Title	Stability Analysis by Using Lie Algebras and Controller Design Based on the Lie Derivative Inclusion for Nonlinear Systems
Author(s)	湯野, 剛史
Citation	大阪大学, 2015, 博士論文
Version Type	VoR
URL	https://doi.org/10.18910/52070
rights	
Note	

Stability Analysis by Using Lie Algebras and Controller Design Based on the Lie Derivative Inclusion for Nonlinear Systems

Tsuyoshi Yuno

March 2015

Stability Analysis by Using Lie Algebras and Controller Design Based on the Lie Derivative Inclusion for Nonlinear Systems

A dissertation submitted to

the Graduate School of Engineering Science
Osaka University

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Engineering

by

Tsuyoshi Yuno

March 2015

Abstract

Many problems in nonlinear control theory have difficulty in computing their exact solutions. In this dissertation, we aim to develop a framework of computing exact solutions to nonlinear control problems by employing the theory of algebras. We require here that the computation consist of a finite number of arithmetic operations. We place particular focus on two problems. The first one concerns the stability analysis of discrete-time nonlinear systems with state-dependent coefficient matrices. A necessary and sufficient condition for the systems to be triangularized is obtained by using the theory of Lie algebras, and sufficient conditions for the local and global stabilities of triangularizable systems are derived. In many cases, the stability conditions can be checked by a finite number of arithmetic operations of matrices and the evaluation of the asymptotic behavior of functions rather than heuristic procedures such as Lyapunov's direct method. The second one concerns controller design. We formulate a particular inclusion of polynomials that represents, in a unified manner, the conditions for controllers to satisfy their respective requirements in several nonlinear control problems, e.g., the realization problem of a vector field and the model matching problem. An algorithm for designing the state feedback controllers satisfying the inclusion is obtained by using the theory of commutative algebras and Gröbner bases. The extension of the algorithm to the output feedback case is not straightforward because it requires solving a linear equation over a subalgebra that lacks a solution method. Even so, by providing a procedure for solving the linear equation, we are able to derive an extended algorithm for designing the output feedback controllers satisfying the inclusion. This also allows us to solve the problem of replacing a state feedback controller with output ones. By using these algorithms, the required controllers in various control problems can be explicitly computed in a finite number of arithmetic operations of polynomials. Since the design of output feedback controllers is very difficult in general, the results here will be of great benefit in control theory.

Notation

$\mathbf{R}$: the field of all real numbers

$\mathbf{C}$: the field of all complex numbers

$\mathbf{N}$: the set of all non-negative integers

$\mathbf{R}^n$: the set (or vector space) of all column vectors with n components in $\mathbf{R}$

$\mathbf{C}^n$: the set (or vector space) of all column vectors with n components in $\mathbf{C}$

$\mathbf{R}^{n\times m}$: the set (or vector space) of all $n \times m$ matrices with entries in $\mathbf{R}$

$\mathbf{C}^{n\times m}$: the set (or vector space) of all $n \times m$ matrices with entries in $\mathbf{C}$

Contents

1 Introduction **1**
1.1 Background and motivation . . . 1
1.2 Composition of this dissertation . . . 4

2 Commutative Algebras, Modules, and Algebraic Geometry **5**
2.1 Commutative algebras and modules . . . 5
2.2 Algebraic geometry . . . 10

3 Stability Analysis by Using Lie Algebras **13**
3.1 Introduction . . . 13
3.2 System description . . . 15
3.3 Upper triangularizability . . . 16
3.3.1 Lie algebras . . . 16
3.3.2 Upper triangularizability condition . . . 18
3.3.3 An algorithm for checking the solvability of $\mathcal{L}(\mathscr{A})$ . . . 19
3.4 Local asymptotic stability . . . 20
3.5 Global asymptotic stability . . . 26
3.6 Examples . . . 29
3.6.1 How to apply the global stability condition . . . 30
3.6.2 The case where one of the conditions in the theorems is not satisfied . . . 32

4 Lie Derivative Inclusion with State Feedback **35**
4.1 Introduction . . . 35
4.2 Notation, system description, and motivating problems . . . 37
4.2.1 Notation and system description . . . 37
4.2.2 Controlled invariance of a prescribed algebraic set . . . 38
4.2.3 Realization of a prescribed vector field . . . 41
4.3 Lie derivative inclusion with state feedback . . . 43
4.4 Solution to the Lie derivative inclusion with state feedback . . . 45
4.5 Behavior outside the algebraic set . . . 50

4.6 Examples 52
4.6.1 Model matching problem 52
4.6.2 Path following control 54

5 Lie Derivative Inclusion with Static Output Feedback 57
5.1 Introduction 57
5.2 Notation, system description, and problem modification 58
5.3 Lie derivative inclusion with static output feedback 60
5.4 Solution to the Lie derivative inclusion with static output feedback 62
5.4.1 Solution algorithm 62
5.4.2 Computing the set $\{\eta_1, ..., \eta_\ell\}$ in Algorithm 5.10 69
5.5 Example 71

6 Conclusions 75
6.1 Summary 75
6.2 Discussion 76
6.3 Future recommendation 77

Appendix 79
A.1 Computing a syzygy module over a factor ring 79
A.2 Intersecting a module with a subalgebra 80
A.3 Computing the kernel of an algebra homomorphism 80
A.4 Solving a linear equation with polynomial coefficients 81

Bibliography 83

List of Publications 95

Chapter 1

Introduction

1.1 Background and motivation

James Watt's steam engines had an indelible impact on the Industrial Revolution. Watt's main contribution was his idea of equipping an engine with a governor so as to regulate its rotational speed. However, although the governor was able to reduce unstable engine motion, it could still cause hunting oscillations. In the middle of the 19th century, James Clerk Maxwell, a Scottish mathematical physicist, tackled the problem of designing a stable governor [1]. He described a governor as a linear ordinary differential equation and mathematically formulated its stability. Moreover, he derived a mathematical condition for the governor to be stable. After his pioneering work, the Laplace transformation was employed for analyzing and controlling linear systems, i.e., systems that can be described by linear ordinary differential equations, and a good number of fundamental results were obtained for the systems [2]. This framework is now known as classical control theory.

In the middle of the 20th century, a new approach to analyze and control linear systems, modern control theory, was pioneered by Rudolf Emil Kalman, an electrical engineer in the U.S. Modern control theory deals with linear systems in the time domain, whereas the classical one does so in the frequency domain. Although modern control theory is less intuitive than the classical one, it allows us to analyze and control linear systems in greater detail. Accordingly, a variety of fundamental concepts, including observability and controllability, have been defined and thoroughly investigated in the time domain. Modern control theory has since spread to control engineering [3].

Although both of these control theories deal with linear systems, most of the practical systems have more or less nonlinearities. Hence, there have been many attempts to establish a control theory for nonlinear systems. Some early works approximated nonlinear systems by linear ones and employed linear control theory for the systems. While this method is effective for systems that have small nonlinearities and is currently used for many practical control problems, it is not sufficient for more precise control of nonlinear systems and cannot be used for systems that have significant nonlinearities.

The principal difficulty in controlling nonlinear systems is that we cannot generally calculate explicit solutions of nonlinear ordinary differential equations. In the late 19th century, the French mathematician Henri Poincaré revealed that the three-body problem is not solvable generally; i.e., we cannot obtain the explicit solution of the nonlinear differential equation describing the motions of three point masses that interact with each other. However, he also showed that we can still figure out the qualitative properties of differential equations and emphasized the importance of this.[1] Many mathematicians and scientists followed his paradigm, which came to be known as dynamical systems theory. Most notably, the Russian mathematician Aleksandr Mikhailovich Lyapunov showed in his work [10, 11] that the stability of a nonlinear system is guaranteed by the existence of a function called the Lyapunov function. Although the dynamical systems theory was developed independently of control theory, it was considerably utilized for nonlinear control theory. In particular, Lyapunov's theory forms the basis of the stability analysis and stabilization of nonlinear systems [12, 13].

The concept of optimality was also introduced to control theory. A general formulation of the optimization problem was first established during World War II in the area of operations research. Richard Ernest Bellman and Lev Semenovich Pontryagin, mathematicians in the U.S. and the U.S.S.R., respectively, introduced the theory of optimization into nonlinear control theory in the late 1950s, and this framework is now known as optimal control theory. They showed [14–16] that an optimal controller, which minimizes the value of a prescribed objective function, can be obtained by solving certain equations. Optimal control theory formed the basis of modern control theory and nonlinear control theory [17–19].

A variety of mathematical theories besides those stated above have been utilized for nonlinear control theory. However, despite a lot of effort, there are still many open problems in nonlinear control theory. For example, we cannot generally check the stability of nonlinear systems because there is no method of explicitly constructing a Lyapunov function. We also cannot design an optimal controller for general nonlinear systems because the corresponding equations cannot be solved generally.

Analysis and (differential) geometry in mathematics have been the main

[1] According to [4, 5], these works are included in [6–9].

tools in nonlinear control theory [12, 13, 20, 21]. In general, analytical approaches tend to yield approximate solutions to mathematical problems, and geometric approaches tend to yield non-constructive procedures for solving the problems. In other words, these approaches are not suitable for computing exact solutions to nonlinear control problems. Since it is difficult in many open problems to compute exact solutions, it seems that another approach should be employed.

Besides the above mathematical tools, the theory of abstract algebras and algebraic geometry have been used for nonlinear control theory. For example, Michael Fliess introduced the theory of differential algebras into nonlinear control theory [22] and also defined the concept of differential flatness [23]. As a result of his work, the differential algebraic framework has been widely utilized for nonlinear control theory, including the analysis of input-output invertibility [24], the analysis of system equivalence [25], and the disturbance decoupling problem [26]. The concept of differential flatness has been associated with many control problems, e.g., controllability analysis [23], feedforward linearization [27], and trajectory planning [28]. Further applications of Fliess's framework are included in [29, 30]. Other algebraic approaches have also been proposed. The nonlinear transfer function was formulated in [31] and utilized for the model matching problem [32] and accessibility/observability analysis [33]. Solutions of the Hamilton-Jacobi equation were characterized within algebraic context by [34]. Nonlinear systems described by partial differential equations were analyzed by using algebraic analysis in [35]. Further algebraic approaches are included in [36], in which many control problems, including the realization problem, the model matching problem, and the input-output linearization, are discussed.

In general, compared to analytic and geometric approaches, algebraic approaches are more suitable for the computation (especially the symbolic calculation) of exact solutions to control problems because they originated in the methodology of symbolic calculation. Indeed, the techniques in computational algebra, including the theories of Gröbner bases [37–39] and quantifier elimination [40], have been employed for the symbolic computations of exact solutions to several control problems [39, 41–43]. Accordingly, in this dissertation, we aim to develop a framework for computing exact solutions to nonlinear control problems by employing the theory of algebras. We require here that the computation consist of a finite number of arithmetic operations. We place particular focus on two problems. The first one concerns the stability analysis of discrete-time nonlinear systems with state-dependent coefficient matrices. We will obtain a necessary and sufficient condition for the triangularizability of the systems by using the theory of Lie algebras and derive sufficient conditions for the local and global stabilities of triangularizable systems. The second one concerns controller design. We will formulate a particular inclusion of polynomials that represents, in a unified manner, the conditions for controllers to satisfy their respective requirements in several nonlinear control problems, e.g., the realization problem of a vector field and the model matching problem. After that, we will derive an algorithm for designing the state feedback controllers satisfying the inclusion by using the theory of commutative algebras and Gröbner bases. The extension of the algorithm to the output feedback case is not straightforward because it requires solving a linear equation over a subalgebra that has no solution method. Even so, by providing a procedure for solving the linear equation, we will derive the extended algorithm for designing the output feedback controllers satisfying the inclusion.

1.2 Composition of this dissertation

This dissertation is composed as follows. In Chapter 2, the basic concepts in commutative algebras, modules, and algebraic geometry are summarized. In Chapter 3, the stability of discrete-time nonlinear systems with state-dependent coefficient matrices is investigated. The theory of Lie algebras is utilized for checking the triangularizability of the systems, and sufficient conditions for the stability of the systems are derived. In Chapter 4, we formulate the Lie derivative inclusion to represent several nonlinear control problems in a unified manner and give an algorithm for solving the inclusion by using the theory of Gröbner bases. In Chapter 5, the Lie derivative inclusion is extended to the case of static output feedback. Although a difficulty in solving the inclusion appears, we provide a new algorithm for solving the inclusion. Chapter 6 presents the conclusions of this dissertation. The supplementary algorithms employed for solving the Lie derivative inclusion are included in the appendix.

Chapter 2

Commutative Algebras, Modules, and Algebraic Geometry

2.1 Commutative algebras and modules

This section summarizes basic concepts in commutative algebras and modules for the sake of completeness. These concepts and further ones are detailed in [44–47].

Definition 2.1. A *law of composition* $\top$ on a set S is a mapping

$$\begin{array}{ccc} (s, s') & \longmapsto & s \top s' \\ \Cap & & \Cap \\ S \times S & \longrightarrow & S. \end{array}$$

Definition 2.2. A law of composition $\top$ is said to be

(i) *associative* if $(s \top t) \top u = s \top (t \top u)$ for all $s, t, u \in S$.

(ii) *commutative* if $s \top t = t \top s$ for all $s, t \in S$.

Definition 2.3. Let T and S be two sets. A *law of (left) action* $\perp$ of T on S is a mapping

$$\begin{array}{ccc} (t, s) & \longmapsto & t \perp s \\ \Cap & & \Cap \\ T \times S & \longrightarrow & S. \end{array}$$

Definition 2.4. An *identity (element)* of a set S under a law of composition $\top$ is an element $e \in S$ satisfying $s \top e = e \top s = s$ for all $s \in S$.

Definition 2.5. A *semigroup* is a set together with an associative law of composition.

Proposition 2.6. *An identity of a semigroup S under its law of composition is unique if it exists.*

Definition 2.7. A *monoid* is a semigroup having an identity under its law of composition.

Definition 2.8. Let S be a monoid with a low of composition $\top$ and $e \in S$ be its identity. An *inverse (element)* of $s \in S$ under $\top$ is an element $s' \in S$ satisfying $s \top s' = s' \top s = e$.

Proposition 2.9. *Let S be a monoid. An inverse of an element $s \in S$ under its law of composition is unique if it exists.*

Definition 2.10. A *group* is a monoid G such that each element in G has its inverse under the law of composition of G.

Definition 2.11. A group is said to be *abelian* if its law of composition is commutative.

Definition 2.12. A *subgroup* of a group G is a subset of G such that it contains the identity of G and is itself a group under the law of composition of G.

Definition 2.13. A *ring* is a set R together with laws of composition $+$ and $\times$ satisfying the following properties:

(i) R is an abelian group w.r.t. $+$,

(ii) R is a monoid w.r.t. $\times$,

(iii) $a \times (b+c) = a \times b + a \times c$ and $(a+b) \times c = a \times c + b \times c$ for all $a, b, c \in R$.

The laws of composition $+$ and $\times$ are called the *addition* and the *multiplication*, respectively.

As usual, for a ring R, its additive identity and multiplicative identity are denoted by 0 and 1, respectively. Moreover, for an element $a \in R$, its additive inverse and multiplicative inverse are denoted by $-a$ and a^{-1}, respectively.

We henceforth let ab denote $a \times b$ as usual.

Definition 2.14. A ring R is called a *commutative ring* if its multiplication is commutative.

Definition 2.15. A *field* is a commutative ring $K \supsetneq \{0\}$ such that any element $a \neq 0 \in K$ has its multiplicative inverse.

Example 2.16. The set of all polynomials over a ring R with several indeterminates $X_1, \ldots, X_n$, denoted by $R[X_1, \ldots, X_n]$, is a commutative ring with its usual addition and multiplication.

Example 2.17. The set of all real numbers and that of all complex numbers are fields and denoted by $\mathbf{R}$ and $\mathbf{C}$, respectively.

We henceforth consider only commutative rings rather than noncommutative ones. Accordingly, in what follows, we let the word "ring" refer to "commutative ring".

Definition 2.18. An *ideal* of a ring R is a subset $I \subset R$ satisfying the following properties:

(i) $x, y \in I \Longrightarrow x + y \in I$,

(ii) $a \in R, x \in I \Longrightarrow ax \in I$.

Definition 2.19. For given elements $r_1, \ldots, r_k$ of a ring R, the *ideal generated by the set of generators* $\{r_1, \ldots, r_k\}$ is the ideal

$$\langle r_1, \ldots, r_k \rangle_R := \{r \in R : \ r = a_1 r_1 + \cdots + a_k r_k, \ a_i \in R\}.$$

Theorem 2.20 (Hilbert's basis theorem)**.** *Let K be a field. Any ideal I of $K[X_1, \ldots, X_n]$ is* finitely generated*, i.e., there exist finitely-many generators $p_1 \ldots, p_k \in K[X_1, \ldots, X_n]$ such that $I = \langle p_1 \ldots, p_k \rangle_{K[X_1, \ldots, X_n]}$.*

Definition 2.21. Let R and R' be rings, and let 1 and $1'$ be their multiplicative identities, respectively. A *ring homomorphism* from R to R' is a map $f : R \to R'$ satisfying the following properties:

(i) $f(a + b) = f(a) + f(b)$ for all $a, b \in R$,

(ii) $f(ab) = f(a)f(b)$ for all $a, b \in R$,

(iii) $f(1) = 1'$.

A bijective ring homomorphism is called a *ring isomorphism*.

Definition 2.22. Let R and R' be two rings. If a ring isomorphism $R \to R'$ exists, the rings R and R' are said to be *isomorphic*, a fact denoted by $R \simeq R'$.

Definition 2.23. An *R-algebra* is a ring A such that a ring homomorphism $\phi : R \to A$ exists. The homomorphism ϕ is called the *structure morphism* of A.

Example 2.24. The polynomial ring $R[X_1, \dots, X_n]$ over a ring R is an R-algebra. The structure morphism is given by the natural embedding $R \hookrightarrow R[X_1, \dots, X_n]$.

Definition 2.25. Let R be a ring. Let A and A' be R-algebras having structure morphisms ϕ and ϕ', respectively. An *R-algebra homomorphism* from A to A' is a ring homomorphism $f : A \to A'$ satisfying $f \circ \phi = \phi'$. A bijective R-algebra homomorphism is called an *R-algebra isomorphism*.

Definition 2.26. Let R be a ring, and let A and A' be R-algebras. If an R-algebra isomorphism $A \to A'$ exists, the algebras are said to be *isomorphic*, a fact denoted by $A \simeq A'$.

Definition 2.27. Let R be a ring and I be an ideal of R. Define the equivalence relation $\sim$ on R by

$$x \sim y \overset{\text{def}}{\iff} x - y \in I.$$

The *factor ring* of R modulo I, denoted by R/I, is the quotient set $R/{\sim}$ together with the ring structure defined by

$$\begin{aligned} \overline{a} + \overline{b} &:= \overline{a + b}, \\ \overline{a}\overline{b} &:= \overline{ab}, \end{aligned}$$

for all $a, b \in R$, where $\overline{\cdot}$ denotes the equivalence class of its argument.

When $\overline{a} = \overline{b} \in R/I$, we write "$a = b \mod I$".

Theorem 2.28 (Homomorphism theorem for rings)**.** *Let R and R' be rings and f be a ring homomorphism $R \to R'$. Then,* $\ker f$ *is an ideal of R, and*

$$R/\ker f \simeq \operatorname{Im} f.$$

Definition 2.29. Let R be a ring and M be an abelian group with a low of composition $\dot{+}$. Suppose there is a law of action $\dot{\times}$ of R on M. The group M is called an *R-module* if $\dot{\times}$ satisfies the following properties:

(i) $1 \dot{\times} x = x$ for all $x \in M$,

(ii) $a \dot{\times} (b \dot{\times} x) = (ab) \dot{\times} x$ for all $a, b \in R$ and $x \in M$,

(iii) $(a+b) \dot{\times} x = a \dot{\times} x \dot{+} b \dot{\times} x$ for all $a, b \in R$ and $x \in M$,

(iv) $a \dot{\times} (x \dot{+} y) = a \dot{\times} x \dot{+} a \dot{\times} y$ for all $a \in R$ and $x, y \in M$.

As usual, for an R-module M, the symbol "+" describing the addition of R is also used to denote the law of composition $\dot{+}$ of M, and it is also called the *addition* of M. The additive inverse of an element $x \in M$ and the additive identity of M are also denoted by $-x$ and 0, respectively, similarly to rings. Moreover, for elements $a \in R$ and $x \in M$, we henceforth let ax denote $a \dot{\times} x$ similarly to rings.

Example 2.30. The set R^n of all column vectors with n components in a ring R is an R-module with its usual addition and operation.

Definition 2.31. Let R be a ring and M be an R-module. An *R-submodule* of M is a subgroup N of M satisfying the property that

$$a \in R, x \in N \Longrightarrow ax \in N.$$

Definition 2.32. Let R be a ring and M be an R-module. For given elements $m_1, \ldots, m_k \in M$, *the R-submodule generated by the set of generators* $\{m_1, \ldots, m_k\}$ is the R-submodule

$$\langle m_1, \ldots, m_k \rangle_R := \{m \in M : \ m = a_1 m_1 + \cdots + a_k m_k, \ a_i \in R\}.$$

Definition 2.33. Let R be a ring, and M and M' be R-modules. An *R-module homomorphism* from M to M' is a map $f : M \to M'$ satisfying the following properties:

(i) $f(x+y) = f(x) + f(y)$ for all $x, y \in M$,

(ii) $f(ax) = af(x)$ for all $a \in R$ and $x \in M$.

A bijective R-module homomorphism is called an *R-module isomorphism*.

Definition 2.34. Let R be a ring, and let M and M' be R-modules. If an R-module isomorphism $M \to M'$ exists, the R-modules are said to be *isomorphic*, a fact denoted by $M \simeq M'$.

Definition 2.35. Let R be a ring and M be an R-module. Let N be an R-submodule of M. Moreover, define the equivalence relation $\sim$ on M by

$$x \sim y \overset{\text{def}}{\Longleftrightarrow} x - y \in N.$$

The *factor module* of M modulo N, denoted by M/N, is the quotient set $M/\sim$ together with the R-module structure defined by

$$\overline{x} + \overline{y} := \overline{x+y},$$
$$a\overline{x} := \overline{ax},$$

for all $a \in R$ and $x \in M$, where $\overline{\cdot}$ denotes the equivalence class of its argument.

When $\overline{x} = \overline{y} \in M/N$, we write "$x = y \mod N$".

Theorem 2.36 (Homomorphism theorem for modules)**.** *Let R be a ring and M be an R-module. Let f be an R-module homomorphism $M \to M'$. Then, $\ker f$ is an R-submodule of M, and*

$$M/\ker f \simeq \operatorname{Im} f.$$

2.2 Algebraic geometry

This section summarizes basic concepts in algebraic geometry for the sake of completeness. These concepts and further ones are detailed in [37, 38, 48].

Proposition 2.37. *Let K be a field and suppose K is of infinite cardinality. Let $K[X] := K[X_1, \ldots, X_n]$ be a polynomial ring over K with indeterminates $X_1, \ldots, X_n$. Then, for any polynomials $p, q \in K[X]$, the following statements are equivalent.*

(i) $p = q$ as polynomials in $K[X]$.

(ii) $p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n)$ for all $[x_1, \ldots, x_n]^{\mathrm{T}} \in K^n$.

In other words, the set $K[X]$ can be identified with the set of all polynomial functions on K^n.

Throughout this section, we let K be a field of infinite cardinality and $K[X] := K[X_1, \ldots, X_n]$ be a polynomial ring over K with indeterminates $X_1, \ldots, X_n$.

Definition 2.38. An *algebraic set* in K^n is a subset of K^n defined by the zeros of several polynomials in $K[X]$.

Proposition 2.39. *Let $p_1, \ldots, p_k$ be polynomials in $K[X]$ and put $I = \langle p_1, \ldots, p_k \rangle_{K[X]}$. Let V be a subset of K^n. Then, the following two statements are equivalent.*

(i) $V = \{x = [x_1, \ldots, x_n]^{\mathrm{T}} \in K^n : \; p_i(x_1, \ldots, x_n) = 0, \; i = 1, \ldots, k\}$.

(ii) $V = \{x = [x_1, \ldots, x_n]^{\mathrm{T}} \in K^n : \; p(x_1, \ldots, x_n) = 0, \; p \in I\}$.

In other words, algebraic sets are characterized by ideals.

Definition 2.40. Let V be an algebraic set in K^n. The *vanishing ideal* of V is the ideal I of $K[X]$ consisting of all the polynomials vanishing on V, i.e.,

$$I := \{p \in K[X] : \; p(x_1, \ldots, x_n) = 0 \; (\forall [x_1, \ldots, x_n]^{\mathrm{T}} \in V)\}.$$

Proposition 2.41. *Let V be an algebraic set in K^n and I be its vanishing ideal. Then, for any polynomials $p, q \in K[X]$, the following statements are equivalent.*

(i) $p = q \mod I$.

(ii) $p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n)$ *for all* $[x_1, \ldots, x_n]^{\mathrm{T}} \in V$.

In other words, the set $K[X]/I$ can be identified with the set of all polynomial functions on V.

Chapter 3

Stability Analysis by Using Lie Algebras

3.1 Introduction

Lyapunov's stability theory forms the basis of the stability analysis and stabilization of nonlinear systems. Lyapunov's direct method [12, 13, 49] is used to check the stability of general nonlinear systems while Lyapunov's indirect method [12, 13], which deals with linearized systems, is used to check local stability. However, the direct method is heuristic with respect to finding a Lyapunov function, and the indirect method cannot check global stability. That is, there is no systematic method of checking the stability of general nonlinear systems.

In this chapter, we focus on discrete-time nonlinear systems with state-dependent coefficient matrices. First, a necessary and sufficient condition for the upper triangularizability of the systems is obtained by using the theory of Lie algebras [44, 50, 51]. Then, sufficient conditions for local and global asymptotic stabilities are derived for upper-triangularizable systems. In many cases, the stability conditions can be checked by a finite number of arithmetic operations of matrices and the evaluation of the asymptotic behavior of functions rather than heuristic procedures such as Lyapunov's direct method.

The idea discussed in this chapter was first proposed in [52] for continuous-time systems, where the upper triangularizability was checked by using the theory of Lie algebras and the local stability of the systems was discussed. Although our results could be seen as discrete-time analogues of the results in [52], we include new results: we give not only a sufficient condition but also a necessary condition for the upper triangularizability of the systems as well as a sufficient condition for global asymptotic stability, which are not given in [52]. Moreover, the stability conditions given in this chapter do not require the transformation matrix for triangularization explicitly, whereas it is necessary in [52] in which the required transformation matrix is not given.

The present local and global stability conditions do not require that a system have differentiability, although they do require a system to have continuity at the origin. In other words, a system is allowed to have nondifferentiability at any point in the state space and to have discontinuity at any point except the origin. Hence, the conditions are applicable to systems in which Lyapunov's indirect method does not work. In fact, under the assumption of the upper triangularizability of systems, the local stability condition is a generalization of Lyapunov's indirect method. In contrast, the differentiability is assumed in [52]. To the best of the author's knowledge, the converse Lyapunov theorem for discontinuous discrete-time systems has not been presented. At least, a certain discontinuous discrete-time system having a globally asymptotically stable equilibrium point at the origin does not admit any smooth Lyapunov function (see [53]). Hence, the present stability conditions might be able to ensure the local or global stability of a system that has no Lyapunov function. Even if the converse Lyapunov theorem held, it would be very difficult to find Lyapunov functions.

Systems with state-dependent coefficient matrices can also be seen as generalizations of switched linear systems. The stability of such systems has been thoroughly investigated in [54] and [55] by Lie algebraic analysis. However, when we regard the nonlinear systems as switched linear systems, the switching signals of the systems may have non-compact images, whereas [54] and [55] assume compactness. Therefore, the results in [54] and [55] cannot be applied to the present problem.

This chapter is composed as follows. In Section 3.2, we describe the system examined in this chapter. In Section 3.3, we give a necessary and sufficient condition for systems to be upper triangularized. In Sections 3.4 and 3.5, we derive sufficient conditions for the local and global asymptotic stabilities for upper-triangularizable systems. Section 3.6 presents several numerical examples. This chapter includes the entire content of journal article 1 in the list of publications, which was written by the author. The copyright of the article is owned by IEEE[1].

