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BLOCK INDUCTION, NORMAL SUBGROUPS AND CHARACTERS OF HEIGHT ZERO

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Introduction

Let G be a finite group and p a prime. Let (K, R, k) be a p -modular system. Let (π) be the maximal ideal of R . We assume that K contains the $|G|$ -th roots of unity and that k is algebraically closed. Let ν be the valuation of K normalized so that $\nu(p)=1$. For an (R -free) RG -module U lying in a block B of G , we define $ht(U)$, the height of U , by $ht(U)=\nu(\text{rank}_R U)-\nu(|G|)+d(B)$, where $d(B)$ is the defect of B . The heights of kG -modules are defined in a similar way, and heights are always nonnegative. In this paper we study indecomposable RG -(or kG -) modules of height zero, especially their behaviors under the block induction. In section 1 we introduce, motivated by Broué [7], the notion of linkage for arbitrary block pairs as a generalization of the one for Brauer pairs, and establish fundamental properties about it. In section 2 we give a condition for a block of a normal subgroup to be induced to the whole group. In section 3 a characterization of RG -(or kG -) modules of height 0 via their vertices and sources is given, which generalizes a result of Knörr [14]. Based on this result it is shown in section 4 that for any irreducible character χ of height 0 in B and any normal subgroup N of G , χ_N contains an irreducible character of height 0. This is well-known when B is weakly regular with respect to N . An answer to the problem of determining which irreducible (Brauer) characters of N appear as irreducible constituents of irreducible (Brauer) characters of height 0 is also obtained (Theorem 4.4). In section 5 a generalization of a theorem of Isaacs and Smith [11] is given. In section 6 an alternative proof of a theorem of Berger and Knörr [1] is given. Throughout this paper an RG -module is assumed to be R -free of finite rank.

1. Block induction and characters of height 0

Throughout this section H is a subgroup of G , and B and b are p -blocks of G and H , respectively.

Let G_p be the set of p -regular elements of G , ZRG the center of RG , and ZRG_p be the R -submodule of ZRG spanned by p -regular conjugacy class sums.

We let

$$Z_0(B) = \{a \in (ZRG_{p'})e_B; \omega_B(a) \not\equiv 0 \pmod{\pi}\},$$

where e_B is the block idempotent of B . An element $a \in (ZRG_{p'})e_B$ is said to be of *height 0* ([7]) if $a \in Z_0(B)$. Let s_H be the R -linear map from RG to RH defined by $s_H(x) = x$ if $x \in H$, and $s_H(x) = 0$ if $x \in G - H$.

DEFINITION 1.1. We say that B and b are *linked* if $s_H(Z_0(B))e_b \subseteq Z_0(b)$.

Let $\text{Chr}(G)$ be the R -module of R -linear combinations of irreducible characters of G and $\text{Chr}(B)$ its submodule of R -linear combinations of irreducible characters lying in B . Put

$$\text{Chr}^0(B) = \{\theta \in \text{Chr}(B); ht(\theta) = 0\},$$

where $ht(\theta)$ is defined as before; so $ht(\theta) = 0$ if and only if $\nu(\theta(1)) = \nu(|G|) - d(B)$. Let $\text{Irr}^0(B)$ (resp. $\text{IBr}^0(B)$) be the set of irreducible characters (resp. irreducible Brauer characters) of height 0 in B . Let $\text{Bch}(G)$ be the R -linear combinations of irreducible Brauer characters of G . $\text{Bch}(B)$ and $\text{Bch}^0(B)$ are defined in a similar way. For $\theta \in \text{Chr}(G)$ (or $\text{Bch}(G)$), put $\theta^* = \sum \theta(x^{-1})x$, where x runs through $G_{p'}$. So $\theta^* \in ZRG_{p'}$.

The following lemma is well-known, cf. Broué [7]. Here we give a direct proof, in this special case.

Lemma 1.2. *We have*

$$(ZRG_{p'})e_B = \{\theta^*; \theta \in \text{Chr}(B)\} = \{\theta^*; \theta \in \text{Bch}(B)\}.$$

Proof. It suffices to show the first equality. For $\theta \in \text{Chr}(G)$ and $\chi \in \text{Irr}(G)$, we have $\chi(\theta^*e_B) = \chi(\theta_B^*)$, where χ is extended linearly over RG and θ_B denotes the B -component of θ . Since χ is arbitrary, we have $\theta^*e_B = \theta_B^*$. Thus the assertion follows, since $ZRG_{p'} = \{\theta^*; \theta \in \text{Chr}(G)\}$.

The following theorem is important for our purpose. (It is a special case of [7, Proposition 3.3.4].) Here we give an alternative proof.

Theorem 1.3. *Let $\theta \in \text{Chr}(B)$ (or $\text{Bch}(B)$). Then θ is of height 0 if and only if θ^* is of height 0.*

Proof. Let $\chi \in \text{Irr}^0(B)$ and define the class function η as follows:

$$\eta(x) = \begin{cases} p^{d(B)} \chi(x) & \text{if } x \in G_{p'}, \\ 0 & \text{otherwise.} \end{cases}$$

We know that $ht(\theta) = 0$ if and only if $(\theta, \eta)_G \not\equiv 0 \pmod{p}$ ([6]). On the other hand, θ^* is of height 0 if and only if $\omega_B(\theta^*) \not\equiv 0 \pmod{\pi}$. Since $\omega_B(\theta^*) \equiv \{ |G| / \chi(1) p^{d(B)} \} (\theta, \eta)_G \pmod{\pi}$, the assertion follows.

Corollary 1.4. *We have*

$$Z_0(B) = \{\theta^*; \theta \in \text{Chr}^0(B)\} = \{\theta^*; \theta \in \text{Bch}^0(B)\}.$$

Proof. This follows from Lemma 1.2 and Theorem 1.3.

For $\theta \in \text{Chr}(G)$ (or $\text{Bch}(G)$), we denote by θ_b the b -component of θ_H . For an RG -(or kG -) module V , V_b is defined in a similar way. Also for $\theta \in \text{Chr}(H)$, we denote by θ^B the B -component of θ^G .

Corollary 1.5. *The following are equivalent.*

- (i) *B and b are linked.*
- (ii) *For every $\theta \in \text{Chr}^0(B)$, $\theta_b \in \text{Chr}^0(b)$.*
- (iii) *For every $\theta \in \text{Bch}^0(B)$, $\theta_b \in \text{Bch}^0(b)$.*

In particular, if B and b are linked, for every RG -(or kG -) module V of height 0 in B , V_b is of height 0.

Proof. The equivalences follow from Corollary 1.4 and the fact that $s_H(\theta^*) e_b = \theta_b^*$ for every $\theta \in \text{Chr}(G)$ (resp. $\text{Bch}(G)$). Let θ be the character (resp. Brauer character) afforded by V . If $ht(V) = 0$, then $ht(\theta_b) = 0$ by (ii) (resp. (iii)). This completes the proof.

The following proposition shows, in particular, that there are many examples of linked pair of blocks in block theory.

