4$ , we have

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B := B_{\pi/4, 1} = \begin{pmatrix} e^{\pi i/4} & i e^{\pi i/4}(-1+i) \\ -i e^{\pi i/4} & e^{\pi i/4} \end{pmatrix}.$$

We set  $G = \langle A, B \rangle$ .

**5.1.** We set  $S := ABAB^{-1}A^{-1}BA^{-1}B^{-1}ABAB^{-1}$ ,  
 $T := ABAB^{-1}A^{-1}BA^{-1}B^{-1}AB^{-1}A^{-1}B$  and  $U := ABAB^{-1}A^{-1}BA^{-1}B^{-1}A$ .

Then we have

$$S = \begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}, \quad T = \begin{pmatrix} -i & -2 \\ 0 & i \end{pmatrix},$$

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad SB = \begin{pmatrix} e^{-\pi i/4} & e^{-\pi i/4}(1+i) \\ -i e^{-\pi i/4} & e^{-\pi i/4} \end{pmatrix}.$$

We easily see the following lemma and so omit the proof.

**Lemma 5.1.** *Let  $A, B, S, T$  and  $U$  be the Möbius transformations represented by the above matrices. Then the following hold:*

- (i)  $A$  is a parabolic transformation with a fixed point  $\infty$ .
- (ii)  $S$  is an elliptic transformation of order two with fixed points  $i$  and  $\infty$ .
- (iii)  $T$  is an elliptic transformation of order two with fixed points  $-i$  and  $\infty$ .
- (iv)  $U$  is an elliptic transformation of order two with fixed points  $i$  and  $-i$ .
- (v)  $SA$  is an elliptic transformation of order two with fixed points  $(-1+2i)/2$  and  $\infty$ .
- (vi)  $TA$  is an elliptic transformation of order two with fixed points  $(-1 - 2i)/2$  and  $\infty$ .
- (vii)  $UA$  is an elliptic transformation of order three with fixed points  $(-1 + \sqrt{3}i)/2$  and  $(-1 - \sqrt{3}i)/2$ .
- (viii)  $U^{-1} \cdot (SB)^{-1} \cdot AB = I$ , where  $I$  is the identity mapping.
- (ix)  $T^{-1} \cdot (SB)^{-1} \cdot B = I$ .
- (x)  $S^{-1} \cdot SB \cdot B^{-1} = I$ .
- (xi)  $U \cdot SB \cdot A^{-1} \cdot B^{-1} = I$ .

We can easily see the following lemma.

**Lemma 5.2.** *Let  $A, B, S, T$  and  $U$  be the Möbius transformations represented by the above matrices. Let  $G$  be the group generated by  $A$  and  $B$ , and  $G^*$  be the group generated by  $A, S, T, U, B$  and  $SB$  :  $G = \langle A, B \rangle$  and  $G^* = \langle A, S, T, U, B, SB \rangle$ . Then  $G = G^*$ .*

**5.2.** We will prove (c) of the main theorem. The part is similar to 6.2 in [4]. Let  $\theta = \pi/4$  and  $k = 1$ . We denote by  $H^3$  and  $\bar{H}^3$  the upper half space and its closure, respectively:  $H^3 = \{(z, t) \in \mathbf{C} \times \mathbf{R} \mid t > 0\}$  and  $\bar{H}^3 = \{(z, t) \in \mathbf{C} \times \mathbf{R} \mid t \geq 0\}$ .

We will define some sides  $F_j$  ( $j = 1, 2, \dots, 12$ ) in  $\bar{H}^3$  as follows:

- $F_1 := \{(z, t) \in \bar{H}^3 \mid x = -1/2, -1 \leq y \leq 1, t \geq 0, |z|^2 + t^2 \geq 1, |z - i|^2 + t^2 \geq 1, |z + i|^2 + t^2 \geq 1\}$ ,
- $F_2 := \{(z, t) \in \bar{H}^3 \mid x = 1/2, -1 \leq y \leq 1, t \geq 0, |z|^2 + t^2 \geq 1, |z - i|^2 + t^2 \geq 1, |z + i|^2 + t^2 \geq 1\}$ ,
- $F_3 := \{(z, t) \in \bar{H}^3 \mid -1/2 \leq x \leq 0, y = 1, t \geq 0, |z - i|^2 + t^2 \geq 1\}$ ,
- $F_4 := \{(z, t) \in \bar{H}^3 \mid 0 \leq x \leq 1/2, y = 1, t \geq 0, |z - i|^2 + t^2 \geq 1\}$ ,
- $F_5 := \{(z, t) \in \bar{H}^3 \mid -1/2 \leq x \leq 0, y = -1, t \geq 0, |z + i|^2 + t^2 \geq 1\}$ ,
- $F_6 := \{(z, t) \in \bar{H}^3 \mid 0 \leq x \leq 1/2, y = -1, t \geq 0, |z + i|^2 + t^2 \geq 1\}$ ,
- $F_7 := \{(z, t) \in \bar{H}^3 \mid -1/2 \leq x \leq 0, -1/2 \leq y \leq 1/2, t \geq 0, |z|^2 + t^2 = 1\}$ ,
- $F_8 := \{(z, t) \in \bar{H}^3 \mid 0 \leq x \leq 1/2, -1/2 \leq y \leq 1/2, t \geq 0, |z|^2 + t^2 = 1\}$ ,
- $F_9 := \{(z, t) \in \bar{H}^3 \mid 0 \leq x \leq 1/2, -1 \leq y \leq -1/2, t \geq 0, |z + i|^2 + t^2 = 1\}$ ,
- $F_{10} := \{(z, t) \in \bar{H}^3 \mid -1/2 \leq x \leq 0, 1/2 \leq y \leq 1, t \geq 0, |z - i|^2 + t^2 = 1\}$ ,

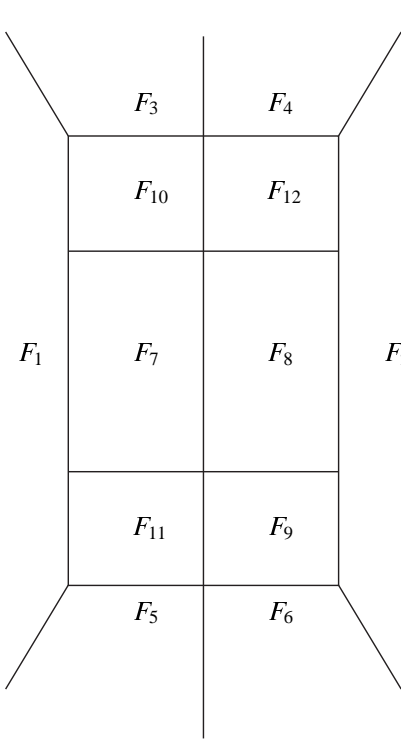


Fig. 5.1.

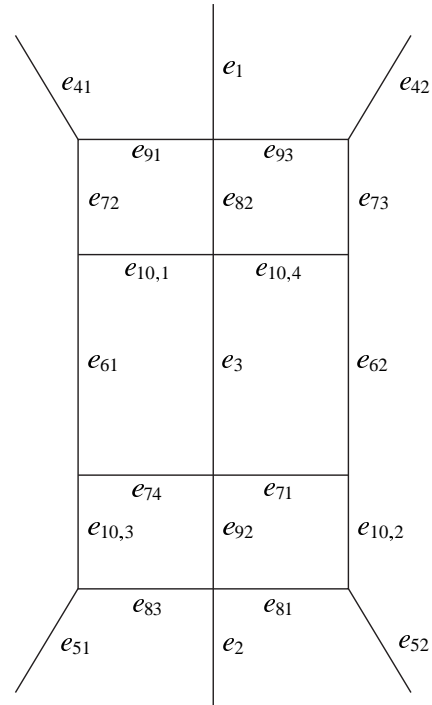


Fig. 5.2.

$F_{11} := \{(z, t) \in \bar{H}^3 \mid -1/2 \leq x \leq 0, -1 \leq y \leq -1/2, t \geq 0, |z + i|^2 + t^2 = 1\}$ ,  
 $F_{12} := \{(z, t) \in \bar{H}^3 \mid 0 \leq x \leq 1/2, 1/2 \leq y \leq 1, t \geq 0, |z - i|^2 + t^2 = 1\}$ ,  
 where  $z = x + iy$ , that is,  $x = \text{Re}(z)$  and  $y = \text{Im}(z)$ .