3.2 System description

This section describes the system treated in this chapter. Consider a discrete time system of the form

$$x(k+1) = A(x(k))x(k), \quad x(0) = x_0 \in \mathbf{C}^n \tag{3.1}$$

with the matrix-valued function $A : \mathbf{C}^n \to \mathbf{C}^{n\times n}$. For example, if a nonlinear system of the form $x(k+1) = f(x(k))$ is smooth at the origin, the system can be described by (3.1) in a neighborhood of the origin. Another example is the closed-loop system of a linear system of the form $x(k+1) = \hat{A}x(k) + \hat{B}u$ under a control input $u = K(x(k))x(k)$ with a state-dependent feedback gain $K(\cdot)$. Note that the continuity of system (3.1) is not required herein.

In the next section, we will discuss the possibility of $A(x)$ to be transformed into an upper-triangular form by a nonsingular linear transformation $x = Py$ using an appropriate constant matrix P. Accordingly, we will deal with an upper-triangular system obtained by such a transformation, i.e., a system of the form

$$y(k+1) = T(y(k))y(k), \quad y(0) = P^{-1}x_0 \in \mathbf{C}^n, \tag{3.2}$$

$$T(y) = \begin{bmatrix} \lambda_1(y) & a_{12}(y) & \cdots & \cdots & a_{1n}(y) \\ 0 & \lambda_2(y) & a_{23}(y) & \cdots & a_{2n}(y) \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & a_{(n-1)n}(y) \\ 0 & & \cdots & & \lambda_n(y) \end{bmatrix},$$

$$\lambda_i : \mathbf{C}^n \to \mathbf{C} \quad (i = 1, 2, \ldots, n),$$
$$a_{ij} : \mathbf{C}^n \to \mathbf{C} \quad (1 \le i < j \le n).$$

It is necessary to consider the entries of $T(y)$ in the complex space $\mathbf{C}$ because the transformation is not always possible over $\mathbf{R}$. If the nonsingular transformation is possible over $\mathbf{R}$, all the following discussions are still valid even if $\mathbf{C}$ is replaced with $\mathbf{R}$.

Note that the solution of system (3.2) is given by

$$y(k) = \left(\prod_{i=0}^{k-1} T(y(i)) \right) y(0), \tag{3.3}$$

where the product $\prod_{i=0}^{k-1} T(y(i))$ is defined by

$$\prod_{i=0}^{k-1} T(y(i)) := T(y(k-1))T(y(k-2)) \cdots T(y(1))T(y(0)).$$

Remark 3.1. The representation of a system in form (3.1) is not unique. Note that all the following discussions may be affected by the choice of $A(x)$. For example, a certain system has two different descriptions such that one is triangularizable while the other is not [52].

3.3 Upper triangularizability

In this section, we discuss the condition for system (3.1) to be transformed into upper-triangular system (3.2). First, we recall several basic concepts in Lie algebras. Next, we give the condition by using Lie algebras. An algorithm for checking the condition is also included.

3.3.1 Lie algebras

In this subsection, we recall several basic concepts in Lie algebras. The concepts in Lie algebras are detailed in [44, 50, 51]. See also Chapter 2 for understanding basic concepts in commutative algebras.

Let K be an arbitrary field. Lie algebras and related concepts are defined as follows.

Definition 3.2. A *Lie algebra* over K is a vector space $\mathfrak{g}$ over K in which a law of composition

$$\begin{array}{ccc} (X,Y) & \longmapsto & [X,Y] \\ \Cap & & \Cap \\ \mathfrak{g}\times\mathfrak{g} & \longrightarrow & \mathfrak{g} \end{array}$$

is defined and satisfies the following properties:

(i) $[\cdot,\cdot]$ is K-bilinear,

(ii) $[X,Y]+[Y,X]=0$ for all $X,Y\in\mathfrak{g}$,

(iii) $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$ for all $X,Y,Z\in\mathfrak{g}$.

The image $[X,Y]$ of elements $X,Y\in\mathfrak{g}$ is called the *Lie bracket* of X and Y.

Definition 3.3. A *subalgebra* of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h}$ of $\mathfrak{g}$ satisfying the property that

$$X,Y\in\mathfrak{h}\Rightarrow[X,Y]\in\mathfrak{h}.$$

Definition 3.4. Let $\mathfrak{g}$ be a Lie algebra and S be a subset of $\mathfrak{g}$. The *subalgebra generated by* S is the smallest subalgebra of $\mathfrak{g}$ containing S w.r.t. set containment.

Definition 3.5. An *ideal* of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{a}$ of $\mathfrak{g}$ satisfying the property that

$$X \in \mathfrak{g}, Y \in \mathfrak{a} \Rightarrow [X, Y] \in \mathfrak{a}.$$

Let $\mathfrak{g}$ be a Lie algebra. For any subspaces $\mathbb{A}$ and $\mathbb{B}$ of $\mathfrak{g}$, let $[\mathbb{A}, \mathbb{B}]$ denote the subspace generated by the set $\{[A, B] \in \mathfrak{g} : \ A \in \mathbb{A}, B \in \mathbb{B}\}$. Moreover, define the symbol $\mathscr{D}^k$ as follows:

$$\mathscr{D}^0\mathfrak{g} := \mathfrak{g}, \quad \mathscr{D}^k\mathfrak{g} := [\mathscr{D}^{k-1}\mathfrak{g}, \mathscr{D}^{k-1}\mathfrak{g}].$$

Definition 3.6. Let $\mathfrak{g}$ be a Lie algebra. The *derived series* of $\mathfrak{g}$ is the series defined by

$$\mathfrak{g} = \mathscr{D}^0\mathfrak{g} \supset \mathscr{D}^1\mathfrak{g} \supset \cdots \supset \mathscr{D}^k\mathfrak{g} \supset \cdots .$$

The Lie algebra $\mathfrak{g}$ is said to be *solvable* if $\mathscr{D}^k\mathfrak{g} = \{0\}$ for some $k \in \mathbf{N}$. The smallest number k satisfying this condition is called the *length* of the derived series of $\mathfrak{g}$.

Proposition 3.7. *If a Lie algebra is solvable, then so are all its subalgebras.*

Definition 3.8. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie algebras over K. A *Lie algebra homomorphism* from $\mathfrak{g}$ to $\mathfrak{g}'$ is a K-linear map $f : \mathfrak{g} \to \mathfrak{g}'$ satisfying the property that

$$f([X, Y]) = [f(X), f(Y)]$$

for all $X, Y \in \mathfrak{g}$. A bijective Lie-algebra homomorphism is called an *Lie algebra isomorphism.*

Definition 3.9. Two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ are said to be *isomorphic* if a Lie algebra isomorphism $\mathfrak{g} \to \mathfrak{g}'$ exists, a fact denoted by $\mathfrak{g} \simeq \mathfrak{g}'$.

Example 3.10. The set of all $n \times n$ matrices with entries in a field K, denoted by $K^{n\times n}$, is a Lie algebra with the Lie bracket $[X, Y] := XY - YX$ $(X, Y \in K^{n\times n})$.

Example 3.11. Let V be a K-vector space and $\mathrm{End}_K(V)$ be the set of all K-linear transformations on V. The set $\mathrm{End}_K(V)$ is a (noncommutative) ring with its natural addition and the multiplication defined by composition of maps. In fact, $\mathrm{End}_K(V)$ is a Lie algebra with the Lie bracket $[X, Y] = XY - YX$. This Lie algebra is called the *general linear Lie algebra* over V, and denoted by $\mathfrak{gl}(V)$. In particular, the Lie algebra $K^{n\times n}$ is naturally isomorphic to $\mathfrak{gl}(K^n)$ as a Lie algebra.

3.3.2 Upper triangularizability condition

This subsection gives a condition for system (3.1) to be transformed into upper-triangular system (3.2). First, we show the following proposition that the simultaneous upper-triangularizability of the set $\mathscr{A} = \{A(x) \in \mathbf{C}^{n\times n} : x \in \mathbf{C}^n\}$ is equivalent to the upper triangularizability of the Lie algebra generated by $\mathscr{A}$. Let $\mathcal{L}(\mathscr{A})$ be the Lie algebra generated by $\mathscr{A}$ over $\mathbf{C}$. Note that the dimension of $\mathcal{L}(\mathscr{A})$ is at most n^2 because $\mathcal{L}(\mathscr{A}) \subset \mathbf{C}^{n\times n}$.

Proposition 3.12. *The following statements are equivalent.*

(i) There exists a nonsingular constant matrix $P \in \mathbf{C}^{n\times n}$ that makes $P^{-1}A(x)P$ upper triangular for all $x \in \mathbf{C}^n$. That is, the set $\mathscr{A}$ is simultaneously upper-triangularizable by P.

(ii) There exists a nonsingular constant matrix $P \in \mathbf{C}^{n\times n}$ that makes $P^{-1}XP$ upper triangular for any element $X \in \mathcal{L}(\mathscr{A})$.

We omit the proof here because it can easily be shown from the definition of the Lie algebra. Note that statement (i) in the above proposition is simply the upper triangularizability of system (3.1).

Remark 3.13. In Proposition 3.12, a matrix P can be chosen that satisfies both (i) and (ii).

Proposition 3.12 and Lie's theorem [50] result in the following theorem.

Theorem 3.14. *System* (3.1) *is upper triangularizable if and only if $\mathcal{L}(\mathscr{A})$ is solvable.*

Proof. (if): Suppose $\mathcal{L}(\mathscr{A})$ is solvable. Then, by Lie's theorem [50], there exists a nonsingular constant matrix $P \in \mathbf{C}^{n\times n}$ making $P^{-1}XP$ upper triangular for all $X \in \mathcal{L}(\mathscr{A})$. By Proposition 3.12, $\mathscr{A}$ is simultaneously upper-triangularizable by P.
(only if): Suppose $\mathscr{A}$ is simultaneously upper-triangularizable by a nonsingular matrix $P \in \mathbf{C}^{n\times n}$. By Proposition 3.12, $P^{-1}XP$ is upper triangular for any $X \in \mathcal{L}(\mathscr{A})$. Let $\mathfrak{t} \subset \mathbf{C}^{n\times n}$ be the Lie algebra consisting of all the upper-triangular matrices in $\mathbf{C}^{n\times n}$. It is well known that $\mathfrak{t}$ is a solvable Lie algebra [50]. Since the map $\sigma : \mathcal{L}(\mathscr{A}) \to \mathfrak{t}$ defined by $X \mapsto P^{-1}XP$ is an injective homomorphism of Lie algebras, $\operatorname{Im}\sigma$ is isomorphic to $\mathcal{L}(\mathscr{A})$. It is well known that any subalgebra of a solvable Lie algebra is again solvable [50]. Since $\operatorname{Im}\sigma$ is a subalgebra of the solvable Lie algebra $\mathfrak{t}$, $\operatorname{Im}\sigma$ is solvable. Thus, $\mathcal{L}(\mathscr{A})$ is solvable. □

Remark 3.15. Let $\mathfrak{gl}(V)$ be the general linear Lie algebra of a finite-dimensional $\mathbf{C}$-vector space V, and let L be a subalgebra of $\mathfrak{gl}(V)$ satisfying $L \neq \{0\}$. Then, there are two types of statement of Lie's theorem in the theory of Lie algebras: the first one claims that if L is solvable, then V contains a common eigenvector for all the endomorphisms in L. The second one claims that if L is solvable, then the representation matrices of L relative to a suitable basis of V are upper triangular. In fact, the second statement is a corollary of the first statement [50]. In the proof of Theorem 3.14, we used Lie's theorem in the sense of the second statement. This form of statement can be found in [50, p.16, Corollary A].

Remark 3.16. The $\mathbf{R}$-scalar version of Theorem 3.14 can also be derived: consider the system $x(k+1) = A(x(k))x(k)$ with the real-matrix valued function $A : \mathbf{R}^n \to \mathbf{R}^{n\times n}$, and assume that all the eigenvalues of $A(x)$ are real for all $x \in \mathbf{R}^n$. Let $\mathcal{L}(\mathscr{A}_{\mathbf{R}})_{\mathbf{R}}$ be the real Lie algebra generated by $\mathscr{A}_{\mathbf{R}} = \{A(x) \in \mathbf{R}^{n\times n} : \ x \in \mathbf{R}^n\}$ over $\mathbf{R}$. Then, the system is upper triangularizable by the real matrix $P \in \mathbf{R}^{n\times n}$ if and only if $\mathcal{L}(\mathscr{A}_{\mathbf{R}})_{\mathbf{R}}$ is solvable.

Remark 3.17. The "(only if)" part of Theorem 3.14 was not given in [52], whereas the "(if)" part has been given in the literature.

As shown above, to check the upper triangularizability of a given system, it suffices to check the solvability of $\mathcal{L}(\mathscr{A})$ of the system. A procedure for checking the solvability of $\mathcal{L}(\mathscr{A})$ is included in the next subsection.

Remark 3.18. This subsection is not concerned with whether the system is of continuous-time or discrete-time. Therefore, the above discussion can also be applied to continuous-time systems.

3.3.3 An algorithm for checking the solvability of $\mathcal{L}(\mathscr{A})$

This subsection gives an algorithm for checking the condition for a system to be upper triangularized. According to the previous subsection, the condition is the solvability of $\mathcal{L}(\mathscr{A})$. Let $V(\mathscr{A})$ denote the vector space generated by $\mathscr{A}$. Let m be the dimension of $\mathcal{L}(\mathscr{A})$ and $\sharp$ denote the cardinality of a set.

First, we obtain a basis of $V(\mathscr{A})$. This can be achieved by the direct method shown in Example 3.36 in Section 3.6 or by the numerical method: let $\mathrm{vec} : \mathbf{C}^{n\times n} \xrightarrow{\sim} \mathbf{C}^{n^2}$ be an isomorphism of vector spaces. Since a basis of $V(\mathscr{A})$ is a maximal linearly independent subset of $\mathscr{A}$, we can find a basis of $V(\mathscr{A})$ by solving the following optimization problem.

$$\begin{array}{ll} \text{maximize} & \operatorname{rank}\left[\mathrm{vec}(A(x^1)) \;\; \mathrm{vec}(A(x^2)) \;\; \cdots \;\; \mathrm{vec}(A(x^{n^2}))\right] \\ \text{such that} & x^i \in \mathbf{C}^n \quad (i = 1, \dots, n^2) \end{array}$$

Second, we calculate a basis of $\mathcal{L}(\mathscr{A})$ as follows. Let L_0 be a basis of $V(\mathscr{A})$. Put $C_0 := \{[X_i, X_j] : X_i, X_j \in L_0\}$ and let L_1 be a maximal linearly independent subset of $L_0 \cup C_0$. Similarly, put $C_1 := \{[X_i, X_j] : X_i, X_j \in L_1\}$ and let L_2 be a maximal linearly independent subset of $L_1 \cup C_1$. Repeating this process, we obtain a maximal linearly independent subset $L_k \subset L_{k-1} \cup C_{k-1}$ for each k. Note that $L_k \subset \mathcal{L}(\mathscr{A})$ and $\sharp L_k \leq \sharp L_{k+1}$ for all k. Since the dimension of $\mathcal{L}(\mathscr{A})$ is finite, this process stops at some p, i.e., $\sharp L_p = \sharp L_{p+1} = \cdots$ for some p. Then, L_p is a basis of $\mathcal{L}(\mathscr{A})$. Indeed, $L_0, \ldots, L_p$ satisfy

$$\mathscr{A} \subset V(\mathscr{A}) \subset \operatorname{span} L_0 \subset \cdots \subset \operatorname{span} L_{p-1} \subset \operatorname{span} L_p \subset \mathcal{L}(\mathscr{A}),$$

and the equality $\sharp L_p = \sharp L_{p+1}$ implies $C_p \subset \operatorname{span} L_p$. That is, $\operatorname{span} L_p = \mathcal{L}(\mathscr{A})$.

Finally, we check the solvability of $\mathcal{L}(\mathscr{A})$. Let B_0 be a basis of $\mathcal{L}(\mathscr{A})$. Then, $\mathscr{D}\mathcal{L}(\mathscr{A})$ is generated by the set $\{[X_i, X_j] : X_i, X_j \in B_0, i < j\}$ as a **C**-vector space. Indeed, any element of $\mathscr{D}\mathcal{L}(\mathscr{A})$ has the form $\sum_{\text{finite sum}} a_{ij}[Y_i, Y_j]$ for some $a_{ij} \in \mathbf{C}$, $Y_i, Y_j \in \mathcal{L}(\mathscr{A})$, and elements Y_i and Y_j are linear combinations of B_0; therefore, we have $\sum_{\text{finite sum}} a_{ij}[Y_i, Y_j] = \sum_{1 \leq i < j \leq m} b_{ij}[X_i, X_j]$ for some $b_{ij} \in \mathbf{C}$ and $X_i, X_j \in B_0$ by applying the bilinearity and anticommutativity of the Lie bracket. Thus, if we choose a maximal linearly independent subset B_1 of $\{[X_i, X_j] : X_i, X_j \in B_0, i < j\}$, then B_1 is a basis of $\mathscr{D}\mathcal{L}(\mathscr{A})$. Similarly, $\mathscr{D}^2\mathcal{L}(\mathscr{A})$ is generated by $\{[X_i, X_j] : X_i, X_j \in B_1, i < j\}$ and we can choose a basis B_2 from the set. Repeating this process, we obtain a basis B_k for $\mathscr{D}^k\mathcal{L}(\mathscr{A})$. When $\mathcal{L}(\mathscr{A})$ is solvable, the length of the derived series of $\mathcal{L}(\mathscr{A})$ does not exceed the dimension m. Indeed, since $\mathcal{L}(\mathscr{A})$ is a vector space, the derived series satisfies $\dim \mathscr{D}^k\mathcal{L}(\mathscr{A}) \geq \dim \mathscr{D}^{k+1}\mathcal{L}(\mathscr{A})$, and the equality $\dim \mathscr{D}^k\mathcal{L}(\mathscr{A}) = \dim \mathscr{D}^{k+1}\mathcal{L}(\mathscr{A})$ implies $\mathscr{D}^k\mathcal{L}(\mathscr{A}) = \mathscr{D}^{k+1}\mathcal{L}(\mathscr{A}) = \cdots$. Hence, to check the solvability of $\mathcal{L}(\mathscr{A})$, it suffices to check if $\mathscr{D}^m\mathcal{L}(\mathscr{A}) = \{0\}$. Since the equality $\mathscr{D}^k\mathcal{L}(\mathscr{A}) = \{0\}$ is equivalent to $B_k = \emptyset$, we can check the solvability of $\mathcal{L}(\mathscr{A})$ by checking if $B_m = \emptyset$.

3.4 Local asymptotic stability

This section gives a local asymptotic stability condition for upper-triangularizable systems. Before describing the condition, let us show several properties of upper-triangular matrices with zero diagonal components, i.e., nilpotent upper-triangular matrices. We also decompose the coefficient matrix $T(y)$ of system (3.2) into suitable form. Moreover, we give a lemma concerning the convergence of a sequence. These are needed to derive the stability condition.

Proposition 3.19. *Nilpotent upper-triangular matrices have the following properties:*

(i) The matrix product of n nilpotent upper-triangular $n \times n$ matrices is a zero matrix,

(ii) The matrix product of a diagonal matrix and a nilpotent upper-triangular matrix is again nilpotent upper-triangular.

We omit the proof because it is obvious.

Remark 3.20. Note that property (i) in the above proposition is different from the nilpotent property of a single matrix.

Next, we decompose the coefficient matrix $T(y)$ of system (3.2) into $T(y) = \Lambda(y) + N(y)$ with $\Lambda(y) = \mathrm{diag}(\lambda_1(y), \lambda_2(y), \dots, \lambda_n(y))$ and $N(y) = T(y) - \Lambda(y)$. Note that $N(y)$ is a nilpotent upper-triangular matrix. Let $\Lambda_i := \Lambda(y(i))$ and $N_i := N(y(i))$ for simplicity. Then, solution (3.3) is written as

$$y(k) = \left(\prod_{i=0}^{k-1} (\Lambda_i + N_i) \right) y(0).$$

We define the coefficient matrix on the right-hand side of the above equation as

$$\Phi_k := \prod_{i=0}^{k-1} (\Lambda_i + N_i). \tag{3.4}$$

The following lemma is used to derive the stability condition. Let $\binom{k}{l} = \frac{k!}{l!(k-l)!}$ be a binomial coefficient.

Lemma 3.21. *For any $\lambda \in [0, 1)$ and any $l \in \mathbf{N}$,*

$$\lim_{k \to \infty} \binom{k}{l} \lambda^k = 0.$$

In particular, for any $\nu \in \mathbf{R}$,

$$\lim_{k \to \infty} \sum_{i=0}^{n-1} \binom{k}{i} \lambda^{k-i} \nu^i = 0.$$

Proof. It is obvious when $\lambda = 0$. Hence, let $\lambda \neq 0$ and $k > l + 1$ without loss of generality. Let $a := 1/\lambda$. Then, we have $a = 1 + h$ for some $h > 0$ because

$a > 1$, and we have

$$\begin{aligned}\binom{k}{l}\lambda^k &= \frac{\binom{k}{l}}{a^k} = \frac{\binom{k}{l}}{(1+h)^k} = \frac{\binom{k}{l}}{\sum_{i=0}^{k}\binom{k}{i}h^i} \\ &< \frac{\binom{k}{l}}{\binom{k}{l+1}h^{l+1}} \\ &= \frac{1}{h^{l+1}}\frac{k!}{l!(k-l)!}\frac{(l+1)!(k-(l+1))!}{k!} = \frac{1}{h^{l+1}}\frac{l+1}{k-l} \to 0 \quad (k\to\infty).\end{aligned}$$

The latter statement follows from

$$\sum_{i=0}^{n-1}\binom{k}{i}\lambda^{k-i}\nu^i = \sum_{i=0}^{n-1}\left(\binom{k}{i}\lambda^k\right)\lambda^{-i}\nu^i.$$

□

Now, we derive a sufficient condition for local stability. The main theorem given later needs the following lemma.

Lemma 3.22. *Consider system* (3.2)*. The origin is locally asymptotically stable if there exists* $r > 0$ *such that the following conditions are satisfied:*

(i) $\sup_{\|y\|<r} |\lambda_i(y)| < 1 \quad (i = 1, \ldots, n),$

(ii) $\sup_{\|y\|<r} \|N(y)\| < \infty,$

where $\|\cdot\|$ *is the Euclidean norm or an operator norm induced from it.*

Proof. By Proposition 3.19, in the case of $k \geq n$, the coefficient matrix Φ_k can be expanded as follows:

$$
\begin{aligned}
\Phi_k = \quad & \Lambda_{k-1}\Lambda_{k-2}\cdots\Lambda_2\Lambda_1\Lambda_0 \\
\binom{k}{1} & \begin{cases} +\Lambda_{k-1}\Lambda_{k-2}\cdots\Lambda_2\Lambda_1 N_0 \\ +\Lambda_{k-1}\Lambda_{k-2}\cdots\Lambda_2 N_1\Lambda_0 \\ \qquad\vdots \\ +N_{k-1}\Lambda_{k-2}\cdots\Lambda_2\Lambda_1\Lambda_0 \end{cases} \\
\binom{k}{2} & \begin{cases} +\Lambda_{k-1}\Lambda_{k-2}\cdots\Lambda_2 N_1 N_0 \\ +\Lambda_{k-1}\Lambda_{k-2}\cdots N_2\Lambda_1 N_0 \\ \qquad\vdots \\ +N_{k-1}N_{k-2}\cdots\Lambda_2\Lambda_1\Lambda_0 \end{cases} \\
& \quad\vdots \\
\binom{k}{n-1} & \begin{cases} +\Lambda_{k-1}\Lambda_{k-2}\cdots\Lambda_{n-1}N_{n-2}\cdots N_1 N_0 \\ \qquad\vdots \\ +N_{k-1}N_{k-2}\cdots N_{k-n+1}\Lambda_{k-n}\cdots\Lambda_1\Lambda_0, \end{cases}
\end{aligned}
\tag{3.5}
$$

where $\binom{k}{l}$ represents the number of terms in the above.

Condition (i) implies $\sup_{\|y\|<r}\|\Lambda(y)\| < 1$ because

$$
\|\Lambda(y)\| = \sqrt{\lambda_{\max}(\Lambda(y)^*\Lambda(y))} = \max_{1\le i\le n}|\lambda_i(y)|
$$

for any $y \in \mathbf{C}^n$. Put

$$
\lambda = \sup_{\|y\|<r}\|\Lambda(y)\| < 1 \quad \text{and} \quad \nu = \sup_{\|y\|<r}\|N(y)\|. \tag{3.6}
$$

Assume that a solution from some initial state satisfies $\|y(i)\| < r$ $(0 \le i \le k)$ for some $k \le n-2$. Then, from (3.4) and (3.6), we have

$$
\begin{aligned}
\|\Phi_{k+1}\| = \left\|\prod_{i=0}^{k}(\Lambda_i + N_i)\right\| &\le \prod_{i=0}^{k}(\|\Lambda_i\| + \|N_i\|) \le (\lambda+\nu)^{k+1} \\
&\le \max\left\{1, (\lambda+\nu)^{n-1}\right\}.
\end{aligned}
$$

Moreover, assuming that the solution satisfies $\|y(i)\| < r$ $(0 \le i \le k)$ for some $k \ge n-1$, we have

$$
\|\Phi_{k+1}\| \le \sum_{i=0}^{n-1}\binom{k+1}{i}\lambda^{k+1-i}\nu^i. \tag{3.7}
$$

Indeed, $\|\Lambda_i\| \leq \lambda$ and $\|N_i\| \leq \nu$ $(0 \leq i \leq k)$ hold by conditions (i) and (ii); thus, we have

$$\begin{aligned} \|\Phi_{k+1}\| &\leq \lambda^{k+1} + \binom{k+1}{1}\lambda^k \nu + \cdots + \binom{k+1}{n-1}\lambda^{(k+1)-(n-1)}\nu^{n-1} \\ &= \sum_{i=0}^{n-1} \binom{k+1}{i}\lambda^{k+1-i}\nu^i, \end{aligned}$$

by applying the triangular inequality and submultiplicativity of the operator norm to (3.5).

Now, put

$$L = \sup_{k \geq n} \sum_{i=0}^{n-1} \binom{k}{i}\lambda^{k-i}\nu^i \quad \text{and} \quad M = \max\left\{1, (\lambda+\nu)^{n-1}, L\right\}.$$

The constant L is well defined because of Lemma 3.21. According to the above discussion, if $\|y(i)\| < r$ $(0 \leq i \leq k)$ for some $k \geq 0$, then $\|\Phi_{k+1}\| \leq M$. If we choose the initial state to be $M\|y_0\| < r$, we find that the solution satisfies

$$\|y(k)\| \leq M\|y_0\| < r \quad (\forall k \geq 0) \tag{3.8}$$

by induction. Moreover, $\sup_{\|y\|<\varepsilon} \|\Lambda(y)\| \leq \lambda$ and $\sup_{\|y\|<\varepsilon} \|N(y)\| \leq \nu$ hold for any $\varepsilon > 0$ such that $\varepsilon < r$; therefore, if we choose the initial condition to be $M\|y_0\| < \varepsilon$, then $\|y(k)\| \leq M\|y_0\| < \varepsilon$ $(\forall k \geq 0)$ holds similarly to the above discussion. Hence, the origin is stable.

Equation (3.7) holds for any $k \geq n-1$ owing to (3.8). Then, by Lemma 3.21, we have

$$\|\Phi_k\| \leq \sum_{i=0}^{n-1} \binom{k}{i}\lambda^{k-i}\nu^i \to 0 \quad (k \to \infty).$$

Therefore, the statement follows from $\|y(k)\| = \|\Phi_k y(0)\| \leq \|\Phi_k\| \|y(0)\|$. □

The proof of Lemma 3.22 also shows that the ball $B = \{y \in \mathbf{C}^n : \|y\| < r/M\}$ is contained in the domain of attraction. We state this fact as a proposition below.