Proposition 1.6. *Assume that b^G is defined. Then B and b are linked if and only if $b^G = B$.*

Proof. Assume $b^G = B$. For $a \in Z_0(B)$, $\omega_b(s_H(a)) \equiv \omega_B(a) \not\equiv 0 \pmod{\pi}$. So $s_H(a) e_b \in Z_0(b)$ and “if part” follows. Conversely assume that B and b are linked. We have $\omega_b(e_B) \equiv \omega_b(s_H(e_B) e_b) \not\equiv 0 \pmod{\pi}$, since $e_B \in Z_0(B)$. Hence $b^G = B$.

For the following, see also [3, Lemma A and Theorem B].

Corollary 1.7. *Assume that b^G is defined and equal to B . Then*

- (i) *For any RG -(or kG -) module V of height 0 in B , V_b is of height 0, and*
- (ii) *for $\theta \in \text{Chr}(G)$ (or $\text{Bch}(G)$), $ht(\theta_b) = 0$ if and only if $ht(\theta) = 0$.*

Proof. (i) follows from Corollary 1.5 and Proposition 1.6. (ii) follows from the fact that $\omega_B(\theta_B^*) \equiv \omega_B(\theta^*) \equiv \omega_b(s_H(\theta^*) e_b) \equiv \omega_b(\theta_b^*) \pmod{\pi}$.

Let T_H^G denote the relative trace map when RG is considered as a G -algebra in the usual way. The following will be needed later.

Proposition 1.8. *Assume that B and b are linked and $d(b) = d(B)$. We have :*

- (i) $T_H^G(Z_0(b)) e_B \subseteq Z_0(B)$, and
- (ii) for any $\xi \in \text{Chr}(b)$, $ht(\xi) = 0$ if and only if $ht(\xi^B) = 0$.

Proof. Let $\chi \in \text{Irr}^0(B)$. For $\xi \in \text{Chr}(b)$, $(\xi^B)^* = T_H^G(\xi^*) e_B$. From this it follows that

$$\omega_B((\xi^B)^*) \equiv \{ |G| \xi(1) / |H| \chi(1) \} \omega_b((\chi_b)^*) \pmod{\pi}.$$

Since $ht(\chi_b) = 0$ by Corollary 1.5 (ii), (ii) follows. Then we get (i) by the above equality and Corollary 1.4.

The following proposition (cf. also [18, Theorem 7]) shows that our terminology is compatible with Brauer's [5]. If (P, b_P) is a Brauer pair (i.e. P is a p -subgroup of G and b_P is a block of $PC_G(P)$ with defect group P), let θ_P be the unique irreducible Brauer character in b_P and b_P^0 the block of $C_G(P)$ covered by b_P .

Proposition 1.9. *Let (P, b_P) and (Q, b_Q) be Brauer pairs such that $P \triangleright Q$ and that b_Q is P -invariant. Then b_P and b_Q are linked (in the sense of Brauer [5]) if and only if b_P^0 and b_Q^0 are linked in our sense.*

Proof. Put $b^* = b_Q^{PC(Q)}$, where $C(Q) = C_G(Q)$. Let ϕ be the unique irreducible Brauer character in b^* . We have $\phi_{b_P} = e \theta_P$, for some integer e . Since $b_P^{PC(Q)}$ is defined, Corollary 1.5 (iii) and Proposition 1.6 yield that $b_P^{PC(Q)} = b^*$ if and only if $e \equiv 0 \pmod{p}$. On the other hand, we must have $(\theta_Q)_{b_P^0} = e \psi$, where $\psi = (\theta_P)_{C(P)}$ is the unique irreducible Brauer character in b_P^0 , since b_P is the unique block of $PC(P)$ covering b_P^0 . By Corollary 1.5 (iii), b_P^0 and b_Q^0 are linked if and only if $e \equiv 0 \pmod{p}$. So the assertion follows.

Now we consider the case where H is normal in G . In this case linked pair has a clear meaning, as the following theorem shows; it shows also that the condition that B and b are linked does not always imply that b^G is defined (and equal to B). See also Blau [4, Theorem 2].

Theorem 1.10. *Assume that H is normal in G . The following conditions are equivalent.*

- (i) B and b are linked.
- (ii) $s_H(e_B) e_b$ is of height 0.
- (iii) B is weakly regular with respect to H and B covers b .

Proof. (i) \Rightarrow (ii): This is obvious.

(ii) \Rightarrow (iii): Put $e_B = \theta^*$, $\theta \in \text{Chr}^0(B)$. Since $s_H(e_B) e_b = \theta_b^*$, we have in particular $\theta_b \neq 0$, so B covers b . Put $s_H(e_B) = \sum_i a_i \hat{K}_i$, \hat{K}_i being conjugacy classes of G (contained in H). We have $\omega_b(s_H(e_B)) \equiv \sum_i a_i \omega_b(\hat{K}_i) \equiv \sum_i a_i \omega_B(\hat{K}_i) \pmod{\pi}$, since B covers b . So we have $a_i \omega_B(\hat{K}_i) \not\equiv 0 \pmod{\pi}$ for some i , which shows that B

is weakly regular.

(iii) \Rightarrow (i): Let $\{B_i\}$ be the blocks of G covering b . We have $\sum e_{B_i} = \sum e_b^g (g \in G/T_b)$, so $\sum s_H(e_{B_i}) e_b = e_b$ and $s_H(e_{B_i}) e_b$ is of height 0 for some i . For such i , put $e_{B_i} = \theta^*$, $\theta \in \text{Chr}^0(B_i)$. We have $\theta_H = \sum \theta_b^g (g \in G/T_b)$, so we get that $d(B_i) = \nu(|T_b|) - \nu(|H|) + d(b)$, because $ht(\theta_b) = 0$. Let $\eta \in \text{Chr}^0(B)$. Similar argument as above shows that $d(B) = \nu(|T_b|) - \nu(|H|) + d(b) - ht(\eta_b)$. So we have $ht(\eta_b) = d(B_i) - d(B)$. On the other hand, $d(B) \geq d(B_i)$, since B is weakly regular. This proves that $ht(\eta_b) = 0$, so B and b are linked (by Corollary 1.5).

The following is [9, (V.3.15)].

Corollary 1.11. *Assume that H is normal in G and that B covers b . Let B be weakly regular with respect to H . For any $\chi \in \text{Irr}^0(B)$, we have $\chi_H = e \sum_i \xi_i$ with $e|T_b/T_{\xi_i}| \not\equiv 0 \pmod{p}$ and $\xi_i \in \text{Irr}^0(b)$, for some i .*

Proof. By Theorem 1.10, χ_b is of height 0, so the assertion follows from the equality $\chi_H = \sum \chi_b^g (g \in G/T_b)$.

By Theorem 1.10 (and Corollary 1.5), we get that when B is weakly regular with respect to the normal subgroup H and B covers b , V_b has an indecomposable summand of height 0 for any RG -(or kG -)module V of height 0 in B . It will be proved in Theorem 4.1 that this is the case for arbitrary blocks covering b .

The rest of this section is devoted to giving alternative proofs of known facts.

Let b be a block of an arbitrary subgroup H as before. For a group X , let $B_0(X)$ be the principal block of X . The following is the Third Main Theorem (as extended by Okuyama [17]). (The present version is due to Blau [3, Corollary 1].) See also Kawai [20, Corollary 2.2].

