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Analysis and Synthesis on Robust Stabilization Problems for Systems with Structured Uncertainties

by

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Abstract

When we design a control system for a real plant, it is necessary to guarantee the stability of the closed-loop system for parameter uncertainties. In this thesis we tackle robust control problems with respect to structured uncertainties, by treating both the qualitative analysis and the quantitative synthesis problems.

We consider two kinds of systems with structured uncertainties. One is the interval system which represents a set of all systems whose uncertain entries are located in some pattern. Another is the system with norm bounded structured uncertainties of one-block. Associated with these systems two notions of stability are treated here. One is the so-called *quadratic stability*, as defined using the fixed Lyapunov function which is invariant for uncertainties. The other is *stability for interval systems*, meaning that all the roots of an interval system are in the strict left half of the complex plane.

First we focus on the structure of systems and analyze system properties. Controllability of systems is one of the most important system properties in control system design and analysis of system dynamics. In control system design we may construct a simpler robust controller by paying attention to system structure. Thus analysis of system properties focusing on structure of systems is meaningful. Hence, we first focus only on the structure of systems and study qualitative analysis problems with respect to robust stabilizability. Then the relations between various kinds of robust stabilizability and controllability are investigated.

Next we treat a design problem of a servo system for systems with norm bounded structured uncertainties of one-block, as a quantitative synthesis problem. The design problem of servo systems, in which system outputs track step reference inputs, is one of the most important problems in control system design. In the design of servo systems it is required that the closed-loop system is stable and the outputs track step reference inputs under parameter variations. Furthermore good characteristics of output responses, for example, small overshoot, short settling time etc., are often required as design specifications. In order to achieve these design specifications, we need control

system design methods by which output responses can be specified quite exactly. We apply the solutions of quadratic stabilization problem already obtained to the design problems of servo systems. The parameterization of feedback gains and positive definite solutions of the Riccati equations are well known in the inverse problem of linear quadratic design problem. With this parameterization, we can construct the control system in which decoupled desirable output responses can be achieved asymptotically in the configuration of one degree of freedom. This design problem is considered first for the state feedback case. However, state variables are not often available in a practical control system, so that this problem is considered for the observer-based output feedback case. Finally robust stability conditions with respect to new design parameters are derived and practical algorithms of robust servo systems are proposed as well based on these conditions.

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Notation

$\boldsymbol{R}$	The set of real numbers.
$\boldsymbol{C}$	The set of complex numbers.
Re x	The real part of $x \in \boldsymbol{C}$.
Im x	The imaginary part of $x \in \boldsymbol{C}$.
$\boldsymbol{C}^-$	Open left half plane of the s plane, $\boldsymbol{C}^- = \{s \mid \text{Re } s < 0\}$.
$\boldsymbol{C}^0$	Imaginary axis of the s plane, $\boldsymbol{C}^0 = \{s \mid \text{Re } s = 0\}$.
$\boldsymbol{C}^+$	Open right half plane of the s plane, $\boldsymbol{C}^+ = \{s \mid \text{Re } s > 0\}$.
$\boldsymbol{R}^n$	The set of real vectors of dimension n.
$\boldsymbol{C}^n$	The set of complex vectors of dimension n.
$\boldsymbol{R}^{m\times n}$	The set of real $m \times n$ matrices.
$\boldsymbol{C}^{m\times n}$	The set of complex $m \times n$ matrices.
$\bar{\sigma}(M)$	The maximum singular value of a matrix M.
M^T	Transpose of a matrix M.
M^*	The complex conjugate transpose of a matrix M.
Range(M)	The range space of a matrix M.
Ker(M)	The kernel space of a matrix M.
$M^\perp$	Annihilator of M, i.e., Range$(M^\perp)$ = Ker(M) and $M^\perp M = 0$ (when M is tall) or $MM^\perp = 0$ (when M is fat).