Proposition 3.23. *Under the conditions in Lemma 3.22, the ball* $B = \{y \in \mathbf{C}^n : \|y\| < r/M\}$ *is contained in the domain of attraction. Here,*

$\|\cdot\|$ *is the Euclidean norm or an operator norm induced from it, and*

$$M = \max\left\{1, (\lambda+\nu)^{n-1}, L\right\}, \quad L = \sup_{k\geq n} \sum_{i=0}^{n-1} \binom{k}{i} \lambda^{k-i}\nu^{i},$$
$$\lambda = \sup_{\|y\|<r} \|\Lambda(y)\|, \quad \nu = \sup_{\|y\|<r} \|N(y)\|.$$

Now we are in a position to give a sufficient condition for the local asymptotic stability of system (3.1). The following theorem follows from Lemma 3.22.

Theorem 3.24. *Consider system* (3.1) *and assume it is upper triangularizable in the sense discussed in Section 3.2. Let $\rho(x)$ be the spectral radius of $A(x)$. The origin is locally asymptotically stable if there exists $s > 0$ such that the following conditions are satisfied:*

(i) $\sup_{\|x\|<s} \rho(x) < 1$,

(ii) $\sup_{\|x\|<s} \|A(x)\| < \infty$,

where $\|\cdot\|$ is the Euclidean norm or an operator norm induced from it.

Proof. System (3.1) can be transformed into (3.2) by setting $x = Py$. Then

$$P^{-1}A(x)P = T(y) = \Lambda(y) + N(y), \quad \rho(x) = \max_{1\leq i\leq n} |\lambda_i(y)|.$$

Since $\|P^{-1}\|^{-1}\|y\| \leq \|x\| \leq \|P\|\|y\|$ and

$$\left(\|P^{-1}\|\|P\|\right)^{-1}\|A(x)\| - \|\Lambda(y)\|$$
$$\leq \|N(y)\| \leq \left(\|P^{-1}\|\|P\|\right)\|A(x)\| + \|\Lambda(y)\|,$$

we have

$$\exists s>0, \sup_{\|x\|<s} \rho(x) < 1 \iff \exists r > 0, \sup_{\|y\|<r} \max_{1\leq i\leq n} |\lambda_i(y)| < 1,$$
$$\exists s>0, \sup_{\|x\|<s} \|A(x)\| < \infty \iff \exists r > 0, \sup_{\|y\|<r} \|N(y)\| < \infty.$$

Therefore, conditions (i) and (ii) are equivalent to the conditions in Lemma 3.22. □

Remark 3.25. Condition (ii) in Theorem 3.24 cannot be removed. See Example 3.37 in Section 3.6.

Remark 3.26. Condition (ii) in Theorem 3.24 implies that $f(x) := A(x)x$ is continuous at the origin, for the following reason: by condition (ii), there exists a constant $b > 0$ such that $\|A(x)\| < b$ for all x satisfying $\|x\| < s$. Therefore, if $\|x\| < s$, we have $\|f(x)\| \leq \|A(x)\| \, \|x\| < b \, \|x\|$, so that f is continuous at the origin. However, the differentiability of f is not required. See Example 3.36 in Section 3.6.

Remark 3.27. Unlike Lyapunov's indirect method, Theorem 3.24 does not require the differentiability of $f(x) := A(x)x$ at the origin. Therefore, under the assumption of the upper triangularizability of systems, Theorem 3.24 is a generalization of Lyapunov's indirect method.

3.5 Global asymptotic stability

This subsection gives a global asymptotic stability condition for upper-triangularizable systems. As stated in the proof of Lemma 3.22, if we choose the initial state of an upper-triangular system as $\|y_0\| < r/M$, then $\|y(k)\| \leq M \, \|y_0\| < r \quad (\forall k \geq 0)$, and the ball $B = \{y \in \mathbf{C}^n : \ \|y\| < r/M\}$ is contained in the domain of attraction. Accordingly, let us consider the expansion of B. If B can be expanded to $\mathbf{C}^n$, the origin is globally asymptotically stable.

Lemma 3.28. *Consider system* (3.2). *The origin is globally asymptotically stable if the following conditions are satisfied:*

(i) $\sup_{y \in \mathbf{C}^n} |\lambda_i(y)| < 1 \quad (i = 1, \ldots, n)$,

(ii) $\displaystyle \lim_{r \to \infty} \frac{\left(\sup_{\|y\|<r} \|N(y)\|\right)^{n-1}}{r} = 0,$

where $\|\cdot\|$ *is the Euclidean norm or an operator norm induced from it.*

Proof. First, we prove two properties of global asymptotic stability excluding the boundedness of the solutions, namely, the local stability of the origin and the attractivity of the solutions.

If we take an arbitrary $r > 0$, the conditions in Lemma 3.22 are satisfied by conditions (i) and (ii); thus, the origin is locally asymptotically stable. Hence, the result of Proposition 3.23 follows. Put

$$\lambda(r) = \sup_{\|y\|<r} \|\Lambda(y)\| < 1 \quad \text{and} \quad \nu(r) = \sup_{\|y\|<r} \|N(y)\| . \tag{3.9}$$

These functions are well defined for any $r > 0$ because of the above conditions (i) and (ii). Note that condition (ii) can be described as $\lim_{r\to\infty} \left(\nu(r)\right)^{n-1}/r =$

0. After replacing λ and ν in Proposition 3.23 with $\lambda(r)$ and $\nu(r)$ of (3.9), respectively, consider the ball $B(r)$ $(r > 0)$ defined by $B(r) = \{y \in \mathbf{C}^n : \|y\| < r/M(r)\}$ with $M(r) = \max\{1, (\lambda(r) + \nu(r))^{n-1}, L(r)\}$ and $L(r) = \sup_{k\geq n} \sum_{i=0}^{n-1} \binom{k}{i} (\lambda(r))^{k-i} (\nu(r))^i$. Since the conditions in Lemma 3.22 hold for arbitrary $r > 0$ because of conditions (i) and (ii), the ball $B(r)$ is contained in the domain of attraction for any $r > 0$. Thus, if

$$\lim_{r\to\infty} \frac{r}{M(r)} = \infty \quad \text{or} \quad \lim_{r\to\infty} \frac{M(r)}{r} = 0 \tag{3.10}$$

is satisfied, then $B(r)$ can be expanded to the whole space $\mathbf{C}^n$. The constant $\lambda_g = \sup_{y\in\mathbf{C}^n} \|\Lambda(y)\| < 1$ is defined because of conditions (i) and (ii). Concerning the factors defining $M(r)$, we have

$$\frac{(\lambda(r) + \nu(r))^{n-1}}{r} \leq \frac{(\lambda_g + \nu(r))^{n-1}}{r} \leq \sum_{i=0}^{n-1} \binom{n-1}{i} \lambda_g^{n-1-i} \frac{(\nu(r))^i}{r} \to 0 \quad (r \to \infty), \tag{3.11a}$$

$$\begin{aligned} \frac{L(r)}{r} &= \frac{1}{r}\left(\sup_{k\geq n} \sum_{i=0}^{n-1} \binom{k}{i} (\lambda(r))^{k-i} (\nu(r))^i\right) = \sup_{k\geq n} \sum_{i=0}^{n-1} \binom{k}{i} (\lambda(r))^{k-i} \frac{(\nu(r))^i}{r} \\ &\leq \sup_{k\geq n} \sum_{i=0}^{n-1} \binom{k}{i} \lambda_g^{k-i} \frac{(\nu(r))^i}{r} \\ &\leq \sum_{i=0}^{n-1} \left(\sup_{k\geq n} \binom{k}{i} \lambda_g^k\right) \lambda_g^{-i} \frac{(\nu(r))^i}{r} \to 0 \quad (r \to \infty). \end{aligned} \tag{3.11b}$$

Indeed, according to Lemma 3.21, the coefficient $\sup_{k\geq n} \binom{k}{i} \lambda_g^k$ in the bottom line of (3.11b) can be defined. If $\nu(r)$ is bounded, then (3.11) is obvious. If $\nu(r)$ is not bounded, $\nu(r)$ is monotonically increasing with respect to r (by definition), and thus $\nu(r) > 1$ $(r \gg 0)$; therefore, $(\nu(r))^i/r \leq (\nu(r))^{n-1}/r \to 0$ $(r \to \infty)$ for $i \leq n-1$, and hence (3.11) follows. Since

$$\frac{M(r)}{r} = \max\left\{\frac{1}{r}, \frac{(\lambda(r) + \nu(r))^{n-1}}{r}, \frac{L(r)}{r}\right\}$$

for any $r > 0$ because of the definition, (3.10) is satisfied by (3.11), and therefore $B(r)$ can be expanded to the whole space $\mathbf{C}^n$. That is, every solution starting from any initial state converges to the origin.

Lastly, we prove the boundedness of solutions. Since (3.10) holds, there exists $r > 0$ such that $M(r)\|y_0\| < r$ for any initial state $y_0 \in \mathbf{C}^n$. Then, as stated in the proof of Lemma 3.22, inequality (3.8) holds and thus the boundedness of solutions is guaranteed. □

The above lemma results in the following theorem, which gives a global asymptotic stability condition.

Theorem 3.29. *Consider system* (3.1) *and assume it is upper triangularizable in the sense discussed in Section 3.2. Let* $\rho(x)$ *be the spectral radius of* $A(x)$. *The origin is globally asymptotically stable if the following conditions are satisfied:*

(i) $\sup_{x\in\mathbf{C}^n} \rho(x) < 1$,

(ii) $\lim_{s\to\infty} \frac{\left(\sup_{\|x\|<s} \|A(x)\|\right)^{n-1}}{s} = 0$,

where $\|\cdot\|$ *is the Euclidean norm or an operator norm induced from it.*

Proof. Similarly to the proof of Theorem 3.24, we obtain

$$\sup_{x\in\mathbf{C}^n} \rho(x) < 1 \iff \sup_{y\in\mathbf{C}^n} \max_{1\le i\le n} |\lambda_i(y)| < 1,$$

$$\lim_{s\to\infty} \frac{\left(\sup_{\|x\|<s} \|A(x)\|\right)^{n-1}}{s} = 0 \iff \lim_{r\to\infty} \frac{\left(\sup_{\|y\|<r} \|N(y)\|\right)^{n-1}}{r} = 0$$

Therefore, conditions (i) and (ii) are equivalent to the conditions in Lemma 3.28. □

Remark 3.30. Let $A(x) = [a_{ij}(x)]_{1\le i,j\le n}$. The eigenvalues of $A(x)$, which are functions of x, are described by the linear combinations of the functions $a_{ij}(x)$ because the eigenvalues align in the diagonal part of the triangularized matrix $T(x) = P^{-1}A(x)P$. Therefore, spectral condition (i) in Theorems 3.24 and 3.29 can be checked by observing particular linear combinations of $a_{ij}(x)$. Moreover, if all the elements $a_{ij}(x)$ are negligible at infinity compared with the $(n-1)$th root of $\|x\|$, then condition (ii) in Theorem 3.29 is satisfied, provided that $\sup_{\|x\|<s} |a_{ij}(x)| < \infty$ for all $1 \le i,j \le n$ and $s > 0$. See Example 3.36 in Section 3.6 for an example.

Remark 3.31. Theorem 3.29 also guarantees the uniqueness of an equilibrium point. Moreover, Theorems 3.24 and 3.29 do not explicitly require the transformation matrix for triangularization, whereas the results in [52] require the transformation matrix, which is not explicitly given.

Remark 3.32. Condition (ii) in Theorem 3.29 cannot be removed. See Example 3.38 in Section 3.6. However, the condition is not necessary. See Example 3.39 in Section 3.6.

Remark 3.33. Theorem 3.29 does not require the compactness of $\mathscr{A}$. This means that Theorem 3.29 can be applied to the problem to which the results in [54] and [55] cannot be applied. Indeed, although the system in Example 3.38 satisfies the Lie-algebraic stability condition in [54] and [55], it is unstable because $\mathscr{A}$ is not compact. See Example 3.38 in Section 3.6.

Remark 3.34. To the best of the author's knowledge, the converse Lyapunov theorem for discontinuous discrete-time systems has not been presented.[2] At least, a certain discontinuous discrete-time system having the globally asymptotically stable equilibrium point at the origin does not admit any smooth Lyapunov function (see [53]). Hence, Theorems 3.24 and 3.29 might be able to ensure the local or global stability of a system that has no Lyapunov function. Even if the converse Lyapunov theorem held, it would be very difficult to find Lyapunov functions.

Theorem 3.29 naturally results in the following corollary applied in the case of a bounded coefficient matrix. To the best of the author's knowledge, this corollary has not yet been reported.

Corollary 3.35. *Consider system* (3.1) *and assume it is upper triangularizable. The origin is globally asymptotically stable if the following conditions are satisfied:*

(i) $\sup_{x \in \mathbf{C}^n} \rho(x) < 1$,

(ii) $\sup_{x \in \mathbf{C}^n} \|A(x)\| < \infty$.

3.6 Examples

This section shows four examples. The first example, included in the first subsection, is the application of the global asymptotic stability condition. The remaining examples, included in the second subsection, investigate the case where one of the conditions in the present theorems is not satisfied.

[2] The paper [56] mentions that the converse Lyapunov theorem for the global asymptotic stability of discontinuous discrete-time systems was proved in [57] in which the stability of sampled-data systems is investigated. However, it is unclear whether or not the result in [57] yields the converse theorem.

3.6.1 How to apply the global stability condition

The following example demonstrates how to apply Theorem 3.29. Let $V(\mathscr{A})$ denote the vector space generated by $\mathscr{A}$.

Example 3.36. Consider the following system:

$$x(k+1) = A(x(k))x(k), \quad x(0) = x_0 \in \mathbf{R}^3, \tag{3.12}$$
$$A(x) = \begin{bmatrix} 1 + \frac{1}{2}\cos x_2 & \sqrt[3]{|x_3|} + h(x_1) & \frac{1}{2}\sin x_2 - \frac{1}{2}\cos x_2 - 1 \\ 0 & \frac{1}{3}h(x_1) & 0 \\ 1 & \sqrt[3]{|x_3|} & \frac{1}{2}\sin x_2 - 1 \end{bmatrix},$$

with $x = [x_1\ x_2\ x_3]^{\mathrm{T}}$ and $h(x_1) = -\operatorname{sgn}(x_1)$. The characteristic polynomial of $A(x)$ is

$$\det\left(A(x) - p(x)I_n\right) = \left(p(x) - \frac{1}{2}\sin x_2\right)\left(p(x) - \frac{1}{2}\cos x_2\right)\left(p(x) - \frac{1}{3}h(x_1)\right). \tag{3.13}$$

Note that, since the system has nondifferentiability at the origin, the linearized system does not exist. Therefore, Lyapunov's indirect method is inapplicable to this system. Note also that, although Lyapunov's direct method can be employed for this system, it is very difficult to find a Lyapunov function. Since the eigenvalues of $A(x)$ are real values because of (3.13), we can consider all discussions to be over $\mathbf{R}$ (see Section 3.3). The matrix $A(x)$ can be decomposed into

$$A(x) = E_1 + \frac{1}{2}(\sin x_2)E_2 + \frac{1}{2}(\cos x_2)E_3 + \sqrt[3]{|x_3|}E_4 + h(x_1)E_5,$$

where $E_1, \ldots, E_5$ are constant matrices:

$$E_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix},\ E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ E_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$E_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ E_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

These matrices are linearly independent, and $V(\mathscr{A})$ is a subset of $\operatorname{span}\{E_i : 1 \le i \le 5\}$. In fact, $\{E_i\}_{1\le i\le 5}$ is a basis of $V(\mathscr{A})$ because it can be verified that $E_i \in V(\mathscr{A})$ for $1 \le i \le 5$ as follows: putting $A(x_1, x_2, x_3) := A(x)$ for $x = [x_1, x_2, x_3]^{\mathrm{T}}$, we have

$$\begin{aligned} E_1 &= \frac{1}{2}A(0,0,0) + \frac{1}{2}A(0,\pi,0) \in V(\mathscr{A}), \\ E_2 &= A(0,\pi/2,0) - A(0,-\pi/2,0) \in V(\mathscr{A}), \\ E_3 &= A(0,0,0) - A(0,\pi,0) \in V(\mathscr{A}), \\ E_4 &= A(0,0,1) - A(0,0,0) \in V(\mathscr{A}), \\ E_5 &= A(1,0,0) - A(0,0,0) \in V(\mathscr{A}). \end{aligned}$$

Since the solvability of $\mathcal{L}(\mathscr{A})$ is verified using the algorithm given in Subsection 3.3.3, the system is upper triangularizable. The spectrum condition $\sup_{x \in \mathbf{R}^n} \rho(x) < 1$ follows from (3.13). Let $\|\cdot\|_2$ be the Euclidean norm of vectors, and $\|\cdot\|_1$ be a matrix norm defined by $\|C\|_1 := \sum_{i,j} |c_{ij}|$ for any square matrix $C = [c_{ij}] \in \mathbf{R}^{n \times n}$. Then, we have

$$\begin{aligned} \|A(x)\|_1 &= \left|1 + \frac{1}{2}\cos x_2\right| + \left|\sqrt[3]{|x_3|} + h(x_1)\right| \\ &\quad + \left|\frac{1}{2}\sin x_2 - \frac{1}{2}\cos x_2 - 1\right| + \left|\frac{1}{3}h(x_1)\right| \\ &\quad + |1| + \left|\sqrt[3]{|x_3|}\right| + \left|\frac{1}{2}\sin x_2 - 1\right| \\ &\le 2\sqrt[3]{|x_3|} + 8 = 2\sqrt[6]{x_3^2} + 8 \le 2\sqrt[6]{\|x\|_2^2} + 8. \end{aligned}$$

Therefore, $\sup_{\|x\|_2 < s} \|A(x)\|_1 \le 2\sqrt[6]{s^2} + 8$ holds, and we have

$$\begin{aligned} \left(\sup_{\|x\|_2 < s} \|A(x)\|_1\right)^2 &\le 2^2 \left(\sqrt[6]{s^2}\right)^2 + (\text{lower order terms}) \\ &= o(s) \ (s \to \infty). \end{aligned}$$

Hence,

$$\lim_{s \to \infty} \frac{\left(\sup_{\|x\|_2 < s} \|A(x)\|_1\right)^2}{s} = 0.$$

The norm equivalence in $\mathbf{R}^{3 \times 3}$ implies

$$\lim_{s \to \infty} \frac{\left(\sup_{\|x\|_2 < s} \|A(x)\|_2\right)^2}{s} = 0,$$

where $\|\cdot\|_2$ is the operator norm induced from the Euclidean norm. Hence, the conditions of Theorem 3.29 are satisfied. Consequently, we can conclude that the system is globally asymptotically stable. See also Remark 3.30 in Section 3.5.

3.6.2 The case where one of the conditions in the theorems is not satisfied

The following examples investigate the case where one of the conditions in the present theorems is not satisfied.

Example 3.37. Consider a system having the following coefficient matrix:

$$A(x) = \begin{bmatrix} 1/2 & h(x) \\ 0 & 1/2 \end{bmatrix}, \quad h(x) := \begin{cases} 0 & \text{if } x_2 = 0, \\ 1/x_2 & \text{if } x_2 \neq 0. \end{cases}$$

This system does not satisfy condition (ii) in Theorem 3.24, whereas it is already triangular and satisfies condition (i). In fact, it can easily be checked that this system is unstable.

Example 3.38. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be the norms defined in Example 3.36. Consider a system with the following coefficient matrix:

$$A(x) = \begin{bmatrix} 1/2 & 0 & (x_1 + x_2x_3)^2 \\ -x_1 & 1/2 & -2x_1x_2 - x_2^2x_3 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

By taking

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

this system can be upper triangularized. Indeed, we have

$$P^{-1}A(x)P = \begin{bmatrix} 1/2 & -x_1 & -2x_1x_2 - x_2^2x_3 \\ 0 & 1/2 & (x_1 + x_2x_3)^2 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

From the above, we see that the eigenvalue of $A(x)$ is $\frac{1}{2}$ (triple root) for all $x \in \mathbf{R}^3$. Now, we evaluate $\|A(x)\|_2$. Since

$$\begin{aligned} \|A(x)\|_1 &= 3\left|\frac{1}{2}\right| + |-x_1| + \left|-2x_1x_2 - x_2^2x_3\right| + \left|(x_1 + x_2x_3)^2\right| \\ &\geq |-x_1|, \end{aligned}$$

we have

$$\sup_{\|x\|_2 < s} \|A(x)\|_1 \geq \sup_{\|x\|_2 < s} |-x_1| = s.$$

Therefore,

$$\lim_{s\to\infty} \frac{\left(\sup_{\|x\|_2 < s} \|A(x)\|_1\right)^2}{s} \geq \lim_{s\to\infty} \frac{s^2}{s} = \infty.$$

The norm equivalence in $\mathbf{R}^{3\times 3}$ implies

$$\lim_{s\to\infty} \frac{\left(\sup_{\|x\|_2 < s} \|A(x)\|_2\right)^2}{s} = \infty.$$

That is, this system does not satisfy condition (ii) in Theorem 3.29, whereas it is triangularizable and satisfies condition (i). In fact, this system is not globally asymptotically stable. This example has been given in [58] as a counterexample to the Markus-Yamabe conjecture [59].

Example 3.39. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be the norms defined in Example 3.36. Consider a system with the following coefficient matrix:

$$A(x) = \begin{bmatrix} 1/2 & x_2 \\ 0 & 1/2 \end{bmatrix}.$$

Since

$$\|A(x)\|_1 = \left|\frac{1}{2}\right| + \left|\frac{1}{2}\right| + |x_2| = 1 + |x_2|,$$

we have

$$\sup_{\|x\|_2 < s} \|A(x)\|_1 = 1 + s.$$

Therefore,

$$\lim_{s\to\infty} \frac{\sup_{\|x\|_2 < s} \|A(x)\|_1}{s} = \lim_{s\to\infty} \frac{1+s}{s} = 1 \neq 0.$$

The norm equivalence in $\mathbf{R}^{2\times 2}$ implies

$$\lim_{s\to\infty} \frac{\sup_{\|x\|_2 < s} \|A(x)\|_2}{s} \neq 0.$$

That is, this system does not satisfy condition (ii) in Theorem 3.29, whereas it is already triangular and satisfies condition (i). However, it can be verified that this system is globally asymptotically stable (see [60, Theorem B]). This implies that the sufficient condition (Theorem 3.29) is not a necessary one.

Chapter 4

Lie Derivative Inclusion with State Feedback

4.1 Introduction

We deal with two problems in this chapter. One aims to control a dynamical system so that a prescribed subset of the state space will be invariant for the resulting closed-loop system. The other aims to control a dynamical system so that the resulting closed-loop system will have a prescribed vector field on a given subset of the state space. It will be shown that the two problems can be represented by a particular inclusion of polynomials and that this inclusion can be solved in a finite number of arithmetic operations of polynomials. By using this representation, all the controllers required in the problems can be exactly computed in a unified manner, even though the problems are totally different from each other. Although the first problem has already been solved in [41, 42], our method of solving it is simpler than that of [41, 42]. Since the above two problems have several control applications, as stated below, the controllers required in the applications can also be exactly computed in a finite number of arithmetic operations of polynomials.

The first problem concerns invariant sets. The notion of an invariant set comes up in many problems in control engineering. For example, it is closely related to Lyapunov theory on differential equations, particularly, LaSalle's invariance principle [12]. It is stated in [61] that a positively invariant set plays an important role in the analysis of dynamical systems described by differential inclusions. Further applications of invariant sets are summarized in the survey paper [62]. There have been many studies [63–65] concerning the computation of invariant sets of given dynamical systems. Meanwhile, the following problem can be posed: 'given a dynamical system, obtain a controller such that a prescribed subset of the state space is invariant for the resulting closed-loop system.' This problem has several control applications. For example, when the state of a controlled system must belong to a given constraint set, the state constraint is always satisfied by the controller rendering the constraint set invariant. Such a controller can also solve the model matching problem.

The second problem concerns the realization of a prescribed vector field on a given subset of the state space. To show that this problem is associated with control problems, we can consider, for example, the realization of a prescribed stable limit cycle. In control engineering, it is important to design a controller such that the resulting closed-loop system has the prescribed stable limit cycle. For example, a controller for a biped robot is often designed to mimic passive dynamic walking [66–69] described by a stable limit cycle in the state space. Realizing the prescribed stable limit cycle consists of reaching two goals: the prescribed periodic orbit and its stabilization. Here, we will focus on the first goal, i.e., the problem of designing a controller such that the resulting closed-loop system has the prescribed periodic orbit. A periodic orbit can be represented by the combination of a closed curve in the state space and a vector field along the curve. Hence, the problem can be solved by designing a controller such that the resulting closed-loop system has the vector field on the closed curve corresponding to the periodic orbit. More generally, we can consider the following problem: 'given a dynamical system and a subset of the state space, obtain a controller such that the resulting closed-loop system has a vector field with prescribed components on the subset.' We consider cases in which all the components of a vector field or only some of them are prescribed. Through the generalization, we can also deal with a path-following control problem (explained in the following sections). The remaining problem of realizing a stable limit cycle, i.e., stabilization of the periodic orbit, is also discussed in this chapter.

In both problems, we assume that the given system is a polynomial dynamical system and that the controller is one of polynomial-type state feedback. Moreover, we assume that the given subset of the state space is algebraic. As is well known, polynomial dynamical systems can represent a variety of objects [70]. In particular, many non-polynomial dynamical systems can be described as polynomial dynamical systems via immersion [71]. Moreover, algebraic sets can represent a variety of typical subsets, including lines, hyperplanes, ellipses, and spheres. Therefore, the above assumptions are reasonable as a first step in tackling the problems.

This chapter is organized as follows. First, the above two problems are mathematically formulated in Section 4.2. The notation used in this chapter is also described. In Section 4.3, we formulate the Lie derivative inclusion with state feedback and show that the problems can be represented by the inclusion. In Section 4.4, we give the procedure (Algorithm 4.23) for solving the inclusion and prove Theorem 4.25 to show that the algorithm indeed solves the inclusion. As a result, all the required state-feedback controllers can be exactly represented by using free polynomial parameters. Section 4.5 outlines the behavior of the resulting closed-loop system outside the given algebraic set when the prescribed vector field is realized on the given set. In particular, if necessary, we can separately deal with the realization of the vector field and the stabilization of the set. Two examples are given in Section 4.6 to demonstrate how to obtain the controllers.

This chapter requires some knowledge of the theories of commutative algebras (see Chapter 2) and Gröbner bases [37–39]. This chapter includes the entire content of journal article 2 in the list of publications, which was written by the author. The copyright of the article is owned by ISCIE.

4.2 Notation, system description, and motivating problems

In this section, we define the notation used in this chapter and describe the system examined herein. After that, we mathematically formulate the problems mentioned in the introduction. The first problem is to obtain a state feedback controller such that a prescribed algebraic set in the state space is invariant for the resulting closed-loop system. The second problem is to obtain a state feedback controller such that the resulting closed-loop system has a prescribed vector field on a given algebraic set in the state space.

4.2.1 Notation and system description

Throughout this chapter, we will use the following notation. Let $R = \mathbf{R}[X_1, \ldots, X_n]$ be a polynomial ring over $\mathbf{R}$ with indeterminates $X_1, \ldots, X_n$. The free R-module of all the column vectors with n components in R is denoted by R^n. We will consider only the standard operations in R^n. The set of all $n \times m$ matrices with entries in R is denoted by $R^{n\times m}$. For given polynomial vectors $m_1, \ldots, m_r \in R^n$ (resp. polynomials $m_1, \ldots, m_r \in R$), let $\langle m_1, \ldots, m_r\rangle_R$ denote the R-submodule of R^n (resp. the ideal of R) generated by the set of generators $\{m_1, \ldots, m_r\}$. For a given polynomial vector $v \in R^n$, we distinguish between a polynomial vector v and a polynomial map $v(\cdot)$ according to these notations. We further distinguish between the indeterminates $X_1, \ldots, X_n$ of v and the variables $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in \mathbf{R}^n$ of $v(\cdot)$.