Proposition 1.12. *Assume that there exists $\chi \in \text{Irr}^0(B)$ such that χ_H is an irreducible character in b . Then*

- (i) *If b_0 is a block of H for which b_0^G is defined and equal to B , then $b_0 = b$.*
- (ii) *If b^G is defined, then $b^G = B$ if and only if $\chi_H \in \text{Irr}^0(b)$.*

Proof. (i) By Corollary 1.7 (ii), χ_{b_0} is of height 0, so $b_0 = b$. (ii) “only if” part follows similarly. “if” part: Since $ht(\chi_b) = 0$, $ht(\chi_{b^G}) = 0$ by Corollary 1.7 (ii). Hence $b^G = B$.

A result similar to the following has appeared in Robinson [19].

Proposition 1.13. *Let u be a central p -element of H . Assume that $b^G = B$. For χ, χ' in $\text{Chr}(B)$, the following are equivalent.*

- (i) $ht(\chi) = ht(\chi') = 0$.
- (ii) $p^{d(b)} |H|^{-1} \sum \chi_b(us) \chi'_b(u^{-1}s^{-1}) \not\equiv 0 \pmod{\pi}$,

where in the summation s runs through $H_{p'}$.

Proof. Define $\psi, \psi' \in \text{Bch}(b)$ by $\psi(s) = \chi_b(us)$ and $\psi'(s) = \chi'_b(u^{-1}s)$, for $s \in H_{p'}$. Put $\chi_b = \sum_i n_i \xi_i$, $\xi_i \in \text{Irr}(b)$. We have $\xi_i(u) = \xi_i(1) \varepsilon_i$, where ε_i is a $|u|$ -th root of unity. Then $\chi_b(u) = \sum_i n_i(\varepsilon_i - 1) \xi_i(1) + \chi_b(1)$. Since $\varepsilon_i - 1 \equiv 0 \pmod{\pi}$, $ht(\psi) = 0 \Leftrightarrow \nu(\chi_b(u)) = \nu(|H|) - d(b) \Leftrightarrow ht(\chi_b) = 0 \Leftrightarrow ht(\chi) = 0$. (For the last equivalence, cf. Corollary 1.7 (ii).) The same holds for ψ' (with u^{-1} in place of u). On the other hand, the number in (ii) is congruent $(\bmod \pi)$ to $p^{d(b)}|H|^{-1} \psi'(1) \omega_b(\psi^*)$, so the assertion follows.

2. Block induction and normal subgroups

Let N be a normal subgroup of G and b a block of N . If B is a block of G covering b , then a defect group D of B is said to be an *inertial defect group* of B if it is a defect group of the Fong-Reynolds correspondent of B in the inertial group T_b of b in G .

In this section we shall prove the following theorem, which settles, in a special case, a question raised by Blau [2]. It has been obtained also by Fan [8, Theorem 2.3] independently. See also Blau [4, Theorem 3].

Theorem 2.1. *Let the notation be as above. The following conditions are equivalent.*

- (i) b^G is defined.
- (ii) (iia) *There exists a unique weakly regular block of G covering b , say B , and*
- (iib) *for a defect group D of B , $Z(D)$ is contained in N .*

We begin with the following lemma, which is due to Berger and Knörr [1, the proof of Corollary], cf. also Fan [8, Proposition 2.1]. Another proof is included here for convenience.

Lemma 2.2. *For a block B of G covering b , let D be an inertial defect group of B and \hat{b} the unique block of DN covering b . Then D is a defect group of \hat{b} .*

Proof. We may assume that b is G -invariant. Put $H = N_G(D)N$. Let \tilde{B} be the Brauer correspondent in H of B . Take a kG -module U in B and a kH -module V in \tilde{B} such that V is a direct summand of U_H . Since b is G -invariant, any direct summand of U_N lies in b , so the same is true for V_N . Hence \tilde{B} covers b . This implies that \tilde{B} covers \hat{b} and a defect group of \hat{b} is by Knörr's theorem (Knörr [13, Proposition 4.2], see also [20, Corollary 2.4]) $DN \cap D = D$, because DN is normal in H and \hat{b} is H -invariant.

We also need the following

Lemma 2.3. (Blau [3, Lemma 2.5 (i)]) *Let H be a subgroup of G and B (resp. b) a block of G (resp. H). Let $\phi \in \text{Irr}(b)$. Suppose that $\phi^G = \tau + \sum_{i=1}^n m_i \chi_i$, where m_i is a nonnegative integer, $\chi_i \in \text{Irr}(B)$ and $\nu(m_i \chi_i(1)) \geq \nu(\phi^G(1))$ for $1 \leq i \leq n$, and τ is a character of G such that for all $g \in G$, $\nu(\tau(g)) > \nu(\phi(1)) + \nu(|C_G(g)|) - \nu(|H|)$ (τ may be 0). Then b^G is defined and equals B .*

Proof of Theorem 2.1. (i) \Rightarrow (iia): This is Lemma 2.6 in Blau [3].

(i) \Rightarrow (iib): This follows from (V.1.6) (i) in Feit [9]. (ii) \Rightarrow (i): We may assume that D is an inertial defect group of B . Let \hat{b} be the unique block of DN covering b . Since D is a defect group of \hat{b} by Lemma 2.2, (iib) implies $b^{pN} = \hat{b}$. In fact, assume that $\omega_b(\hat{K}) \not\equiv 0 \pmod{\pi}$ for a conjugacy class K of DN . Let $x \in K$ and let u and s be the p -part and p' -part of x , respectively. Since \hat{b} is induced from a root of it, $u \in {}_{DN}Z(D) \leq N$. We get $s \in N$, since DN/N is a p -group. So $K \subseteq N$, as required. Now let ϕ be an irreducible character of height 0 in \hat{b} . Any irreducible constituent χ of ϕ^G lies in a block covering b . So $\nu(\chi(1)) \geq \nu(\phi^G(1))$, and the inequality is strict if χ does not lie in B by (iia). From this it follows that $\hat{b}^G = B$ by Lemma 2.3. So $b^G = B$, completing the proof.

3. Characterization of modules of height 0

In this section we shall characterize RG -(or kG -) modules of height 0 via their vertices and sources. In the following, let \mathfrak{o} denote either R or k .

Lemma 3.1. *Let T be a subgroup of G and N a normal subgroup of T such that T/N is a p' -group. Let Y be an indecomposable $\mathfrak{o}T$ -module and W an indecomposable $\mathfrak{o}N$ -module. If $Y_N \cong eW$ for some integer e , then e is prime to p .*

Proof. Since k is algebraically closed, e is equal to the dimension of some projective indecomposable $k^\alpha[T/N]$ -module for some $\alpha \in Z^2(T/N, k^*)$ (cf. Theorem 7.8 in [12]). Since k is algebraically closed and T/N is a p' -group, e is prime to p .

The following theorem generalizes Corollary 4.6 in Knörr [14].

Theorem 3.2. *Let U be an indecomposable $\mathfrak{o}G$ -module lying in a block B with defect group D . The following are equivalent.*

- (i) $ht(U) = 0$.
- (ii) $\text{vx}(U) = {}_G D$ and the rank of a source of U is prime to p .

