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<td><strong>Author(s)</strong></td>
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<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 18(1) P.257–P.288</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1981</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/5189">https://doi.org/10.18910/5189</a></td>
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<td><strong>DOI</strong></td>
<td>10.18910/5189</td>
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Osaka University
ON THE FUNDAMENTAL SOLUTION FOR
A DEGENERATE HYPERBOLIC SYSTEM

KENZO SHINKAI

(Received November 2, 1979)

Introduction. Let $S[m_1, m_2]$ denote the set of all $C^\infty$-symbols $a(t, x, \xi)$ on $[0, T] \times R_+ \times R$ ($0 < T \leq 1$) such that

\begin{equation}
|D_i^j D_\xi^j D_\xi^j a(t, x, \xi)| \leq C_{j, a, \beta} \langle \xi \rangle^{m_1 - \alpha_l (t + \langle \xi \rangle^{-\alpha}) m_2 - j}
\end{equation}

for constants $C_{j, a, \beta}$, where $\langle \xi \rangle = (1 + |\xi|^2)^{l/2}$ and $\omega = 1/(l+1)$ with an integer $l > 0$.

Consider a hyperbolic operator of first order:

\begin{equation}
L = D_t - t' \left[ \begin{array}{cccc}
\mu_1(t, X, D_x) & 0 \\
0 & \ddots \\
& & \mu_m(t, X, D_x)
\end{array} \right] + B(t),
\end{equation}

where $\mu_j, j = 1, \ldots, m$ are real valued and satisfy

\begin{equation}
\left\{ \begin{array}{l}
i) \quad \mu_j(t, x, \xi) \in S[1, 0] \\
ii) \quad |\mu_j(t, x, \xi) - \mu_k(t, x, \xi)| \leq \epsilon \langle \xi \rangle \quad (j \neq k)
\end{array} \right.
\end{equation}

for a constant $\epsilon > 0$, and the symbol $\sigma(B(t))(x, \xi)$ of the lower order operator $B(t)$ satisfies

\begin{equation}
\sigma(B(t))(x, \xi) \in S[0, -1].
\end{equation}

The purpose of the present paper is to construct the fundamental solution $E(t, s) (0 \leq s \leq t \leq T_0)$ of the Cauchy problem

\begin{equation}
\left\{ \begin{array}{l}
LU = \Phi(t) \quad \text{on} \quad [s, T_0], \\
U|_{t=s} = \Psi
\end{array} \right.
\end{equation}

for a small constant $T_0 (0 < T_0 \leq T)$. It should be noted that the operator $L$ is degenerate at $t=0$ and $B(t)$ is not uniformly bounded on $[0, T]$ as a family of pseudo-differential operators with parameter $t \in [0, T]$.

To construct $E(t, s)$, we find first the perfect diagonalizer $N(t)$ such that the symbol $\sigma(N(t))(x, \xi)$ belongs to $S[0, 0]$ and
(0.6) \[ LN(t) \equiv N(t)L_1 \mod \mathcal{B}_i(S^{-\omega}) , \]

where \( L_1 \) is an operator of the form

\[
L_1 = \begin{bmatrix}
\mu_1(t, X, D_x) & 0 \\
0 & \mu_m(t, X, D_x)
\end{bmatrix} + \begin{bmatrix}
f_1(t, X, D_x) & 0 \\
0 & f_m(t, X, D_x)
\end{bmatrix} + R(t)
\]

such that \( f_j(t, x, \xi) \in S[0, -1] \) and \( \sigma(R(t))(x, \xi) \in \mathcal{S} = \bigcup_{\omega} S[\omega - \nu \omega, -\nu] \).

Then, for \( L_1 \) we can construct the fundamental solution \( E_1(t, s) \), and, by using \( E_1(t, s) \), the fundamental solution \( E(t, s) \) for \( L \) can be found in the form

\[
E(t, s) = N(t)E_1(t, s)N^t(s) + R_{-\omega}(t, s) ,
\]

where \( N^t(s) \) is a parametrix of \( N(s) \) and \( \sigma(R_{-\omega}(t, s))(x, \xi) \in \mathcal{B}_i(S^{-\omega}) \).

We note that \( E(t, s) \) is represented as the sum of Fourier integral operators which have phase functions \( \phi_j(t, s, x, \xi) \) defined as the solutions of eiconal equations:

\[
\begin{align*}
\partial_t \phi_j - t^1 \mu_j(t, X, V, \phi, \xi) &= 0 \quad (0 \leq s \leq t \leq T_0) , \\
\phi_j(s, s) &= x \cdot \xi ,
\end{align*}
\]

and have symbols in \( \bigcap_{0 \leq \epsilon < 1} S[0, M + \epsilon, -M - \epsilon] \). The constant \( M \) is defined by

\[
M = \max \limsup_{1 \leq t \leq m \to \infty} \sup_{x, \xi, \omega, \varrho, \rho \geq R} \{ t \mathcal{A}_1 f_j(t, x, \xi) \} ,
\]

and indicates the order of regularity-loss of the solution of the Cauchy problem.

Concerning the problem (0.5) Kumano-go [7] constructed the fundamental solution without condition (0.3) ii) by using Fourier integral operators of multi-phase. It should be emphasized that our fundamental solution \( E(t, s) \) is represented by Fourier integral operators of single phase, and \( M \) is determined explicitly by (0.10). The perfect diagonalization of (0.6) for \( L \) enable us such a construction of \( E(t, s) \).

In §1 we define some classes of pseudo-differential operators and Fourier integral operators as variants of classes in Boutet de Monvel [2], and summarize fundamental theorems on operators of these classes. In §2, using a similar method to that of Kumano-go [6], we construct the perfect diagonalizer \( N(t) \) such
that (0.6) holds. We note that \( \sigma(R(t))(x, \xi) \in \mathcal{A}^n \) and \( \in S^{-m} \) for any fixed \( t > 0 \), but that \( \mathcal{A}^n \subset \mathcal{B}_l(S^{-m}) \) on \([0, T]\). So we can not apply the method in Kumano-go [6] directly. From this reason, in §3, we first treat a single operator and then construct the fundamental solution \( E_2(t, s) \) for a purely diagonal operator \( L_2 = L_0 - R(t) \). In §4 the fundamental solution \( E_1(t, s) \) for \( L_1 \) is constructed in the form

\[
E_1(t, s) = E_2(t, s)(I + Q(t, s)) + Q_\infty(t, s),
\]

and by using \( E_1(t, s) \) the fundamental solution \( E(t, s) \) for the general \( L \) can be constructed. The crucial point in the discussions of §4 is in finding \( Q(t, s) \). Finally in §5 we consider a higher order operator \( L \) of the form:

\[
L = D_t^n + \sum_{k=1}^n a_k(t, X, D_x)D_x^{n-k},
\]

where \( a_k(t, x, \xi) \) have the forms

\[
a_k(t, x, \xi) = \sum_{j=0}^k \sigma^{(j,k)}a_{k,j}(t, x, \xi)
\]

with differential polynomials \( a_{k,j}(t, x, \xi) \) of order \( k-j \) in \( \xi \) and \( \sigma(j, k) = \max \{0, (k-j)(l+1) - k\} \). We assume that the roots \( \mu_1, \ldots, \mu_m \) of the equation

\[
\lambda^m + a_1 \lambda^{m-1} + \cdots + a_m = 0
\]

are real and satisfy (0.3). Then, we show that the Cauchy problem:

\[
\begin{cases}
    Lu = \varphi(t) & \text{on } [s, T_0] \\
    D_j |_{t=s} = \psi_j, & j=0, \ldots, m-1
\end{cases}
\]

is reduced to the system (0.5) by modifying the method in Shinaki [11]. We note that the operator \( L \) of this type is a generalization of operators which have been treated by Alinhac [1], Chi Min-You [3], Nakamura [8], Nakamura and Uryu [9], Oleinik [10], Uryu [13] and Yoshikawa [14], [15].

The author would like to express his gratitude to Professor H. Kumano-go, Professor M. Ikawa and Mr. K. Taniguchi for their kind suggestions and a number of stimulating conversations.

1. Preliminaries. For \( x \in \mathbb{R}^n_x, \xi \in \mathbb{R}^n_\xi \) and multi-indices \( \alpha, \beta \) we use the following notation:

\[
\begin{align*}
    x \cdot \xi &= x_1 \xi_1 + \cdots + x_n \xi_n, & \langle \xi \rangle &= (1 + |\xi|^2)^{1/2}, \\
    |\alpha| &= \alpha_1 + \cdots + \alpha_n, & \alpha! &= \alpha_1! \cdots \alpha_n!, \\
    d\xi &= (2\pi)^{-n} d\xi, & D_t &= -i\partial / \partial t, & \partial_{\xi_j} &= \partial / \partial \xi_j, \\
    D_{\xi j} &= -i \partial / \partial x_j.
\end{align*}
\]
\[ \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}, \quad D_x^\alpha = D_{x_1}^\alpha \cdots D_{x_n}^\alpha, \]
\[ a_{\alpha}(x, \xi) = \partial_\xi^\alpha \partial_x^\alpha a(x, \xi), \quad \nabla_x f(x) = (\partial_1 f(x), \cdots, \partial_n f(x)). \]

Let \( S^\gamma(=S^\gamma_{1,0}) \) denote Hörmander's class of symbols \( a(x, \xi) \) on \( \mathbb{R}^n \) which satisfy
\[ |a_{\alpha}(x, \xi)| \leq C_{\alpha} \rho^\gamma \langle \xi \rangle^{\gamma+|\alpha|} \quad \text{on} \quad \mathbb{R}^n_x \times \mathbb{R}^n_\xi, \]
and the associated pseudo-differential operators \( a(x, D_x) \) are defined by
\[ a(X, D_x)u(x) = \int e^{-iy \cdot x} a(y, \xi) u(x+y) d\xi dy \]
\[ = \lim_{t \to 0} \int e^{-iy \cdot x} \chi(\xi) a(x, \xi) u(x+y) d\xi dy \]
\[ (u \in \mathcal{B}(\mathbb{R}^n)), \]
where \( \chi(\xi) \in \mathcal{S} \) (the Schwartz space of rapidly decreasing functions on \( \mathbb{R}^{2n} \)) such that \( \chi(0, 0) = 1 \) and \( \mathcal{B}(\mathbb{R}^n) \) denotes the space of \( C^\infty \)-functions in \( \mathbb{R}^n \) whose derivatives of any order are all bounded.

