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ON THE FUNDAMENTAL SOLUTION FOR A DEGENERATE HYPERBOLIC SYSTEM

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Introduction. Let $S[m_1, m_2]$ denote the set of all C^∞-symbols $a(t, x, \xi)$ on $[0, T]\times R_x^n\times R_\xi^n$ $(0<T\leqq 1)$ such that

$$(0.1)\qquad |D_t^j D_\xi^\alpha D_x^\beta a(t, x, \xi)|\leqq C_{j,\alpha,\beta}\langle\xi\rangle^{m_1-|\alpha|}(t+\langle\xi\rangle^{-\omega})^{m_2-j}$$

for constants $C_{j,\alpha,\beta}$, where $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$ and $\omega=1/(l+1)$ with an integer $l>0$.

Consider a hyperbolic operator of first order:

$$(0.2)\qquad \boldsymbol{L} = \boldsymbol{D}_t - t^l \begin{bmatrix} \mu_1(t, X, D_x) & & 0 \\ & \ddots & \\ 0 & & \mu_m(t, X, D_x) \end{bmatrix} + \boldsymbol{B}(t),$$

where μ_j, $j=1, \cdots, m$ are real valued and satisfy

$$(0.3)\qquad \begin{cases} \text{i)} \quad \mu_j(t, x, \xi)\in S[1, 0] \\ \text{ii)} \quad |\mu_j(t, x, \xi)-\mu_k(t, x, \xi)|\geqq c\langle\xi\rangle \qquad (j\neq k) \end{cases}$$

for a constant $c>0$, and the symbol $\sigma(\boldsymbol{B}(t))(x, \xi)$ of the lower order operator $\boldsymbol{B}(t)$ satisfies

$$(0.4)\qquad \sigma(\boldsymbol{B}(t))(x, \xi)\in S[0, -1].$$

The purpose of the present paper is to construct the fundamental solution $\boldsymbol{E}(t, s)$ $(0\leqq s\leqq t\leqq T_0)$ of the Cauchy problem

$$(0.5)\qquad \begin{cases} \boldsymbol{L}U = \Phi(t) \qquad \text{on} \quad [s, T_0], \\ U|_{t=s} = \Psi \end{cases}$$

for a small constant T_0 $(0<T_0\leqq T)$. It should be noted that the operator $\boldsymbol{L}$ is degenerate at $t=0$ and $\boldsymbol{B}(t)$ is not uniformly bounded on $[0, T]$ as a family of pseudo-differential operators with parameter $t\in[0, T]$.

To construct $\boldsymbol{E}(t, s)$, we find first the perfect diagonalizer $\boldsymbol{N}(t)$ such that the symbol $\sigma(\boldsymbol{N}(t))(x, \xi)$ belongs to $S[0, 0]$ and

$$LN(t) \equiv N(t)L_1 \bmod \mathcal{B}_t(S^{-\infty}), \tag{0.6}$$

where L_1 is an operator of the form

$$L_1 = D_t - t^l \begin{bmatrix} \mu_1(t, X, D_x) & & 0 \\ & \ddots & \\ 0 & & \mu_m(t, X, D_x) \end{bmatrix} + \begin{bmatrix} f_1(t, X, D_x) & & 0 \\ & \ddots & \\ 0 & & f_m(t, X, D_x) \end{bmatrix} + R(t) \tag{0.7}$$

such that $f_j(t, x, \xi) \in S[0, -1]$ and $\sigma(R(t))(x, \xi) \in \mathcal{H}^\omega = \bigcap_{\nu=0}^{\infty} S[\omega - \nu\omega, -\nu]$. Then, for L_1 we can construct the fundamental solution $E_1(t, s)$, and, by using $E_1(t, s)$, the fundamental solution $E(t, s)$ for L can be found in the form

$$E(t, s) = N(t)E_1(t, s)N^\sharp(s) + R_{-\infty}(t, s), \tag{0.8}$$

where $N^\sharp(s)$ is a parametrix of $N(s)$ and $\sigma(R_{-\infty}(t, s))(x, \xi) \in \mathcal{B}_{t,s}(S^{-\infty})$.

We note that $E(t, s)$ is represented as the sum of Fourier integral operators which have phase functions $\phi_j(t, s, x, \xi)$ defined as the solutions of eiconal equations:

$$\begin{cases} \partial_t \phi_j - t^l \mu_j(t, x, \nabla_x \phi_j) = 0 & (0 \leqq s \leqq t \leqq T_0), \\ \phi_j(s, s) = x \cdot \xi, \end{cases} \tag{0.9}$$

and have symbols in $\bigcap_{0<\varepsilon<1} S[0, M+\varepsilon, -M-\varepsilon]$. The constant M is defined by

$$M = \max_{1 \leqq i \leqq m} \lim_{R \to \infty} \sup_{\substack{x, t\langle\xi\rangle^\omega \geqq R \\ 0 \leqq t \leqq R^{-1}}} \{t\, \mathcal{I}m\, f_j(t, x, \xi)\}, \tag{0.10}$$

and indicates the order of regularity-loss of the solution of the Cauchy problem.

Concerning the problem (0.5) Kumano-go [7] constructed the fundamental solution without condition (0.3) ii) by using Fourier integral operators of multi-phase. It should be emphasized that our fundamental solution $E(t, s)$ is represented by Fourier integral operators of single phase, and M is determined explicitly by (0.10). The perfect diagonzalization of (0.6) for L enable us such a construction of $E(t, s)$.

In §1 we define some classes of pseudo-differential operators and Fourier integral operators as variants of classes in Boutet de Monvel [2], and summarize fundamental theorems on operators of these classes. In §2, using a similar method to that of Kumano-go [6], we construct the perfect diagonalizer $N(t)$ such that (0.6) holds. We note that $\sigma(\boldsymbol{R}(t))(x, \xi)\in\mathcal{H}^{\infty}$ and $\in S^{-\infty}$ for any fixed $t>0$, but that $\mathcal{H}^{\infty}\not\subset\mathcal{B}_t(S^{-\infty})$ on $[0, T]$. So we can not apply the method in Kumano-go [6] directly. From this reason, in §3, we first treat a single operator and then construct the fundamental solution $\boldsymbol{E}_2(t, s)$ for a purely diagonal operator $\boldsymbol{L}_2=\boldsymbol{L}_1-\boldsymbol{R}(t)$. In §4 the fundamental solution $\boldsymbol{E}_1(t, s)$ for $\boldsymbol{L}_1$ is constructed in the form

$$\boldsymbol{E}_1(t, s) = \boldsymbol{E}_2(t, s)(I+\boldsymbol{Q}(t, s))+\boldsymbol{Q}_{\infty}(t, s)\,,$$

and by using $\boldsymbol{E}_1(t, s)$ the fundamental solution $\boldsymbol{E}(t, s)$ for the general $\boldsymbol{L}$ can be constructed. The crucial point in the discussions of §4 is in finding $\boldsymbol{Q}(t, s)$. Finally in §5 we consider a higher order operator L of the form:

$$L = D_t^m+\sum_{k=1}^{m} a_k(t, X, D_x)D_t^{m-k}\,, \tag{0.11}$$

where $a_k(t, x, \xi)$ have the forms

$$a_k(t, x, \xi) = \sum_{j=0}^{k} t^{\sigma(j,k)}a_{k,j}(t, x, \xi) \tag{0.12}$$

with differential polynomials $a_{k,j}(t, x, \xi)$ of order $k-j$ in ξ and $\sigma(j, k)=\max\{0, (k-j)(l+1)-k\}$. We assume that the roots $\mu_1, \cdots, \mu_m$ of the equation

$$\lambda^m+a_{1,0}\lambda^{m-1}+\cdots+a_{m,0} = 0 \tag{0.13}$$

are real and satisfy (0.3). Then, we show that the Cauchy problem:

$$\begin{cases} Lu = \varphi(t) & \text{on } [s, T_0] \\ D_t^j|_{t=s} = \psi_j\,, & j=0, \cdots, m-1 \end{cases} \tag{0.14}$$

is reduced to the system (0.5) by modifying the method in Shinaki [11]. We note that the operator L of this type is a generalization of operators which have been treated by Alinhac [1], Chi Min-You [3], Nakamura [8], Nakamura and Uryu [9], Oleinik [10], Uryu [13] and Yoshikawa [14], [15].

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1. Preliminaries. For $x\in R_x^n$, $\xi\in R_\xi^n$ and multi-indices α, β we use the following notation:

$$x\cdot\xi = x_1\xi_1+\cdots+x_n\xi_n\,,\quad \langle\xi\rangle = (1+|\xi|^2)^{1/2}\,,$$
$$|\alpha| = \alpha_1+\cdots+\alpha_n\,,\qquad \alpha! = \alpha_1!\cdots\alpha_n!\,.$$
$$đ\xi = (2\pi)^{-n}d\xi\,,\quad D_t = -i\partial/\partial t\,,\quad \partial_{\xi_j} = \partial/\partial_{\xi_j}\,,$$
$$D_{x_j} = -i\partial/\partial x_j\,,$$

$$\partial_\xi^\alpha = \partial_{\xi_1}^{\alpha_1}\cdots\partial_{\xi_n}^{\alpha_n}, \qquad D_x^\beta = D_{x_1}^{\beta_1}\cdots D_{x_n}^{\beta_n},$$
$$a_{(\beta)}^{(\alpha)}(x,\xi) = \partial_\xi^\alpha D_x^\beta a(x,\xi),$$
$$\nabla_x f(x) = (\partial_{x_1} f(x), \cdots, \partial_{x_n} f(x)).$$

Let $S^\nu(=S^\nu_{1,0})$ denote Hörmander's class of symbols $a(x,\xi)$ on R^n which satisfy

$$|a_{(\beta)}^{(\alpha)}(x,\xi)| \leqq C_{\alpha\,\beta}\langle\xi\rangle^{\nu-|\alpha|} \quad \text{on} \quad R_x^n\times R_\xi^n, \tag{1.1}$$

and the associated pseudo-differential operators $a(X, D_x)$ are defined by

$$\begin{aligned} a(X,D_x)u(x) &= Os-\iint e^{-iy\cdot\xi}a(x,\xi)u(x+y)d\xi dy \\ &= \lim_{\varepsilon\to 0}\iint e^{-iy\cdot\xi}\chi(\varepsilon\xi,\varepsilon y)a(x,\xi)u(x+y)d\xi dy \\ &\qquad (u\in\mathcal{B}(R^n)), \end{aligned} \tag{1.2}$$

where $\chi(\xi, y)\in\mathcal{S}$ (the Schwartz space of rapidly decreasing functions on R^{2n}) such that $\chi(0,0)=1$ and $\mathcal{B}(R^n)$ denotes the space of C^∞-functions in R^n whose derivatives of any order are all bounded.

Let $\chi(t)$ be a C^∞-function in R^1 such that

$$\begin{cases} 0\leqq\chi(t)\leqq 1 & \text{on} \quad R^1, \\ \chi(t) = 1 \ (|t|\leqq 1), & = 0 \ (|t|\geqq 2). \end{cases} \tag{1.3}$$

Set $\omega=1/(l+1)$ for a positive integer l and define a function η by

$$\eta(t) = \eta(t,\xi) = t+\langle\xi\rangle^{-\omega}\chi(t\langle\xi\rangle^{\omega}). \tag{1.4}$$

Then we have

$$\begin{cases} (t+\langle\xi\rangle^{-\omega})/2\leqq\eta(t,\xi)\leqq t+\langle\xi\rangle^{-\omega}, \\ \langle\xi\rangle^{-\omega}\leqq\eta(t,\xi)\leqq 2 \qquad (0\leqq t\leqq T\leqq 1) \end{cases} \tag{1.5}$$

and by easy calculation

$$|D_t^j\partial_\xi^\alpha\eta(t,\xi)|\leqq C_{j,\alpha}\langle\xi\rangle^{-|\alpha|}\eta(t,\xi)^{1-j}. \tag{1.6}$$

Following Boutet de Monvel [2] we define classes of symbols of pseudo-differential operators.

DEFINITION 1.1. i) For real m_1, m_2 we denote by $S[m_1, m_2]$ the space of all C^∞-symbols $a(t,x,\xi)$ on $[0,T]\times R_x^n\times R_\xi^n$ $(0\leqq T\leqq 1)$ such that for any non-negative integer j and multi-indices α, β we have

$$|D_t^j a_{(\beta)}^{(\alpha)}(t,x,\xi)|\leqq C_{j,\alpha,\beta}\langle\xi\rangle^{m_1-|\alpha|}\eta(t,\xi)^{m_2-j}. \tag{1.7}$$

ii) For real m_1, m_2, m_3 we denote by $S[m_1, m_2, m_3]$ the space of all C^∞-

symbols $a(t, s, x, \xi)$ on $[0, T]\times[0, T]\times R^n_x\times R^n_\xi$ $(0\leqq T\leqq 1)$ such that for any non-negative integers j, k and multi-indices α, β we have

$$|D_t^j D_s^k a_{(\beta)}^{(\alpha)}(t, s, x, \xi)|\leqq C_{j,k,\alpha,\beta}\langle\xi\rangle^{m_1-|\alpha|}\eta(t, \xi)^{m_2-j}\eta(s, \xi)^{m_3-k}. \tag{1.8}$$

iii) We set

$$\mathcal{B}_t(S^{-\infty}) = \bigcap_\nu S[m_1-\nu, m_2],$$
$$\mathcal{B}_{t,s}(S^{-\infty}) = \bigcap_\nu S[m_1-\nu, m_2, m_3],$$
$$\mathcal{H}^m = \bigcap_\nu S[m-\nu, -\nu(l+1)].$$

REMARK. 1°. From (1.5) and (1.6) we have

$$\eta(t, \xi)^\nu\in S[0, \nu] \qquad \text{for real } \nu \tag{1.9}$$

and

$$\begin{cases} a(t, x, \xi)\in S[m_1, m_2] \Rightarrow a(t, x, \xi)\in S^{m,\omega m_2^-}, \\ a(t, s, x, \xi)\in S[m_1, m_2, m_3] \\ \Rightarrow a(t, s, x, \xi)\in S^{m_1+\omega m_2^-+\omega m_3^-}\ (m_j^- = \max\{0, -m_j\}) \\ \text{for any fixed } t \text{ and } s\in[0, T]. \end{cases} \tag{1.10}$$

2°. We can consider $a(t, x, \xi)\in S[m_1, m_2]$ as an element of $S[m_1, m_2, 0]$. So by this identification we write $S[m_1, m_2]\subset S[m_1, m_2, 0]$, and the statements for the symbols of $S[m_1, m_2, m_3]$ often hold for symbols of $S[m_1, m_2]$.