Proof. Since $ht(U) = 0$ implies $\text{vx}(U) = {}_G D$, it suffices to prove that for an $\mathfrak{o}G$ -module U with vertex D , $ht(U) = 0$ if and only if the rank of a source is prime to p . Let V be the Green correspondent of U with respect to $(G, N_G(D), D)$. V lies in the Brauer correspondent \tilde{B} of B and $ht(V) = 0$ if and

only if $ht(U)=0$. Let W be an indecomposable summand of V_N , where $N=DC_G(D)$. W lies in a block b covered by \tilde{B} . Let T be the inertial group of W in $N_G(D)$. For some $\mathfrak{o}T$ -module Y , $W|Y_N$ and $V=Y^{N_G(D)}$. Since Y belongs to b^r , $ht(Y)=0$ if and only if $ht(Y)=0$. Put $Y_N \cong eW$. Since T/N is a p -group, $e \not\equiv 0 \pmod{p}$ by Lemma 3.1. So $ht(Y)=0$ if and only if $ht(W)=0$. From the explicit Morita equivalence between b and $\mathfrak{o}D$ (b is, as a ring, isomorphic to a full matrix ring over $\mathfrak{o}D$), it follows that $ht(W)=0$ if and only if the rank of the corresponding $\mathfrak{o}D$ -module (which is a source of U) is prime to p . This completes the proof.

REMARK 3.3. Theorem 2.1 in Kawai [20] follows (in the special case when the residue field k is algebraically closed, as we are assuming) from the above theorem and Corollary 1.7(i).

4. Normal subgroups and characters of height 0

Throughout this section, we use the following notation: N is a normal subgroup of G , B is a block of G with defect group D , and b is a block of N covered by B .

Theorem 4.1. *For any indecomposable $\mathfrak{o}G$ -module U of height 0 lying in B , some indecomposable summand of U_N is a module of height 0 lying in b .*

Proof. We may assume that b is G -invariant. Let D be a defect group of B , \tilde{B} the Brauer correspondent of B in $N_G(D)N$, and V the Green correspondent of U with respect to $(G, N_G(D)N, D)$. Since V lies in \tilde{B} , $ht(U)=0$ implies $ht(V)=0$. Let \hat{b} be the unique block of DN covering b . D is a defect group of \hat{b} by Lemma 2.2. Let W be an indecomposable summand of V_{DN} lying in \hat{b} . (Note that \tilde{B} covers \hat{b} , cf. the proof of Lemma 2.2) Since V is DN -projective, V and W have vertex and source in common, so $ht(W)=0$ by Theorem 3.2. Since $\nu(|DN|)-d(\hat{b})=\nu(|N|)-d(b)$, some indecomposable summand of W_N is of height 0 (in b). This completes the proof.

Corollary 4.2.

- (i) *For any $\chi \in \text{Irr}^0(B)$, $\xi \in \text{Irr}^0(b)$ for some irreducible constituent ξ of χ_N .*
- (ii) *(Kawai [20, Corollary 2.5]) For any $\phi \in \text{IBr}^0(B)$, $\psi \in \text{IBr}^0(b)$ for some irreducible constituent ψ of ϕ_N .*

Proof. It suffices to prove (i). Let U be an R -form of a KG -module affording χ . By Theorem 4.1 some indecomposable summand V of U_N is of height 0 in b , so some irreducible constituent of $K \otimes_R V$ is of height 0.

Let $\text{Irr}^0(b \setminus B)$ be the set of irreducible characters in b appearing as an irreducible constituent of χ_N for some $\chi \in \text{Irr}^0(B)$. We define $\text{IBr}^0(b \setminus B)$ in a

similar way. To determine these sets, we need the following

Lemma 4.3. *Assume that b is G -invariant. Let D and δ be defect groups of B and b , respectively, such that $\delta \leq D$. If $\xi \in \text{Irr}(b)$ extends to QN for some subgroup Q with $\delta \leq Q \leq D$, then there is $\chi \in \text{Irr}(B)$ such that $(\chi, \xi)_N \neq 0$ and that $ht(\chi) \leq d(B) - \nu(|Q|) + ht(\xi)$.*

Proof. Let $\hat{\xi}$ be an extension of ξ to QN . Let \hat{b} and \tilde{B} be as in the proof of Theorem 4.1. Any irreducible constituent of $\hat{\xi}_{DN}$ belongs to \hat{b} . By the degree comparison it follows that there is $\eta \in \text{Irr}(\hat{b})$ such that $(\hat{\xi}_{DN}, \eta)_{DN} \neq 0$ and that $(\eta(1))_p \leq |DN/QN|_p (\xi(1))_p$. There is $\tilde{\chi} \in \text{Irr}(\tilde{B})$ such that $(\tilde{\chi}, \eta^{N_G(D)N})_{N_G(D)N} \neq 0$. Then we have $(\tilde{\chi}(1))_p \leq |N_G(D)N/DN|_p (\eta(1))_p$. Since \tilde{B} induces B , $(\tilde{\chi}^B(1))_p = (\tilde{\chi}^G(1))_p$, cf. [9, (V.1.3)]. Thus there is $\chi \in \text{Irr}(B)$ such that $(\chi(1))_p \leq |G/N_G(D)N|_p (\tilde{\chi}(1))_p$ and that $(\tilde{\chi}^G, \chi)_G \neq 0$. Since $Q \cap N = \delta$ by Knörr's theorem, this χ is a required character.

Theorem 4.4. *With the notations as above, we have :*

- (i) $\text{Irr}^0(b \setminus B) = \{\xi \in \text{Irr}^0(b) ; \xi \text{ extends to } DN \text{ for some inertial defect group } D \text{ of } B\}$.
- (ii) $\text{IBr}^0(b \setminus B) = \{\psi \in \text{IBr}^0(b) ; \psi \text{ is } D\text{-invariant for some inertial defect group } D \text{ of } B\}$.

Proof. We may assume that b is G -invariant. To prove (i), let $\xi \in \text{Irr}^0(b \setminus B)$ and take $\chi \in \text{Irr}^0(B)$ with $(\chi, \xi)_N \neq 0$. Let U be an R -form of a KG -module affording χ . As in the proof of Theorem 4.1, some indecomposable summand of U_{DN} is of height 0 in \hat{b} (with \hat{b} as above). So there is $\eta \in \text{Irr}^0(\hat{b})$ with $(\chi, \eta)_{DN} \neq 0$. Put $\eta_N = e \sum_{i=1}^n \xi_i$. We have $\eta(1) = en \xi_1(1)$. Since ξ_1 is G -conjugate to ξ , $\nu(\eta(1)) = \nu(\xi(1)) = \nu(\xi_1(1))$. So $\eta_N = \xi_1$, because e and n are powers of p . If $\xi_1 = \xi^x$, $x \in G$, then ξ extends to $D^{x^{-1}}N$, as required. The reverse inclusion follows from Lemma 4.3 (with D in place of Q). (ii) It is proved in a similar way that $\text{IBr}^0(b \setminus B)$ is contained in the right side. Assume that $\psi \in \text{IBr}^0(b)$ is D -invariant for a defect group D of B . Let W be a kN -module affording ψ . Let \hat{b} and \tilde{B} be as in the proof of Theorem 4.1. Then W extends to a kDN -module \hat{W} . Let V be a $kN_G(D)N$ -module lying in \tilde{B} such that $\hat{W} \mid V_{DN}$. As in the proof of Theorem 4.1, $ht(V) = 0$. Let U be the Green correspondent of V as before, so U lies in B and $ht(U) = 0$. From the above and Mackey decomposition, U_N is a sum of G -conjugates of W . Some irreducible constituent M of U is of height 0, because $ht(U) = 0$, and we have $W \mid M_N$. This completes the proof.