Let \( \chi(t) \) be a \( C^\infty \)-function in \( \mathbb{R}^l \) such that
\[ \begin{cases} 0 \leq \chi(t) \leq 1 & \text{on} \quad \mathbb{R}^l, \\ \chi(t) = 1 & (|t| \leq 1), \\ = 0 & (|t| \geq 2). \end{cases} \]

Set \( \omega = 1/(l+1) \) for a positive integer \( l \) and define a function \( \gamma \) by
\[ \gamma(t) = \gamma(t, \xi) = t + \langle \xi \rangle^{-\omega} \chi(t + \langle \xi \rangle^\omega). \]
Then we have
\[ \begin{cases} (t + \langle \xi \rangle^{-\omega})/2 \leq \gamma(t, \xi) \leq t + \langle \xi \rangle^{-\omega}, \\ \langle \xi \rangle^{-\omega} \leq \gamma(t, \xi) \leq 2 & (0 \leq t \leq T \leq 1). \end{cases} \]
and by easy calculation
\[ \left| D_\xi^\alpha \gamma(t, \xi) \right| \leq C_{j,\alpha} \langle \xi \rangle^{\gamma + |\alpha|} \gamma(t, \xi)^{1-j}. \]


**Definition 1.1.**
\begin{enumerate}
\item For real \( m_1, m_2 \) we denote by \( S[m_1, m_2] \) the space of all \( C^m \)-symbols \( a(t, x, \xi) \) on \([0, T] \times \mathbb{R}_x^* \times \mathbb{R}_\xi^* \) \((0 \leq T \leq 1)\) such that for any non-negative integer \( j \) and multi-indices \( \alpha, \beta \) we have
\[ |D_\xi^\alpha a_{\beta}(t, x, \xi)| \leq C_{j,\alpha} \langle \xi \rangle^{m_1 + |\alpha|} \gamma(t, \xi)^{m_2 - j}. \]
\item For real \( m_1, m_2, m_3 \) we denote by \( S[m_1, m_2, m_3] \) the space of all \( C^m \)-
symbols \( a(t, s, x, \xi) \) on \([0, T] \times [0, T] \times R_x^+ \times R_x^+ \) \((0 \leq T \leq 1)\) such that for any non-negative integers \( j, k \) and multi-indices \( \alpha, \beta \) we have

\[
|D_t D_s a_{(s)}(t, s, x, \xi)| \leq C_{j,k,\alpha,\beta} \langle \xi \rangle^{m_z^{-1}j} \eta(t, \xi)^{m_z^{-1}j} \eta(s, \xi)^{m_z^{-1}k}.
\]

iii) We set

\[
\mathcal{B}_i(S^{-\nu}) = \bigcup_{\nu} S[m_1 - \nu, m_2],
\]

\[
\mathcal{B}_{i,s}(S^{-\nu}) = \bigcup_{\nu} S[m_1 - \nu, m_2, m_3],
\]

\[
\mathcal{B}^\nu = \bigcup_{\nu} S[m - \nu, -\nu(l+1)].
\]

**Remark.** 1°. From (1.5) and (1.6) we have

\[
\eta(t, \xi)^{\nu} \in S[0, \nu] \quad \text{for real } \nu
\]

and

\[
a(t, x, \xi) \in S[m_1, m_2] \Leftrightarrow a(t, s, x, \xi) \in S[m - m_z^{-1}, m_2] \quad \text{for any fixed } t \text{ and } s \in [0, T].
\]

2°. We can consider \( a(t, x, \xi) \in S[m_1, m_2] \) as an element of \( S[m_1, m_2, 0] \). So by this identification we write \( S[m_1, m_2] \subset S[m_1, m_2, 0] \), and the statements for the symbols of \( S[m_1, m_2, m_3] \) often hold for symbols of \( S[m_1, m_2] \).

3°. It is easily proved that

\[
\bigcup_{\nu} S[m_1 - \nu, m_2] = \bigcup_{\nu} \mathcal{B}_i(S^{-\nu})
\]

and

\[
\bigcup_{\nu} S[m_1 - \nu, m_2, m_3] = \bigcup_{\nu} \mathcal{B}_{i,s}(S^{-\nu}).
\]

**Proposition 1.2.** i) \( S[m_1, m_2] \cap S[m'_1, m'_2], \) if \( m_1 \leq m'_1 \) and \( m_1 - m_z \omega \leq m'_1 - m_z \omega \).

ii) \( S[m_1, m_2, m_3] \subset S[m'_1, m'_2, m'_3] \) if \( m_1 \leq m'_1, m_1 - m_z \omega \leq m'_1 - m_z \omega \) and \( m_3 \geq m_3' \) or \( m_1 \leq m'_1, m_1 - m_z \omega \leq m'_1 - m_z \omega \) and \( m_2 \geq m_3' \).

Proof is omitted.

**Proposition 1.3.** i) Let \( a(t, s, x, \xi) \in S[m_1, m_2, m_3] \). Then, for any non-negative integers \( j, k \) we have

\[
t^{j+k} a(t, s, x, \xi) \in S[m_1, m_2 + j, m_3 + k]
\]

and
(1.12) \[ D_{\xi}^j D_s a(t, s, x, \xi) \in \mathcal{S}[m_1 - j, m_3 - k]. \]

ii) Let \( a(t, s, x, \xi) \in \mathcal{S}[m_1, m_2, m_3] \) and \( b(t, s, x, \xi) \in \mathcal{S}[m'_1, m'_2, m'_3]. \) Then, we have

(1.13) \[ a(t, s, x, \xi) b(t, s, x, \xi) \in \mathcal{S}[m_1 + m'_1, m_2 + m'_2, m_3 + m'_3]. \]

iii) Let \( a(t, x, \xi) \in \mathcal{H}^m \) and \( b(t, x, \xi) \in \mathcal{S}[m_1, m_2]. \) Then, we have

(1.14) \[ a(t, x, \xi) b(t, x, \xi) \in \mathcal{H}^{m + m_1 - m_2}. \]

Proof. i) and ii) are clear. Writing \( \gamma(t, \xi) = \langle \xi \rangle^{-m_2} \langle \xi \rangle^m \eta(t, \xi) \rangle^m \), we get (1.14).

**Lemma 1.4.** Set

(1.15) \[ h(t, \xi) = \gamma(t, \xi) \langle \xi \rangle. \]

Then, we have

i) \( h(t, \xi) \in \mathcal{S}[\nu, \nu'] \) for any real \( \nu, \nu' \),

ii) \( h(t, \xi) - t \langle \xi \rangle \in \mathcal{H}^m \),

iii) \( ih(t, \xi) h(t, \xi) - l/\gamma(t, \xi) \in \mathcal{H}^m \),

where \( h(t, \xi) = D_t h(t, \xi) \).

Proof. i) is clear. Since \( I(t, \xi) = h(t, \xi) - t \langle \xi \rangle \in \mathcal{H}^m \) when \( t \langle \xi \rangle \geq 2 \), \( \gamma(t, \xi) \langle \xi \rangle^m \) is bounded on \( \text{supp } I(t, \xi) \). So we have ii). Since \( i \gamma(t, \xi) = 1 + i \chi_d(t \langle \xi \rangle^m) \) and \( \chi_d(t \langle \xi \rangle^m) \in \mathcal{H}^0 \), we have by Proposition 1.3-iii)

\[
\begin{align*}
  ih(t, \xi) h(t, \xi) - l/\gamma(t, \xi) &= i \chi_d(t \langle \xi \rangle^m) / \gamma(t, \xi) \\
  &= i \chi_d(t \langle \xi \rangle^m) / \gamma(t, \xi) \in \mathcal{H}^m.
\end{align*}
\]

**Proposition 1.5.** i) Let \( a_v(t, s, x, \xi) \in \mathcal{S}[m_1 - v, m_2, m_3] \) for \( v = 0, 1, \ldots \). Then, there exists an \( a(t, s, x, \xi) \in \mathcal{S}[m_1, m_2, m_3] \) such that

\[ a \sim a_0 + a_1 + \cdots \mod \mathcal{B}_1(S^{-\infty}) \]

in the sense

\[ a - \sum_{v=0}^{\infty} a_v \in \mathcal{S}[m_1 - N, m_2, m_3] \]

for all \( N \).

Two such symbols differ by an element of \( \mathcal{B}_1(S^{-\infty}). \)

ii) Let \( b_v(t, x, \xi) \in \mathcal{S}[m_1 - v, m_2 - v(l+1)] \) for \( v = 0, 1, \ldots \). Then, there exists a \( b(t, x, \xi) \in \mathcal{S}[m_1, m_2] \) such that

\[ b \sim b_0 + b_1 + \cdots \mod \mathcal{H}^{m_1 - m_2}. \]

in the sense...
Fundamental Solution for a Degenerate Hyperbolic System

\[ b - \sum_{v=0}^{N-1} b_v \in S[m_1 - N, m_2 - N(l+1)] \quad \text{for all } N. \]

Two such symbols differ by an element of \( \mathcal{I}^{m_1 - m_2}. \)

Proof. Using \( \chi(\xi) \) of (1.3) we set

\[
\begin{align*}
\psi_\varepsilon(\xi) &= 1 - \chi(\varepsilon \xi) \\
g_\varepsilon(t, \xi) &= 1 - \chi(\varepsilon g(t, \xi)^{i+1} \xi). 
\end{align*}
\]

Then, setting

\[ a(t, s, x, \xi) = \sum_{\varepsilon=0}^{\infty} \psi_\varepsilon(\xi) a_\varepsilon(t, s, x, \xi) \]

and

\[ b(t, s, x, \xi) = \sum_{\varepsilon=0}^{\infty} g_\varepsilon(t, \xi) b_\varepsilon(t, s, x, \xi) \]

for appropriate \( 1 \geq \varepsilon_0 > \varepsilon_1 > \cdots > \varepsilon_v > \cdots \to 0, \) we get i) and ii) by usual method.

**Proposition 1.6.** Let \( a(t, s, x, \xi) \in S[m_1, m_2, m_3] \) and \( b(t, s, x, \xi) \in S[m_1', m_2', m_3'] \) and define \( a \circ b(t, s, x, \xi) \) by

\[
a \circ b(t, s, x, \xi) = 0 - \int e^{-i\varepsilon \xi'} a(t, s, x, \xi + \xi') b(t, s, x + y, \xi') d\xi' dy.
\]

Then, we have

\[
a \circ b(t, s, x, \xi) \in S[m_1 + m_1', m_2 + m_2', m_3 + m_3']
\]

and for \( A = a(t, s, X, D_x), B = b(t, s, X, D_x) \) we have

\[ AB = a \circ b(t, s, X, D_x). \]

Moreover, we have

\[
a \circ b(t, s, x, \xi) \sim \sum_{\alpha} \frac{1}{|\alpha|!} a^{(\alpha)}(t, s, x, \xi) b^{(\beta)}(t, s, x, \xi) \mod \mathcal{B}_{i,s}(S^{-\infty}).
\]

Proof. If we note Remark 1° of Definition 1.1, the proof is clear.

**Corollary 1.7.** Let \( a(t, x, \xi) \in S[m_1, m_2] \) and \( b(t, x, \xi) \in \mathcal{I}^m. \) Then, both \( a \circ b(t, x, \xi) \) and \( b \circ a(t, x, \xi) \) belong to \( \mathcal{I}^{m_1 + m_1 - m_2}. \)

When \( A(t) \) is an \( m \times m \) matrix of pseudo-differential operators with symbols in \( S[m_1, m_3], \) we also write \( \sigma(A(t)) \in S[m_1, m_2]. \) We define \( |\sigma(A(t))| \) by

\[ |\sigma(A(t))| = \max_{1 \leq i, j \leq m} |a_{i,j}(t, x, \xi)|. \]
where \( a_{jk}(t, x, \xi) \) is the \((j, k)\)-element of \( \sigma(A(t))(x, \xi) \).

**Lemma 1.8.** Let \( \sigma(N^{(v)}(t))(x, \xi) \in \mathcal{S}[-\nu, -\nu(l+1)] \) \( v = 1, 2, \ldots \), be \( m \times m \) matrices. Then, there exists \( N(t) \) such that \( \sigma(N(t))(x, \xi) \in \mathcal{S}[0, 0] \) and

\[
N(t) \sim I + N^{(1)}(t) + N^{(2)}(t) + \cdots \mod \mathcal{S}^0.
\]

Moreover, \( N(t) \) has a parametrix \( N(t)^\dagger \) such that \( \sigma(N(t)^\dagger)(x, \xi) \in \mathcal{S}[0, 0] \) and

\[
\sigma(N(t)N(t)^\dagger - I), \quad \sigma(N(t)^\dagger N(t) - I) \in \mathcal{B}_1(S^{-\infty}).
\]

**Proof.** Let \( \gamma_s(t, \xi) \) be the symbol defined by (1.16). Then, by Proposition 1.5-ii) we see that

\[
\sigma(N(t))(x, \xi) = I + \sum_{v=1}^\infty \gamma_{s,v}(t, \xi) \sigma(N^{(v)}(t))(x, \xi)
\]

belongs to \( \mathcal{S}[0, 0] \) for appropriate \( 1 \geq \varepsilon_1 \geq \cdots \geq \varepsilon_v \geq \cdots \to 0 \) and (1.20) holds. Furthermore, noting

\[
|D|^{1/2}\gamma_s^{(v)}(t, \xi) \leq C_{1,v} \varepsilon^{|x|} \gamma(t, \xi)^{(1+\varepsilon)} \langle \langle \xi \rangle \rangle^{-|\xi|} \gamma(t, \xi)^{-1}
\]

and \( \gamma(t, \xi)^{(1+\varepsilon)} \sigma(N^{(v)}(t))(x, \xi) \in \mathcal{S}[-(\nu-1), -(\nu-1)(l+1)] \in \mathcal{S}[0, 0], \nu = 1, 2, \ldots, \)

we get \( |\det \sigma(N(t))(x, \xi)| \geq c \) for a constant \( c > 0 \), if we choose small \( \varepsilon_v > 0 \).

