MINIMAL IMMERSIONS OF SOME CIRCLE BUNDLES
OVER HOLOMORPHIC CURVES
IN COMPLEX QUADRIC TO SPHERE

To the memory of Yuko

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0. Introduction

Minimal surfaces in a space of constant curvature has been studied by many mathematicians (cf. [3], [7], [8], [9]). In particular, for minimally immersed 2-sphere in the standard sphere, Calabi [4] and Barbosa [1] showed that: *There exists a canonical 1-1 correspondence between the set of generalized minimal immersions* $\chi : S^2 \to S^{2m}(1)$ *which are not contained in any lower dimensional subspace of* $\mathbb{R}^{2m+1}$, *and the set of totally isotropic holomorphic curves* $\Xi : S^2 \to \mathbb{CP}^{2m}$ *which are not contained in any complex hyperplane of* $\mathbb{CP}^{2m}$. *The correspondence is the one that associates with minimal immersion* χ *its directrix curve* (§2). Note that this fact is valid for *pseudo holomorphic map* [4] from a compact Riemann surface Σ^2 instead of S^2, and that the image of the directrix curve is contained in a complex quadric Q^{2m-1} of $\mathbb{CP}^{2m}$.

On the other hand one of the most interesting 3-dimensional minimal submanifolds in a sphere is the *minimal Cartan hypersurface* (MCH) of S^4, i.e. the minimal hypersurface with 3 distinct constant principal curvatures in a 4-sphere (cf. [5]). MCH is obtained from the directrix curve of the Veronese surface as follows: Let $\chi : S^2(1/3) \to S^4(1)$ be the Veronese immersion from the 2-sphere with constant Gauss curvature 1/3 to the unit 4-sphere, and let $\Xi : S^2(1/3) \to \mathbb{CP}^4$ be the directrix curve of χ. Then χ is congruent to the fourth order Veronese embedding $\mathbb{CP}^1 \to \mathbb{CP}^4$ and the image $\Xi(S^2(1/3))$ is contained in a complex quadric Q^3 in $\mathbb{CP}^4$. Let $P = SO(5)/SO(3)$ be the set of ordered two orthonormal vectors in $\mathbb{R}^5$, and let $P(Q^3, S^1)$ be the circle bundle over Q^3, which is given as the pullback bundle of the Hopf fibration $S^9(\mathbb{CP}^4, S^1)$ with respect to the natural inclusion $Q^3 \subset \mathbb{CP}^4$. Then MCH is identified with the pullback bundle $\pi_\Xi : \Xi^* P \to S^2(1/3)$ such that each fiber $\pi_\Xi^{-1}(p)$ for $p \in S^2(1/3)$ is corresponding to the great circle which is determined by $\Xi(p) \in Q^3$

in $S^4(1)$. In other words, MCH is realized as a tube of radius $\pi/2$ over the Veronese surface in S^4, so MCH is diffeomorphic to the unit normal bundle of the Veronese surface in the 4-sphere.

In this paper we will study, as a generalization of minimal Cartan hypersurface, minimal immersion of some circle bundle over a Riemann surface Σ^2 which is immersed in complex quadric $Q^{n-1} = SO(n + 1)/SO(n - 1) \times SO(2)$ to sphere S^n. More precisely let $P(Q^{n-1}, S^1)$ be the circle bundle over Q^{n-1}, where $P = SO(n + 1)/SO(n - 1)$ is the set of ordered two orthonormal vectors in $\mathbb{R}^{n+1}$. Here P is the pullback bundle of the Hopf fibration $S^{2n+1}(\mathbb{CP}^n, S^1)$ with respect to the natural inclusion $Q^{n-1} \subset \mathbb{CP}^n$. Let $\varphi : \Sigma^2 \to Q^{n-1}$ be a conformal immersion from a Riemann surface Σ^2 to the complex quadric, and let $\pi_\varphi : \varphi^* P \to \Sigma^2$ be the pullback bundle over Σ^2 with respect to φ. Then each fiber $\pi_\varphi^{-1}(p)$ for $p \in \Sigma^2$ is naturally identified with the great circle of S^n determined by the 2-plane $\varphi(p) \in Q^{n-1}$. We can define the map $\Phi : \varphi^* P \to S^n(1)$ by this identification.

In §1 we review complex quadric Q^{n-1} and construct the circle bundle P over Q^{n-1}, and in §2 we see some surfaces and holomorphic curves in Q^{n-1}. In §3 we show that if a three dimensional submanifold M in a sphere S^n is foliated by great circles of S^n, then there is an associated surface Σ^2 in Q^{n-1}. Conversely we construct the map $\Phi : \varphi^* P \to S^n(1)$ from the surface $\varphi : \Sigma^2 \to Q^{n-1}$ explicitly and, on the set of regular points of Φ, we determine the condition with respect to φ for which the pullback bundle $\varphi^* P$ is minimal in $S^n(1)$ (Proposition 3.9 and Corollary 3.10). In §4 we show that if the immersion $\varphi : \Sigma^2 \to Q^{n-1}$ is holomorphic, then the corresponding map $\Phi : \varphi^* P \to S^n(1)$ is regular at each point in $\pi_\varphi^{-1}(x)$ for $x \in \Sigma^2$ if and only if x is not a real point (Definition 2.7) of φ. Moreover we can see that Φ is minimal if and only if either Φ is totally geodesic or the corresponding holomorphic curve $\varphi(\Sigma^2)$ in Q^{n-1} is first order isotropic (Theorem 1). As a consequence, we can construct full and minimal immersion $\Phi : \Xi^* P \to S^{2m}(1)$ from the directrix curve $\Xi : S^2 \to Q^{2m-1}$ of fully immersed minimal 2-sphere $\chi : S^2 \to S^{2m}(1)$ (Theorem 2). We also discover relations of the curvatures between holomorphic curve $\varphi : \Sigma^2 \to Q^{n-1}$ and the immersion $\Phi : \varphi^* P \to S^n(1)$.

1. Preliminaries

First of all, we recall the Fubini-Study metric on the complex projective space $\mathbb{CP}^n$. The Euclidean metric $\langle \ , \ \rangle$ on $\mathbb{C}^{n+1}$ is given by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{x} \cdot \mathbf{u} + \mathbf{y} \cdot \mathbf{v},$$

where $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, $\mathbf{w} = \mathbf{u} + i\mathbf{v} \in \mathbb{C}^{n+1}$ ($i = \sqrt{-1}$), $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^{n+1}$ and $\mathbf{x} \cdot \mathbf{y}$ denotes the standard inner product on $\mathbb{R}^{n+1}$. The sphere $S^{2n+1}(1/c)$ of radius $\sqrt{c}$ ($c > 0$) in $\mathbb{C}^{n+1}$ is the principal fiber bundle over $\mathbb{C}P^n$ with the structure group S^1 and the projection map π (*the Hopf fibration*). The tangent space of S^{2n+1} at a point $\mathbf{z}$ is

$$T_{\mathbf{z}}S^{2n+1} = \{\mathbf{w} \in \mathbb{C}^{n+1}; \langle \mathbf{z}, \mathbf{w} \rangle = 0\}.$$

Let

$$T'_{\mathbf{z}} = \{\mathbf{w} \in \mathbb{C}^{n+1}; \langle \mathbf{z}, \mathbf{w} \rangle = \langle \mathbf{z}, i\mathbf{w} \rangle = 0\}.$$

Then the distribution $T'_{\mathbf{z}}$ defines a connection in the principal fiber bundle $S^{2n+1}(\mathbb{C}P^n,$ $S^1)$, because $T'_{\mathbf{z}}$ is complementary to the subspace $\{i\mathbf{z}\}$ tangent to the fiber through $\mathbf{z}$, and invariant under the S^1-action. The Fubini-Study metric $\overline{g}$ of constant holomorphic sectional curvature $4/c$ is then given by $\overline{g}(X, Y) = \langle X^*, Y^* \rangle$, where $X, Y \in T_x\mathbb{C}P^n$ and X^*, Y^* are respectively their horizontal lifts at a point $\mathbf{z}$ with $\pi(\mathbf{z}) = x$. The complex structure on $T'_{\mathbf{z}}$ defined by multiplication of $i = \sqrt{-1}$ induces a canonical complex structure J on $\mathbb{C}P^n$ through π_*.

Given a vector field X on $\mathbb{C}P^n$, there is a corresponding *basic vector field* X' on S^{2n+1} such that at $z \in S^{2n+1}$, $X'_z \in T'_z$ and $(\pi_*)_z X'_z = X_{\pi(z)}$. If X, Y are vector fields on $\mathbb{C}P^n$, the Kählerian covariant derivative takes the form

$$\overline{\nabla}^{\mathbb{C}P}_X Y = (\pi_*)\nabla'_{X'}Y',$$

where X', Y' are the basic vector fields corresponding to X, Y and ∇' is the Levi-Civita connection on S^{2n+1}.