For a given polynomial $\Phi \in R$, the derivation of Φ w.r.t. X_i is denoted by $\partial\Phi/\partial X_i$. Let $\partial\Phi/\partial X$ be the row vector $[\partial\Phi/\partial X_1, \ldots, \partial\Phi/\partial X_n]$. For a given polynomial $\Phi \in R$ and a polynomial vector $f \in R^n$, let $L_f\Phi := (\partial\Phi/\partial X)f$. For given integers s_1 and s_2 such that $s_1 > s_2 \geq 1$, let $\mathrm{Prj}_{s_2}^{s_1} : R^{s_1} \to R^{s_2}$ be the projection map defined by $R^{s_1} \ni [\omega_1, \ldots, \omega_{s_1}]^{\mathrm{T}} \mapsto [\omega_1, \ldots, \omega_{s_2}]^{\mathrm{T}} \in R^{s_2}$.

The following notation is commonly used in (real) algebraic geometry [72].

Definition 4.1. Let J be a given ideal of R. The *real radical* of J is the ideal

$$\sqrt[\mathrm{R}]{J} := \Big\{\rho \in R \ : \ \exists N \in \mathbf{N} \qquad \exists c_1, \ldots, c_r \in R, \\ \rho^{2N} + c_1^2 + \cdots + c_r^2 \in J\Big\}.$$

Let J be an ideal of R representing a given algebraic set $\mathscr{A} \subset \mathbf{R}^n$, i.e., an ideal such that $\mathscr{A} = \{x \in \mathbf{R}^n : \ \rho(x) = 0 \ (\forall \rho \in J)\}$. It is well known [72] that the vanishing ideal I of $\mathscr{A}$ is the real radical of J, i.e., $I = \sqrt[\mathrm{R}]{J}$. Several algorithms for computing the real radial of a given ideal are given in [73–76]. We henceforth assume that the vanishing ideal I of a given algebraic set $\mathscr{A}$ can be computed.

We will deal with a system of the form

$$\dot{x} = f^{\mathrm{o}}(x, u) = f(x) + G(x)u \tag{4.1}$$

with $f \in R^n$ and $G = [g_1, \ldots, g_m]$, $g_i \in R^n$ $(i = 1, \ldots, m)$. We write $f^{\mathrm{o}}(\cdot, \cdot) = [f_1^{\mathrm{o}}(\cdot, \cdot), \ldots, f_n^{\mathrm{o}}(\cdot, \cdot)]^{\mathrm{T}}$.

4.2.2 Controlled invariance of a prescribed algebraic set

Consider an autonomous system of the form

$$\dot{x} = F(x) \tag{4.2}$$

with $F \in R^n$. Let $\phi(t, x_0)$ denote the solution of (4.2) at time t with the initial condition $x(0) = x_0 \in \mathbf{R}^n$. Denote by $\mathcal{E}(x_0)$ the maximal existence interval of $\phi(\cdot, x_0)$. We define the notion of the invariance of a set as follows.

Definition 4.2. A set $\mathscr{A} \subset \mathbf{R}^n$ is said to be *invariant* for system (4.2) if $x_0 \in \mathscr{A}$ implies $\phi(t, x_0) \in \mathscr{A}$ for all $t \in \mathcal{E}(x_0)$. A set $\mathscr{A} \subset \mathbf{R}^n$ is said to be *positively invariant* for system (4.2) if $x_0 \in \mathscr{A}$ implies $\phi(t, x_0) \in \mathscr{A}$ for all $t \geq 0$ such that $t \in \mathcal{E}(x_0)$.

Remark 4.3. We do not require the existence of $\phi(t, x_0)$ for all $t \in (-\infty, \infty)$, whereas a lot of literature (e.g. [12, 62]) does require it for the definition of invariance. The same sort of definition appears in [41, 42, 77]. When $\mathscr{A}$ is compact, our definition is equivalent to those ones in [12, 62].

Now, let us return to system (4.1) and define the notion of the controlled invariance of a set.

Definition 4.4. A set $\mathscr{A} \subset \mathbf{R}^n$ is said to be *controlled invariant* (resp. *positively controlled invariant*) for system (4.1) if there exists a state feedback controller $u \in R^n$ such that $\mathscr{A}$ is invariant (resp. positively invariant) for the resulting closed-loop system of (4.1).

We will prove the following lemma to show that invariance and positive invariance are equivalent for algebraic sets.

Lemma 4.5. *Let $\mathscr{A} \subset \mathbf{R}^n$ be a given algebraic set. Then, the following statements are equivalent:*

(i) $\mathscr{A}$ is invariant for system (4.2).

(ii) $\mathscr{A}$ is positively invariant for system (4.2).

Proof. The implication (i) $\Rightarrow$ (ii) is obvious. Suppose that statement (ii) holds. Then, $x_0 \in \mathscr{A}$ implies $\phi(t, x_0) \in \mathscr{A}$ for all $t \geq 0$ such that $t \in \mathcal{E}(x_0)$. Fix an initial state $x_0 \in \mathscr{A}$. Let $I = \langle \rho_1, \ldots, \rho_k \rangle_R \subset R$ be the vanishing ideal of $\mathscr{A}$. Then, we have that $\rho_i(\phi(t, x_0)) = 0$ for $i = 1, \ldots, k$ and for all $t \geq 0$ such that $t \in \mathcal{E}(x_0)$. Since $F(\cdot)$ is real analytic, so is the function $\phi(\cdot, x_0)$ in $\mathcal{E}(x_0)$ (see, e.g., [78]). In particular, the function $\rho_i(\phi(\cdot, x_0))$ is real analytic in $\mathcal{E}(x_0)$. Therefore, from the identity theorem, it follows that $\rho_i(\phi(t, x_0)) = 0$ for $i = 1, \ldots, k$ and for all $t \in \mathcal{E}(x_0)$. This means $\phi(t, x_0) \in \mathscr{A}$ for all $t \in \mathcal{E}(x_0)$, which implies that statement (i) holds. $\square$

According to the above lemma, we do not need to distinguish between the invariance and the positive invariance of an algebraic set. Hence, in what follows, we will consider only the invariance of an algebraic set.

We shall consider the following problem.

Problem 4.6. *Let $I = \langle \rho_1, \ldots, \rho_k \rangle_R \subset R$ be the vanishing ideal of a given algebraic set $\mathscr{A} \subset \mathbf{R}^n$. Then, given a system of form* (4.1)*, determine whether $\mathscr{A}$ is controlled invariant for system (4.1). Moreover, if $\mathscr{A}$ is controlled invariant, obtain a state feedback controller $u \in R^m$ for system (4.1) such that $\mathscr{A}$ is invariant for the resulting closed-loop system.*

Definition 4.7. A *solution* of Problem 4.6 is a state feedback controller $u \in R^m$ rendering $\mathscr{A}$ invariant for the resulting closed-loop system of (4.1). Problem 4.6 is said to be *solvable* if a solution to it exists.

We give two examples indicating the relation of the above problem to control problems. Example 4.8 concerns the state constraint, and Example 4.9 concerns the model matching problem.

Example 4.8 (State constraint)**.** Let $\mathcal{X}$ be a given subset of $\mathbf{R}^n$. Suppose that the state of system (4.1) must satisfy the constraint $x(t) \in \mathcal{X}$. Assume that $\mathcal{X}$ is an algebraic set; i.e., the constraint is given by algebraic equations. Then, a solution of Problem 4.6 with $\mathscr{A} = \mathcal{X}$ is a feasible input for system (4.1).

Example 4.9 (Model matching problem)**.** Let $R_{\mathrm{p}} = \mathbf{R}[X_1, \ldots, X_{n_{\mathrm{p}}}]$ and $R_{\mathrm{r}} = \mathbf{R}[X_1, \ldots, X_{n_{\mathrm{r}}}]$ with given positive integers n_{p} and n_{r}. Suppose we have a controlled plant of the form

$$\begin{cases} \dot{\xi} = f^{\mathrm{p}}(\xi) + G^{\mathrm{p}}(\xi)u, \\ y^{\mathrm{p}} = h^{\mathrm{p}}(\xi) \end{cases} \tag{4.3}$$

with $\xi = [\xi_1, \ldots, \xi_{n_{\mathrm{p}}}]^{\mathrm{T}} \in \mathbf{R}^{n_{\mathrm{p}}}$, $f^{\mathrm{p}} \in R_{\mathrm{p}}^{n_{\mathrm{p}}}$, $G^{\mathrm{p}} \in R_{\mathrm{p}}^{n_{\mathrm{p}} \times m}$, and $h^{\mathrm{p}} = [h_1^{\mathrm{p}}, \ldots, h_p^{\mathrm{p}}]^{\mathrm{T}}$, $h_i^{\mathrm{p}} \in R_{\mathrm{p}}$ $(i = 1, \ldots, p)$. On the other hand, suppose the reference model is

$$\begin{cases} \dot{\zeta} = f^{\mathrm{r}}(\zeta), \\ y^{\mathrm{r}} = h^{\mathrm{r}}(\zeta) \end{cases} \tag{4.4}$$

with $\zeta = [\zeta_1, \ldots, \zeta_{n_{\mathrm{r}}}]^{\mathrm{T}} \in \mathbf{R}^{n_{\mathrm{r}}}$, $f^{\mathrm{r}} \in R_{\mathrm{r}}^{n_{\mathrm{r}}}$, and $h^{\mathrm{r}} = [h_1^{\mathrm{r}}, \ldots, h_p^{\mathrm{r}}]^{\mathrm{T}}$, $h_i^{\mathrm{r}} \in R_{\mathrm{r}}$ $(i = 1, \ldots, p)$. Now, let us consider the problem of obtaining a dynamic state feedback controller for plant (4.3) such that $y^{\mathrm{p}}(\xi(0)) = y^{\mathrm{r}}(\zeta(0))$ implies $y^{\mathrm{p}}(\xi(t)) = y^{\mathrm{r}}(\zeta(t))$ for all $t \geq 0$. The problem can be solved as follows. Put $n := n_{\mathrm{p}} + n_{\mathrm{r}}$, and let $R := \mathbf{R}[X_1, \ldots, X_n]$ be a polynomial ring with indeterminates $X_1, \ldots, X_n$. Let $x = [x_1, \ldots, x_n]^{\mathrm{T}} := [\xi^{\mathrm{T}}, \zeta^{\mathrm{T}}]^{\mathrm{T}} \in \mathbf{R}^n$ be the state of the extended system

$$\dot{x} = f(x) + G(x)u \tag{4.5}$$

with

$$f(x) = \begin{bmatrix} f^{\mathrm{p}}(\xi) \\ f^{\mathrm{r}}(\zeta) \end{bmatrix}, \quad G(x) = \begin{bmatrix} G^{\mathrm{p}}(\xi) \\ O \end{bmatrix}.$$

We regard f as an element of R^n and G as an element of $R^{n\times m}$. The polynomials h_i^{p} and h_i^{r} with $i = 1,\ldots,p$ are regarded as elements of R. Then, define the ideal J of R as

$$J := \left\langle h_1^{\mathrm{p}} - h_1^{\mathrm{r}}, \ldots, h_p^{\mathrm{p}} - h_p^{\mathrm{r}} \right\rangle_R$$

and define the algebraic set $\mathscr{A} = \{x \in \mathbf{R}^n : \ \rho(x) = 0 (\forall \rho \in J)\}$. We can now consider Problem 4.6 with system (4.5) and the above $\mathscr{A}$. Accordingly, a solution $u = \tilde{u} \in R^m$ of Problem 4.6 can be seen as a dynamic state feedback controller for system (4.3) defined by

$$\begin{cases} \dot{\zeta} = f^{\mathrm{r}}(\zeta), \\ u = \tilde{u}(\xi, \zeta) \end{cases} \tag{4.6}$$

such that $y^{\mathrm{p}}(\xi(0)) = y^{\mathrm{r}}(\zeta(0))$ implies $y^{\mathrm{p}}(\xi(t)) = y^{\mathrm{r}}(\zeta(t))$ for all $t \geq 0$. Hence, a solution of Problem 4.6 leads to a solution to the model matching problem mentioned above.

4.2.3 Realization of a prescribed vector field

The second problem is the realization of a prescribed vector field. It is described as follows.

Problem 4.10. *Let $I = \langle \rho_1, \ldots, \rho_k \rangle_R \subset R$ be the vanishing ideal of a given algebraic set $\mathscr{A} \subset \mathbf{R}^n$. Given a system of form* (4.1) *and a polynomial vector $f^{\mathrm{d}} = [f_1^{\mathrm{d}}, \ldots, f_{n'}^{\mathrm{d}}]^{\mathrm{T}} \in R^{n'}$ with a positive integer $n' \leq n$, determine whether there exists a state feedback controller $u \in R^m$ such that*

$$\begin{cases} f_1^{\mathrm{d}}(x) = f_1^{\mathrm{o}}(x, u(x)), \\ \qquad \vdots \\ f_{n'}^{\mathrm{d}}(x) = f_{n'}^{\mathrm{o}}(x, u(x)), \end{cases} \tag{4.7}$$

for all $x \in \mathscr{A}$. Moreover, if such a controller exists, it should be able to be computed explicitly.

Definition 4.11. A *solution* of Problem 4.10 is a state feedback controller $u \in R^m$ satisfying (4.7). Problem 4.10 is said to be *solvable* if a solution to it exists.

Remark 4.12. Let $f^{\mathrm{d}\prime} \in R^{n'}$ be another polynomial vector such that $f^{\mathrm{d}\prime}(x) = f^{\mathrm{d}}(x)$ on $\mathscr{A}$. Then, a solution of Problem 4.10 with $f^{\mathrm{d}\prime}$ is identical to that with f^{d}. Hence, the use of a specific polynomial vector f^{d} does not lose generality.

Remark 4.13. When $I = \{0\}$ and $n' = n$, Problem 4.10 is one of obtaining a state feedback controller such that the vector field of the resulting closed-loop system of (4.1) coincides with a prescribed one on the whole space $\mathbf{R}^n$.

Remark 4.14. Here, the prescribed vector field need not be tangential to the given algebraic set.

Let us now give examples indicating how Problem 4.10 is related to control problems. Example 4.15 concerns the realization of a prescribed periodic orbit, and Example 4.17 concerns a path following control problem.

Example 4.15 (Realization of a periodic orbit)**.** Consider a system of form (4.1) with $n = 2$, and put

$$f^{\mathrm{d}} := \begin{bmatrix} x_2 \\ -\omega^2 x_1 \end{bmatrix} \quad \text{and} \quad I := \left\langle \frac{X_1^2}{A^2} + \frac{X_2^2}{A^2\omega^2} - 1 \right\rangle_R$$

with arbitrary parameters $A, \omega \in \mathbf{R}$. Then, Problem 4.10 is one of realizing a simple harmonic oscillation with amplitude A and frequency $\omega/2\pi$ via state feedback.

Remark 4.16. The above example differs from the synthesis problem of systems having a prescribed limit cycle (e.g., [79–81]) and the controller design for systems having some limit cycle (e.g., [82, 83]). In the above example, an arbitrary periodic orbit in the state space can be prescribed, instead of the harmonic oscillation, under the assumptions that the dynamical system is of polynomial and the orbit is an algebraic set.

Example 4.17 (Path-following control)**.** Consider a system of the form

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, \theta) + G_1(x_1, x_2, \theta)u, \\ \dot{x}_2 = f_2(x_1, x_2, \theta) + G_2(x_1, x_2, \theta)u, \\ \dot{\theta} = f_3(x_1, x_2, \theta) + G_3(x_1, x_2, \theta)u. \end{cases} \tag{4.8}$$

The above system is intended to describe some object moving on the plane $\mathbf{R}^2$. Here, $[x_1, x_2]^{\mathrm{T}} \in \mathbf{R}^2$ and $\theta \in \mathbf{R}$ represent the position and orientation of the object, respectively. The control input is denoted by u. This sort of system can be found in the literature [84–86]. Let $R_3 := \mathbf{R}[X_1, X_2, X_3]$ and $R_2 := \mathbf{R}[X_1, X_2]$ be polynomial rings with indeterminates X_1, X_2, X_3, and assume that $f_i \in R_3$ and $G_i \in R_3^{1\times 3}$ for $i = 1, 2, 3$. Let $\mathcal{C} \subset \mathbf{R}^2$ be a given geometric path defined by polynomials $\sigma_i \in R_2$ $(i = 1, \dots, k)$, i.e., $\mathcal{C} = \big\{[x_1, x_2]^{\mathrm{T}} \in \mathbf{R}^2 : \ \sigma_i(x_1, x_2) = 0, \ i = 1, \dots, k\big\}$, and $v(\cdot) : \mathcal{C} \to \mathbf{R}^2$ be the desired velocity on the path. The path-following control aims to design a controller such that the controlled object converges to and follows a prescribed geometric path at the desired velocity. Accordingly, we will consider the problem of determining whether the path and velocity are realizable by state feedback and of computing the feedback controller explicitly. That is, if it exists, the state feedback controller $u \in R_3^3$ is such that the controlled object indeed moves exactly along $\mathcal{C}$ at velocity $v(\cdot)$ provided that the object starts from a point $[x_{01}, x_{02}]^{\mathrm{T}}$ in $\mathcal{C}$ at velocity $v(x_{01}, x_{02})$. From a practical standpoint, it might be sufficient for the desired path and velocity to be approximately realized. However, ideally, the above situation is the main goal of the path-following control; hence, the problem certainly has theoretical importance. Suppose that v is an element of R_2^2 and write $v = [v_1, v_2]^{\mathrm{T}}$ with $v_i \in R_2$ for $i = 1, 2$. Regard v_i and σ_j as elements of R_3 for $i = 1, 2$ and $j = 1, \ldots, k$. Define the algebraic set $\mathscr{A} = \left\{[x_1, x_2, \theta]^{\mathrm{T}} \in \mathbf{R}^3 : \ \sigma_i(x_1, x_2, \theta) = 0, \ i = 1, \ldots, k\right\} \subset \mathbf{R}^3$. Then, the above problem can be translated into one of determining the existence of a feedback controller $u \in R_3^3$ satisfying

$$\begin{aligned} v_1(x_1, x_2, \theta) &= f_1(x_1, x_2, \theta) + G_1(x_1, x_2, \theta) u(x_1, x_2, \theta), \\ v_2(x_1, x_2, \theta) &= f_2(x_1, x_2, \theta) + G_2(x_1, x_2, \theta) u(x_1, x_2, \theta), \end{aligned}$$

for all $[x_1, x_2, \theta]^{\mathrm{T}} \in \mathscr{A}$, and of computing the controller if it exists. This problem can be represented by Problem 4.10 with

$$f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}, \quad f^{\mathrm{d}} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Remark 4.18. The above two examples do not include the stabilization of the given algebraic set. Stabilization is discussed in Section 4.5.

4.3 Lie derivative inclusion with state feedback

In this section, we formulate the Lie derivative inclusion and show that Problems 4.6 and 4.10 can be represented by the inclusion.

The Lie derivative inclusion is formulated as follows.

Definition 4.19. Let Φ_i, Ψ_i $(i = 1, \ldots, \ell)$ be given polynomials in R and I_i $(i = 1, \ldots, \ell)$ be given ideals of R. Then, consider the system of inclusions

$$\begin{cases} L_{f+Gu}\Phi_1 \in \Psi_1 + I_1, \\ \qquad \vdots \\ L_{f+Gu}\Phi_\ell \in \Psi_\ell + I_\ell \end{cases} \tag{4.9}$$

with $u \in R^m$ to be found. We call the above system the *Lie derivative inclusion with state feedback.* A *solution* of (4.9) is a state feedback controller $u \in R^m$ satisfying (4.9). Inclusion (4.9) is said to be *solvable* if a solution to it exists.

To figure out what the above inclusion means, let us consider the simple equation

$$L_{f+Gu}\Phi_i = \Psi_i. \tag{4.10}$$

The meaning of the equation is clear. If, for example, $\Phi_i(\cdot)$ is positive definite and $\Psi_i(\cdot)$ is negative definite, then u satisfying (4.10) is a stabilizing controller for system (4.1) such that the resulting closed-loop system has the Lyapunov pair $(\Phi_i(\cdot), -\Psi_i(\cdot))$. If Φ_i is an arbitrary polynomial and Ψ_i is the zero polynomial, then u satisfying (4.10) is a controller such that the resulting closed-loop system has $\Phi_i(\cdot)$ as a first integral. Now, consider the algebraic set $\mathscr{A}_i = \{x \in \mathbf{R}^n : \ \rho(x) = 0, \ (\forall \rho \in I_i)\}$. The inclusion

$$L_{f+Gu}\Phi_i \in \Psi_i + I_i$$

means that the equation

$$L_{f+Gu}\Phi_i(x) = \Psi_i(x) \tag{4.11}$$

holds for all x in $\mathscr{A}_i$. That is, inclusion (4.9) means that (4.11) holds on the algebraic set $\mathscr{A}_i$ for $i = 1, \ldots, \ell$.

Remark 4.20. It is possible but difficult to utilize the Lie derivative inclusion for the stabilization problem because a (control) Lyapunov pair of the system must be prescribed.

Next, we show that Problem 4.6 is represented by the Lie derivative inclusion. Let $\mathscr{A} \subset \mathbf{R}^n$ be a given algebraic set and $I = \langle \rho_1, \ldots, \rho_k \rangle_R$ be the vanishing ideal of $\mathscr{A}$. A necessary and sufficient condition for the invariance of an algebraic set is given by the following theorem [41, 42, 77].

Theorem 4.21. *A given algebraic set $\mathscr{A}$ is invariant for system (4.2) if and only if*

$$L_F\rho_i \in I$$

for $i = 1, \ldots, k$.

The following corollary is a straightforward consequence of the above theorem.

Corollary 4.22. *A given algebraic set $\mathscr{A}$ is invariant for the resulting closed-loop system of* (4.1) *with a controller $u \in R^m$ if and only if u satisfies the inclusion*

$$L_{f+Gu}\rho_i \in I$$

for $i = 1, \ldots, k$.

From this corollary, we have that the solutions of Problem 4.6 coincide with the solutions of (4.9) with $\Phi_i = \rho_i$, $\Psi_i = 0$, and $I_i = I$ for $i = 1, \ldots, k$.

Finally, we show that Problem 4.10 can be represented by the Lie derivative inclusion. Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbf{R}^n$ and rewrite (4.7) as follows:

$$\begin{cases} f_1^{\mathrm{d}}(x) = e_1^{\mathrm{T}}\left(f(x) + G(x)u(x)\right), \\ \qquad\vdots \\ f_{n'}^{\mathrm{d}}(x) = e_{n'}^{\mathrm{T}}\left(f(x) + G(x)u(x)\right). \end{cases}$$

Problem 4.10 requires the above set of equations to be satisfied for all $x \in \mathscr{A}$. This is equivalent to the following inclusions of polynomials:

$$\begin{cases} e_1^{\mathrm{T}}\left(f + Gu\right) & \in f_1^{\mathrm{d}} + I, \\ & \vdots \\ e_{n'}^{\mathrm{T}}\left(f + Gu\right) & \in f_{n'}^{\mathrm{d}} + I. \end{cases}$$

The above inclusions can be derived from (4.9) by setting $\Phi_i = X_i$, $\Psi_i = f_i^{\mathrm{d}}$, and $I_i = I$ for $i = 1, \ldots, n'$. That is, the solutions of Problem 4.10 coincide with the solutions of (4.9) with $\Phi_i = X_i$, $\Psi_i = f_i^{\mathrm{d}}$, and $I_i = I$ for $i = 1, \ldots, n'$.

4.4 Solution to the Lie derivative inclusion with state feedback

In this section, we describe a procedure for solving the Lie derivative inclusion with state feedback. Algorithm 4.23 is the procedure for solving the inclusion. Theorem 4.25 shows that Algorithm 4.23 indeed solves the inclusion. As a result, all the required state feedback controllers can be represented by using free polynomial parameters. As shown in the previous section, the procedure can be used to solve Problems 4.6 and 4.10.

Write $I_i = \langle \rho_{i1}, \ldots, \rho_{ik_i} \rangle_R$ for $i = 1, \ldots, \ell$ and put $d := \sum_{i=1}^{\ell} k_i$. Inclusion (4.9) is redescribed as follows:

$$\begin{cases} \dfrac{\partial \Phi_1}{\partial X}(f + Gu) \in \Psi_1 + \langle \rho_{11}, \ldots, \rho_{1k_1} \rangle_R\,, \\ \qquad\qquad\vdots \\ \dfrac{\partial \Phi_\ell}{\partial X}(f + Gu) \in \Psi_\ell + \langle \rho_{\ell 1}, \ldots, \rho_{\ell k_\ell} \rangle_R\,. \end{cases}$$

The above inclusion can be transformed into

$$\begin{cases} \Psi_1 - \dfrac{\partial \Phi_1}{\partial X} f \in \left(\dfrac{\partial \Phi_1}{\partial X} G \right) u + \langle \rho_{11}, \ldots, \rho_{1k_1} \rangle_R\,, \\ \qquad\qquad\vdots \\ \Psi_\ell - \dfrac{\partial \Phi_\ell}{\partial X} f \in \left(\dfrac{\partial \Phi_\ell}{\partial X} G \right) u + \langle \rho_{\ell 1}, \ldots, \rho_{\ell k_\ell} \rangle_R\,. \end{cases} \tag{4.12}$$

Now, put

$$\begin{aligned} \overline{\Psi} &:= \begin{bmatrix} \Psi_1 - \frac{\partial \Phi_1}{\partial X} f \\ \vdots \\ \Psi_\ell - \frac{\partial \Phi_\ell}{\partial X} f \end{bmatrix} \in R^\ell, \\ \overline{G} &:= [\overline{g}_1, \ldots, \overline{g}_m] := \begin{bmatrix} \frac{\partial \Phi_1}{\partial X} G \\ \vdots \\ \frac{\partial \Phi_\ell}{\partial X} G \end{bmatrix} \in R^{\ell \times m}, \\ u &= \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \in R^m. \end{aligned} \tag{4.13}$$

Let $\{e'_1, \ldots, e'_\ell\}$ be the canonical basis of $\mathbf{R}^\ell$ and put

$$\overline{\rho}_{ij} := \rho_{ij} e'_i \in R^\ell \quad (i = 1, \ldots, \ell \text{ and } j = 1, \ldots, k_i). \tag{4.14}$$

Then, it can easily be verified that a solution u of (4.12) is nothing but $u = [u_1, \ldots, u_m]^{\mathrm{T}}$ satisfying the equation

$$\begin{aligned} \overline{\Psi} = \overline{g}_1 u_1 + \cdots + \overline{g}_m u_m &+ \overline{\rho}_{11} z_{11} + \cdots + \overline{\rho}_{1k_1} z_{1k_1} \\ &\vdots \\ &+ \overline{\rho}_{\ell 1} z_{\ell 1} + \cdots + \overline{\rho}_{\ell k_\ell} z_{\ell k_\ell} \end{aligned}$$

for some $z_{ij} \in R$ ($i = 1, \ldots, \ell$ and $j = 1, \ldots, k_i$). Indeed, from the definition of ideals, u satisfies (4.12) if and only if there exist polynomials $z_{ij} \in R$ ($i = 1, \ldots, \ell$ and $j = 1, \ldots, k_i$) such that

$$\begin{cases} \Psi_1 - \dfrac{\partial \Phi_1}{\partial X} f = \left(\dfrac{\partial \Phi_1}{\partial X} G \right) u + \rho_{11} z_{11} + \cdots + \rho_{1k_1} z_{1k_1}, \\ \qquad\qquad \vdots \\ \Psi_\ell - \dfrac{\partial \Phi_\ell}{\partial X} f = \left(\dfrac{\partial \Phi_\ell}{\partial X} G \right) u + \rho_{\ell 1} z_{\ell 1} + \cdots + \rho_{\ell k_\ell} z_{\ell k_\ell}. \end{cases}$$

The above equation can be described in vector form,

$$\begin{aligned} \overline{\Psi} = \overline{G} u + \overline{\rho}_{11} z_{11} + \cdots + \overline{\rho}_{1k_1} z_{1k_1} \\ \vdots \qquad\qquad \\ + \overline{\rho}_{\ell 1} z_{\ell 1} + \cdots + \overline{\rho}_{\ell k_\ell} z_{\ell k_\ell}, \end{aligned}$$

which is nothing but (4.12). Summarizing the above discussion, we find that a solution $u = [u_1, \ldots, u_m]^{\mathrm{T}}$ of inclusion (4.9) can be obtained by solving

$$\begin{aligned} \overline{\Psi} = \overline{g}_1 \omega_1 + \cdots + \overline{g}_m \omega_m + \overline{\rho}_{11} \omega_{m+1} + \cdots + \overline{\rho}_{1k_1} \omega_{m+k_1} \\ \vdots \qquad\qquad \\ + \overline{\rho}_{\ell 1} \omega_{m + (\sum_{i=1}^{\ell-1} k_i) + 1} + \cdots + \overline{\rho}_{\ell k_\ell} \omega_{m+d} \end{aligned} \tag{4.15}$$

with variables $\omega = [\omega_1, \ldots, \omega_{m+d}]^{\mathrm{T}} \in R^{m+d}$ and by projecting $u := \mathrm{Prj}_m^{m+d}(\omega)$. Since this is a linear equation with coefficients in the polynomial ring R, we can determine whether there exists a solution and exactly compute the set of all solutions of (4.15) as

$$\omega^* + \langle m_1^\omega, \ldots, m_r^\omega \rangle_R \,,$$

where $\omega^* \in R^{m+d}$ and $m_i^\omega \in R^{m+d}$ ($i = 1, \ldots, r$) with some $r \geq 1$, by using Algorithm A.8 in the appendix. Putting $u^* := \mathrm{Prj}_m^{m+d}(\omega^*) \in R^m$ and $m_i^u := \mathrm{Prj}_m^{m+d}(m_i^\omega) \in R^m$ ($i = 1, \ldots, r$), we obtain the set $u^* + \langle m_1^u, \ldots, m_r^u \rangle_R$ of all the solutions of inclusion (4.9).