Noting Remark 1° of Definition 1.1, the parametrix \( N(t)^\dagger \) of \( N(t) \) can be constructed by usual procedure.

According to Kumano-go [5] we call a real valued \( C^\infty \)-function \( \phi(x, \xi) \) in \( \mathbb{R}^*_x \times \mathbb{R}^*_\xi \) a *phase function*, when it satisfies conditions:

\[
\begin{aligned}
i) \quad & \phi(x, \xi) - x \cdot \xi \in \mathcal{S}^1 \\
ii) \quad & |\nabla_x \phi(x, \xi) - \xi| \leq (1 - \varepsilon_0) |\xi| + c \\
iii) \quad & |\nabla_x \nabla_\xi \phi(x, \xi) - I| \leq 1 - \varepsilon_0 \\
& (0 < \varepsilon_0 \leq 1, \ 0 < \varepsilon_0 \leq 1, \ c > 0).
\end{aligned}
\]

Then the Fourier integral operator \( A_\phi = a_\phi(X, D_x) \) with phase function \( \phi(x, \xi) \) and symbol \( a(x, \xi) \in \mathcal{S}^{m} \) is defined by

\[
A_\phi u(x) = \mathcal{O}_0 - \int \hat{a}^{(\phi(x,t) - x \cdot \xi)} a(x, \xi) u(x') \xi d\xi dx' (u \in \mathcal{B}(\mathbb{R}^*_x)).
\]

Concerning fundamental theorems on Fourier integral operators, we refer to §2 of [5].

Let \( \lambda(t, x, \xi) \in \mathcal{S}[1, l] \) be real valued. Consider the Hamilton equation

\[
\begin{align*}
\frac{dt}{dq} &= -\nabla_\xi \lambda(t, q, p), \quad \frac{dp}{dt} = \nabla_q \lambda(t, q, p) \quad \text{on } 0 \leq s, t \leq T_0, \\
\{q, p\}_{t=s} &= \{y, \xi\}
\end{align*}
\]
and the eiconal equation

\begin{equation}
\begin{cases}
\partial_t \phi - \lambda(t, x) = 0 & \text{on } 0 \leq s, t \leq T_0, \\
\phi(s, s, x, \xi) = x \cdot \xi
\end{cases}
\end{equation}

for a small $T_0 (0 < T_0 \leq T)$. Then, we can prove the following statements by the same procedure to §3 in [5].

**Lemma 1.9.** For a small $T_1 (0 < T_1 \leq T)$ the initial value problem (1.23) has the solution \( \{q, p\} \) \((t, s, y, \xi')\) on \(0 \leq s, t \leq T_1\) such that

\begin{equation}
\begin{cases}
q(t, s, y, \xi') - y \in S[0, l + 1, 0] & (0 \leq s \leq t \leq T_1), \\
p(t, s, y, \xi') - \xi' \in S[1, l + 1, 0] & (0 \leq s \leq t \leq T_1)
\end{cases}
\end{equation}

and

\begin{equation}
\begin{cases}
q(t, s, y, \xi') - y \in S[0, 0, l + 1] & (0 \leq t \leq s \leq T_1), \\
p(t, s, y, \xi') - \xi' \in S[1, 0, l + 1] & (0 \leq t \leq s \leq T_1)
\end{cases}
\end{equation}

**Lemma 1.10.** Let $T_2 (0 < T_2 \leq T_1)$ and $\varepsilon_1 (0 < \varepsilon_1 \leq 1)$ be constants such that

\[|\partial q / \partial y - I| \leq (1 - \varepsilon_1) \quad 0 \leq s, t \leq T_1\]

Then, for the mapping $x = q(t, s, y, \xi) : R^n \mapsto R^n$ with \((t, s, \xi)\) as parameters, there exists the inverse $y = y(t, s, x, \xi)$ such that

\begin{equation}
\begin{cases}
y(t, s, x, \xi) - x \in S[0, l + 1, 0] & 0 \leq s \leq t \leq T_2 \\
y(t, s, x, \xi) - x \in S[0, 0, l + 1] & 0 \leq t \leq s \leq T_2
\end{cases}
\end{equation}

\[|\partial y / \partial x - I| \leq (1 - \varepsilon_1) / \varepsilon_1.\]

**Theorem 1.11.** There exists $T_0 (0 < T_0 \leq T)$ such that the initial value problem (1.24) has the unique solution $\phi(t, s) = \phi(t, s, x, \xi)$ on $0 \leq s, t \leq T_0$ which satisfies (1.21) and

\begin{equation}
\begin{cases}
\phi(t, s, x, \xi) - x \cdot \xi \in S[1, l + 1, 0] & (0 \leq t \leq s \leq T_0), \\
\phi(t, s, x, \xi) - x \cdot \xi \in S[1, 0, l + 1] & (0 \leq t \leq s \leq T_0)
\end{cases}
\end{equation}

**Corollary 1.12.** For a $C^\infty$-function $f(t, x, \xi)$ on $[0, T_0] \times R^n \times R^n$ set

\[\tilde{f}(t, s, y, \xi) = f(t, q(t, s, y, \xi), \xi).\]

Then, we have

\begin{equation}
D_s \tilde{f}(t, s, y, \xi) = \left\{D_s f(t, x, \xi) - \sum_{[\alpha]} \lambda^{(\alpha)}(t, x, \nabla_s \phi(t, x, \xi)) f^{(\alpha)}(t, x, \xi)\right\}_{x = q(t, s, y, \xi)}.
\end{equation}

The following lemma is important in the proof of Theorem 4.2 in §4.
Lemma 1.13. Let \( a(t, s, x, \xi) \in S[m_1, m_2, m_3] \) and \( r(t, s, x, \xi) \in \bigcap_{\nu=0}^{\infty} S[m'_1 - \nu, m'_2 - \nu(l+1), m'_3] \). Set \( A_\phi = a_\phi(t, s, X, D_x) \) with \( \phi(t, s, x, \xi) \) of Theorem 1.11 and \( R = r(t, s, X, D_x) \). Then both \( R_1 = A_\phi R \) and \( R_2 = RA_\phi \) are pseudo-differential operators with symbols

\[
(1.29) \quad r_j(t, s, x, \xi) \in \bigcap_{\nu=0}^{\infty} S[m_1 + m'_1 - \nu, m_2 + m'_2 - \nu(l+1), m_3 + m'_3] \\
(j = 1, 2),
\]

where

\[
(1.30) \quad r_1(t, s, x, \xi') = O_{m} - \int e^{i\varphi_1(t, s, x, \xi)} r(t, s, x, \xi') d\xi dx' 
\]

with

\[
\varphi_1 = \phi(t, s, x, \xi) - x \cdot \xi + (x - x') \cdot (\xi - \xi')
\]

and

\[
(1.31) \quad r_2(t, s, x, \xi') = O_{m} - \int e^{i\varphi_2(t, s, x, \xi)} a(t, s, x, \xi') d\xi dx' 
\]

with

\[
\varphi_2 = (x - x') \cdot (\xi - \xi') + \phi(t, s, x', \xi') - x'' \cdot \xi'.
\]

Moreover, we have

\[
(1.32) \quad r_j(t, s, x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \tilde{a}^{(\alpha)}(t, s, x, \xi) r_{(\alpha)}(t, s, x, \xi) \mod B_{t,s}(S^{-m})
\]

and

\[
(1.33) \quad r_j(t, s, x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \tilde{a}^{(\alpha)}(t, s, x, \xi) a_{(\alpha)}(t, s, x, \xi) \mod B_{t,s}(S^{-m}),
\]

where

\[
(1.34) \quad \tilde{a}(t, s, x, \xi) = e^{i\phi(t, s, x, \xi)} a(t, s, x, \xi).
\]

Proof. It is clear that \( r_1 \) and \( r_2 \) are defined by (1.30) and (1.31), respectively. By Theorem 1.11 we have

\[
(1.35) \quad |D_j^k D_s^\alpha \tilde{a}^{(\alpha)}(t, s, x, \xi)| \leq C_{j,s,s} \vee \vee (\eta(t, \xi)^{i+1} \langle \xi \rangle)^{\alpha + j + \beta} \eta(t, \xi)^{-j} \eta(s, \xi)^{m_j - k}.
\]

On the other hand by the assumption for \( r(t, s, x, \xi) \) we have

\[
(1.36) \quad \eta((t, \xi)^{i+1} \langle \xi \rangle) r(t, s, x, \xi) \in \bigcap_{\nu=0}^{\infty} S[m'_1 - \nu, m'_2 - \nu(l+1), m'_3]
\]

for any \( \tau \).

Then, from (1.35) and (1.36) we see that
Now we write

\[ r_1(t, s, x, \xi') = O S - \int \int e^{-i s t} \bar{a}(t, s, x, \xi') r(t, s, x + y, \xi') d\xi'' dy . \]

Then, by Taylor’s expansion

\[
\bar{a}(t, s, x, \xi'+\xi'') = \sum_{|\alpha|<\infty} \frac{\xi'^{\alpha}}{\alpha!} \bar{a}^{(\alpha)}(t, s, x, \xi') \\
+ N \sum_{|\alpha|<\infty} \frac{\xi'^{\alpha}}{\alpha!} \int_0^1 (1-\theta)^{N-1} \bar{a}^{(\alpha)}(t, s, x, \xi'+\theta \xi'') d\theta ,
\]

we have

\[
r_1(t, s, x, \xi') = \sum_{|\alpha|<\infty} \frac{1}{\alpha!} \bar{a}^{(\alpha)}(t, s, x, \xi') r_2(t, s, x, \xi') \\
+ N \sum_{|\alpha|<\infty} \frac{1}{\alpha!} \int_0^1 (1-\theta)^{N-1} h_{\alpha,0}(t, s, x, \xi') d\theta ,
\]

where

\[
h_{\alpha,0}(t, s, x, \xi') \\
= O S - \int \int e^{-i s t} \bar{a}^{(\alpha)}(t, s, x, \xi'+\theta \xi'') r_2(t, s, x + y, \xi') d\xi'' dy \\
= O S - \int \int e^{-i s t} \langle \xi'' \rangle^{-\frac{n}{2}} \bar{a}^{(\alpha)}(t, s, x, \xi'+\theta \xi'') \\
\times \langle D_y \langle r_2 \rangle(t, s, x + y, \xi') d\xi'' dy
\]

for any even integer \( \alpha \geq 0 \). Then, noting

\[
\begin{align*}
\{ \langle \xi' + \theta \xi'' \rangle^{\pm1} \leq 2 \langle \xi'' \rangle \langle \xi' \rangle^{\pm1} , \\
\eta(t, \xi' + \xi'')^{\pm1} \leq 2^{\alpha} \langle \xi'' \rangle^{\alpha} \eta(t, \xi')^{\pm1}
\end{align*}
\]

and using (1.35) we see from the assumption for \( r(t, s, x, \xi) \) that

\[
\{ h_{\alpha,0}(t, s, x, \xi) \}_{|\alpha|-N,0\leq\theta\leq1} \quad \text{is bounded in} \\
\int_{\nu=0}^{\infty} S[m_1+m'_1-\nu, m_2+m'_2-\nu(l+1), m_3+m'_3] .
\]

Hence, from (1.38), (1.39) we get (1.29) for \( j=1 \) and (1.32). By the same method we get the statement for \( r_2(t, s, x, \xi) \).