Next we recall a description of a complex quadric Q^{n-1} in $\mathbb{C}P^n$ (cf. [13]). Let P be the space of ordered two orthonormal vectors in $\mathbb{R}^{n+1}$, i.e.,

$$(1.1) \qquad P = \{Z \in M(n+1, 2, \mathbb{R}); \ {}^t ZZ = E_2\}.$$

As a homogeneous space, P is isomorphic to $SO(n+1)/SO(n-1)$ (Stiefel manifold) with $\dim_{\mathbb{R}} P = 2n - 1$. Denote $Z = (\mathbf{e}, \mathbf{f}) \in P$, where $\mathbf{e}$ and $\mathbf{f}$ are column vectors of Z. Then the tangent space of P at the point Z is

$$T_Z P = \{X \in M(n+1, 2, \mathbb{R}); \ {}^t XZ + {}^t ZX = 0\},$$
$$= \mathbb{R}(-\mathbf{f}, \mathbf{e}) \oplus \{(\mathbf{x}, \mathbf{y}); \ \mathbf{x}, \mathbf{y} \perp \text{span}\{\mathbf{e}, \mathbf{f}\}\},$$

and the Riemannian metric $\widetilde{g}$ on P is given by

$$\widetilde{g}(X, Y) = \text{trace}({}^t XY), \quad X, Y \in T_Z P \subset M(n+1, 2, \mathbb{R}).$$

Let Q^{n-1} be the space of oriented 2-planes in $\mathbb{R}^{n+1}$. Then P is the principal fiber bundle over Q^{n-1} with the structure group S^1 and the projection map $\pi' : P \to Q^{n-1}$

defined by

(1.2) $$\pi'((\mathbf{e}, \mathbf{f})) = \mathrm{span}\{\mathbf{e}, \mathbf{f}\}.$$

Let

$$T'(\mathbf{e}, \mathbf{f}) = \{(\mathbf{x}, \mathbf{y}) \in M(n + 1, 2, \mathbb{R}); \ \mathbf{x}, \mathbf{y} \perp \mathrm{span}\{\mathbf{e}, \mathbf{f}\}\}.$$

Then the distribution $T'_{(\mathbf{e}, \mathbf{f})}$ defines a connection in the principal fiber bundle $P(Q^{n-1}, S^1)$, because $T'_{(\mathbf{e}, \mathbf{f})}$ is complementary to the subspace $\mathbb{R}(-\mathbf{f}, \mathbf{e})$ tangent to the fiber through $(\mathbf{e}, \mathbf{f})$, and invariant under the S^1-action.

The metric g is then given by $g(X, Y) = \tilde{g}(X^*, Y^*)$, where $X, Y \in T_z Q^{n-1}$ and X^*, Y^* are respectively their horizontal lifts at a point $Z = (\mathbf{e}, \mathbf{f})$ with $\pi'(Z) = z$. The complex structure on $T'_{(\mathbf{e}, \mathbf{f})}$ defined by

(1.3) $$(\mathbf{x}, \mathbf{y}) \mapsto (-\mathbf{y}, \mathbf{x})$$

induces a canonical complex structure J' on Q^{n-1} through π_*. Given a vector field X on Q^{n-1}, there is a corresponding *basic vector field* X' on P such that at $Z = (\mathbf{e}, \mathbf{f}) \in P$, $X'_Z \in T'_Z$ and $(\pi'_*)_Z X'_Z = X_{\pi'(Z)}$. If X, Y are vector fields on Q^{n-1}, the Kählerian covariant derivative takes the form

(1.4) $$\overline{\nabla}^Q_X Y = (\pi'_*)\nabla^P_{X'} Y'$$

where X', Y' are the basic vector fields corresponding to X, Y and ∇^P is the Levi-Civita connection on P.

We consider an injective map $\tilde{\imath}$ from P to a $2n + 1$-dimensional sphere S^{2n+1} of radius $\sqrt{2}$ in $\mathbb{C}^{n+1}$, defined by

$$\tilde{\imath}((\mathbf{e}, \mathbf{f})) = \mathbf{e} + i\mathbf{f}.$$

For tangent vectors $(-\mathbf{f}, \mathbf{e})$ and $(\mathbf{x}, \mathbf{y})$ $(\mathbf{x}, \mathbf{y} \perp \mathrm{span}\{\mathbf{e}, \mathbf{f}\})$ in $T_{(\mathbf{e}, \mathbf{f})}P$, the differential map of $\tilde{\imath}$ is

$$(\tilde{\imath}_*)_{(\mathbf{e}, \mathbf{f})}(-\mathbf{f}, \mathbf{e}) = -\mathbf{f} + i\mathbf{e},$$
$$(\tilde{\imath}_*)_{(\mathbf{e}, \mathbf{f})}(\mathbf{x}, \mathbf{y}) = \mathbf{x} + i\mathbf{y},$$

so $\tilde{\imath}$ is an embedding. Now we can define a holomorphic embedding $\imath : Q^{n-1} \to \mathbb{CP}^n$ as

$$\imath(\mathrm{span}\{\mathbf{e}, \mathbf{f}\}) = \pi(\mathbf{e} + i\mathbf{f}).$$

Hence we have the following commutative diagram:

$$
\begin{array}{ccc}
P & \xrightarrow{\ \tilde{\imath}\ } & S^{2n+1} \\
{\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\
Q^{n-1} & \xrightarrow{\ \imath\ } & \mathbb{CP}^n
\end{array}
$$

Q^{n-1} is also defined by the quadratic equation $z_0^2 + z_1^2 + \cdots + z_n^2 = 0$, where $z_0, z_1, \ldots, z_n$ is a homogeneous coordinate of $\mathbb{CP}^n$.

REMARK 1.1.   Note that $P(Q^{n-1}, S^1)$ is nothing but the pullback bundle of the Hopf fibration $S^{2n+1}(\mathbb{CP}^n, S^1)$ with respect to $\imath$. Clearly we have the following identification:

$$
\begin{array}{ccc}
\mathrm{span}\{\mathbf{e}, \mathbf{f}\} & \mapsto & \{\cos\theta\,\mathbf{e} + \sin\theta\,\mathbf{f} \mid \theta \in S^1\} \\
\cap & & \cap \\
Q^{n-1} & & \{\text{oriented great circles } S^1 \subset S^n\}.
\end{array}
$$

Then for each oriented great circle $C \in Q^{n-1}$, the fiber of C with respect to π' is identified with C itself as

$$
\begin{array}{ccc}
(\cos\theta\,\mathbf{e} + \sin\theta\,\mathbf{f},\ -\sin\theta\,\mathbf{f} + \cos\theta\,\mathbf{e}) & \mapsto & (\cos\theta\,\mathbf{e} + \sin\theta\,\mathbf{f}). \\
\cap & & \cap \\
(\pi')^{-1}(C) & & C
\end{array}
$$

With respect to the metric induced by $\tilde{\imath}$ and $\imath$, P and Q^n become Riemannian manifolds, respectively, and the projection $\pi' : P \to Q^{n-1}$ becomes a Riemannian submersion. The normal space of P in S^{2n+1} (resp. Q^{n-1} in $\mathbb{CP}^n$) at the point $(\mathbf{e}, \mathbf{f})$ is spanned by the following orthonormal vectors:

$$
N_1' = \frac{\mathbf{e} - i\mathbf{f}}{\sqrt{2}}, \qquad N_2' = \frac{\mathbf{f} + i\mathbf{e}}{\sqrt{2}},
$$

$$
(\text{resp. } N_1 = (\pi_*)N_1', \qquad N_2 = (\pi_*)N_2').
$$

The shape operators $A_{N_1}^{\imath}$ and $A_{N_2}^{\imath}$ of Q^{n-1} in $\mathbb{CP}^n$ with respect to unit normal vectors N_1 and N_2 at $\pi(\mathbf{e}, \mathbf{f})$ are given by

(1.5)
$$
\langle A_{N_1}^{\imath} \pi_*'(\mathbf{x}, \mathbf{y}), \pi_*'(\mathbf{u}, \mathbf{v}) \rangle = \frac{-\mathbf{x} \cdot \mathbf{u} + \mathbf{y} \cdot \mathbf{v}}{\sqrt{2}},
$$

$$
\langle A_{N_2}^{\imath} \pi_*'(\mathbf{x}, \mathbf{y}), \pi_*'(\mathbf{u}, \mathbf{v}) \rangle = -\frac{\mathbf{x} \cdot \mathbf{v} + \mathbf{y} \cdot \mathbf{u}}{\sqrt{2}},
$$

where $(\mathbf{x}, \mathbf{y})$ and $(\mathbf{u}, \mathbf{v}) \in T_{(\mathbf{e}, \mathbf{f})}'$.

2. Surfaces and holomorphic curves of complex quadric

Let Σ^2 be a Riemann surface and let $\varphi : \Sigma^2 \to Q^{n-1}$ be a conformal immersion to the complex quadric. Then there exists a local lift $\widetilde{\varphi} : U \to P$ (U is an open set in Σ^2) of φ, i.e., $\widetilde{\varphi}(p) = (\mathbf{e}(p), \mathbf{f}(p)) \in P$ for $p \in U$ (cf. §1), where $(\mathbf{e}(p), \mathbf{f}(p))$ is an ordered orthonormal frame of the 2-plane $\varphi(p) \in Q^{n-1}$. Put $\widetilde{\psi} = \widetilde{\imath} \circ \widetilde{\varphi} : \Sigma^2 \to S^{2n+1} \subset \mathbb{C}^{n+1}$ and $\psi = \imath \circ \varphi : \Sigma^2 \to \mathbb{C}P^n$, respectively. Then $\widetilde{\psi}$ is written as

$$(2.1) \qquad \widetilde{\psi}(p) = \mathbf{e}(p) + i\mathbf{f}(p),$$

where $\mathbf{e}$ and $\mathbf{f}$ are both $\mathbb{R}^{n+1}$-valued function on some open set of Σ^2.