3°. It is easily proved that

$$\bigcap_\nu S[m_1-\nu, m_2] = \bigcap_\nu \mathcal{B}_t(S^{-\nu})$$

and

$$\bigcap_\nu S[m_1-\nu, m_2, m_3] = \bigcap_\nu \mathcal{B}_{t,s}(S^{-\nu}).$$

Proposition 1.2. i) *$S[m_1, m_2]\cap S[m_1', m_2']$, if $m_1\leqq m_1'$ and $m_1-m_2\omega\leqq m_1'-m_2'\omega$.*

ii) *$S[m_1, m_2, m_3]\subset S[m_1', m_2', m_3']$ if "$m_1\leqq m_1'$, $m_1-m_2\omega\leqq m_1'-m_2'\omega$ and $m_3\geqq m_3'$" or "$m_1\leqq m_1'$, $m_1-m_3\omega\leqq m_1'-m_3'\omega$ and $m_2\geqq m_2'$".*

Proof is omitted.

Proposition 1.3. i) *Let $a(t, s, x, \xi)\in S[m_1, m_2, m_3]$. Then, for any non-negative integers j, k we have*

$$t^j s^k a(t, s, x, \xi)\in S[m_1, m_2+j, m_3+k] \tag{1.11}$$

and

$$D_t^j D_s^k a(t, s, x, \xi) \in S[m_1, m_2-j, m_3-k]\,. \tag{1.12}$$

ii) *Let* $a(t, s, x, \xi) \in S[m_1, m_2, m_3]$ *and* $b(t, s, x, \xi) \in S[m_1', m_2', m_3']$. *Then, we have*

$$a(t, s, x, \xi) b(t, s, x, \xi) \in S[m_1+m_1', m_2+m_2', m_3+m_3']\,. \tag{1.13}$$

iii) *Let* $a(t, x, \xi) \in \mathcal{H}^m$ *and* $b(t, x, \xi) \in S[m_1, m_2]$. *Then, we have*

$$a(t, x, \xi) b(t, x, \xi) \in \mathcal{H}^{m+m_1-m_2\omega}\,. \tag{1.14}$$

Proof. i) and ii) are clear. Writing $\eta(t, \xi)^{m_2} = \langle\xi\rangle^{-m_2\omega}(\langle\xi\rangle^{\omega}\eta(t, \xi))^{m_2}$ we get (1.14).

Lemma 1.4. *Set*

$$h(t, \xi) = \eta(t, \xi)^l \langle\xi\rangle\,. \tag{1.15}$$

Then, we have

i) $h(t, \xi)^\nu \in S[\nu, \nu l]$ *for any real* ν,
ii) $h(t, \xi) - t^l\langle\xi\rangle \in \mathcal{H}^{\omega}$,
iii) $ih_t(t, \xi)/h(t, \xi) - l/\eta(t, \xi) \in \mathcal{H}^{\omega}$,

where $h_t(t, \xi) = D_t h(t, \xi)$.

Proof. i) is clear. Since $I(t, \xi) = h(t, \xi)) - t^l\langle\xi\rangle = 0$ when $t\langle\xi\rangle^{\omega} \geqq 2$, $\eta(t, \xi)\langle\xi\rangle^{\omega}$ is bounded on supp $I(t, \xi)$. So we have ii). Since $i\eta_t(t, \xi) = 1 + i\chi_t(t\langle\xi\rangle^{\omega})$ and $\chi_t(t\langle\xi\rangle^{\omega}) \in \mathcal{H}^0$, we have by Proposition 1.3-iii)

$$\begin{aligned} &ih_t(t, \xi)/h(t, \xi) - l/\eta(t, \xi) \\ &= il\chi_t(t\langle\xi\rangle^{\omega})/\eta(t, \xi) \in \mathcal{H}^{\omega}\,. \end{aligned}$$

Proposition 1.5. i) *Let* $a_\nu(t, s, x, \xi) \in S[m_1-\nu, m_2, m_3]$ *for* $\nu = 0, 1, \cdots$. *Then, there exists an* $a(t, s, x, \xi) \in S[m_1, m_2, m_3]$ *such that*

$$a \sim a_0 + a_1 + \cdots \qquad \text{mod } \mathcal{B}_{t,s}(S^{-\infty})$$

in the sense

$$a - \sum_{\nu=0}^{N-1} a_\nu \in S[m_1-N, m_2, m_3] \qquad \text{for all } N\,.$$

Two such symbols differ by an element of $\mathcal{B}_{t,s}(S^{-\infty})$.

ii) *Let* $b_\nu(t, x, \xi) \in S[m_1-\nu, m_2-\nu(l+1)]$ *for* $\nu = 0, 1, \cdots$. *Then, there exists a* $b(t, x, \xi) \in S[m_1, m_2]$ *such that*

$$b \sim b_0 + b_1 + \cdots \qquad \text{mod } \mathcal{H}^{m_1-m_2\omega}$$

in the sense

$$b-\sum_{\nu=0}^{N-1} b_\nu \in S[m_1-N,\ m_2-N(l+1)] \qquad \text{for all } N.$$

Two such symbols differ by an element of $\mathcal{H}^{m_1-m_2\omega}$.

Proof. Using $\chi(t)$ of (1.3) we set

$$\text{(1.16)} \qquad \begin{cases} \psi_\varepsilon(\xi) = 1-\chi(\varepsilon\langle\xi\rangle) \\ \gamma_\varepsilon(t,\xi) = 1-\chi(\varepsilon\eta(t,\xi)^{l+1}\langle\xi\rangle). \end{cases}$$

Then, setting

$$a(t, s, x, \xi) = \sum_{\nu=0}^{\infty} \psi_{\varepsilon_\nu}(\xi) a_\nu(t, s, x, \xi)$$

and

$$b(t, x, \xi) = \sum_{\nu=0}^{\infty} \gamma_{\varepsilon_\nu}(t, \xi) b_\nu(t, x, \xi)$$

for appropriate $1 \geqq \varepsilon_0 > \varepsilon_1 > \cdots > \varepsilon_\nu > \cdots \to 0$, we get i) and ii) by usual method.

Proposition 1.6. *Let* $a(t, s, x, \xi) S \in [m_1, m_2, m_3]$ *and* $b(t, s, x, \xi) \in S[m_1', m_2', m_3']$ *and define* $a \circ b(t, s, x, \xi)$ *by*

$$\text{(1.17)} \qquad \begin{aligned} &a\circ b(t, s, x, \xi) \\ &= 0s-\int e^{-iy\cdot\xi'} a(t, s, x, \xi+\xi') b(t, s, x+y, \xi) d\!\!{}^-\xi' dy. \end{aligned}$$

Then, we have

$$\text{(1.18)} \qquad a\circ b(t, s, x, \xi) \in S[m_1+m_1', m_2+m_2', m_3+m_3']$$

and for $A = a(t, s, X, D_x)$, $B = b(t, s, X, D_x)$ *we have*

$$AB = a\circ b(t, s, X, D_x).$$

Moreover, we have

$$\text{(1.19)} \qquad a\circ b(t, s, x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} a^{(\alpha)}(t, s, x, \xi) b_{(\alpha)}(t, s, x, \xi) \qquad \text{mod } \mathcal{B}_{t,s}(S^{-\infty}).$$

Proof. If we note Remark 1° of Definition 1.1, the proof is clear.

Corollary 1.7. *Let* $a(t, x, \xi) \in S[m_1, m_2]$ *and* $b(t, x, \xi) \in \mathcal{H}^m$. *Then, both* $a\circ b(t, x, \xi)$ *and* $b\circ a(t, x, \xi)$ *belong to* $\mathcal{H}^{m+m_1-m_2\omega}$.

When $\boldsymbol{A}(t)$ is an $m\times m$ matrix of pseudo-differential operators with symbols in $S[m_1, m_3]$, we also write $\sigma(\boldsymbol{A}(t)) \in S[m_1, m_2]$. We denfie $|\sigma(\boldsymbol{A}(t))|$ by

$$|\sigma(\boldsymbol{A}(t))| = \max_{1\leqq j,k\leqq m} |a_{j,k}(t, x, \xi)|,$$

where $a_{j,k}(t, x, \xi)$ is the (j, k)-element of $\sigma(A(t))(x, \xi)$.

Lemma 1.8. *Let $\sigma(N^{(\nu)}(t))(x, \xi)\in S[-\nu, -\nu(l+1)]$ $\nu=1, 2, \cdots$, be $m\times m$ matrices. Then, there exists $N(t)$ such that $\sigma(N(t))(x, \xi)\in S[0, 0]$ and*

$$N(t)\sim I+N^{(1)}(t)+N^{(2)}(t)+\cdots \quad \mathrm{mod}\ \mathcal{H}^0 . \tag{1.20}$$

Moreover, $N(t)$ has a parametrix $N(t)^{\sharp}$ such that $\sigma(N(t)^{\sharp})(x, \xi)\in S[0, 0]$ and

$$\sigma(N(t)N(t)^{\sharp}-I), \quad \sigma(N(t)^{\sharp}N(t)-I)\in\mathcal{B}_t(S^{-\infty}) .$$

Proof. Let $\gamma_\varepsilon(t, \xi)$ be the symbol defined by (1.16). Then, by Proposition 1.5-ii) we see that

$$\sigma(N(t))(x, \xi) = I+\sum_{\nu=1}^{\infty}\gamma_{\varepsilon_\nu}(t, \xi)\sigma(N^{(\nu)}(t))(x, \xi)$$

belongs to $S[0, 0]$ for appropriate $1\geqq\varepsilon_1>\cdots>\varepsilon_\nu>\cdots\rightarrow 0$ and (1.20) holds. Furthermore, noting

$$|D_t^j\gamma_{\varepsilon_\nu}^{(\alpha)}(t, \xi)|\leqq C_{j,\alpha}\varepsilon_\nu\eta(t, \xi)^{l+1}\langle\xi\rangle\langle\xi\rangle^{-|\alpha|}\eta(t, \xi)^{-j}$$

and $\eta(t, \xi)^{l+1}\langle\xi\rangle\sigma(N^{(\nu)}(t))(x, \xi)\in S[-(\nu-1), -(\nu-1)(l+1)]\subset S[0, 0], \nu=1, 2, \cdots$, we get $|\det\sigma(N(t))(x, \xi)|\geqq c$ for a constant $c>0$, if we choose small $\varepsilon_\nu>0$. Noting Remark 1° of Definition 1.1, the parametrix $N(t)^{\sharp}$ of $N(t)$ can be constructed by usual procedure.

According to Kumano-go [5] we call a real valued C^∞-function $\phi(x, \xi)$ in $R_x^n\times R_\xi^n$ a *phase function*, when it satisfies conditions:

$$\left\{\begin{array}{ll} \text{i)} & \phi(x, \xi)-x\cdot\xi\in S^1 \\ \text{ii)} & |\nabla_x\phi(x, \xi)-\xi|\leqq(1-\varepsilon_0)|\xi|+c \\ \text{iii)} & |\nabla_x\nabla_\xi\phi(x, \xi)-I|\leqq 1-\varepsilon_0' \\ & (0<\varepsilon_0\leqq 1,\ 0<\varepsilon_0'\leqq 1,\ c>0) . \end{array}\right. \tag{1.21}$$

Then the Fourier integral operator $A_\phi=a_\phi(X, D_x)$ with phase function $\phi(x, \xi)$ and symbol $a(x, \xi)\in S^m$ is defined by.

$$A_\phi u(x) = Os-\iint e^{i(\phi(x,\xi)-x'\cdot\xi)}a(x, \xi)u(x')d\xi dx'(u\in\mathcal{B}(R_x^n)) . \tag{1.22}$$

Concerning fundamental theorems on Fourier integral operators, we refer to §2 of [5].

Let $\lambda(t, x, \xi)\in S[1, l]$ be real valued. Consider the Hamilton equation

$$\left\{\begin{array}{l} \dfrac{dt}{dq} = -\nabla_\xi\lambda(t, q, p), \quad \dfrac{dp}{dt} = \nabla_x\lambda(t, q, p) \qquad \text{on } 0\leqq s,\ t\leqq T_0 , \\ \{q, p\}_{t=s} = \{y, \xi'\} \end{array}\right. \tag{1.23}$$

and the eiconal equation

$$(1.24)\qquad \begin{cases} \partial_t\phi-\lambda(t,x,\nabla_x\phi)=0 & \text{on } 0\leqq s,\ t\leqq T_0, \\ \phi(s,s,x,\xi)=x\cdot\xi \end{cases}$$

for a small $T_0\,(0<T_0\leqq T)$. Then, we can prove the following statements by the same procedure to §3 in [5].

