<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>A criterion for hypoellipticity of second order differential operators</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Morimoto, Yoshinori</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 24(3) P.651–P.675</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1987</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/5212">https://doi.org/10.18910/5212</a></td>
</tr>
<tr>
<td><strong>DOI</strong></td>
<td>10.18910/5212</td>
</tr>
<tr>
<td><strong>rights</strong></td>
<td></td>
</tr>
</tbody>
</table>
A CRITERION FOR HYPOELLIPTICITY OF
SECOND ORDER DIFFERENTIAL OPERATORS

YOSHINORI MORIMOTO

(Received December 19, 1985)

Introduction and results. In this paper we give a sufficient condition for second order differential operators to be hypoelliptic. The condition is also necessary for a special class of differential operators.

Let Ω be an open set in $\mathbb{R}^n$ and let $P=p(x, D_x)$ be a second order differential operator with coefficients in $C^\infty(\Omega)$, that is,

$$p(x, D_x) = \sum_{j,k=1}^n a_{jk}(x) D_x^j D_x^k + \sum_{j=1}^n b_j D_x^j + c, \quad D_x^j = -i \partial_x^j,$$

where coefficients $a_{jk}(x)$, $b_j(x)$ and $c(x)$ belong to $C^\infty(\Omega)$. We assume that $a_{jk}(x)$, $b_j(x)$ are real valued and $a_{jk}(x)$ satisfy for any $x$ in $\Omega$

$$(2) \quad \sum_{j,k=1}^n a_{jk}(x) \xi_j \xi_k \geq 0 \quad \text{for all } \xi \in \mathbb{R}^n.$$

Let $\log \langle D_x \rangle$ denote a pseudodifferential operator with symbol $\log \langle \xi \rangle$, where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

Theorem 1. Assume that for any $\varepsilon > 0$ and any compact set $K$ of $\Omega$ there exists a constant $C_{\varepsilon, K}$ such that

$$(3) \quad ||\log \langle D_x \rangle u||^2 \leq \varepsilon ||Pu|| + C_{\varepsilon, K} ||u||, \quad u \in C^\infty(\Omega).$$

Then $P$ is hypoelliptic in $\Omega$. Furthermore we have

$$(4) \quad WF P\psi = WF \psi \quad \text{for } \psi \in \mathcal{D}'(\Omega).$$

Corollary 2. Assume that for any $\varepsilon > 0$ and any compact set $K$ of $\Omega$ there exists a constant $C_{\varepsilon, K}$ such that

$$(5) \quad ||\log \langle D_x \rangle u||^2 \leq \varepsilon \text{ Re}(Pu, u) + C_{\varepsilon, K} ||u||^2, \quad u \in C^\infty(\Omega).$$

Then we have (4).

Proof of Corollary. For $u \in C^\infty(\Omega)$ take $\phi, \psi \in C^\infty(\Omega)$ such that $\phi=1$ on $K$ and $\psi=1$ on supp $\phi$. Note

$$(\log \langle D_x \rangle) u = \psi (\log \langle D_x \rangle) u + (1-\psi) (\log \langle D_x \rangle) \phi u$$
and \((1-\psi)(\log\langle D_x \rangle)\phi \in \mathcal{S}^{-\infty}\) (see Chapter 2 of [5]). Since \(\psi(\log\langle D_x \rangle)u\) belongs to \(C_0^\infty\), in view of the above formula we may replace \(u\) in (5) by \((\log\langle D_x \rangle)u\). Since the principal symbol of \([P, \log\langle D_x \rangle]\) is purely imaginary we have

\[
\text{Re}([P, \log\langle D_x \rangle]u, (\log\langle D_x \rangle)u) \leq C(\|\langle D_x \rangle^2u\|^2 + \|u\|^2).
\]

In view of this it is clear that (5) implies (3). Q.E.D.

The estimate (3) is not always necessary for the hypoellipticity. We have a counter example \(\mathcal{A}_0(x, D_x)=D_x^2+\exp(-1/|x_1|^{8})D_x^2\) for \(\delta \geq 1\) given by [Fedii [2] (cf. [8]). In fact, \(\mathcal{A}_0\) is hypoelliptic for any \(\delta \geq 0\), but when \(\delta \geq 1\) the estimate (3) does not hold for some small \(\varepsilon > 0\) (see Remark 3.1 in Section 3). However, for a class of differential operators, the estimate (3) is necessary to be hypoelliptic.

The result concerning the necessity can be discussed for some class of operators of higher order. Let \(m\) be an even positive integer and let \(P_0\) be a differential operator of the form

\[
P_0 = D_t^m + \mathcal{A}(x, D_x) \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x^m,
\]

where \(\mathcal{A}(x, D_x)\) is a differential operator of order \(m\) with \(C^\infty\)-coefficients. We assume that \(\mathcal{A}(x, D_x)\) is formally self-adjoint in an open set \(\Omega\) of \(\mathbb{R}_x^m\) and bounded from below, that is, there exists a real \(c_0\) such that

\[
(\mathcal{A}(x, D_x)u, u) \geq c_0\|u\|^2 \quad \text{for } u \in L^2(\Omega) \text{ satisfying } \mathcal{A}u \in L^2(\Omega).
\]

**Theorem 3.** Let \(P_0\) be the above operator. Assume that \(P_0\) is hypoelliptic in \(R_t \times \Omega\). Then for any \(x_0 \in \Omega\) there exists a neighborhood \(\omega\) of \(x_0\) such that for any \(\varepsilon > 0\) the estimate

\[
\|((\log\langle D_t, D_x \rangle)^{m/2}u\|^2 \leq \varepsilon \text{Re}(P_0u, u) + C_\varepsilon\|u\|^2, \quad u \in C_0^\infty(R_t \times \omega)
\]

holds with a constant \(C_\varepsilon\).

**Remark.** When \(m=2\) the estimate (5) follows from (7). In fact, for any compact set \(K\) of \(R_t \times \Omega\), let \(K'\) be the projection of \(K\) to \(\Omega\), and take the partition of unity \(\sum \phi_j(x)=1\) over \(K'\). Since \(\text{Re}([P_0, \phi_j]u, \phi_ju)\) is majorated by a constant times of \(\|u\|^2\), we have (7) for \(u \in C_0^\infty(R_t \times K')\), which implies (5). In view of the proof of Corollary 2, the estimate (3) also follows from (7).

Our two theorems are applicable to the hypoellipticity of operators considered in [10] and [11]. Especially, an application shows that \(D_t^2+D_x^2+\exp(-1/|x_1|^{8})D_x^2\) is hypoelliptic in \(\mathbb{R}^3\) if and only if \(\delta\) satisfies \(\delta < 1\) (cf. Theorem 8.41 of [6]). As another application we give:

**Proposition 4.** Set \(P_1=D_t^2+x_1^2D_x^2+D_x^2+D_x(\sigma(x_1)\zeta(x_2)^2)D_x\), where \(\sigma, \zeta \in C^\infty, \sigma(s) > 0 (s \neq 0), \zeta > 0, \sigma(0) = 0\) and \(\sigma'(s) \geq 0\). Then \(P_1\) is hypoelliptic in \(\mathbb{R}^3\) if
and only if $\sigma(s)$ satisfies

$$\lim_{s \to 0} |s^{1/2} \log \sigma(s)| = 0.$$  

**Remark.** If $\sigma(s) = \exp(-1/|s|^\delta)$ for $\delta > 0$ then (8) means $\delta < 1/2$.

The plan of this paper is as follows: In Section 1 we prove Theorem 1. The idea of the proof is the same as in Section 5 of [11], though we employ the microlocalization method by Hörmander [4]. In Section 2 we prove Theorem 3 by using the interpolation method similar to the one in Métivier [7], where nonanalytic hypoellipticity for operators of the same form as (6) was studied (cf. Baouendi-Goulaouic [1]). The proof given in Section 2 is nothing but $C^\infty$-version of [7]. Section 3 is devoted to the proof of Proposition 4.

We finally remark that the criterion of Theorem 1 applies to second order differential operators with finite degeneracy studied by Hörmander [3] and Oleinik-Radkevich [13], because for such operators we have the sub-elliptic estimate $||u||_8 \leq C(||Pu||+||u||)$, $\delta > 0$.

### 1. Proof of Theorem 1

Before proving the theorem we introduce some notations. When $\phi, \psi \in C^\infty(R^n)$ satisfy $\psi = 1$ in a neighborhood of supp $\phi$, we write $\phi \ll \psi$. For a pseudodifferential operator $Q = q(x, D_x)$ we denote by $\sigma(Q)$ the symbol $q(x, \xi)$. We denote by $Q^{(\alpha)}$ a pseudodifferential operator with symbol $q^{(\alpha)}(x, \xi) = D^\alpha \partial^\alpha q(x, \xi)$ for multi-indices $\alpha$ and $\beta$.

Here and throughout the present paper $P = p(x, D_x)$ denotes the second order differential operator in Introduction satisfying the condition (2). For the brevity we assume $\Omega = R^n$. Without loss of generality we may assume that coefficients of $P$ are defined in $R^n$ and belong to $C^\infty(R^n)$. As proved by [13], it follows from (2) that

$$(1.1) \quad \sum_{|\alpha|=1} ||Q^{(\alpha)}u||^2 \leq C \text{Re}(Pu, u) + ||u||^2, \quad u \in C^\infty_0(K),$$

and

$$(1.2) \quad \sum_{|\alpha|=1} ||D_x \gamma P^{(\alpha)}u||^2 \leq C \text{Re} \left( \sum_{j=1}^n (D_j < D_x \gamma)^{-1} Pu, D_j < D_x \gamma)^{-1} u \right) + ||u||^2, \quad u \in C^\infty_0(K),$$

hold with some constant $C = C_K$, where $D_j = D_{x_j}$. In fact, (1.1) follows from (2.6.6) and (2.6.9) of [13], and (1.2) follows from (2.6.14) of [13].

Write $p(x, \xi) = \sum_{k=0}^\infty p_k(x, \xi)$, where $p_k$ is positively homogeneous in $\xi$ of degree $k$. Let $h(x) \in C^\infty_0(R^n)$ be 1 for $|x| \leq 1/5$ and vanish for $|x| \geq 7/24$. For $\gamma = (x_0, \xi_0) \in R^n \times S^{n-1}$ we consider a microlocalized pseudodifferential operator

$$P_\gamma = p_{\gamma}(\lambda y, \lambda D_y) = \sum_{k=0}^\infty p_k(x_0 + \lambda y, \xi_0 + \lambda D_y) h(\lambda D_y/3) \lambda^{-2k}$$
with a small parameter $\lambda > 0$ (see [4] and Section 2 of [9]).