We can summarize to obtain the following algorithm for solving the Lie derivative inclusion, which is a modification of Algorithm A.8 in the appendix.

Algorithm 4.23 (For solving the Lie derivative inclusion with state feedback)**.**

Given: the polynomial vector $f \in R^n$ and polynomial matrix $G \in R^{n\times m}$ of system (4.1). Polynomials $\Phi_i \in R$ and $\Psi_i \in R$, and ideals $I_i = \langle \rho_{i1}, \dots, \rho_{ik_i} \rangle_R \subset R$ for $i = 1, \dots \ell$.

Obtain: a polynomial vector $u^* \in R^m$ and a set $\{m_1^u, \dots, m_r^u\}$ of polynomial vectors $m_i^u \in R^m$ $(i = 1, \dots, r)$, where the set $u^* + \langle m_1^u, \dots, m_r^u \rangle_R$ represents the set of all solutions of inclusion (4.9).

Step 1. Define the polynomial vectors $\overline{\Psi} \in R^\ell$, $\overline{g}_i \in R^\ell$ $(i = 1, \dots, m)$, and $\overline{\rho}_{ij} \in R^\ell$ $(i = 1, \dots, \ell$ and $j = 1, \dots, k_i)$ as (4.13) and (4.14).

Step 2. Compute a Gröbner basis $\{\tilde{\gamma}_1, \dots, \tilde{\gamma}_q\}$ of the R-submodule

$$\left\langle \overline{g}_1, \dots, \overline{g}_m, \overline{\rho}_{11}, \dots, \overline{\rho}_{1k_1}, \dots, \overline{\rho}_{\ell 1}, \dots, \overline{\rho}_{\ell k_\ell} \right\rangle_R$$

by using Buchberger's algorithm [37, 48].

Step 3. Compute a transformation matrix $Q \in R^{(m+d)\times q}$ such that

$$[\tilde{\gamma}_1, \dots, \tilde{\gamma}_q] = \left[\overline{g}_1, \dots, \overline{g}_m, \overline{\rho}_{11}, \dots, \overline{\rho}_{1k_1}, \dots, \overline{\rho}_{\ell 1}, \dots, \overline{\rho}_{\ell k_\ell}\right] Q.$$

This matrix can be obtained by tracing Buchberger's algorithm in Step 2 (see, e.g., [87]).

Step 4. Divide $\overline{\Psi}$ by $(\tilde{\gamma}_1, \dots, \tilde{\gamma}_q)$, and let $\omega' \in R^q$ and $\omega'' \in R^\ell$ be the quotient and the remainder of $\overline{\Psi}$ in the division, respectively.

Step 5. If $\omega'' \neq 0$, terminate the algorithm because inclusion (4.9) is not solvable; otherwise, proceed to the next step because inclusion (4.9) is solvable.

Step 6. Put $\omega^* := Q\omega'$ and compute a set of generators $\{m_1^\omega, \dots, m_r^\omega\}$ of the syzygy module of the $(m + d)$-tuple

$$(\overline{g}_1, \dots, \overline{g}_m, \overline{\rho}_{11}, \dots, \overline{\rho}_{1k_1}, \dots, \overline{\rho}_{\ell 1}, \dots, \overline{\rho}_{\ell k_\ell})$$

by executing the algorithm presented in [37].

Step 7. Put $u^* := \mathrm{Prj}_m^{m+d}(\omega^*) \in R^m$ and $m_i^u := \mathrm{Prj}_m^{m+d}(m_i^\omega) \in R^m$ $(i = 1, \dots, r)$.

Remark 4.24. All the calculations in the above algorithm are not approximate but exact. Moreover, the algorithm consists of a finite number of arithmetic operations of polynomials.

The foregoing discussion proves the following theorem.

Theorem 4.25. *The Lie derivative inclusion with state feedback is solvable if and only if* $\omega'' = 0$. *Let* $u^*, m_1^u, \ldots,$ *and* m_r^u *be the polynomial vectors computed by Algorithm 4.23. Then,* $u^* + \langle m_1^u, \ldots, m_r^u \rangle_R$ *is the set of all solutions of the Lie derivative inclusion with state feedback.*

Remark 4.26. The above result can be interpreted as follows: let $u^*, m_1^u, \ldots$ and m_r^u be as above. Then, any state feedback controller $u \in R^m$ satisfying (4.9) can be parametrized in the form

$$u = u^* + c_1 m_1^u + \ldots + c_r m_r^u \tag{4.16}$$

with arbitrary polynomial parameters $c_i \in R$ $(i = 1, \ldots, r)$. Conversely, any controller u of the above form satisfies (4.9).

Remark 4.27. Problem 4.6 was first solved in [41,42]. While the procedure in the literature consists of computing all systems having a prescribed invariant set and of solving a particular system of linear equations, our procedure only needs to solve a single system of linear equations. Moreover, we solved the problem in the framework of a more general concept, the Lie derivative inclusion.

Remark 4.28. Consider Problem 4.6 and put $h = [\rho_1, \ldots, \rho_k]^{\mathrm{T}} \in R^k$. Then, Problem 4.6 can be regarded as one of obtaining a state feedback controller for the system

$$\begin{aligned} \dot{x} &= f(x) + G(x)u, \\ y &= h(x), \end{aligned}$$

such that the set $\mathscr{A} = \{x \in \mathbf{R}^n \ : \ y = 0\}$ is invariant for the closed-loop system. This problem can also be handled in the framework of the *zero dynamics algorithm* [88]. Although our approach limits the systems to polynomial dynamical ones, it has several advantages over the zero dynamics algorithm. First, the zero dynamics algorithm works only locally. That is, the resulting controller only renders the set $\mathscr{A} \cap U$ invariant, where $U \subset \mathbf{R}^n$ is some neighborhood of an a priori fixed point in $\mathscr{A}$. Second, the zero dynamics algorithm assumes several regularity conditions that ours does not. For example, when $\mathscr{A}$ is not smooth at the origin, the zero dynamics algorithm cannot work there because no locally maximal controlled-invariant manifold exists there. Third, our approach can *compute* the set of *all* required controllers, unlike the zero dynamics algorithm.

4.5 Behavior outside the algebraic set

The previous section described a procedure for solving Problem 4.10 by solving the Lie derivative inclusion, that is, an algorithm for obtaining all state feedback controllers realizing a prescribed vector field on a given algebraic set. Although the procedure specifies the behavior of a closed-loop system on the set, it does not specify anything about the behavior outside the set. In this section, we discuss the behavior of a closed-loop system outside a given algebraic set, especially the stability of the algebraic set, when the prescribed vector field is realized on the algebraic set. Regarding stabilizability, we assume that the given algebraic set is compact and the prescribed vector field is such that the set is invariant for the resulting closed-loop system.

Consider Problem 4.10 and let $u^* \in R^m$ and $m_i^u \in R^m$ $(i = 1, \ldots, r)$ be the polynomial vectors computed by Algorithm 4.23. Then, the set $u^* + \langle m_1^u, \ldots, m_r^u \rangle_R$ represents the set of all state feedback controllers such that the resulting closed-loop systems have the prescribed vector field on the given algebraic set $\mathscr{A}$. By describing $u^* + \langle m_1^u, \ldots, m_r^u \rangle_R$ in parametrized form (4.16), we can modify the behavior of a closed-loop system of (4.1) outside $\mathscr{A}$ by changing $c_1, \ldots, c_r$ in R. Note that the vector field on $\mathscr{A}$ will still be realized even after the c_i's have changed. In particular, if one wants to locally (resp. globally) asymptotically stabilize the set $\mathscr{A}$ at the same time as realizing a prescribed vector field on $\mathscr{A}$, it suffices to find $c_1, \ldots, c_r$ so that $\mathscr{A}$ is a locally (resp. globally) asymptotically stable set for the resulting closed-loop system. Now, substituting $u^*(x) + c_1 m_1^u(x) + \cdots + c_r m_r^u(x)$ into u of (4.1), we obtain the closed-loop system

$$\begin{aligned} \dot{x} &= f(x) + G(x)\big\{u^*(x) + c_1 m_1^u(x) + \cdots + c_r m_r^u(x)\big\} \\ &= \{f(x) + G(x)u^*(x)\} + \{G(x)m_1^u(x)\}\, c_1 + \cdots + \{G(x)m_r^u(x)\}\, c_r \\ &= \tilde{f}(x) + \tilde{G}(x)c, \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} \tilde{f} &= f + Gu^* \in R^n, \\ \tilde{G} &= [Gm_1^u, \ldots, Gm_r^u] \in R^{n\times r}, \\ c &= [c_1, \ldots, c_r]^{\mathrm{T}} \in R^r. \end{aligned}$$

Since c is arbitrary, it can be seen as a new input to system (4.17). The above discussion can be summarized into the following proposition.

Proposition 4.29. *Consider Problem 4.10 with a system of form* (4.1)*, a polynomial vector* $f^{\mathrm{d}} = [f_1^{\mathrm{d}}, \ldots, f_{n'}^{\mathrm{d}}]^{\mathrm{T}} \in R^{n'}$ *with* $n' \leq n$*, and an algebraic set* $\mathscr{A} \in \mathbf{R}^n$*. Assume that* $\mathscr{A}$ *is compact, and* f^{d} *is such that* $\mathscr{A}$ *is invariant for* *the resulting closed-loop system. Let* $u^* \in R^m$ *and* $m_i^u \in R^m$ $(i = 1, \dots, r)$ *be the polynomial vectors computed by Algorithm 4.23. Then, the state feedback controller* $u = \hat{u} \in R^m$ *for* (4.1) *satisfies the following conditions:*

(i) the controller $u = \hat{u}$ *satisfies* (4.7) *on* $\mathscr{A}$*, and*

(ii) $\mathscr{A}$ *is a locally (resp. globally) asymptotically stable set for the closed-loop system,*

if and only if there exist polynomials $c_1, \dots, c_r \in R$ *satisfying*

(a) the controller $\hat{u}$ *is described as* $\hat{u} = u^* + c_1 m_1^u + \cdots + c_r m_r^u$*, and*

(b) $c = [c_1, \dots, c_r]^{\mathrm{T}} \in R^r$ *is a state feedback controller for* (4.17) *such that* $\mathscr{A}$ *is a locally (resp. globally) asymptotically stable set for* (4.17).

The above proposition does not describe how to check the stabilizability of $\mathscr{A}$ or how to stabilize $\mathscr{A}$. Nevertheless, it has a certain significance in the sense that we can separately deal with the realization of the prescribed vector field on $\mathscr{A}$ and the stabilization of $\mathscr{A}$. That is, if one wants to design a state feedback controller $u = \hat{u} \in R^m$ for (4.1) such that the resulting closed-loop system $\dot{x} = f(x) + G(x)\hat{u}(x)$ satisfies (4.7) on $\mathscr{A}$ and that $\mathscr{A}$ is locally (resp. globally) asymptotically stable for the closed-loop system, it suffices to follow this procedure:

Step 1. Compute $u^* \in R^m$ and $m_i^u \in R^m$ $(i = 1, \dots, r)$ using Algorithm 4.23 with $\Phi_i = X_i$, $\Psi_i = f_i^{\mathrm{d}}$, and $I_i = I$ for $i = 1, \dots, n'$, where $u^* + \langle m_1^u, \dots, m_r^u \rangle_R$ is the set of all solutions of Problem 4.10.

Step 2. Define system (4.17) and check the local (resp. global) asymptotic stabilizability of $\mathscr{A}$ for system (4.17). If it is locally (resp. globally) asymptotically stabilizable, proceed to the next step because there exists a controller which locally (resp. globally) asymptotically stabilizes $\mathscr{A}$ for the original system (4.1) at the same time as realizing the prescribed vector field on $\mathscr{A}$; otherwise, terminate the algorithm because such a controller does not exist.

Step 3. Obtain a controller $c = [c_1, \dots, c_r]^{\mathrm{T}} \in R^r$ such that $\mathscr{A}$ is a locally (resp. globally) asymptotically stable set for (4.17).

Step 4. Put $\hat{u} := u^* + c_1 m_1^u + \cdots + c_r m_r^u$.

Moreover, since Steps 2 and 3 are the stabilization of $\mathscr{A}$ for the input-affine polynomial system (4.17), various stabilization methods (e.g. [12,89–92]) can be employed.

4.6 Examples

Two examples are given here to demonstrate how to solve the Lie derivative inclusion with state feedback. The first example concerns Problem 4.6, the second Problem 4.10.

4.6.1 Model matching problem

Let us return to Example 4.9, and suppose we have a controlled plant of form (4.3) with

$$f^{\mathrm{p}}(\xi) = \begin{bmatrix} \xi_1^2\xi_2 \\ \xi_1^2 + \xi_3 \\ \xi_2\xi_3 \end{bmatrix}, \quad G^{\mathrm{p}}(\xi) = \begin{bmatrix} \xi_1 - 1 & 0 \\ 0 & 1 \\ 0 & \xi_3 \end{bmatrix},$$
$$h^{\mathrm{p}}(\xi) = \begin{bmatrix} h_1^{\mathrm{p}}(\xi) \\ h_2^{\mathrm{p}}(\xi) \end{bmatrix} = \begin{bmatrix} \xi_1^2 - 1 \\ \xi_2^2 + \xi_3^2 \end{bmatrix}.$$

On the other hand, suppose we have the reference model of form (4.4) with

$$f^{\mathrm{r}}(\zeta) = \begin{bmatrix} \zeta_1\zeta_2 \\ \zeta_2^2 \end{bmatrix}, \quad h^{\mathrm{r}}(\zeta) = \begin{bmatrix} h_1^{\mathrm{r}}(\zeta) \\ h_2^{\mathrm{r}}(\zeta) \end{bmatrix} = \begin{bmatrix} -\zeta_1^2 \\ -\zeta_2^2 \end{bmatrix}.$$

Now, let $R := \mathbf{R}[X_1, \ldots, X_5]$ be a polynomial ring with indeterminates $X_1, \ldots, X_5$ and $x = [x_1, \ldots, x_5]^{\mathrm{T}} := [\xi_1, \xi_2, \xi_3, \zeta_1, \zeta_2]^{\mathrm{T}} \in \mathbf{R}^5$ be the state of the extended system (4.5) with

$$\begin{aligned} f(x) &= \begin{bmatrix} f^{\mathrm{p}}(\xi) \\ f^{\mathrm{r}}(\zeta) \end{bmatrix} = \begin{bmatrix} x_1^2x_2 \\ x_1^2 + x_3 \\ x_2x_3 \\ x_4x_5 \\ x_5^2 \end{bmatrix}, \\ G(x) &= \begin{bmatrix} G^{\mathrm{p}}(\xi) \\ O \end{bmatrix} = \begin{bmatrix} x_1 - 1 & 0 \\ 0 & 1 \\ 0 & x_3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

The polynomials h_i^{p} and h_i^{r} with $i = 1, 2$ are regarded as elements of R, that is,

$$h_1^{\mathrm{p}} = X_1^2 - 1, \quad h_2^{\mathrm{p}} = X_2^2 + X_3^2, \quad h_1^{\mathrm{r}} = -X_4^2, \quad h_2^{\mathrm{r}} = -X_5^2.$$

Next, define the ideal J of R as

$$J := \langle h_1^{\mathrm{p}} - h_1^{\mathrm{r}}, h_2^{\mathrm{p}} - h_2^{\mathrm{r}} \rangle_R = \left\langle X_1^2 + X_4^2 - 1, X_2^2 + X_3^2 + X_5^2 \right\rangle_R$$

and define the algebraic set $\mathscr{A} = \left\{ x \in \mathbf{R}^5 \ : \ \rho(x) = 0 \ (\forall \rho \in J) \right\}$. The vanishing ideal I of $\mathscr{A}$ is computed as

$$I = \sqrt[\mathrm{R}]{J} = \left\langle X_1^2 + X_4^2 - 1, X_2, X_3, X_5 \right\rangle_R$$

by using the algorithms presented in [73–76]. Now consider Problem 4.6 with system (4.5) and the above $\mathscr{A}$. A solution $u = \tilde{u} \in R^m$ of the problem can be regarded as a dynamic state feedback controller defined by (4.6) such that $y^{\mathrm{p}}(\xi(0)) = y^{\mathrm{r}}(\zeta(0))$ implies $y^{\mathrm{p}}(\xi(t)) = y^{\mathrm{r}}(\zeta(t))$ for all $t \geq 0$. We can solve Problem 4.6 as follows. First, put

$$\Phi_1 := X_1^2 + X_4^2 - 1, \quad \Phi_2 := X_2, \quad \Phi_3 := X_3, \quad \Phi_4 := X_5,$$

and $\Psi_i := 0$ and $I_i := I$ for $i = 1, \dots, 4$. Next, consider inclusion (4.9) and execute Algorithm 4.23 as follows. In Step 1, we define

$$\overline{\Psi} := [-2X_1^3 X_2 - 2X_4^2 X_5, \ -X_1^2 - X_3, -X_2 X_3, \ -X_5^2]^{\mathrm{T}}$$

and

$$\begin{aligned}
\overline{g}_1 &:= \begin{bmatrix} 2X_1^2 - 2X_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \overline{g}_2 := \begin{bmatrix} 0 \\ 1 \\ X_3 \\ 0 \end{bmatrix}, \\
\overline{\rho}_{11} &:= \begin{bmatrix} X_1^2 + X_4^2 - 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \overline{\rho}_{12} := \begin{bmatrix} X_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
\overline{\rho}_{13} &:= \begin{bmatrix} X_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \overline{\rho}_{14} := \begin{bmatrix} X_5 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
\overline{\rho}_{21} &:= \begin{bmatrix} 0 \\ X_1^2 + X_4^2 - 1 \\ 0 \\ 0 \end{bmatrix}, \quad \overline{\rho}_{22} := \begin{bmatrix} 0 \\ X_2 \\ 0 \\ 0 \end{bmatrix}, \quad \dots \\
& \qquad \dots, \quad \overline{\rho}_{43} := \begin{bmatrix} 0 \\ 0 \\ 0 \\ X_3 \end{bmatrix}, \quad \overline{\rho}_{44} := \begin{bmatrix} 0 \\ 0 \\ 0 \\ X_5 \end{bmatrix}.
\end{aligned}$$

We will omit the details of Steps 3–6 because of lack of space. In executing those steps, we confirm that $\omega'' = 0$, which indicates the problem is solvable, and obtain a particular solution of (4.15):

$$\begin{aligned}\omega^* \;=\; & [X_5,\ -X_1^2 - X_3,\ -2X_5,\ -2X_1^3,\ 0,\ 2X_1 - 2, \\ & 0,\ \dots,\ 0,\ X_1^2 - X_2 + X_3,\ 0,\ \dots,\ 0,\ -X_5]^{\mathrm{T}} \in R^{18}.\end{aligned}$$

We also obtain a set of generators $\{m_1^\omega, \dots, m_r^\omega\}$ in Step 6. However, we will not write all of them down here and give only an example:

$$m_i^\omega = [X_2,\ 0,\ -2X_2, 2X_1 + 2X_4^2 - 2,\ 0,\ \dots,\ 0]^{\mathrm{T}} \in R^{18}.$$

In Step 7, the projection $\mathrm{Prj}_2^{18}(\cdot)$ yields the set $u^* + \langle m_1^u, \dots, m_8^u \rangle_R$ of all solutions of Problem 4.6 as follows:

$$\begin{aligned}
u^* &= \begin{bmatrix} X_5 \\ -X_1^2 - X_3 \end{bmatrix}, \\
m_1^u &= \begin{bmatrix} X_1^2 + X_4^2 - 1 \\ 0 \end{bmatrix}, m_2^u = \begin{bmatrix} 0 \\ X_1^2 + X_4^2 - 1 \end{bmatrix}, \\
m_3^u &= \begin{bmatrix} X_2 \\ 0 \end{bmatrix}, m_4^u = \begin{bmatrix} 0 \\ X_2 \end{bmatrix}, m_5^u = \begin{bmatrix} X_3 \\ 0 \end{bmatrix}, \\
m_6^u &= \begin{bmatrix} 0 \\ X_3 \end{bmatrix}, m_7^u = \begin{bmatrix} X_5 \\ 0 \end{bmatrix}, m_8^u = \begin{bmatrix} 0 \\ X_5 \end{bmatrix}.
\end{aligned}$$

According to Remark 4.26, any $\tilde{u}$ in (4.6) solving the present problem can be parametrized as

$$\begin{aligned}
\tilde{u}(\xi, \zeta) = \begin{bmatrix} \zeta_2 \\ -\xi_1^2 - \xi_3 \end{bmatrix} \;&+\; c_1(\xi, \zeta) \begin{bmatrix} \xi_1^2 + \zeta_1^2 - 1 \\ 0 \end{bmatrix} \\
&+\; c_2(\xi, \zeta) \begin{bmatrix} 0 \\ \xi_1^2 + \zeta_1^2 - 1 \end{bmatrix} \\
&\;\;\vdots \\
&+\; c_8(\xi, \zeta) \begin{bmatrix} 0 \\ \zeta_2 \end{bmatrix}
\end{aligned}$$

with arbitrary polynomial functions $c_i(\cdot, \cdot)$ $(i = 1, \dots, 8)$. Conversely, any $\tilde{u}$ of the above form solves the present problem.

4.6.2 Path following control

Let us return to Example 4.17, and put $R := R_3 = \mathbf{R}[X_1, X_2, X_3]$ and $x := [x_1, x_2, \theta]^{\mathrm{T}} \in \mathbf{R}^3$. Suppose we have a system of form (4.8) defined by

$$\dot{x} = f^{\mathrm{o}}(x, u) = f(x) + G(x)u \tag{4.18}$$

with

$$f(x) := \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} = \begin{bmatrix} x_1 + x_2^2 \\ x_1^2 + x_1 - 1 \\ x_2 x_3 \end{bmatrix},$$

$$G(x) := \begin{bmatrix} G_1(x) \\ G_2(x) \\ G_3(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & x_2 \\ 0 & x_3 \end{bmatrix}.$$

Let the geometric path $\mathcal{C}$ and desired velocity v be given by

$$\mathcal{C} := \{[x_1, x_2]^{\mathrm{T}} \in \mathbf{R}^2 : \ x_1^2 + x_2^2 - 1 = 0\}, \quad v(x_1, x_2) := [-x_2, \ x_1]^{\mathrm{T}}.$$

These define the motion such that the controlled object moves along the unit circle with unit speed. As is done in Example 4.17, we define the algebraic set $\mathscr{A} = \{x \in \mathbf{R}^3 : \ x_1^2 + x_2^2 - 1 = 0\}$ and the polynomials $v_1 = -X_2$ and $v_2 = X_1$. The vanishing ideal I of $\mathscr{A}$ is computed as

$$I = \sqrt[\mathrm{R}]{\langle X_1^2 + X_2^2 - 1\rangle_R} = \left\langle X_1^2 + X_2^2 - 1\right\rangle_R$$

by using the algorithms presented in [73–76]. Put

$$f^{\mathrm{d}} := \begin{bmatrix} f_1^{\mathrm{d}} \\ f_2^{\mathrm{d}} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -X_2 \\ X_1 \end{bmatrix}.$$

Now, consider Problem 4.10 with (4.18) and the above f^{d} and $\mathscr{A}$. The aim is to determine whether the path and velocity are realizable by state feedback and to compute the feedback controller explicitly. We can solve Problem 4.10 as follows. First, put

$$\Phi_1 := X_1, \quad \Phi_2 := X_2, \quad \Psi_1 := f_1^{\mathrm{d}} = -X_2, \quad \Psi_2 := f_2^{\mathrm{d}} = X_1,$$

and $I_i := I$ for $i = 1, 2$. Next, consider inclusion (4.9) and execute Algorithm 4.23 as follows. In Step 1, we define

$$\overline{\Psi} := [-X_2^2 - X_1 - X_2, \ -X_1^2 + 1]^{\mathrm{T}}$$

and

$$\overline{g}_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \overline{g}_2 := \begin{bmatrix} 0 \\ X_2 \end{bmatrix},$$

$$\overline{\rho}_{11} := \begin{bmatrix} X_1^2 + X_2^2 - 1 \\ 0 \end{bmatrix}, \quad \overline{\rho}_{21} := \begin{bmatrix} 0 \\ X_1^2 + X_2^2 - 1 \end{bmatrix}.$$

We will omit the details of Steps 3–6 because of lack of space. In executing those steps, we confirm that $\omega'' = 0$, which indicates the problem is solvable, and obtain the set $\omega^* + \langle m_1^\omega, m_2^\omega \rangle_R$ of all solutions of (4.15):

$$\begin{aligned} \omega^* &= [-X_2^2 - X_1 - X_2,\ X_2,\ 0,\ -1]^{\mathrm{T}}, \\ m_1^\omega &= [X_1^2 + X_2^2 - 1,\ 0,\ -1,\ 0]^{\mathrm{T}}, \\ m_2^\omega &= [0,\ X_1^2 + X_2^2 - 1,\ 0,\ -X_2]^{\mathrm{T}}. \end{aligned}$$

In Step 7, the projection $\mathrm{Prj}_2^4(\cdot)$ yields the set $u^* + \langle m_1^u, m_2^u \rangle_R$ of all solutions of Problem 4.10 as follows:

$$u^* = \begin{bmatrix} -X_2^2 - X_1 - X_2 \\ X_2 \end{bmatrix}, \quad m_1^u = \begin{bmatrix} X_1^2 + X_2^2 - 1 \\ 0 \end{bmatrix}, \quad m_2^u = \begin{bmatrix} 0 \\ X_1^2 + X_2^2 - 1 \end{bmatrix}.$$

According to Remark 4.26, any state feedback controller u solving the present problem can be parametrized as

$$u(x) = \begin{bmatrix} -x_2^2 - x_1 - x_2 \\ x_2 \end{bmatrix} + c_1(x) \begin{bmatrix} x_1^2 + x_2^2 - 1 \\ 0 \end{bmatrix} + c_2(x) \begin{bmatrix} 0 \\ x_1^2 + x_2^2 - 1 \end{bmatrix}$$

with arbitrary polynomial functions $c_i(\cdot)$ $(i = 1, 2)$. Conversely, any controller u of the above form solves the present problem.