Let (t_1, t_2) be an isothermal coordinate on some coordinate neighborhood U of Σ^2. We put the differential of $\mathbf{e}$ and $\mathbf{f}$ with respect to (t_1, t_2) as

$$(2.2) \qquad \mathbf{e}_j := \partial \mathbf{e}/\partial t_j = \lambda_j \mathbf{f} + \mathbf{p}_j, \quad \mathbf{f}_j := \partial \mathbf{e}/\partial t_j = -\lambda_j \mathbf{e} + \mathbf{q}_j \quad (j = 1, 2),$$

where $\lambda_j : \Sigma^2 \to \mathbb{R}$ ($j = 1, 2$) is a function, and $\mathbf{p}_j, \mathbf{q}_j \perp \mathrm{span}\{\mathbf{e}, \mathbf{f}\}$. Then the differential map $(\widetilde{\varphi}_*)_p : T_p(\Sigma^2) \to T_{\widetilde{\varphi}(p)}(P)$ is

$$(2.3) \qquad (\widetilde{\varphi}_*)_p(\partial/\partial t_j) = (\lambda_j \mathbf{f} + \mathbf{p}_j, -\lambda_j \mathbf{e} + \mathbf{q}_j),$$

and the horizontal part with respect to $\pi' : P \to Q^{n-1}$ is

$$(2.4) \qquad \mathcal{H}(\widetilde{\varphi}_*)_p(\partial/\partial t_j) = (\mathbf{p}_j, \mathbf{q}_j).$$

Since (t_1, t_2) is an isothermal coordinate of Σ^2, we have

$$(2.5) \qquad \rho := \|\mathbf{p}_1\|^2 + \|\mathbf{q}_1\|^2 = \|\mathbf{p}_2\|^2 + \|\mathbf{q}_2\|^2, \quad \mathbf{p}_1 \cdot \mathbf{p}_2 + \mathbf{q}_1 \cdot \mathbf{q}_2 = 0.$$

(1.3) and (2.4) imply

Proposition 2.1. *Let $\varphi : \Sigma^2 \to Q^{n-1}$ be an immersion from a Riemann surface to a complex quadric. Then*

$$(2.6) \qquad \varphi \text{ is holomorphic} \iff \mathbf{p}_1 = \mathbf{q}_2 \text{ and } \mathbf{p}_2 = -\mathbf{q}_1,$$

$$\varphi \text{ is anti-holomorphic} \iff \mathbf{p}_1 = -\mathbf{q}_2 \text{ and } \mathbf{p}_2 = \mathbf{q}_1.$$

Note that the Kähler angle α of the immersion $\varphi : \Sigma^2 \to Q^{n-1}$ is given by

$$\cos \alpha = \rho^{-1} \langle J\varphi_*(\partial/\partial t_1), \varphi_*(\partial/\partial t_2) \rangle$$
$$= \rho^{-1}(\mathbf{p}_1 \cdot \mathbf{q}_2 - \mathbf{p}_2 \cdot \mathbf{q}_1).$$

Suppose that the immersion $\varphi : \Sigma^2 \to Q^{n-1}$ is holomorphic, i.e., (2.6) holds. Then (2.3) is written as

$$(2.7) \qquad (\widetilde{\varphi}_*)_p(\partial/\partial t_1) = (\lambda_1 \mathbf{f} + \mathbf{p}_1, -\lambda_1 \mathbf{e} - \mathbf{p}_2),$$

$$(\widetilde{\varphi}_*)_p(\partial/\partial t_2) = (\lambda_2 \mathbf{f} + \mathbf{p}_2, -\lambda_2 \mathbf{e} + \mathbf{p}_1),$$

and the horizontal part of these vectors are

$$\mathcal{H}(\widetilde{\varphi}_*)_p(\partial/\partial t_1) = (\mathbf{p}_1, -\mathbf{p}_2),$$
$$\mathcal{H}(\widetilde{\varphi}_*)_p(\partial/\partial t_2) = (\mathbf{p}_2, \mathbf{p}_1).$$

EXAMPLE 2.2.   Let $\psi_n : \mathbb{CP}^1 \to \mathbb{CP}^n$ be the Veronese embedding of order n given by

$$\psi_n(z) = \left[1, \sqrt{\binom{n}{1}}z, \sqrt{\binom{n}{2}}z^2, \dots, \sqrt{\binom{n}{k}}z^k, \dots, \sqrt{\binom{n}{n-1}}z^{n-1}, z^n \right],$$

where z is an inhomogeneous coordinate of $\mathbb{CP}^1$. Then $\psi_n(\mathbb{CP}^1)$ is contained in some Q^{n-1} in $\mathbb{CP}^n$ if and only if n is even. When $n = 4m$ ($m \geq 1$), $\psi_{4m} : \mathbb{CP}^1 \to Q^{4m-1} \subset \mathbb{CP}^{4m}$ is represented as:

$$
\begin{aligned}
\psi_{4m}(z) = \Bigg[& 1 + z^{4m}, i(1 - z^{4m}), \\
& \sqrt{\binom{4m}{1}}(z - z^{4m-1}), i\sqrt{\binom{4m}{1}}(z + z^{4m-1}), \\
& \sqrt{\binom{4m}{2}}(z^2 + z^{4m-2}), i\sqrt{\binom{4m}{2}}(z^2 - z^{4m-2}), \\
& \sqrt{\binom{4m}{3}}(z^3 - z^{4m-3}), i\sqrt{\binom{4m}{3}}(z^3 + z^{4m-3}), \\
& \dots, \\
& \sqrt{\binom{4m}{2m-1}}(z^{2m-1} - z^{2m+1}), i\sqrt{\binom{4m}{2m-1}}(z^{2m-1} + z^{2m+1}), \\
& \sqrt{2\binom{4m}{2m}}z^{2m} \Bigg],
\end{aligned}
$$

and when $n = 4m - 2$ ($m \geq 1$), $\psi_{4m-2} : \mathbb{CP}^1 \to Q^{4m-3} \subset \mathbb{CP}^{4m-2}$ is represented as:

$$
\begin{aligned}
\psi_{4m-2}(z) = \Bigg[& 1 + z^{4m-2}, i(1 - z^{4m-2}), \\
& \sqrt{\binom{4m-2}{1}}(z - z^{4m-3}), i\sqrt{\binom{4m-2}{1}}(z + z^{4m-3}),
\end{aligned}
$$

$$\sqrt{\binom{4m-2}{2}}(z^2 + z^{4m-4}), \, i\sqrt{\binom{4m-2}{2}}(z^2 - z^{4m-4}),$$

$$\sqrt{\binom{4m-2}{3}}(z^3 - z^{4m-5}), \, i\sqrt{\binom{4m-2}{3}}(z^3 + z^{4m-5}),$$

$$\cdots,$$

$$\sqrt{\binom{4m-2}{2m-2}}(z^{2m-2} + z^{2m}), \, i\sqrt{\binom{4m-2}{2m-2}}(z^{2m-2} - z^{2m}),$$

$$i\sqrt{2\binom{4m-2}{2m-1}}z^{2m-1} \Bigg].$$

EXAMPLE 2.3. Let f be a holomorphic immersion from a Riemann surface Σ^2 to $\mathbb{CP}^m$, and let ι' be the inclusion of $\mathbb{CP}^m$ to Q^{2m} defined by $\pi(\mathbf{z}) \mapsto \pi'((\mathbf{z}, i\mathbf{z}))$. Then the composition $\iota' \circ f$ gives a holomorphic curve of Q^{2m}.

EXAMPLE 2.4. Let f be an immersion from a Riemann surface Σ^2 to $\mathbb{R}^{n+1}$. Then the Gauss map $G : \Sigma^2 \to Q^{n-1}$ of f is anti-holomorphic if and only if the immersion f is minimal (cf. [8]). So from a (non-flat) minimal surface in $\mathbb{R}^{n+1}$, we can find a holomorphic curve $\overline{G} : \Sigma^2 \to Q^{n-1}$ by taking the complex conjugate of G.

Theorem 2.5 ([1, 4]). *There exists a canonical* 1-1 *correspondence between the set of generalized minimal immersions* $\chi : S^2 \to S^{2m}(1)$ *which are not contained in any lower dimensional subspace of* $\mathbb{R}^{2m+1}$ *and the set of totally isotropic holomorphic curves* $\Xi : S^2 \to \mathbb{CP}^{2m}$ *which are not contained in any complex hyperplane of* $\mathbb{CP}^{2m}$. *The correspondence is the one that associates with minimal immersion* χ *its directrix curve.*

This theorem holds for *pseudo-holomorphic maps* [4] χ from a Riemann surface Σ^2 to $\to S^{2m}(1)$, i.e.

$$((\partial^j \chi, \partial^k \chi)) = 0, \quad j + k > 0,$$

where $\partial^j \chi = \partial^j \chi / \partial z^j$, z is a local isothermal parameter of Σ^2, and $((\ ,\ ))$ denotes the symmetric product of $\mathbb{C}^{2m+1}$.

A holomorphic curve $\Xi : \Sigma^2 \to \mathbb{CP}^{2m}$ is *totally isotropic* if and only if $\Xi(\Sigma^2)$ is not contained in any complex hyperplane of $\mathbb{CP}^{2m}$ and for a local expression ξ of Ξ,

$$((\xi, \xi)) = ((\xi', \xi')) = \cdots = ((\xi^{m-1}, \xi^{m-1})) = 0,$$

where $\xi^k = \partial^k \xi$. In particular, the image of a totally isotropic holomorphic curve $\Xi : S^2 \to \mathbb{P}^{2m}(\mathbb{C})$ is contained in Q^{2m-1}, for $((\xi, \xi)) = 0$. So Ξ gives a holomorphic curve

of the complex quadric.

The directrix curve of a minimal immersion $\chi : S^2 \to S^{2m}(1)$ is nothing but the map $\Xi : \Sigma^2 \to Q^{2m-1}$ defined by $\Xi(p) =$ the $(m-1)$-th normal space at p with respect to χ.

EXAMPLE 2.6. Let $\chi : S^2(1/3) \to S^4(1)$ be the Veronese immersion from the sphere of constant Gaussian curvature $1/3$ to the unit 4-sphere. Then the directrix curve of χ is congruent to $\psi_4 : S^2 \to Q^3 \subset \mathbb{CP}^4$ of Example 2.2.

