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REGULAR ORBIT CLOSURES IN MODULE VARIETIES

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Abstract

Let A be a finitely generated associative algebra over an algebraically closed field. We characterize the finite dimensional modules over A whose orbit closures are regular varieties.

1. Introduction and the main result

Throughout the paper k denotes a fixed algebraically closed field. By an algebra we mean an associative finitely generated k -algebra with identity, and by a module a finite dimensional left module. Let d be a positive integer and denote by $\mathbb{M}(d)$ the algebra of $d \times d$ -matrices with coefficients in k . For an algebra A the set $\text{mod}_A(d)$ of the A -module structures on the vector space k^d has a natural structure of an affine variety. Indeed, if $A \simeq k\langle X_1, \dots, X_t \rangle / J$ for $t > 0$ and a two-sided ideal J , then $\text{mod}_A(d)$ can be identified with the closed subset of $(\mathbb{M}(d))^t$ given by vanishing of the entries of all matrices $\rho(X_1, \dots, X_t)$ for $\rho \in J$. Moreover, the general linear group $\text{GL}(d)$ acts on $\text{mod}_A(d)$ by conjugation and the $\text{GL}(d)$ -orbits in $\text{mod}_A(d)$ correspond bijectively to the isomorphism classes of d -dimensional A -modules. We shall denote by \mathcal{O}_M the $\text{GL}(d)$ -orbit in $\text{mod}_A(d)$ corresponding to (the isomorphism class of) a d -dimensional A -module M . It is an interesting task to study geometric properties of the Zariski closure $\overline{\mathcal{O}}_M$ of \mathcal{O}_M . We note that using a geometric equivalence described in [4], this is closely related to a similar problem for representations of quivers. We refer to [2], [3], [4], [5], [6], [9], [10], [11], [12], [13] and [14] for results concerning geometric properties of orbit closures in module varieties or varieties of representations.

The main result of the paper concerns the global regularity of such varieties. Let $\text{Ann}(M)$ denote the annihilator of a module M . It is the kernel of the algebra homomorphism $A \rightarrow \text{End}_k(M)$ induced by the module M , and therefore the algebra $B = A/\text{Ann}(M)$ is finite dimensional. Obviously M can be considered as a B -module.

Theorem 1.1. *Let M be an A -module and let $B = A/\text{Ann}(M)$. Then the orbit closure $\overline{\mathcal{O}}_M$ is a regular variety if and only if the algebra B is hereditary and $\text{Ext}_B^1(M, M) = 0$.*

Let $d = \dim_k M$. Observe that $\text{mod}_B(d)$ is a closed $\text{GL}(d)$ -subvariety of $\text{mod}_A(d)$ containing $\overline{\mathcal{O}}_M$. Moreover, M is faithful as a B -module. Hence we may reformulate Theorem 1.1 as follows:

Theorem 1.2. *Let M be a faithful module over a finite dimensional algebra B . Then the orbit closure $\overline{\mathcal{O}}_M$ is a regular variety if and only if the algebra B is hereditary and $\text{Ext}_B^1(M, M) = 0$.*

The next section contains a reduction of the proof of Theorem 1.2 to Theorem 2.1 presented in terms of properties of regular orbit closures for representations of quivers. Sections 3 and 4 are devoted to the proof of Theorem 2.1. For basic background on the representation theory of algebras and quivers we refer to [1].

2. Representations of quivers

Let $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1; s, t : \mathcal{Q}_1 \rightarrow \mathcal{Q}_0)$ be a finite quiver, i.e. \mathcal{Q}_0 is a finite set of vertices, and \mathcal{Q}_1 is a finite set of arrows $\alpha : s(\alpha) \rightarrow t(\alpha)$. By a representation of \mathcal{Q} we mean a collection $V = (V_i, V_\alpha)$ of finite dimensional k -vector spaces V_i , $i \in \mathcal{Q}_0$, together with linear maps $V_\alpha : V_{s(\alpha)} \rightarrow V_{t(\alpha)}$, $\alpha \in \mathcal{Q}_1$. The dimension vector of the representation V is the vector

$$\mathbf{dim} V = (\dim_k V_i) \in \mathbb{N}^{\mathcal{Q}_0}.$$

By a path of length $m \geq 1$ in \mathcal{Q} we mean a sequence of arrows in \mathcal{Q}_1 :