2. **Diagonalization.** In this section we consider a hyperbolic \( m \times m \) system

\[
(1.37) \quad \bar{a}^{(\alpha)}(t, s, x, \xi)r_2(t, s, x, \xi)
\]
(2.1) \[ L_0 = D_t - A_1(t) - A_0(t) \] on \([0, T]\) of pseudo-differential operators of first order, where

\[ D_t = \begin{bmatrix} D_1 & \cdots & 0 \\ 0 & \ddots & 0 \\ \end{bmatrix} \]

and

\[ \sigma(A_1(t))(x, \xi) \in \mathcal{S}[1, I], \quad \sigma(A_0(t))(x, \xi) \in \mathcal{S}[0, -1] \]

for an integer \(I > 0\). We assume the eigenvalues \(\lambda_i(t, x, \xi), \cdots, \lambda_m(t, x, \xi)\) of \(\sigma(A_1(t))(x, \xi)\) are all real and belong to \(\mathcal{S}[1, I]\). Modifying the notion 'perfectly diagonalizable' in Kumano-go [6] we introduce the following notion.

**Definition 2.1.** i) For \(\eta(t) = \eta(t, \xi)\) defined in (1.4) the operator \(L_0\) is said to be \(\eta(t)\)-diagonalizable, when there exists \(N_0(t)\) such that \(\sigma(N_0(t)) \in \mathcal{S}[0, 0]\) and \(|\det \sigma(N_0(t))| \geq c\) on \([0, T] \times R^*_+ \times R^*_\), for a constant \(c > 0\), and we can write

\[ (2.2) \quad L_0 N_0(t) \equiv N_0(t)L \mod \mathcal{B}(\mathcal{S}^\omega) \]

for some \(L\) of the form

\[ (2.3) \quad L = D_t - \mathcal{D}(t) + B(t) \quad \text{on} \quad [0, T], \]

where

\[ (2.4) \quad \sigma(\mathcal{D}(t))(x, \xi) = \begin{bmatrix} \lambda_1(t, x, \xi) & 0 \\ 0 & \cdots & 0 \\ \end{bmatrix} \]

and \(\sigma(B(t))(x, \xi) \in \mathcal{S}[0, -1]\).

ii) The operator \(L_0\) is said to be \(\eta(t)\)-perfectly diagonalizable, when there exists \(N(t)\) such that \(\sigma(N(t)) \in \mathcal{S}[0, 0]\) and \(|\det \sigma(N(t))| > c\) on \([0, T] \times R^*_+ \times R^*_\) for a constant \(c > 0\), and we can write

\[ (2.5) \quad L_0 N(t) \equiv N(t)L_1 \mod \mathcal{B}(\mathcal{S}^\omega) \]

for some \(L_1\) of the form

\[ (2.6) \quad L_1 = D_t - \mathcal{D}(t) + F(t) + R(t) \quad \text{on} \quad [0, T], \]

where \(\sigma(F(t))\) is a diagonal matrix of the form

\[ (2.7) \quad \sigma(F(t)) = \begin{bmatrix} f_1(t, x, \xi) & 0 \\ 0 & \cdots & 0 \\ \end{bmatrix} \in \mathcal{S}[0, -1] \]

and \(\sigma(R(t)) \in \mathcal{S}^\omega\).

\(N_0(t), N(t)\) are called the diagonalizer, the perfect diagonalizer for \(L_0\), respec-
Theorem 2.2. For $L$ of (2.3), assume that there exists a constant $c_0 > 0$ such that

$$|\lambda_j(t, x, \xi) - \lambda_k(t, x, \xi)| \geq c_0 \eta(t)^{\langle \xi \rangle} \quad (j \neq k).$$

Then, $L$ is $\eta(t)$-perfectly diagonalizable.

Proof. According to Kumano-go [6] we find the perfect diagonalizer $N(t)$ such that

$$\text{mod } \mathcal{H}^0$$

$$\text{mod } \mathcal{H}^{\nu}$$

$$\text{mod } \mathcal{H}^{\nu-1} \ (\nu = 0, 1, \ldots).$$

Let $b_{j,k}(k)$ be the $(j, k)$-elements of $\sigma(B(t))$, and set

$$\begin{align*}
\left\{ \begin{array}{l}
F^{(0)} = \text{diag} [B] \\
\sigma(N^{(1)}) = (n^1_{j,k})
\end{array} \right.
\text{by}
\end{align*}$$

$$\begin{align*}
\begin{cases}
b_{j,k}(\lambda_j - \lambda_k) & (j \neq k) \\
0 & (j = k)
\end{cases}
\end{align*}$$

$$B^{(1)} = (D_t - \mathcal{D}) N(t)(D_t - \mathcal{D}) + F(t) + R(t)$$

where by $\text{diag} [B]$ we denote a diagonal matrix with the same diagonal with $B$'s.

Then, we have

$$B^{(1)} = B - [\mathcal{D}, N^{(1)}] - F^{(0)} + N^{(1)} - BN^{(1)} - N^{(1)} F^{(0)},$$

where $[\mathcal{D}, N^{(1)}] = \mathcal{D} N^{(1)} - N^{(1)} \mathcal{D}$ and $\sigma(N^{(1)}) = D_t \sigma(N^{(1)})$.

Since $\sigma(B - [\mathcal{D}, N^{(1)}] - F^{(0)}) \in S[-1, -1]$, we have

$$\begin{align*}
\left\{ \begin{array}{l}
\sigma(F^{(0)}) \in S[0, -1], \\
\sigma(N^{(0)}) \in S[-1, -(l+1)], \\
\sigma(B^{(0)}) \in S[-1, -(l+1)-1].
\end{array} \right.
\end{align*}$$

Now, we assume that $F^{(0)}, N^{(l+1)}, B^{(l+1)}, \mu = 0, 1, \ldots, \nu - 1 (\nu \geq 1)$ are determined as

$$\begin{align*}
\left\{ \begin{array}{l}
\sigma(F^{(0)}) \in S[-\mu, -\mu(l+1)-1], \\
\sigma(N^{(l+1)}) \in S[-(\mu+1), -(\mu+1)(l+1)], \\
\sigma(B^{(l+1)}) \in S[-(\mu+1), -(\mu+1)(l+1)-1],
\end{array} \right.
\end{align*}$$
and define $F^{(V)}, N^{(V+1)}, B^{(V+1)}$ by

\[
F^{(V)} = \text{diag} [B^{(V)}],
\]

\[
\sigma(N^{(V+1)}) = (n_{j,k}^{(V+1)}) \quad \text{by}
\]

\[
n_{j,k}^{(V+1)} = \begin{cases} b_{j,k}^{(V)}(\lambda_j - \lambda_k) & (j \neq k) \\ 0 & (j = k) \end{cases}
\]

\[
B^{(V+1)} = (D_t - \mathcal{D} + B)(I + \sum_{\mu=1}^{\nu+1} N^{(\mu)})
\]

\[
- (I + \sum_{\mu=1}^{\nu+1} N^{(\mu)})(D_t - \mathcal{D} + \sum_{\mu=0}^{\nu} F^{(\mu)}),
\]

where $b_{j,k}^{(V)}$ are the $(j, k)$-elements of $\sigma(B^{(V)})$.

Then, we have

\[
B^{(V+1)} = (B^{(V)} - [\mathcal{D}, N^{(V+1)}] - F^{(V)}) + N^{(V+1)}
\]

\[
+ BN^{(V+1)} - \sum_{\mu=1}^{\nu+1} N^{(\mu)} F^{(V)} - N^{(V+1)} \sum_{\mu=0}^{\nu} F^{(\mu)},
\]

and by the definition of $F^{(V)}$ and $N^{(V+1)}$ we have

\[
\sigma(B^{(V)} - [\mathcal{D}, N^{(V+1)}] - F^{(V)}) \in \mathcal{S}(-(\nu+1), -\nu(l+1)-1).
\]

Hence we get (2.13) for $\mu = \nu$, and by induction, for any $\mu = 0, 1, \ldots$.

Now, by Proposition 1.5-ii) there exist $N(t)$ and $F(t)$ such that (2.9) and (2.11) hold. We set

\[
\mathbf{R} = LN - N(D_t - \mathcal{D} + F).
\]

Then, we have $\sigma(\mathbf{R}) \subseteq \mathcal{H}^\mu$. Let $N^\mu$ be a parametrix of $N$ which exists by Lemma 1.8, and set $R = N^\mu \mathbf{R}$. Then $\sigma(R) \subseteq \mathcal{H}^\mu$ and

\[
\sigma(LN(t) - N(t)(D_t - \mathcal{D} + F(t) + R(t))) \subseteq \mathcal{B}_I(S^{-\infty}).
\]

**Corollary 2.3.** Let $L_0$ be $\gamma(t)$-diagonalizable. Assume that the eigenvalues $\lambda_1(t, x, \xi), \ldots, \lambda_m(t, x, \xi)$ of $\sigma(A(t))$ satisfy (2.8). Then $L_0$ is perfectly diagonalizable.

**3. Construction of fundamental solution. The first order single operator case.** Let $L$ be a single hyperbolic operator of the form

\[
L = D_t - \lambda(t, X, D_x) + j(t, X, D_x) \quad \text{on } [0, T] \quad (0 < T \leq 1),
\]

where

\[
\{ \lambda(t, x, \xi) \in S[1, 1] \quad \text{real valued,} \}
\]

\[
\{ j(t, x, \xi) \in S[0, -1] \}.
\]
Consider the Cauchy problem

\[
\begin{cases}
Lu = \varphi(t) & \text{on } [s, T_0] \\
u|_{t=s} = \psi & (0 \leq s \leq T_0)
\end{cases}
\]

for a small \(T_0\) \((0 < T_0 \leq T)\).