It is clear that for any multi-indices $\alpha$ and $\beta$ we have

$$|\delta_\alpha^*D_\beta^p(\lambda y, \lambda \gamma)| \leq C_{a,\beta}\lambda^{-|\alpha|+|\beta|}, \ 0<\lambda \leq 1,$$

with a constant $C_{a,\beta}$ independent of $\lambda$.

Let $(P^{(a)})$ and $(P^{(a)})$ be microlocalized operators defined from symbols of $P^{(a)}$ and $P^{(a)}$ by the similar formula as (1.3). From estimates (3), (1.1) and (1.2) we have the following:

**Lemma 1.1.** For any real $s > 0$ and any $\gamma = (\tau_0, \zeta_0)^\mathbb{R}_n \chi_\mathbb{R}_n$ there exists a constant $C(s, \gamma)$ such that for $0<\lambda \leq 1$

$$\sum_{|\alpha| = 1} ||H^{(P^{(a)})}_\gamma || \leq C_\gamma (||H_P\gamma || + ||\gamma||), \ \gamma \in S_1,$$

and

$$\sum_{|\alpha| = 1} ||H^{(P^{(a)})}_\gamma || \leq C_\gamma (||H_P\gamma || + ||\gamma||), \ \gamma \in S_1.$$

**Proof.** Set $\psi(y) = (\exp(-i\lambda^{-2}x \cdot \xi_0)\sigma(x))|_{x=\lambda y + \xi_0}$ for $\psi \in S_1$. Then we have

$$\exp(-i\lambda^{-2}x \cdot \xi_0)\sigma(x) = \psi(x)$$

and for real $s'$ we have

$$\exp(-i\lambda^{-2}x \cdot \xi_0) \psi(x) = \lambda^{-2s'}(\xi_0 + \lambda \xi_0)\psi(x) \lambda^{-1}(x-x_0).$$

Indeed, both formulas are easily seen if we note the change of variables

$$x-x_0 = \lambda y, \ \xi - \xi_0 = \lambda^{-1}\eta.$$

Furthermore we have

$$\exp(-i\lambda^{-2}x \cdot \xi_0) \log(Dx)\psi(x) = \psi(x)$$

where $r(\eta; \lambda) = (\log(\lambda^4 + |\eta + \xi_0|^4))/4$. It is clear that $\{r(\eta; \lambda)\psi(\eta); 0<\lambda \leq 1\}$ and $\{r(\eta; \lambda)\psi(\eta); 0<\lambda \leq 1\}$ are bounded sets in $S^m_{\mathbb{R}}$, as pseudodifferential operators in $R^m_{\mathbb{R}}$. Note that $\{h(\lambda\xi - \xi_0); 0<\lambda \leq 1\}$ is a bounded set in $S^m_{\mathbb{R}}$, as a pseudodifferential operator in $R^m_{\mathbb{R}}$, because $\lambda^2 \leq (31/24)|\xi|^{-1}$ on supp $h(\lambda\xi - \xi_0)$. We shall prove (1.6). Substitute $u = h(x-x_0)h(\lambda^2Dx - \xi_0)\psi(x) = h(x-x_0)h(\lambda^2Dx - \xi_0)$.
CRITERION FOR HYPOELLIPTICITY

655

(h(λ^2D_x−ξ_0)/3)w for w∈𝑆_*$ into (1.1). Then we have

\[ \sum_{|α|=0} ||h(λ^2D_x−ξ_0)h(x−x_0)P(α)h((λ^2D_x−ξ_0)/3)w||^2 \]
\[ \leq C(Re(h(λ^2D_x−ξ_0))h(x−x_0)P((λ^2D_x−ξ_0)/3)w, h(λ^2D_x−ξ_0)h(x−x_0)w) \]
\[ + ||w||, \ w∈𝑆_* , \]

In fact, we can majorate the terms concerning commutators among \( h(x−x_0) \), \( h(λ^2D_x−ξ_0) \) and \( P \) appearing in the right hand side by a constant times of \( ||w||^2 \), because their symbols are purely imaginary. In view of (1.8) and the same formula with \( P \) replaced by \( P(α) \) we have

\[ \sum_{|α|=0} ||H(P(α))∀v||^2 \leq C(Re(HP∀v, Hv)+||v||^2), \ v∈𝑆_* , \]

which gives (1.6) together with Schwartz inequality. Similarly it follows from (1.2) that

\[ \sum_{|α|=0} ||D_x||^{-1}h(λ^2D_x−ξ_0)h(x−x_0)P(α)h((λ^2D_x−ξ_0)/3)w||^2 \]
\[ \leq C(Re(\sum_{|α|=0} (D_j)(D_α)^{-1}h(λ^2D_x−ξ_0)h(x−x_0)P((λ^2D_x−ξ_0)/3)w, \]
\[ D_j)(D_α)^{-1}h(λ^2D_x−ξ_0)h(x−x_0)w||+||w||^2) . \]

From this we obtain (1.7) if we note (1.9) and

\[ \exp(-iλ^{-2}x−ξ_0)D_j)(D_α)^{-1}h(λ^2D_x−ξ_0)w(x) \]
\[ = (r_j(λ, α)v)(λ^{-1}(x−x_0)) , \]

where \( r_j(η; λ)≡(λη_j−ξ_0)(λ^4+|λη_j+ξ_0|^2)^{-1/2}h(λη) \) belongs to \( S_{0, 0}^\circ \) uniformly with respect to \( 0<λ≤1 \). We shall prove (1.5). Substituting \( u=h(x−x_0) \)
\[ h(λ^2D_x−ξ_0)h((λ^2D_x−ξ_0)/3)w \]
\[ (1.9') \]
\[ ||(log(D_α))^βh(λ^2D_x−ξ_0)h(x−x_0)w|| \]
\[ \leq ε(||h(λ^2D_x−ξ_0)h(x−x_0)P((λ^2D_x−ξ_0)/3)w|| \]
\[ + ||[(h(λ^2D_x−ξ_0), h(x−x_0)]P((λ^2D_x−ξ_0)/3)w|| \]
\[ + ||[P, h(x−x_0)h(λ^2D_x−ξ_0)]h((λ^2D_x−ξ_0)/3)w||) \]
\[ + C_ε||w||)≡ε(I_1+I_2+I_3+C_ε||w||) . \]

Note \( h(λ^2D_x−ξ_0)h(x−x_0)=h(λ^2D_x−ξ_0)h(x−x_0)h((x−x_0)/2) \) and

\[ \sigma(h(λ^2D_x−ξ_0)h(x−x_0))h(x−x_0)h(λ^2D_x−ξ_0) \]
\[ − \sum_{|α|=1} D_jh(x−x_0)∂^αh(λ^2D_x−ξ_0)∈S_{1/3}^\circ , \]

uniformly with respect to \( 0<λ≤1 \). Then we see that \( I_1 \) is estimated above by a constant times of

\[ J≡||h((λ^2D_x−ξ_0)/2)h((x−x_0)/2)P((λ^2D_x−ξ_0)/3)w||+||w|| , \]
because \( h(\lambda^2 D_x - \xi_0)/2 = 1 \) on \( \text{supp} \partial^\alpha h(\lambda^2 D_x - \xi_0) \). Since \( D^\alpha h(x-x_0)\partial^\alpha h(\lambda^2 D_x - \xi_0) \equiv D^\alpha h(x-x_0)\partial^\alpha h(\lambda^2 D_x - \xi_0)h((x-x_0)/2) \) mod \( S^{-\infty} \), \( I_2 \) is also estimated above by \( J \) with a constant factor. Noting

\[
[P, h(x-x_0)h(\lambda^2 D_x - \xi_0)] = \sum_{|\alpha| + |\beta| = 1} (-1)^{\beta}|D^\alpha h(x-x_0)|\partial^\alpha h(\lambda^2 D_x - \xi_0)P(\alpha) \in S^0_{1,0}
\]

we see that \( I_2 \) is estimated above by a constant times of

\[
||h(\lambda^2 D_x - \xi_0)/2)h((x-x_0)/2)P(\alpha)h(\lambda^2 D_x - \xi_0)/3)\nu|| + ||D_x^{-1}h(\lambda^2 D_x - \xi_0)/2)h((x-x_0)/2)P(\alpha)h(\lambda^2 D_x - \xi_0)/3)\nu|| + ||\nu||.
\]

By substituting \( u = h((x-x_0)/2)h(\lambda^2 D_x - \xi_0)/2)h(\lambda^2 D_x - \xi_0)/3)\nu \) into (1.1) and (1.2), we have (1.11) and (1.13) with \( h(\lambda^2 D_x - \xi_0)h(x-x_0) \) replaced by \( h(\lambda^2 D_x - \xi_0)h(x-x_0/2) \). Using these estimates together with Schwartz's inequality, from the estimations for \( I_j \) \( (j=1, 2, 3) \) we have with a constant \( c > 0 \) independent of \( \epsilon \)

\[
||h((\lambda^2 D_x - \xi_0)/2)h((x-x_0)/2)P(\alpha)h((\lambda^2 D_x - \xi_0)/3)\nu|| + C'_\nu ||\nu||.
\]

In view of (1.8) and (1.10), we obtain

\[
(log \lambda^{-1})||H\nu|| \leq \epsilon ||H_0P_\gamma \nu|| + C'_\nu ||\nu|| \quad \text{if} \quad 0 < \lambda \leq \lambda_1,
\]

where \( \lambda_1 > 0 \) is a sufficiently small number such that for \( 0 < \lambda \leq \lambda_1 \)

\[
||r(D_x; \lambda)h(\lambda D_x)\nu|| \leq (log \lambda^{-1})||h(\lambda D_x)\nu||.
\]

Taking \( s = \epsilon^{-2} \) we obtain (1.5) when \( 0 < \lambda \leq \lambda_1 \). The estimate (1.5) for \( \lambda_1 < \lambda \leq 1 \) is obvious.

Q.E.D.

Estimates (1.6) and (1.7) can be strengthened to the following form:

**Corollary.** For any \( \gamma \in R^n \times S^{n-1} \) there exists a constant \( C'_\gamma \) such that for any \( s > 0 \) estimates

(1.6) \[ \sum_{|\alpha| = 1} ||H(P(\alpha))\gamma \nu|| \leq C'_\gamma ((log \lambda^{-1})||HP_\gamma \nu|| + (log \lambda^{-1})||H\nu|| + ||\nu||), \]

and

(1.7) \[ \sum_{|\alpha| = 1} ||\lambda^2 H(P(\alpha))\gamma \nu|| \leq C'_\gamma ((log \lambda^{-1})||HP_\gamma \nu|| + (log \lambda^{-1})||H\nu|| + ||\nu||) \]

holds if \( 0 < \lambda < 1 \).