DEFINITION 2.7. For a holomorphic curve $\varphi : \Sigma^2 \to Q^{n-1}$, $x \in \Sigma^2$ is called a *real point* [10, p. 131] if

$$\mathbf{p}_1 \wedge \mathbf{p}_2 = 0,$$

at x, and $x \in \Sigma^2$ is called an *isotropic point* [10, p. 130], if

$$(2.8) \qquad \|\mathbf{p}_1\|^2 = \|\mathbf{p}_2\|^2 \neq 0, \text{ and } \mathbf{p}_1 \cdot \mathbf{p}_2 = 0.$$

at x, respectively. φ is called *first order isotropic* [10, p. 134]) if every point $x \in \Sigma^2$ is isotropic.

With respect to the above notation, a holomorphic curve $\varphi : \Sigma^2 \to Q^{n-1}$ is first order isotropic if and only if $(\xi', \xi') = 0$. On the other hand, if every point of Σ^2 is real, then $\varphi(\Sigma^2)$ is contained in a totally geodesic Q^1 in Q^{n-1} [10, Theorem 3.1]. These definitions do not depend on the choice of the section $(\mathbf{e}, \mathbf{f})$.

For a holomorphic curve $\varphi : \Sigma^2 \to Q^{n-1}$ (which in not necessary first order isotropic), put

$$(2.9) \qquad \mathbf{p}^*_{j,k} = \text{orthogonal component of } \frac{\partial \mathbf{p}_j}{\partial t_k} \text{ to span}\{\mathbf{e}, \mathbf{f}\} \text{ in } \mathbb{R}^{n+1},$$

$$\mathbf{p}^{**}_{j,k} = \text{orthogonal component of } \frac{\partial \mathbf{p}_j}{\partial t_k} \text{ to span}\{\mathbf{e}, \mathbf{f}, \mathbf{p}_1, \mathbf{p}_2\} \text{ in } \mathbb{R}^{n+1}.$$

Since

$$\overline{\nabla}^Q_{\varphi_*(\partial/\partial t_2)} \varphi_*(\partial/\partial t_1) = \pi'_*(\mathbf{p}^*_{1,2} + \lambda_1 \mathbf{p}_1, -\mathbf{p}^*_{2,2} - \lambda_1 \mathbf{p}_2)$$

and

$$\overline{\nabla}^Q_{\varphi_*(\partial/\partial t_1)} \varphi_*(\partial/\partial t_2) = \pi'_*(\mathbf{p}^*_{2,1} - \lambda_2 \mathbf{p}_2, \mathbf{p}^*_{1,1} - \lambda_2 \mathbf{p}_1)$$

are equal (cf. (1.4)), we have

$$(2.10) \qquad \mathbf{p}^{**}_{1,1} + \mathbf{p}^{**}_{2,2} = 0, \quad \mathbf{p}^{**}_{1,2} = \mathbf{p}^{**}_{2,1}.$$

Suppose the holomorphic curve $\varphi : \Sigma^2 \to Q^{n-1}$ is *first order isotropic*. Then the second fundamental form σ^φ of φ is written as

$$\sigma^\varphi(\partial/\partial t_1, \partial/\partial t_1) = -\sigma^\varphi(\partial/\partial t_2, \partial/\partial t_2) = \pi'_*(\mathbf{p}^{**}_{1,1}, -\mathbf{p}^{**}_{1,2}),$$
$$\sigma^\varphi(\partial/\partial t_1, \partial/\partial t_2) = \pi'_*(\mathbf{p}^{**}_{1,2}, \mathbf{p}^{**}_{1,1}).$$

Using (1.5) and the Gauss equation, the Gauss curvature K^{Σ^2} of Σ^2 with respect to the metric induced by the first order isotropic holomorphic curve $\psi = \imath \circ \varphi : \Sigma^2 \to \mathbb{CP}^n$ is given by

$$(2.11) \quad K^{\Sigma^2} = 2 - \frac{1}{2}(\|\sigma^\varphi\|^2 + \|A^\imath_{N_1}\|^2 + \|A^\imath_{N_2}\|^2)$$

$$= 2 - \frac{1}{\rho^2}\Big(\|\sigma^\varphi(\partial/\partial t_1, \partial/\partial t_1)\|^2 + \|\sigma^\varphi(\partial/\partial t_1, \partial/\partial t_2)\|^2$$

$$+ \langle A^\imath_{N_1}\varphi_*(\partial/\partial t_1), \varphi_*(\partial/\partial t_1)\rangle^2 + \langle A^\imath_{N_1}\varphi_*(\partial/\partial t_1), \varphi_*(\partial/\partial t_2)\rangle^2$$

$$+ \langle A^\imath_{N_2}\varphi_*(\partial/\partial t_1), \varphi_*(\partial/\partial t_1)\rangle^2 + \langle A^\imath_{N_2}\varphi_*(\partial/\partial t_1), \varphi_*(\partial/\partial t_2)\rangle^2 \Big)$$

$$= 2 - \frac{1}{\rho^2}\Big(2(\|\mathbf{p}^{**}_{1,1}\|^2 + \|\mathbf{p}^{**}_{1,2}\|^2) + (\|\mathbf{p}_1\|^2 - \|\mathbf{p}_2\|^2)^2 + 4(\mathbf{p}_1 \cdot \mathbf{p}_2)^2 \Big)$$

$$= 1 - \frac{2}{\rho^2}\Big(\|\mathbf{p}^{**}_{1,1}\|^2 + \|\mathbf{p}^{**}_{1,2}\|^2 - 2\big(\|\mathbf{p}_1\|^2\|\mathbf{p}_2\|^2 - (\mathbf{p}_1 \cdot \mathbf{p}_2)^2\big) \Big).$$

3. Immersions of some circle bundles over surfaces in complex quadric to sphere

Let M^3 be a 3-dimensional submanifolds foliated by (oriented) great circles of unit sphere $S^n(1)$ with an immersion $\Phi : M^3 \to S^n(1)$, and let $C(p)$ be the great circle of the foliation through $p \in M^3$. We note that the foliation on M^3 is regular in the sense of Palais [16, p. 13] (i.e. every point has a coordinate chart distinguished by the foliation, such that each leaf intersects the chart in at most one 2-dimensional slice). This implies that the space of leaves Σ^2 is an 2-dimensional manifold, for each $C(p)$ is complete. Since $C(p)$ is an element of Q^{n-1}, we have a map $\widetilde{\varphi} : M^3 \to Q^{n-1}$ defined by $\widetilde{\varphi}(p) = C(p)$. Then we can easily see that $\widetilde{\varphi}$ factors through an immersion $\varphi : \Sigma^2 \to Q^{n-1}$ (cf. [5, p. 142, Theorem 4.6]).

EXAMPLE 3.1. Let M^3 be a hypersurface of $S^4(1)$ on which type number (i.e., rank of shape operator A) of M is 2. Then each integral curve of 1-dimensional distribution $\ker A$ on M^3 is a part of great circle of $S^4(1)$. In particular, minimal Cartan hypersurface (i.e., the minimal isoparametric hypersurface with 3 distinct constant principal curvatures c, 0, $-c$ and $c \neq 0$) of unit 4-sphere is foliated by great circles of S^4.

EXAMPLE 3.2. As a generalization of Example 3.1, let M^3 be a 3-dimensional submanifold of $S^n(1)$ with $n \geq 5$. Suppose dimension of subspace $V(p)$ of tangent space $T_p(M^3)$, defined by

$$V(p) = \{X \in T_p(M^3) | \sigma^M(X, Y) = 0 \text{ for any } Y \in T_p(M^3)\},$$

is 1 for each $p \in M$, where σ^M denotes second fundamental form of M^3 in $S^n(1)$. Then each integral curve of 1-dimensional distribution $V(p)$ on M^3 is a part of great circle of $S^n(1)$.

EXAMPLE 3.3. Let Σ^2 be a 2-dimensional surface of $\mathbb{CP}^m$. Then $M^3 = \pi^{-1}(\Sigma^2)$ is a 3-dimensional submanifold foliated by great circles $\pi^{-1}(p)$ for $p \in \Sigma^2$ of $S^{2m+1}(1)$.

EXAMPLE 3.4. Let $S^1(c_1) \times S^2(c_2)$ be a Riemann product of the circle of radius $1/\sqrt{c_1}$ in $\mathbb{R}^2$ and the round 2–sphere of radius $1/\sqrt{c_2}$ on which $1/c_1 + 1/c_2 = 1$ holds. We parameterize the immersion $\Phi : S^1(c_1) \times S^2(c_2) \to S^4(1)$ into the unit 4-sphere as:

$$\left(\frac{1}{\sqrt{c_1}}(\cos\theta, \sin\theta), \frac{1}{\sqrt{c_2}}(\cos u \cos v, \cos u \sin v, \sin u) \right) \mapsto$$

$$\left(\frac{1}{\sqrt{c_1}}(\cos\theta, \sin\theta), \frac{1}{\sqrt{c_2}}\left(\cos(\theta + u)\cos v, \cos(\theta + u)\sin v, \sin(\theta + u) \right) \right).$$

Then integral curves of the vector field $\partial/\partial\theta$ are great circles of S^4.