**Theorem 3.1.** Set

\[
M = \lim_{R \to \infty} \sup \left\{ t Jm f(t, x, \xi) \right\}.
\]

Then, there exists uniquely a symbol \(e(t, s, x, \xi)\) in the class \(\bigcap_{0 < \zeta < 1} S[0, M + \varepsilon, -M - \varepsilon]\) on \(0 \leq s \leq t \leq T_0\) (with \(T_0\) of Theorem 1.11) such that the Fourier integral operator \(E_\phi(t, s) = e_\phi(t, s, X, D_x)\) with phase function \(\phi(t, s, x, \xi)\) given by Theorem 1.11 is the fundamental solution of the Cauchy problem (3.1) for \(L\), i.e.,

\[
\begin{cases}
LE_\phi(t, s) = 0 & \text{on } 0 \leq s \leq t \leq T_0, \\
E_\phi(s, s) = I & \text{(identity operator)}.
\end{cases}
\]

**Remark.** Since \((t + \langle \xi \rangle^{-\eta})(1 - 1/(t + \langle \xi \rangle^{-\eta} + 1) \leq (t + \langle \xi \rangle^{-\eta} - \langle \xi \rangle^{-\eta}) \leq \eta(t, \xi) \leq t + \langle \xi \rangle^{-\eta}\) and \(\langle \xi \rangle^{-\eta} \leq t \leq R\) when \(t \leq R\), we have

\[
M = \lim_{R \to \infty} \sup \left\{ t + \langle \xi \rangle^{-\eta} \right\}.
\]

**Proof.** The uniqueness will be proved after Theorem 3.2. Solving transport equations we first construct an approximate fundamental solution \(E_\phi(t, s)\) in the sense

\[
\begin{cases}
LE_\phi(t, s) \equiv 0 \mod B_{t,s}(S^{-\eta}) & \text{on } 0 \leq s \leq t \leq T_0, \\
E_\phi(s, s) = I.
\end{cases}
\]

We assume that the symbol \(\hat{e}(t, s, x, \xi)\) of \(E_\phi(t, s)\) has the form:

\[
\hat{e}(t, s, x, \xi) = \sum_{\nu=0}^{\infty} e_\nu(t, s, x, \xi) \mod B_{t,s}(S^{-\eta})
\]

and

\[
e_\nu(t, s, x, \xi) \in \bigcap_{0 < \zeta < 1} S[-\nu, M + \varepsilon, -M - \varepsilon] \quad (\nu = 0, 1, 2, \ldots).
\]

Set

\[
g(t, s, x, \xi) = -i \sum_{|\alpha| = 1} \frac{1}{\alpha!} \lambda^{(\alpha)}(t, x, \nabla \phi(t, s, x, \xi)) \times \partial_{s}^\alpha \phi(t, s, x, \xi) + f(t, s, x, \nabla \phi(t, s, x, \xi))
\]
and consider

\[(3.10) \quad \mathcal{L} = D_t - \sum_{|\alpha|=1} \lambda^{(\alpha)}(t, x, \nabla_x \phi) D_x^\alpha + g(t, s, x, \xi) .\]

Then, by the usual expansion formula of Fourier integral operators (See [5]), we have by using (1.24)

\[(3.11) \quad \sigma(Le_{\nu,\phi}(t, s))(x, \xi) = \mathcal{L}e_{\nu,\phi} + r_{\nu}(t, s, x, \xi) .\]

Here

\[(3.12) \quad r_{\nu}(t, s, x, \xi) \sim -\sum_{|\alpha|=1} \frac{1}{\alpha!} \{D_x^\alpha \lambda^{(\alpha)}((t, x, \nabla_x \phi)(t, s, x, x', \xi)) \times e_{\nu}(t, s, x', \xi))\}_{x'=x} \mod \mathcal{B}_t, (S^{-\infty})\]

and

\[(3.13) \quad \nabla_x \phi(t, s, x, x', \xi) = \int_0^1 \nabla_x \phi(t, s, x, x', \xi + \theta(x-x'), \xi) d\theta .\]

Then, from (1.27), (3.2) and (3.8) we see that

\[(3.14) \quad r_{\nu}(t, s, x, \xi) \in \bigcap_{0 \leq t < 1} S[-\nu, M+\epsilon-1, -M-\epsilon] \quad (\nu=0, 1, \cdots) .\]

Hence, if we can determine \(e_{\nu}(t, s)\) as the solution of

\[(3.15) \quad \begin{cases} \mathcal{L}e_0 = 0 & \text{on } 0 \leq s \leq t \leq T_0, \\ e_0(s, s) = 1 \end{cases} \]

and

\[(3.16) \quad \begin{cases} \mathcal{L}e_{\nu} + r_{\nu-1} = 0 & \text{on } 0 \leq s \leq t \leq T_0, \\ e_{\nu}(s, s) = 0 & (\nu=1, 2, \cdots) , \end{cases} \]

then we have

\[(3.17) \quad \sigma(L \sum_{\nu=0}^N e_{\nu,\phi}(t, s, X, D_x)) = \sum_{\nu=0}^N (\mathcal{L}e_\nu + r_\nu) = \mathcal{L}e_0 + \sum_{\nu=1}^N (\mathcal{L}e_{\nu} + r_{\nu-1}) + r_N \quad (\nu=0, 1, \cdots) .\]

Thus, if we determine \(\tilde{e}(t, s, x, \xi)\) so that (3.7) holds and \(e(s, s)=1\), then we get (3.6).

Now, we solve (3.15) and (3.16) in what follows. Let \(q(t, s, y, \xi)\) be the solution of (1.23) given by Lemma 1.9. Then, by Corollary 1.12 the equations (3.15) and (3.16) are reduced, respectively, to
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\[
(3.18) \quad \begin{cases}
   D_{\xi} \tilde{\phi}_0(t, s, y, \xi) + \bar{g}(t, s, y, \xi) \tilde{\phi}_0(t, s, y, \xi) = 0, \\
   \tilde{\psi}_0(s, s, y, \xi) = 1
\end{cases}
\]

and

\[
(3.19) \quad \begin{cases}
   D_{\xi} \tilde{\phi}_0(t, s, y, \xi) + \bar{g}(t, s, y, \xi) \tilde{\phi}_0(t, s, y, \xi) + \bar{r}_{\nu-1}(t, s, y, \xi) = 0, \\
   \tilde{\psi}_0(s, s, y, \xi) = 0 \quad (\nu = 1, 2, \ldots)
\end{cases}
\]

where

\[
(3.20) \quad \begin{cases}
   \tilde{\phi}_0(t, s, y, \xi) = e_0(t, s, q(t, s, y, \xi), \xi), \\
   \bar{g}(t, s, y, \xi) = g(t, s, q(t, s, y, \xi), \xi), \\
   \bar{r}_\nu(t, s, y, \xi) = r_\nu(t, s, q(t, s, y, \xi), \xi).
\end{cases}
\]

Hence we have

\[
(3.21) \quad \tilde{\psi}_0(t, s, y, \xi) = \exp[-i \int_s^t \bar{g}(\sigma, s, y, \xi) d\sigma]
\]

and

\[
(3.22) \quad \tilde{\phi}_0(t, s, y, \xi) = -i \int_s^t \bar{r}_{\nu-1}(\sigma, s, y, \xi) \exp[-i \int_{\sigma}^{t} \bar{g}(\sigma', s, y, \xi) d\sigma'] d\sigma.
\]

Consequently, setting

\[
(3.23) \quad \begin{cases}
   \bar{g}(t, \sigma, s, x, \xi) = g(\sigma, s, q(\sigma, s, t, s, x, \xi), \xi, \xi) \\
   \bar{r}_\nu(t, \sigma, s, x, \xi) = r_\nu(\sigma, s, q(\sigma, s, t, s, x, \xi), \xi, \xi)
\end{cases}
\]

for the inverse \( y = y(t, s, x, \xi) \) of \( x = q(t, s, y, \xi) \) given by Lemma 1.10, we have

\[
(3.24) \quad e_0(t, s, x, \xi) = \exp[-i \int_s^t \bar{g}(t, \sigma, s, x, \xi) d\sigma]
\]

and

\[
(3.25) \quad e_\nu(t, s, x, \xi) = -i \int_s^t \bar{r}_{\nu-1}(t, \sigma, s, x, \xi) \exp[-i \int_{\sigma}^{t} \bar{g}(t, \sigma', s, x, \xi) d\sigma'] d\sigma.
\]

Now, we first note that from Remark of Theorem 3.1 we have

\[
(3.26) \quad (t + \langle \xi \rangle^{-n}) \left( 1 - 1/(t\langle \xi \rangle^{-n} + 1) \right) \leq \gamma(t, \xi) \leq t + \langle \xi \rangle^{-n}.
\]

By the definition (3.4)' of \( M \) there exists for any \( \epsilon > 0 \) a constant \( C_\epsilon \) such that

\[
\begin{align*}
\Im f(t, s, \xi) & \leq (M + \epsilon/2) (t + \langle \xi \rangle^{-n}) + C_\epsilon \{ 1 + \langle \xi \rangle^{-n}/(t + \langle \xi \rangle^{-n}) \}.
\end{align*}
\]

Then, using (3.9), (3.23), (3.26), (3.27) and Theorem 1.11, we have
for another constant $C'$. On the other hand, by (1.5)

\[
3.29 \quad \log((t+\langle\xi\rangle^\alpha)/(s+\langle\xi\rangle^\alpha)) \leq \log(\gamma(t,\xi)/(\gamma(s,\xi)) + \log 2.
\]

Hence, using

\[
\int_s^t \langle\xi\rangle^{-\sigma+\langle\xi\rangle^\alpha} d\sigma \leq 1
\]

and (3.28), we have

\[
3.30 \quad |\exp[-i \int_s^t \tilde{g}(t,\sigma,s,\xi) d\sigma] | \leq C'' \gamma(t,\xi)/(\gamma(s,\xi))^{M+\epsilon}
\]

for a constant $C''$. We have

\[
|D^\alpha D^\beta D^\gamma \tilde{g}(t,\sigma,s,\xi) | \leq C \langle\xi\rangle^{-\sigma+\langle\xi\rangle^\alpha} \gamma(t,\xi)^{-1-\epsilon \gamma(s,\xi)^{-1}}.
\]

Then, noting

\[
\int_s^t \gamma(t,\xi)^{-1} d\sigma \leq 2 \log((t+\langle\xi\rangle^\alpha)/(s+\langle\xi\rangle^\alpha))
\]

for a constant $C_\epsilon$, we have

\[
3.31 \quad |D^\alpha D^\beta D^\gamma \tilde{g}(t,\sigma,s,\xi) d\sigma | \leq C \langle\xi\rangle^{-\sigma+\langle\xi\rangle^\alpha} \gamma(t,\xi)^{-1-\epsilon \gamma(s,\xi)^{-1}}.
\]

Thus, together with (3.30) we see that

\[
3.32 \quad e_\epsilon(t,s,\xi) \subseteq \bigcap_{0 < \epsilon < 1} S[0,M+\epsilon,-M-\epsilon].
\]

We already checked (3.14) for $r_\epsilon$ if (3.8) holds for $e_\epsilon$. Hence, if we prove (3.8) for $e_\epsilon$, assuming (3.14) for $\nu-1$, then (3.8) holds for any $\nu$. And this fact is clear by (3.25).

Now, from (3.17) we see that there exists $r_\infty(t,s,x,\xi) \subseteq \mathcal{B}_{1}(S^{-\infty})$ such that

\[
3.33 \quad L \tilde{e}_\epsilon(t,s,X,D_x) = R_{\infty}(t,s) (= r_{\infty}(t,s,x,D_x)).
\]

Then, setting

\[
3.34 \quad \begin{cases}
W_{1}(t,s) = -iR_{\infty}(t,s),
W_{\nu+1}(t,s) = \int_{s}^{t} W_{1}(t,\theta) W_{\nu}(\theta,s) d\theta \quad (\nu = 1,2,\cdots),
\end{cases}
\]
we get the fundamental solution $E(t, s)$ in the form

$$
E(t, s) = q(Lt, s) + \int_{t}^{s} \mathcal{E}_{0}(t, \theta) \sum_{\nu = 1} W_{\nu}(\theta, s) d\theta.
$$

From the theory of pseudo-differential operators of multiple symbols, there exists a symbol $\tilde{e}_{\omega}(t, s, x, \xi) \in B_{t, s}(S^{-\infty})$ such that

$$
E(t, s) = e_{\phi}(t, s, X, D_{x}) + \tilde{e}_{\omega}(t, s, X, D_{x})
$$

(cf. [5], [12]). Then, setting

$$
e(t, s, x, \xi) = \tilde{e}(t, s, x, \xi) + e_{\omega}(t, s, x, \xi),
$$

we get the desired result.