Let  $\mathbf{P}$  be the polyhedron in  $H^3$  bounded by the above twelve sides  $F_j$  ( $j = 1, 2, \dots, 12$ ). Then there are six side pairing transformations  $A, S, T, U, B$  and  $SB$  of  $\mathbf{P}$  as follows:  $F_2 = A(F_1)$ ,  $F_3 = S(F_4)$ ,  $F_5 = T(F_6)$ ,  $F_7 = U(F_8)$ ,  $F_{10} = B(F_9)$ ,  $F_{12} = SB(F_{11})$  (see Fig. 5.1 and Fig. 5.3).

Next we denote the edges  $e_j$  and  $e_{jk}$  ( $j = 1, 2, \dots, 10$ ;  $k = 1, 2, 3, 4$ ) of  $\mathbf{P}$  as follows:

$e_1 = F_3 \cap F_4$ ;  $e_2 = F_5 \cap F_6$ ;  $e_3 = F_7 \cap F_8$ ;  $e_{41} = F_1 \cap F_3$ ,  $e_{42} = F_2 \cap F_4$ ;  $e_{51} = F_1 \cap F_5$ ,  
 $e_{52} = F_2 \cap F_6$ ;  $e_{61} = F_1 \cap F_7$ ,  $e_{62} = F_2 \cap F_8$ ;  $e_{71} = F_8 \cap F_9$ ,  $e_{72} = F_1 \cap F_{10}$ ,  $e_{73} = F_2 \cap F_{12}$ ,  
 $e_{74} = F_7 \cap F_{11}$ ;  $e_{81} = F_6 \cap F_9$ ,  $e_{82} = F_{10} \cap F_{12}$ ,  $e_{83} = F_5 \cap F_{11}$ ;  $e_{91} = F_3 \cap F_{10}$ ,  $e_{92} = F_9 \cap F_{11}$ ,  
 $e_{93} = F_4 \cap F_{12}$ ;  $e_{10,1} = F_7 \cap F_{10}$ ,  $e_{10,2} = F_2 \cap F_9$ ,  $e_{10,3} = F_1 \cap F_{11}$ ,  $e_{10,4} = F_8 \cap F_{12}$  (see Fig. 5.2).

Let  $\theta_j = \alpha(e_j)$  ( $j = 1, 2, \dots, 8$ ) be the angles measured from inside  $\mathbf{P}$  at  $e_j$ . Then it is easily seen that  $\theta_1 = \pi$ ,  $\theta_2 = \pi$ ,  $\theta_3 = \pi$ ,  $\theta_{41} = \theta_{42} = \pi/2$ ,  $\theta_{51} = \theta_{52} = \pi/2$ ,  $\theta_{61} = \theta_{62} = \pi/3$ ,  $\theta_{71} = \theta_{74} = 2\pi/3$ ,  $\theta_{72} = \theta_{73} = \pi/3$ ,  $\theta_{81} = \theta_{83} = \pi/2$ ,  $\theta_{82} = \pi$ ,

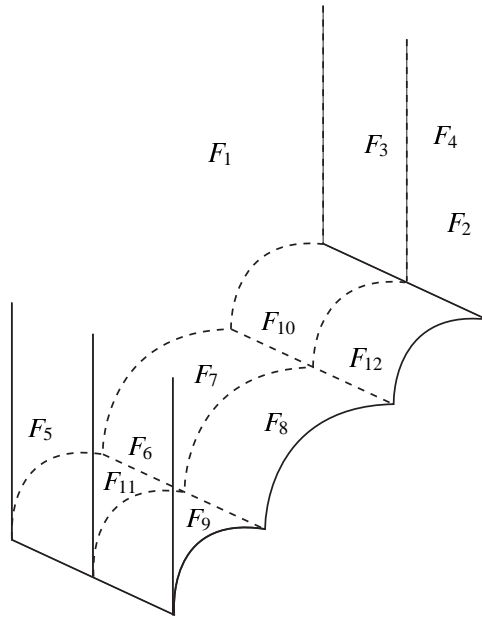


Fig. 5.3.