Corollary 4.5. *Let B_m be a weakly regular block of G covering b . Then $\text{Irr}^0(b \setminus B_m) \subseteq \text{Irr}^0(b \setminus B)$ and $\text{IBr}^0(b \setminus B_m) \subseteq \text{IBr}^0(b \setminus B)$. In particular, the sets $\text{Irr}^0(b \setminus B_m)$ and $\text{IBr}^0(b \setminus B_m)$ do not depend on the choice of B_m .*

Proof. We may assume that b is G -invariant. Since there is a defect group of B_n containing a defect group of B , the assertion follows from Theorem 4.4.

Corollary 4.6. *Assume that B covers $B_0(N)$, then there is $\chi \in \text{Irr}^0(B)$ such that $N \leq \text{Ker}(\chi)$.*

Proof. Since 1_N extends to any overgroups, this follows from Theorem 4.4 (or simply from Lemma 4.3).

REMARK 4.7. The above corollary is the same as saying that if B covers $B_0(N)$, some block of G/N dominated by B has defect group DN/N . This fact has been known for special N , cf. Chap. V, section 4 of Feit [9].

Put $\text{mod-Ker}(B) = \bigcap \text{Ker}(\phi)$, where ϕ runs through $\text{IBr}(B)$. The following corollary gives a characterization of $\text{mod-Ker}(B)$ via the (ordinary) irreducible characters in B , which extends Theorem 2.4 in [15]. Let $\mathcal{N}(B)$ be the set of normal subgroups N of G such that $B_0(N)$ is covered by B and that for any $\chi \in \text{Irr}^0(B)$, χ_N is a sum of linear characters.

Corollary 4.8. *$\text{mod-Ker}(B)$ is the unique maximal member of $\mathcal{N}(B)$.*

Proof. Put $N = \text{mod-Ker}(B)$. For any $\chi \in \text{Irr}^0(B)$, χ_N is a sum of irreducible characters of height 0 in $B_0(N)$, by Corollary 4.2. This shows that $N \in \mathcal{N}(B)$, since N is p -nilpotent. Now conversely let $N \in \mathcal{N}(B)$. Let D be a defect group of B and $\xi \in \text{Irr}^0(B_0(N))$ be D -invariant and assume that the determinantal order $o(\det \xi)$ is prime to p . Then ξ extends to DN (cf. [10]), so by Theorem 4.4 there is $\chi \in \text{Irr}^0(B)$ with $(\chi, \xi)_N \neq 0$. By definition of $\mathcal{N}(B)$, ξ must be linear, and then $o(\det \xi) \equiv 0 \pmod{p}$ implies that the decomposition number $d(\xi, 1_N) = 0$ unless $\xi = 1_N$. This implies that N is p -nilpotent, cf. [15, Lemma 2.1 (ii)]. Since B covers $B_0(N)$, $N \leq \text{Ker}(\chi)$ for some $\chi \in \text{Irr}(B)$. Then $O_p(N) \leq O_p(G) \cap \text{Ker}(\chi) = \text{Ker}(B)$, so $N \leq \text{mod-Ker}(B)$. This completes the proof.

In the rest of this section we prove the following theorem. Put $\delta = D \cap N$ for an inertial defect group D of B . (So δ is a defect group of b .)

Theorem 4.9. *Assume that $D = C_D(\delta) \delta$. Then we have $\text{Irr}^0(b \setminus B) = \text{Irr}^0(b)$, if one of the following conditions holds.*

- (i) $C_D(\delta)$ is abelian.
- (ii) D is abelian.
- (iii) There is a complement for δ in D .

The condition (ii) above is quite natural in view of the height zero conjecture. By Theorem 4.4, we have $\text{Irr}^0(b \setminus B) = \text{Irr}^0(b)$, if there is an (inertial) defect

group D of B with the following properties.

- (I) Every $\xi \in \text{Irr}^0(b)$ is D -invariant, and
- (II) every D -invariant $\xi \in \text{Irr}^0(b)$ extends to DN .

We first consider the condition (II). For this purpose we may assume that $G=DN$, where D is a defect group of B and b is G -invariant. We have:

Lemma 4.10. *For a suitable root b_0 in $\delta C_N(\delta)$ of b , the unique block B_0 of $DC_N(\delta)$ covering b_0 has defect group D and b_0 is D -invariant.*

Proof. Let \tilde{b} be the block of $N_N(\delta)$ such that $\tilde{b}^N=b$. Since $N_G(D) \subseteq N_G(\delta)$, there is a block \tilde{B} of $N_G(\delta)$ such that $\tilde{B}^G=B$ and that D is a defect group of \tilde{B} . Since the block idempotents corresponding to B and b are the same, it follows that \tilde{B} covers \tilde{b} . By the First Main Theorem, \tilde{b} is $N_G(\delta)$ -invariant. Put $C=\delta C_N(\delta)$ and $H=DC_N(\delta)$. Let b_1 be a block of C covered by \tilde{b} and B_1 the unique block of H covering b_1 . Let V be an indecomposable $kN_G(\delta)$ -module in \tilde{B} of height 0. It is easy to see that C is normal in $N_G(\delta)$ and that \tilde{B} is a unique block of $N_G(\delta)$ covering b_1 . So V_{b_1} is of height 0 by Theorem 1.10 (and Corollary 1.5). Since $V_{b_1}=(V_{B_1})_C$ and $\nu(|H|)-d(B_1) \geq \nu(|C|)-d(b_1)$ (with equality only when b_1 is H -invariant), consideration of dimension shows that b_1 is H -invariant and that some indecomposable summand W of V_{B_1} is of height 0. Hence $\text{vx}(W)$ is a defect group of B_1 and $|\text{vx}(W)|=|D|$. Since $\text{vx}(W) \leq_{N_G(\delta)} D$, we get that $\text{vx}(W)=D^n$ for some $n \in N_G(\delta)$. Then $n \in N_G(H)$, so $b_0=b_1^{n^{-1}}$ is the required root of b .

The following clarifies the condition (II) completely.

Proposition 4.11. *The following conditions are equivalent.*

- (i) Every D -invariant $\xi \in \text{Irr}^0(b)$ extends to DN .
- (ii) Every D -invariant linear character of δ extends to D .
- (iii) $[D, \delta]=[D, D] \cap \delta$.

Proof. Let B_0 and b_0 be chosen as in Lemma 4.10 and H, C be as in the proof of Lemma 4.10. We prove that (i) is equivalent to:

- (iv) Every D -invariant $\xi_0 \in \text{Irr}^0(b_0)$ extends to H .

(iv) \Rightarrow (i): For any D -invariant $\xi \in \text{Irr}^0(b)$, there is $\xi_0 \in \text{Irr}^0(b_0)$ such that ξ_0 is D -invariant and that $(\xi, \xi_0)_C \not\equiv 0 \pmod{p}$, because ξ_{b_0} is D -invariant and $ht(\xi_{b_0})=0$. Now it is easy to see that ξ extends to G if (and only if) ξ_0 extends to H . So (iv) implies (i).