**Proof.** The estimate (1.6) is a direct consequence of (1.12) because

\[
\text{Re}(HP_\gamma \nu, H\nu) \leq (log \lambda^{-1})||HP_\gamma \nu||^2 + (log \lambda^{-1})||H\nu||^2.
\]
We also have (1.7)' by the similar estimate as (1.12) that is derived from (1.13). Q.E.D.

Note that \( ||v|| \leq ||He|| + ||(1 - H)v|| \) and that for any \( s > 0 \) and any \( \gamma \in \mathbb{R}^s \times S^{s-1} \) there exists a small positive number \( \lambda_0 = \lambda_0(s, \gamma) < 1 \) such that

\[
(1.14) \quad (\log \lambda^{-\eta})^2 \geq 1 + 2(C(s, \gamma) + C' \log \lambda^{-\eta}) \quad \text{if } 0 < \lambda \leq \lambda_0,
\]

where \( C(s, \gamma) \) and \( C' \) are the same constants as in (1.5) and (1.6)', respectively. Then it follows from (1.5) that

\[
(1.15) \quad (\log \lambda^{-\eta}) ||Hv|| \leq 2(||H_0P_\gamma v|| + C(s, \gamma)||1 - H)v||), \quad v \in S_\gamma,
\]

if \( 0 < \lambda \leq \lambda_0 \).

Note that for \( |\alpha| = 1 \)

\[
(1.16) \quad \left\{\begin{array}{ll}
\sigma(P_\gamma^{(s)}) = \delta_\alpha P_\gamma(\lambda \eta, \lambda \gamma) = \lambda^{-1} \sigma((P_\gamma^{(s)})_\gamma) & \text{if } |\lambda \eta| \leq 3/5, \\
\sigma(P_\gamma^{(s)}) = D_\gamma P_\gamma(\lambda \eta, \lambda \gamma) = \lambda \sigma((P_\gamma^{(s)})_\gamma). & \text{if } |\lambda \eta| > 3/5,
\end{array}\right.
\]

Since supp \( h(\lambda \eta) \) is contained in \( \{\eta; \lambda |\eta| \leq 3/5\} \), we see that for \( |\alpha| = 1 \), \( H(\lambda P_\gamma^{(s)} - (P_\gamma^{(s)})_\gamma) \) is \( L^2 \)-bounded uniformly with respect to \( 0 < \lambda \leq \lambda_0 \). Together with (1.14) and (1.15), the estimate (1.6)' gives

\[
(1.17) \quad \sum_{|\alpha|=1} ||H\lambda P_\gamma^{(s)}v|| \leq C'_{1}(\log \lambda^{-\eta})^{-1}(||Hv|| + 4||H_0P_\gamma v||) + (2C(s, \gamma) + C')||1 - H)v||, \quad v \in S_\gamma, \quad \text{if } 0 < \lambda \leq \lambda_0.
\]

From (1.7)' we also have

\[
(1.18) \quad \sum_{|\alpha|=1} ||H\lambda P_{(\gamma, \alpha)}v|| \leq G, \quad v \in S_\gamma, \quad \text{if } 0 < \lambda \leq \lambda_0,
\]

where \( G \) denotes the right hand side of (1.17). For a while we assume \( 0 < \lambda \leq \lambda_0(s, \gamma) \) for fixed \( s > 0 \) and \( \gamma \in \mathbb{R}^s \times S^{s-1} \).

For a real \( \kappa > 0 \) and an integer \( k > 0 \) we denote by \( \Delta_{\kappa, k} \) a pseudodifferential operator with a symbol \( (1 + \kappa \langle \xi \rangle)^{-k} \). It is easy to check that for any \( \alpha \) the estimate

\[
(1.19) \quad |\partial_\xi((1 + \kappa \langle \xi \rangle)^{-k})| \leq C_\alpha \langle \xi \rangle^{-|\alpha| - k}(1 + \kappa \langle \xi \rangle)^{-k}
\]

holds with a constant \( C_\alpha \) independent of \( \kappa \). Set

\[
(1.20) \quad k_\alpha(\eta; \lambda) = (1 + \kappa \langle \xi \rangle)^{-k} h(\lambda \eta).
\]

Then it follows from (1.19) that for any \( \alpha \) the estimate

\[
(1.21) \quad |\partial_\xi k_\alpha(\lambda; \eta)| \leq C_\alpha \lambda^{1/2} k_\alpha(\eta; \lambda), \quad \lambda |\eta| \leq 1/5,
\]

holds with another constant \( C' \) independent of \( \kappa \) and \( \lambda \).
Set \( h_\delta(x) = h(x/\delta) \) for a small \( 0 < \delta \leq 1/10 \). Fix an integer \( N \geq 3 \). Take a sequence \( \{ h^j \}_{j=1}^N \subset C^\infty_0(\mathbb{R}^n) \) such that

\[
(1.22) \quad h_\delta = h_\delta \subset h_\delta^{\beta-1} \subset h_\delta^{\beta-2} \subset \ldots = h_\delta = h_\delta
\]

and for any \( \alpha \) the estimate

\[
(1.23) \quad |D^\alpha h^j| \leq C'\delta^j N^{|\alpha|}
\]

holds with a constant \( C' \) independent of \( N \) and \( j \) (\( C' = 1 \)).

**Lemma 1.2.** Write

\[
(1.24) \quad h^j(\lambda D_y)k_\delta(D_x; \lambda)h^j(\lambda y)h^j(\lambda D_x)h^j(\lambda x) = h^j(\lambda D_y)k_\delta(D_x; \lambda)h^j(\lambda y) + r(y, D_x; \lambda).
\]

Then for any integer \( l \) there exists a constant \( C_1 \) independent of \( \lambda \) and \( N \) such that

\[
|D^\lambda \psi^j(x, y)\varphi| \leq C_1 \lambda^{2l+3} N^{l}|\psi^j\varphi|,
\]

\( \psi^j \in S \).

Proof. Consider the expansion formula of the simplified symbol of \( h^j(\lambda D_y)k_\delta(D_x; \lambda)h^j(\lambda y)\varphi(\lambda D_x)h^j(\lambda x) \) (See Chapter 2 of [5]). Noting (1.21) and (1.23) we obtain (1.25) by means of the Calderón Vaillancourt theorem (See Chapter 7 of [5]). Q.E.D.

To make clear the discussion below we prove the following simple lemma.

**Lemma 1.3.** Let \( N \) be a fixed positive integer and let \( \lambda \) satisfy \( 0 < \lambda \leq 1 \). For any finite sequence of positive numbers \( \{ C_j \}_{j=1}^r \) there exists a constant \( C'_1 \) such that

\[
(1.26) \quad \sum_{j=1}^r C_j(N\lambda)^{2l} \leq 1 + C'_1(N\lambda)^{2l}.
\]

Proof. Set \( R = \max \{ C_j, 1 \} \). When \( N\lambda \leq 1/2^l \), we have

\[
\sum_{j=1}^r C_j(N\lambda)^{2l} \leq R \sum_{j=1}^r (1/2^l)^{2l} \leq \sum_{j=1}^r (1/2^l)^{2l} < 1.
\]

If \( N\lambda \geq 1/2^l \) then we have

\[
\sum_{j=1}^r C_j(N\lambda)^{2l} \leq R \sum_{j=1}^r (N\lambda)^{-2l} \leq (R \sum_{j=1}^r (2)^{2l-2j}) (N\lambda)^{2l}.
\]

It suffices to set \( C'_1 = R \sum_{j=1}^r (2)^{2l-2j} \). Q.E.D.

Set \( \tilde{E}_v^j = \tilde{h}(\lambda D_y)k_\delta(D_x; \lambda)h^j(\lambda y) \) and substitute \( \tilde{E}_v^j \varphi \) into (1.15). Then for any \( s > 0 \) there exists a constant \( C_s \) independent of \( \kappa, \lambda \) and \( N \) such that

\[
(1.27) \quad (\log \lambda^{-s})^j||\tilde{E}^j_v\varphi|| \leq 2||P_s \tilde{E}^j_v\varphi|| + C_s \lambda^s N^{s+2l+4}||\varphi||, \quad \varphi \in S.
\]
Indeed, considering the expansion formulas of the simplified symbols of \((1-H)\) \(\tilde{H}_i\) and \((1-H_0)P_\gamma \tilde{H}_i\) and using (1.23), (1.21) and (1.4), by Calderon-Vaillantcourt theorem we see that for any \(s>0\) there exists a constant \(C_s\) such that

\[
\|((1-H)\tilde{H}_i\| + \|(1-H_0)P_\gamma \tilde{H}_i\| \leq C_s N^{s-2n+i} \|v\|, \quad v \in \mathcal{S},
\]

Similarly, it follows from (1.17) and (1.18) that for any \(s>0\) estimates

\[
\sum_{\theta \in \mathcal{A}} \|\lambda P_\gamma \Phi_i \tilde{H}_i \| \leq 5C_s (\log \lambda^{-s})^{-1} \|P_\gamma \tilde{H}_i\| + C_s N^{s+2n+i} \|v\|, \quad v \in \mathcal{S},
\]

and

\[
\sum_{\theta \in \mathcal{A}} \|\lambda P_\gamma \Phi_i \tilde{H}_i \| \leq 5C_s (\log \lambda^{-s})^{-1} \|P_\gamma \tilde{H}_i\| + C_s N^{s+2n+i} \|v\|, \quad v \in \mathcal{S},
\]

hold with a constant \(C_s\).

**Lemma 1.4.** There exists a constant \(M\) independent of \(\lambda, \kappa\) and \(N\) such that for any \(s>0\)

\[
\|P_\gamma \tilde{H}_i\| \leq M \|\tilde{H}_i\| + MN (\log \lambda^{-s})^{-1} \|P_\gamma \tilde{H}_i\| + C_s N^{s+2n+i} \|v\|, \quad v \in \mathcal{S}, \quad \text{if} \quad \log \lambda^{-s} \geq MN,
\]

where \(C_s\) is a constant independent of \(\lambda, \kappa\) and \(N\). Hence \(\tilde{H}_i = h_2(\lambda D_x)k_2(D_y; \lambda) h_1(\lambda y)\).