Chapter 5

Lie Derivative Inclusion with Static Output Feedback

5.1 Introduction

In the previous chapter, we formulated and solved the Lie derivative inclusion with state feedback controllers. However, from the practical standpoint, output feedback controllers are more desirable because the state of a system cannot always be sensed in practical situations. Therefore, in this chapter, we extend the results given in the previous chapter to the case of static output feedback. However, the solution method presented in the previous chapter cannot be applied to solving the Lie derivative inclusion within the context of static output feedback because a solution of a linear equation over a subalgebra is needed to solve the inclusion, unlike the case of state feedback (see Section 5.3). Although an algorithm for partially solving such an equation has been proposed as a SINGULAR [93, 94] package, it requires very strict assumptions regarding the cardinality of a set of generators of the subalgebra and the algebraic independence of the set. To the best of the author's knowledge, no algorithm for fully solving a linear equation over a subalgebra has been proposed. Hence, in this chapter, we give an algorithm for solving the Lie derivative inclusion with static output feedback through more involved techniques than the previous chapter. Similarly to the state feedback case, the algorithm consists of a finite number of arithmetic operations of polynomials. Moreover, the algorithm implicitly gives a procedure for solving a linear equation over a subalgebra. As a result, all the static output-feedback controllers solving the Lie derivative inclusion can be exactly computed, even though the design of an output feedback controller is well known to be a difficult problem [95]. We also show that the realization of a prescribed vector field via static output feedback can be applied to the replacement of a state feedback controller with a static output-feedback controller. This application does not appear in the case of state feedback.

As in the previous chapter, we assume that the given system is a polynomial dynamical one and that the controller is of polynomial-type static output feedback. We also assume that the given subset of the state space is algebraic.

This chapter is organized as follows. In Section 5.2, we define the notation and describe the system treated in this chapter. After that, the two problems posed in the previous chapter, which were associated with the Lie derivative inclusion with state feedback, are reformulated in output feedback form. In Section 5.3, we formulate the Lie derivative inclusion with static output feedback and show that the two problems can be represented as the inclusion. After that, we explain the difficulty in solving the Lie derivative inclusion with static output feedback. In Section 5.4, we present the algorithm (Algorithm 5.10) for solving the inclusion and prove Theorem 5.14 to show that the algorithm indeed solves the inclusion. As a result, all the static output-feedback controllers solving the inclusion can be exactly represented by using free polynomial parameters. We also give the algorithm for computing the polynomial vectors required in Algorithm 5.10 and give the lemma required for proving Theorem 5.14. In Section 5.5, an example is given to demonstrate the implementation of Algorithm 5.10.

This chapter requires some knowledge of the theories of commutative algebras (see Chapter 2) and Gröbner bases [37–39]. This chapter includes the entire content of journal article 3 in the list of publications, which was written by the author. The copyright of the article is owned by ISCIE.

5.2 Notation, system description, and problem modification

This section defines notation, describes the system treated in this chapter, and modifies the problems posed in Chapter 4 into static output-feedback form.

First, let us define the notation. Let $S = \mathbf{R}[Y_1, \ldots, Y_p]$ be a polynomial ring over $\mathbf{R}$ with indeterminates $Y_1, \ldots, Y_p$. For a given positive integer s, let S^s denote the free S-module of all the column vectors with s components in S. For given positive integers s_1 and s_2, let $S^{s_1 \times s_2}$ denote the set of all $s_1 \times s_2$ matrices with entries in S. For a given polynomial vector $v \in S^s$, we distinguish between a polynomial vector v and a polynomial map $v(\cdot)$ according to these notations. We further distinguish between the indeterminates $Y_1, \dots, Y_n$ of v and the variables $y = [y_1, \dots, y_n]^{\mathrm{T}} \in \mathbf{R}^p$ of $v(\cdot)$. Moreover, the notation defined in Subsection 4.2.1 is also used in this chapter.

Next, let us define the system examined in this chapter as follows:

$$\begin{aligned} \dot{x} &= f^{\mathrm{o}}(x,u) = f(x) + G(x)u, \\ y &= h(x). \end{aligned} \tag{5.1}$$

Here, $f \in R^n$, $G \in R^{n\times m}$, and $h = [h_1, \dots, h_p]^{\mathrm{T}},\ h_i \in R\ (i = 1, \dots, p)$. We write $f^{\mathrm{o}}(\cdot,\cdot) = [f_1^{\mathrm{o}}(\cdot,\cdot), \dots, f_n^{\mathrm{o}}(\cdot,\cdot)]^{\mathrm{T}}$.

Now, let us modify the problems posed in Chapter 4 into static output-feedback form. The first one is to render a prescribed subset invariant, and the second one is to realize a prescribed vector field.

Consider system (5.1) and let us define the notion of controlled invariance by static output feedback.

Definition 5.1. A set $\mathscr{A} \subset \mathbf{R}^n$ is said to be *controlled invariant by static output feedback* for system (5.1) if there exists a static output-feedback controller $u \in S^m$ such that $\mathscr{A}$ is invariant for the resulting closed-loop system of (5.1).

We shall consider the following problem associated with Problem 4.6.

Problem 5.2. *Let $I = \langle \rho_1, \dots, \rho_k \rangle_R \subset R$ be the vanishing ideal of a given algebraic set $\mathscr{A} \subset \mathbf{R}^n$. Then, given a system of form* (5.1)*, determine whether $\mathscr{A}$ is controlled invariant by static output feedback for system (5.1). Moreover, if $\mathscr{A}$ is controlled invariant by static output feedback, obtain a static output-feedback controller $u \in S^m$ for system (5.1) such that $\mathscr{A}$ is invariant for the resulting closed-loop system.*

Definition 5.3. A *solution* of Problem 5.2 is a static output-feedback controller $u \in S^m$ such that $\mathscr{A}$ is invariant for the resulting closed-loop system of (5.1). Problem 5.2 is said to be *solvable* if a solution to it exists.

As stated in Subsection 4.2.2, the problem of state constraint and that of model matching can be represented by Problem 5.2.

The second problem is associated with Problem 4.10, which is described as follows.

Problem 5.4. *Let $I = \langle \rho_1, \dots, \rho_k \rangle_R \subset R$ be the vanishing ideal of a given algebraic set $\mathscr{A} \subset \mathbf{R}^n$. Given a system of form* (5.1) *and a polynomial vector*

$f^{\mathrm{d}} = [f^{\mathrm{d}}_1, \ldots, f^{\mathrm{d}}_{n'}]^{\mathrm{T}} \in R^{n'}$ with a given positive integer $n' \leq n$, determine whether there exists a static output-feedback controller $u \in S^m$ such that

$$\begin{aligned} f^{\mathrm{d}}_1(x) &= f^{\mathrm{o}}_1(x, u(h(x))), \\ &\vdots \\ f^{\mathrm{d}}_{n'}(x) &= f^{\mathrm{o}}_{n'}(x, u(h(x))), \end{aligned} \tag{5.2}$$

for all $x \in \mathscr{A}$. Moreover, if such a controller exists, it should be able to be computed explicitly.

Definition 5.5. A *solution* of Problem 5.4 is a static output-feedback controller $u \in S^m$ satisfying (5.2). Problem 5.4 is said to be *solvable* if a solution to it exists.

As stated in Subsection 4.2.3, the problem of realizing a prescribed periodic orbit and that of a path-following control can be represented by Problem 5.4. We give another problem represented by Problem 5.4 in the following example, which is the replacement of a state feedback controller with a static output-feedback controller.

Example 5.6. (Replacement of a state feedback controller with a static output-feedback controller) Consider system (5.1) and suppose that a state feedback controller $u_X \in R^m$ for the system has already been given. Put

$$f^{\mathrm{d}} := f + Gu_X \quad \text{and} \quad I := \{0\}.$$

Then, Problem 5.4 represents the problem of obtaining a static output-feedback controller $u \in S^m$ such that the vector field of the resulting closed-loop system coincides with that of the closed-loop system with u_X.

5.3 Lie derivative inclusion with static output feedback

In this section, we formulate the Lie derivative inclusion with static output feedback, show that Problems 5.2 and 5.4 can be represented by the inclusion, and explain the difficulty in solving the inclusion.

First, we formulate the Lie derivative inclusion with static output feedback. For a given positive integer r, let $\mathrm{Sbs}^r_{Y \mapsto h} : S^r \to R^r$ denote the map defined by the substitution $Y_i \mapsto h_i$ $(i = 1, \ldots, p)$. The Lie derivative inclusion is defined as follows.

Definition 5.7. Let Φ_i, Ψ_i $(i = 1, \ldots, \ell)$ be given polynomials in R and I be a given ideal of R. Then, consider the system of inclusions

$$\begin{aligned} L_{f+G\mathrm{Sbs}^m_{Y\mapsto h}(u)}\Phi_1 &\in \Psi_1 + I, \\ &\vdots \\ L_{f+G\mathrm{Sbs}^m_{Y\mapsto h}(u)}\Phi_\ell &\in \Psi_\ell + I \end{aligned} \tag{5.3}$$

with $u \in S^m$ to be found. We call the above system the *Lie derivative inclusion with static output feedback*. A *solution* of (5.3) is a static output-feedback controller $u \in S^m$ satisfying (5.3). Inclusion (5.3) is said to be *solvable* if a solution to it exists.

Remark 5.8. In the state feedback case, ideals are given independently for each inclusion in (5.3). However, to solve Problems 5.2 and 5.4, it suffices to use a single ideal I as inclusion (5.3). This treatment allows (5.3) to be calculated in the factor ring $\overline{R} = R/I$. See the proof of Theorem 5.14 in Section 5.4.

Now, we demonstrate how Problems 5.2 and 5.4 can be represented by the Lie derivative inclusion with static output feedback.

Let $\mathscr{A} \subset \mathbf{R}^n$ be a given algebraic set and $I = \langle \rho_1, \ldots, \rho_k \rangle_R$ be the vanishing ideal of $\mathscr{A}$. The following corollary is a straightforward consequence of Theorem 4.21.

Corollary 5.9. *A given algebraic set $\mathscr{A}$ is invariant for the resulting closed-loop system of* (5.1) *with a controller $u \in S^m$ if and only if u satisfies the inclusion*

$$L_{f+G\mathrm{Sbs}^m_{Y\mapsto h}(u)}\rho_i \in I$$

for $i = 1, \ldots, k$.

From this corollary, we have that the solutions of Problem 5.2 coincide with the solutions of inclusion (5.3) with $\Phi_i = \rho_i$ and $\Psi_i = 0$ for $i = 1, \ldots, k$.

Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbf{R}^n$ and rewrite (5.2) as follows:

$$\begin{aligned} f^{\mathrm{d}}_1(x) &= e^{\mathrm{T}}_1 \left(f(x) + G(x)u(h(x))\right), \\ &\vdots \\ f^{\mathrm{d}}_{n'}(x) &= e^{\mathrm{T}}_{n'} \left(f(x) + G(x)u(h(x))\right). \end{aligned}$$

Problem 5.4 requires the above set of equations to be satisfied for all $x \in \mathscr{A}$. This is equivalent to the following inclusions of polynomials:

$$\begin{aligned} e^{\mathrm{T}}_1 \left(f + G\mathrm{Sbs}^m_{Y\mapsto h}(u)\right) &\in f^{\mathrm{d}}_1 + I, \\ &\vdots \\ e^{\mathrm{T}}_{n'} \left(f + G\mathrm{Sbs}^m_{Y\mapsto h}(u)\right) &\in f^{\mathrm{d}}_{n'} + I. \end{aligned}$$

The above inclusions can be derived from inclusion (5.3) by setting $\Phi_i = X_i$ and $\Psi_i = f_i^{\mathrm{d}}$ for $i = 1, \dots, n'$. That is, the solutions of Problem 5.4 coincide with the solutions of inclusion (5.3) with $\Phi_i = X_i$ and $\Psi_i = f_i^{\mathrm{d}}$ for $i = 1, \dots, n'$.

Finally, we explain the difficulty in solving the Lie derivative inclusion with static output feedback. Given positive integers s and r, consider the linear equation

$$\gamma_0 = z_1\gamma_1 + \cdots + z_r\gamma_r$$

with given polynomial vectors $\gamma_i \in R^s$ $(i = 0, \dots, r)$, where $z := [z_1, \dots, z_r]^{\mathrm{T}} \in R^r$ is to be found. In Chapter 4, it was shown that solving the Lie derivative inclusion with state feedback reduces to solving the above equation. Moreover, the above equation can be solved by using the technique of Gröbner bases. On the other hand, in the case of static output feedback, the z_i's are taken to be elements of S rather than R. Although an algorithm for partially solving such an equation has been proposed as a Singular package (see [93]), it requires very strict assumptions that $p = n$ holds and $\{h_1, \dots, h_p\}$ is algebraically independent. To the best of the author's knowledge, no algorithm for fully solving such an equation has been proposed. Hence, in this chapter, we will give an algorithm for solving the Lie derivative inclusion with static output feedback through more involved techniques than that with state feedback. The algorithm implicitly gives a procedure for solving such equations where the z_i's are taken to be elements in S.

5.4 Solution to the Lie derivative inclusion with static output feedback

In this section, we will describe an algorithm for solving the Lie derivative inclusion with static output feedback. Algorithm 5.10 is the algorithm for solving the inclusion. Theorem 5.14 shows that the algorithm indeed solves the inclusion. As a result, all the required static output-feedback controllers can be represented by using free polynomial parameters. The algorithm can also be used to solve Problems 5.2 and 5.4, as explained in the previous sections.

5.4.1 Solution algorithm

Before we describe the algorithm, let us define the following notation. Let $\overline{R} = R/I$ be the factor ring of R modulo I, and $H = \mathbf{R}[h_1, \dots, h_p]$ be the subalgebra of R generated by $\{h_1, \dots, h_p\}$. For any element $a \in R$, let $\overline{a}$ denote the equivalence class $[a] \in \overline{R}$. Similarly, for any element $\xi = [\xi_1, \ldots, \xi_s]^{\mathrm{T}} \in R^s$ with some integer $s \geq 1$, we use the same notation $\overline{\xi}$ to describe the element $[\overline{\xi}_1, \ldots, \overline{\xi}_s]^{\mathrm{T}} \in \overline{R}^s$. Let $\overline{H}$ be the subalgebra $\mathbf{R}[\overline{h}_1, \ldots, \overline{h}_p]$ of $\overline{R}$ generated by $\{\overline{h}_1, \ldots, \overline{h}_p\}$. Furthermore, let $\mathrm{Fct}^I : R^{m+1} \to \overline{R}^{m+1}$ be the map defined by $R^{m+1} \ni \xi \mapsto \overline{\xi} \in \overline{R}^{m+1}$. For a given positive integer r, let $\mathrm{Sbs}^r_{Y \mapsto h} : S^r \to R^r$ denote the map defined by the substitution $Y_i \mapsto h_i$ $(i = 1, \ldots, p)$, as was defined before.

Let P be either of the rings R, S, or $\overline{R}$. We further define the following notation. For a given ideal $J \subset P$ and a given integer $s \geq 1$, let $J^{\oplus s}$ be the direct sum of s copies of J, i.e.,

$$J^{\oplus s} = \{\xi = [\xi_1, \ldots, \xi_s]^{\mathrm{T}} \in P^s : \ \xi_i \in J \ (i = 1, \ldots, s)\}.$$

Note that, when J is described by $J = \langle \mu_1, \ldots, \mu_k \rangle_P$ with generators $\{\mu_1, \ldots, \mu_k\}$, we can describe $J^{\oplus s}$ in the explicit form

$$J^{\oplus s} = \langle \mu_1 e_1, \ldots, \mu_k e_1 \rangle_P + \cdots + \langle \mu_1 e_s, \ldots, \mu_k e_s \rangle_P ,$$

where $\{e_1, \ldots, e_s\}$ is the canonical basis of P^s. In particular, $J^{\oplus s}$ is a P-submodule of P^s. For given polynomial vectors $v_i \in P^s$ $(i = 1, \ldots, r)$ with some $r \geq 1$, let $\mathrm{Syz}^P(v_1, \ldots, v_r) \subset P^r$ be the syzygy module of the r-tuple $(v_1, \ldots, v_r)$, i.e.,

$$\mathrm{Syz}^P(v_1, \ldots, v_r) := \Big\{\zeta = [\zeta_1, \ldots, \zeta_r]^{\mathrm{T}} \in P^r : \zeta_1 v_1 + \cdots + \zeta_r v_r = 0\Big\}.$$

For given integers s_1 and s_2 such that $s_1 \geq s_2 \geq 1$, let $\mathrm{Prj}^{s_1}_{s_2} : P^{s_1} \to P^{s_2}$ be the projection map defined by $P^{s_1} \ni [\omega_1, \ldots, \omega_{s_1}]^{\mathrm{T}} \mapsto [\omega_1, \ldots, \omega_{s_2}]^{\mathrm{T}} \in P^{s_2}$.

We will give the algorithm for solving inclusion (5.3) in what follows. As we will see in Theorem 5.14, the algorithm checks the solvability of (5.3) and computes the set of all solutions of (5.3). The proof of Theorem 5.14 indicates that the solutions $u = [u_1, \ldots, u_m]^{\mathrm{T}} \in S^m$ of (5.3) coincide with the solutions of the inclusion

$$e_1 \in u_1 e_2 + \cdots + u_m e_{m+1} + \langle \eta_1, \ldots, \eta_\nu \rangle_S + (\langle \kappa_1, \ldots, \kappa_{\nu'} \rangle_S)^{\oplus m+1} ,$$

where $\{e_1, \ldots, e_{m+1}\}$ is the canonical basis of S^{m+1}, $\eta_i \in S^{m+1}$ $(i = 1, \ldots, \nu)$ are the polynomial vectors computed in Step 1 in Algorithm 5.10, and $\kappa_i \in S$ $(i = 1, \ldots, \nu')$ are the polynomials computed in Steps 2 and 3 in the algorithm. Hence, the algorithm for solving (5.3) is based on solving the above inclusion. The algorithm is as follows.

Algorithm 5.10 (For solving the Lie derivative inclusion with static output feedback)**.**

Given: the polynomial vector $f \in R^n$, the polynomial matrix $G \in R^{n\times m}$, and the polynomials $h_i \in R$ $(i = 1, \ldots, p)$ of system (5.1). Polynomials $\Phi_i \in R$ and $\Psi_i \in R$ with $i = 1, \ldots, \ell$, and an ideal $I = \langle \rho_1, \ldots, \rho_k \rangle_R \subset R$.

Obtain: a polynomial vector $u^* \in S^m$ and a set $\{m_1^u, \ldots, m_r^u\}$ of polynomial vectors $m_i^u \in S^m$ $(i = 1, \ldots, r)$, where $u^* + \langle m_1^u, \ldots, m_r^u \rangle_S$ is the set of all solutions of (5.3).

Step 1. Compute a set $\{\eta_1, \ldots, \eta_\nu\}$ of polynomial vectors $\eta_i \in S^{m+1}$ $(i = 1, \ldots, \nu)$ by using Algorithm 5.16 in Subsection 5.4.2. Write $\eta_i = [\eta_{(i,1)}, \ldots, \eta_{(i,m+1)}]^{\mathrm{T}}$ $(i = 1, \ldots, \nu)$.

Step 2. Define the $\mathbf{R}$-algebra homomorphism $\pi : S \to \overline{R}$ by

$$S \ni \alpha(Y_1, \ldots, Y_p) \longmapsto \overline{\alpha(h_1, \ldots, h_p)} \in \overline{R}. \tag{5.4}$$

Step 3. Compute a set of generators $\{\kappa_1, \ldots, \kappa_{\nu'}\}$ of $\ker \pi$ by using Algorithm A.6 in the appendix.

Step 4. Check if

$$1 \in \left\langle \eta_{(1,1)}, \ldots, \eta_{(\nu,1)} \right\rangle_S + \langle \kappa_1, \ldots, \kappa_{\nu'} \rangle_S . \tag{5.5}$$

If the above inclusion holds, proceed to the next step because (5.3) is solvable; otherwise, terminate the algorithm because (5.3) is not solvable.

Step 5. Let $\{e_1, \ldots, e_{m+1}\}$ be the canonical basis in S^{m+1}.

Step 6. Set $d = \nu + (m+1)\nu'$, and define the equation

$$\begin{aligned} e_1 = {} & z_1 e_2 + \cdots + z_m e_{m+1} + (z_{m+1}\eta_1 + \cdots + z_{m+\nu}\eta_\nu) \\ & + (z_{m+\nu+1}\kappa_1 e_1 + \cdots + z_{m+\nu+\nu'}\kappa_{\nu'} e_1) \\ & \vdots \\ & + (z_{m+\nu+m\nu'+1}\kappa_1 e_{m+1} + \cdots + z_{m+d}\kappa_{\nu'} e_{m+1}) \end{aligned}$$

with the variables $z := [z_1, \ldots, z_{m+d}]^{\mathrm{T}} \in S^{m+d}$.

Step 7. By solving the above equation using Algorithm A.8 in the appendix, obtain polynomial vectors $z^* \in S^{m+d}$ and $m_i^z \in S^{m+d}$ $(i = 1, \ldots, r)$, where $z^* + \langle m_1^z, \ldots, m_r^z \rangle_S$ is the set of all solutions $z \in S^{m+d}$ of the equation.

Step 8. Put $u^* := \mathrm{Prj}_m^{m+d}(z^*) \in S^m$ and $m_i^u := \mathrm{Prj}_m^{m+d}(m_i^z) \in S^m$ $(i = 1, \ldots, r)$.

Remark 5.11. Checking if inclusion (5.5) holds is the ideal membership problem, which can be exactly solved [37, 48].

Remark 5.12. All the calculations in the above algorithm are not approximate but exact. Moreover, the algorithm consists of a finite number of arithmetic operations of polynomials.

Remark 5.13. Although the above algorithm is applicable to the Lie derivative inclusion with state feedback, Algorithm 4.23 is more preferable to solve it because the computation of the above one is more complicated than that of Algorithm 4.23.

We shall prove that the above algorithm indeed solves the Lie derivative inclusion.

Theorem 5.14. *The Lie derivative inclusion with static output feedback is solvable if and only if* (5.5) *holds. Moreover,* $u^* + \langle m_1^u, \ldots, m_r^u \rangle_S$ *is the set of all solutions of the Lie derivative inclusion with static output feedback.*

Proof. Let $\pi : S \to \overline{R}$ be the $\mathbf{R}$-algebra homomorphism defined by (5.4) as is in Algorithm 5.10. For a given integer $s \geq 1$, the $\overline{R}$-module $\overline{R}^s$ can be regarded as an S-module with the law of action $S \times \overline{R}^s \ni (\alpha, \zeta) \mapsto \pi(\alpha)\zeta \in \overline{R}^s$. We henceforth regard $\overline{R}^s$ as an S-module in such a way. The $\mathbf{R}$-algebra homomorphism π induces the S-module homomorphism $\Pi : S^{m+1} \to \overline{R}^{m+1}$ defined by

$$[\zeta_1, \ldots, \zeta_{m+1}]^{\mathrm{T}} \longmapsto [\pi(\zeta_1), \ldots, \pi(\zeta_{m+1})]^{\mathrm{T}}.$$

Note that $\Pi = \mathrm{Fct}^I \circ \mathrm{Sbs}_{Y \mapsto h}^{m+1}$, $\mathrm{Im}\,\Pi = \overline{H}^{m+1}$, and $\ker \Pi = (\ker \pi)^{\oplus m+1}$. Inclusion (5.3) can be transformed into

$$\Psi_1 - \frac{\partial \Phi_1}{\partial X} f \in \left(\frac{\partial \Phi_1}{\partial X} G \right) \mathrm{Sbs}_{Y \mapsto h}^m(u) + I,$$
$$\vdots$$
$$\Psi_\ell - \frac{\partial \Phi_\ell}{\partial X} f \in \left(\frac{\partial \Phi_\ell}{\partial X} G \right) \mathrm{Sbs}_{Y \mapsto h}^m(u) + I.$$

This is equivalent to the equation

$$\begin{aligned} \overline{\Psi_1 - \frac{\partial \Phi_1}{\partial X} f} &= \overline{\left(\frac{\partial \Phi_1}{\partial X} G\right)} \overline{\mathrm{Sbs}^m_{Y \mapsto h}(u)}, \\ &\vdots \\ \overline{\Psi_\ell - \frac{\partial \Phi_\ell}{\partial X} f} &= \overline{\left(\frac{\partial \Phi_\ell}{\partial X} G\right)} \overline{\mathrm{Sbs}^m_{Y \mapsto h}(u)}. \end{aligned} \tag{5.6}$$

Now, put

$$\begin{aligned} \widetilde{\Psi} &:= \begin{bmatrix} \overline{\Psi_1 - \frac{\partial \Phi_1}{\partial X} f} \\ \vdots \\ \overline{\Psi_\ell - \frac{\partial \Phi_\ell}{\partial X} f} \end{bmatrix} \in \overline{R}^\ell, \\ \widetilde{G} &:= [\widetilde{g}_1, \ldots, \widetilde{g}_m] := \begin{bmatrix} \overline{\frac{\partial \Phi_1}{\partial X} G} \\ \vdots \\ \overline{\frac{\partial \Phi_\ell}{\partial X} G} \end{bmatrix} \in \overline{R}^{\ell \times m}, \\ u &= \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \in S^m. \end{aligned} \tag{5.7}$$

Then, (5.6) is transformed into

$$\widetilde{\Psi} = \overline{\mathrm{Sbs}^1_{Y \mapsto h}(u_1)} \widetilde{g}_1 + \cdots + \overline{\mathrm{Sbs}^1_{Y \mapsto h}(u_m)} \widetilde{g}_m = \pi(u_1)\widetilde{g}_1 + \cdots + \pi(u_m)\widetilde{g}_m.$$

Hence, the solutions of (5.3) are identical to the solutions $u = [u_1, \ldots, u_m]^{\mathrm{T}} \in S^m$ satisfying the equation

$$\widetilde{\Psi} = u_1 \widetilde{g}_1 + \cdots + u_m \widetilde{g}_m \tag{5.8}$$

with the law of action of the S-module $\overline{R}^\ell$. Now, consider the S-submodule $M := \left\langle \widetilde{\Psi}, \widetilde{g}_1, \ldots, \widetilde{g}_m \right\rangle_S \subset \overline{R}^\ell$. Let $\{e_1, \ldots, e_{m+1}\}$ be the canonical basis of S^{m+1}, and define the surjective S-module homomorphism $\chi : S^{m+1} \to M$ by $e_1 \mapsto \widetilde{\Psi}$, $e_2 \mapsto \widetilde{g}_1, \ldots,$ and $e_{m+1} \mapsto \widetilde{g}_m$. By the homomorphism theorem, we have $S^{m+1}/N \simeq M$, where $N = \ker \chi$. Using this isomorphism, we have that (5.8) is equivalent to the equation

$$e_1 = u_1 e_2 + \cdots + u_m e_{m+1} \mod N.$$

That is, the solutions of (5.3) are identical to the solutions $u = [u_1, \ldots, u_m]^{\mathrm{T}} \in S^m$ satisfying the inclusion

$$e_1 \in u_1 e_2 + \cdots + u_m e_{m+1} + N. \tag{5.9}$$

Now, let $\{\eta_1, \ldots, \eta_\nu\}$ and $\{\kappa_1, \ldots, \kappa_{\nu'}\}$ be the sets computed in Step 1 and Steps 2–3 in Algorithm 5.10, respectively. Note that $\langle \kappa_1, \ldots, \kappa_{\nu'} \rangle_S = \ker \pi$. Then, it can be verified that

$$N = \langle \eta_1, \ldots, \eta_\nu \rangle_S + (\langle \kappa_1, \ldots, \kappa_{\nu'} \rangle_S)^{\oplus m+1} \tag{5.10}$$

as follows. By definition, we have

$$\begin{aligned}
N &= \ker \chi \\
&= \Big\{ [\zeta_1, \ldots, \zeta_{m+1}]^{\mathrm{T}} \in S^{m+1} : \zeta_1 \widetilde{\Psi} + \zeta_2 \widetilde{g}_1 + \cdots + \zeta_{m+1} \widetilde{g}_m = 0 \\
&\qquad\qquad \text{under the law of action of the } S\text{-module } \overline{R}^{\ell} \Big\} \\
&= \Big\{ [\zeta_1, \ldots, \zeta_{m+1}]^{\mathrm{T}} \in S^{m+1} : \ \pi(\zeta_1) \widetilde{\Psi} + \pi(\zeta_2) \widetilde{g}_1 + \cdots + \pi(\zeta_{m+1}) \widetilde{g}_m = 0 \\
&\qquad\qquad \text{under the law of action of the } \overline{R}\text{-module } \overline{R}^{\ell} \Big\} \\
&= \Big\{ \zeta \in S^{m+1} : \ \Pi(\zeta) \in \mathrm{Syz}^{\overline{R}}(\widetilde{\Psi}, \widetilde{g}_1, \ldots, \widetilde{g}_m) \cap \overline{H}^{m+1} \Big\} \\
&= \Pi^{-1} \Big(\mathrm{Syz}^{\overline{R}}(\widetilde{\Psi}, \widetilde{g}_1, \ldots, \widetilde{g}_m) \cap \overline{H}^{m+1} \Big).
\end{aligned}$$