$\theta_{91} = \theta_{93} = \pi/2$ ,  $\theta_{92} = \pi$ ,  $\theta_{10,1} = \theta_{10,4} = 2\pi/3$ ,  $\theta_{10,2} = \theta_{10,3} = \pi/3$ . Thus we have the following:  $\theta_1 = 2\pi/2$ ,  $\theta_2 = 2\pi/2$ ,  $\theta_3 = 2\pi/2$ ,  $\theta_{41} + \theta_{42} = 2\pi/2$ ,  $\theta_{51} + \theta_{52} = 2\pi/2$ ,  $\theta_{61} + \theta_{62} = 2\pi/3$ ,  $\theta_{71} + \theta_{72} + \theta_{73} + \theta_{74} = 2\pi/1$ ,  $\theta_{81} + \theta_{82} + \theta_{83} = 2\pi/1$ ,  $\theta_{91} + \theta_{92} + \theta_{93} = 2\pi/1$  and  $\theta_{10,1} + \theta_{10,2} + \theta_{10,3} + \theta_{10,4} = 2\pi/1$ .

By Lemma 5.1 we can see the following. Since  $e_1 = S(e_1)$  (resp.  $e_2 = T(e_2)$  and  $e_3 = U(e_3)$ ), we have that  $S$  (resp.  $T$  and  $U$ ) are the cycle transformations at  $e_1$  (resp.  $e_2$  and  $e_3$ ) and that all of the cycle transformations are elliptic of order two. Since  $e_{42} = A(e_{41})$  and  $e_{41} = S(e_{42})$  (resp.  $e_{52} = A(e_{51})$  and  $e_{51} = T(e_{52})$ ;  $e_{62} = A(e_{61})$  and  $e_{61} = U(e_{62})$ ) we have that  $SA$  (resp.  $TA$  and  $UA$ ) are the cycle transformations at  $e_{41}$  (resp.  $e_{51}$  and  $e_{61}$ ) and that the cycle transformations are elliptic of order two (resp. two and three). Since  $e_{72} = B(e_{71})$ ,  $e_{73} = A(e_{72})$ ,  $e_{74} = (SB)^{-1}(e_{73})$  and  $e_{71} = U^{-1}(e_{74})$  (resp.  $e_{82} = B(e_{81})$ ,  $e_{83} = (SB)^{-1}(e_{82})$  and  $e_{81} = T^{-1}(e_{83})$ ;  $e_{92} = B^{-1}(e_{91})$ ,  $e_{93} = (SB)(e_{92})$  and  $e_{91} = S^{-1}(e_{93})$ ;  $e_{10,2} = B^{-1}(e_{10,1})$ ,  $e_{10,3} = A^{-1}(e_{10,2})$ ,  $e_{10,4} = (SB)(e_{10,3})$  and  $e_{10,1} = U(e_{10,4})$ ) we have that  $U^{-1} \cdot (SB)^{-1} \cdot A \cdot B$  (resp.  $T^{-1} \cdot (SB)^{-1} \cdot B$ ,  $S^{-1} \cdot SB \cdot B^{-1}$ ,  $U \cdot SB \cdot A^{-1} \cdot B^{-1}$ ) are the cycle transformations at  $e_{71}$  (resp.  $e_{81}$ ,  $e_{91}$  and  $e_{10,1}$ ) and that all cycle transformations are the identity mapping.

By the above we see that the conditions (1) through (5) in Theorem E are satisfied. The completeness condition (6) in Theorem E is easily shown by using Ratcliffe [7, Theorem 11.1.2]. Thus  $G^*$  is a Kleinian group of the first kind. By Lemma 5.2,  $G$  is a Kleinian group of the first kind and so a Jørgensen group.

We can easily calculate the volume of the polyhedron  $\mathbf{P}$  constructed above, and so

we omit the calculations. □

**5.3.** We will prove that  $G_{\theta,k}$  is not discrete in the case of  $\theta = \pi/4$  and  $\sqrt{3}/2 < k < 1$ . We have

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_k := B_{\pi/4,k} = \begin{pmatrix} ke^{\pi i/4} & i(k^2 e^{\pi i/4} - e^{-\pi i/4}) \\ -ie^{\pi i/4} & ke^{\pi i/4} \end{pmatrix} \quad (k \in \mathbf{R}).$$

We set  $G_k = \langle A, B_k \rangle$ .