**Proof.** It is clear that

\[
\|P_\gamma \tilde{H}_i\| \leq \|H_i\| + \|[P_\gamma, H_i]\|.
\]

Noting \(h_1(x) = h_2(x)h_3(x)\) and considering the expansion formula of the simplified symbol of \(\tilde{H}_i\), we have

\[
\|H_i\| = \|h_2(\lambda y)P_\gamma v\| \leq (1 + \sum_{r=1}^{t/2+1} C^\gamma_r(N\lambda)^{2s}) \|h_2(\lambda D_x)k_2h_3(\lambda y)P_\gamma v\|
\]

for some constant \(C^\gamma_r\) and \(C_s\). Here we used the estimate

\[
\|(h_2(\lambda D_x)k_2)^e v\| \leq C_\alpha (N\lambda)^{2s} \|h_2(\lambda D_x)k_2v\|,
\]

which follows from (1.21), (1.23) and the fact that \(h_2(x) = 1\) on \(\text{supp} D^e h_2(x)\). Using Lemma 1.3 we have

\[
\|H_i\| \leq 2\|H_2\| + C_s N^{s+2n+i} \|v\|.
\]

Here and throughout the proof of the lemma we denote by the same notation \(C_s\) different constants independent of \(\lambda, \kappa\) and \(N\) (depending on \(s\)). We shall estimate the second term of the right hand side of (1.31). In view of Lemma 1.2 it suffices to estimate \(\|[P_\gamma, H_i]H_i^{s+1}\|\), where \(H_i^{s+1} = h_2(\lambda D_x)h_1^{s+1}(\lambda y)\). Write
\[
[P_\gamma, \hat{H}^s] = [P_\gamma, h(\lambda D_y)k_x]h(\lambda y) + k_x h(\lambda D_y) [P_\gamma, h(\lambda y)].
\]

Note that the expansion formula
\[
[P_\gamma, h(\lambda D_y)k_x] = \sum_{\alpha \in \Lambda, |\alpha| \leq s+1} (-1)^{|\alpha|} (h(\lambda D_y)k_x)^{\alpha} P_{\gamma(\alpha)}/\alpha! + R(y, D_y; \lambda),
\]
where \( R \) is a negligible operator, in the sense of
\[
||Rv|| \leq C_2 \lambda^s N^{s+2n+6} ||v||.
\]

In view of (1.4), (1.21) and (1.23), we see that there exist constants \( M_1 \) and \( M_2 \) independent of \( s, \kappa, \lambda \) and \( N \) such that
\[
||[P_\gamma, h(\lambda D_y)k_x]h(\lambda y)H^{s+1}v|| \leq M_1 N \sum_{|\alpha| \leq s} ||k_x \lambda P_{\gamma(\alpha)}h(\lambda y)H^{s+1}v||
\]
\[
+ M_2 N^2 (1 + \sum_{\ell \geq 1} C_{\ell}' N^{\lambda s}) ||k_x h(\lambda y)H^{s+1}v|| + C_3 \lambda^s N^{s+2n+6} ||v||.
\]

holds with some constants \( C_{\ell}' \). Consider the expansion formula of the simplified symbol of \( k_x h(\lambda y) \) and use Lemma 1.3. Then the second term of the right hand side of (1.32) is estimated above by
\[
2M_2 N^2 ||\hat{H}^{s+1}v|| + C_3 \lambda^s N^{s+2n+6} ||v||.
\]

For \( |\alpha| = 1 \) the estimate
\[
||k_x \lambda P_{\gamma(\alpha)}h(\lambda y)H^{s+1}v|| \leq ||k_x \lambda P_{\gamma(\alpha)}H^{s+1}v|| + M_2 N ||\hat{H}^{s+1}v|| + C_3 \lambda^s N^{s+2n+6} ||v||
\]
holds with a constant \( M_2 \) independent of \( s, \kappa, \lambda \) and \( N \). Here we used Lemma 1.3 to estimate terms corresponding to \( [k_x \lambda P_{\gamma(\alpha)}, h(\lambda y)] \) and \( [\lambda P_{\gamma(\alpha)}, k_x] \). From (1.29) we obtain
\[
||k_x \lambda P_{\gamma(\alpha)}h(\lambda y)H^{s+1}v|| \leq 2C_{\ell}' (\log \lambda^{-s})^{-1} ||P_\gamma \hat{H}^{s+1}v|| + M_3 N ||\hat{H}^{s+1}v||
\]
\[
+ C_3 \lambda^s N^{s+2n+6} ||v||,
\]
\( |\alpha| = 1 \)
\[
(1.34)
\]

From (1.32)–(1.34) we see that the estimate
\[
||[P_\gamma, h(\lambda D_y)k_x]h(\lambda y)H^{s+1}v|| \leq (M_4 N)^2 ||\hat{H}^{s+1}v|| + M_4 N (\log \lambda^{-s})^{-1} ||P_\gamma \hat{H}^{s+1}v|| + C_3 \lambda^s N^{s+2n+6} ||v||
\]
holds with a suitable constant \( M_4 \) larger than \( C_{\ell}' \) and \( M_j \) (\( j = 1, 2, 3 \)). If we use (1.27) with \( j \) replaced by \( j+1 \) to estimate the first term of the right hand side, we can estimate \( [P_\gamma, h(\lambda D_y)k_x]h(\lambda D_y)H^{s+1}v|| by the right hand side of (1.30) with another suitable \( M \) larger than \( M_4 \), because \((MN)^2(\log \lambda^{-s})^{p} \leq MN/\log \lambda^{-s} \) if \( \log \lambda^{-s} \geq MN \). Noting \( ||k_x h(\lambda D_y)\theta|| \leq ||k_x h_{28}(\lambda D_y)\theta|| \) for \( \theta \in S_y \), we can also estimate the term \( ||k_x h(\lambda D_y) [P_\gamma, h(\lambda y)]H^{s+1}v|| \) by using (1.28) instead of (1.29). We have estimated the second term of the right hand side of (1.31). So we ob-
tain the desired estimate.

Q.E.D.

From (1.27) and (1.30) we have

**Lemma 1.5.** For any integer \( N \geq 3 \) there exists a constant \( M \) independent of \( N, \lambda \) and \( \kappa \) such that for any \( s > 0 \)

\[
(\log \lambda - \gamma)^N \| \mathcal{H}_s \mathcal{V} \| \leq (\log \lambda - \gamma)^N \| \mathcal{H}_{2s} \mathcal{P}_s \mathcal{V} \| + (MN)^N \| \mathcal{V} \| + C_s N^{1+2s} \| \mathcal{V} \|,
\]

\( v \in S_y \), if \( 0 < \lambda \leq \lambda_0 (s, \gamma) \),

where \( C_s \) is a constant independent of \( \kappa, \lambda \) and \( N \).

**Proof.** In view of \( \mathcal{H}_s = \mathcal{H}_3 \) it follows from (1.27) that

\[
(\log \lambda - \gamma)^N \| \mathcal{H}_s \mathcal{V} \| \leq 2 \leq (\log \lambda - \gamma)^N \| \mathcal{H}_{2s} \mathcal{P}_s \mathcal{V} \| + C_s N^{1+2s} \| \mathcal{V} \|.
\]

Applying (1.30) to the first term of the right hand side. Then we have

\[
(\log \lambda - \gamma)^N \| \mathcal{H}_s \mathcal{V} \| \leq 2 \leq (\log \lambda - \gamma)^N \| \mathcal{H}_{2s} \mathcal{P}_s \mathcal{V} \| + (MN)^N \| \mathcal{V} \| + 2C_s (\log \lambda - \gamma)^N \| \mathcal{V} \| \quad \text{ if } \log \lambda - \gamma \geq MN.
\]

Use (1.30) for the second term of the right hand side and use repeatedly \((N-3)\) times. Then we obtain

\[
(\log \lambda - \gamma)^N \| \mathcal{H}_s \mathcal{V} \| \leq 2 \leq M \sum_{j=0}^{N-3} (\log \lambda - \gamma)^N \| \mathcal{H}_{2s} \mathcal{P}_s \mathcal{V} \| + (MN)^N \| \mathcal{V} \| + 2C_s (\log \lambda - \gamma)^N \| \mathcal{V} \| \quad \text{ if } \log \lambda - \gamma \geq MN.
\]

Note that \( \mathcal{H}_s^{-1} = \mathcal{H}_{3s/3} \) and \( \mathcal{H}_{3s/3} \subseteq \mathcal{H}_{2s} \). By means of similar formulas as (1.6) and (1.7) (together with (1.16) we have

\[
\| \mathcal{H}_s \mathcal{V} \| \leq M (\| \mathcal{H}_{2s} \mathcal{P}_s \mathcal{V} \| + \| \mathcal{V} \|),
\]

taking another larger \( M \) if necessary. If \( \log \lambda - \gamma \geq MN \), it follows from (1.36) that

\[
(\log \lambda - \gamma)^N \| \mathcal{H}_s \mathcal{V} \| \leq (\log \lambda - \gamma)^N \| \mathcal{H}_{2s} \mathcal{P}_s \mathcal{V} \| + (MN)^N \| \mathcal{V} \| + C_s (\log \lambda - \gamma)^N \lambda^sN^{2s+6} \| \mathcal{V} \|.
\]

When \( \log \lambda - \gamma \leq MN \) this estimate still holds because of the second term of the right hand side. Noting \( (\log \lambda - \gamma)^N \lambda^s = (\log \lambda - \gamma)^N \exp(-\log \lambda - \gamma) \leq N! \), we obtain (1.35). Q.E.D.

The estimate (1.35) with \( N \leq 2 \) also follows from (1.27) with \( j=1 \) because for a suitable constant \( M' \) we have