$$\omega = \alpha_m \alpha_{m-1} \cdots \alpha_2 \alpha_1,$$

such that $s(\alpha_{l+1}) = t(\alpha_l)$ for $l = 1, \dots, m - 1$. In the above situation we write $s(\omega) = s(\alpha_1)$ and $t(\omega) = t(\alpha_m)$. We agree to associate to each $i \in \mathcal{Q}_0$ a path ε_i in \mathcal{Q} of length zero with $s(\varepsilon_i) = t(\varepsilon_i) = i$. The paths of \mathcal{Q} form a k -linear basis of the path algebra $k\mathcal{Q}$. We define

$$V_\omega = V_{\alpha_m} \circ V_{\alpha_{m-1}} \circ \cdots \circ V_{\alpha_2} \circ V_{\alpha_1} : V_{s(\omega)} \rightarrow V_{t(\omega)}$$

for a path $\omega = \alpha_m \cdots \alpha_1$ and extend easily this definition to $V_\rho : V_i \rightarrow V_j$ for any ρ in $\varepsilon_j \cdot k\mathcal{Q} \cdot \varepsilon_i$, where $i, j \in \mathcal{Q}_0$, as ρ is a k -linear combination of paths ω with $s(\omega) = i$ and $t(\omega) = j$. Finally, we set

$$\text{Ann}(V) = \{ \rho \in k\mathcal{Q} \mid V_{\varepsilon_j \cdot \rho \cdot \varepsilon_i} = 0 \text{ for all } i, j \in \mathcal{Q}_0 \},$$

which is a two-sided ideal in $k\mathcal{Q}$. In fact, it is the annihilator of the $k\mathcal{Q}$ -module induced by V with underlying k -vector space $\bigoplus_{i \in \mathcal{Q}_0} V_i$.

Let $\mathbf{d} = (d_i)_{i \in Q_0} \in \mathbb{N}^{Q_0}$ be a dimension vector. Then the representations $V = (V_i, V_\alpha)$ of Q with $V_i = k^{d_i}$, $i \in Q_0$, form a vector space

$$\text{rep}_Q(\mathbf{d}) = \bigoplus_{\alpha \in Q_1} \text{Hom}_k(V_{s(\alpha)}, V_{t(\alpha)}) = \bigoplus_{\alpha \in Q_1} \mathbb{M}(d_{t(\alpha)} \times d_{s(\alpha)}),$$

where $\mathbb{M}(d' \times d'')$ stands for the space of $d' \times d''$ -matrices with coefficients in k . For abbreviation, we denote the representations in $\text{rep}_Q(\mathbf{d})$ by $V = (V_\alpha)$. The group $\text{GL}(\mathbf{d}) = \bigoplus_{i \in Q_0} \text{GL}(d_i)$ acts regularly on $\text{rep}_Q(\mathbf{d})$ via

$$(g_i)_{i \in Q_0} * (V_\alpha)_{\alpha \in Q_1} = (g_{t(\alpha)} \cdot V_\alpha \cdot g_{s(\alpha)}^{-1})_{\alpha \in Q_1}.$$

Given a representation $W = (W_i, W_\alpha)$ of Q with $\mathbf{dim} W = \mathbf{d}$, we denote by \mathcal{O}_W the $\text{GL}(\mathbf{d})$ -orbit in $\text{rep}_Q(\mathbf{d})$ of representations isomorphic to W .

Let M be a faithful module over a finite dimensional algebra B . It is well known that the algebra B is Morita-equivalent to the quotient algebra kQ/I , where Q is a finite quiver and I an admissible ideal in kQ , i.e. I is a two-sided ideal such that $(\mathcal{R}_Q)^r \subseteq I \subseteq (\mathcal{R}_Q)^2$ for some positive integer r , where \mathcal{R}_Q denotes the two-sided ideal of kQ generated by the paths of length one (arrows) in Q . Furthermore, the algebra B is hereditary if and only if $I = \{0\}$ (in particular, the quiver Q has no oriented cycles, i.e. paths ω of positive lengths with $s(\omega) = t(\omega)$). According to the above equivalence, the faithful B -module M corresponds to a representation $N = (N_\alpha)$ in $\text{rep}_Q(\mathbf{d})$ for some \mathbf{d} , such that $\text{Ann}(N) = I$. Applying the geometric version of the Morita equivalence described by Bongartz in [4], $\overline{\mathcal{O}}_M$ is isomorphic to an associated fibre bundle $\text{GL}(d) \times^{\text{GL}(\mathbf{d})} \overline{\mathcal{O}}_N$. In particular, $\overline{\mathcal{O}}_M$ is regular if and only if $\overline{\mathcal{O}}_N$ is. By the Artin-Voigt formula (see [8]):

$$\text{codim}_{\text{rep}_Q(\mathbf{d})} \overline{\mathcal{O}}_N = \dim_k \text{Ext}_Q^1(N, N),$$

the vanishing of $\text{Ext}_Q^1(N, N)$ means that $\overline{\mathcal{O}}_N = \text{rep}_Q(\mathbf{d})$. Consequently, one implication in Theorem 1.2 is proved and it suffices to show the following fact:

Theorem 2.1. *Let N be a representation in $\text{rep}_Q(\mathbf{d})$ such that $\text{Ann}(N)$ is an admissible ideal in kQ and $\overline{\mathcal{O}}_N$ is a regular variety. Then $\text{Ann}(N) = \{0\}$ and $\overline{\mathcal{O}}_N = \text{rep}_Q(\mathbf{d})$.*

3. Tangent spaces of orbit closures and nilpotent representations

From now on, N is a representation in $\text{rep}_Q(\mathbf{d})$ such that $\text{Ann}(N)$ is an admissible ideal in kQ and $\overline{\mathcal{O}}_N$ is a regular variety. The aim of the section is to prove that the quiver Q has no oriented cycles.

Let $S[j] = (S[j]_i, S[j]_\alpha)$ stand for the simple representation of Q such that $S[j]_j = k$ is the only non-zero vector space and all linear maps $S[j]_\alpha$ are zero, for any vertex

$j \in Q_0$. Observe that the point 0 in $\text{rep}_Q(\mathbf{d})$ is the semisimple representation $\bigoplus_{i \in Q_0} S[i]^{d_i}$. A representation $W = (W_i, W_\alpha)$ of Q is said to be nilpotent if one of the following equivalent conditions is satisfied:

- (1) The endomorphism $W_\omega \in \text{End}_k(W_{s(\omega)})$ is nilpotent for any oriented cycle ω in Q .
- (2) The ideal $\text{Ann}(W)$ contains $(\mathcal{R}_Q)^r$ for some positive integer r .
- (3) Any composition factor of W is isomorphic to some $S[i]$, $i \in Q_0$.
- (4) The orbit closure $\overline{\mathcal{O}}_W$ in $\text{rep}_Q(\mathbf{dim} W)$ contains 0 .

Obviously the representation N is nilpotent. Thus the set $\mathcal{N}_Q(\mathbf{d})$ of nilpotent representations in $\text{rep}_Q(\mathbf{d})$ is a closed $\text{GL}(\mathbf{d})$ -invariant subset which contains $\overline{\mathcal{O}}_N$. Furthermore, $\mathcal{N}_Q(\mathbf{d})$ is a cone, i.e. it is invariant under multiplication by scalars in the vector space $\text{rep}_Q(\mathbf{d})$.

We shall identify the tangent space $\mathcal{T}_{\text{rep}_Q(\mathbf{d}),0}$ of $\text{rep}_Q(\mathbf{d})$ at 0 with $\text{rep}_Q(\mathbf{d})$ itself. Thus the tangent space $\mathcal{T}_{\overline{\mathcal{O}}_N,0}$ is a subspace of $\text{rep}_Q(\mathbf{d})$ and is invariant under the action of $\text{GL}(\mathbf{d})$, i.e. it is a $\text{GL}(\mathbf{d})$ -subrepresentation of $\text{rep}_Q(\mathbf{d})$. Since $\overline{\mathcal{O}}_N$ is a regular variety, the tangent space $\mathcal{T}_{\overline{\mathcal{O}}_N,0}$ is the tangent cone of $\overline{\mathcal{O}}_N$ at 0 (see [7, III. 4]), and the latter is contained in the tangent cone of $\mathcal{N}_Q(\mathbf{d})$ at 0 . Therefore

$$(3.1) \qquad \mathcal{T}_{\overline{\mathcal{O}}_N,0} \subseteq \mathcal{N}_Q(\mathbf{d}).$$

Lemma 3.1. *Let $W = (W_\alpha)$ be a tangent vector in $\mathcal{T}_{\overline{\mathcal{O}}_N,0}$. Then $W_\gamma = 0$ for any loop $\gamma \in Q_1$.*

Proof. Suppose that the nilpotent matrix $W_\gamma \in \mathbb{M}(d_j)$ is non-zero for some loop $\gamma : j \rightarrow j$ in Q_1 . Then there are two linearly independent vectors $v_1, v_2 \in k^{d_j}$ such that $W_\gamma \cdot v_1 = v_2$ and $W_\gamma \cdot v_2 = 0$. We choose $g = (g_i)$ in $\text{GL}(\mathbf{d})$ such that $g_j \cdot v_1 = v_2$ and $g_j \cdot v_2 = v_1$. Then $U = W + g * W$ belongs to $\mathcal{T}_{\overline{\mathcal{O}}_N,0}$. Observe that $U_\gamma \cdot v_1 = v_2$ and $U_\gamma \cdot v_2 = v_1$. Hence the representation U is not nilpotent, contrary to (3.1). \square