Let $\varphi : \Sigma^2 \to Q^{n-1}$ be a conformal immersion from a Riemann surface Σ^2 to the complex quadric Q^{n-1} as in §2, and let $P(Q^{n-1}, S^1)$ be the circle bundle over Q^{n-1} (cf. §1), which is the pullback bundle of the Hopf fibration $S^{2n+1}(\mathbb{CP}^n, S^1)$, where P is the space of ordered two orthonormal vectors in $\mathbb{R}^{n+1}$. We denote the pullback bundle over Σ^2 with respect to φ as $\pi_\varphi : \varphi^*P \to \Sigma^2$, and let $M^3 = \varphi^*P$. By the definition, there is a bundle chart $\{(U_\alpha, \varphi_\alpha)\}$ $(\alpha \in \Lambda)$ of φ^*P such that

$$u \in \pi_\varphi^{-1}(U_\alpha) \to (\pi_\varphi(u), \varphi_\alpha(u)) \in U_\alpha \times S^1$$

gives a homeomorphism. If $U_\alpha \cap U_\beta \neq \emptyset$, then $\varphi_\beta(u)(\varphi_\alpha(u))^{-1}$ gives rise to the transition function

$$x \in U_\alpha \cap U_\beta \to \Theta_{\beta\alpha}(x) \in S^1 = \mathbb{R}/2\pi\mathbb{Z}.$$

Note that if $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, then we can see that

$$\Theta_{\gamma\alpha}(p) = \Theta_{\gamma\beta}(p) + \Theta_{\beta\alpha}(p), \quad p \in U_\alpha \cap U_\beta \cap U_\gamma.$$

For each $\alpha \in \Lambda$, we can take the (differentiable) section $\rho_\alpha : U_\alpha \to \pi_\varphi^{-1}(U_\alpha)$ as

$$\pi_\varphi(\rho_\alpha(p)) = p, \quad \varphi_\alpha(\rho_\alpha(p)) = e \quad \text{(identity)}.$$

If $U_\alpha \cap U_\beta \neq \emptyset$, then we have

$$(3.1) \qquad \rho_\alpha(p) = \rho_\beta(p)\Theta_{\beta\alpha}(p), \quad p \in U_\alpha \cap U_\beta.$$

Since $\rho_\alpha(p)$ and $\rho_\beta(p)$ are viewed as elements in P, we may write them as

$$\rho_\alpha(p) = (\mathbf{e}_\alpha(p), \mathbf{f}_\alpha(p)), \quad \rho_\beta(p) = (\mathbf{e}_\beta(p), \mathbf{f}_\beta(p)),$$

where $\mathbf{e}_\alpha(p)$, $\mathbf{f}_\alpha(p)$ and $\mathbf{e}_\beta(p)$, $\mathbf{f}_\beta(p)$ are oriented orthonormal basis of the two-plane $\varphi(p) \in Q^{n-1}$. Then (3.1) is written as

$$(3.2) \qquad (\mathbf{e}_\alpha(p), \mathbf{f}_\alpha(p)) = (\cos\Theta_{\beta\alpha}(p)\mathbf{e}_\beta(p) + \sin\Theta_{\beta\alpha}(p)\mathbf{f}_\beta(p),$$
$$- \sin\Theta_{\beta\alpha}(p)\mathbf{e}_\beta(p) + \cos\Theta_{\beta\alpha}(p)\mathbf{f}_\beta(p)).$$

Note that the pullback bundle $M^3 = \varphi^* P$ is also realized as the quotient space $\Lambda \times \Sigma^2 \times S^1 / \sim$, where

$$(3.3) \qquad (\alpha, p, \theta) \sim (\beta, q, \zeta) \Longleftrightarrow p = q \in U_\alpha \cap U_\beta, \text{ and } \zeta = \theta + \Theta_{\beta\alpha}.$$

For each $p \in \Sigma^2$, the fiber $\pi_\varphi^{-1}(p)$ with respect to $\pi_\varphi : \varphi^* P \to \Sigma^2$ is identified with the great circle $\varphi(p) \in Q^{n-1}$ (cf. Remark 1.1).

We define the map $\Phi : M^3 = \varphi^* P \to S^n(1)$ as

$$(3.4) \qquad \Phi([\alpha, p, \theta]) = \cos\theta\mathbf{e}_\alpha(p) + \sin\theta\mathbf{f}_\alpha(p),$$

where $[\alpha, p, \theta]$ is the equivalence class of $(\alpha, p, \theta) \in \Lambda \times \Sigma^2 \times S^1$. By (3.2) and (3.3), Φ is well-defined. We can see that Φ maps each fiber $\pi_\varphi^{-1}(p)$ for $p \in \Sigma^2$ to the corresponding great circle $\varphi(p) \in Q^{n-1}$

If $p \in U_\alpha \subset \Sigma^2$, then $p \mapsto (\mathbf{e}_\alpha(p), \mathbf{f}_\alpha(p))$ gives a lift of $\varphi|_{U_\alpha} : U_\alpha \to Q^{n-1}$ to P. For simplicity we denote $(\mathbf{e}(p), \mathbf{f}(p))$ instead of $(\mathbf{e}_\alpha(p), \mathbf{f}_\alpha(p))$, and we use the same notations as §1. We may view Φ as a $\mathbb{R}^{n+1}$-valued function on M^3. Using (2.2), we get that the first order differential of Φ is

$$(3.5) \qquad \Phi_\theta = \frac{\partial\Phi}{\partial\theta} = -\sin\theta\mathbf{e} + \cos\theta\mathbf{f},$$

$$(3.6) \qquad \Phi_j = \frac{\partial\Phi}{\partial t_j} = \cos\theta(\lambda_j\mathbf{f} + \mathbf{p}_j) + \sin\theta(-\lambda_j\mathbf{e} + \mathbf{q}_j) \quad (j = 1, 2),$$

and

$$(3.7) \qquad \Phi \wedge \Phi_\theta = \mathbf{e} \wedge \mathbf{f}.$$

Denote

$$(3.8) \qquad \Psi_j := \Phi_j - \lambda_j\Phi_\theta$$

$$= \cos \theta \mathbf{p}_j + \sin \theta \mathbf{q}_j, \quad (j = 1, 2).$$

So

$$\Psi_1 \wedge \Psi_2 = \cos^2 \theta (\mathbf{p}_1 \wedge \mathbf{p}_2) + \cos \theta \sin \theta (\mathbf{p}_1 \wedge \mathbf{q}_2 - \mathbf{p}_2 \wedge \mathbf{q}_1)$$
$$+ \sin^2 \theta (\mathbf{q}_1 \wedge \mathbf{q}_2).$$

Hence we have

Proposition 3.5. *Let* $\varphi : \Sigma^2 \to Q^{n-1}$ *be a conformal immersion from a Riemann surface* Σ^2 *to the complex quadric* Q^{n-1}, *and let* $P(Q^{n-1}, S^1)$ *be the circle bundle over* Q^{n-1}, *where* $P = SO(n+1)/SO(n-1)$ *is the space of ordered two orthonormal vectors in* $\mathbb{R}^{n+1}$. *Then*

(1) *The map* Φ *from the pullback bundle* $\varphi^* P$ *to* $S^n(1)$ *defined by* (3.4) *maps each fiber* $\pi_\varphi^{-1}(p)$ *for* $p \in \Sigma^2$ *of the circle bundle* $\pi_\varphi : P \to \Sigma^2$ *to the corresponding great circle* $\varphi(p) \in Q^{n-1}$ *of* $S^n(1)$.

(2) Φ *is regular at* $[\alpha, p, \theta] \in \varphi^* P$ *if and only if at* $p \in U_\alpha \subset \Sigma^2$, φ *satisfies*

$$\cos^2 \theta (\mathbf{p}_1 \wedge \mathbf{p}_2) + \cos \theta \sin \theta (\mathbf{p}_1 \wedge \mathbf{q}_2 - \mathbf{p}_2 \wedge \mathbf{q}_1) + \sin^2 \theta (\mathbf{q}_1 \wedge \mathbf{q}_2) \neq 0.$$

(3) *If* $\varphi(\Sigma^2)$ *is not contained in a totally geodesic* Q^{n-2} *in* Q^{n-1}, *then* $\Phi(\varphi^* P)$ *is not contained in a totally geodesic* $S^{n-1}(1)$ *in* $S^n(1)$.

We suppose that Φ is an immersion, i.e., with respect to a basis $\{\Phi_\theta, \Psi_1, \Psi_2\}$ of the tangent space $T_{(\mathbf{p}, \theta)} M$, the metric of M induced by Φ is given as follows:

$$\|\Phi_\theta\|^2 = 1, \quad \Phi_\theta \cdot \Psi_1 = \Phi_\theta \cdot \Psi_2 = 0,$$
$$\|\Psi_1\|^2 = \|\mathbf{p}_1\|^2 \cos^2 \theta + 2\mathbf{p}_1 \cdot \mathbf{q}_1 \cos \theta \sin \theta + \|\mathbf{q}_1\|^2 \sin^2 \theta,$$
$$\Psi_1 \cdot \Psi_2 = \mathbf{p}_1 \cdot \mathbf{p}_2 \cos^2 \theta + (\mathbf{p}_1 \cdot \mathbf{q}_2 + \mathbf{p}_2 \cdot \mathbf{q}_1) \cos \theta \sin \theta + \mathbf{q}_1 \cdot \mathbf{q}_2 \sin^2 \theta,$$
$$\|\Psi_2\|^2 = \|\mathbf{p}_2\|^2 \cos^2 \theta + 2\mathbf{p}_2 \cdot \mathbf{q}_2 \cos \theta \sin \theta + \|\mathbf{q}_2\|^2 \sin^2 \theta.$$