\[
\| \mathcal{P}_s \mathcal{H}_s \mathcal{V} \| \leq M' (\| \mathcal{H}_{2s} \mathcal{P}_s \mathcal{V} \| + \| \mathcal{V} \|)
\]
similarly as the estimate after (1.36). Thus, (1.35) holds for any \( N=0,1,2,\ldots \).
Let $\tau$ be a small parameter chosen later on. Multiply both sides of (1.35) by $\tau^N/N!$ and sum up with respect to $N=0, 1, 2, \cdots$. Then we obtain
\[
\lambda^{-\sigma}||H_\theta v|| \leq \lambda^{-\sigma}||H_{2\theta} P_\tau v|| + \left( \sum_{N=0}^{\infty} \left( MN\tau \right)^N / N! + C \sum_{N=0}^{\infty} \tau^{N+2s+\theta} \right)||v||
\]
because $\sum (\log \lambda)^N / N! = \lambda^{-\sigma}$. Choose $\tau$ such that $Me^\tau < 1$ and $0 < \tau < 1$. Then, by using Stirling formula $N^N / N! \leq e^N$ we have
\[
(1.37) \quad \lambda^{-\sigma}||H_\theta v|| \leq \lambda^{-\sigma}||H_{2\theta} P_\tau v|| + C'||v||, \quad v \in S_y,
\]
for another constant $C'_\sigma$. Note that $\tau$ is independent of $s$ because $M$ is so. Hence we can replace $s\tau$ in (1.37) by $2s' + 2s''$ for any real $s', s'' > 0$. Multiply $\lambda^{-s''}$ by (1.37) with $s\tau$ replaced by $2s' + 2s''$. Then we see that there exists a constant $C_0 = C_0(s', s'', \gamma)$ independent of $s$ and $\lambda$ such that
\[
(1.38) \quad \lambda^{-s''}||H_\theta v|| \leq \lambda^{-s''}||H_{2\theta} P_\tau v|| + C_0 \lambda^{2s''}||v||, \quad v \in S_y \quad \text{if } 0 < \lambda > 0 \text{ is sufficiently small.}
\]
Taking another large $C_0$ if necessary, we may assume that (1.38) holds for $0 < \lambda \leq 1$. Note that for any $\xi_0 \in S^{s-1}$, any $0 < s' \leq 1$ and any real $s$ the estimate
\[
C^{-1}||h_\theta(\lambda D_y) v|| \leq ||\xi_0 + \lambda D_y \bar{\phi} h_\theta(\lambda D_y) v|| \leq C ||h_\theta(\lambda D_y) v||, \quad v \in S_y
\]
holds for some constant $C = C_{\xi, s'}$ because
\[
C^{-1} \leq |\xi_0 + \xi| \leq C \text{ on } \text{supp } h_\theta(\xi).
\]
Substitute $v(y) = h(\lambda D_y) \theta(y)$ into (1.38) for $\theta(y) = \exp(-i\lambda^{-2} x \cdot \xi_0) w(x) |x - \lambda y + x_0|$, $\tau \in S_x$. Then in view of (1.8), (1.20), (1.9) and the above estimate, we see that there exists a constant $C_0$ such that
\[
(1.39) \quad \|h_\theta(\lambda^2 D_x - \xi_0) |D_x|^{\sigma} \Lambda_{n,s} \tilde{h}_\theta(x - x_0) w||^2
\]
\[
\leq C_0 \left( \|h_{2\theta}(\lambda^2 D_x - \xi_0) |D_x|^{\sigma} \Lambda_{n,s} h_{2\theta}(x - x_0) P(x, D_x) w||^2
\right.
\]
\[
+ \|h(\lambda^2 D_x - \xi_0) |D_x|^{\sigma} \Lambda_{n,s} \tilde{h}_\theta(x - x_0) P(x, D_x) w||^2 + \lambda ||w||^2 ||x||^{\sigma''}, \quad v \in S_x, \quad \text{if } 0 < \lambda \leq 1.
\]
Here we used the fact that
\[
\{ \lambda^{-1/2} h_{2\theta}(\lambda^2 D_x - \xi_0) h_\theta(x - x_0) (1 - h(\lambda^2 D_x - \xi_0)); 0 < \lambda \leq 1\}
\]
\[
\{ \lambda^{-1/2} h_{2\theta}(\lambda^2 D_x - \xi_0) h_{2\theta}(x - x_0) P(x, D_x) (1 - h((\lambda^2 D_x - \xi_0)/3)); 0 < \lambda \leq 1\}
\]
are contained in a bounded set of $S_{\xi, s''}$. To complete the proof of Theorem 1 we prepare the followings:

**DEFINITION 1.6.** For $\delta > 0$ and $\xi_0 \in S^{s-1}$ we say that a function $\psi(\xi) \in C^\infty(\mathbb{R}^n)$ belongs to $\Psi_{\delta, \xi_0}$ if $\psi$ satisfies

...
Proof. Set $r=|\xi|$, $\theta=|\xi|/|\xi|$. Then

$$
\int_0^\infty |h_0(\lambda^2x_0-\xi_0')u|^2/\lambda d\lambda = \int_{r-\delta}^\infty \int_0^\infty |h_0(\lambda^2\theta-\xi_0')|^2/\lambda (r^{s-1} dr d\lambda).
$$

This gives the first inequality of (1.40). Another inequality easily follows if we note that supp $h_0(\lambda^2\theta-\xi_0')$ is contained in $\Sigma$. From now on we shall prove (4). Let $(x_0, \xi_0)\in T^*R^n\setminus 0$ and let $u\in \mathcal{D}'(R^n)$. Set $\xi_0=\xi_0/|\xi_0|$. Suppose that $(x_0, \xi_0)\in \text{WF} P\mu$. Then there exists a $\delta>0$ such that $\psi_0(\xi_0)h_0(x-x_0)P\mu(x, D_x)\omega \in H_{s'}$ for any real $s'>0$ if $\psi_0(\xi_0)\in \Psi_{148, \xi_0}$. Since $h_0(x-x_0)u\in S'$ we have $h_0(x-x_0)u\in H_{s''}$. Choose $k>0$ in (1.41) such that $\lambda^2 r - 1 \leq \delta/10$. Then, by taking a sequence $\{w_j\}_{j=1}^\infty \subset S_x$ such that

$$
w_j \to h_0(x-x_0)u \text{ in } H_{s''},
$$

we have

$$
||\psi_0(\xi_0)h_0(x-x_0)P\mu(x, D_x)\omega||^2_{L^2} \leq C''(||\psi_0(\xi_0)h_0(x-x_0)P\mu(x, D_x)\omega||^2_{L^2} + ||\omega||^2_{L^2}) \text{ for } \omega \in S_x.
$$

(1.41)
from (1.41) we see that

\[(1.42) \quad ||\Delta, \psi \partial_0 (D_2) \partial_3 (x_0 - x_0) u||^2 \leq C^\prime(\partial \psi_0 (D_2) \partial_3 (x_0 - x_0) P u \partial^2 \psi_0 (x_0 - x_0) u||^2 + ||h_4 \partial_3 (x_0 - x_0) u||^2),\]

if \(\psi_0 (\xi) \in \Psi, \xi_0 \) and \(\psi_0 (\xi) \in \Psi_{\text{loc}}, \xi_0 \). Here we used the fact that \(||\Delta, \psi \partial_0 w|| \leq ||w||\) for \(w \in L^2\). Letting \(\kappa\) tend to 0 in (1.42), we have \(\psi_0 (D_2) \partial_3 (x_0 - x_0) u \in H^\prime\). Since \(s'\) is arbitrary, we have \((x_0, \xi_0) \in WF u\). Now the proof of Theorem 1 has been completed.

2. Proof of Theorem 3

As stated in Introduction, the method used here is only a version of the one in [7]. Let \(P_0\) be the differential operator in Introduction, that is,

\[(2.1) \quad P_0 = D_{\xi} + \mathcal{A}(x, D_2), \quad \text{in } R^t \times R^n,\]

where \(\mathcal{A}(x, D_2)\) is formally self-adjoint in an open set \(\Omega\) of \(R^n\) and bounded below. Following [7], for \(s \geq 1\) we introduce \(G^t (\Omega; \mathcal{A})\) the space of \(u \in L^2 (\Omega)\) such that \(\mathcal{A}^k u \in L^2 (\Omega)\) for \(k = 1, 2, \ldots\) and moreover there exists a constant \(M\) satisfying

\[(2.2) \quad ||\mathcal{A}^k u||_{L^2 (\Omega)} \leq M^{k+1} (k!)m, \quad k = 1, 2, \ldots.\]

We also introduce the space \(G^t (\Omega; \mathcal{A})\) of \(u \in L^2_{\text{loc}} (\Omega)\) whose restriction in any \(\Omega_1 \subseteq \Omega\) is in \(G^t (\Omega_1; \mathcal{A})\).

**Proposition 2.1.** Assume that \(G^t (\Omega; \mathcal{A}) \subseteq C^\infty (\Omega)\). Then \(P_0\) is not hypoelliptic in \(R_t \times \Omega\) (cf. Corollaries 3.6–3.7 of [7] and see also [1]).

**Proof.** There exists a \(u_0 \in G^t (\Omega; \mathcal{A})\) such that \(u_0 \in C^\infty (\Omega)\). The series

\[u(t, x) = \sum_{k=0}^{\infty} (it)^{mk} (-\mathcal{A})^k u_0 (x)/(mk)!\]

is strongly convergent in \(L^2 (\Omega)\) for some \(\Omega = I_\delta \times \Omega_1\), where \(I_\delta = (-\delta, \delta) \subseteq R_t\) and \(\Omega_1 \subseteq \Omega\). We have \(P_0 u = 0\) and \(u\) is not \(C^\infty\) in \(\Omega\) because \(u_0 = u(0, \cdot)\) is not \(C^\infty\) in \(\Omega\). Q.E.D.

Note that for any open set \(\omega \subseteq \Omega\)

\[\text{Re}(P_0 u, u) = ||D_{\xi}^{mk} u||^2 + (\mathcal{A} u, u), \quad u \in C^\infty (R_t \times \omega).\]

For the proof of Theorem 3 it suffices to show:

**Proposition 2.2.** Assume that \(G^t (\Omega; \mathcal{A}) \subseteq C^\infty (\Omega)\). Then for any \(x_0 \in \Omega\) there exists a neighborhood of \(\omega\) of \(x_0\) such that for any \(s' > 0\) the estimate

\[(2.3) \quad ||(\log <D_2>)^{mk} u||^2 \leq C_0 ||u||^2, \quad u \in C^\infty (\omega)\]
CRITERION FOR HYPOELLIPTICITY

holds with a constant \( C_r \). (cf. Theorem 3.5 of [7]).

In the proof of Proposition 2.2 we may replace \( \mathcal{A} \) by \( \mathcal{A} + \mu \) for any real \( \mu \) because \( G'(\Omega; \mathcal{A}) = G'(\Omega; \mathcal{A} + \mu) \). Taking a large \( \mu > 0 \), in view of (6) we may assume that \( (\mathcal{A}u, u) > 0 \) for \( u \in L^2(\Omega) \) satisfying \( \mathcal{A}u \in L^2(\Omega) \). Therefore, we have the Friedrichs extension \( (A, D(A)) \) in \( L^2(\Omega) \) of \( \mathcal{A}(x, D_x) \), as a positive self-adjoint realization.