**Theorem 3.2.** The fundamental solution $E_{\phi}(t, s) (0 \leq s \leq t \leq T_{0})$ given in Theorem 3.1 has the meaning even when $0 \leq t \leq s \leq T_{0}$, and $E_{\phi}(t, s) (0 \leq t \leq s \leq T_{0})$ is the fundamental solution of the backward initial value problem for $L$, i.e.,

$$
\begin{cases}
LE_{\phi}(t, s) = 0 & \text{on } 0 \leq t \leq s \leq T_{0}, \\
E_{\phi}(s, s) = I.
\end{cases}
$$

Furthermore, we have

$$
e(t, s, x, \xi) \in \bigcap_{0 < \epsilon < 1} S[0, -M' - \epsilon, M' + \epsilon] \quad (0 \leq t \leq s \leq T_{0}),
$$

where $M'$ is defined by

$$
\lim_{R \to \infty} \sup_{x, \xi \in \mathbb{R}^{n}, y \in \mathbb{R}^{n}} \{ -t \text{Im} f(t, x, \xi) \}.
$$

**Proof.** We check the proof of Theorem 3.1. We have by Lemma 1.9, 1.10 and Theorem 1.11 that

$$
\begin{cases}
q(t, s, y, \xi') - y \in S[0, 0, 1] \\
y(t, s, x, \xi') - x \in S[0, 0, 1] \\
\phi(t, s, x, \xi) - x \cdot \xi \in S[1, 0, 1]
\end{cases}
$$
on $0 \leq t \leq s \leq T_{0}$. Noting (3.24) we write

$$
e_{0}(t, s, x, \xi) = \exp \left[ i \int_{t}^{s} \mathcal{g}(t, \sigma, s, x, \xi) d\sigma \right]
$$
on $0 \leq t \leq s \leq T_{0}$. Then, from (3.9) and (3.38)

$$
e_{0}(t, s, x, \xi) \in \bigcap_{0 < \epsilon < 1} S[0, -M' - \epsilon, M' + \epsilon],
$$

and, following the similar procedure to the proof of Theorem 3.1 by keeping in mind the fact $0 \leq t \leq s \leq T_{0}$, we complete the proof.
Proof of the uniqueness of $E_\phi(t,s)$ in Theorem 3.1. Set

$$L^* = D_t - \lambda^*(t,X,D_x) + f^*(t,X,D_x),$$

where $\lambda^*$ and $f^*$ are the formal adjoints of $\lambda$ and $f$, respectively. Then, $L^*$ is the formal adjoint of $L$. Since $\lambda(t,x,\xi)$ is real valued, we see that $\lambda^*(t,x,\xi) - \lambda(t,x,\xi) \in S[0, l]$, and, there exists a $\tilde{f}^*(t,x,\xi) \in S[0, -1]$ such that

$$L^* = D_t - \lambda(t,X,D_x) + \tilde{f}^*(t,X,D_x).$$

Therefore, we can apply Theorem 3.2 to $L^*$. Let $E_{\xi}^\phi(t,s) (0 \leq t \leq s \leq T_0)$ be the fundamental solution of the backward initial value problem for $L^*$.

Now, assume that there exist two fundamental solutions $E_\phi(t,s)$ and $E_{\psi}^\phi(t,s) (0 \leq s \leq t \leq T_0)$ of $L$ in $\bigcap_{0 < \varepsilon < 1} S[0, M+\varepsilon, -M-\varepsilon]$. For $w \in \mathcal{S}$ we set

$$u(t,s,x) = (E_\phi(t,s) - E_{\psi}^\phi(t,s))w \ (0 \leq s \leq t \leq T_0).$$

Then, $u(t,s,x)$ satisfies

$$\begin{cases}
    Lu(t,s,x) = 0 & \text{on } 0 \leq s \leq t \leq T_0, \\
    u(s,s,x) = 0.
\end{cases}$$

On the other hand, for $b(t,x) \in \mathcal{B}_1(S)$ on $[0,T_0]$ set

$$v(t,x) = \int_0^t E^* \xi(t,\sigma)b(\sigma,x)d\sigma.$$

Then, we have

$$L^*v = b(t,x) \ (0 \leq t \leq T_0), \ v(T_0,x) = 0.$$

Hence, we have

$$0 = \int_0^{T_0} (Lu,v)d\sigma = \int_0^{T_0} (u,L^*v)d\sigma
= \int_0^{T_0} (u,b)d\sigma \ \text{for all } b \in \mathcal{B}_1(S).$$

This means that

$$0 = u(t,s,x) = (E_\phi(t,s) - E_{\psi}^\phi(t,s))w \ \text{for all } w \in \mathcal{S} \ (0 \leq s \leq t \leq T_0).$$

Thus we have $e(t,s,x,\xi) = e'(t,s,x,\xi)$.

**Corollary 3.3.** i) The solution $u(t,x) \in \mathcal{B}_1(S)$ on $[0,T_0]$ of the Cauchy problem (3.3) for $\phi(t) \in \mathcal{B}_1(S)$ on $[0,T_0]$ and $\psi \in \mathcal{S}$ exists uniquely and is represented by

$$(3.40) \quad u(t,s,x) = E_\phi(t,s)\psi + i \int_s^t E_\phi(t,\theta)\psi(\theta)d\theta.$$
ii) We have

\( E_\phi(t, \tau)E_\phi(\tau, s) = E_\phi(t, s) (0 \leq s, \tau, t \leq T_0) . \)

Proof. i) It is clear that \( u(t, s, x) \) defined by (3.40) is the solution of (3.3).

Let \( v(t, x) \in \mathcal{B}_0(S) \) on \([0, T_0]\) be the solution of (3.3) for \( \phi(t) = 0 \) and \( \psi = 0 \), and let \( E_\phi^*(t, s) \) be the fundamental solution for the formal adjoint \( L^* \) of \( L \).

Set

\[ w(t, s) = i \int_{r_0}^{t} E_\phi^*(t, \theta) v(\theta, x) d\theta . \]

Then, we have

\[ L^* w = v \quad \text{on} \quad [0, T_0], \quad w(T_0, x) = 0 . \]

Hence we have

\[ \int_{0}^{T_0} (v, v) dt = \int_{0}^{T_0} (v, L^* w) dt = \int_{0}^{T_0} (Lv, w) dt = 0 , \]

and \( v(t, x) = 0 \) on \([0, T_0] \). This proves the uniqueness of the solution of (3.3).

ii) Set \( u(t, \tau, x) = E_\phi(t, \tau)E_\phi(\tau, s) \psi \) for \( \psi \in S \). Then, \( u \) satisfies

\[ \begin{cases} 
Lu = 0 & \text{on} \quad [0, T_0] , \\
u(\tau, \tau, x) = E_\phi(\tau, s) \psi .
\end{cases} \]

On the other hand \( \bar{u}(t, x) = E_\phi(t, s) \psi \) also satisfies (3.42). Hence, by i) we have \( u = \bar{u} \) on \([0, T_0] \) which proves (3.41).

**Corollary 3.4.** For the operator \( L_1 \) of (2.6) let \( L_2 \) be the operator of the form

\[ L_2 = D_t - D(t) + F(t) . \]

Let \( E_{1, \phi}(t, s), (0 \leq s, t \leq T_0) \) be the fundamental solution for \( L_1 = D_t - \chi_j(t, X, D_x) + f_j(t, X, D_x) \). Then, the fundamental solution \( E_{2}(t, s) (0 \leq s, t \leq T_0) \) of the Cauchy problem

\[ \begin{cases} 
L_2 U = \Phi(t) & \text{on} \quad [0, T_0] , \\
U|_{t=s} = \Psi & (0 \leq s \leq T_0)
\end{cases} \]

exists uniquely in the form

\[ E_{2}(t, s) = \begin{bmatrix} E_{1, \phi}(t, s) & 0 \\ 0 & E_{m, \phi}(t, s) \end{bmatrix} . \]

**4. Construction of fundamental solution. The first order system case.** In the first place we prove the fundamental lemma.
Lemma 4.1. Let $L_1$ be the operator of the form (2.6). Then, the fundamental solution $E(t,s)$ of the Cauchy problem

\[
\begin{align*}
L_1 U &= \Phi(t) \text{ on } [s, T_0], \\
U|_{t=s} &= \Psi \quad (0 \leq s \leq T_0)
\end{align*}
\]

exists in the form

\[
E(t,s) = E_2(t,s)(I+Q(t,s)) + Q_\omega(t,s),
\]

where $E_2(t,s)$ is the fundamental solution for $L_2$ in Corollary 3.4, and $Q(t,s), Q_\omega(t,s)$ satisfy

\[
\begin{align*}
Q(s,s) &= 0, \\
\sigma(Q(t,s)) (x, \xi) &\in \mathcal{S}[0,0,0], \\
Q_\omega(s,s) &= 0, \quad \sigma(Q_\omega(t,s)) (x, \xi) \in \mathcal{B}_{t,s}(S^{-\infty}), \\
(0 \leq s \leq t \leq T_0).
\end{align*}
\]

Proof. If we find $Q$ such that $E_1(t,s) = E_2(t,s)(I+Q(t,s))$ satisfies

\[
\sigma(D_tQ(t,s)+R(t,s)Q(t,s)+\tilde{R}(t,s)) \in \mathcal{B}_{t,s}(S^{-\infty})
\]

then $E_1(t,s)$ is an approximate fundamental solution for $L_1$. Hence by the usual procedure, which also used in the proof of Theorem 3.1, we can find $E_1(t,s)$ in the form (4.2).

Set

\[
\tilde{R}(t,s) = E_2(s,t)R(t)E_2(t)
\]

Then, we see that (4.4) is equivalent to

\[
\sigma(D_tQ(t,s)+\tilde{R}(t,s)Q(t,s)+\tilde{R}(t,s)) \in \mathcal{B}_{t,s}(S^{-\infty})
\]

$\quad (0 \leq s \leq t \leq T_0).$

We find such $Q(t,s)=q(t,s,X,D_x)$ in the form

\[
\begin{align*}
q(t,s) &= 0, \\
q(0,s) &= 0
\end{align*}
\]

and

\[
q(t,s) \approx q_0(t,s) + q_1(t,s) + \cdots \text{ mod } \mathcal{B}_{t,s}(S^{-\infty}).
\]

We first note that from Theorem 3.1, 3.2 and Corollary 3.4

\[
\begin{align*}
\sigma(E_2(t,s)) (x, \xi) &\in \bigcap_{0<\xi<1} S[0, M+\xi, -M-\xi] \\
(0 \leq s \leq t \leq T_0), \\
\sigma(E_2(s,t)) (x, \xi) &\in \bigcap_{0<\xi<1} S[0, M'+\xi, -M'-\xi] \\
(0 \leq s \leq t \leq T_0),
\end{align*}
\]
where

\[
\begin{align*}
M &= \max_{1 \leq i \leq m} \{M_i\}, \quad M' = \max_{1 \leq i \leq m} \{M'_i\}, \\
M_f &= \max_{1 \leq i \leq m} \{M_f\}, \\
M_{f'} &= \max_{1 \leq i \leq m} \{M_{f'}\}, \\
M'' &= \max_{1 \leq i \leq m} \{M''\}.
\end{align*}
\]

(4.10)

Hence, setting \( \tilde{r}(t,s,x,\xi) = \sigma(\tilde{R}(t,s)) (x,\xi) \), we have by Lemma 1.13

\[
\tilde{r}(t,s,x,\xi) \in \bigcap_{j \geq 1} \bigcap_{j \geq 1} S[0, -j, M + M' + \varepsilon - j(l+1), -M - M' - \varepsilon].
\]

(4.11)

Then, noting

\[
(\gamma(t,\xi)/\gamma(s,\xi))^\nu \leq (\gamma(t,\xi)^{i+1}<\xi>^\nu(\gamma(s,\xi)^{\varepsilon} - \varepsilon)^{-\gamma}.
\]

we see that

\[
\tilde{r}(t,s,x,\xi) \in \bigcap_{j \geq 1} \bigcap_{j \geq 1} S[0, -j, -j(l+1), 0].
\]

(4.12)

If we assume (4.7), we can write for \( Q_\nu(t,s) = q_\nu(t,s,X,D_x) \)

\[
\sigma(D_t Q_\nu + \tilde{R}Q_\nu) = D_t q_\nu + \tilde{r}q_\nu + r_\nu \quad (\nu = 0, 1, \ldots),
\]

(4.13)

where

\[
r_\nu(t,s,x,\xi) \in S[0, -\nu - 1, 0, 0].
\]

(4.14)

Now, using (4.13) we define \( q_\nu \) by

\[
\begin{align*}
\begin{cases}
D_t q_\nu + \tilde{r}q_\nu + r_{\nu-1} = 0 \quad (\nu - 1 = \tilde{r}), \\
q_\nu(s,s) = 0 \quad (\nu = 0, 1, \ldots)
\end{cases}
\end{align*}
\]

(4.15)

inductively. Then, if we check (4.7) for \( q_\nu \), we get \( q(t,s,x,\xi) \) by (4.8).