Lemma 1.9. *For a small* $T_1\,(0<T_1\leqq T)$ *the initial value problem* (1.23) *has the solution* $\{q,p\}\ (t,s,y,\xi')$ *on* $0\leqq s,\ t\leqq T_1$ *such that*

$$(1.25)\qquad \begin{cases} q(t,s,y,\xi')-y\in S[0,\,l+1,\,0] & (0\leqq s\leqq t\leqq T_1), \\ p(t,s,y,\xi')-\xi'\in S[1,\,l+1,\,0] & (0\leqq s\leqq t\leqq T_1) \end{cases}$$

and

$$(1.25)'\qquad \begin{cases} q(t,s,y,\xi')-y\in S[0,\,0,\,l+1] & (0\leqq t\leqq s\leqq T_1), \\ p(t,s,y,\xi')-\xi'\in S[1,\,0,\,l+1] & (0\leqq t\leqq s\leqq T_1). \end{cases}$$

Lemma 1.10. *Let* $T_2\,(0<T_2\leqq T_1)$ *and* $\varepsilon_1\,(0<\varepsilon_1\leqq 1)$ *be constants such that*

$$|\partial q/\partial y-I|\leqq(1-\varepsilon_1)\qquad 0\leqq s,\ t\leqq T_1$$

Then, for the mapping $x=q(t,s,y,\xi)\colon R^n_y\ni y\to x\in R^n_x$ *with* (t,s,ξ) *as parameters, there exists the inverse* $y=y(t,s,x,\xi)$ *such that*

$$(1.26)\qquad \begin{cases} y(t,s,x,\xi)-x\in S[0,\,l+1,\,0] & 0\leqq s\leqq t\leqq T_2 \\ y(t,s,x,\xi)-x\in S[0,\,0,\,l+1] & 0\leqq t\leqq s\leqq T_2 \\ |\partial y/\partial x-I|\leqq(1-\varepsilon_1)/\varepsilon_1. \end{cases}$$

Theorem 1.11. *There exists* $T_0\,(0<T_0\leqq T)$ *such that the initial value problem* (1.24) *has the unique solution* $\phi(t,s)=\phi(t,s,x,\xi)$ *on* $0\leqq s,\ t\leqq T_0$ *which satisfies* (1.21) *and*

$$(1.27)\qquad \begin{cases} \phi(t,s,x,\xi)-x\cdot\xi\in S[1,\,l+1,\,0] & (0\leqq s\leqq t\leqq T_0), \\ \phi(t,s,x,\xi)-x\cdot\xi\in S[1,\,0,\,l+1] & (0\leqq t\leqq s\leqq T_0). \end{cases}$$

Corollary 1.12. *For a* C^∞*-function* $f(t,x,\xi)$ *on* $[0,T_0]\times R^n_x\times R^n_\xi$ *set*

$$\tilde f(t,s,y,\xi)=f(t,q(t,s,y,\xi),\xi).$$

Then, we have

$$(1.28)\qquad \begin{aligned} &D_t\tilde f(t,s,y,\xi)\\ &=\{D_tf(t,x,\xi)-\sum_{|\alpha|=1}\lambda^{(\alpha)}(t,x,\nabla_x\phi(t,s,x,\xi))f_{(\alpha)}(t,x,\xi)\}_{x=q(t,s,y,\xi)}. \end{aligned}$$

The following lemma is important in the proof of Theorem 4.2 in §4.

Lemma 1.13. *Let* $a(t, s, x, \xi)\in S[m_1, m_2, m_3]$ *and* $r(t, s, x, \xi)\in \bigcap_{\nu=0}^{\infty} S[m_1'-\nu, m_2'-\nu(l+1), m_3']$. *Set* $A_\phi=a_\phi(t, s, X, D_x)$ *with* $\phi(t, s, x, \xi)$ *of Theorem* 1.11 *and* $R=r(t, s, X, D_x)$. *Then both* $R_1=A_\phi R$ *and* $R_2=RA_\phi$ *are pseudo-differential operators with symbols*

$$(1.29)\qquad r_j(t, s, x, \xi)\in \bigcap_{\nu=0}^{\infty} S[m_1+m_1'-\nu, m_2+m_2'-\nu(l+1), m_3+m_3'] \qquad (j=1, 2),$$

where

$$(1.30)\qquad r_1(t, s, x, \xi') = Os-\iint e^{i\varphi_1}a(t, s, x, \xi)r(t, s, x', \xi')d\!\!\!^-\xi dx'$$

with

$$\varphi_1 = \phi(t, s, x, \xi)-x\cdot\xi+(x-x')\cdot(\xi-\xi')$$

and

$$(1.31)\qquad r_2(t, s, x, \xi') = Os-\iint e^{i\varphi_2}r(t, s, x, \xi)a(t, s, x', \xi')d\!\!\!^-\xi dx'$$

with

$$\varphi_2 = (x-x')\cdot(\xi-\xi')+\phi(t, s, x', \xi')-x'\cdot\xi' .$$

Moreover, we have

$$(1.32)\qquad r_1(t, s, x, \xi)\sim\sum_\alpha \frac{1}{\alpha!}\tilde{a}^{(\alpha)}(t, s, x, \xi)r_{(\alpha)}(t, s, x, \xi) \qquad \bmod \mathcal{B}_{t,s}(S^{-\infty})$$

and

$$(1.33)\qquad r_2(t, s, x, \xi)\sim\sum_\alpha \frac{1}{\alpha!}r^{(\alpha)}(t, s, x, \xi)\tilde{a}_{(\alpha)}(t, s, x, \xi) \qquad \bmod \mathcal{B}_{t,s}(S^{-\infty}),$$

where

$$(1.34)\qquad \tilde{a}(t, s, x, \xi) = e^{i(\phi(t,s,x,\xi)-x\cdot\xi)}a(t, s, x, \xi) .$$

Proof. It is clear that r_1 and r_2 are defined by (1.30) and (1.31), respectively. By Theorem 1.11 we have

$$(1.35)\qquad |D_t^j D_s^k \tilde{a}_{(\beta)}^{(\alpha)}(t, s, x, \xi)| \\ \leq C_{j,k,\alpha,\beta}\langle\xi\rangle^{m_1-|\alpha|}(\eta(t, \xi)^{l+1}\langle\xi\rangle)^{|\alpha+\beta|+j+k}\eta(t, \xi)^{m_2-j}\eta(s, \xi)^{m_3-k} .$$

On the other hand by the assumption for $r(t, s, x, \xi)$ we have

$$(1.36)\qquad \eta((t, \xi)^{l+1}\langle\xi\rangle)^{\tau}r(t, s, x, \xi)\in \bigcap_{\nu=0}^{\infty} S[m_1'-\nu, m_2'-\nu(l+1), m_3'] \quad \text{for any } \tau .$$

Then, from (1.35) and (1.36) we see that

(1.37) $$\tilde{a}^{(\alpha)}(t, s, x, \xi) r_{(\alpha)}(t, s, x, \xi) \in \bigcap_{\nu=0}^{\infty} S[m_1+m_1'-\nu-|\alpha|, m_2+m_2'-\nu(l+1), m_3+m_3'] .$$

Now we write

$$r_1(t, s, x, \xi') = Os-\iint e^{-iy\cdot\xi''} \tilde{a}(t, s, x, \xi'+\xi'') r(t, s, x+y, \xi') d\xi'' dy .$$

Then, by Taylor's expansion

$$\tilde{a}(t, s, x, \xi'+\xi'') = \sum_{|\alpha|<N} \frac{\xi''^{\alpha}}{\alpha!} \tilde{a}^{(\alpha)}(t, s, x, \xi') + N \sum_{|\alpha|=N} \frac{\xi''^{\alpha}}{\alpha!} \int_0^1 (1-\theta)^{N-1} \tilde{a}^{(\alpha)}(t, s, x, \xi'+\theta\xi'') d\theta ,$$

we have

(1.38) $$r_1(t, s, x, \xi') = \sum_{|\alpha|<N} \frac{1}{\alpha!} \tilde{a}^{(\alpha)}(t, s, x, \xi') r_{(\alpha)}(t, s, x, \xi') + N \sum_{|\alpha|=N} \frac{1}{\alpha!} \int_0^1 (1-\theta)^{N-1} h_{\alpha\,\theta}(t, s, x, \xi') d\theta ,$$

where

(1.39) $$\begin{aligned} &h_{\alpha,\theta}(t, s, x, \xi') \\ &= Os-\iint e^{-iy\cdot\xi''} \tilde{a}^{(\alpha)}(t, s, x, \xi'+\theta\xi'') r_{(\alpha)}(t, s, x+y, \xi') d\xi'' dy \\ &= Os-\iint e^{-iy\cdot\xi'} \langle\xi''\rangle^{-\tilde{n}} \tilde{a}^{(\alpha)}(t, s, x, \xi'+\theta\xi'') \\ &\times \langle D_y\rangle^{\tilde{n}} r_{(\alpha)}(t, x, s+y, \xi') d\xi'' dy \end{aligned}$$

for any even integer $\tilde{n} \geqq 0$. Then, noting

$$\begin{cases} \langle\xi'+\theta\xi''\rangle^{\pm 1} \leqq 2\langle\xi''\rangle\langle\xi'\rangle^{\pm 1} , \\ \eta(t, \xi'+\xi'')^{\pm 1} \leqq 2^{\omega}\langle\xi''\rangle^{\omega}\eta(t, \xi')^{\pm 1} \end{cases}$$

and using (1.35) we see from the assumption for $r(t, s, x, \xi)$ that

$$\{h_{\alpha,\theta}(t, s, x, \xi)\}_{|\alpha|=N, 0\leqq\theta\leqq 1} \text{ is bounded in } \bigcap_{\nu=0}^{\infty} S[m_1+m_1'-N-\nu, m_2+m_2'-\nu(l+1), m_3+m_3'] .$$

Hence, from (1.38), (1.39) we get (1.29) for $j=1$ and (1.32). By the same method we get the statement for $r_2(t, s, x, \xi)$.

2. Diagonalization. In this section we consider a hyperbolic $m\times m$ system

$$(2.1)\qquad \boldsymbol{L}_0=\boldsymbol{D}_t-\boldsymbol{A}_1(t)-\boldsymbol{A}_0(t)\qquad \text{on } [0, T]$$

of pseudo-differential operators of first order, where

$$\boldsymbol{D}_t=\begin{bmatrix} D_t & & 0\\ & \ddots & \\ 0 & & D_t\end{bmatrix}$$

and

$$\sigma(\boldsymbol{A}_1(t))(x, \xi)\in S[1, l]\,,\quad \sigma(\boldsymbol{A}_0(t))(x, \xi)\in S[0, -1]$$

for an integer $l>0$. We assume the eigenvalues $\lambda_1(t, x, \xi), \cdots, \lambda_m(t, x, \xi)$ of $\sigma(\boldsymbol{A}_1(t))(x, \xi)$ are all real and belong to $S[1, l]$. Modifying the notion *'prefectly diagonalizable'* in Kumano-go [6] we introduce the following notion.

DEFINITION 2.1. i) For $\eta(t)=\eta(t, \xi)$ defined in (1.4) the operator $\boldsymbol{L}_0$ is said to be $\eta(t)$-diagonalizable, when there exists $\boldsymbol{N}_0(t)$ such that $\sigma(\boldsymbol{N}_0(t))\in S[0, 0]$ and $|\det\sigma(\boldsymbol{N}_0(t))|\geqq c$ on $[0, T]\times R^n_x\times R^n_\xi$ for a constant $c>0$, and we can write

$$(2.2)\qquad \boldsymbol{L}_0\boldsymbol{N}_0(t)\equiv\boldsymbol{N}_0(t)\boldsymbol{L}\qquad \text{mod } \mathcal{B}_t(S^{-\infty})$$

for some $\boldsymbol{L}$ of the form

$$(2.3)\qquad \boldsymbol{L}=\boldsymbol{D}_t-\mathcal{D}(t)+\boldsymbol{B}(t)\qquad \text{on } [0, T]\,,$$

where

$$(2.4)\qquad \sigma(\mathcal{D}(t))(x, \xi)=\begin{bmatrix}\lambda_1(t, x, \xi) & & 0\\ & \ddots & \\ 0 & & \lambda_m(t, x, \xi)\end{bmatrix}$$

and $\sigma(\boldsymbol{B}(t))(x, \xi)\in S[0, -1]$.

ii) The operator $\boldsymbol{L}_0$ is said to be *$\eta(t)$-perfectly diagonalizable*, when there exists $\boldsymbol{N}(t)$ such that $\sigma(\boldsymbol{N}(t))\in S[0, 0]$ and $|\det\sigma(\boldsymbol{N}(t))|>c$ on $[0, T]\times R^n_x\times R^n_\xi$ for a constant $c>0$, and we can write

$$(2.5)\qquad \boldsymbol{L}_0\boldsymbol{N}(t)\equiv\boldsymbol{N}(t)\boldsymbol{L}_1\qquad \text{mod } \mathcal{B}_t(S^{-\infty})$$

for some $\boldsymbol{L}_1$ of the form

$$(2.6)\qquad \boldsymbol{L}_1=\boldsymbol{D}_t-\mathcal{D}(t)+\boldsymbol{F}(t)+\boldsymbol{R}(t)\qquad \text{on } [0, T]\,,$$

where $\sigma(\boldsymbol{F}(t))$ is a diagonal matrix of the form

$$(2.7)\qquad \sigma(\boldsymbol{F}(t))=\begin{bmatrix} f_1(t, x, \xi) & & 0\\ & \ddots & \\ 0 & & f_m(t, x, \xi)\end{bmatrix}\in S[0, -1]$$

and $\sigma(\boldsymbol{R}(t))\in\mathcal{H}^\infty$.

$\boldsymbol{N}_0(t)$, $\boldsymbol{N}(t)$ are called the *diagonalizer*, the *perfect diagonalizer* for $\boldsymbol{L}_0$, respectively.