For the proof of (2.3) it suffices to show that for any \( \epsilon > 0 \) and any \( r > 0 \) there exists a \( C_{\epsilon, r} \) such that

\[
(\log < D_x >)^{m_r} |u|^2 \leq \epsilon |A'u|^2 + C_{\epsilon, r} |u|^2, \quad u \in C_0(\omega).
\]

In fact, the estimate (2.3) follows immediately from (2.4) with \( r = 1/2 \). From now on we shall prove (2.4). We may assume that \( x_0 \) is the origin. We use the same notation as in [7] p. 840–849. Let \( \psi \in C_0^\infty(\Omega) \) equal 1 in \( \Omega = (-\alpha, \alpha)^n \subset \Omega \). The hypothesis of the proposition implies that \( u \in D_0(A) \Rightarrow \psi u \in S \) for a fixed \( \delta > 0 \) because \( D'(\pi) \equiv (\cup D_0(A)) \subset G'(\Omega; \mathcal{A}) \). The Banach closed graph theorem shows that for any integer \( k > 0 \) there exists a constant \( M_k \) such that

\[
\sup_{\xi} |< \xi >^k \hat{u}(\xi)| \leq M_k(N_k(u))^k, \quad u \in D_0(A).
\]

In view of (3.4) of [7], it is clear that for any \( k \) there exists a constant \( M_k(\geq 1) \) such that

\[
J_k(u) \leq e^{20\delta} \|(L + 1)^k u\|_{L^2(\Omega)} \leq M_k(\|< \xi >^k \hat{u}(\xi)|^2,
\]

where \( J_k(u) \) denotes \( J_k(u) \) defined from the spectrum resolution of \( L \). Here \( (L, D(L)) \) is the realization of Legendre operator

\[
\mathcal{L} = \sum_{j=1}^n \partial_{x_j} (x_j^2 - a^2) \partial_{x_j}
\]

defined in [5] p. 845. In what follows, to make clear the correspondence we often use the superscript \( A \) or \( L \) such as \( J_k^A(u), J_k^L(u) \). Set \( K_k = \{ \xi; \langle \xi \rangle \leq M_k^2 M_{k+2} \} \). Then from (2.5) and (2.6) we have

\[
J_k^L(u) \leq ||(M_k^2 M_{k+2}/\langle \xi \rangle)M_{k+2}^2 \langle \xi >^{2k+2} \hat{u} \langle \xi >^{-1} ||_{L^2(\Omega)} + M_k^2 ||< \xi >^k \hat{u}||_{L^2(\Omega)} \leq N_k(u) + C_k ||u||_{L^2(\Omega)}, \quad u \in D_0(A),
\]

with a constant \( C_k \). Set \( u(t) = F^A(t)u \). Then the estimate (2.7) and Lemma 3.1 of [7] show that for any \( r > 0 \) and \( k > 0 \)

\[
I_{r,k}(\cdot) = \left\{ \begin{array}{ll}
\exp (-\delta (et)^{1/2} J_k^L(u(t)) + ||u(t)||_{L^2(\Omega)}) t^{\epsilon_r} dt / t \\
\leq 2 J_k^L(u) + C_k ||u||_{L^2(\Omega)}, \quad u \in D(A')
\end{array} \right.
\]

holds with a constant \( C'_k \). We need replace Lemma 3.2 of [7] by
Lemma 2.3. Let $t \mapsto u(t)$ be a measurable mapping from $[1, \infty)$ to $L^2(\Pi)$ and let $I_{r,k}(u(\cdot))$ denote the integral defined by the formula (2.8). Assume that for reals $\delta > 0$, $r > 0$ and an integer $k > 0$ the integral $I_{r,k}(u(\cdot))$ is bounded. Then the integral $u = \int_1^\infty u(t) \frac{dt}{t}$ is convergent, $u \in D((\log(L+1))^{mr})$ and for a constant $C$ independent of $k$ we have

$$k^{mr}||\log(L+1))^{mr}u||^2_{L^2(\Pi)} \leq CI_{r,k}(u(\cdot)).$$

The proof of the lemma is parallel to the one of Lemma 3.2 of [7] if we set $\sigma(t, \lambda) = \exp(2k \log \lambda - \delta e^{t/m} \ln(m))$ and $\tau(\lambda) = e^{-t}((k/\delta) \log \lambda)^m$. We remark that the estimate

$$||\log(L+1))^{mr}u||^2_{L^2(\Pi)} \leq \int_1^\infty (\log \lambda)^{mr}||F^L(\lambda)u||^2_{L^2(\Pi)} d\lambda$$

holds similarly to (3.4) of [7]. The detail is omitted.

Set $\omega = ((-a/2, a/2))^a$. Then for the proof of Proposition 2.2 it remains to show

$$||(\log(L+1))^{mr}u||^2_{L^2(\Pi)} \leq C(||(\log(L+1))^{mr}u||^2 + ||u||^2), \; u \in C^0(\omega).$$

In fact, from (2.8)–(2.10) we have (2.4) since we can take any large $k$.

From now on we shall prove (2.10). Let $\{\lambda_j; 0 < \lambda_j < \lambda_{j+1} < \cdots\}$ be the set of eigenvalues of $(L, D(L))$ and let $P_j(x)$ be the normalized eigenfunction (Legendre polynomial) associated with $\lambda_j$. Then, for $u \in C^0(\omega)$, $(\log(L+1))^{mr}u$ is defined by

$$\begin{equation}
(\log(L+1))^{mr}u = \sum_{j=1}^\infty (\log(\lambda_j+1))^{mr}(P_j, u)_{L^2(\Pi)}P_j(x).
\end{equation}$$

Here we remark that $v \in C^\infty(\Pi)$ belongs to $D^s(L) \equiv \cap_{j=1}^\infty D(L^j)$ and hence to $D((\log(L+1))^{mr})$ because $\sum_{j=1}^\infty (\log(\lambda_j+1))^{sr}|(P_j, v)|^2 \leq C \sum_{j=1}^\infty \lambda_j |(P_j, v)|^2$. Note that $(\log(L+1))^{mr} = (L+1)(L+1)^{-1}((\log(L+1))^{mr}$ and let $\Gamma$ be a contour of Figure 1:

![Fig. 1](image-url)
Since the similar formulas to (2.11) hold for \((L+1)^{-1}(\log(L+1))^{\nu}\) and \((L-\xi)^{-1}\), by the residue calculus we have

\[
(L+1)^{-1}(\log(L+1))^{\nu} u = \frac{1}{2\pi i} \int_{\gamma} (\xi+1)^{-1}(\log(\xi+1))^{\nu}(L-\xi)^{-1} ud\xi, \quad u \in C_0^\infty(\omega).
\]

We shall approximate \((L-\xi)^{-1}\) by a pseudodifferential operator by using the argument in Chapter 8 of [5]. Let \(\tilde{\mathcal{L}}\) be a second order differential operator with real valued \(\mathcal{B}^{m}\)-coefficients such that \(\tilde{\mathcal{L}} = \mathcal{L}\) in a neighborhood of \(\omega\). We may assume that the symbol \(I(x, \xi)\) of \(\tilde{\mathcal{L}}\) satisfies \(C_0^{-1}\langle \xi \rangle^{2} < I(x, \xi) < C_0 \langle \xi \rangle^{2}\) for large \(\xi\). Then we have

\[
|I_{(\beta)}(x, \xi) (I(x, \xi)-\xi)^{-1}| \leq C_{ab} \langle \xi \rangle^{-1} |\xi| \quad \text{for large } |\xi| \quad \text{and} \quad \xi \in \Omega_\xi \equiv C \setminus \Omega_\xi
\]

(cf. (1.4) of Chapter 8 of [5]).

Here \(\Omega_\xi\) denotes the interior of clockwise-oriented Jordan curve \(\Gamma_\xi^0\) that is defined as in Figure 2:

![Figure 2](image_url)

By means of Lemma 2.2 of Chapter 8 of [5] we have a parametrix \(Q(\xi) = Q(x, D_x; \xi)\) of \(\tilde{\mathcal{L}} - \xi\) such that

\[
(\tilde{\mathcal{L}} - \xi) Q(\xi) = I + R(x, D_x; \xi),
\]

where symbols \(q(\xi) = q(x, \xi; \xi)\) and \(r(\xi) = r(x, \xi; \xi)\) are analytic with respect to \(\xi \in \Omega_\xi\) and satisfy for large \(|\xi|\) and \(\xi \in \Omega_\xi\)

\[
q(\xi) = (I(x, \xi)-\xi)^{-1}(1+q(x, \xi; \xi)),
\]

\[
r(\xi) = O(\xi^{-2} \langle \xi \rangle^{0}).
\]
for any $N > 0$. In $\Pi$ we have

\begin{equation}
(L-\zeta) ((L-\zeta)^{-1} u - Q(\zeta) u) = u - (L-\zeta) Q(\zeta) u \\
= (L-L) Q(\zeta) u - R(\zeta) u = R(\zeta) u,
\end{equation}

where the symbol of $R(\zeta)$ satisfies the inequality similar to (2.16). In fact, for
\[\phi(x) \in C^\infty_0(\Pi)\] such that $\phi = 1$ in a neighborhood of $\omega$, we see that $(X-X) Q(\zeta) \phi$ is a regularizer in the sense of (2.16). It follows from (2.17) that

\begin{equation}
(L-\zeta)^{-1} u = Q(\zeta) u + (L-\zeta)^{-1} R(\zeta) u \quad \text{in } \Pi.
\end{equation}

From this and (2.12) we have

\begin{equation}
\frac{1}{2\pi i} \int_{\tau} (\log(\xi + 1))^{-1} (\log(\xi + 1)) Q(\zeta) u \ d\xi
\end{equation}

\begin{equation}
+ \frac{1}{2\pi i} \int_{\tau} (\log(\xi + 1))^{mr} (L-\zeta)^{-1} R(\zeta) u \ d\xi \quad \text{in } \Pi.
\end{equation}

Since it follows from (2.16) for $\sigma(R(\zeta))$ that

\[\| (L-\zeta)^{-1} R(\zeta) u \|_{L^2(\Pi)} \leq C |\zeta|^{-2} ||u||,\]

the $L^2(\Pi)$ norm of the second term of the right hand side of (2.18) is estimated above by constant times of $||u||$. In view of (2.14) and (2.15), the residue calculus shows that

\begin{equation}
\frac{1}{2\pi i} \int_{\tau} (\log(\xi + 1))^{-1} (\log(\xi + 1))^{mr} Q(\zeta) \ d\xi
\end{equation}

is a pseudodifferential operator with principal symbol $(I_0 + 1)^{-1} (\log(I_0 + 1))^{mr}$, where $I_0(x, \xi)$ is the principal symbol of $L$ and $I_0(x, \xi) = \sum_{j=1}^{n}(a^2 - x^2)_{\xi j}$ in a neighborhood of $\omega$. Therefore, noting the product formula of pseudodifferential operators, from (2.18) we obtain

\[\| \psi (\log(D_x))^{mr} u \| \leq C (\| \psi (\log(L+1))^{mr} u \| + ||u||), \ u \in C^\infty_0(\omega),\]

where $\psi \in C^\infty_0(\Pi)$ satisfies $0 \leq \psi \leq 1$ and $\psi = 1$ in a neighborhood $\omega_1$ of $\omega$. Since $(1-\psi)(\log(D_x))^{mr} \phi \in S^{-\infty}$ for $\phi \in C^\infty_0(\omega_1)$ satisfying $\phi = 1$ on $\omega$, we have (2.10).