On the other hand, by Lemma 5.17 in Subsection 5.4.2, we have

$$\Pi \left(\langle \eta_1, \ldots, \eta_\nu \rangle_S \right) = \mathrm{Syz}^{\overline{R}}(\widetilde{\Psi}, \widetilde{g}_1, \ldots, \widetilde{g}_m) \cap \overline{H}^{m+1}.$$

Hence, we have

$$N = \langle \eta_1, \ldots, \eta_\nu \rangle_S + \ker \Pi = \langle \eta_1, \ldots, \eta_\nu \rangle_S + (\ker \pi)^{\oplus m+1},$$

which implies (5.10). Now, let us consider (5.9). The above expression for N implies that there exists a solution u satisfying (5.9) if and only if the following inclusion holds:

$$\begin{aligned}
e_1 \in\ & \langle e_2, \ldots, e_{m+1} \rangle_S + \langle \eta_1, \ldots, \eta_\nu \rangle_S \\
& + \langle \kappa_1 e_1, \ldots, \kappa_{\nu'} e_1 \rangle_S + \cdots + \langle \kappa_1 e_{m+1}, \ldots, \kappa_{\nu'} e_{m+1} \rangle_S .
\end{aligned} \tag{5.11}$$

Noting that

$$\langle e_2, \dots, e_{m+1} \rangle_S \supset \langle \kappa_1 e_2, \dots, \kappa_{\nu'} e_2 \rangle_S + \cdots + \langle \kappa_1 e_{m+1}, \dots, \kappa_{\nu'} e_{m+1} \rangle_S \,,$$

we have that (5.11) is equivalent to

$$e_1 \in \langle e_2, \dots, e_{m+1} \rangle_S + \langle \eta_1, \dots, \eta_\nu \rangle_S + \langle \kappa_1 e_1, \dots, \kappa_{\nu'} e_1 \rangle_S \,.$$

Write $\eta_i = [\eta_{(i,1)}, \dots, \eta_{(i,m+1)}]^{\mathrm{T}}$ $(i = 1, \dots, \nu)$. Then, by comparing the positions of components, we have that the above inclusion is equivalent to the inclusion

$$1 \in \left\langle \eta_{(1,1)}, \dots, \eta_{(\nu,1)} \right\rangle_S + \langle \kappa_1, \dots, \kappa_{\nu'} \rangle_S \,.$$

Therefore, the first statement of the theorem follows. In what follows, we assume that (5.3) is solvable. Put $d := \nu + (m+1)\nu'$, and consider the equation

$$\begin{aligned} e_1 = {} & z_1 e_2 + \ldots + z_m e_{m+1} + (z_{m+1}\eta_1 + \cdots + z_{m+\nu}\eta_\nu) \\ & + (z_{m+\nu+1}\kappa_1 e_1 + \cdots + z_{m+\nu+\nu'}\kappa_{\nu'} e_1) \\ & \vdots \\ & + (z_{m+\nu+m\nu'+1}\kappa_1 e_{m+1} + \cdots + z_{m+d}\kappa_{\nu'} e_{m+1}) \end{aligned} \tag{5.12}$$

with the variables $z := [z_1, \dots, z_{m+d}]^{\mathrm{T}} \in S^{m+d}$. A solution of (5.9) is nothing but the image of the projection $\mathrm{Prj}_m^{m+d} : S^{m+d} \to S^m$ of a solution z of (5.12). Since (5.12) is a linear equation with coefficients in the polynomial ring S, we can exactly compute the set of all solutions of (5.12) by Algorithm A.8 in the appendix as

$$z^* + \langle m_1^z, \dots, m_r^z \rangle_S \,,$$

where $z^* \in S^{m+d}$ and $m_i^z \in S^{m+d}$ $(i = 1, \dots, r)$ with some $r \geq 1$. Putting $u^* := \mathrm{Prj}_m^{m+d}(z^*) \in S^m$ and $m_i^u := \mathrm{Prj}_m^{m+d}(m_i^z) \in S^m$ $(i = 1, \dots, r)$, we obtain the set $u^* + \langle m_1^u, \dots, m_r^u \rangle_S$ of all solutions of (5.9), i.e., the set of all solutions of (5.3). □

Remark 5.15. Theorem 5.14 can be interpreted as follows: let $u^*, m_1^u, \dots$ and m_r^u be as above. Then, any static output-feedback controller $u \in S^m$ satisfying (5.3) can be parametrized in the form

$$u = u^* + c_1 m_1^u + \ldots + c_r m_r^u \tag{5.13}$$

with arbitrary polynomial parameters $c_i \in S$ $(i = 1, \dots, r)$. Conversely, any controller u of the above form satisfies inclusion (5.3).

5.4.2 Computing the set $\{\eta_1, ..., \eta_\ell\}$ in Algorithm 5.10

We use the notation defined so far. The following algorithm, which is a modification of Algorithms A.2 and A.4 in the appendix, is used to derive Algorithm 5.10 in the previous subsection.

Algorithm 5.16.

Given: the polynomial vector $f \in R^n$, the polynomial matrix $G \in R^{n\times m}$, and the polynomials $h_i \in R$ $(i = 1, \dots, p)$ of system (5.1). Polynomials $\Phi_i \in R$ and $\Psi_i \in R$ with $i = 1, \dots, \ell$, and an ideal $I = \langle \rho_1, \dots, \rho_k\rangle_R \subset R$.

Obtain: a set $\{\eta_1, \dots, \eta_\nu\}$ of polynomial vectors $\eta_i \in S^{m+1}$ $(i = 1, \dots, \nu)$.

Step 1. Put

$$\Psi' := \begin{bmatrix} \Psi_1 - \frac{\partial \Phi_1}{\partial X} f \\ \vdots \\ \Psi_\ell - \frac{\partial \Phi_\ell}{\partial X} f \end{bmatrix} \in R^\ell,$$

$$G' := [g'_1, \dots, g'_m] := \begin{bmatrix} \frac{\partial \Phi_1}{\partial X} G \\ \vdots \\ \frac{\partial \Phi_\ell}{\partial X} G \end{bmatrix} \in R^{\ell\times m}.$$

Step 2. Let $\{e_1, \dots, e_\ell\}$ be the canonical basis of R^ℓ, and compute a set of generators $\{b'_1, \dots, b'_{\ell'}\}$ of the syzygy module

$$\mathrm{Syz}^R(\Psi', g'_1, \dots, g'_m, \rho_1 e_1, \dots, \rho_k e_1, \rho_1 e_2, \dots, \rho_k e_2, \dots, \rho_1 e_\ell, \dots, \rho_k e_\ell),$$

using the algorithm given in [37].

Step 3. Put $b_i := \mathrm{Prj}_{m+1}^{m+1+k\ell}(b'_i) \in R^{m+1}$ $(i = 1, \dots, \ell')$.

Step 4. Let $\{e_1, \dots, e_{m+1}\}$ be the canonical basis of R^{m+1}, and define the R-submodule

$$\begin{aligned} L &= \langle b_1, \dots, b_{\ell'}\rangle_R + I^{\oplus m+1} \\ &= \left\langle b_1, \dots, b_{\ell'}, \rho_1 e_1, \dots, \rho_k e_1, \dots, \rho_1 e_{m+1}, \dots, \rho_k e_{m+1}\right\rangle_R. \end{aligned}$$

Step 5. Set $D = \mathbf{R}[X_1, \dots, X_n, Y_1, \dots, Y_p]$.

<u>Step 6.</u> Let $\{e_1, \ldots, e_{m+1}\}$ be the canonical basis of D^{m+1}, and define the D-submodule

$$\begin{aligned} L' = &\left\langle b_1, \ldots, b_{\ell'}, \rho_1 e_1, \ldots, \rho_k e_1, \ldots, \rho_1 e_{m+1}, \ldots, \rho_k e_{m+1} \right\rangle_D \\ &+ \langle (Y_1 - h_1)e_1, \ldots, (Y_p - h_p)e_1 \rangle_D \\ &\;\vdots \\ &+ \langle (Y_1 - h_1)e_{m+1}, \ldots, (Y_p - h_p)e_{m+1} \rangle_D . \end{aligned}$$

<u>Step 7.</u> Compute a Gröbner basis $\mathcal{L}$ of L' w.r.t. an elimination ordering for $\{X_1, \ldots, X_n\}$.

<u>Step 8.</u> Put $\{\eta_1, \ldots, \eta_\nu\} := \mathcal{L} \cap S^{m+1}$.

Lemma 5.17. *Let $\{\eta_1, \ldots, \eta_\nu\}$ be the set computed by Algorithm 5.16. Then,*

$$\Pi\left(\langle \eta_1, \ldots, \eta_\nu \rangle_S\right) = \mathrm{Syz}^{\overline{R}}(\widetilde{\Psi}, \widetilde{g}_1, \ldots, \widetilde{g}_m) \cap \overline{H}^{m+1}. \tag{5.14}$$

Proof. Note that $\Pi = \mathrm{Fct}^I \circ \mathrm{Sbs}^{m+1}_{Y \mapsto h}$. Algorithm 5.16 can be divided into two parts: the first half (Steps 1–3) corresponds to Algorithm A.2 and the latter half (Steps 3–7) corresponds to Algorithm A.4. Consider the set $\{b_1, \ldots, b_{\ell'}\}$ in Algorithm 5.16. Since the construction of $\{b_1, \ldots, b_{\ell'}\}$ in Algorithm 5.16 corresponds to Algorithm A.2 with $(v_1, \ldots, v_{m+1}) := (\Psi', g'_1, \ldots, g'_m)$, we have

$$\left\langle \overline{b}_1, \ldots, \overline{b}_{\ell'} \right\rangle_{\overline{R}} = \mathrm{Syz}^{\overline{R}}(\overline{\Psi'}, \overline{g'_1}, \ldots, \overline{g'_m}) = \mathrm{Syz}^{\overline{R}}(\widetilde{\Psi}, \widetilde{g}_1, \ldots, \widetilde{g}_m). \tag{5.15}$$

On the other hand, since the construction of $\{\eta_1, \ldots, \eta_\nu\}$ in Algorithm 5.16 corresponds to Algorithm A.4 with $L := \langle b_1, \ldots, b_{\ell'} \rangle_R + I^{\oplus m+1}$, we have

$$\begin{aligned} \mathrm{Sbs}^{m+1}_{Y \mapsto h}\left(\langle \eta_1, \ldots, \eta_\nu \rangle_S\right) &= \left\langle \mathrm{Sbs}^{m+1}_{Y \mapsto h}(\eta_1), \ldots, \mathrm{Sbs}^{m+1}_{Y \mapsto h}(\eta_\nu) \right\rangle_H \\ &= L \cap H^{m+1} \\ &= \left(\langle b_1, \ldots, b_{\ell'} \rangle_R + I^{\oplus m+1}\right) \cap H^{m+1}. \end{aligned} \tag{5.16}$$

Now, we prove that

$$\mathrm{Fct}^I\left(L \cap H^{m+1}\right) = \mathrm{Syz}^{\overline{R}}(\widetilde{\Psi}, \widetilde{g}_1, \ldots, \widetilde{g}_m) \cap \overline{H}^{m+1} \tag{5.17}$$

as follows. Take an arbitrary $\xi \in R^{m+1}$ such that $\xi \in L \cap H^{m+1}$. Since $\xi \in H^{m+1}$, it is obvious that $\mathrm{Fct}^I(\xi) = \overline{\xi} \in \overline{H}^{m+1}$. Moreover, since

$$\xi \in L = \langle b_1, \ldots, b_{\ell'} \rangle_R + I^{\oplus m+1},$$

there exist $a_i \in R$ $(i = 1, \ldots, \ell')$ and $\theta \in I^{\oplus m+1}$ such that

$$\xi = a_1 b_1 + \cdots + a_{\ell'} b_{\ell'} + \theta.$$

Hence, we have

$$\mathrm{Fct}^I(\xi) = \overline{\xi} = \overline{a}_1 \overline{b}_1 + \cdots + \overline{a}_{\ell'} \overline{b}_{\ell'} \in \left\langle \overline{b}_1, \ldots, \overline{b}_{\ell'} \right\rangle_{\overline{R}},$$

which implies $\mathrm{Fct}^I(\xi) \in \mathrm{Syz}^{\overline{R}}(\widetilde{\Psi}, \widetilde{g}_1, \ldots, \widetilde{g}_m)$. Thus, we have the inclusion

$$\mathrm{Fct}^I \left(L \cap H^{m+1} \right) \subset \mathrm{Syz}^{\overline{R}}(\widetilde{\Psi}, \widetilde{g}_1, \ldots, \widetilde{g}_m) \cap \overline{H}^{m+1}.$$

Conversely, take an arbitrary $\zeta \in \overline{R}^{m+1}$ such that

$$\zeta \in \mathrm{Syz}^{\overline{R}}(\widetilde{\Psi}, \widetilde{g}_1, \ldots, \widetilde{g}_m) \cap \overline{H}^{m+1}.$$

Since (5.15) holds, there exist $a_i \in R$ $(i = 1, \ldots, \ell')$ such that

$$\zeta = \overline{a}_1 \overline{b}_1 + \cdots + \overline{a}_{\ell'} \overline{b}_{\ell'}.$$

Moreover, since $\zeta \in \overline{H}^{m+1}$, there exists $\theta \in I^{\oplus m+1}$ such that

$$a_1 b_1 + \cdots + a_{\ell'} b_{\ell'} + \theta \in H^{m+1}.$$

Setting $\zeta' := a_1 b_1 + \cdots + a_{\ell'} b_{\ell'} + \theta$, we have

$$\zeta' \in \left(\langle b_1, \ldots, b_{\ell'} \rangle_R + I^{\oplus m+1} \right) \cap H^{m+1} = L \cap H^{m+1}$$

and $\mathrm{Fct}^I(\zeta') = \zeta$, which imply the inclusion

$$\mathrm{Fct}^I \left(L \cap H^{m+1} \right) \supset \mathrm{Syz}^{\overline{R}}(\widetilde{\Psi}, \widetilde{g}_1, \ldots, \widetilde{g}_m) \cap \overline{H}^{m+1}.$$

Therefore, we have that (5.17) holds. Thus, from (5.16) and (5.17), we have

$$\mathrm{Fct}^I \circ \mathrm{Sbs}^{m+1}_{Y \mapsto h} \left(\langle \eta_1, \ldots, \eta_\nu \rangle_S \right) = \mathrm{Syz}^{\overline{R}}(\widetilde{\Psi}, \widetilde{g}_1, \ldots, \widetilde{g}_m) \cap \overline{H}^{m+1},$$

which implies (5.14). □

5.5 Example

We give an example that demonstrates the implementation of Algorithm 5.10 as follows.

Example 5.18. Suppose we have the system of form (5.1) defined by

$$f(x) = \begin{bmatrix} x_1x_2 + x_2x_3^2 \\ x_1^3 + x_2^3 \\ -x_1 + x_3 + x_1^3 \end{bmatrix}, \quad G(x) = \begin{bmatrix} x_1+1 & x_3^2 \\ 0 & x_2 \\ -x_1x_2 & 0 \end{bmatrix},$$
$$h(x) = \begin{bmatrix} x_2 + x_1x_3 \\ x_1x_2 \end{bmatrix}, \tag{5.18}$$

the polynomial vector

$$f^{\mathrm{d}}(x) = \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix}, \tag{5.19}$$

and the algebraic set $\mathscr{A} = \{x \in \mathbf{R}^3 : \ x_1^2 + x_2^2 = 1, x_3 = 0\}$, and consider Problem 5.4. The set $\mathscr{A}$ is the unit circle on the x_1-x_2 plane, and its vanishing ideal of $\mathscr{A}$ is computed as

$$I = \sqrt[\mathrm{R}]{\langle X_1^2 + X_2^2 - 1, X_3\rangle_R} = \left\langle X_1^2 + X_2^2 - 1, X_3\right\rangle_R \tag{5.20}$$

by using the algorithms presented in [73–76]. Put

$$\Phi_1 = X_1,\ \Phi_2 = X_2,\ \Phi_3 = X_3,$$
$$\Psi_1 = f_1^{\mathrm{d}} = -X_2,\ \Psi_2 = f_2^{\mathrm{d}} = X_1,\ \Psi_3 = f_3^{\mathrm{d}} = 0$$

as are in Subsection 5.3. Now, we execute Algorithm 5.10 under the above settings. In Step 1, we obtain the set $\{\eta_1, \eta_2, \eta_3\}$ as

$$\eta_1 = [1, Y_1, Y_1^2 - Y_2]^{\mathrm{T}},$$
$$\eta_2 = [Y_1^3 + Y_1Y_2 - Y_1, Y_1^2Y_2 - Y_2^2, -2Y_1Y_2^2 + Y_1Y_2]^{\mathrm{T}},$$
$$\eta_3 = [Y_1^2 - Y_2, Y_1^3 - Y_1Y_2, -2Y_1^2Y_2 + Y_1^2]^{\mathrm{T}}.$$

In Steps 2 and 3, we obtain the set $\{\kappa_1\}$ as

$$\kappa_1 = Y_1^4 - Y_1^2 + Y_2^2.$$

In Step 4, we can verify that the following inclusion holds:

$$1 \in \left\langle 1, Y_1^3 + Y_1Y_2 - Y_1, Y_1^2 - Y_2\right\rangle_S + \left\langle Y_1^4 - Y_1^2 + Y_2^2\right\rangle_S,$$

which indicates the present problem is solvable. In Steps 5 and 6, we define the equation

$$\begin{bmatrix}1\\0\\0\end{bmatrix} = z_1\begin{bmatrix}0\\1\\0\end{bmatrix} + z_2\begin{bmatrix}0\\0\\1\end{bmatrix} + z_3\begin{bmatrix}1\\Y_1\\Y_1^2-Y_2\end{bmatrix} + z_4\begin{bmatrix}Y_1^3+Y_1Y_2-Y_1\\Y_1^2Y_2-Y_2^2\\-2Y_1Y_2^2+Y_1Y_2\end{bmatrix}$$
$$+ z_5\begin{bmatrix}Y_1^2-Y_2\\Y_1^3-Y_1Y_2\\-2Y_1^2Y_2+Y_1^2\end{bmatrix} + z_6\begin{bmatrix}Y_1^4-Y_1^2+Y_2^2\\0\\0\end{bmatrix} + z_7\begin{bmatrix}0\\Y_1^4-Y_1^2+Y_2^2\\0\end{bmatrix}$$
$$+ z_8\begin{bmatrix}0\\0\\Y_1^4-Y_1^2+Y_2^2\end{bmatrix}$$

with the variables $z := [z_1, \dots, z_8] \in S^8$. In Step 7, we obtain the polynomial vectors $z^* \in S^8$, $m_1^z \in S^8, \dots,$ and $m_6^z \in S^8$ as

$$\begin{aligned}
z^* &= [-Y_1, -Y_1^2+Y_2, 1, 0, 0, 0, 0, 0]^{\mathrm{T}},\\
m_1^z &= [0, 0, 0, Y_1, -Y_2, -1, 0, 0]^{\mathrm{T}},\\
m_2^z &= [0, 0, 2Y_1Y_2-Y_1, -1, -Y_1, 0, -1, 0]^{\mathrm{T}},\\
m_3^z &= [0, 0, Y_1^2-Y_2, 0, -1, 0, 0, -1]^{\mathrm{T}},\\
m_4^z &= [0, 0, 2Y_2^2-Y_2, 0, Y_1^2+Y_2-1, -1, -Y_1, 2Y_2-1]^{\mathrm{T}},\\
m_5^z &= [Y_1^4-Y_1^2+Y_2^2, 0, 0, 0, 0, 0, -1, 0]^{\mathrm{T}},\\
m_6^z &= [0, Y_1^4-Y_1^2+Y_2^2, 0, 0, 0, 0, 0, -1]^{\mathrm{T}}.
\end{aligned}$$

Finally, in Step 8, we obtain by projection the polynomial vectors $u^* \in S^2$, $m_1^u \in S^2$, and $m_2^u \in S^2$ as

$$u^* = \begin{bmatrix}-Y_1\\-Y_1^2+Y_2\end{bmatrix}, \quad m_1^u = \begin{bmatrix}Y_1^4-Y_1^2+Y_2^2\\0\end{bmatrix}, \quad m_2^u = \begin{bmatrix}0\\Y_1^4-Y_1^2+Y_2^2\end{bmatrix}.$$

According to Remark 5.15, any static output-feedback controller u satisfying (5.2) is parametrized as

$$u(y) = \begin{bmatrix}-y_1\\-y_1^2+y_2\end{bmatrix} + c_1(y)\begin{bmatrix}y_1^4-y_1^2+y_2^2\\0\end{bmatrix} + c_2(y)\begin{bmatrix}0\\y_1^4-y_1^2+y_2^2\end{bmatrix}$$

with arbitrary polynomial functions $c_i(\cdot)$ $(i = 1, 2)$. Conversely, any controller u of the above form satisfies (5.2).

Chapter 6

Conclusions

6.1 Summary

In this dissertation, we aimed to develop a framework of computing exact solutions to nonlinear control problems by employing the theory of algebras. We required here that the computation consist of a finite number of arithmetic operations.

In Chapter 3, we tackled stability analysis by using Lie algebras. We focused on discrete-time nonlinear systems with state-dependent coefficient matrices and derived sufficient conditions for local and global asymptotic stabilities for upper-triangularizable systems. Although this result could be seen as a discrete analogue of [52], we included new results: a necessary condition for upper triangularizability and a sufficient condition for global asymptotic stability. Moreover, the transformation matrix for triangularization is not required explicitly for checking the conditions.

In Chapter 4, we dealt with two problems of input-affine polynomial dynamical systems. One aims to obtain a state feedback controller such that a prescribed algebraic set is invariant for the resulting closed-loop system. The other aims to obtain a state feedback controller such that the resulting closed-loop system has a prescribed vector field on a given algebraic set. We formulated the Lie derivative inclusion with state feedback to represent the problems and solved it in a finite number of arithmetic operations of polynomials by using the theory of Gröbner bases. As a result, all the state feedback controllers required in the problems can be exactly described by using free polynomial parameters. Since the above two problems have several control applications, the controllers required in the applications can also be exactly computed in a finite number of arithmetic operations of polynomials.

In Chapter 5, we extended the results in Chapter 4 to the case of static output feedback. Although there had been a certain difficulty in solving the Lie derivative inclusion with static output feedback, we derived an algorithm for solving the inclusion by removing the difficulty. By using the algorithm, all the static output-feedback controllers solving the inclusion can be exactly computed in a finite number of arithmetic operations of polynomials.

6.2 Discussion

We review the results given in this dissertation in regard to the following points of view.

Theoretical importance
Since the stability conditions given in Chapter 3 do not require the differentiability of a system, they are applicable to systems with which Lyapunov's indirect method does not work. In fact, under the assumption of the upper triangularizability of systems, the local stability condition is a generalization of Lyapunov's indirect method. Moreover, to the best of the author's knowledge, a converse Lyapunov theorem for discontinuous discrete-time systems has not been presented. At least, a certain discontinuous discrete-time system having a globally asymptotically stable equilibrium point at the origin does not admit any smooth Lyapunov function. Hence, the present stability conditions might be able to ensure the local or global stability of a system that has no Lyapunov function. Even if the converse Lyapunov theorem held, it would be very difficult to find Lyapunov functions. The results in the chapter also revealed that the present stability conditions can be applied to systems that the results in [54] and [55] applied for switched linear systems cannot, in which the image of the coefficient matrix of a system is assumed to be compact.

The importance of the results given in Chapters 4 and 5 is that a particular inclusion of polynomials can represent several nonlinear control problems in a unified manner and that the inclusion can be solved even within the context of static output feedback. Moreover, the algorithm given in Chapter 5 can solve linear equations over a subalgebra, which had never before been done.

Computability
Although the results of Chapter 3 have theoretical importance, there is a lack of computability in the sense that the Lie algebraic condition for the triangularizability of a system is not always checkable symbolically. Nevertheless, in many cases, the triangularizability condition can be checked by a finite

number of arithmetic operations of matrices. Moreover, to check the stability conditions, it is enough to evaluate the asymptotic behavior of functions, unlike heuristic approaches such as Lyapunov's direct method.

On the other hand, a solution of the Lie derivative inclusion is completely computable in a finite number of arithmetic operations by using the algorithms given in Chapters 4 and 5, which are based on the theory of Gröbner bases. This computability is highly significant in nonlinear control theory because the computation of exact solutions of nonlinear control problems is very difficult in general.

Practical applications

The stability conditions given in Chapter 3 are restrictive because they require the triangularizability of a system and are only sufficient conditions. Hence, practical applications of the conditions have not yet been found.

In Chapters 4 and 5, the Lie derivative inclusion is defined for polynomial dynamical systems and algebraic sets. As stated in the chapters, polynomial dynamical systems can represent a lot of systems, and algebraic sets can represent a variety of typical subsets of the state space. Hence, the results given in these chapters have a certain generality and applicability. A disadvantage is that the proposed algorithms require heavy computational cost for higher dimensional systems because the algorithms are based on the computation of Gröbner bases.

6.3 Future recommendation

Since the Lie algebraic stability analysis in Chapter 3 depends on the choice of the description of a system, the stability conditions do not reflect the essential properties of systems. This suggest that the Lie-algebraic stability conditions cannot be significantly extended in the theoretical sense. One possible extension is to relax the stability conditions. For example, the stability of *almost* triangularizable systems in some sense can be investigated. Another recommendation is the practical application of the stability analysis of linear closed-loop systems equipped with a state-dependent feedback gain, to which the results in Chapter 3 are applicable.

The Lie derivative inclusion can be formulated in more general rings, e.g., the rings of analytic functions, rational functions, and meromorphic functions. One direction for future study on the Lie derivative inclusion is to obtain an algorithm for solving the inclusion in these rings. Another direction is to find more practical problems that can be represented by the Lie derivative inclusion. In particular, it is recommended to establish an algorithm that can compute approximate solutions, rather than exact ones, of the Lie derivative inclusion. It is also recommended to formulate the inclusion for systems having parameters.

Appendix

This appendix includes the supplementary algorithms employed for solving the Lie derivative inclusion. In the following, the indexes of the symbols are associated with those in Chapter 5.

A.1 Computing a syzygy module over a factor ring

Consider the following problem.

Problem A.1. *Given an $(m+1)$-tuple $(v_1, \ldots, v_{m+1})$ of polynomial vectors $v_i \in R^\ell$ $(i = 1, \ldots, m+1)$, compute a set of generators $\{\overline{b}_1, \ldots, \overline{b}_{\ell'}\}$ of the syzygy module* $\mathrm{Syz}^{\overline{R}}(\overline{v}_1, \ldots, \overline{v}_{m+1})$.

The algorithm for solving the above problem is given below.

Algorithm A.2. (Computing a syzygy module over a factor ring) [94, Remark 2.5.6]

Given: an $(m+1)$-tuple $(v_1, \ldots, v_{m+1})$ of polynomial vectors $v_i \in R^\ell$ $(i = 1, \ldots, m+1)$.

Obtain: a set of generators $\{\overline{b}_1, \ldots, \overline{b}_{\ell'}\}$ of the syzygy module

$$\mathrm{Syz}^{\overline{R}}(\overline{v}_1, \ldots, \overline{v}_{m+1}).$$

Step 1. Let $\{e_1, \ldots, e_\ell\}$ be the canonical basis of R^ℓ, and compute a set of generators$\{b'_1, \ldots, b'_{\ell'}\}$ of the syzygy module

$$\mathrm{Syz}^R(v_1, \ldots, v_{m+1}, \rho_1 e_1, \ldots, \rho_k e_1, \rho_1 e_2, \ldots, \rho_k e_2, \ldots, \rho_1 e_\ell, \ldots, \rho_k e_\ell),$$

using the algorithm given in [37].

Step 2. Put $b_i := \mathrm{Prj}_{m+1}^{m+1+k\ell}(b'_i) \in R^{m+1}$ $(i = 1, \ldots, \ell')$.