Theorem 2.2. *For $\boldsymbol{L}$ of* (2.3), *assume that there exists a constant $c_0>0$ such that*

$$|\lambda_j(t, x, \xi)-\lambda_k(t, x, \xi)|\geqq c_0\eta(t)^l\langle\xi\rangle \qquad (j\neq k)\,. \tag{2.8}$$

Then, $\boldsymbol{L}$ is $\eta(t)$-perfectly diagonalizable.

Proof. According to Kumano-go [6] we find the perfect diagonalizer $\boldsymbol{N}(t)$ such that

$$\begin{cases} \boldsymbol{N}(t)\sim I+\boldsymbol{N}^{(1)}(t)+\boldsymbol{N}^{(2)}(t)+\cdots & \text{mod } \mathcal{H}^0 \\ \sigma\boldsymbol{N}^{(\nu)}((t))\in S[-\nu,\ -\nu(l+1)] & (\nu=1, 2, \cdots)\,, \end{cases} \tag{2.9}$$

$$(\boldsymbol{D}_t-\mathcal{D}(t)+\boldsymbol{B}(t))\boldsymbol{N}(t)\equiv\boldsymbol{N}(t)(\boldsymbol{D}_t-\mathcal{D}(t)+\boldsymbol{F}(t)+\boldsymbol{R}(t)) \quad \text{mod } \mathcal{B}_t(S^{-\infty})\,, \tag{2.10}$$

and

$$\begin{cases} \boldsymbol{F}(t)\sim\boldsymbol{F}^{(0)}(t)+\boldsymbol{F}^{(1)}(t)+\cdots & \text{mod } \mathcal{H}^\infty \\ \sigma(\boldsymbol{F}^{(\nu)}(t))\in S[-\nu,\ -\nu(l+1)-1] & (\nu=0, 1, \cdots)\,. \end{cases} \tag{2.11}$$

Let $b_{j,k}(k)$ bet the (j, k)-elements of $\sigma(\boldsymbol{B}(t))$, and set

$$\begin{cases} \boldsymbol{F}^{(0)}=\text{diag}\,[\boldsymbol{B}] \\ \sigma(\boldsymbol{N}^{(1)})=(n_{j,k}^{(1)}) \quad \text{by} \\ \qquad n_{j,k}^{(1)}=\begin{cases} b_{j,k}/(\lambda_j-\lambda_k) & (j\neq k) \\ 0 & (j=k) \end{cases} \\ \boldsymbol{B}^{(1)}=(\boldsymbol{D}_t-\mathcal{D}+\boldsymbol{B})(I+\boldsymbol{N}^{(1)})-(I+\boldsymbol{N}^{(1)})(\boldsymbol{D}_t-\mathcal{D}+\boldsymbol{F}^{(0)})\,, \end{cases} \tag{2.12$_0$}$$

where by diag $[\boldsymbol{B}]$ we denote a diagonal matrix with the same diagonal with $\boldsymbol{B}$'s. Then, we have

$$\boldsymbol{B}^{(1)}=\boldsymbol{B}-[\mathcal{D},\ \boldsymbol{N}^{(1)}]-\boldsymbol{F}^{(0)}+\boldsymbol{N}_t^{(1)}+\boldsymbol{B}\boldsymbol{N}^{(1)}-\boldsymbol{N}^{(1)}\boldsymbol{F}^{(0)}\,,$$

where $\quad [\mathcal{D},\ \boldsymbol{N}^{(1)}]=\mathcal{D}\boldsymbol{N}^{(1)}-\boldsymbol{N}^{(1)}\mathcal{D}$ and $\sigma(\boldsymbol{N}_t^{(1)})=D_t\sigma(\boldsymbol{N}^{(1)})$.

Since $\sigma(\boldsymbol{B}-[\mathcal{D},\ \boldsymbol{N}^{(1)}]-\boldsymbol{F}^{(0)})\in S[-1,\ -1]$, we have

$$\begin{cases} \sigma(\boldsymbol{F}^{(0)})\in S[0,\ -1]\,, \\ \sigma(\boldsymbol{N}^{(1)})\in S[-1,\ -(l+1)]\,, \\ \sigma(\boldsymbol{B}^{(1)})\in S[-1,\ -(l+1)-1]\,. \end{cases} \tag{2.13$_0$}$$

Now, we assume that $\boldsymbol{F}^{(\mu)}$, $\boldsymbol{N}^{(\mu+1)}$, $\boldsymbol{B}^{(\mu+1)}$, $\mu=0, 1, \cdots, \nu-1$ $(\nu\geqq1)$ are determined as

$$\begin{cases} \sigma(\boldsymbol{F}^{(\mu)})\in S[-\mu,\ -\mu(l+1)-1]\,, \\ \sigma(\boldsymbol{N}^{(\mu+1)})\in S[-(\mu+1),\ -(\mu+1)(l+1)]\,, \\ \sigma(\boldsymbol{B}^{(\mu+1)})\in S[-(\mu+1),\ -(\mu+1)(l+1)-1]\,, \end{cases} \tag{2.13$_\mu$}$$

and define $F^{(\nu)}$, $N^{(\nu+1)}$, $B^{(\nu+1)}$ by

$$(2.12)_\nu \qquad \begin{cases} F^{(\nu)} = \operatorname{diag}[B^{(\nu)}]\,, \\ \sigma(N^{(\nu+1)}) = (n_{j,k}^{(\nu+1)}) \quad \text{by} \\ \qquad n_{j,k}^{(\nu+1)} = \begin{cases} b_{j,k}^{(\nu)}/(\lambda_j-\lambda_k) & (j\neq k) \\ 0 & (j=k) \end{cases} \\ B^{(\nu+1)} = (D_t-\mathcal{D}+B)(I+\sum_{\mu=1}^{\nu+1} N^{(\mu)}) \\ \qquad -(I+\sum_{\mu=1}^{\nu+1} N^{(\mu)})(D_t-\mathcal{D}+\sum_{\mu=0}^{\nu} F^{(\mu)})\,, \end{cases}$$

where $b_{j,k}^{(\nu)}$ are the (j, k)-elements of $\sigma(B^{(\nu)})$.

Then, we have

$$B^{(\nu+1)} = (B^{(\nu)}-[\mathcal{D}, N^{(\nu+1)}]-F^{(\nu)})+N_t^{(\nu+1)} + BN^{(\nu+1)} - \sum_{\mu=1}^{\nu} N^{(\mu)}F^{(\nu)} - N^{(\nu+1)}\sum_{\mu=0}^{\nu} F^{(\mu)}\,,$$

and by the definition of $F^{(\nu)}$ and $N^{(\nu+1)}$ we have

$$\sigma(B^{(\nu)}-[\mathcal{D}, N^{(\nu+1)}]-F^{(\nu)})\in S[-(\nu+1), -\nu(l+1)-1]\,.$$

Hence we get $(2.13)_\mu$ for $\mu=\nu$, and by induction, for any $\mu=0, 1, \cdots$.

Now, by Proposition 1.5-ii) there exist $N(t)$ and $F(t)$ such that (2.9) and (2.11) hold. We set

$$\tilde{R} = LN - N(D_t-\mathcal{D}+F)\,.$$

Then, we have $\sigma(\tilde{R})\in\mathcal{H}^\infty$. Let $N^\sharp$ be a parametrix of N which exists by Lemma 1.8, and set $R=N^\sharp\tilde{R}$. Then $\sigma(R)\in\mathcal{H}^\infty$ and

$$\sigma(LN(t)-N(t)(D_t-\mathcal{D}+F(t)+R(t)))\in\mathcal{B}_t(S^{-\infty})\,.$$

Corollary 2.3. *Let L_0 be $\eta(t)$-diagonalizable. Assume that the eigenvalues $\lambda_1(t, x, \xi), \cdots, \lambda_m(t, x, \xi)$ of $\sigma(A_1(t))$ satisfy* (2.8). *Then L_0 is perfectly diagonalizable.*

3. Construction of fundamental solution. The first order single operator case.

Let L be a single hyperbolic operator of the form

$$(3.1) \qquad L = D_t-\lambda(t, X, D_x)+f(t, X, D_x) \qquad \text{on } [0, T] \quad (0<T\leqq 1)\,,$$

where

$$(3.2) \qquad \begin{cases} \lambda(t, x, \xi)\in S[1, l] & \text{real valued,} \\ f(t, x, \xi)\in S[0, -1]\,. \end{cases}$$

Consider the Cauchy problem

$$(3.3)\qquad \begin{cases} Lu=\varphi(t) & \text{on } [s, T_0] \\ u|_{t=s}=\psi & (0\leqq s\leqq T_0) \end{cases}$$

for a small T_0 $(0<T_0\leqq T)$.

Theorem 3.1. *Set*

$$(3.4)\qquad M=\lim_{R\to\infty}\sup_{\substack{x,t\langle\xi\rangle^{\omega}\geqq R\\ 0\leqq t\leqq R^{-1}}}\{t\,\mathcal{I}m\,f(t, x, \xi)\}\,.$$

Then, there exists uniquely a symbol $e(t, s, x, \xi)$ *in the class* $\bigcap_{0<\varepsilon<1} S[0, M+\varepsilon, -M-\varepsilon]$ *on* $0\leqq s\leqq t\leqq T_0$ *(with* T_0 *of Theorem* 1.11*) such that the Fourier integral operator* $E_\phi(t, s)=e_\phi(t, s, X, D_x)$ *with phase function* $\phi(t, s, x, \xi)$ *given by Theorem* 1.11 *is the fundamental solution of the Cauchy problem*(3.1) *for* L*, i.e.,*

$$(3.5)\qquad \begin{cases} LE_\phi(t, s)=0 & \text{on } 0\leqq s\leqq t\leqq T_0\,, \\ E_\phi(s, s)=I & \text{(identity operator)}\,. \end{cases}$$

REMARK. Since $(t+\langle\xi\rangle^{-\omega})(1-1/(t\langle\xi\rangle^{\omega}+1))\leqq(t+\langle\xi\rangle^{-\omega})-\langle\xi\rangle^{-\omega}\leqq\eta(t, \xi)\leqq t+\langle\xi\rangle^{-\omega}$, and $\langle\xi\rangle^{-\omega}\leqq t/R$ when $t\langle\xi\rangle^{\omega}\geqq R$, we have

$$(3.4)'\qquad M=\lim_{R\to\infty}\sup_{\substack{x,t\langle\xi\rangle^{\omega}\geqq R\\ 0\leqq t\leqq R^{-1}}}\{(t+\langle\xi\rangle^{-\omega})\,\mathcal{I}m\,f(t, x, \xi)\}\,,$$

$$(3.4)''\qquad M=\lim_{R\to\infty}\sup_{\substack{x,t\langle\xi\rangle^{\omega}\geqq R\\ 0\leqq t\leqq R^{-1}}}\{\eta(t, \xi)\,\mathcal{I}m\,f(t, x, \xi)\}\,.$$

Proof. The uniqueness will be proved after Theorem 3.2. Solving transport equations we first construct an approximate fundamental solution $\tilde{E}_\phi(t, s)$ in the sence

$$(3.6)\qquad \begin{cases} L\tilde{E}_\phi(t, s)\equiv 0 & \text{mod } \mathcal{B}_{t,s}(S^{-\infty}) \text{ on } 0\leqq s\leqq t\leqq T_0\,, \\ \tilde{E}_\phi(s, s)=I\,. \end{cases}$$

We assume that the symbol $\tilde{e}(t, s, x, \xi)$ of $\tilde{E}_\phi(t, s)$ has the form:

$$(3.7)\qquad \tilde{e}(t, s, x, \xi)\sim\sum_{\nu=0}^{\infty}e_\nu(t, s, x, \xi)\qquad \text{mod } \mathcal{B}_{t,s}(S^{-\infty})$$

and

$$(3.8)\qquad e_\nu(t, s, x, \xi)\in\bigcap_{0<\varepsilon<1}S[-\nu, M+\varepsilon, -M-\varepsilon]\qquad (\nu=0, 1, 2, \cdots)\,.$$

Set

$$(3.9)\qquad g(t, s, x, \xi)=-i\sum_{|\alpha|=2}\frac{1}{\alpha!}\lambda^{(\alpha)}(t, x, \nabla_x\phi(t, s, x, \xi))\times\partial_x^\alpha\phi(t, s, x, \xi)+f(t, s, x, \nabla_x\phi(t, s, x, \xi))$$

and consider

$$\mathcal{L} = D_t - \sum_{|\alpha|=1} \lambda^{(\alpha)}(t, x, \nabla_x\phi) D_x^\alpha + g(t, s, x, \xi)\,. \tag{3.10}$$

Then, by the usual expansion formula of Fourier integral operators (See [5]), we have by using (1.24)

$$\sigma(Le_{\nu,\phi}(t, s))(x, \xi) = \mathcal{L}e_{\nu,\phi} + r_\nu(t, s, x, \xi)\,. \tag{3.11}$$

Here

$$r_\nu(t, s, x, \xi) \sim -\sum_{|\alpha|\geqq 2} \frac{1}{\alpha!} \{D_{x'}^\alpha \lambda^{(\alpha)}((t, x, \tilde{\nabla}_x\phi(t, s, x, x', \xi)) \times e_\nu(t, s, x', \xi))\}_{x'=x} \quad \text{mod } \mathcal{B}_{t,s}(S^{-\infty}) \tag{3.12}$$

and

$$\tilde{\nabla}_x\phi(t, s, x, x', \xi) = \int_0^1 \nabla_x\phi(t, s, x'+\theta(x-x'), \xi)d\theta\,. \tag{3.13}$$

Then, from (1.27), (3.2) and (3.8) we see that

$$r_\nu(t, s, x, \xi) \in \bigcap_{0<\varepsilon<1} S[-\nu, M+\varepsilon-1, -M-\varepsilon] \qquad (\nu=0, 1, \cdots)\,. \tag{3.14}$$

Hence, if we can determine $e_\nu(t, s)$ as the solution of

$$\begin{cases} \mathcal{L}e_0 = 0 & \text{on } 0\leqq s\leqq t\leqq T_0\,, \\ e_0(s, s) = 1 \end{cases} \tag{3.15}$$

and

$$\begin{cases} \mathcal{L}e_\nu + r_{\nu-1} = 0 & \text{on } 0\leqq s\leqq t\leqq T_0\,, \\ e_\nu(s, s) = 0 & (\nu=1, 2, \cdots)\,, \end{cases} \tag{3.16}$$

then we have

$$\begin{aligned} &\sigma(L\sum_{\nu=0}^N e_{\nu,\phi}(t, s, X, D_x)) \\ &= \sum_{\nu=0}^N (\mathcal{L}e_\nu + r_\nu) \\ &= \mathcal{L}e_0 + \sum_{\nu=1}^N (\mathcal{L}e_\nu + r_{\nu-1}) + r_N \\ &= r_N \in \bigcap_{0<\varepsilon<1} S[-N, M+\varepsilon-1, -M-\varepsilon]\,. \end{aligned}$$

Thus, if we determine $\tilde{e}(t, s, x, \xi)$ so that (3.7) holds and $e(s, s)=1$, then we get (3.6).