(i) \Rightarrow (iv): For any D -invariant $\xi_0 \in \text{Irr}^0(b_0)$, ξ_0^b is D -invariant and of height 0, cf. Proposition 1.8, so similar argument applies.

Next we show that (ii) and (iv) are equivalent. Note that every D -invariant $\xi_0 \in \text{Irr}^0(b_0)$ is written as $\xi_0=\tilde{\xi}$ for a D -invariant linear character ζ of δ (and vice versa), where $\tilde{\xi}$ is defined as in Feit [9, (V.4.7)]. We show that ξ_0 extends to H if

and only if ζ extends to D . First assume that there is an extension η of ξ_0 . Since $ht(\eta)=0$, $(\eta, \lambda)_D \neq 0$ for some linear character λ of D . (Apply Theorem 3.2). Since $(\xi_0)_\delta$ is a multiple of ζ , this implies $\lambda_\delta=\zeta$. Conversely let λ be an extension of ζ . Let b_1 be a root of B_0 in $DC_H(D)$. We have $\lambda^{DC_H(D)}=\tilde{\lambda}+\theta$ for some character θ , where $\tilde{\lambda} \in \text{Irr}^0(b_1)$ is defined as above. So $\zeta^c=(\lambda^H)_c=(\tilde{\lambda}^{B_0})_c+\psi$ for some character ψ . Since ζ^c is a sum of a multiple of ξ_0 and characters lying outside b_0 , it follows that $(\tilde{\lambda}^{B_0})_c$ is a multiple of ξ_0 . Now $ht(\tilde{\lambda}^{B_0})=0$ by Proposition 1.8, so for some irreducible constituent χ of $\tilde{\lambda}^{B_0}$, $\chi_c=\xi_0$.

The equivalence of (ii) and (iii) is obvious.

REMARK 4.12. Theorem 8.26 in [10] reads: Let N be a normal subgroup of G with G/N a p -group. For a p -Sylow subgroup P of G , assume (a) $P \cap N \leq Z(P)$, and (b) every irreducible character of $P \cap N$ extends to P . Then every G -invariant irreducible character of N extends to G .

The above proposition is related to this theorem as follows: Let $\xi \in \text{Irr}(N)$ be G -invariant. Let b be the block of N (with defect group δ) containing ξ . If $ht(\xi)=0$, then (b) implies that ξ extends to G by Proposition 4.11. (On the other hand, δ is abelian by (a). So $ht(\xi)=0$ would follow from the height zero conjecture.)

To consider the condition (I), we let $T'_b = \cap I_G(\xi)$, where ξ runs through $\text{Irr}(b)$. T'_b is normal in T_b . We first extend Lemma 2.2 as follows:

Lemma 4.13. *Assume that b is G -invariant. Let Q be a subgroup such that $\delta \leq Q \leq D$ and let $b(Q)$ be the block of QN covering b . Then Q is a defect group of $b(Q)$.*

Proof. By Lemma 2.2, D is a defect group of $b(D)$. By induction on $|D/Q|$, we may assume $|D/Q|=p$. Since $b(Q)$ is DN -invariant and covered by $b(D)$, $D \cap QN=Q$ is a defect group of $b(Q)$ by Knörr's theorem.

Lemma 4.14. *Assume that b is G -invariant. Let B_1 be a block of T'_b covered by B . Then we have*

(i) $B_1^G=B$.

(ii) $\delta C_D(\delta)$ is contained in a defect group of a G -conjugate of B_1 . In particular, $Z(D) \leq T'_b$.

Proof. Let $\xi_1 \in \text{Irr}(b)$ and take $\zeta_1 \in \text{Irr}(I_G(\xi_1) | \xi_1)$ such that $\zeta_1^G \in \text{Irr}(B) \cap \text{Irr}(G | \xi_1)$. If b_1 is the block containing ζ_1 , then $b_1^G=B$, cf. [9, (V.1.2)]. Take another $\xi_2 \in \text{Irr}(b)$, if any, and take $\zeta_2 \in \text{Irr}(I_G(\xi_1) \cap I_G(\xi_2) | \xi_2)$ such that $\zeta_2^{G(\xi_1)} \in \text{Irr}(b_1) \cap \text{Irr}(I_G(\xi_1) | \xi_2)$. If b_2 is the block of $I_G(\xi_1) \cap I_G(\xi_2)$ containing ζ_2 , then $b_2^{G(\xi_1)}=b_1$. Hence $b_2^G=B$. Repeating this process, we finally get a block B' of T'_b such that $B'^G=B$. Then B' is G -conjugate to B_1 , so $B_1^G=B$. This implies $Z(D) \leq T'_b$, cf. Theorem 2.1. Now for any $x \in C_D(\delta)$, put $Q=\langle x, \delta \rangle$ and let

$b(Q)$ be the block of QN covering b . By the above (with $b(Q)$, QN in place of B, G) and Lemma 4.13, we get that $x \in Z(Q) \leq T'_b \cap QN$, so $C_D(\delta) \leq T'_b$. Let $D^x, x \in G$, be a defect group of the Fong-Reynolds correspondent of B in the inertial group of B_1 in G . Then $\delta C_D(\delta) \leq (D^x \cap T'_b)^{x^{-1}}$, which is a defect group of $B_1^{x^{-1}}$. This completes the proof.

Proposition 4.15. *Assume that b is G -invariant. Let A be a subgroup of $C_D(\delta)$ such that (1) A is abelian, or (2) δ is complemented in $A\delta$. Then for every $\xi \in \text{Irr}^0(b)$,*

- (i) ξ extends to AN , and
- (ii) there is $\chi \in \text{Irr}(B)$ such that $(\chi, \xi)_N \neq 0$ and that $ht(\chi) \leq d(B) - \nu(|A\delta|)$.

Proof. (i) Put $Q = A\delta$ and let $b(Q)$ be as in Lemma 4.13. So Q is a defect group of $b(Q)$. In either case, the condition (ii) in Proposition 4.11 is satisfied (with Q in place of D ; in case (2), use Wigner's method.) and any $\xi \in \text{Irr}^0(b)$ is Q -invariant by Lemma 4.14, so the conclusion follows from Proposition 4.11. (ii) follows from (i) and Lemma 4.3.

Proof of Theorem 4.9. Since we may assume that b is G -invariant, the assertion follows from Proposition 4.15 (ii) (with $A = C_D(\delta)$).

5. A generalization of a theorem of Isaacs and Smith

In [11] Isaacs and Smith have given a characterization of groups of p -length 1 ([11], Theorem 2). Here we prove a generalization of their result.

For a block B of G , let $\text{mod-Ker}(B)$ be as in section 4 and let $\text{Ker}^0(B) = \cap \text{Ker}(\chi)$, where χ runs through $\text{Irr}^0(B)$. Let $\text{Ker}(B)$ be defined in the usual way.

Lemma 5.1. *Let B be a block of G with defect group D .*

- (i) *If B covers the principal block of a normal subgroup N of G , D is a p -Sylow subgroup of DN .*
- (ii) $\text{Ker}^0(B) \leq \text{Ker}(B)D$ and $\text{mod-Ker}(B) \leq \text{Ker}(B)D$.