Let $V_i = k^{d_i}$ and $R_{i,j}$ be the vector space of formal linear combinations of arrows $\alpha \in Q_1$ with $s(\alpha) = i$ and $t(\alpha) = j$, for any $i, j \in Q_0$. We shall identify:

$$\text{rep}_Q(\mathbf{d}) = \bigoplus_{i,j \in Q_0} \text{Hom}_k(R_{i,j}, \text{Hom}_k(V_i, V_j)) \quad \text{and} \quad \text{GL}(\mathbf{d}) = \bigoplus_{i \in Q_0} \text{GL}(V_i).$$

Applying Lemma 3.1 we get

$$\mathcal{T}_{\overline{\mathcal{O}}_N,0} \subseteq \bigoplus_{\substack{i,j \in Q_0 \\ i \neq j}} \text{Hom}_k(R_{i,j}, \text{Hom}_k(V_i, V_j)).$$

Since the $\text{GL}(\mathbf{d})$ -representations $\text{Hom}_k(V_i, V_j)$, $i \neq j$, are simple and pairwise non-isomorphic, we have

$$\mathcal{T}_{\overline{\mathcal{O}}_{N,0}} = \bigoplus_{\substack{i,j \in Q_0 \\ i \neq j}} \{\varphi: R_{i,j} \rightarrow \text{Hom}_k(V_i, V_j) \mid \varphi(U_{i,j}) = 0\}$$

for some subspaces $U_{i,j}$ of $R_{i,j}$, $i \neq j$.

The spaces $U_{i,j}$ are not necessarily spanned by arrows $\alpha: i \rightarrow j$ in Q_1 , and we are going to replace N by a ‘‘better’’ representation in $\text{rep}_Q(\mathbf{d})$. The group $\tilde{G} = \bigoplus_{i,j \in Q_0} \text{GL}(R_{i,j})$ can be identified naturally with a subgroup of automorphisms of the path algebra kQ which change linearly the paths of length 1 but do not change the paths of length 0. Let $\tilde{g} = (\tilde{g}_{i,j})$ be an element of \tilde{G} . Then $\tilde{g} \star (\mathcal{R}_Q)^p = (\mathcal{R}_Q)^p$ for any positive integer p , where \star denotes the action of \tilde{G} on kQ . For a representation W of Q presented in the form

$$W = (W_i, W_{i,j}: R_{i,j} \rightarrow \text{Hom}_k(W_i, W_j))_{i,j \in Q_0},$$

we define the representation

$$\tilde{g} \star W = (W_i, W_{i,j} \circ (\tilde{g}_{i,j})^{-1})_{i,j \in Q_0}.$$

Hence \tilde{G} acts regularly on $\text{rep}_Q(\mathbf{d})$ and this action commutes with the $\text{GL}(\mathbf{d})$ -action. Therefore the orbit closure $\overline{\mathcal{O}}_{\tilde{g} \star N} = \tilde{g} \star \overline{\mathcal{O}}_N$ is a regular variety, $\mathcal{T}_{\overline{\mathcal{O}}_{\tilde{g} \star N,0}} = \tilde{g} \star \mathcal{T}_{\overline{\mathcal{O}}_{N,0}}$ and the ideal $\text{Ann}(\tilde{g} \star N) = \tilde{g} \star \text{Ann}(N)$ is admissible as

$$(\mathcal{R}_Q)^r = \tilde{g} \star (\mathcal{R}_Q)^r \subseteq \tilde{g} \star \text{Ann}(N) \subseteq \tilde{g} \star (\mathcal{R}_Q)^2 = (\mathcal{R}_Q)^2.$$

Hence, replacing N by $\tilde{g} \star N$ for an appropriate \tilde{g} , we may assume that the spaces $U_{i,j}$, $i \neq j$, are spanned by arrows in Q_1 . Consequently,

$$(3.2) \quad \mathcal{T}_{\overline{\mathcal{O}}_{N,0}} = \text{rep}_{Q'}(\mathbf{d}) \subseteq \text{rep}_Q(\mathbf{d})$$

for some subquiver Q' of Q such that $Q'_0 = Q_0$ and Q'_1 has no loops.