The solution of (4.15) can be written in the form

\[
q_\nu(t,s) = -i\int_s^t r_{\nu-1}(s_1,s)ds_1 + \sum_{\mu=2}^\nu (-i)^{\mu-1}\int_s^{s_1}ds_1\int_{s_1}^{s_2}ds_2 \cdots \int_{s_{\mu-1}}^{s_\mu} \tilde{r}(s_{\mu-1}, s).
\]

(4.16)

By (4.12) we have

\[
|\tilde{r}(t,s)| \leq C|\xi|^\omega(|t<\xi|^\omega + 1)^{-2}
\]

(4.17)

and get
(4.18) \[ \int_0^t \left| \tilde{F}(\sigma, s) \right| d\sigma \leq \int_0^t C \left\langle \xi \right\rangle^\mu \left( \left\langle \xi \right\rangle + 1 \right)^{-1} d\sigma \]
\[ = -\left[ C \left( \sigma \left\langle \xi \right\rangle + 1 \right) \right]^t_0 \]
\[ \leq C (t-s) \left( t + \left\langle \xi \right\rangle^{-\nu} \right). \]

Set

(4.19) \[ I_\mu(t, s, x, \xi) \]
\[ = \int_s^t d_1 \int_s^{d_2} \cdots \int_s^{d_{\mu-1}} |\tilde{F}(s_1, s)| \cdots |\tilde{F}(s_{\mu}, s)| ds_\mu \]

and assume

\[ I_\mu(t, s, x, \xi) \leq \frac{C^\mu (t-s)^\mu}{\mu! \left( t + \left\langle \xi \right\rangle^{-\nu} \right)^\mu}. \]

Then, we have

\[ I_{\mu+1}(t, s, x, \xi) \leq \int_s^t \frac{C \left\langle \xi \right\rangle^\mu \sigma^\mu}{\left( \sigma \left\langle \xi \right\rangle + 1 \right)^\mu} \frac{C^\mu (\sigma-s)^\mu}{\mu! \left( \sigma + \left\langle \xi \right\rangle^{-\nu} \right)^\mu} d\sigma \]
\[ = \frac{C^\mu (\sigma-s)^\mu}{\mu!} \int_s^t \frac{(\sigma-s)^\mu}{\left( \sigma + \left\langle \xi \right\rangle^{-\nu} \right)^{\mu+2}} d\sigma \]

and setting \( z = (\sigma-s)/(\sigma + \left\langle \xi \right\rangle^{-\nu}) \) we have

\[ \int_s^t \frac{(\sigma-s)^\mu}{(\sigma + \left\langle \xi \right\rangle^{-\nu})^{\mu+2}} d\sigma = (s + \left\langle \xi \right\rangle^{-\nu})^{-1} \int_0^{(t-s)/(t + \left\langle \xi \right\rangle^{-\nu})} z^\mu dz \]
\[ = \frac{(t-s)^{\mu+1}}{(\mu+1) \left( s + \left\langle \xi \right\rangle^{-\nu} \right) \left( t + \left\langle \xi \right\rangle^{-\nu} \right)^{\mu+1}}. \]

So we have

\[ I_{\mu+1}(t, s, x, \xi) \leq \frac{C^\mu (t-s)^{\mu+1}}{(\mu+1) \left( t + \left\langle \xi \right\rangle^{-\nu} \right)^{\mu+1}}. \]

With (4.19) we have

(4.20) \[ I_\mu(t, s, x, \xi) \leq \frac{C^\mu (t-s)^\mu}{\mu! \left( t + \left\langle \xi \right\rangle^{-\nu} \right)^\mu} \quad (\mu = 1, 2, \cdots). \]

Thus, from (4.16) for \( \nu = 0 \) we have

(4.21) \[ |q_0(t, s)| \leq \sum_{\mu=1}^\infty \frac{C^\mu (t-s)^\mu}{\mu! \left( t + \left\langle \xi \right\rangle^{-\nu} \right)^\mu} \]
\[ \leq \exp \left[ \frac{C (t-s)}{t + \left\langle \xi \right\rangle^{-\nu}} \right] - 1. \]

Differentiate the both sides of (4.16) and estimate similarly. Then, we see that

(4.22) \[ q_0(t, s, x, \xi) \in S[0, 0, 0], \quad q_0(s, s) = 0. \]
Now, assume that (4.7) holds for some \( \nu \geq 0 \). Then, by (4.14)

\[
(4.23) \quad \int_{\sigma} |r_\nu(\sigma, s)| d\sigma \leq C(t-s) \langle \xi \rangle^{\nu-\nu-1}.
\]

Hence, in (4.16) we use

\[
\int_{s}^{\nu-1} |r_{\nu-1}(s_{\nu}, s)| ds_{\nu} \leq C(s_{\nu}-s) \langle \xi \rangle^{\nu-\nu-1} 
\leq C(t-s) \langle \xi \rangle^{\nu-\nu-1}.
\]

Then, by (4.20) we have

\[
(4.24) \quad |q_\nu(t, s)| \leq C \langle \xi \rangle^{\nu-\nu-1} \exp \left[ \frac{C(t-s)}{t+\langle \xi \rangle^{-\nu}} \right] 
\leq C \langle \xi \rangle^{\nu-\nu-1},
\]

and finally get (4.7) for all \( \nu \geq 0 \). Q.E.D.

Now, we shall state the main theorem of the present paper.

**Theorem 4.2.** Let \( L \) and \( L_0 \) be the operators of the form (2.3) and (2.1), respectively. Let \( N(t) \) and \( \tilde{N}(t) \) be the perfect diagonalizers for \( L \) and \( L_0 \), respectively, and let \( \Omega(t,s) \) be the fundamental solution for \( L_0 \) of (2.6).

Then, the fundamental solutions \( E(t,s) \) and \( E_0(t,s) \) for \( L \) and \( L_0 \) can be found in the forms

\[
E(t,s) = N(t) \Omega(t,s) \Omega(t,s) + R_\nu(t,s),
\]

\[
(4.25)
\]

and

\[
E_0(t,s) = \tilde{N}(t) \tilde{E}_0(t,s) \Omega(t,s) + \tilde{R}_\nu(t,s),
\]

\[
(4.26)
\]

respectively, where \( N(t) \) and \( \tilde{N}(t) \) are the parametrices of \( N(t) \) and \( \tilde{N}(t) \), respectively.

Furthermore, both \( E(t,s) \) and \( E_0(t,s) \) are represented as the sums of Fourier integral operators with phase functions \( \phi_j(t,s) \), \( j = 1, \ldots, m \) and symbols of class

\[
(4.27) \quad \cup_{0<\epsilon<1} S[0, M+\epsilon, -M-\epsilon].
\]

Proof. It is easy to see that

\[
N(t) \Omega(t,s) N(t) + (I - N(t) N(t))
\]

is an approximate fundamental solution for \( L \). Then, noting \( \sigma(I - N(t) N(t)) \in B(S^{-\infty}) \) and solving the integral equation as in (3.35) we get (4.25). Since by Lemma 4.1, \( \Omega(t,s) \) is the sum of Fourier integral operators with phase func-
tions $\phi_j(t,s)$ and symbols of class stated in (4.27), the rest of the proof for $L$ is clear. Similarly we get for $L_0$.

**Theorem 4.3.** The Cauchy problem

\[
\begin{cases}
    LU = 0 \text{ on } [s, T_0] \\
    U|_{t=s} = \Psi \in H_\sigma, (0 \leq s \leq T_0)
\end{cases}
\]

and

\[
\begin{cases}
    L_0 U = 0 \text{ on } [s, T_0] \\
    U|_{t=s} = \Psi \in H_\sigma, (0 \leq s \leq T_0)
\end{cases}
\]

have the unique solutions $U(t,s)$ and $U_0(t,s)$ in the form

\[
U(t,s) = E(t,s)\Psi, \quad U_0(t,s) = E_0(t,s)\Psi,
\]

respectively, where $H_\sigma$ is the usual Sobolev space for real $\sigma$. Furthermore, for any $\varepsilon > 0$ we have

\[
\begin{align*}
    \|\gamma_{M+\varepsilon}U\|_{\sigma} &\leq C_\varepsilon\|\Psi\|_{\sigma} \quad (0 \leq s \leq t \leq T_0) \\
    \|U\|_{\sigma-(M+\varepsilon)\omega} &\leq C'_\varepsilon\|\Psi\|_{\sigma} \quad (0 \leq s \leq t \leq T_0)
\end{align*}
\]

where $\gamma_{M+\varepsilon} = \gamma_{M+\varepsilon}(t,s,X,D_x)$ is defined by

\[
\gamma_{M+\varepsilon}(t,s,x,\xi) = (\gamma(s,\xi)/\gamma(t,\xi))^{M+\varepsilon}.
\]

**Remark.** From (4.32) we see that $M\omega$ denotes the supremum of regularity loss of the solution. It should be noted that in Kumano-go [7] the constant $M$ is determined as a sufficiently large number depending on $L$, and that constants $C_\varepsilon$ and $C'_\varepsilon$ are independent of $t$ and $s$ for $0 \leq s \leq t \leq T_0$.

**Proof.** Since $\gamma_{M+\varepsilon}E(t,s)$ is the sum of Fourier integral operators with symbols of class $S[0,0,0]$, we have (4.31) for $0 \leq s \leq t \leq T_0$ and $U(t,s)$. Since

\[
\|\gamma_{M+\varepsilon}U\|_{\sigma} \geq 2^{-\omega(M+\varepsilon)}\|\gamma(s,D_x)M+\varepsilon U\|_{\sigma}
\]

\[
\geq 2^{-\omega(M+\varepsilon)}\|U\|_{\sigma-(M+\varepsilon)\omega},
\]

we get (4.32) for $0 \leq s \leq t \leq T_0$ and $U(t,s)$. The rest of the proof is done similarly.

5. **The higher order case.** In this section we consider a single higher order operator of the following type:

\[
\begin{align*}
    L &= D_t^\sigma + \sum_{k=1}^m a_k(t,x,D_x)D_t^{\sigma-k}, \\
    a_k(t,x,\xi) &= \sum_{j=0}^{k-1} \gamma(t)^{(k-j)(j+1)} a_{k,j}(t,x,\xi) \\
    a_{k,j}(t,x,\xi) &\in S[k-j,0] \text{ on } [0,T] \times R_t^* \times R^*_x,
\end{align*}
\]
and consider the Cauchy problem

\begin{equation}
Lu = \varphi(t) \quad \text{on } [s, T_0], \tag{5.2}
\end{equation}

Set

\begin{equation}
P(\lambda) = \lambda^m + a_{1,0}\lambda^{m-1} + \cdots + a_{m,0} \tag{5.3}
\end{equation}

and

\begin{equation}
G(\lambda) = a_{1,1}\lambda^{m-1} + a_{2,1}\lambda^{m-2} + \cdots + a_{m,1} \tag{5.4}
\end{equation}

**Theorem 5.1.** Let the roots \( \mu_1(t,x,\xi), \ldots, \mu_m(t,x,\xi) \) of \( P(\lambda) = 0 \) be real valued and satisfy (0.3) for a constant \( c > 0 \).