Now, we solve (3.15) and (3.16) in what follows. Let $q(t,s,y,\xi)$ be the solution of (1.23) given by Lemma 1.9. Then, by Corollary 1.12 the equations (3.15) and (3.16) are reduced, respectively, to

$$(3.18)\qquad \begin{cases} D_t\tilde{e}_0(t,s,y,\xi)+\tilde{g}(t,s,y,\xi)\tilde{e}_0(t,s,y,\xi)=0\,, \\ \tilde{e}_0(s,s,y,\xi)=1 \end{cases}$$

and

$$(3.19)\qquad \begin{cases} D_t\tilde{e}_\nu(t,s,y,\xi) \\ \quad +\tilde{g}(t,s,y,\xi)\tilde{e}_\nu(t,s,y,\xi)+\tilde{r}_{\nu-1}(t,s,y,\xi)=0\,, \\ \tilde{e}_\nu(s,s,y,\xi)=0 \quad (\nu=1,2,\cdots)\,, \end{cases}$$

where

$$(3.20)\qquad \begin{cases} \tilde{e}_\nu(t,s,y,\xi)=e_\nu(t,s,q(t,s,y,\xi),\xi)\,, \\ \tilde{g}(t,s,y,\xi)=g(t,s,q(t,s,y,\xi),\xi)\,, \\ \tilde{r}_\nu(t,s,y,\xi)=r_\nu(t,s,q(t,s,y,\xi),\xi)\,. \end{cases}$$

Hence we have

$$(3.21)\qquad \tilde{e}_0(t,s,y,\xi)=\exp\Big[-i\int_s^t\tilde{g}(\sigma,s,y,\xi)d\sigma\Big]$$

and

$$(3.22)\qquad \begin{aligned} &\tilde{e}_\nu(t,s,y,\xi) \\ &=-i\int_s^t\tilde{r}_{\nu-1}(\sigma,s,y,\xi)\exp\Big[-i\int_\sigma^t\tilde{g}(\sigma',s,y,\xi)d\sigma'\Big]d\sigma\,. \end{aligned}$$

Consequently, setting

$$(3.23)\qquad \begin{cases} \tilde{\tilde{g}}(t,\sigma,s,x,\xi)=g(\sigma,s,q(\sigma,s,y(t,s,x,\xi),\xi),\xi) \\ \tilde{\tilde{r}}_\nu(t,\sigma,s,x,\xi)=r_\nu(\sigma,s,q(\sigma,s,y(t,s,x,\xi),\xi),\xi) \end{cases}$$

for the inverse $y=y(t,s,x,\xi)$ of $x=q(t,s,y,\xi)$ given by Lemma 1.10, we have

$$(3.24)\qquad e_0(t,s,x,\xi)=\exp\Big[-i\int_s^t\tilde{\tilde{g}}(t,\sigma,s,x,\xi)d\sigma\Big]$$

and

$$(3.25)\qquad \begin{aligned} &e_\nu(t,s,x,\xi) \\ &=-i\int_s^t\tilde{\tilde{r}}_{\nu-1}(t,\sigma,s,x,\xi)\exp\Big[-i\int_\sigma^t\tilde{\tilde{g}}(t,\sigma',s,x,\xi)d\sigma'\Big]d\sigma\,. \end{aligned}$$

Now, we first note that from Remark of Theorem 3.1 we have

$$(3.26)\qquad \begin{aligned} &(t+\langle\xi\rangle^{-\omega})\,(1-1/(t\langle\xi\rangle^{\omega}+1)) \\ &\leqq\eta(t,\xi)\leqq t+\langle\xi\rangle^{-\omega}\,. \end{aligned}$$

By the definition (3.4)′ of M there exists for any $\varepsilon>0$ a constant C_ε such that

$$\begin{aligned} &\mathcal{I}m\,f(t,s,\xi) \\ &\leqq(M+\varepsilon/2)\,(t+\langle\xi\rangle^{-\omega})+C_\varepsilon\{1+\langle\xi\rangle^{-\omega}/(t+\langle\xi\rangle^{-\omega})^2\}\,. \end{aligned}$$

Then, using (3.9), (3.23), (3.26), (3.27) and Theorem 1.11, we have

$$\begin{aligned}&\mathcal{R}e(-i\tilde{\tilde{g}}(t,\sigma,s,x,\xi))\\&\leqq(M+\varepsilon)/(\sigma+\langle\xi\rangle^{-\omega})+C'_\varepsilon\{1+\langle\xi\rangle^{-\omega}/(\sigma+\langle\xi\rangle^{-\omega})^2\}\end{aligned}$$

for another constant C'_ε. On the other hand, by (1.5)

$$\log((t+\langle\xi\rangle^{-\omega})/(s+\langle\xi\rangle^{-\omega}))\leqq\log(\eta(t,\xi)/(\eta(s,\xi))+\log 2. \tag{3.29}$$

Hence, using

$$\int_s^t\langle\xi\rangle^{-\omega}(\sigma+\langle\xi\rangle^{-\omega})^2d\sigma\leqq 1$$

and (3.28), we have

$$|\exp[-i\int_s^t\tilde{\tilde{g}}(t,\sigma,s,x,\xi)d\sigma]|\leqq C''_\varepsilon(\eta(t,\xi)/\eta(s,\xi))^{M+\varepsilon} \tag{3.30}$$

for a constant C''_ε. We have

$$\begin{aligned}&|D_t^jD_\sigma^kD_s^{j'}\partial_\xi^\alpha D_x^\beta\tilde{\tilde{g}}(t,\sigma,s,x,\xi)|\\&\leqq C_{j,j',k,\alpha,\beta}\langle\xi\rangle^{-|\alpha|}\eta(t,\xi)^{-j}\eta(\sigma,\xi)^{-1-k}\eta(s,\xi)^{-j'}.\end{aligned}$$

Then, noting

$$\begin{aligned}&\int_s^t\eta(\sigma,\xi)^{-1}d\sigma\leqq 2\log((t+\langle\xi\rangle^{-\omega})/(s+\langle\xi\rangle^{-\omega}))\\&\leqq\tilde{C}_\varepsilon(\eta(t,\xi)/\eta(s,\xi))^\varepsilon\end{aligned}$$

for a constant $\tilde{C}_\varepsilon$, we have

$$\begin{aligned}&|D_t^jD_s^k\partial_\xi^\alpha D_x^\beta\int_s^t\tilde{\tilde{g}}(t,\sigma,s,x,\xi)d\sigma|\\&\leqq C_{\varepsilon,j,k,\alpha,\beta}\langle\xi\rangle^{-|\alpha|}\eta(t,\xi)^{-j}\eta(s,\xi)^{-k}(\eta(t,\xi)/\eta(s,\xi))^\varepsilon.\end{aligned} \tag{3.31}$$

Thus, together with (3.30) we see that

$$e_0(t,s,x,\xi)\in\bigcap_{0<\varepsilon<1}S[0,M+\varepsilon,-M-\varepsilon]. \tag{3.32}$$

We already checked (3.14) for r_ν if (3.8) holds for e_ν. Hence, if we prove (3.8) for e_ν assuming (3.14) for $\nu-1$, then (3.8) holds for any ν. And this fact is clear by (3.25).

Now, from (3.17) we see that there exists $r_\infty(t,s,x,\xi)\in\mathcal{B}_{t,s}(S^{-\infty})$ such that

$$L\tilde{e}_\phi(t,s,X,D_x)=R_\infty(t,s)\,(=r_\infty(t,s,X,D_x)). \tag{3.33}$$

Then, setting

$$\begin{cases}W_1(t,s)=-iR_\infty(t,s),\\W_{\nu+1}(t,s)=\int_s^tW_1(t,\theta)W_\nu(\theta,s)d\theta\quad(\nu=1,2,\cdots),\end{cases} \tag{3.34}$$

we get the fundamental solution $E(t,s)$ in the form

$$E(t,s) = {}_{\phi}(\tilde{E}t,s) + \int_s^t \tilde{E}_{\phi}(t,\theta) \sum_{\nu=1}^{\infty} W_{\nu}(\theta,s)d\theta \,. \tag{3.35}$$

From the theory of pseudo-differential operators of multiple symbols, there exists a symbol $\tilde{e}_{\infty}(t,s,x,\xi) \in \mathcal{B}_{t,s}(S^{-\infty})$ such that

$$E(t,s) = \tilde{e}_{\phi}(t,s,X,D_x) + \tilde{e}_{\infty,\phi}(t,s,X,D_x)$$

(cf. [5], [12]). Then, setting

$$e(t,s,x,\xi) = \tilde{e}(t,s,x,\xi) + \tilde{e}_{\infty}(t,s,x,\xi)\,,$$

we get the desired result.

Theorem 3.2. *The fundamental solution $E_{\phi}(t,s)$ $(0 \leqq s \leqq t \leqq T_0)$ given in Theorem 3.1 has the meaning even when $0 \leqq t \leqq s \leqq T_0$, and $E_{\phi}(t,s)$ $(0 \leqq t \leqq s \leqq T_0)$ is the fundamental solution of the backward initial value problem for L, i.e.,*

$$\begin{cases} LE_{\phi}(t,s) = 0 \quad \text{on } 0 \leqq t \leqq s \leqq T_0\,, \\ E_{\phi}(s,s) = I\,. \end{cases} \tag{3.36}$$

Furthermore, we have

$$e(t,s,x,\xi) \in \bigcap_{0<\varepsilon<1} S[0, -M'-\varepsilon, M'+\varepsilon] \quad (0 \leqq t \leqq s \leqq T_0)\,, \tag{3.37}$$

where M' is defined by

$$\lim_{R\to\infty} \sup_{\substack{x,t\langle\xi\rangle^{\omega} \geqq R \\ 0 \leqq t \leqq R^{-1}}} \{-t\mathcal{Im} f(t,x,\xi)\}\,. \tag{3.38}$$

Proof. We check the proof of Theorem 3.1. We have by Lemma 1.9, 1.10 and Theorem 1.11 that

$$\begin{cases} q(t,s,y,\xi') - y \in S[0,0,l+1] \\ y(t,s,x,\xi') - x \in S[0,0,l+1] \\ \phi(t,s,x,\xi) - x\cdot\xi \in S[1,0,l+1] \end{cases} \tag{3.39}$$

on $0 \leqq t \leqq s \leqq T_0$. Noting (3.24) we write

$$e_0(t,s,x,\xi) = \exp[i\int_t^s \tilde{\tilde{g}}(t,\sigma,s,x,\xi)d\sigma]$$

on $0 \leqq t \leqq s \leqq T_0$. Then, from (3.9) and (3.38)

$$e_0(t,s,x,\xi) \in \bigcap_{0<\varepsilon<1} S[0, -M'-\varepsilon, M'+\varepsilon]\,,$$

and, following the similar procedure to the proof of Theorem 3.1 by keeping in mind the fact $0 \leqq t \leqq s \leqq T_0$, we complete the proof.

Proof of the uniqueness of $E_\phi(t,s)$ in Theorem 3.1. Set

$$L^* = D_t - \lambda^*(t,X,D_x) + f^*(t,X,D_x),$$

where λ^* and f^* are the formal adjoints of λ and f, respectively. Then, L^* is the formal adjoint of L. Since $\lambda(t,x,\xi)$ is real valued, we see that $\lambda^*(t,x,\xi)-\lambda(t,x,\xi)\in S[0,l]$, and, there exists a $\tilde{f}^*(t,x,\xi)\in S[0,-1]$ such that

$$L^* = D_t - \lambda(t,X,D_x) + \tilde{f}^*(t,X,D_x).$$

Therefore, we can apply Theorem 3.2 to L^*. Let $E^*_\phi(t,s)$ $(0\leqq t\leqq s\leqq T_0)$ be the fundamental solution of the backward initial value problem for L^*.