Lemma 3.2. *The quiver Q' has no oriented cycles.*

Proof. Suppose there is an oriented cycle ω in Q' . Let $W = (W_\alpha)$ be a tangent vector in $\mathcal{T}_{\overline{\mathcal{O}}_{N,0}} = \text{rep}_{Q'}(\mathbf{d})$ such that each W_α , $\alpha \in (Q')_1$, is the matrix whose $(1, 1)$ -entry is 1, while the other entries are 0. Then the matrix W_ω has the same form, contrary to (3.1). \square

Let $W = (W_i, W_\alpha)$ be a representation of Q . We denote by $\text{rad}(W)$ the radical of W . In case W is nilpotent, $\text{rad}(W) = \sum_{\alpha \in Q_1} \text{Im}(W_\alpha)$. We write $\langle w \rangle$ for the subrepresentation of W generated by a vector $w \in \bigoplus_{i \in Q_0} W_i$.

Lemma 3.3. *Let $\alpha : i \rightarrow j$ be an arrow in Q_1 such that $N_\alpha(v)$ does not belong to $\text{rad}^2\langle v \rangle$ for some $v \in V_i$. Then $\alpha \in Q'_1$.*

Proof. Let $d = \sum_{i \in Q_0} d_i$ and $c = \dim_k \langle v \rangle$. Then $\dim_k \text{rad}\langle v \rangle = c - 1$ and $d \geq c \geq 2$. Since $N_\alpha(v)$ does not belong to $\text{rad}(\text{rad}\langle v \rangle)$, there is a codimension one subrepresentation W of $\text{rad}\langle v \rangle$ which does not contain $N_\alpha(v)$. We choose a basis $\{\epsilon_1, \dots, \epsilon_d\}$ of the vector space $\bigoplus_{i \in Q_0} V_i$ such that:

- the vector ϵ_b belongs to V_{i_b} for some vertex $i_b \in Q_0$, for any $b \leq d$;
- the vectors $\epsilon_1, \dots, \epsilon_b$ span a subrepresentation, say $N(b)$, of N for any $b \leq d$;
- $N(c - 2) = W$, $\epsilon_{c-1} = N_\alpha(v)$, $N(c - 1) = \text{rad}\langle v \rangle$, $\epsilon_c = v$ and $N(c) = \langle v \rangle$.

In fact, $0 = N(0) \subset N(1) \subset N(2) \subset \dots \subset N(d) = N$ is a composition series of N . In particular, $N_\beta(\epsilon_b)$ belongs to $N(b - 1)$, for any $b \leq d$ and any arrow $\beta : i_b \rightarrow j$ in Q_1 . We take a decreasing sequence of integers

$$p_1 > p_2 > \dots > p_d$$

and define a group homomorphism $\varphi : k^* \rightarrow \text{GL}(\mathbf{d}) = \bigoplus_{i \in Q_0} \text{GL}(V_i)$ such that $\varphi(t)(\epsilon_b) = t^{p_b} \cdot \epsilon_b$ for any $b \leq d$. Observe that

$$N_\beta(\epsilon_b) = \sum_{i < b} \lambda_i \cdot \epsilon_i, \quad \lambda_i \in k, \quad \text{implies} \quad (\varphi(t) * N)_\beta(\epsilon_b) = \sum_{i < b} t^{p_i - p_b} \lambda_i \cdot \epsilon_i$$

for any $b \leq d$ and any arrow $\beta : i_b \rightarrow j$ in Q_1 . This leads to a regular map $\psi : k \rightarrow \overline{O}_N$ such that $\psi(t) = \varphi(t) * N$ for $t \neq 0$ and $\psi(0) = 0$.

Assume now that $p_{c-1} - p_c = 1$. Applying the induced linear map $T_{\psi,0} : T_{k,0} \rightarrow T_{\overline{O}_N,0}$ and using the fact that $N_\alpha(\epsilon_c) = \epsilon_{c-1}$, we obtain a tangent vector $W = (W_\alpha) \in T_{\overline{O}_N,0}$ such that $W_\alpha(\epsilon_c) = \epsilon_{c-1} \neq 0$. Thus $\alpha \in Q'_1$. \square

Lemma 3.4. *For any arrow $\alpha : i \rightarrow j$ in Q_1 , there exists a path ω in Q' of positive length such that $s(\omega) = i$ and $t(\omega) = j$.*