Proof. If B covers the principal block of N , $D \cap N$ is a p -Sylow subgroup of N , by Knörr's theorem. So (i) follows. By Corollary 4.8 (or more simply, by [15, Theorem 2.3]), $\text{Ker}^0(B) \leq \text{mod-Ker}(B)$. As is well-known, $(\text{mod-Ker}(B))D$ is p -nilpotent and its normal p -complement is $\text{Ker}(B)$. Since D is a p -Sylow subgroup of $(\text{mod-Ker}(B))D$ by (i), $(\text{mod-Ker}(B))D = \text{Ker}(B)D$. This completes the proof.

Let K be a normal subgroup of G such that B covers the principal block of K , and put $\bar{G} = G/K$ and let $\{\bar{B}_i; 1 \leq i \leq s\}$ be the blocks of \bar{G} dominated by B . Put $\bar{D} = DK/K$. Then we have the following

Proposition 5.2. *Assume that there is a defect group D of B such that $\Phi(D)$ (the Frattini subgroup of D) contains a p -Sylow subgroup of K . Then for exactly one value of i , \bar{B}_i has defect group \bar{D} .*

Proof. There is a block \bar{B}_i with defect group \bar{D} by Remark 4.7. Let b be the Brauer correspondent of B in $N_G(D)$. Let \bar{b} be a block of $\overline{N_G(D)}$ dominated by b . (Since D is a p -Sylow subgroup of DK by Lemma 5.1, $\overline{N_G(D)} = N_{\bar{G}}(\bar{D})$, by the Frattini argument.) We claim that \bar{b} is unique. Let Q be a p -Sylow subgroup of K such that $Q \leq \Phi(D)$. Put $L = N_G(D) \cap K$. Then $N_{\bar{G}}(\bar{D}) \cong N_G(D)/L$. We note that b covers $B_0(L)$. In fact, there is $\chi \in \text{Irr}^0(B)$ such that $\text{Ker}(\chi) \geq K$ by Corollary 4.6. Since $ht(\chi_b) = 0$, $\chi_b \neq 0$. So b covers $B_0(L)$. Thus it suffices to show that b does not “decompose” in $N_G(D)/L$. We see that $L \subset \text{mod-Ker}(b)$ is p -nilpotent and that $L/L \cap \text{mod-Ker}(b)$ is a p' -group, since $Q \leq D \leq \text{mod-Ker}(b)$. So the claim follows from [16, Problem 9 on p. 389], since $Q \leq \Phi(D)$. Now assume that \bar{B}_i has defect group \bar{D} . We show that $\bar{B}_i = \bar{b}^{\bar{G}}$ with \bar{b} as above, which proves the uniqueness of i . Let \bar{U} be a $k\bar{G}$ -module in \bar{B}_i with vertex \bar{D} and \bar{V} the Green correspondent of \bar{U} with respect to $(\bar{G}, N_{\bar{G}}(\bar{D}), \bar{D})$. Let U (resp. V) be the inflation of \bar{U} (resp. \bar{V}) to G (resp. $N_G(D)$). D is a vertex of U , since D is a p -Sylow subgroup of DK . Similarly D is a vertex of V . So V is the Green correspondent of U with respect to $(G, N_G(D), D)$. Hence V must lie in b . So \bar{V} lies in \bar{b} , which shows that \bar{b} induces \bar{B}_i , as required.

Theorem 5.3. *Let B be a block of G with defect group D . If every $\chi \in \text{Irr}^0(B)$ restricts irreducibly to $N_G(D)$, then $G = N_G(D) \text{Ker}(B)$.*

Proof. We first consider the case where D is abelian. Let b be the Brauer correspondent of B in $N_G(D)$. For any $\xi \in \text{Irr}^0(b)$, $ht(\xi^B) = 0$ by Proposition 1.8, so it follows from the assumption that there is $\chi \in \text{Irr}^0(B)$ such that $\chi_{N_G(D)} = \xi$. Let $I = \{\xi \in \text{Irr}^0(b); D \leq \text{Ker}(\xi)\}$. For each $\xi \in I$, take $\chi(\xi) \in \text{Irr}^0(B)$ whose restriction to $N_G(D)$ equals ξ and let $K = \cap \text{Ker} \{\chi(\xi)\}$, where ξ runs through I . Clearly $K \cap N_G(D) \leq \text{mod-Ker}(b)$ and, by Lemma 5.1, $\text{mod-Ker}(b) \leq \text{Ker}(b)D$. Since $\text{Ker}(b)$ is a normal p' -subgroup, $\text{Ker}(b) \leq C_G(D)$. Hence $K \cap N_G(D) \leq C_G(D)$. On the other hand, D is a p -Sylow subgroup of K by Lemma 5.1. Hence K is p -nilpotent, by Burnside's theorem. By the Frattini argument, $G = N_G(D)K$. Since $K = O_{p'}(K)D \leq \text{Ker}(B)D$, we get $G = N_G(D)\text{Ker}(B)$, as required. For the general case, put $\bar{G} = G/\text{Ker}^0(B)$. We claim that $\text{Ker}^0(B)$ satisfies the assumption of Proposition 5.2 with $K = \text{Ker}^0(B)$. Put $Q = D \cap \text{Ker}^0(B)$. Then Q is a p -Sylow subgroup of $\text{Ker}^0(B)$, cf. Lemma 5.1. For any linear character λ of D , define $\tilde{\lambda} \in \text{Irr}(DC_G(D))$ as in the proof of Proposition 4.11. Then $ht(\tilde{\lambda}^B) = 0$, so there is $\chi \in \text{Irr}^0(B)$ such that λ is an irreducible constituent of χ_D . This shows $Q \leq \text{Ker}(\lambda)$, and hence $Q \leq [D, D]$. So the claim

follows. Now let \bar{B} be the block of \bar{G} as in Proposition 5.2. Since every $\chi \in \text{Irr}^0(B)$ comes then from \bar{B} , $\text{Ker}^0(\bar{B})=1$. Since $\overline{N_G(D)}=N_{\bar{G}}(\bar{D})$ by the Frattini argument, \bar{B} satisfies the same assumption as B . On the other hand, since (by Corollary 1.7 (ii)) $\chi_{N_G(D)} \in \text{Irr}^0(b)$ for any $\chi \in \text{Irr}^0(B)$, it follows that χ_D is a sum of linear characters (by Corollary 4.2 (i)). Hence $[D, D] \leq \text{Ker}^0(B)$ and \bar{D} is abelian. So $\bar{G}=N_{\bar{G}}(\bar{D})/\text{Ker}(\bar{B})$, by the above. Thus $\bar{G}=N_{\bar{G}}(\bar{D})$, since $\text{Ker}(\bar{B}) \leq \text{Ker}^0(\bar{B})=1$. Hence we get $G=N_G(D)/\text{Ker}^0(B)=N_G(D)/\text{Ker}(B)=D=N_G(D)/\text{Ker}(B)$, by Lemma 5.1. This completes the proof.

6. The height zero conjecture

The following is a well-known conjecture of Brauer:

(*) Blocks with abelian defect groups contain only characters of height 0.
Berger and Knörr [1] have proved the following

Theorem 6.1. *If (*) is true for all quasi-simple groups, it is true for all finite groups.*

We prove this theorem by applying some results in section 4 and a theorem of Knörr [14, Corollary 3.7].