Now, assume that thr there exist two fundamental solutions $E_\phi(t,s)$ and $E'_\phi(t,s)$ $(0\leqq s\leqq t\leqq T_0)$ of L in $\bigcap_{0<\varepsilon<1} S[0,M+\varepsilon, -M-\varepsilon]$. For $w\in\mathcal{S}$ we set

$$u(t,s,x) = (E_\phi(t,s)-E'_\phi(t,s))w \quad (0\leqq s\leqq t\leqq T_0).$$

Then, $u(t,s,x)$ satisfies

$$\begin{cases} Lu(t,s,x) = 0 \text{ on } 0\leqq s\leqq t\leqq T_0, \\ u(s,s,x) = 0. \end{cases}$$

On the other hand, for $b(t,x)\in\mathcal{B}_t(\mathcal{S})$ on $[0,T_0]$ set

$$v(t,x) = i\int_{T_0}^t E^*(t,\sigma)b(\sigma,x)d\sigma.$$

Then, we have

$$L^*v = b(t,x)\ (0\leqq t\leqq T_0),\ v(T_0,x) = 0.$$

Hence, we have

$$0 = \int_s^{T_0}(Lu,v)d\sigma = \int_s^{T_0}(u,L^*v)d\sigma$$
$$= \int_s^{T_0}(u,b)d\sigma \text{ for all } b\in\mathcal{B}_t(\mathcal{S}).$$

This means that

$$0 = u(t,s,x) = (E_\phi(t,s)-E'_\phi(t,s))w \text{ for all}$$
$$w\in\mathcal{S}\ (0\leqq s\leqq t\leqq T_0).$$

Thus we have $e(t,s,x,\xi)=e'(t,s,x,\xi)$.

Corollary 3.3. i) *The solution $u(t,x)\in\mathcal{B}_t(\mathcal{S})$ on $[0,T_0]$ of the Cauchy problem* (3.3) *for $\varphi(t)\in\mathcal{B}_t(\mathcal{S})$ on $[0,T_0]$ and $\psi\in\mathcal{S}$ exists uniquely and is represented by*

$$u(t,s,x) = E_\phi(t,s)\psi + i\int_s^t E_\phi(t,\theta)\varphi(\theta)d\theta. \tag{3.40}$$

ii) *We have*

$$E_\Phi(t,\tau)E_\Phi(\tau,s) = E_\Phi(t,s)\ (0\leqq s,\tau,t\leqq T_0)\,. \tag{3.41}$$

Proof. i) It is clear that $u(t,s,x)$ defined by (3.40) is the solution of (3.3). Let $v(t,x)\in\mathcal{B}_t(\mathcal{S})$ on $[0,T_0]$ be the solution of (3.3) for $\varphi(t)=0$ and $\psi=0$, and let $E_\Phi^*(t,s)$ be the fundameintal solution for the formal adjoint L^* of L. Set

$$w(t,s) = i\int_{T_0}^{t} E_\Phi^*(t,\theta)v(\theta,x)d\theta\,.$$

Then, we have

$$L^*w = v \ \text{ on } [0,T_0],\ w(T_0,x) = 0\,.$$

Hence we have

$$\int_0^{T_0}(v,v)dt = \int_0^{T_0}(v,L^*w)dt = \int_0^{T_0}(Lv,w)dt = 0\,,$$

and $v(t,x)=0$ on $[0,T_0]$. This proves the uniqueness of the solution of (3.3).

ii) Set $u(t,\tau,x)=E_\Phi(t,\tau)E_\Phi(\tau,s)\psi$ for $\psi\in\mathcal{S}$. Then, u satisfies

$$\begin{cases} Lu = 0 \ \text{ on } [0,T_0]\,, \\ u(\tau,\tau,x) = E_\Phi(\tau,s)\psi\,. \end{cases} \tag{3.42}$$

On the other hand $\tilde{u}(t,x)=E_\Phi(t,s)\psi$ also satisfies (3.42). Hence, by i) we have $u=\tilde{u}$ on $[0,T_0]$ which proves (3.41).

Corollary 3.4. *For the operator* $\boldsymbol{L}_1$ *of* (2.6) *let* $\boldsymbol{L}_2$ *be the operator of the form*

$$\boldsymbol{L}_2 = \boldsymbol{D}_t - \mathcal{D}(t) + \boldsymbol{F}(t)\,. \tag{3.43}$$

Let $E_{j,\Phi_j}(t,s)$, $(0\leqq s,t\leqq T_0)$ *be the fundamental solution for* $L_j=D_t-\lambda_j(t,X,D_x)+f_j(t,X,D_x)$. *Then, the fundamental solution* $\boldsymbol{E}_2(t,s)\ (0\leqq s,t\leqq T_0)$ *of the Cauchy problem*

$$\begin{cases} \boldsymbol{L}_2 U = \Phi(t) \ \text{ on } [0,T_0]\,, \\ U|_{t=s} = \Psi \ \ (0\leqq s\leqq T_0) \end{cases} \tag{3.44}$$

exists uniquely in the form

$$\boldsymbol{E}_2(t,s) = \begin{bmatrix} E_{1,\Phi_1}(t,s) & & 0 \\ & \ddots & \\ 0 & & E_{m,\Phi_m}(t,s) \end{bmatrix}. \tag{3.45}$$

4. Construction of fundamental solution. The first order system case. In the first place we prove the fundamental lemma.

Lemma 4.1. *Let $\boldsymbol{L}_1$ be the operator of the form* (2.6). *Then, the fundamental solution $\boldsymbol{E}_1(t,s)$ of the Cauchy problem*

$$(4.1) \qquad \begin{cases} \boldsymbol{L}_1 U = \Phi(t) \text{ on } [s, T_0], \\ U|_{t=s} = \Psi \quad (0 \leqq s \leqq T_0) \end{cases}$$

exists in the form

$$(4.2) \qquad \boldsymbol{E}_1(t,s) = \boldsymbol{E}_2(t,s)(I+\boldsymbol{Q},(ts))+\boldsymbol{Q}_\infty(t,s),$$

where $\boldsymbol{E}_2(t,s)$ is the fundamental solution for $\boldsymbol{L}_2$ in Corollary 3.4, *and $\boldsymbol{Q}(t,s)$, $\boldsymbol{Q}_\infty(t,s)$ satisfy*

$$(4.3) \qquad \begin{cases} \boldsymbol{Q}(s,s) = 0, \ \sigma(\boldsymbol{Q}(t,s))(x,\xi) \in S[0,0,0], \\ \boldsymbol{Q}_\infty(s,s) = 0, \ \sigma(\boldsymbol{Q}_\infty(t,s))(x,\xi) \in \mathcal{B}_{t,s}(S^{-\infty}), \end{cases}$$

$(0 \leqq s \leqq t \leqq T_0)$.

Proof. If we find $\boldsymbol{Q}$ such that $\tilde{\boldsymbol{E}}_1(t,s) = \boldsymbol{E}_2(t,s)(I+\boldsymbol{Q}(t,s))$ satisfies

$$(4.4) \qquad \sigma(\boldsymbol{L}_1\tilde{\boldsymbol{E}}_1) = \sigma(\boldsymbol{R}\boldsymbol{E}_2 + \boldsymbol{R}\boldsymbol{E}_2\boldsymbol{Q} + \boldsymbol{E}_2\boldsymbol{D}_t\boldsymbol{Q}) \in \mathcal{B}_{t,s}(S^{-\infty}),$$

then $\tilde{\boldsymbol{E}}_1(t,s)$ is an approximate fundamental solution for $\boldsymbol{L}_1$. Hence by the usual procedure, which also used in the proof of Theorem 3.1, we can find $\boldsymbol{E}_1(t,s)$ in the form (4.2).

Set

$$(4.5) \qquad \tilde{\boldsymbol{R}}(t,s) = \boldsymbol{E}_2(s,t)\boldsymbol{R}(t)\boldsymbol{E}_2(t) \quad \text{with } \boldsymbol{R}(t) \text{ of } (2.6).$$

Then, we see that (4.4) is equivalent to

$$(4.6) \qquad \sigma(\boldsymbol{D}_t\boldsymbol{Q}(t,s) + \tilde{\boldsymbol{R}}(t,s)\boldsymbol{Q}(t,s) + \tilde{\boldsymbol{R}}(t,s)) \in \mathcal{B}_{t,s}(S^{-\infty}) \quad (0 \leqq s \leqq t \leqq T_0).$$

We find such $\boldsymbol{Q}(t.s) = \boldsymbol{q}(t,s,X,D_x)$ in the form

$$(4.7) \qquad \begin{cases} \boldsymbol{q}_\nu(t,s) \in S[-\nu,0,0] \quad (0 \leqq s \leqq t \leqq T_0), \\ \boldsymbol{q}_\nu(s,s) = 0 \end{cases}$$

and

$$(4.8) \qquad \boldsymbol{q}(t,s) \sim \boldsymbol{q}_0(t,s) + \boldsymbol{q}_1(t,s) + \cdots \mod \mathcal{B}_{t,s}(S^{-\infty}).$$

We first note that from Theorem 3.1, 3.2 and Corollary 3.4

$$(4.9) \qquad \begin{cases} \sigma(\boldsymbol{E}_2(t,s))(x,\xi) \in \bigcap_{0<\varepsilon<1} S[0, \boldsymbol{M}+\varepsilon, -\boldsymbol{M}-\varepsilon] \quad (0 \leqq s \leqq t \leqq T_0), \\ \sigma(\boldsymbol{E}_2(s,t))(x,\xi) \in \bigcap_{0<\varepsilon<1} S[0, \boldsymbol{M}'+\varepsilon, -\boldsymbol{M}'-\varepsilon] \quad (0 \leqq s \leqq t \leqq T_0), \end{cases}$$

where

$$(4.10)\qquad \begin{cases} \boldsymbol{M} = \max_{1\leq j\leq m} \{M_j\},\ \boldsymbol{M}' = \max_{1\leq j\leq m} \{M'_j\}\,, \\ M_j = \lim_{R\to\infty} \sup_{\substack{x,t\langle\xi\rangle^{\omega}\geq R\\ 0\leq t\leq R^{-1}}} \{t\mathcal{I}m f_j(t,x,\xi)\}\,, \\ M'_j = \lim_{R\to\infty} \sup_{\substack{x,t\langle\xi\rangle^{\omega}\geq R\\ 0\leq t\leq R^{-1}}} \{t\mathcal{I}m f_j(t,x,\xi)\}\,. \end{cases}$$

Hence, setting $\tilde{\boldsymbol{r}}(t,s,x,\xi)=\sigma(\tilde{\boldsymbol{R}}(t,s))\,(x,\xi)$, we have by Lemma 1.13

$$(4.11)\qquad \tilde{\boldsymbol{r}}(t,s,x,\xi) \in \bigcap_{0<\varepsilon<1}\bigcap_{\nu\geq 1} S[\omega-j,\ \boldsymbol{M}+\boldsymbol{M}'+\varepsilon-j(l+1),\ -\boldsymbol{M}-\boldsymbol{M}'-\varepsilon]\,.$$

Then, noting

$$(\eta(t,\xi)/\eta(s,\xi))^{\nu}\leq(\eta(t,\xi)^{l+1}\langle\xi\rangle)^{\nu\omega}(\eta(s,\xi)\langle\xi\rangle^{\omega})^{-\nu} \leq(\eta(t,\xi)^{l+1}\langle\xi\rangle)^{\nu\omega}\,,$$

we see that

$$(4.12)\qquad \tilde{\boldsymbol{r}}(t,s,x,\xi)\in\bigcap_{j\geq 1} S[\omega-j,\ -j(l+1),\ 0]\,.$$

If we assume (4.7), we can write for $\boldsymbol{Q}_\nu(t,s)=\boldsymbol{q}_\nu(t,s,X,D_x)$

$$(4.13)\qquad \sigma(\boldsymbol{D}_t\boldsymbol{Q}_\phi+\tilde{\boldsymbol{R}}\boldsymbol{Q}_\nu) = \boldsymbol{D}_t\boldsymbol{q}_\nu+\tilde{\boldsymbol{r}}\boldsymbol{q}_\nu+\boldsymbol{r}_\nu \quad (\nu = 0,1,\cdots)\,,$$

where

$$(4.14)\qquad \boldsymbol{r}_\nu(t,s,x,\xi)\in S[\omega-\nu-1,0,0]\,.$$

Now, using (4.13) we define $\boldsymbol{q}_\nu$ by

$$(4.15)\qquad \begin{cases} \boldsymbol{D}_t\boldsymbol{q}_\nu+\tilde{\boldsymbol{r}}\boldsymbol{q}_\nu+\boldsymbol{r}_{\nu-1} = 0 \quad (\boldsymbol{r}_{-1} = \tilde{\boldsymbol{r}})\,, \\ \boldsymbol{q}_\nu(s,s) = 0 \quad (\nu = 0,1,\cdots) \end{cases}$$

inductively. Then, if we check (4.7) for $\boldsymbol{q}_\nu$, we get $\boldsymbol{q}(t,s,x,\xi)$ by (4.8).