We find the condition whether the tangent vectors Φ_θ of each great circle corresponding to a two-plane $\varphi(p) \subset \mathbb{R}^n$ is a null direction of the second fundamental form σ^Φ of $\Phi : M^3 \to S^n(1)$ or not. Since $D_{\Phi_\theta} \Phi_\theta = -\Phi$, clearly

$$(3.9) \qquad\qquad \sigma^\Phi(\Phi_\theta, \Phi_\theta) = 0,$$

where D is a flat connection of $\mathbb{R}^{n+1}$. By the fact that

$$(3.10) \qquad\qquad D_{\Phi_\theta} \Psi_j = -\sin \theta \mathbf{p}_j + \cos \theta \mathbf{q}_j \quad (j = 1, 2)$$

is orthogonal to Φ and Φ_θ, we have

$$\sigma^\Phi(\Phi_\theta, \Psi_j) = 0 \Leftrightarrow \Psi_1 \wedge \Psi_2 \wedge D_{\Phi_\theta} \Psi_j = 0.$$

Hence

(3.11) $\sigma^{\Phi}(\Phi_{\theta}, \Psi_1) = 0 \iff -\mathbf{p}_1 \wedge \mathbf{q}_1 \wedge (\cos\theta \mathbf{p}_2 + \sin\theta \mathbf{q}_2) = 0$

$$\iff \begin{cases} \mathbf{p}_1 \wedge \mathbf{q}_1 \wedge \mathbf{p}_2 = 0 \\ \mathbf{p}_1 \wedge \mathbf{q}_1 \wedge \mathbf{q}_2 = 0, \end{cases}$$

and

(3.12) $\sigma^{\Phi}(\Phi_{\theta}, \Psi_2) = 0 \iff \mathbf{p}_2 \wedge \mathbf{q}_2 \wedge (\cos\theta \mathbf{p}_1 + \sin\theta \mathbf{q}_1) = 0$

$$\iff \begin{cases} \mathbf{p}_2 \wedge \mathbf{q}_2 \wedge \mathbf{p}_1 = 0 \\ \mathbf{p}_2 \wedge \mathbf{q}_2 \wedge \mathbf{q}_1 = 0. \end{cases}$$

Consequently we have

Proposition 3.6. *Let $\varphi : \Sigma^2 \to Q^{n-1}$ be an immersion from a surface to a complex quadric, and let $\Phi : M^3 = \varphi^* P \to S^n(1)$ be the corresponding immersion defined by (3.4). Then the tangent vectors Φ_{θ} of each great circle corresponding to a two-plane $\varphi(p) \subset \mathbb{R}^n$ is a null direction of the second fundamental form σ^{Φ} of Φ if and only if*

$$\dim \operatorname{span}\{\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2\} \le 2.$$

Because of Proposition 2.1, we get

Corollary 3.7. *Under the same assumption as in* Proposition 3.6, *if φ is either holomorphic or anti-holomorphic, then the tangent vectors Φ_{θ} of each great circle corresponding to a two-plane $\varphi(p) \subset \mathbb{R}^n$ is a null direction of the second fundamental form σ^{Φ} of Φ.*

Remark 3.8. In Example 3.4, generalized Clifford torus $\Phi : S^1(c_1) \times S^2(c_2) \to S^4(1) \subset S^n(1)$ is given by a *totally real* surface Σ^2 in Q^{n-1}. But there is no null direction of the second fundamental form σ^{Φ} of Φ.

Next we try to find the condition such that the immersion $\Phi : M^3 = \varphi^* P \to S^n(1)$ is minimal. Since $\Phi \wedge \Phi_{\theta} = \mathbf{e} \wedge \mathbf{f}$, Φ is minimal if and only if

(3.13) $\Psi_1 \wedge \Psi_2 \wedge \{\|\Psi_2\|^2 D_{\Psi_1}\Psi_1 - (\Psi_1 \cdot \Psi_2)(D_{\Psi_1}\Psi_2 + D_{\Psi_2}\Psi_1)$

$+ \|\Psi_1\|^2 D_{\Psi_2}\Psi_2\} \equiv 0 \mod (\mathbf{e}, \mathbf{f})$.

Differentiating Ψ_j by $\Psi_k = \Phi_k - \lambda_k \Phi_{\theta}$ (see (3.5)), we get

$$D_{\Psi_k}\Psi_j = \cos\theta(\mathbf{p}_{j,k} - \lambda_k \mathbf{q}_j) + \sin\theta(\mathbf{q}_{j,k} + \lambda_k \mathbf{p}_j),$$

where $\mathbf{p}_{j,k} = \partial\mathbf{p}_j/\partial t_k$ and $\mathbf{q}_{j,k} = \partial\mathbf{q}_j/\partial t_k$ in $\mathbb{R}^{n+1}$, respectively. Put

$$A_{j,k} = \mathbf{p}_{j,k}^* - \lambda_k\mathbf{q}_j, \quad \mathbf{p}_{j,k}^* = \mathbf{p}_{j,k} + (\mathbf{p}_j\cdot\mathbf{p}_k)\mathbf{e} + (\mathbf{p}_j\cdot\mathbf{q}_k)\mathbf{f},$$
$$B_{j,k} = \mathbf{q}_{j,k}^* + \lambda_k\mathbf{p}_j, \quad \mathbf{q}_{j,k}^* = \mathbf{q}_{j,k} + (\mathbf{q}_j\cdot\mathbf{p}_k)\mathbf{e} + (\mathbf{q}_j\cdot\mathbf{q}_k)\mathbf{f},$$

i.e., $\mathbf{p}_{j,k}^*$ and $\mathbf{q}_{j,k}^*$ are orthogonal components of $\mathbf{p}_{j,k}$ and $\mathbf{q}_{j,k}$ to span$\{\mathbf{e},\mathbf{f}\}$ in $\mathbb{R}^{n+1}$, respectively. Then we obtain

$$D_{\Psi_k}\Psi_j \equiv \cos\theta A_{j,k} + \sin\theta B_{j,k} \quad \mod (\mathbf{e},\mathbf{f}).$$

Note that (2.2), (3.2) and $\partial^2\Phi/\partial t_j\partial t_k = \partial^2\Phi/\partial t_k\partial t_j$ imply

$$A_{j,k} = A_{k,j} \text{ and } B_{j,k} = B_{k,j}.$$

By direct calculations, we have

$$\|\Psi_2\|^2 D_{\Psi_1}\Psi_1 - (\Psi_1\cdot\Psi_2)(D_{\Psi_1}\Psi_2 + D_{\Psi_2}\Psi_1) + \|\Psi_1\|^2 D_{\Psi_2}\Psi_2$$
$$\equiv \cos^3\theta C_0 + \cos^2\theta\sin\theta C_1 + \cos\theta\sin^2\theta C_2 + \sin^3\theta C_3, \quad \mod (\mathbf{e},\mathbf{f})$$

where

$$C_0 = \|\mathbf{p}_2\|^2 A_{1,1} - 2(\mathbf{p}_1\cdot\mathbf{p}_2)A_{1,2} + \|\mathbf{p}_1\|^2 A_{2,2},$$
$$C_1 = \|\mathbf{p}_2\|^2 B_{1,1} - 2(\mathbf{p}_1\cdot\mathbf{p}_2)B_{1,2} + \|\mathbf{p}_1\|^2 B_{2,2},$$
$$+ 2(\mathbf{p}_2\cdot\mathbf{q}_2)A_{1,1} - 2(\mathbf{p}_1\cdot\mathbf{q}_2 + \mathbf{p}_2\cdot\mathbf{q}_1)A_{1,2} + 2(\mathbf{p}_1\cdot\mathbf{q}_1)A_{2,2}$$
$$C_2 = \|\mathbf{q}_2\|^2 A_{1,1} - 2(\mathbf{q}_1\cdot\mathbf{q}_2)A_{1,2} + \|\mathbf{q}_1\|^2 A_{2,2},$$
$$+ 2(\mathbf{p}_2\cdot\mathbf{q}_2)B_{1,1} - 2(\mathbf{p}_1\cdot\mathbf{q}_2 + \mathbf{p}_2\cdot\mathbf{q}_1)B_{1,2} + 2(\mathbf{p}_1\cdot\mathbf{q}_1)B_{2,2}$$
$$C_3 = \|\mathbf{q}_2\|^2 B_{1,1} - 2(\mathbf{q}_1\cdot\mathbf{q}_2)B_{1,2} + \|\mathbf{q}_1\|^2 B_{2,2}.$$

Since $\cos^5\theta, \cos^4\theta\sin\theta, \ldots, \sin^5\theta$ are independent functions in (3.13), we get

Proposition 3.9. *Let* $\varphi : \Sigma^2 \to Q^{n-1}$ *be an immersion from a surface to a complex quadric, and let* $\Phi : M^3 = \varphi^*P \to S^n(1)$ *be the corresponding immersion defined by* (3.4). *Then* Φ *is minimal if and only if the following equations hold:*

$$\mathbf{p}_1\wedge\mathbf{p}_2\wedge C_0 = 0,$$
$$(\mathbf{p}_1\wedge\mathbf{q}_2 - \mathbf{p}_2\wedge\mathbf{q}_1)\wedge C_0 + \mathbf{p}_1\wedge\mathbf{p}_2\wedge C_1 = 0,$$
$$\mathbf{q}_1\wedge\mathbf{q}_2\wedge C_0 + (\mathbf{p}_1\wedge\mathbf{q}_2 - \mathbf{p}_2\wedge\mathbf{q}_1)\wedge C_1 + \mathbf{p}_1\wedge\mathbf{p}_2\wedge C_2 = 0,$$
$$\mathbf{q}_1\wedge\mathbf{q}_2\wedge C_1 + (\mathbf{p}_1\wedge\mathbf{q}_2 - \mathbf{p}_2\wedge\mathbf{q}_1)\wedge C_2 + \mathbf{p}_1\wedge\mathbf{p}_2\wedge C_3 = 0,$$
$$\mathbf{q}_1\wedge\mathbf{q}_2\wedge C_2 + (\mathbf{p}_1\wedge\mathbf{q}_2 - \mathbf{p}_2\wedge\mathbf{q}_1)\wedge C_3 = 0,$$
$$\mathbf{q}_1\wedge\mathbf{q}_2\wedge C_3 = 0.$$

Corollary 3.10. *Under the same assumption as* Proposition 3.9, *suppose* $n \geq 5$ *and* $\dim \mathrm{span}\{\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2\} = 4$. *Then* Φ *is minimal if and only if the following equations hold*:

$$
\begin{aligned}
C_0 &= \mu_0 \mathbf{p}_1 && +\nu_0 \mathbf{p}_2, \\
C_1 &= \mu_1 \mathbf{p}_1 + \mu_0 \mathbf{q}_1 +\nu_1 \mathbf{p}_2 +\nu_0 \mathbf{q}_2, \\
C_2 &= \mu_2 \mathbf{p}_1 + \mu_1 \mathbf{q}_1 +\nu_2 \mathbf{p}_2 +\nu_1 \mathbf{q}_2, \\
C_3 &= \qquad \mu_2 \mathbf{q}_1 && +\nu_2 \mathbf{q}_2,
\end{aligned}
$$

where $\mu_0, \mu_1, \mu_2, \nu_0, \nu_1, \nu_2$ *are some functions on* Σ^2.