Proof. Since $\text{Ann}(N)$ is an admissible ideal in kQ , there is a vector $v \in V_i$ such that $N_\alpha(v) \neq 0$. Let $\omega = \alpha_m \cdot \dots \cdot \alpha_2 \alpha_1$ be a longest path from i to j with $N_\omega(v) \neq 0$. Hence $N_\rho(v) = 0$ for any $\rho \in \epsilon_j \cdot (\mathcal{R}_Q)^{m+1} \cdot \epsilon_i$. We show that the path ω satisfies the claim. Let $v_0 = v$ and $v_l = N_{\alpha_l}(v_{l-1})$ for $l = 1, \dots, m$. According to Lemma 3.3, it is enough to show that $v_l \notin \text{rad}^2\langle v_{l-1} \rangle$ for any $1 \leq l \leq m$. Indeed, if $v_l \in \text{rad}^2\langle v_{l-1} \rangle$ for some l , then $v_m \in \text{rad}^{m+1}\langle v_0 \rangle$, or equivalently, $N_\omega(v) = N_\rho(v)$ for some $\rho \in \epsilon_j \cdot (\mathcal{R}_Q)^{m+1} \cdot \epsilon_i$, a contradiction. \square

Combining Lemmas 3.2 and 3.4, we get

Corollary 3.5. *The quiver Q does not contain oriented cycles.*

4. Gradings of polynomials on $\text{rep}_Q(\mathbf{d})$

Let $\pi: \text{rep}_Q(\mathbf{d}) \rightarrow \text{rep}_{Q'}(\mathbf{d})$ denote the obvious $\text{GL}(\mathbf{d})$ -equivariant linear projection and let $N' = \pi(N)$. Then $\pi(\mathcal{O}_N) = \mathcal{O}_{N'}$ and we get a dominant morphism

$$\eta = \pi|_{\overline{\mathcal{O}}_N}: \overline{\mathcal{O}}_N \rightarrow \overline{\mathcal{O}}_{N'}.$$

Lemma 4.1. $\overline{\mathcal{O}}_{N'} = \text{rep}_{Q'}(\mathbf{d})$.

Proof. Since $\text{Ker}(\pi) \cap \mathcal{T}_{\overline{\mathcal{O}}_N,0} = \{0\}$, the morphism η is étale at 0. This implies that the variety $\overline{\mathcal{O}}_{N'}$ is regular at $\eta(0) = 0$ (see [7, III. 5] for basic information about étale morphisms). Since it is contained in $\text{rep}_{Q'}(\mathbf{d})$, it suffices to show that $\mathcal{T}_{\overline{\mathcal{O}}_{N'},0} = \text{rep}_{Q'}(\mathbf{d})$. The latter can be concluded from the induced linear map $\mathcal{T}_{\eta,0}: \mathcal{T}_{\overline{\mathcal{O}}_N,0} \rightarrow \mathcal{T}_{\overline{\mathcal{O}}_{N'},0}$, which is the restriction of $\mathcal{T}_{\pi,0} = \pi$. \square

Let $R = k[X_{\alpha,p,q}]_{\alpha \in Q_1, p \leq d_{t(\alpha)}, q \leq d_{s(\alpha)}}$ denote the algebra of polynomial functions on the vector space $\text{rep}_Q(\mathbf{d})$ and $\mathfrak{m} = (X_{\alpha,p,q})$ be the maximal ideal in R generated by variables. Here, $X_{\beta,p,q}$ maps a representation $W = (W_\alpha)$ to the (p, q) -entry of the matrix W_β . Using π , the polynomial functions on $\text{rep}_{Q'}(\mathbf{d})$ form the subalgebra $R' = k[X_{\alpha,p,q}]_{\alpha \in Q'_1, p \leq d_{t(\alpha)}, q \leq d_{s(\alpha)}}$ of R . By Lemma 4.1,

$$(4.1) \quad I(\overline{\mathcal{O}}_N) \cap R' = \{0\},$$

where $I(\overline{\mathcal{O}}_N)$ stands for the ideal of the set $\overline{\mathcal{O}}_N$ in R .

Let X_α denote the $d_{t(\alpha)} \times d_{s(\alpha)}$ -matrix whose (p, q) -entry is the variable $X_{\alpha,p,q}$, for any arrow α in Q_1 . We define the $d_j \times d_i$ -matrix X_ρ for $\rho \in \varepsilon_j \cdot kQ \cdot \varepsilon_i$, with coefficients in R , in a similar way as for representations of Q .