Then, the equation \( Lu = \varphi \) can be reduced to a system \( LU = (0, \ldots, 0, \varphi) \), where \( L \) has the form (2.3) with \( \nu_j(t,x,\xi) = \eta(t)\mu_j(t,x,\xi), j = 1, \ldots, m \). The constant \( M \) of (4.10) is given by

\begin{equation}
M = \max_{1 \leq k \leq m} \lim_{R \to 0} \sup_{0 \leq t \leq R} \{ J_k(t, x, \xi) \} + l(m-1) \tag{5.5}
\end{equation}

where

\( J_k(t, x, \xi) = \frac{g_m G(\mu_k) - \eta((t)d\mu_k + l\mu_k)P''(\mu_k)/2}{P'(\mu_k)} \)

\( P' = \partial_\lambda P \) and \( P'' = \partial^2_\lambda P \).

**Remark.** It is easily verified that the differential operator of the form (0.11) satisfies (5.1).

**Proof.** I (Reduction to first order system). Let

\begin{equation}
H(t) = \begin{bmatrix}
h(t, D_t)^{1-m} & 0 \\
h(t, D_t) & 0 \\
\vdots & \ddots \\
0 & \cdots & 1
\end{bmatrix}, \tag{5.6}
\end{equation}

where \( h(t, \xi) \) is the symbol defined in Lemma 1.4, and set

\begin{equation}
U = H(t) \begin{bmatrix}
u \\
D_t u \\
\vdots \\
D_t^{m-1} u
\end{bmatrix} \tag{5.7}
\end{equation}

Then, \( Lu = \varphi \) is reduced to a first order system \( L_0 U = \Phi \), where

\begin{equation}
L_0 = D_t - A(t), \quad \sigma(A(t)) = \begin{bmatrix}
(m-1)h, h^{-1}, h, 0 \\
(m-2)h, h^{-1}, h, & \ddots \\
0, \ddots, \ddots, h^{-1}, h \\
-a_m h^{1-m}, -a_{m-1} h^{2-m}, \ldots, -a_1 h^{-1}, -a_1
\end{bmatrix} \tag{5.8}
\end{equation}
and $\Phi = \phi(0, 0, \ldots, 0, \phi)$.

By Lemma 1.2, 1.4, Proposition 1.6 and its Corollary we have $\sigma(A(t)) \subseteq S[1, l]$.  

II (Principal and sub-principal part). We set 

$$\sigma(A_i(t)) = \begin{bmatrix} 0 & \cdots & h & 0 \\ \vdots & \ddots & \vdots & \vdots \\ -\gamma a_{m,0} \langle \xi \rangle^{1-m} & \cdots & -\gamma a_{m-1,0} \langle \xi \rangle^{2-m} & \cdots & -\gamma a_{1,0} \end{bmatrix}$$

and 

$$\sigma(A_0(t)) = \frac{1}{\eta} \begin{bmatrix} -i(m-1)l & \cdots & 0 \\ \vdots & \ddots & \vdots \\ -a_{m-1,1} \langle \xi \rangle^{1-m} & \cdots & -a_{1,1} \end{bmatrix}$$

Then, we have 

$$\left\{ \begin{array}{l} \sigma(A_i(t)) \subseteq S[1, l] \\
\sigma(A_0(t)) \subseteq S[0, -1] \\
\sigma(A(t) - A_i(t) - A_0(t)) \subseteq S[-1, -(l+1)-1] \end{array} \right.$$  

(5.9)

This follows from 

$$a_b h^{1-k} - \gamma a_{b,0} \langle \xi \rangle^{1-k} - \gamma a_{b,1} \langle \xi \rangle^{1-k} = \sum_{l=1}^k \gamma^{(k-j)(l+1)-k} a_{b,l} h^{l-k} \subseteq S[-1, -(l+1)-1]$$

and 

$$-(m-j)h h^{-1} - i(m-j)\eta^{-1} \in \mathcal{B}^m.$$  

III (Diagonalizer). The diagonalizer $N_0(t)$ of $L_0$ is given by 

$$\sigma(N_0(t)) = \begin{bmatrix} 1 & \cdots & 1 \\ \mu_1/\langle \xi \rangle & \cdots & \mu_m/\langle \xi \rangle \\ \cdots \\ (\mu_1/\langle \xi \rangle)^{m-1} & \cdots & (\mu_m/\langle \xi \rangle)^{m-1} \end{bmatrix}$$  

(5.10)

To prove this it is enough to show that 

$$\sigma(N_0^{-1}) \sigma(A_i) \sigma(N_0) = \delta_{j,k} \gamma^j \mu_k,$$

where $\delta_{j,k} = 1$ if $j = k$, $= 0$ if $j \neq k$. And, this follows from 

the $(j, k)$-element of $\sigma(A_i) \sigma(N_0) = \gamma^j \mu_k \langle \xi \rangle^{i-1}$ for $j = 1, \ldots, m-1.$
the \((m, k)\)-element of \(\sigma(A_0)\sigma(N_0)\)

\[
= -\eta_l \langle \xi \rangle^{1-n} \{a_{m, 0} + a_{m-1, 0} \mu_k + \cdots + a_{1, 0} \mu_k^{m-1} \}
\]

and

\[
\sum_{j=1}^{m} q_{j, \nu}(\mu_k/\langle \xi \rangle)^{1-n} = \delta_{j, k},
\]

where \(q_{j, \nu}\) is the \((j, \nu)\)-element of \(\sigma(N_0)^{-1}\).

IV (Computation of \(M\)). We have

the \((j, k)\)-element of \(\eta \sigma(A_0)\sigma(N_0) = -i(m-j)l(\mu_k/\langle \xi \rangle)^{j-1}\) for \(j=1, \ldots, m-1\) and

the \((m, k)\)-element of \(\eta \sigma(A_0)\sigma(N_0) = -G(\mu_k)\langle \xi \rangle^{1-m}\).

We define polynomials \(Q_j(\lambda)\) of \(\lambda (j=1, \ldots, m)\) by

\[
(5.12) \quad Q_j(\lambda) = \sum_{\nu=1}^{m} q_{j, \nu}(\lambda)^{1-n} \lambda^{j-1}.
\]

Then we have

\[
Q_j(\mu_k) = \delta_{j, k}.
\]

Thus we have

\[
Q_j(\lambda) = \prod_{k \neq j} (\lambda - \mu_k)/(\mu_j - \mu_k)
\]

and

\[
\partial_\lambda Q_j(\mu_k) = \frac{1}{2} P''(\mu_k)/P'(\mu_k).
\]

Since \(q_{j, m} = \langle \xi \rangle^{1-m}/P'(\mu_k)\), we have

\[
(5.13) \quad \text{the \((k, k)\)-element of } \eta \sigma(N_0)^{-1}\sigma(A_0)\sigma(N_0)
\]

\[
= -i \sum_{\nu=1}^{m} q_{j, \nu}(m-\nu)l(\mu_k/\langle \xi \rangle)^{y-1} - q_{j, m} G(\mu_k)\langle \xi \rangle^{m-1}
\]

\[
= -i m \sum_{\nu=1}^{m} q_{j, \nu}(\mu_k/\langle \xi \rangle)^{y-1} + il \sum_{\nu=1}^{m} \nu q_{j, \nu}(\mu_k/\langle \xi \rangle)^{y-1}
\]

\[
-G(\mu_k)/P'(\mu_k)
\]

\[
= -i m \{ \partial_{\lambda} (\lambda Q_j(\lambda)) \}_{\lambda = \mu_k} - G(\mu_k)/P'(\mu_k)
\]

\[
= -i(m-1) - (G(\mu_k) - \frac{i}{2} l \mu_k P''(\mu_k))/P'(\mu_k).
\]

We also have

\[
(5.14) \quad \text{the \((k, k)\)-element of } \sigma(N_0)^{-1}\sigma(N_0)
\]
V (End of the proof of Theorem 5.1). If we set
\[(5.15)\]
\[L = D_t - N^0 N_0 + N^1 N_0,\]
then we have
\[N_0 L \equiv L_0 N_0 \mod \mathcal{B}_1(S^{-\infty}).\]

By (5.9) and (5.11) we have
\[L = D_t - \mathcal{D} + \mathcal{B},\]
where
\[\sigma(\mathcal{D}) = \eta' \begin{bmatrix} \mu(t, x, \xi) & 0 \\ 0 & \mu(t, x, \xi) \end{bmatrix},\]
and \(\sigma(\mathcal{B}) \in \mathbb{S}[0, -1] \). We set
\[c_0(t, x, \xi) = \text{the (k, k)-element of } \sigma(\mathcal{B})\]
\[-\{il(m-1) + (G(\mu_k) - \frac{i}{2} (\eta \partial_0 \mu_k + l_0 \mu_k) P''(\mu_k)) / \eta \}.\]

Since
\[\sigma(N^0 A_0 N_0) - \sigma(N_0)^{-1} \sigma(A_0) \sigma(N_0) \in \mathbb{S}[0, 0],\]
\[\sigma(N^0 A_0 N_0) - \sigma(N_0)^{-1} \sigma(A_0) \sigma(N_0) \in \mathbb{S}[-1, -1],\]
and
\[\sigma(N_0 N_{0,t}) - \sigma(N_0)^{-1} \sigma(N_{0,t}) \in \mathbb{S}[-1, -1],\]
we have by (5.13) and (5.14)
\[\lim_{R \to \infty} \sup_{0 \leq t \leq R} \{\eta(t, \xi)c_0(t, x, \xi)\} = 0,\]
Thus the \(M\) of (4.10) is given by (5.5).

**Theorem 5.2.** Let \(L\) satisfy the condition of Theorem 5.1. Then the solution \(u\) of the Cauchy problem (5.2) with \(\varphi(t) \in \mathcal{B}_1(S)\) and \(\psi_j \in \mathcal{S} j=0, 1, \ldots, m-1,\)
exists uniquely in \([s, T_0]\) and it is given by
\[(5.16)\]
\[u(t, x) = \sum_{j=0}^{\infty} E_0^{i,j+1}(t, s, X, D_x) \psi_j\]
\[+ i \int_{s}^{t} E_0^{i,m}(t, \sigma, X, D_x) \varphi(\sigma) d\sigma,\]
where $E_0^{1,j}$ is the $(1,j)$-element of the fundamental solution $E_0$ of the operator

$$(5.17) \quad \mathbf{D} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_m & -a_{m-1} & \cdots & -a_0 & -a_1 \end{bmatrix}. $$

The regularity loss caused by $E_0^{1,k}$ in the sense of Remark to Theorem 4.3 is equal to

$$(5.18) \quad m_k = \omega(M-l(m-1)-k+1).$$

Proof. An approximate fundamental solution of (5.17) is given by

$$(5.19) \quad \tilde{E}_0 = H(t)N_0(t)E(t, s)N_0^*(s)H^*(s) + (I-H(s)N_0(s)N_0^*(s)H^*(s)),$$

where $E(t, s)$ is the fundamental solution of $L$ of (5.15). Using $\tilde{E}_0$ we can construct the fundamental solution $E_0$ as in the proof of Theorem 3.1. By (5.19) we have

$$(5.20) \quad \sigma(E_0^{1,k}(t, s))(x, \xi) \in \bigcap_{0<\epsilon<1} S[1-k, M+\epsilon-l(m-1), -M-\epsilon-l(k-m)].$$

Thus, we have (5.18).

References


equation with variable multiplicity, unpublished.