4. Three dimensional submanifolds of the sphere given by holomorphic curves of the complex quadric

In this section, as a special case of §2, we investigate 3-dimensional submanifold M^3 of $S^n(1)$ given by *holomorphic curve* Σ^2 of Q^{n-1}. We use the same notation as §2 and §3.

Let $\varphi : \Sigma^2 \to Q^{n-1}$ be a holomorphic immersion from a Riemann surface Σ^2 to the complex quadric Q^{n-1} and let $\pi_\varphi : \varphi^* P \to \Sigma^2$ be the pullback bundle of the circle bundle $P(Q^{n-1}, S^1)$ ($P = SO(n+1)/SO(n-1)$ is the set of ordered orthonormal 2-vectors in $\mathbb{R}^{n+1}$) with respect to φ. We consider the map $\Phi : \varphi^* P \to S^n(1)$ defined by (3.4). Using (2.2), we get that the first order differential of Φ is

$$
\begin{aligned}
\Phi_\theta &= \frac{\partial \Phi}{\partial \theta} = -\sin \theta \mathbf{e} + \cos \theta \mathbf{f}, \\
\Phi_1 &= \frac{\partial \Phi}{\partial t_1} = \cos \theta (\lambda_1 \mathbf{f} + \mathbf{p}_1) + \sin \theta (-\lambda_1 \mathbf{e} - \mathbf{p}_2), \\
\Phi_2 &= \frac{\partial \Phi}{\partial t_2} = \cos \theta (\lambda_2 \mathbf{f} + \mathbf{p}_2) + \sin \theta (-\lambda_2 \mathbf{e} + \mathbf{p}_1).
\end{aligned}
$$

As in §3, we denote

$$
\begin{aligned}
\Psi_1 &:= \Phi_1 - \lambda_1 \Phi_\theta = \cos \theta \mathbf{p}_1 - \sin \theta \mathbf{p}_2, \\
\Psi_2 &:= \Phi_2 - \lambda_2 \Phi_\theta = \cos \theta \mathbf{p}_2 + \sin \theta \mathbf{p}_1.
\end{aligned}
$$

Using Proposition 3.5, we get

Proposition 4.1. *Let* $\varphi : \Sigma^2 \to Q^{n-1}$ *be a holomorphic immersion from a Riemann surface* Σ^2 *to the complex quadric* Q^{n-1}, *and let* $\varphi^* P(\Sigma^2, S^1)$ *be the pullback bundle of the circle bundle* $P(Q^{n-1}, S^1)$ ($P = SO(n+1)/SO(n-1)$) *with respect to* φ. *Then the map* $\Phi : \varphi^* P \to S^n(1)$ *defined by (3.4) is regular at each point in* $\pi_\varphi^{-1}(x)$ *for* $x \in \Sigma^2$ *if and only if* x *is not a real point for* φ (*Definition 2.7*). *Consequently if*

the holomorphic curve $\varphi : \Sigma^2 \to Q^{n-1}$ *is first order isotropic, then the corresponding map* $\Phi : \varphi^* P \to S^n(1)$ *is always an immersion.*

With respect to a basis $\{\Phi_\theta, \Psi_1, \Psi_2\}$ of the tangent space $T_{[\alpha,\mathbf{p},\theta]}M$, the metric of $\varphi^* P$ induced by Φ is given as follows:

$$\|\Phi_\theta\|^2 = 1, \quad \Phi_\theta \cdot \Psi_1 = \Phi_\theta \cdot \Psi_2 = 0,$$
$$\|\Psi_1\|^2 = \|\mathbf{p}_1\|^2 \cos^2\theta - 2\mathbf{p}_1 \cdot \mathbf{p}_2 \cos\theta \sin\theta + \|\mathbf{p}_2\|^2 \sin^2\theta,$$
$$\Psi_1 \cdot \Psi_2 = \mathbf{p}_1 \cdot \mathbf{p}_2(\cos^2\theta - \sin^2\theta) + (\|\mathbf{p}_1\|^2 - \|\mathbf{p}_2\|^2)\cos\theta \sin\theta,$$
$$\|\Psi_2\|^2 = \|\mathbf{p}_2\|^2 \cos^2\theta + 2\mathbf{p}_1 \cdot \mathbf{p}_2 \cos\theta \sin\theta + \|\mathbf{p}_1\|^2 \sin^2\theta.$$

Put

(4.1)
$$\rho = \|\mathbf{p}_1\|^2 + \|\mathbf{p}_2\|^2,$$
$$\rho_1 = \|\mathbf{p}_1\|^2 - \|\mathbf{p}_2\|^2,$$
$$\rho_2 = 2\mathbf{p}_1 \cdot \mathbf{p}_2.$$

Note that the holomorphic immersion $\varphi : \Sigma^2 \to Q^{n-1}$ is first order isotropic (cf. Definition 2.7) if and only if $\rho_1 = \rho_2 = 0$. Then we have

$$\|\Psi_1\|^2 = \frac{1}{2}(\rho + \rho_1 \cos 2\theta - \rho_2 \sin 2\theta),$$
$$\Psi_1 \cdot \Psi_2 = \frac{1}{2}(\rho_1 \sin 2\theta + \rho_2 \cos 2\theta),$$
$$\|\Psi_2\|^2 = \frac{1}{2}(\rho - \rho_1 \cos 2\theta + \rho_2 \sin 2\theta),$$

and

$$\Delta := \|\Psi_1\|^2 \|\Psi_2\|^2 - (\Psi_1 \cdot \Psi_2)^2 = \frac{\rho^2 - \rho_1^2 - \rho_2^2}{4}$$
$$= \|\mathbf{p}_1\|^2 \|\mathbf{p}_2\|^2 - (\mathbf{p}_1 \cdot \mathbf{p}_2)^2 > 0.$$

Next, we calculate the second fundamental form of $\Phi : M^3 \to S^n(1)$. By (3.9), (3.11) and (3.12), we have

$$\sigma^\Phi(\Phi_\theta, \Phi_\theta) = \sigma^\Phi(\Phi_\theta, \Psi_j) = 0 \quad (j = 1, 2).$$

(3.10) yields that

$$D_{\Psi_j}\Psi_1 = \cos\theta(\mathbf{p}_{1,j} + \lambda_j \mathbf{p}_2) + \sin\theta(-\mathbf{p}_{2,j} + \lambda_j \mathbf{p}_1),$$
$$D_{\Psi_j}\Psi_2 = \cos\theta(\mathbf{p}_{2,j} - \lambda_j \mathbf{p}_1) + \sin\theta(\mathbf{p}_{1,j} + \lambda_j \mathbf{p}_2).$$

Since span$\{\Phi\} + T_\Phi(M^3)$ is spanned by $\mathbf{e}, \mathbf{f}, \mathbf{p}_1, \mathbf{p}_2$, second fundamental form of Φ is (cf. (2.9) and (2.10))

$$\sigma_{11} := \sigma^\Phi(\Psi_1, \Psi_1) = \cos\theta \mathbf{p}_{1,1}^{**} - \sin\theta \mathbf{p}_{1,2}^{**},$$

$$\sigma_{12} := \sigma^\Phi(\Psi_1, \Psi_2) = \cos\theta \mathbf{p}_{1,2}^{**} + \sin\theta \mathbf{p}_{1,1}^{**},$$

$$\sigma_{22} := \sigma^\Phi(\Psi_2, \Psi_2) = -\cos\theta \mathbf{p}_{1,1}^{**} + \sin\theta \mathbf{p}_{1,2}^{**}.$$

Hence the mean curvature vector H^Φ of $\Phi : M^3 \to S^n(1)$ is

$$
\begin{aligned}
H^\Phi &= \frac{1}{\Delta}(\|\Psi_2\|^2 \sigma_{11} - 2\Psi_1 \cdot \Psi_2 \sigma_{12} + \|\Psi_1\|^2 \sigma_{22}) \\
&= \frac{2}{\Delta}\left(-\cos\theta(\rho_1 \mathbf{p}_{1,1}^{**} + \rho_2 \mathbf{p}_{1,2}^{**}) + \sin\theta(-\rho_1 \mathbf{p}_{1,2}^{**} + \rho_2 \mathbf{p}_{1,1}^{**})\right).
\end{aligned}
$$