The action of $\text{GL}(\mathbf{d})$ on $\text{rep}_Q(\mathbf{d})$ induces an action on the algebra R by $(g * f)(W) = f(g^{-1} * W)$ for $g \in \text{GL}(\mathbf{d})$, $f \in R$ and $W \in \text{rep}_Q(\mathbf{d})$. We choose a standard maximal torus T in $\text{GL}(\mathbf{d})$ consisting of $g = (g_i)$, where all $g_i \in \text{GL}(d_i)$ are diagonal matrices. Let \tilde{Q}_0 denote the set of pairs (i, p) with $i \in Q_0$ and $1 \leq p \leq d_i$. Then the action of T on R leads to a $\mathbb{Z}^{\tilde{Q}_0}$ -grading on R with

$$(4.2) \quad \text{deg}(X_{\alpha,p,q}) = e_{s(\alpha),q} - e_{t(\alpha),p},$$

where $\{e_{i,p}\}_{(i,p) \in \tilde{Q}_0}$ is the standard basis of $\mathbb{Z}^{\tilde{Q}_0}$.

Proposition 4.2. $Q' = Q$.

Proof. Suppose the contrary, which means there is an arrow β in $Q_1 \setminus Q'_1$. Since the quiver Q has no oriented cycles, we can choose β minimal in the sense that any path ω in Q of length greater than 1 with $s(\omega) = s(\beta)$ and $t(\omega) = t(\beta)$ is in fact a path

in Q' . We conclude from (3.2) that $X_{\beta,u,v} \in \mathfrak{m}^2 + I(\overline{\mathcal{O}}_N)$ for $u \leq d_{t(\beta)}$ and $v \leq d_{s(\beta)}$. Since the polynomials $X_{\beta,u,v}$ as well as the ideals \mathfrak{m}^2 and $I(\overline{\mathcal{O}}_N)$ are homogeneous with respect to the above grading, there are homogeneous polynomials $f_{\beta,u,v}$ in the ideal \mathfrak{m}^2 such that

$$X_{\beta,u,v} - f_{\beta,u,v} \in I(\overline{\mathcal{O}}_N) \quad \text{and} \quad \deg(f_{\beta,u,v}) = e_{s(\beta),v} - e_{t(\beta),u}.$$

Let $\prod_{l \leq n} X_{\alpha_l, p_l, q_l}$ be a monomial in R of degree $e_{s(\beta),v} - e_{t(\beta),u}$. Then

$$\begin{aligned} & \#\{1 \leq l \leq n \mid s(\alpha_l) = i, q_l = r\} - \#\{1 \leq l \leq n \mid t(\alpha_l) = i, p_l = r\} \\ &= \begin{cases} 1 & (i, r) = (s(\beta), v), \\ -1 & (i, r) = (t(\beta), u), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus by (4.2), up to a permutation of the above variables, we get that $\omega = \alpha_m \cdots \alpha_1$ is a path in Q for some $m \leq n$ such that $(s(\alpha_1), q_1) = (s(\beta), v)$, $(t(\alpha_m), p_m) = (t(\beta), u)$ and $q_l = p_{l-1}$ for $l = 2, \dots, m$. Consequently, $\deg(X_{\alpha_{m+1}, p_{m+1}, q_{m+1}} \cdots X_{\alpha_n, p_n, q_n}) = 0$. Since Q has no oriented cycles, the only monomial in R with degree zero is the constant function 1. Hence $m = n$ and the homogenous polynomial $f_{\beta,u,v}$ is the following linear combination:

$$\begin{aligned} f_{\beta,u,v} = & \sum \lambda(u, \alpha_m, p_{m-1}, \alpha_{m-1}, \dots, p_1, \alpha_1, v) \\ & \cdot X_{\alpha_m, u, p_{m-1}} \cdot X_{\alpha_{m-1}, p_{m-1}, p_{m-2}} \cdots X_{\alpha_2, p_2, p_1} \cdot X_{\alpha_1, p_1, v}, \end{aligned}$$

where the sum runs over all paths $\omega = \alpha_m \cdots \alpha_1$ in Q with $s(\omega) = s(\beta)$, $t(\omega) = t(\beta)$ and positive integers $p_l \leq d_{t(\alpha_l)}$ for $l = 1, \dots, m - 1$. Since $f_{\beta,u,v}$ belongs to the ideal \mathfrak{m}^2 , we may assume that $m \geq 2$. Then the arrows $\alpha_1, \dots, \alpha_m$ belong to Q' , by the minimality of β . In particular, $f_{\beta,u,v}$ belongs to R' .