The solution of (4.15) can be written in the form

$$(4.16)\qquad \boldsymbol{q}_\nu(t,s) = -i\int_s^t \boldsymbol{r}_{\nu-1}(s_1,s)ds_1 +\sum_{\mu=2}^{\infty}(-i)^{\mu}\int_s^t ds_1\int_{s2}^{s_1} ds_2\cdots\int_s^{s_{\mu-1}}\tilde{\boldsymbol{r}}(s_1,s) \cdots\tilde{\boldsymbol{r}}(s_{\mu-1},s)\boldsymbol{r}_{\nu-1}(s_\mu,s)ds_\mu\,.$$

By (4.12) we have

$$(4.17)\qquad |\tilde{\boldsymbol{r}}(t,s)|\leq C\langle\xi\rangle^{\omega}(t\langle\xi\rangle^{\omega}+1)^{-2}$$

and get

$$\int_s^t |\tilde{r}(\sigma,s)|\,d\sigma \leqq \int_s^t C\langle\xi\rangle^{\omega}(\sigma\langle\xi\rangle^{\omega}+1)^{-2}d\sigma \tag{4.18}$$
$$= -[C/(\sigma\langle\xi\rangle^{\omega}+1)]_s^t$$
$$\leqq C(t-s)/(t+\langle\xi\rangle^{-\omega}).$$

Set

$$I_\mu(t,s,x,\xi) \tag{4.19}$$
$$= \int_s^t ds_1 \int_s^{s_1} ds_2 \cdots \int_s^{s_{\mu-1}} |\tilde{r}(s_1,s)| \cdots |\tilde{r}(s_\mu,s)|\, ds_\mu$$

and assume

$$I_\mu(t,s,x,\xi) \leqq \frac{C^\mu(t-s)^\mu}{\mu!\,(t+\langle\xi\rangle^{-\omega})^\mu}.$$

Then, we have

$$I_{\mu+1}(t,s,x,\xi) \leqq \int_s^t \frac{C\langle\xi\rangle^{\omega}}{(\sigma\langle\xi\rangle^{\omega}+1)^2}\,\frac{C^\mu(\sigma-s)^\mu}{\mu!(\sigma+\langle\xi\rangle^{-\omega})^\mu}\,d\sigma$$
$$= \frac{C^{\mu+1}\langle\xi\rangle^{-\omega}}{\mu!}\int_s^t \frac{(\sigma-s)^\mu}{(\sigma+\langle\xi\rangle^{-\omega})^{\mu+2}}\,d\sigma$$

and setting $z=(\sigma-s)/(\sigma+\langle\xi\rangle^{-\omega})$ we have

$$\int_s^t \frac{(\sigma-s)^\mu}{(\sigma+\langle\xi\rangle^{-\omega})^{\mu+2}}\,d\sigma = (s+\langle\xi\rangle^{-\omega})^{-1}\int_0^{(t-s)/(t+\langle\xi\rangle^{-\omega})} z^\mu dz$$
$$= \frac{(t-s)^{\mu+1}}{(\mu+1)\,(s+\langle\xi\rangle^{-\omega})\,(t+\langle\xi\rangle^{-\omega})^{\mu+1}}.$$

So we have

$$I_{\mu+1}(t,s,x,\xi) \leqq \frac{C^{\mu+1}(t-s)^{\mu+1}}{(\mu+1)!\,(t+\langle\xi\rangle^{-\omega})^{\mu+1}}.$$

With (4.19) we have

$$I_\mu(t,s,x,\xi) \leqq \frac{C^\mu(t-s)^\mu}{\mu!\,(t+\langle\xi\rangle^{-\omega})^\mu} \quad (\mu = 1,2,\cdots). \tag{4.20}$$

Thus, from (4.16) for $\nu=0$ we have

$$|q_0(t,s)| \leqq \sum_{\mu=1}^{\infty} \frac{C^\mu(t-s)^\mu}{\mu!\,(t+\langle\xi\rangle^{-\omega})^\mu} \tag{4.21}$$
$$\leqq \exp\left[\frac{C(t-s)}{t+\langle\xi\rangle^{-\omega}}\right]-1.$$

Differentiate the both sides of (4.16) and estimate similarly. Then, we see that

$$q_0(t,s,x,\xi) \in S[0,0,0],\; q_0(s,s) = 0. \tag{4.22}$$

Now, assume that (4.7) holds for some $\nu \geqq 0$. Then, by (4.14)

$$\int_s^t |\boldsymbol{r}_\nu(\sigma, s)| d\sigma \leqq C(t-s)\langle\xi\rangle^{\omega-\nu-1}. \tag{4.23}$$

Hence, in (4.16) we use

$$\int_s^{s_{\mu-1}} |\boldsymbol{r}_{\nu-1}(s_\mu, s)| ds_\mu \leqq C(s_{\mu-1}-s)\langle\xi\rangle^{\omega-\nu-1}$$
$$\leqq C(t-s)\langle\xi\rangle^{\omega-\mu-1}.$$

Then, by (4.20) we have

$$|\boldsymbol{q}_\nu(t,s)| \leqq C_\nu \langle\xi\rangle^{\omega-\nu-1} \exp\left[\frac{C(t-s)}{t+\langle\xi\rangle^{-\omega}}\right] \tag{4.24}$$
$$\leqq C_\nu e \langle\xi\rangle^{\omega-\nu-1},$$

and finally get (4.7) for all $\nu \geqq 0$. Q.E.D.

Now, we shall state the main theorem of the present paper.

Theorem 4.2. *Let $\boldsymbol{L}$ and $\boldsymbol{L}_0$ be the operators of the form* (2.3) *and* (2.1), *respectively. Let $\boldsymbol{N}(t)$ and $\tilde{\boldsymbol{N}}(t)$ be the perfect diagonalizers for $\boldsymbol{L}$ and $\boldsymbol{L}_0$, respectively, and let $\boldsymbol{E}_1(t,s)$ be the fundamental solution for $\boldsymbol{L}_1$ of* (2.6).

Then, the fundamental solutions $\boldsymbol{E}(t,s)$ and $\boldsymbol{E}_0(t,s)$ for $\boldsymbol{L}$ and $\boldsymbol{L}_0$ can be found in the forms

$$\begin{cases} \boldsymbol{E}(t,s) = \boldsymbol{N}(t)\boldsymbol{E}_1(t,s)\boldsymbol{N}^\sharp(s) + \boldsymbol{R}_\infty(t,s), \\ \sigma(\boldsymbol{R}_\infty(t,s))(x,\xi) \in \mathscr{B}_{t,s}(S^{-\infty}) \quad (0 \leqq s \leqq t \leqq T_0) \end{cases} \tag{4.25}$$

and

$$\begin{cases} \boldsymbol{E}_0(t,s) = \tilde{\boldsymbol{N}}(t)\boldsymbol{E}_1(t,s)\tilde{\boldsymbol{N}}^\sharp(s) + \tilde{\boldsymbol{R}}_\infty(t,s), \\ \sigma(\tilde{\boldsymbol{R}}_\infty(t,s))(x,\xi) \in \mathscr{B}_{t,s}(S^{-\infty}) \quad (0 \leqq s \leqq t \leqq T_0), \end{cases} \tag{4.26}$$

respectively, where $\boldsymbol{N}^\sharp(s)$ and $\tilde{\boldsymbol{N}}^\sharp(s)$ are the parametrices of $\boldsymbol{N}(s)$ and $\tilde{\boldsymbol{N}}(s)$, respectively.

Furthermore, both $\boldsymbol{E}(t,s)$ and $\boldsymbol{E}_0(t,s)$ are represented as the sums of Fourier integral operators with phase functions $\phi_j(t,s)$, $j=1,\cdots,m$ and symbols of class

$$\bigcap_{0<\varepsilon<1} S[0, \boldsymbol{M}+\varepsilon, -\boldsymbol{M}-\varepsilon]. \tag{4.27}$$

Proof. It is easy to see that

$$\boldsymbol{N}(t)\boldsymbol{E}_1(t,s)\boldsymbol{N}^\sharp(s) + (I - \boldsymbol{N}(s)\boldsymbol{N}^\sharp(s))$$

is an approximate fundamental solution for $\boldsymbol{L}$. Then, noting $\sigma(I-\boldsymbol{N}(s)\boldsymbol{N}^\sharp(s)) \in \mathscr{B}_s(S^{-\infty})$ and solving the integral equation as in (3.35) we get (4.25). Since by Lemma 4.1, $\boldsymbol{E}_1(t,s)$ is the sum of Fourier integral operators with phase functions $\phi_j(t,s)$ and symbols of class stated in (4.27), the rest of the proof for $\boldsymbol{L}$ is clear. Similarly we get for $\boldsymbol{L}_0$.

Theorem 4.3. *The Cauchy problem*

$$(4.28)\qquad \begin{cases} \boldsymbol{L}U = 0 \ \text{ on } [s, T_0] \\ U|_{t=s} = \Psi \in H_\sigma, \ (0 \leqq s \leqq T_0) \end{cases}$$

and

$$(4.29)\qquad \begin{cases} \boldsymbol{L}_0 U = 0 \ \text{ on } [s, T_0] \\ U|_{t=s} = \Psi \in H_\sigma, \ (0 \leqq s \leqq T_0) \end{cases}$$

have the unique solutions $U(t,s)$ and $U_0(t,s)$ in the form

$$(4.30)\qquad U(t,s) = \boldsymbol{E}(t,s)\Psi, \ U_0(t,s) = \boldsymbol{E}_0(t,s)\Psi\,,$$

respectively, where H_σ is the usual Sobolev space for real σ. Furthermore, for any $\varepsilon > 0$ we have

$$(4.31)\qquad ||\gamma_{M+\varepsilon}U||_\sigma,\ ||\gamma_{M+\varepsilon}U_0||_\sigma \leqq C_\varepsilon ||\Psi||_\sigma \qquad (0 \leqq s \leqq t \leqq T_0)$$

and

$$(4.32)\qquad ||U||_{\sigma-(M+\varepsilon)\omega},\ ||U_0||_{\sigma-(M+\varepsilon)\omega} \leqq C'_\varepsilon ||\Psi||_\sigma \qquad (0 \leqq s \leqq t \leqq T_0)$$

where $\gamma_{M+\varepsilon} = \gamma_{M+\varepsilon}(t,s,X,D_x)$ is defined by

$$\gamma_{M+\varepsilon}(t,s,x,\xi) = (\eta(s,\xi)/\eta(t,\xi))^{M+\varepsilon}\,.$$

REMARK. From (4.32) we see that $M\omega$ denotes the supremum of regularity loss of the solution. It should be noted that in Kumano-go [7] the constant M is determind as a sufficiently large number depending on $\boldsymbol{L}$, and that constants C_ε and C'_ε are independent of t and s for $0 \leqq s \leqq t \leqq T_0$.

Proof. Since $\gamma_{M+\varepsilon}\boldsymbol{E}(t,s)$ is the sum of Fourier integral operators with symbols of class $S[0,0,0]$, we have (4.31) for $0 \leqq s \leqq t \leqq T_0$ and $U(t,s)$. Since

$$\begin{aligned} ||\gamma_{M+\varepsilon}U||_\sigma &\geqq 2^{-(M+\varepsilon)}||\eta(s,D_x)^{M+\varepsilon}U||_\sigma \\ &\geqq 2^{-(M+\varepsilon)}||U||_{\sigma-(M+\varepsilon)\omega}\,, \end{aligned}$$

we get (4.32) for $0 \leqq s \leqq t \leqq T_0$ and $U(t,s)$. The rest of the proof is done similarly.

5. The higher order case. In this section we consider a single higher order operator of the following type:

$$(5.1)\qquad \begin{cases} L = D_t^m + \sum_{k=1}^{m} a_k(t,X,D_x) D_t^{m-k}\,, \\ a_k(t,x,\xi) = \sum_{j=0}^{k} \eta(t)^{(k-j)(l+1)-k} a_{k,j}(t,x,\xi) \\ a_{k,j}(t,x,\xi) \in S[k-j,0] \ \text{ on } [0,T]\times R_x^n \times R_\xi^n\,, \end{cases}$$

and consider the Cauchy problem

$$(5.2) \qquad \begin{cases} Lu = \varphi(t) & \text{on } [s, T_0]\,, \\ D_t^j u|_{t=s} = \psi_j & (j=0, \cdots, m-1)\,. \end{cases}$$

Set

$$(5.3) \qquad P(\lambda) = \lambda^m + a_{1,0}\lambda^{m-1} + \cdots + a_{m,0}$$

and

$$(5.4) \qquad G(\lambda) = a_{1,1}\lambda^{m-1} + a_{2,1}\lambda^{m-2} + \cdots + a_{m,1}\,.$$

Theorem 5.1. *Let the roots $\mu_1(t, x, \xi), \cdots, \mu_m(t, x, \xi)$ of $P(\lambda)=0$ be real valued and satisfy* (0.3) *for a constant $c>0$.*

Then, the equation $Lu=\varphi$ can be reduced to a system $\boldsymbol{L}U={}^t(0, \cdots, 0, \varphi)$, where $\boldsymbol{L}$ has the form (2.3) *with $\lambda_j(t, x, \xi)=\eta(t)^l\mu_j(t, x, \xi)$, $j=1, \cdots, m$. The constant $\boldsymbol{M}$ of* (4.10) *is given by*

$$(5.5) \qquad \boldsymbol{M} = \max_{1\leq k\leq m} \lim_{R\to\infty} \sup_{\substack{x,t\langle\xi\rangle^{\omega}\geq R \\ 0\leq t\leq R^{-1}}} \{J_k(t, x, \xi)\} + l(m-1)$$

where

$$J_k(t, x, \xi) = \frac{\mathcal{I}m\, G(\mu_k) - \eta((t)\partial_t\mu_k + l\mu_k)P''(\mu_k)/2}{P'(\mu_k)}\,,$$

$$P' = \partial_\lambda P \qquad \text{and} \qquad P'' = \partial_\lambda^2 P\,.$$

REMARK. It is easily verified that the differential operator of the form (0.11) satisfies (5.1).

Proof. I (Reduction to first order system). Let

$$(5.6) \qquad \boldsymbol{H}(t) = \begin{bmatrix} h(t, D_x)^{1-m} & & & 0 \\ & h(t, D_x)^{2-m} & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix},$$

where $h(t, \xi)$ is the symbol defined in Lemma 1.4, and set

$$(5.7) \qquad U = \boldsymbol{H}(t) \begin{bmatrix} u \\ D_t u \\ \cdots\cdots \\ D_t^{m-1} u \end{bmatrix}.$$

Then, $Lu=\varphi$ is reduced to a first order system $\boldsymbol{L}_0 U = \Phi$, where

$$\boldsymbol{L}_0 = \boldsymbol{D}_t - \boldsymbol{A}(t)\,,$$

$$(5.8) \qquad \sigma(\boldsymbol{A}(t)) = \begin{bmatrix} (m-1)h_t h^{-1}, & h & & & 0 \\ & (m-2)h_t h^{-1}, & h & & \\ & & \ddots & \ddots & \\ 0 & & & h_t h^{-1}, & h \\ -a_m h^{1-m}, & -a_{m-1}h^{2-m}, & \cdots, & -a_2 h^{-1}, & -a_1 \end{bmatrix}$$

and $\Phi={}^t(0, 0, \cdots, 0, \varphi)$.