3. Proof of Proposition 4

First we shall prove the sufficiency of (8). For the proof it suffices to show the estimate (5) by Corollary 2 in Introduction. Note that
CRITERION FOR HYPOELLIPTICITY

(3.1) \[ \text{Re}(P_1 u, u) = \|x_2 D_{x_2} u\|^2 + \|D_{x_2} u\|^2 + \|\sigma(x_1) \xi(x) D_{x_1} u\|^2 + \|D_{x_1} u\|^2, \quad u \in S, \]

Here and in what follows we denote the variable \( t \) of \( P \) by \( x_4 \). Let \( \|u\|^2 \) denote the right hand side of (3.1) and let \( f(\xi) \in C^\infty(R^4) \) be a symbol in \( S^0_{1,0} \) such that

\[
\begin{cases}
0 \leq f \leq 1, & f = 1 \text{ on } \{ |\xi| \leq 2 |\xi_3| \} \cap \{ |\xi| \geq 1 \}, \\
\text{supp } f \subset \{ |\xi| \leq 3 |\xi_3| \} \cap \{ |\xi| \geq 1/2 \}.
\end{cases}
\]

For any compact set \( K \) of \( R^4 \) the estimate

\[ \|\langle D_{x_2}\rangle^{1/2} (1 - f(D_{x_2})) u\| \leq C_K(\|u\|^2 + \|u\|^3), \quad u \in C_0^\infty(K), \]

holds with a constant \( C_K \) because for some constants \( C_K' \) and \( C_K'' \) we have

(3.2) \[ \|fu\|^2 \leq \|u\|^2 + C_K' \|u\|^2, \quad u \in C_0^\infty(K), \]

(3.3) \[ \|\langle D_{x_2}\rangle^{1/2} u\|^2 \leq C_K''(\|x_2 D_{x_2} u\|^2 + \|D_{x_2} u\|^2 + \|u\|^3), \quad u \in C_0^\infty(K), \]

(see [3], [14]). Hence, to derive (1) it suffices to show that for any \( \varepsilon > 0 \) and compact set any \( K \) there exists a constant \( C_{\varepsilon,K} \) such that

(3.4) \[ \|\langle D_{x_2}\rangle^{1/2} f u\|^2 \leq \varepsilon \|u\|^2 + C_{\varepsilon,K} \|u\|^3, \quad u \in C_0^\infty(K). \]

In deriving (3.4) for a fixed \( K \) we may assume that \( \sigma \) and \( \xi \) belong to \( B^\infty \), and \( \sigma(x_1) \geq \sigma_0 (|x_1| \geq 1), \xi(x) \geq \xi_0 \) for constants \( \sigma_0, \xi_0 > 0 \). Let \( \phi_0(t), \phi_1(t), \phi_2(t) \) and \( \phi_3(t) \) be \( C^\infty \)-functions in \( [0, \infty) \) such that \( 0 \leq \phi_j \leq 1, \)

\[
\begin{align*}
\text{supp } \phi_0 &\subset [0, 1), & \phi_0 &= 1 \text{ on } [0, 1/2], \\
\text{supp } \phi_1 &\subset [0, 2), & \phi_1 &= 1 \text{ on } [0, 3/2], \\
\text{supp } \phi_2 &\subset (3/2, \infty), & \phi_2 &= 1 \text{ in } [2, \infty), \\
\text{supp } \phi_3 &\subset (1, \infty), & \phi_3 &= 1 \text{ in } [3/2, \infty)
\end{align*}
\]

and

(3.5) \[ \phi_1 + \phi_2 = 1 \text{ in } [0, \infty). \]

Let \( \kappa \) be a small positive constant such that \( \kappa \leq 1/4 \) and set \( \chi_j(x_1, \xi) = \phi_j(\sigma(x_1) \langle \xi \rangle^\kappa) \) \( (j = 0, \cdots, 3) \).

**Lemma 3.1.** It follows that \( \chi_j(x_1, D_x) \) belongs to \( S^0_{1,k} \). Furthermore we have

(3.6) \[ \chi_1 + \chi_2 = I. \]
Proof. (3.6) is the direct consequence of (3.5). Since \( \sigma \) is non-negative we have

\[
|d\sigma(x_j)/dx_j| \leq C_0 \sigma(x_j)^{1/2}
\]

and hence for any \( j \)

\[
|d^j \sigma(x_j)/dx_j^j| \leq C_j \sigma(x_j)^{(1-i/2)_+}
\]

where \( a_+ \) denotes \( \max \{a, 0\} \). From the Leibniz formula we have \( |\alpha + \beta| \neq 0 \)

\[
\chi_{j, j}^{(a)}(x_j, \xi) = \sum_{0 < d_1 + d_2 + \cdots + d_n = d} C_k \phi_j^{(b)}(\sigma(x_j) h) \times \sum_{0 < d_1 + d_2 + \cdots + d_n} C_{a_1}^{(a)} \cdots \sigma(\beta^1) \cdots \sigma(\beta^k) h^{(\alpha^1)} \cdots h^{(\alpha^k)},
\]

where \( h = \langle \xi \rangle^{2\nu} \). Using (3.8) and \( |h^{(a)}| \leq C_2 h^{(\nu)^{-1/2}} \), we obtain

\[
|\chi_{j, j}^{(a)}| \leq C \sum_{0 < d_1 + d_2 + \cdots + d_n} \phi_j^{(b)}(\sigma(x_j) h) \sigma(x_j)^{(k-1/2)_+} h^{(\nu)^{-1/2}}.
\]

Since for \( k \neq 0 \) we have

\[
1/2 \leq \sigma(x_j) h \leq 2 \quad \text{on } \supp \phi_j^{(b)}(\sigma(x_j) h),
\]

we see \( \chi_j(x_j, \xi) \in S^0_{1, x} \). Q.E.D.

**Lemma 3.2.** For any real \( s \) and any \( N > 0 \) there exists a constant \( C = C(s, N) \) such that for \( j = 2, 3 \)

\[
||\chi_j u||_s \leq C(||\sigma(x_j) \xi(x) u||_{s+2\nu} + ||u||_{-N}), u \in S.
\]

Proof. Let \( a(x, \xi) \) denote the simplified symbol of a pseudo-differential operator

\[
\langle D_x \rangle^{\nu} \sigma(x_j) \xi(x) + \chi_{0}(x_j, D_x).
\]

Then \( a(x, \xi) \) belongs to \( S^2_{1, x} \) and satisfies the \( (H) \)-condition in the following sense:

i) There exists a constant \( C_0 > 0 \) such that

\[
a(x, \xi) \geq C_0 \quad \text{for large } |\xi|.
\]

ii) For any \( \alpha \) and \( \beta \) there exists a constant \( C_{\alpha \beta} \) such that

\[
|a^{(a)}_{j}(x, \xi)|/a(x, \xi)| \leq C_{\alpha \beta} \langle \xi \rangle^{(\nu)\alpha - \beta} \quad \text{for large } |\xi|.
\]

Indeed, if \( a_0(x, \xi) = \sigma(x_j) \xi(x) \langle \xi \rangle^{2\nu} + \chi_{0}(x_j, \xi) \) then \( a(x, \xi) - a_0(x, \xi) \in S^2_{1, x-1} \) and hence it suffices to show that \( a_0(x, \xi) \) satisfies the \( (H) \)-condition. In view of (3.7) and (3.8) it is not difficult to check (3.11) for \( a_0 \), by the same way as in the proof.
of Proposition 5.3 of [8]. Since $a(x, \xi)$ satisfies the $(H)$-condition there exists a parametrix $b(x, D_x) \in \mathcal{S}^0_{\text{loc}}$ such that

$$b(x, D_x) a(x, D_x) \equiv I \mod \mathcal{S}^{-\infty}$$

(cf. Theorem 5.4 in Chapter 2 of [5]). Note that for $j=2, 3$

$$\chi_j \equiv \chi_j b a \equiv \chi_j b \langle D_x \rangle^{\sigma(x)} \xi(x) \mod \mathcal{S}^{-\infty}$$

because $\text{supp } \phi_0 \cap \text{supp } \phi_j = \emptyset$. Since $\chi_j b \langle D_x \rangle^{\sigma(x)} \in \mathcal{S}^\infty_{\text{loc}}$, the estimate (3.9) is the direct consequence of (3.12). Q.E.D.

Substituting $D_x u$ instead of $u$ into (3.9) with $s=-2\kappa$ we have

$$||\langle D_x \rangle^{-\sigma(x)} D_x \chi u||^2 \leq C(||\sigma(x) \xi(x) D_x u||^2 + ||u||^2), \ j=2, 3, u \in \mathcal{S},$$

for some constant $C$. In order to show for a fixed compact set $K$

$$||\langle \log \langle D_x \rangle \rangle \chi f u||^2 \leq C ||u||^2 + C||u||, \ u \in C_0^\infty(K)$$

we prepare the following lemma.

**Lemma 3.3.** Set $I_\mu = \{t \in \mathbb{R}^1; |t| < \mu\}$ for $\mu > 0$. Then for any $s > 0$ there exists a constant $C_s$ independent of $\mu$ such that

$$||v|| \leq C_s \mu^s ||v||_s \ for \ v \in C_s^\infty(I_\mu).$$

The lemma seems to be fairly well-known, but we give the proof for the convenience of the reader.

Proof. First we shall prove that for any $\varepsilon > 0$ there exists a $\mu_0 > 0$ such that

$$||v|| \leq \varepsilon ||v||_s \ for \ v \in C_s^\infty(I_{\mu_0}).$$

Suppose that there exist an $\varepsilon_0 > 0$ and $\{v_j\}_{j=1}^\infty \subset C_s^\infty$ such that

$$\text{supp } v_j \subset I_{\mu_j}, \ \mu_j \to 0 (j \to \infty),$$

$$||v_j||_s = 1, \ ||v_j|| > \varepsilon_0.$$
\( v_0 = 0 \), which is contradictory to \( \|v_0\| \geq \varepsilon_0 \). From (3.16) we have for some \( \mu_0 > 0 \)
\begin{equation}
\|\hat{\theta}(\tau)\| \leq \|\tau\|^\frac{1}{2} \|\hat{\theta}(\tau)\| \quad \text{for} \quad \psi \in C_0^0(I_{\mu_0}),
\end{equation}
where \( \theta \) is the Fourier transform of \( v \). The estimate (3.15) easily follows from (3.17) if we take the change of variable from \( t \) to \( \mu_0 t / \mu \). Q.E.D.