We claim that the scalars $\lambda(u, \alpha_m, p_{m-1}, \alpha_{m-1}, \dots, p_1, \alpha_1, v)$ do not depend on the integers u, p_{m-1}, \dots, p_1 and v . Indeed, take $u' \leq d_{t(\beta)}$, $v' \leq d_{s(\beta)}$ and $p'_l \leq d_{t(\alpha_l)}$ for $l = 1, \dots, m - 1$. We choose $g = (g_i)$ in $\text{GL}(\mathbf{d})$ with each g_i being the permutation matrix associated to a specific permutation $\sigma_i \in S_{d_i}$. Then the multiplication by g in the algebra R permutes the monomials in R . We assume that

$$\begin{aligned} \sigma_{s(\beta)}(v) &= v', \quad \sigma_{s(\beta)}(v') = v, \quad \sigma_{t(\beta)}(u) = u', \quad \sigma_{t(\beta)}(u') = u, \\ \sigma_{t(\alpha_l)}(p_l) &= p'_l \quad \text{and} \quad \sigma_{t(\alpha_l)}(p'_l) = p_l, \quad \text{for } l = 1, \dots, m - 1. \end{aligned}$$

Since $g * X_{\beta,u',v'} = X_{\beta,u,v}$, the polynomial

$$f_{\beta,u,v} - g * f_{\beta,u',v'} = g * (X_{\beta,u',v'} - f_{\beta,u',v'}) - (X_{\beta,u,v} - f_{\beta,u,v})$$

belongs to the ideal $I(\overline{\mathcal{O}}_N)$, as the latter is $\text{GL}(\mathbf{d})$ -invariant. Thus $f_{\beta,u,v} = g * f_{\beta,u',v'}$, by (4.1). Hence the claim follows from the fact that the monomial

$$X_{\alpha_m, u, p_{m-1}} \cdot X_{\alpha_{m-1}, p_{m-1}, p_{m-2}} \cdots \cdots X_{\alpha_2, p_2, p_1} \cdot X_{\alpha_1, p_1, v}$$

appears in $g * f_{\beta,u',v'}$ with coefficient $\lambda(u', \alpha_m, p'_{m-1}, \alpha_{m-1}, \dots, p'_1, \alpha_1, v')$.

Let Ξ denote the set of all paths ξ in Q' of length greater than 1 with $s(\xi) = s(\beta)$ and $t(\xi) = t(\beta)$. Then there are scalars $\lambda(\xi)$, $\xi \in \Xi$, such that

$$f_{\beta,u,v} = \sum_{\xi = \alpha_m \dots \alpha_1 \in \Xi} \lambda(\xi) \cdot \sum_{p_1 \leq d_{t(\alpha_1)}} \cdots \sum_{p_{m-1} \leq d_{t(\alpha_{m-1})}} X_{\alpha_m, u, p_{m-1}} \cdots \cdots X_{\alpha_1, p_1, v}$$

for any $u \leq d_{t(\beta)}$ and $v \leq d_{s(\beta)}$. This equality means that $f_{\beta,u,v}$ is the (u, v) -entry of the matrix X_ρ , where $\rho = \sum_{\xi \in \Xi} \lambda(\xi) \cdot \xi \in kQ'$. Consequently, the entries of the matrix $X_{\beta-\rho}$ belong to the ideal $I(\overline{\mathcal{O}}_N)$. This implies that $\beta - \rho$ belongs to $\text{Ann}(N)$. Since $\beta - \rho$ does not belong to $(\mathcal{R}_Q)^2$, the ideal $\text{Ann}(N)$ is not admissible, a contradiction. □

Combining Lemma 4.1 and Proposition 4.2 we get

$$(4.3) \quad \overline{\mathcal{O}}_N = \text{rep}_Q(\mathbf{d}).$$

Hence the following lemma finishes the proof of Theorem 2.1.

Lemma 4.3. $\text{Ann}(N) = \{0\}$.

Proof. Suppose the contrary, that there is a non-zero element ρ in $\varepsilon_j \cdot \text{Ann}(N) \cdot \varepsilon_i$ for some vertices i and j . Observe that the set of representations $W = (W_\alpha)$ in $\text{rep}_Q(\mathbf{d})$ such that $W_\rho = 0$ is closed and $\text{GL}(\mathbf{d})$ -invariant. Hence $W_\rho = 0$ for any representation $W = (W_\alpha)$ in $\text{rep}_Q(\mathbf{d})$, by (4.3). Of course, ρ is a linear combination of paths in Q of length greater than 1 with $s(\omega) = i$ and $t(\omega) = j$. Let ω_0 be a path appearing in ρ with coefficient $\lambda \neq 0$. We choose a representation $W = (W_\alpha)$ in $\text{rep}_Q(\mathbf{d})$ such that W_α is the matrix whose $(1, 1)$ -entry is 1 and the other entries are 0 if the arrow α appears in the path ω_0 , and $W_\alpha = 0$ otherwise. Then the $(1, 1)$ -entry of W_ρ equals λ , a contradiction. □

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