We set  $P_k := B_k^{-1} A B_k B_k A B_k^{-1}$ . Then we have  $J(A, P_k) = |2k - 2|^2$ . Thus  $0 \leq J(A, P_k) < 1$  if and only if  $1/2 < k < 3/2$ . We easily see that the group  $G_k$  is an elementary group for  $k$  with  $k = 1$  and  $G_k$  is a non-elementary group for  $k$  with  $1/2 < k < 3/2$  ( $k \neq 1$ ). By Theorem A we can see that  $G_k$  is a not discrete group for  $k$  with  $\sqrt{3}/2 < k < 1$ . □

**6. Proof of the case of  $\theta = \pi/6$**

In this section we will prove that  $G_{\theta,k}$  is not discrete in the case of  $\theta = \pi/6$ . We have

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_k := B_{\pi/6,k} = \begin{pmatrix} ke^{\pi i/6} & i(k^2 e^{\pi i/6} - e^{-\pi i/6}) \\ -ie^{\pi i/6} & ke^{\pi i/6} \end{pmatrix} \quad (k \in \mathbf{R}).$$

We set  $G_k = \langle A, B_k \rangle$ .

We set  $E_k := A B_k^{-1} A^{-1} B_k A^{-1} B_k^{-1} A B_k A^{-1} B_k A B_k^{-1}$ . Then we have  $J(A, E_k) = |-1/2 - i(3\sqrt{3}/2 - 2k)|^2$ . Thus  $0 \leq J(A, E_k) < 1$  if and only if  $\sqrt{3}/2 < k < \sqrt{3}$ .

We note that  $G_k$  is a non-elementary group for  $k$  with  $\sqrt{3}/2 < k \leq 1$ . By Theorem A we can see that  $G_k$  is not a discrete group for  $k$  with  $\sqrt{3}/2 < k \leq 1$ . □

**7. Proof of the case  $\theta = \pi/3$**

In this section we will prove that  $G_{\theta,k}$  is not discrete in the case of  $\theta = \pi/3$ . We have

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_k := B_{\pi/3,k} = \begin{pmatrix} ke^{\pi i/3} & i(k^2 e^{\pi i/3} - e^{-\pi i/3}) \\ -ie^{\pi i/3} & ke^{\pi i/3} \end{pmatrix} \quad (k \in \mathbf{R}).$$

We set  $G_k = \langle A, B_k \rangle$ .

We set  $F_k := A B_k A^{-1} B_k^{-1} \cdot A B_k A B_k^{-1} \cdot A B_k^{-1} A^{-1} B_k$ . Then we have  $J(A, F_k) = |(\sqrt{3}k - 2) + i(k - \sqrt{3})|^2$ . Thus  $0 \leq J(A, F_k) < 1$  if and only if  $\sqrt{3}/2 < k < \sqrt{3}$ .

We note that  $G_k$  is a non-elementary group for  $k$  with  $\sqrt{3}/2 < k \leq 1$ . By Theorem A we can see that  $G_k$  is not a discrete group for  $k$  with  $\sqrt{3}/2 < k \leq 1$ . □

**8. Proof of the case  $\theta = \pi/2$**

The main theorem (d) and (e) are proved by Sato-Yamada [9]. In this section we will prove that  $G_{\theta,k}$  is not discrete in the case of  $\theta = \pi/2$ . For  $\theta = \pi/2$ , we have

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_k := B_{\pi/2,k} = \begin{pmatrix} ik & -(k^2 + 1) \\ 1 & ik \end{pmatrix} \quad (k \in \mathbf{R}).$$

We set  $G_k = \langle A, B_k \rangle$ .

We set  $S_k := AB_k^{-1}AB_k$ ,  $Q_k := AB_kAB_k^{-1}$ ,  $V_k := A^{-1}S_k^{-1}A^{-1}Q_k^{-1}$ . We can easily calculate  $V_k$ :

$$V_k = \begin{pmatrix} 2k^2 - 1 & 2ik - 2ik^3 \\ 2ik & 2k^2 - 1 \end{pmatrix}.$$

We set

$$T_k = \left( \frac{1}{2\sqrt{1-k^2}} \right)^{1/2} \begin{pmatrix} 1 & -\sqrt{1-k^2} \\ 1 & \sqrt{1-k^2} \end{pmatrix} \quad (0 < k < 1).$$