Consequently we obtain

Theorem 1. *Let $\varphi : \Sigma^2 \to Q^{n-1}$ be a holomorphic immersion from a Riemann surface Σ^2 to complex quadric Q^{n-1}, and let $\pi_\varphi : \varphi^* P \to \Sigma^2$ be the pullback bundle of the circle bundle $P(Q^{n-1}, S^1)$ with respect to φ, where $P = SO(n+1)/SO(n-1)$ is the set of ordered two orthonormal vectors. Suppose the map $\Phi : \varphi^* P \to S^n(1)$ defined by (3.4), i.e. each fiber $\pi_\varphi^{-1}(p)$ for $p \in \Sigma^2$ is mapped to the corresponding great circle $\varphi(p) \in Q^{n-1}$, is an immersion. Then Φ is minimal (i.e. $H^\Phi = 0$) if and only if either Φ is totally geodesic ($\mathbf{p}_{1,1}^{**} = \mathbf{p}_{1,2}^{**} = 0$) or the holomorphic curve φ is first order isotropic ($\rho_1 = \rho_2 = 0$).*

Theorem 1 and Proposition 4.1 imply

Theorem 2. *Let χ be a full minimal immersion from 2-sphere S^2 (resp. a pseudo holomorphic map [4] from a Riemann surface Σ^2) to $S^{2m}(1)$ and let $\pi_\Xi : \Xi^* P \to S^2$ (resp. Σ^2) be the pullback bundle of the circle bundle $P(Q^{2m-1}, S^1)$ ($P = SO(2m+1)/SO(2m-1)$) with respect to the directrix curve $\Xi : S^2$ (resp. Σ^2) $\to Q^{2m-1}$. Then the immersion $\Phi : \varphi^* P \to S^{2m}(1)$ defined by (3.4), i.e. each fiber $\pi_\Xi^{-1}(p)$ for $p \in S^2$ (resp. Σ^2) is mapped to the corresponding great circle $\Xi(p) \in Q^{2m-1}$, is full and minimal.*

REMARK 4.2. In Theorem 2, the minimal immersion $\Phi : \Xi^* P \to S^{2m}(1)$ is realized as a *tube* of radius $\pi/2$ over the minimal 2-sphere S^2 or the pseudo-holomorphic map Σ^2 with respect to the $(m-1)$-th normal space. More precisely, let e_{2m-1}, e_{2m} be a *local* orthonormal frame field of the $(m-1)$-th normal space on some open neighborhood U of either a minimal S^2 or a pseudo holomorphic Σ^2. Then on $\pi_\Xi^{-1}(U) = U \times S^1$, Φ is given by

$$\Phi(x, \theta) = \cos\theta e_{2m-1} + \sin\theta e_{2m}.$$

EXAMPLE 4.3. Let $\psi_4 : \mathbb{CP}^1 \to Q^3 \subset \mathbb{CP}^4$ be the Veronese curve of order 4 in Example 2.2. Then the minimal immersion Φ from the pullback bundle over $\mathbb{CP}^1$ with respect to ψ_4 to $S^4(1)$ given by (3.4) is nothing but the Cartan minimal hypersurface (cf. Example 3.1).

Put

$$g^{11} = \|\Psi_2\|^2/\Delta, \quad g^{12} = -\Psi_1 \cdot \Psi_2/\Delta, \quad g^{22} = \|\Psi_1\|^2/\Delta.$$

Then the square of the length of H^Φ is

$$
\begin{aligned}
\|H^\Phi\|^2 &= (g^{22})^2\|\sigma_{11}\|^2 + 4(g^{12})^2\|\sigma_{12}\|^2 + (g^{11})^2\|\sigma_{22}\|^2 \\
&\quad + 4g^{22}g^{12}\sigma_{11} \cdot \sigma_{12} + 4g^{11}g^{12}\sigma_{12} \cdot \sigma_{22} + 2g^{11}g^{22}\sigma_{11} \cdot \sigma_{22} \\
&= \frac{4}{\Delta^2}\Big(\cos^2\theta\big(\rho_1^2\|\mathbf{p}_{1,1}^{**}\|^2 + 2\rho_1\rho_2(\mathbf{p}_{1,1}^{**} \cdot \mathbf{p}_{1,2}^{**}) + \rho_2^2\|\mathbf{p}_{1,2}^{**}\|^2\big) \\
&\quad + 2\sin\theta\cos\theta\big((\rho_1^2 - \rho_2^2)(\mathbf{p}_{1,1}^{**} \cdot \mathbf{p}_{1,2}^{**}) + \rho_1\rho_2(\|\mathbf{p}_{1,2}^{**}\|^2 - \|\mathbf{p}_{1,1}^{**}\|^2)\big) \\
&\quad + \sin^2\theta\big(\rho_1^2\|\mathbf{p}_{1,2}^{**}\|^2 - 2\rho_1\rho_2(\mathbf{p}_{1,1}^{**} \cdot \mathbf{p}_{1,2}^{**}) + \rho_2^2\|\mathbf{p}_{1,1}^{**}\|^2\big)\Big).
\end{aligned}
$$

The square of the length of the second fundamental form $\|\sigma^\Phi\|^2$ is given by

$$
\begin{aligned}
\|\sigma^\Phi\|^2 &= (g^{11})^2\|\sigma_{11}\|^2 + (g^{22})^2\|\sigma_{22}\|^2 \\
&\quad + 2(g^{11}g^{22} + (g^{12})^2)\|\sigma_{12}\|^2 + 2(g^{12})^2\sigma_{11} \cdot \sigma_{22} \\
&\quad + 4g^{11}g^{12}\sigma_{11} \cdot \sigma_{12} + 4g^{12}g^{22}\sigma_{12} \cdot \sigma_{22}.
\end{aligned}
$$

Hence, using $\sigma_{11} = -\sigma_{22}$, we get

$$
\begin{aligned}
\|H^\Phi\|^2 - \|\sigma^\Phi\|^2 &= 2\big((g^{12})^2 - g^{11}g^{22}\big)(\|\sigma_{11}\|^2 + \|\sigma_{12}\|^2) \\
&\quad + 8(g^{22} - g^{11})g^{12}\sigma_{11} \cdot \sigma_{12} \\
&= -\frac{1}{\Delta}(\|\mathbf{p}_{1,1}^{**}\|^2 + \|\mathbf{p}_{1,2}^{**}\|^2) - \frac{1}{\Delta^2}\big(2\rho_1\rho_2\cos 4\theta + (\rho_1^2 - \rho_2^2)\sin 4\theta\big) \\
&\quad \cdot \big(2\mathbf{p}_{1,1}^{**} \cdot \mathbf{p}_{1,2}^{**}\cos 2\theta + (\|\mathbf{p}_{1,1}^{**}\|^2 - \|\mathbf{p}_{1,2}^{**}\|^2)\sin 2\theta\big).
\end{aligned}
$$

Since $\cos 4\theta\cos 2\theta$, $\cos 4\theta\sin 2\theta$, $\sin 4\theta\cos 2\theta$ and $\sin 4\theta\sin 2\theta$ are independent functions, we finally obtain

Theorem 3. *Let $\varphi : \Sigma^2 \to Q^{n-1}$ be a holomorphic immersion from a Riemann surface Σ^2 to the complex quadric Q^{n-1}, and let Φ be the immersion from of the pullback bundle $\pi_\varphi : \varphi^*P \to \Sigma^2$ of the circle bundle $P(Q^{n-1}, S^1)$ ($P = SO(n+1)/SO(n-1)$) with respect to φ to sphere defined by which each fiber $\pi_\varphi^{-1}(p)$ for $p \in \Sigma^2$ is mapped to the corresponding great circle $\varphi(p) \in Q^{n-1}$ (cf. (3.4)).*
(1)   *If the length of the mean curvature vector $\|H^\Phi\|$ with respect to Φ is constant along each great circles $\varphi(p)$ for $p \in \Sigma^2$, then M is minimal.*

(2)   The scalar curvature $R^M = 6 + \|H^\Phi\|^2 - \|\sigma^\Phi\|^2$ of M^3 is constant along each
great circles $\varphi(p)$ for $p \in \Sigma^2$ if and only if the corresponding holomorphic
curve φ satisfies either

   (i) $\rho_1 = \rho_2 = 0$, i.e., first order isotropic, or

   (ii) $\|\mathbf{p}_{1,1}^{**}\|^2 = \|\mathbf{p}_{1,2}^{**}\|^2$ and $\mathbf{p}_{1,1}^{**} \cdot \mathbf{p}_{1,2}^{**} = 0$.

(3)   The scalar curvature R^M is constant on M^3, if and only if the corresponding
holomorphic curve φ satisfies either

   (i)   First order isotropic and the Gauss curvature K^{Σ^2} is constant, or

   (ii) Not first order isotropic, $\|\mathbf{p}_{1,1}^{**}\|^2 = \|\mathbf{p}_{1,2}^{**}\|^2$, $\mathbf{p}_{1,1}^{**} \cdot \mathbf{p}_{1,2}^{**} = 0$ and $\|\mathbf{p}_{1,1}^{**}\|^2 +$
         $\|\mathbf{p}_{1,2}^{**}\|^2 = C\left(\|\mathbf{p}_1\|^2\|\mathbf{p}_2\|^2 - (\mathbf{p}_1 \cdot \mathbf{p}_2)^2\right)$ for some constant C.

(4)   Suppose the holomorphic immersion $\varphi : \Sigma^2 \to Q^{n-1}$ is of first order isotropic,
and so the immersion from $M^3 = \varphi^* P$ to $S^n(1)$ defined by (3.4) is minimal. Then
the scalar curvature R^M of M^3 is constant if and only if the Gauss curvature
K^{Σ^2} of the corresponding holomorphic curve φ is constant.

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