Step 3. Put $\overline{b}_i := \mathrm{Fct}^I(b_i) \in \overline{R}^{m+1}$ $(i = 1, \ldots, \ell')$.

A.2 Intersecting a module with a subalgebra

Consider the following problem.

Problem A.3. *Given an R-submodule $L = \langle w_1, \ldots, w_{\ell'+k(m+1)} \rangle_R \subset R^{m+1}$ and a subring $H = \mathbf{R}[h_1, \ldots, h_p] \subset R$, obtain a set of generators $\{\eta'_1, \ldots, \eta'_\nu\}$ of the H-module $L \cap H^{m+1}$.*

The algorithm for solving the above problem is given below.

Algorithm A.4. (Intersecting a module with a subalgebra) [96]

Given: a set of generators $\{w_1, \ldots, w_{\ell'+k(m+1)}\}$ of an R-submodule $L \subset R^{m+1}$, and a set of generators $\{h_1, \ldots, h_p\}$ of a subalgebra $H = \mathbf{R}[h_1, \ldots, h_p] \subset R$.

Obtain: a set of generators $\{\eta'_1, \ldots, \eta'_\nu\}$ of the H-module $L \cap H^{m+1}$.

Step 1. Set $D := \mathbf{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_p]$.

Step 2. Let $\{e_1, \ldots, e_{m+1}\}$ be the canonical basis of R^{m+1}, and form the D-submodule

$$\begin{aligned} L' = \left\langle w_1, \ldots, w_{\ell'+k(m+1)} \right\rangle_D &+ \langle (Y_1 - h_1)e_1, \ldots, (Y_p - h_p)e_1 \rangle_D \\ &\vdots \\ &+ \langle (Y_1 - h_1)e_{m+1}, \ldots, (Y_p - h_p)e_{m+1} \rangle_D . \end{aligned}$$

Step 3. Compute a Gröbner basis $\mathcal{L}$ of L' w.r.t. an elimination ordering for $\{X_1, \ldots, X_n\}$.

Step 4. Put $\{\eta_1, \ldots, \eta_\nu\} := \mathcal{L} \cap S^{m+1}$.

Step 5. Put $\eta'_i := \mathrm{Sbs}^{m+1}_{Y \mapsto h}(\eta_i)$ $(i = 1, \ldots, \nu)$.

A.3 Computing the kernel of an algebra homomorphism

Consider the following problem.

Problem A.5. *Given an $\mathbf{R}$-algebra homomorphism $\pi : S \to \overline{R}$ defined by $Y_i \mapsto \overline{h}_i$, compute a set of generators $\{\kappa_1, \ldots, \kappa_{\nu'}\}$ of $\ker \pi$.*

The algorithm for solving the above problem is given below.

Algorithm A.6. (Computing the kernel of an algebra homomorphism) [94]

Given: an $\mathbf{R}$-algebra homomorphism $\pi : S \to \overline{R}$ defined by $Y_i \mapsto \overline{h}_i$.

Obtain: a Gröbner basis $\{\kappa_1, \ldots, \kappa_{\nu'}\}$ of $\ker \pi$.

Step 1. Set $D := \mathbf{R}[X_1, \ldots, X_n, Y_1, \ldots, Y_p]$.

Step 2. Put $\beta_i := Y_i - h_i \in D$ $(i = 1, \ldots, p)$.

Step 3. Compute a Gröbner basis $\mathcal{B}$ of the ideal

$$\langle \rho_1, \ldots, \rho_k, \beta_1, \ldots, \beta_p \rangle_D \subset D$$

w.r.t. an elimination ordering for $\{X_1, \ldots, X_n\}$.

Step 4. Put $\{\kappa_1, \ldots, \kappa_{\nu'}\} := \mathcal{B} \cap S$.

A.4 Solving a linear equation with polynomial coefficients

Let P be either of the rings R or S, and let k and l be given positive integers. Then, consider the following problem.

Problem A.7. *Given $\gamma_0 \in P^k$ and $\gamma_i \in P^k$ $(i = 1, \ldots, l)$, obtain a solution $z := [z_1, \ldots, z_l]^{\mathrm{T}} \in P^l$ of the equation*

$$\gamma_0 = z_1\gamma_1 + \cdots + z_l\gamma_l. \tag{A.1}$$

The algorithm for solving the above problem is given below.

Algorithm A.8. (Computing the set of all solutions of a linear equation with polynomial coefficients) [37, 39, 87]

Given: polynomial vectors $\gamma_i \in P^k$ $(i = 0, \ldots, l)$.

Obtain: a polynomial vector $z^* \in P^l$ and a set $\{m_1^z, \ldots, m_r^z\}$ of polynomial vectors $m_i^z \in P^l$ $(i = 1, \ldots, r)$ such that $z^* + \langle m_1^z, \ldots, m_r^z \rangle_S$ is the set of all solutions of (A.1).

Step 1. Compute a Gröbner basis $\{\tilde{\gamma}_1, \ldots, \tilde{\gamma}_q\}$ of $\langle \gamma_1, \ldots, \gamma_l \rangle_S$ by Buchberger's algorithm [37, 48].

Step 2. Compute a transformation matrix $Q \in P^{l \times q}$ such that $[\tilde{\gamma}_1, \ldots, \tilde{\gamma}_q] = [\gamma_1, \ldots, \gamma_l]Q$. This matrix can be obtained by tracing Buchberger's algorithm in Step 1.

Step 3. Divide γ_0 by $(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_q)$, and let $z' \in P^q$ and $z'' \in P^k$ be the quotient and the remainder of γ_0, respectively.

Step 4. If $z'' \neq 0$, then terminate the algorithm because (A.1) does not have a solution; otherwise, proceed to the next step because (A.1) has a solution.

Step 5. Compute a set of generators $\{m'^z_1, \ldots, m'^z_r\}$ of $\mathrm{Syz}^P(\gamma_1, \ldots, \gamma_l)$ by the algorithm given in [37].

Step 6. Put $z^* := Qz'$ and $\{m^z_1, \ldots, m^z_r\} := \{m'^z_1, \ldots, m'^z_r\}$.

Bibliography

[1] J. C. Maxwell, “On governors,” in *Proceedings of the Royal Society of London*, vol. 16, 1867, pp. 270–283.

[2] W. S. Levine, Ed., *The Control Handbook: Control System Fundamentals*, 2nd ed. Boca Raton, Florida: CRC Press, 2010.

[3] T. Kailath, *Linear Systems*. Upper Saddle River, New Jersey: Prentice-Hall, 1980.

[4] I. Kubo and K. Yano, *Dynamical Systems*. Tokyo: Iwanami Shoten, 2006, (in Japanese).

[5] T. Saitoh, *Topological Dynamics*. Tokyo: Kyoritsu Shuppan, 2002, (in Japanese).

[6] H. Poincaré, “Mémoire sur les courbes définies par une équation différentielle,” *Journal de mathématiques pures et appliquées*, pp. 375–422, 1881.

[7] ——, “Mémoire sur les courbes définies par une équation différentielle,” *Journal de mathématiques pures et appliquées*, pp. 251–296, 1882.

[8] ——, “Sur les courbes définies par les équations différentielles,” *Journal de mathématiques pures et appliquées*, pp. 167–244, 1885.

[9] ——, “Sur les courbes définies par les équations différentielles,” *Journal de mathématiques pures et appliquées*, pp. 151–217, 1886.

[10] A. M. Lyapunov, *The General Problem of the Stability of Motion*. Kharkov, Russia: Kharkov Mathematical Society, 1892, (in Russian).

[11] ——, “The general problem of the stability of motion,” *International Journal of Control*, vol. 55, no. 3, pp. 531–773, 1992, (translation).

[12] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, New Jersey: Prentice-Hall, 2002.

[13] W. M. Haddad and V. Chellaboina, *Nonlinear Dynamical Systems and Control*. Princeton, New Jersey: Princeton University Press, 2008.

[14] R. E. Bellman, *Dynamic Programming*. Princeton, New Jersey: Princeton University Press, 1957.

[15] L. S. Pontryagin, "Optimal control processes," *Progress of Mathematical Science*, vol. 14, no. 1, pp. 3–20, 1959, (in Russian).

[16] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes*. New York: Interscience Publishers, 1962, (translation).

[17] A. A. Fel'dbaum, *Optimal Control Systems*. New York: Academic Press, 1965.

[18] A. E. Bryson, Jr. and Y.-C. Ho, *Applied Optimal Control*. New York: Taylor & Francis, 1975.

[19] B. D. O. Anderson and J. B. Moore, *Optimal Control*. Mineola, New York: Dover Publications, 1990.

[20] A. Isidori, *Nonlinear Control Systems*, 3rd ed. London: Springer-Verlag, 1995.

[21] H. Nijmeijer and A. van der Schaft, *Nonlinear Dynamical Control Systems*. New York: Springer-Verlag, 1990.

[22] M. Fliess, "A new approach to the noninteracting control problem in nonlinear systems theory," in *Proceedings of the 23rd Annual Allerton Conference on Communication, Control, and Computing*, 1985, pp. 123–129.

[23] M. Fliess, J. Lévine, P. Martin, and P. Rouchon, "Flatness and defect of non-linear systems: introductory theory and examples," *International Journal of Control*, vol. 61, no. 6, pp. 1327–1361, 1995.

[24] M. Fliess, "A note on the invertibility of nonlinear input-output differential systems," *Systems & Control Letters*, vol. 8, no. 2, pp. 147–151, 1986.

[25] J. Rudolph, "Viewing input-output system equivalence from differential algebra," *Journal of Mathematical Systems, Estimation, and Control*, vol. 4, no. 3, pp. 353–383, 1994.

[26] M. Bröcker, J. Polzer, and M. Lemmen, "An advanced algorithm based on differential algebra for disturbance decoupling of nonlinear systems," in *Proceedings of the 39th IEEE Conference on Decision and Control*, vol. 5, 2000, pp. 4367–4372.

[27] V. Hagenmeyer and E. Delaleau, "Exact feedforward linearization based on differential flatness," *International Journal of Control*, vol. 76, no. 6, pp. 537–556, 2003.

[28] N. Faiz, S. K. Agrawal, and R. M. Murray, "Trajectory planning of differentially flat systems with dynamics and inequalities," *Journal of Guidance, Control, and Dynamics*, vol. 24, no. 2, pp. 219–227, 2001.

[29] J. Franch and S. K. Agrawal, "Design of differentially flat planar space robots and their planning and control," *International Journal of Control*, vol. 81, no. 3, pp. 407–416, 2008.

[30] V. Hagenmeyer and M. Nohr, "Flatness-based two-degree-of-freedom control of industrial semi-batch reactors using a new observation model for an extended Kalman filter approach," *International Journal of Control*, vol. 81, no. 3, pp. 428–438, 2008.

[31] M. Halás, "An algebraic framework generalizing the concept of transfer functions to nonlinear systems," *Automatica*, vol. 44, no. 5, pp. 1181–1190, 2008.

[32] M. Halás, Ü. Kotta, and C. H. Moog, "Transfer function approach to the model matching problem of nonlinear systems," in *Proceedings of the 17th IFAC World Congress*, 2008, pp. 15 197–15 202.

[33] M. Halás, "Nonlinear systems: A polynomial approach," in *Computer Aided Systems Theory - EUROCAST 2009*, ser. Lecture Notes in Computer Science, R. Moreno-Díaz, F. Pichler, and A. Quesada-Arencibia, Eds. Berlin: Springer, 2009, vol. 5717, pp. 595–602.

[34] T. Ohtsuka, "Solutions to the Hamilton-Jacobi equation with algebraic gradients," *IEEE Transactions on Automatic Control*, vol. 56, no. 8, pp. 1874–1885, 2011.

[35] J.-F. Pommaret, *Partial Differential Control Theory.* Dordrecht: Kluwer Academic Publishers, 2001.

[36] G. Conte, C. H. Moog, and A. M. Perdon, *Algebraic Methods for Nonlinear Control Systems*, 2nd ed. London: Spronger-Verlag, 2007.

[37] D. A. Cox, J. Little, and D. O'Shea, *Using Algebraic Geometry*, 2nd ed. New York: Springer Science+Business Media, 2005.

[38] ——, *Ideals, Varieties, and Algorithms*, 2nd ed. New York: Springer-Verlag, 1997.

[39] Z. Lin, L. Xu, and N. K. Bose, "A tutorial on Gröbner bases with applications in signals and systems," *IEEE Transactions on Circuits and Systems I*, vol. 55, no. 1, pp. 445–461, 2008.

[40] S. Basu, R. Pollack, and M.-F. Roy, *Algorithms in Real Algebraic Geometry*, 2nd ed. Berlin: Springer-Verlag, 2006.

[41] E. Zerz and S. Walcher, "Controlled invariant hypersurfaces of polynomial control systems," *Qualitative Theory of Dynamical Systems*, vol. 11, no. 1, pp. 145–158, 2012.

[42] E. Zerz, S. Walcher, and F. Güçlü, "Controlled invariant varieties of polynomial control systems," in *Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems*, 2010, pp. 1663–1667.

[43] M. Jirstrand, "Nonlinear control system design by quantifier elimination," *Journal of Symbolic Computation*, vol. 24, no. 2, pp. 137–152, 1997.

[44] N. Bourbaki, *Elements of Mathematics: Algebra I.* Berlin: Springer-Verlag, 1989.

[45] ——, *Elements of Mathematics: Algebra II.* Berlin: Springer-Verlag, 1990.

[46] M. Artin, *Algebra.* Upper Suddle River, New Jersey: Prentice-Hall, 1991.

[47] M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra.* MA: Addison-Wesley, 1969.

[48] M. Kreuzer and L. Robbiano, *Computational Commutative Algebra 1.* Berlin: Springer-Verlag, 2000.

[49] J.-J. E. Slotine and W. Li, *Applied Nonlinear Control.* Englewood Cliffs, New Jersey: Prentice-Hall, 1991.

[50] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory.* New York: Springer-Verlag, 1972.

[51] N. Bourbaki, *Elements of Mathematics: Lie Groups and Lie Algebras.* Berlin: Springer-Verlag, 1999.

[52] S. P. Banks, "The Lie algebra of a nonlinear dynamical system and its application to control," *International Journal of Systems Science*, vol. 32, no. 2, pp. 157–174, 2001.

[53] C. M. Kellett and A. R. Teel, "On robustness of stability and Lyapunov functions for discontinuous difference equations," in *Proceedings of the 41st IEEE Conference on Decision and Control*, vol. 4, 2002, pp. 4282–4287.

[54] D. Liberzon, J. P. Hespanha, and A. S. Morse, "Stability of switched systems: a Lie-algebraic condition," *Systems & Control Letters*, vol. 37, pp. 117–122, 1999.

[55] A. A. Agrachev and D. Liberzon, "Lie-algebraic stability criteria for switched systems," *SIAM Journal on Control and Optimization*, vol. 40, no. 1, pp. 253–269, 2001.

[56] M. Lazar, W. P. Maurice, H. Heemels, and A. R. Teel, "Lyapunov functions, stability and input-to-state stability subtleties for discrete-time discontinuous systems," *IEEE Transactions on Automatic Control*, vol. 54, no. 10, pp. 2421–2425, 2009.

[57] D. Nešić, A. R. Teel, and P. V. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations," *Systems & Control Letters*, vol. 38, no. 4–5, pp. 259–270, 1999.

[58] A. Cima, A. van den Essen, A. Gasull, E. Hubbers, and F. Mañosas, "A polynomial counterexample to the Markus-Yamabe conjecture," *Advances in Mathematics*, vol. 131, no. 2, pp. 453–457, 1997.

[59] L. Markus and H. Yamabe, "Global stability criteria for differential systems," *Osaka Mathematical Journal*, vol. 12, pp. 305–317, 1960.

[60] A. Cima, A. Gasull, and F. Mañosas, "The discrete Markus-Yamabe problem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 35, no. 3, pp. 343–354, 1999.

[61] G. Colombo, "Weak flow-invariance for non-convex differential inclusions," *Differential Integral Equations*, vol. 5, no. 1, pp. 173–180, 1992.

[62] F. Blanchini, "Set invariance in control," *Automatica*, vol. 35, pp. 1747–1767, 1999.

[63] S. V. Raković, E. C. Kerrigan, K. I. Kouramas, and D. Q. Mayne, "Invariant approximations of the minimal robust positively invariant set," *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 406–410, 2005.

[64] A. Alessio, M. Lazar, A. Bemporad, and W. Heemels, "Squaring the circle: An algorithm for generating polyhedral invariant sets from ellipsoidal ones," *Automatica*, vol. 43, no. 12, pp. 2096–2103, 2007.

[65] M. A. B. Sassi and A. Girard, "Computation of polytopic invariants for polynomial dynamical systems using linear programming," *Automatica*, vol. 48, no. 12, pp. 3114–3121, 2012.

[66] R. Katoh and M. Mori, "Control method of biped locomotion giving asymptotic stability of trajectory," *Automatica*, vol. 20, no. 4, pp. 405–414, 1984.

[67] A. Goswami, B. Espiau, and A. Keramane, "Limit cycles and their stability in a passive bipedal gait," in *Proceedings of the 1996 IEEE International Conference on Robotics and Automation*, vol. 1, 1996, pp. 246–251.

[68] ——, "Limit cycles in a passive compass gait biped and passivity-mimicking control laws," *Journal of Autonomous Robots*, vol. 4, no. 3, pp. 273–286, 1997.

[69] Y. Hu, G. Yan, and Z. Lin, "Feedback control of planar biped robot with regulable step length and walking speed," *IEEE Transactions on Robotics*, vol. 27, no. 1, pp. 162–169, 2011.

[70] C. Ebenbauer and F. Allgöwer, "Analysis and design of polynomial control systems using dissipation inequalities and sum of squares," *Computers & Chemical Engineering*, vol. 30, no. 10–12, pp. 1590–1602, 2006.

[71] T. Ohtsuka, "Model structure simplification of nonlinear systems via immersion," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 607–618, 2005.

[72] J. Bochnak, M. Coste, and M.-F. Roy, *Real Algebraic Geometry.* Berlin: Springer-Verlag, 1998.

[73] E. Becker and R. Neuhaus, "Computation of real radicals of polynomial ideals," in *Computational Algebraic Geometry*, F. Eyssette and A. Galligo, Eds. Boston: Birkhäuser, 1993, pp. 1–20.

[74] A. Galligo and N. Vorobjov, "Complexity of finding irreducible components of a semialgebraic set," *Journal of Complexity*, vol. 11, no. 1, pp. 174–193, 1995.

[75] E. Becker and J. Schmid, "On the real nullstellensatz," in *Algorithmic Algebra and Number Theory*, B. H. Matzat, G.-M. Greuel, and G. Hiss, Eds. Berlin: Springer-Verlag, 1999, pp. 173–185.

[76] S. J. Spang, "A zero-dimensional approach to compute real radicals," *Computer Science Journal of Moldova*, vol. 16, pp. 64–92, 2008.

[77] N. Motee, B. Bamieh, and M. Khammash, "Model reduction of polynomial dynamical systems using differential algebra," in *the 49th IEEE Conference on Decision and Control*, 2010, pp. 6195–6200.

[78] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations.* McGraw-Hill, 1955.

[79] L. Chua and D. Green, "Synthesis of nonlinear periodic systems," *IEEE Transactions on Circuits and Systems*, vol. 21, no. 2, pp. 286–294, 1974.

[80] D. Green, "Synthesis of systems with periodic solutions satisfying $\mathscr{V}(x) = 0$," *IEEE Transactions on Circuits and Systems*, vol. 31, no. 4, pp. 317–326, 1984.

[81] K. Hirai and K. Maeda, "A method of limit-cycle synthesis and its applications," *IEEE Transactions on Circuit Theory*, vol. 19, no. 6, pp. 631–633, 1972.

[82] C. Aguilar-Ibánez, J. C. Martinez, J. de Jesus Rubio, and M. S. Suarez-Castanon, "Inducing sustained oscillations in feedback-linearizable single-input nonlinear systems," *ISA Transactions*, 2014, (in press).

[83] J. Aracil, F. Gordillo, and E. Ponce, "Stabilization of oscillations through backstepping in high-dimensional systems," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 705–710, 2005.

[84] A. Morro, A. Sgorbissa, and R. Zaccaria, "Path following for unicycle robots with an arbitrary path curvature," *IEEE Transactions on Robotics*, vol. 27, no. 5, pp. 1016–1023, 2011.

[85] F. D. del Río, G. Jiménez, J. L. Sevillano, S. Vicente, and A. C. Balcells, "A path following control for unicycle robots," *Journal of Robotic Systems*, vol. 18, no. 7, pp. 325–342, 2001.

[86] M. Aicardi, G. Casalino, G. Indiveri, A. Aguiar, P. Encarnação, and A. Pascoal, "A planar path following controller for underactuated marine vehicles," in *Proceedings of the 9th IEEE Mediterranean Conference on Control and Automation*, 2001.

[87] A. Furukawa, T. Sasaki, and H. Kobayashi, "Gröbner basis of a module over $K[x_1, \ldots, x_n]$ and polynomial solutions of a system of linear equations," in *the 5th ACM Symposium on Symbolic and Algebraic Computation*, Waterloo, Ontario, Canada, 1986, pp. 222–224.

[88] C. I. Byrnes, F. D. Priscoli, and A. Isidori, *Output Regulation of Uncertain Nonlinear Systems.* Boston: Birkäuser, 1997.

[89] A. S. Shiriaev, "Stabilization of compact sets for passive affine nonlinear systems," *Automatica*, vol. 36, no. 9, pp. 1373–1379, 2000.

[90] A. S. Shiriaev and A. L. Fradkov, "Stabilization of invariant sets for nonlinear non-affine systems," *Automatica*, vol. 36, no. 11, pp. 1709–1715, 2000.

[91] M. I. El-Hawwary and M. Maggiore, "Passivity-based stabilization of non-compact sets," in *Proceedings of the 46th IEEE Conference on Decision and Control*, 2007, pp. 1734–1739.

[92] S. Prajna, P. Parrilo, and A. Rantzer, "Nonlinear control synthesis by convex optimization," *IEEE Transactions on Automatic Control*, vol. 49, no. 2, pp. 310–314, 2004.

[93] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann, "Singular." [Online]. Available: http://www.singular.uni-kl.de/

[94] G.-M. Greuel and G. Pfister, *A Singular Introduction to Commutative Algebra*, 2nd ed. Berlin: Springer-Verlag, 2008.

[95] V. L. Syrmos, C. T. Abdallah, P. Dorato, and K. Grigoriadis, "Static output feedback–a survey," *Automatica*, vol. 33, no. 2, pp. 125–137, 1997.

[96] G. Kemper, "Calculating invariant rings of finite groups over arbitrary fields," *Journal of Symbolic Computation*, vol. 21, pp. 351–366, 1996.

Acknowledgments

I am grateful to Professor Toshiyuki Ohtsuka at Kyoto University for his supervision on this work. He gave me a lot of counsel on both my research and my daily living as a researcher. This has guided me through the Ph.D. program and has formed the basis of my research activities. I am grateful to Professor Koh Hosoda at Osaka University for his valuable comments on this dissertation, which helped refine the motivation of this work. I am also grateful to Associate Professor Kenji Kashima at Kyoto University for the great deal of advice he gave me on my research and on my career as a researcher. It has greatly helped me to conduct my research and to deliberate upon my career. I highly appreciate the valuable comments on this dissertation from Professor Toshimitsu Ushio and Professor Youji Iiguni at Osaka University. They thoroughly reviewed this dissertation and pointed out ways it could be extended and improved. I also appreciate the valuable advice from Assistant Professor Tomoaki Hashimoto at Osaka University and Assistant Professor Toshihiro Hiraoka at Kyoto University.

I am thankful to my colleague, Special Assistant Professor Yu Kawano at Kyoto University, for a good deal of advice about my research activity. He has greatly helped me in my daily research activities through the Ph.D. program. The discussions I have had with him have been very significant for my research. I am also thankful to Professor Miroslav Halás at the Slovak University of Technology in Bratislava (STU) for supporting me on my visit to STU and for giving me advice on my research. Owing to his support, I successfully gave a seminar at STU. I am grateful to the lab secretaries, Ms. Kayoko Ushio, Ms. Yoshika Kohiyama, and Ms. Miki Yasunaga, and to all members of the administrative staff. Their work has greatly helped me to conduct my research smoothly. I am also thankful to all my friends for their help and encouragement.

I am also grateful for the financial support with which I was provided. The loan from the Japan Student Services Organization (JASSO) and the stipend from the Japan Society for the Promotion of Science (JSPS) were both of great help for me through my student years. The research grant from

JSPS gave me crucial support in conducting my research.

Finally, I would like to express my sincere gratitude to my family for their unwavering support of me. My family gracefully bore their economic hardship and have always encouraged me in my study. Without their constant support and encouragement, I could never have done this work.

List of Publications

Journal articles

1. T. Yuno and T. Ohtsuka, "A Sufficient Condition for the Stability of Discrete-Time Systems with State-Dependent Coefficient Matrices," *IEEE Transactions on Automatic Control*, vol. 59, no. 1, pp. 243–248, 2014.

2. T. Yuno and T. Ohtsuka, "Lie Derivative Inclusion for a Class of Polynomial State Feedback Controls," *Transactions of the Institute of Systems, Control and Information Engineers*, vol. 27, no. 11, pp. 423–433, 2014.

3. T. Yuno and T. Ohtsuka, "Lie Derivative Inclusion with Polynomial Output Feedback," *Transactions of the Institute of Systems, Control and Information Engineers*, vol. 28, no. 1, pp. 22–31, 2015.

International conference proceedings (refereed)

1. T. Yuno and T. Ohtsuka, "Realization of a Vector Field via State Feedback for Polynomial Dynamical Systems," in *Proceedings of the 13th European Control Conference*, Strasbourg, France, June 24–27, 2014, pp. 2454–2459.

Domestic conference proceedings (in Japanese)

1. T. Yuno and Y. Taki, "Object-Oriented Design of Sub Motion Space Components for Redundant Serial Robots," in *Proceedings of the 9th SICE System Integration Division Annual Conference*, Gifu, December 5–7, 2008, 2L3-5, pp. 895–896.

2. T. Yuno and T. Ohtsuka, "The Stability of Discrete Time Systems with an Upper-Triangular and State-Dependent Coefficient Matrix," in *Proceedings of the SICE 11th Annual Conference on Control Systems*, Okinawa, March 16–18, 2011, 185-1-1.

3. T. Yuno and T. Ohtsuka, "On the Stability of Discrete-Time Systems with State-Dependent Coefficient Matrices," in *Proceedings of the 41st SICE Symposium on Control Theory*, Hayama, September 18–20, 2012, pp. 279–282.

4. T. Yuno and T. Ohtsuka, "Stability of Discrete-Time Systems with Triangularizable State-Dependent Coefficient Matrices," in *Proceedings of the 55th Japan Joint Automatic Control Conference*, Kyoto University, November 17–18, 2012, 1G402, pp. 510–512.

5. T. Yuno and T. Ohtsuka, "Model Matching of Polynomial Systems via State Feedback," in *Proceedings of the SICE 13th Annual Conference on Control Systems*, Fukuoka, March 5–8, 2013, 7A1-1.

6. T. Yuno and T. Ohtsuka, "State Feedback Control for Polynomial Systems Using Energy Functions," in *Proceedings of the SICE 13th Annual Conference on Control Systems*, Fukuoka, March 5–8, 2013, 7A1-2.

7. T. Yuno and T. Ohtsuka, "Solution and Applications of the Lie Derivative Inclusion for Polynomial Dynamical Systems," in *Proceedings of the SICE Multi Symposium on Control Systems 2014*, Tokyo, March 4–7, 2014, 7C5-3.

Other presentations and lectures

1. T. Yuno, "Making a Vector Field," *1st Study Camp on Control Theory*, Osaka, September 9–10, 2013. (in Japanese)

2. T. Yuno, "Realization of a Vector Field via State Feedback for Polynomial Dynamical Systems," *a seminar at the Slovak University of Technology in Bratislava*, Bratislava, Slovakia, July 2, 2014.

3. T. Yuno, "Solutions and Applications of Linear Equations over Subalgebras," *2nd Study Camp on Control Theory*, Niigata, September 23–24, 2014. (in Japanese)