Let \( \phi_4(t) \) be a \( C^\infty \)-function in \([0, \infty)\) such that \( \phi_4 = 1 \) in \([0,2]\) and \( \operatorname{supp} \phi_4 \subseteq [0,3) \). Set
\[ \chi_4(x_1, \xi_3) = \phi_4(\sigma(x_1)\sqrt{\xi_3^{2n}}). \]
It follows from the condition (8) that for any \( \varepsilon > 0 \) there exists a \( M_\varepsilon > 0 \) such that
\begin{equation}
|x_1| \leq \varepsilon (\log\langle \xi_3 \rangle)^{-2} \quad \text{on} \quad \chi_4(x_1, \xi_3) \quad \text{if} \quad |\xi_3| \geq M_\varepsilon,
\end{equation}
because \( (x_1, \xi_3) \in \operatorname{supp} \chi_4 \) implies \( \sigma(x_1)\sqrt{\xi_3^{2n}} \leq 3 \). Let \( \hat{u} \) denote the Fourier transform of \( u \in S_\varepsilon \) with respect to \( x_3 \) variable. Setting \( v(x_1) = \chi_4(x_1, \cdot) \hat{u}(x_1, \cdot) \) in (3.15) \( s = 1/2 \), in view of (3.18) we have
\[ ||\chi_4(x_1, \xi_3) \hat{u}||^2 \chi^2_{(x_1, \xi_3)} \leq C_1 \varepsilon (\log\langle \xi_3 \rangle)^{-2} ||\langle D_{x_3}\rangle^{1/2} \chi_4 \hat{u}||^{2} \chi^2_{(x_1, \xi_3)}, \]
if \( |\xi_3| \geq M_\varepsilon \), for some constant \( C_1 \) independent of \( \varepsilon \). Multiplying both sides by \( (\log\langle \xi_3 \rangle)^{2} \) and integrating with respect to \( x_2, x_4 \) and \( \xi_3 \) we have
\[ ||(\log\langle D_{x_3}\rangle) \chi_4(x_1, D_{x_3}) u||^2 \leq C_1 \varepsilon ||\langle D_{x_3}\rangle^{1/2} \chi_4(x_1, D_{x_3}) u||^2 
\]
\[ + C_2 ||u||^2. \]
Noting \( \chi_4(x_1, D_{x_3}) \chi_4(x_1, D_{x_3}) = \chi_4(x_1, D_{x_3}) \), we obtain
\begin{equation}
|| (\log \langle D_{x_3}\rangle) \chi_4 f u ||^2 \leq C_1 \varepsilon ||\langle D_{x_3}\rangle^{1/2} \chi_4 f u ||^2 + C_2 ||u||^2, \quad u \in S.
\end{equation}
In view of (3.3) we have for a fixed compact set \( K \)
\begin{equation}
|| \langle D_{x_3}\rangle^{1/2} \chi_4 f u ||^2 \leq C_2 (||x_2 D_{x_3} f u||^2 + ||D_{x_3} f u||^2 
\]
\[ + ||(D_{x_1} \chi_4 f u||^2 + ||u||^2), \quad u \in C_0^0(K). \]
Since \( (D_{x_3} \chi_4)(x, \xi) \in S^1_{1, \infty} \) and \( \chi_3 = 1 \) on \( \operatorname{supp} D_{x_3} \chi_4 \), we obtain
\begin{equation}
|| (D_{x_3} \chi_4) f u ||^2 \leq C (||\langle D_{x_3}\rangle^{2} \chi_3 f u||^2 + ||u||^2) 
\]
\[ \leq C' (||\langle D_{x_3}\rangle^{1-2n} D_{x_3} \chi_3 f u||^2 + ||u||^2), \quad u \in S. \]
Using (3.13) with \( j = 3 \) and (3.2), from (3.20) and (3.21) we have
\[ ||\langle D_{x_3}\rangle^{1/2} \chi_4 f u||^2 \leq C_K (||u||^2 + ||u||^2), \quad u \in C_0^0(K). \]
Combining this and (3.19) we obtain (3.14). The estimate (3.4) follows from (3.14) and (3.13) with \( j=2 \). We have proved the estimate (5) for \( P_x \).

From now on we shall prove the necessity of (8). Suppose that (8) is not satisfied but \( P_x \) is hypoelliptic. Then there exists a \( \delta>0 \) and a sequence \( \{s_k\}_{k=1}^\infty \) such that

\[
\begin{aligned}
&\{s_k \to 0 \quad (k \to \infty)\, , \\
&\sigma(s_k) \exp(2\delta^{-1} |s_k|^{-1/2}) \leq 1 .
\end{aligned}
\]

Without loss of generality, we may assume \( s_k>0 \). Set \( \lambda_k = \exp(-\delta^{-1} s_k^{1/2}) \). Then it follows from (3.22) that

\[
\lambda_k^2 \sigma(s/(\log \lambda_k^{-\delta})) \leq 1 \quad \text{for} \quad 0 \leq s \leq 1 ,
\]

because \( \sigma \) is non-decreasing in \( R_+ \). By means of Theorem 3 and its remark we see that for any \( \varepsilon>0 \) and any compact set \( K \) of \( \mathbb{R}^4 \) there exists a constant \( C_{s,K} \) such that

\[
||\log(D_x^4) u||^2 \leq \varepsilon \Re(P_1 u, u) + C_{s,K}||u||^2 , \quad u \in C_0^\infty(K) .
\]

(Recall that the variable \( t \) of \( P_x \) is denoted by \( x_4 \) in this section.) Note

\[
||\log(D_x^4) \phi_0(\lambda_k^2 D_{s_x} - 1) u|| \leq ||\log(D_x^4) u|| , \quad u \in C_0^\infty(K) .
\]

Since \( \xi \) is bounded on \( K \), in view of (3.1) we have

\[
||\log(D_x^4) \phi_0(\lambda_k^2 D_{s_x} - 1) u|| \\
\leq \varepsilon (||x_2 D_{s_x} u|| + ||D_{s_x} u|| + ||\sigma(x_1) D_{s_x} u|| + ||D_{x_4} u||) \\
+ C_{s,K}||u|| , \quad u \in C_0^\infty(K) .
\]

Set \( v(y) = \prod_{j=1}^4 \phi_0(2 |y_j - 1/2|) \) and consider the change of variables

\[
\begin{align*}
y_1 &= (\log \lambda_k^{-\delta})^2 x_1 , \quad y_2 = (\log \lambda_k^{-\delta}) x_2 , \\
y_3 &= \lambda_k^{-1} x_3 , \quad y_4 = x_4 .
\end{align*}
\]

Let \( u_0(x) \) denote the function \( v \) after the above change of variables. Then the support of \( u_0(x) \) is contained in \( \{ |x| \leq 4 \} \) if \( \lambda_k \) is small enough. Substitute \( \exp(i\lambda_k^{-2} x_3) u_0(x) \) into (3.25) and take the change of variables from \( x \) to \( y \). Then, by using the similar formula as (1.10) we have

\[
2 \log \lambda_k^{-1} ||\phi_0(\lambda_k D_{s_x}) v|| \\
\leq \varepsilon (\delta \log \lambda_k^{-1} (||y_2 D_{s_x} v|| + ||D_{s_x} v||) \\
+ \lambda_k^{-1} ||\sigma(y_1/(\log \lambda_k^{-\delta})) D_{s_x} v|| \\
+ \lambda_k^{-2} ||\sigma(y_1/(\log \lambda_k^{-\delta})) v|| + ||D_{x_4} v||) + C_s ||v|| ,
\]

(3.26)
because a pseudodifferential operator in $\mathbb{R}^3$ with a symbol $(\log(\lambda_\epsilon + (\lambda_\epsilon \eta_3 + 1)^\delta))^{1/2}$ $\phi_\delta(\lambda_\epsilon \eta_3)$ is $L^2$-bounded uniformly with respect to $\lambda_\epsilon$. Note $\phi_\delta(\lambda_\epsilon D_\eta) v$ converges $v$ in $L^2$ when $\lambda_\epsilon$ tends to 0. Then, there exists a $c_0 > 0$ such that

$$||\phi_\delta(\lambda_\epsilon D_\eta) v|| \leq c_0$$

if $\lambda_\epsilon \leq \lambda_0$ for a sufficiently small $\lambda_0$.

Since it follows from (3.23) that $\lambda_\epsilon^{-2} \sigma(\gamma_3/(\log \lambda_\epsilon^{-\delta})) \leq 1$ on $\text{supp} \ v$ and also on $\text{supp} \ D_\eta v$, there exist constants $c_1, c_2$ independent of $\epsilon$ and $C_\epsilon'$ such that

$$2c_0 \log \lambda_\epsilon^{-1} \leq c_1 \epsilon \log \lambda_\epsilon^{-1} + c_2 \epsilon + C_\epsilon' \quad \text{if} \quad \lambda_\epsilon \leq \lambda_0.$$

Setting $\epsilon = c_0 / c_1$ we have a contradiction when $\lambda_\epsilon$ tends to 0.

REMARK 3.1. By the similar way as in the proof of the necessity of (8), we can show that the estimate (3) dose not hold with some small $\epsilon_0 > 0$ for $\mathcal{A}_\delta (x, D_x) = D_{\eta_1}^2 + \exp(-1/|x_1|^\delta)D_{\eta_2}^2$ when $\delta \geq 1$. This fact also can be seen by considering the eigenvalue problem for a differential operator $-d^2/dx^2 + \exp(-1/|x|^\delta) \eta^2$ with Dirichlet boundary condition. It was proved in [10] that the smallest eigenvalue is estimated above by $(\log \eta)^2$ with a constant factor.

REMARK 3.2. Let $P_3$ be a differential operator

$$D_{\eta_1}^2 + x_1^2 D_{\eta_1}^2 + D_{\eta_2}^2 + \sigma(x_1)^2 (x_1 D_{\eta_2} + D_{\eta_2} x_1)^2 \quad \text{in} \quad \mathbb{R}^6,$$

where $\sigma \in C^\infty$, $\sigma(0) = 0$, $\sigma(s) > 0 (s \neq 0)$ and $s \sigma'(s) \geq 0$. By the same way as in this section we can prove that $P_3$ is hypoelliptic in $\mathbb{R}^6$ if and only if $\sigma$ satisfies (8).

Acknowledgement. The study of the present paper originates in the excellent work of Kusuoka-Strook [6], who studied the hypoellipticity of differential operators (even with infinite degeneracy) by using the Malliavin calculus. The author would like to thank Prof. S. Kusuoka for useful discussions. The author wishes to express his hearty gratitude to Prof. M. Ikawa and Prof. H. Tamura for their suggestions to the proof of (2.10). The author also would like to thank Prof. T. Matsuzawa for many valuable discussions about the hypoellipticity.

References


Adding to the proof: After the present paper had been submitted, the author received a preprint [15] from Mr. Hoshiro. In [15], one can see almost the same result as Proposition 2.2.

Institute of Mathematics
Yoshida College
Kyoto University
Sakyo-ku, Kyoto 606
Japan