Then for  $0 < k < 1$  we have

$$A^* := T_kAT_k^{-1} = \frac{1}{2\sqrt{1-k^2}} \begin{pmatrix} 2\sqrt{1-k^2} - 1 & 1 \\ -1 & 2\sqrt{1-k^2} + 1 \end{pmatrix}$$

and

$$V_k^* := T_kV_kT_k^{-1} = \begin{pmatrix} (2k^2 - 1) - 2ik\sqrt{1-k^2} - 1 & 0 \\ 0 & (2k^2 - 1) + 2ik\sqrt{1-k^2} - 1 \end{pmatrix}.$$

We set  $\cos \theta = k$ . Then  $e^{2i\theta} = (2k^2 - 1) + 2ik\sqrt{1 - k^2}$  and  $e^{-2i\theta} = (2k^2 - 1) - 2ik\sqrt{1 - k^2}$ .

Thus we have  $J(A, V_k^n) = J(A^*, (V_k^*)^n) = |(-1 + \cos 4n\theta)/(1 - \cos 2\theta)|^2$ . We have that  $0 \leq J(A, V_k^n) < 1$  if and only if  $\cos(\pi/(2n - 1)) < k < \cos(\pi/(2n + 1))$  ( $k \neq \cos \pi/2n$ ) ( $n = 3, 4, \dots$ ).

We note that if  $k = \cos(\pi/2n)$  ( $n = 3, 4, \dots$ ), then  $G_k = \langle A, V_k^n \rangle$  is an elementary group.  $G_k$  is a non-elementary group for  $k$  with  $\cos(\pi/(2n - 1)) < k < \cos(\pi/(2n + 1))$  and  $k \neq \cos(\pi/2n)$  ( $n = 3, 4, \dots$ ). By Theorem A, we can see that  $G_k$  is not a discrete group for  $k$  with  $\cos(\pi/(2n - 1)) < k < \cos(\pi/(2n + 1))$  and  $k \neq \cos(\pi/2n)$  ( $n = 3, 4, \dots$ ). □

**9. Proof of the other cases**

In this section we will see that  $G_{\theta,k}$  is not discrete in the case where  $0 < \theta < \pi/6$ ,  $\pi/6 < \theta < \pi/4$ ,  $\pi/4 < \theta < \pi/3$ ,  $\pi/3 < \theta < \pi/2$ , and  $\sqrt{3}/2 < k \leq 1$ . The part follows from the following proposition.



**Proposition** (Li-Oichi-Sato [4]). *Let  $G_{\theta,k} = \langle A, B_{\theta,k} \rangle$  be the group generated by  $A$  and  $B_{\theta,k}$ . If  $0 < \theta < \pi/6$ ,  $\pi/6 < \theta < \pi/4$ ,  $\pi/4 < \theta < \pi/3$ ,  $\pi/3 < \theta < \pi/2$ , then  $G_{\theta,k} = \langle A, B_{\theta,k} \rangle$  is not a discrete group for  $k \in \mathbf{R}$ .  $\square$*

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### References

- [1] T. Jørgensen: *On discrete groups of Möbius transformations*, Amer. J. Math. **98** (1976), 739–749.
- [2] T. Jørgensen and M. Kiikka: *Some extreme discrete groups*, Ann. Acad. Sci. Fenn. **1** (1975), 245–248.
- [3] T. Jørgensen, A. Lascurain and T. Pignataro: *Translation extensions of the classical modular group*, Complex Variables **19** (1992), 205–209.
- [4] C. Li, M. Oichi and H. Sato: *Jørgensen groups of parabolic type I (Finite case)*, preprint.
- [5] C. Li, M. Oichi and H. Sato: *Jørgensen groups of parabolic type III (Uncountably infinite case)*, preprint.
- [6] B. Maskit: *Kleinian Groups*, Springer-Verlag, New York-Berlin-Heidelberg, 1987.
- [7] J.G. Ratcliffe: *Foundations of Hyperbolic Manifolds*, Springer-Verlag, Berlin-Heidelberg-New York, 1994.
- [8] H. Sato: *One-parameter families of extreme groups for Jørgensen's inequality*, Contemporary Math. **256** (The First Ahlfors-Bers Colloquium) edited by I. Kra and B. Maskit, 2000, 271–287.
- [9] H. Sato and R. Yamada: *Some extreme Kleinian groups for Jørgensen's inequality*, Rep. Fac. Sci. Shizuoka Univ. **27** (1993), 1–8.

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