Lemma 6.2. *If (*) is true for all quasi-simple groups, it is true for any group H with H/C simple for a central subgroup C of H .*

Proof. The proof is done by induction on the group order. If $H=[H, H]$, then H is quasi-simple and (*) is true by assumption. If $H \neq [H, H]$, let K be such that $[H, H] \triangleleft K \triangleleft H$ with $|H/K|=q$, a prime. Let B be a block of H with abelian defect group D and let $\chi \in \text{Irr}(B)$. We consider the case when $q=p$ and $\chi_K=\sum_{i=1}^b \zeta_i$, where all ζ_i are distinct. If b is the block of K containing ζ_1 , then $b^G=B$, since $\zeta_1^G=\chi$. So D is G -conjugate to a defect group of b , cf. Theorem 2.1. Since $ht(\zeta_1)=0$ by induction, $ht(\chi)=0$. Other cases are treated similarly. This completes the proof.

Proof of Theorem 6.1. The proof is done by induction on the group order. Let B be a block of a group G with an abelian defect group D and let $\chi \in \text{Irr}(B)$. Let N be a maximal normal subgroup of G . So G/N is simple. Let $\zeta \in \text{Irr}(N)$ be such that $(\chi, \zeta)_N \neq 0$. Let b be the block of N containing ζ and δ a defect group of b . We may assume that b is G -invariant. Let T be the inertial group of ζ in G . If $T \neq G$, let $\eta \in \text{Irr}(T|\zeta)$ be such that $\eta^G=\chi$ and let B' be the block of T to which η belongs and D' a defect group of B' . Then $D' \leq_G D$, since $B'^G=B$. On the other hand, $D' \geq_G Z(D)=D$. (In fact, the proof of Lemma 4.14 shows that B' is induced from a G -conjugate of B_1 , B_1 being the same as in Lemma 4.14. So the assertion follows.) Hence $D'=G D$. By induction $ht(\eta)=0$, so $ht(\chi)=0$. So we may assume ζ is G -invariant. Now take a central ex-

tension of G ,

$$1 \rightarrow Z \rightarrow \hat{G} \xrightarrow{f} G \rightarrow 1,$$

such that $f^{-1}(N) = N_1 \times Z$, $N_1 \triangleleft \hat{G}$ and that ζ extends to a character of \hat{G} , say $\hat{\zeta}$, under the identification of N_1 with N through f , and that Z is a finite cyclic group. Here we note the following. Since δ is abelian, $ht(\zeta) = 0$ by induction. So ζ extends to DN by Proposition 4.11, since D is abelian. Thus the above central extension may be taken so that

$$(\#) \text{ the subextension } 1 \rightarrow Z \rightarrow f^{-1}(DN) \xrightarrow{f} DN \rightarrow 1 \text{ splits.}$$

Let $\hat{\chi}$ be the inflation of χ to \hat{G} . Let \hat{B} be the block of \hat{G} to which $\hat{\chi}$ belongs. There is a unique irreducible character ψ of $\bar{G} = \hat{G}/N$ such that $\hat{\chi} = \xi \psi$. Let \bar{B} be the block of \bar{G} to which ψ belongs. Let \hat{D} and \bar{D} be defect groups of \hat{B} and \bar{B} , respectively. We have

$$(I) \quad \hat{D}Z/Z =_G D.$$

Proof. Since B is dominated by \hat{B} and \hat{G} is a central extension of G , the result follows.

$$(II) \quad \hat{D} \text{ is abelian.}$$

Proof. We have $f^{-1}(DN) = \hat{D}ZN = H \times Z$ for a subgroup H by $(\#)$ and (I). So $\hat{D}Z = K \times Z$ for a subgroup K . Then $K \simeq \hat{D}Z/Z \simeq D$ is abelian, so \hat{D} is abelian.

$$(III) \quad \hat{D}N/N =_{\bar{G}} \bar{D}.$$

Proof. We first show $\hat{D}N/N \leq_{\bar{G}} \bar{D}$. We have $\omega_{\hat{\chi}}(\hat{K}) = \xi(x) \psi(x) |\hat{G}| / |\xi(1) \psi(1) |C_{\hat{G}}(x)|$, where $x \in \hat{G}$ and K is the conjugacy class of \hat{G} containing x . From this we get that $\omega_{\hat{\chi}}(\hat{K}) = \omega_{\psi}(\hat{L}) m_x(\xi(x) |N| / |\xi(1) |C_N(x)|)$, where $m_x = |C_{\bar{G}}(\bar{x}) : C_{\hat{G}}(x) N/N|$ and L is the conjugacy class of \bar{G} containing \bar{x} , the image of x in \bar{G} . Here $\xi(x) |N| / |\xi(1) |C_N(x)|$ is an integer. In fact, let A be the \mathbf{Z} -linear combinations of the N -conjugacy class sums of \hat{G} , where \mathbf{Z} is the ring of rational integers. If T is a matrix representation affording ξ , then $T(A)$ is a commutative ring (with finite \mathbf{Z} -rank), since ζ_N is irreducible. If C is the N -conjugacy class containing x , $T(\hat{C}) = \alpha I$, a scalar matrix, where α equals the number in question. Hence the assertion follows. Hence, if $\omega_{\hat{\chi}}(\hat{K}) \not\equiv 0 \pmod{\pi}$, then $m_x \omega_{\psi}(\hat{L}) \not\equiv 0 \pmod{\pi}$. This implies $\hat{D}N/N \leq_{\bar{G}} \bar{D}$. Hence \bar{D} is abelian by (II), and $ht(\psi) = 0$ by assumption and Lemma 6.2. Let V (resp. W) be an R -form of $\hat{\chi}$ (resp. ψ). Thus $V \otimes_R \text{Inf } W$ is an R -form of $\hat{\chi}$. Since $ht(\psi) = 0$, \bar{D} is a vertex of W . So, if we let Δ be the inverse image of \bar{D} in \hat{G} , $V \otimes_R \text{Inf } W$ is Δ -projective. But \hat{D} must be a vertex of it, by Knörr's theorem [14]. Hence $\hat{D} \leq_{\hat{G}} \Delta$, and $\hat{D}N/N \leq_{\bar{G}} \bar{D}$. This completes the proof of (III).

Now we show $ht(\chi) = 0$. Since $\hat{\chi} = \xi \psi$, $\hat{\chi}(1) = \chi(1)$, $\xi(1) = \zeta(1)$, and $ht(\zeta) = ht(\psi) = 0$, $ht(\chi) = d(B) - d(b) + \nu(|Z|) - d(\bar{B})$. Since $d(\bar{B}) = d(\hat{B}) - \nu(|\hat{D} \cap$

$N|)$ by (III), and $d(\hat{B})=d(B)+\nu(|\hat{D} \cap Z|)$ by (I), it follows that $ht(\chi)=\nu(|\hat{D} \cap N|)-d(b)+\nu(|Z|)-\nu(|\hat{D} \cap Z|)$. Since $\hat{D} \cap N$ is a defect group of b and a p -Sylow subgroup of Z is contained in \hat{D} , we get $ht(\chi)=0$, completing the proof.

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