By Lemma 1.2, 1.4, Proposition 1.6 and its Corollary we have $\sigma(A(t))\in S[1, l]$.

II (Principal and sub-principal part). We set

$$\sigma(A_1(t))=\begin{bmatrix} 0, & h & & 0 \\ & \ddots & \ddots & \\ 0 & & 0, & h \\ -\eta^l a_{m,0}\langle\xi\rangle^{1-m}, & -\eta^l a_{m-1,0}\langle\xi\rangle^{2-m}, & \cdots, & -\eta^l a_{1,0} \end{bmatrix}$$

and

$$\sigma(A_0(t))=\frac{1}{\eta}\begin{bmatrix} -i(m-1)l & & & 0 \\ & -i(m-2)l & & \\ 0 & & \ddots & -il \\ -a_{m,1}\langle\xi\rangle^{1-m}, & -a_{m-1,1}\langle\xi\rangle^{2-m}, & \cdots, & -a_{1,1} \end{bmatrix}$$

Then, we have

$$(5.9)\qquad \begin{cases} \sigma(A_1(t))\in S[1, l] \\ \sigma(A_0(t))\in S[0, -1] \\ \sigma(A(t)-A_1(t)-A_0(t))\in S[-1, -(l+1)-1]. \end{cases}$$

This follows from

$$\begin{aligned} & a_k h^{1-k}-\eta^l a_{k,0}\langle\xi\rangle^{1-k}-\eta^{-1}a_{k,1}\langle\xi\rangle^{1-k} \\ & =\sum_{j=2}^{k}\eta^{(k-j)(l+1)-k}a_{k,j}h^{1-k}\in S[-1, -(l+1)-1] \end{aligned}$$

and

$$-(m-j)h_t h^{-1}-i(m-j)l\eta^{-1}\in\mathcal{H}^\omega.$$

III (Diagonalizer). The diagonalizer $N_0(t)$ of L_0 is given by

$$(5.10)\qquad \sigma(N_0(t))=\begin{bmatrix} 1 & , \cdots, & 1 \\ \mu_1/\langle\xi\rangle & , \cdots, & \mu_m/\langle\xi\rangle \\ \cdots\cdots & & \\ (\mu_1/\langle\xi\rangle)^{m-1}, & \cdots, & (\mu_m/\langle\xi\rangle)^{m-1} \end{bmatrix}.$$

To prove this it is enough to show that

$$(5.11)\qquad \text{the } (j, k)\text{-element of } \sigma(N_0)^{-1}\sigma(A_1)\sigma(N_0)=\delta_{j,k}\eta^l\mu_k,$$

where $\delta_{j,k}=1$ if $j=k$, $=0$ if $j\neq k$. And, this follows from

$$\text{the } (j, k)\text{-element of } \sigma(A_1)\sigma(N_0)=\eta^l\mu_k(\mu_k/\langle\xi\rangle)^{j-1} \quad \text{for } j=1, \cdots, m-1,$$

$$\begin{aligned}&\text{the } (m, k)\text{-element of } \sigma(A_1)\sigma(N_0)\\&= -\eta^l\langle\xi\rangle^{1-m}\{a_{m,0}+a_{m-1,0}\mu_k+\cdots+a_{1,0}\mu_k^{m-1}\}\\&= \eta^l\mu_k(\mu_k/\langle\xi\rangle)^{m-1}\end{aligned}$$

and

$$\sum_{\nu=1}^{m} q_{j,\nu}(\mu_k/\langle\xi\rangle)^{\nu-1} = \delta_{j,k},$$

where $q_{j,\nu}$ is the (j, ν)-element of $\sigma(N_0)^{-1}$.

IV (Computation of M). We have

$$\text{the } (j, k)\text{-element of } \eta\sigma(A_0)\sigma(N_0) = -i(m-j)l(\mu_k/\langle\xi\rangle)^{j-1} \quad \text{for } j=1, \cdots, m-1$$

and

$$\text{the } (m, k)\text{-element of } \eta\sigma(A_0)\sigma(N_0) = -G(\mu_k)\langle\xi\rangle^{1-m}.$$

We define polynomials $Q_j(\lambda)$ of $\lambda (j=1, \cdots, m)$ by

$$Q_j(\lambda) = \sum_{\nu=1}^{m} q_{j,\nu}\langle\xi\rangle^{1-\nu}\lambda^{\nu-1}. \tag{5.12}$$

Then we have

$$Q_j(\mu_k) = \delta_{i,k}.$$

Thus we have

$$Q_j(\lambda) = \prod_{k\neq j}(\lambda-\mu_k)/(\mu_j-\mu_k)$$

and

$$\partial_\lambda Q_k(\mu_k) = \frac{1}{2}P''(\mu_k)/P'(\mu_k).$$

Since $q_{j,m}=\langle\xi\rangle^{m-1}/P'(\mu_k)$, we have

$$\begin{aligned}&\text{the } (k, k)\text{-element of } \eta\sigma(N_0)^{-1}\sigma(A_0)\sigma(N_0)\\&= -i\sum_{\nu=1}^{m-1} q_{k,\nu}(m-\nu)l(\mu_k/\langle\xi\rangle)^{\nu-1}-q_{k,m}G(\mu_k)\langle\xi\rangle^{m-1}\\&= -ilm\sum_{\nu=1}^{m} q_{k,\nu}(\mu_k/\langle\xi\rangle)^{\nu-1}+il\sum_{\nu=1}^{m}\nu q_{k,\nu}(\mu_k/\langle\xi\rangle)^{\nu-1}\\&\quad -G(\mu_k)/P'(\mu_k)\\&= -ilm+il\{\partial_\lambda(\lambda Q_k(\lambda))\}_{\lambda=\mu_k}-G(\mu_k)/P'(\mu_k)\\&= -il(m-1)-(G(\mu_k)-\frac{i}{2}l\mu_kP''(\mu_k))/P'(\mu_k).\end{aligned} \tag{5.13}$$

We also have

$$\text{the } (k, k)\text{-element of } \sigma(N_0)^{-1}\sigma(N_{0,t}) \tag{5.14}$$

$$= \mu_{k,t} \sum_{\nu=2}^{m} q_{k,\nu}(\nu-1)\mu_k^{\nu-2}\langle\xi\rangle^{1-\nu}$$
$$= \mu_{k,t}\partial_\lambda Q_k(\mu_k)$$
$$= -\frac{i}{2}(\partial_t\mu_k)P''(\mu_k)/P'(\mu_k)\,.$$

V (End of the proof of Theorem 5.1). If we set

$$L = D_t - N_0^\sharp A N_0 + N_0^\sharp N_{0,t}\,, \tag{5.15}$$

then we have

$$N_0 L \equiv L_0 N_0 \qquad \bmod \mathcal{B}_t(S^{-\infty})\,.$$

By (5.9) and (5.11) we have

$$L = D_t - \mathcal{D} + B\,,$$

where

$$\sigma(\mathcal{D}) = \eta^l \begin{bmatrix} \mu_1(t, x, \xi) & & 0 \\ & \ddots & \\ 0 & & \mu_m(t, x, \xi) \end{bmatrix}$$

and $\sigma(B) \in S[0, -1]$. We set

$$\begin{aligned} &c_k(t, x, \xi) \\ &= \text{the } (k, k)\text{-element of } \sigma(B) \\ &-\{il(m-1)+(G(\mu_k)-\frac{i}{2}(\eta\partial_t\mu_k+l\mu_k)P''(\mu_k))/P'(\mu_k)\}/\eta\,. \end{aligned}$$

Since

$$\sigma(N_0^\sharp A_1 N_0) - \sigma(N_0)^{-1}\sigma(A_1)\sigma(N_0) \in S[0, 0]\,,$$
$$\sigma(N_0^\sharp A_0 N_0) - \sigma(N_0)^{-1}\sigma(A_0)\sigma(N_0) \in S[-1, -1]$$

and

$$\sigma(N_0 N_{0,t}) - \sigma(N_0)^{-1}\sigma(N_{0,t}) \in S[-1, -1]\,,$$

we have by (5.13) and (5.14)

$$\lim_{R\to\infty} \sup_{\substack{x,t\langle\xi\rangle^{\omega}\geqq R \\ 0\leqq t\leqq R^{-1}}} \{\eta(t, \xi)c_k(t, x, \xi)\} = 0\,.$$

Thus the M of (4.10) is given by (5.5).

Theorem 5.2. *Let L satisfy the condition of Theorem 5.1. Then the solution u of the Cauchy problem (5.2) with $\varphi(t) \in \mathcal{B}_t(\mathcal{S})$ and $\psi_j \in \mathcal{S}$ $j=0, 1, \cdots, m-1$, exists uniquely in $[s, T_0]$ and it is given by*

$$u(t, x) = \sum_{j=0}^{m-1} E_0^{1,j+1}(t, s, X, D_x)\psi_j + i\int_s^t E_0^{1,m}(t, \sigma, X, D_x)\varphi(\sigma)d\sigma\,, \tag{5.16}$$

where $E_0^{1,j}$ *is the* $(1, j)$*-element of the fundamental solution* $\boldsymbol{E}_0$ *of the operator*

$$\boldsymbol{D}-\begin{bmatrix} 0, & 1 & & & \\ & & & 0 & \\ & 0 & & 0, & 1 \\ -a_m, & -a_{m-1}, & \cdots, & -a_2, & -a_1 \end{bmatrix}. \tag{5.17}$$

The regularity loss caused by $E_0^{1,k}$ *in the sense of Remark to Theorem* 4.3 *is equalt to*

$$m_k = \omega(\boldsymbol{M}-l(m-1)-k+1). \tag{5.18}$$

Proof. An approximate fundamental solution of (5.17) is given by

$$\begin{aligned}\tilde{\boldsymbol{E}}_0 = & \boldsymbol{H}(t)\boldsymbol{N}_0(t)\boldsymbol{E}(t, s)\boldsymbol{N}_0^\sharp(s)\boldsymbol{H}^\sharp(s) \\ & +(I-\boldsymbol{H}(s)\boldsymbol{N}_0(s)\boldsymbol{N}_0^\sharp(s)\boldsymbol{H}^\sharp(s)),\end{aligned} \tag{5.19}$$

where $\boldsymbol{E}(t, s)$ is the fundamental solution of $\boldsymbol{L}$ of (5.15). Using $\tilde{\boldsymbol{E}}_0$ we can construct the fundamental solution $\boldsymbol{E}_0$ as in the proof of Theorem 3.1. By (5.19) we have

$$\begin{aligned}& \sigma(E_0^{1,k}(t, s))(x, \xi) \\ & \in \bigcap_{0<\varepsilon<1} S[1-k, \boldsymbol{M}+\varepsilon-l(m-1), -\boldsymbol{M}-\varepsilon-l(k-m)].\end{aligned} \tag{5.20}$$

Thus, we have (5.18).

References

[1] S. Alinhac: *Paramétrix et propagation des singularités pour un problème de Cauchy a multiplicité variable,* Soc. Math. France Astérisque **34–35** (1976), 3–26.

[2] L. Blutet de Monvel: *Hypoelliptic operators with double characteristics and related pseudo-differntial operators,* Comm. Pure Appl. Math. **27** (1974), 585–639.

[3] Chi Min-You: *On the Cauchy problem for a class of hyperbolic equations with data given on the degenerate parabolic line,* Acta Math. Sinica **8** (1958), 521–529.

[4] H. Kumano-go: *Oscillatory integrals of symbols of pseudo-differential operators and local solvability theorem of Nirenberg and Treves,* Katada Symp. on Partial Diff. Equations, (1972), 166–191.

[5] H. Kumano-go: *A calculus of Fourier integral operators on R^n and the fundamental solution for an operator of hyperbolic type,* Comm. Partial Differential Equation **1** (1976), 1–44.

[6] H. Kumano-go: *Fundamental solutions for operators of regularly hyperbolic type,* J. Math. Soc. Japan **29** (1977), 339–406.

[7] H. Kumano-go: *Fundamental solution for a hyperbolic system with diagonal principal part,* Comm. Partial Differential Equation **4** (1979), 959-1015.

[8] G. Nakamura: *A parametrix of a certain linear hyperbolic partial differential*

equation with variable multiplicity, unpublished.

[9] G. Nakamura and H. Uryu: *Parametrix of certain weakly hyperbolic operators*, to appear.

[10] O.A. Oleinik: *On the Cauchy problem for weakly hyperbolic equations*, Comm. Pure Appl. Math. **23** (1970), 569–586.

[11] K. Shinkai: *The symbol calculus for the fundamental solution of a degenerate parabolic system with applications*, Osaka J. Math. **14** (1977), 55–84.

[12] C. Tsutsumi: *The fundamental solution for a degenerate parabolic pseudo-differential operator*, Proc. Japan Acad. **49** (1974), 11–15.

[13] H. Uryu: *The Cauchy problem for weakly hyperbolic equations and systems*, Master thesis (1978), (in Japanese).

[14] A. Yoshikawa: *Construction of a parametrix for the Cauchy problem of some weakly hyperbolic equation* I, II, III, Hokkaido Math. J. **6** (1977), 313–344; **7** (1978), 1–26, 127–141.

[15] A. Yoshikawa: *Fundamental solutions to the Cauchy problem of some weakly hyperbolic equation*, Proc. Japan Acad. **53** (1977), 103–107.

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