$M > 0$	M is symmetric positive definite.
$M \geq 0$	M is symmetric positive semidefinite.
I_n	The $n \times n$ identity matrix. The subscript is omitted when n can be determined form context.
$0_{m\times n}$	The $m \times n$ entirely zero matrix. The subscript is omitted when n can be determined form context.
$\text{conv}\{*\}$	convex hull of a set $*$
$\in$	belong to
$\lvert\alpha\rvert$	absolute value of $\alpha \in \boldsymbol{C}$
$\det M$	determinant of M
$\lambda(M)$	The eigenvalue of a matrix M.
$\lambda_{\max}(M)$	The maximum eigenvalue of a matrix $M = M^*$.
$\lambda_{\min}(M)$	The minimum eigenvalue of a matrix $M = M^*$.
$\lVert x\rVert$	Euclidean norm of x in $\boldsymbol{R}^n$ or $\boldsymbol{C}^n$.
$\lVert M\rVert$	The spectral norm of a matrix or vector, i.e., $\sqrt{\lambda_{\max}(M^TM)}$
$:=$	Define.

$$\text{diag}\{A, \ldots, Z\} := \begin{bmatrix} A & & 0 \\ & \ddots & \\ 0 & & Z \end{bmatrix}.$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := D + C(sI - A)^{-1}B.$$

$\boldsymbol{RH}^\infty$	The set of proper stable real rational transfer functions.
$\lVert G(s)\rVert_\infty$	The H_∞ norm of a transfer function $G(s) \in \boldsymbol{RH}^\infty$, i.e., $:= \sup_{\omega\in\boldsymbol{R}} \bar{\sigma}(G(j\omega)) = \sup_{\text{Re}s\geq 0} \bar{\sigma}(G(s))$.

Contents

Chapter 1

Introduction

1.1 Motivation

A mathematical model is needed in order to apply a control theory to a real plant. For this purpose, a complex model is constructed in order to describe the behavior of the real plant quite exactly, and often becomes nonlinear or time varying, or has high orders because the real plant in general is a complex dynamical system and its environments change the plant characteristics. Hence, it is difficult to design a controller for the complex model. Even if we can design a controller for the model, it becomes complex and is difficult to implement. Therefore, it has to be simplified or its order has to be reduced, and as a result the performance of the closed-loop system becomes worse and sometimes unstable. For this reason, it is necessary to construct a simpler model for design in order to make a reasonable controller. Then there always exists a gap between the real plant to be controlled and the model for design. Hence, we need the theory of control system design which guarantees stability and specified performance of the closed-loop system under the gap, i.e., uncertainty. It is the theme of *robust control* to establish such a theory.

This thesis tackles robust control problems stated above. In the robust control design, at least *robust stability* as well as *nominal stability* has to be guaranteed. The former implies that the closed-loop system remains stable under uncertainties and the latter implies that the closed-loop system is stable for a nominal model. There are two kinds of uncertainties, that is, *unstructured uncertainty* and *structured uncertainty.* The cause of the former is not so clear, for example, unmodelled dynamics at high frequencies, neglected dynamics due to lack of understanding of the physical processes, the difference between frequency response test and a model for design, etc. The cause of the latter is quite clear; for example, uncertainties of plant parameters in the plant whose structure is known, a discrete set of plants etc. In this thesis robust control problems with respect to structured uncertainties are treated through the qualitative analysis problem among them followed by the quantitative synthesis one.

We consider these problems with two notions of stability for systems with structured uncertainties. One is called *quadratic stability*, which is defined using the fixed Lyapunov function invariant for uncertainties. The other is called *stability for interval systems*; this means that all the roots of the interval system, i.e., the system whose uncertain entries are located in some pattern, are in the strict left half of the complex plane.

When we focus on the structure of systems, some basic properties regarding controllability, stability etc. can be clarified and focusing on it often makes analysis of systems easy. Controllability of systems is one of the most important system properties in control system design and analysis of system dynamics. In control system design we may construct a simpler robust controller by paying attention to the system structure. Thus, analysis of system properties focusing only on the structure of systems is meaningful. Hence, we consider the structured system, i.e., the system which consists of fixed zero entries, independently varying entries or sign-invariant varying entries and sizes of varying entries are arbitrary. We first focus on the structure of systems and study qualitative analysis problems with respect to robust stabilizability. Then the relations between various kinds of robust stabilizability and controllability are investigated.

Next we treat a design problem of a control system for an uncertain system, as a quantitative synthesis problem. The objective of the design problem for servo systems, one of the most important problems in control system design, is to make the outputs track step reference inputs. In this design it is required that the closed-loop system is stable and outputs track step reference inputs under parameter uncertainties. Furthermore good characteristics of output responses, for example, small overshoot, short settling time etc., are often required as design specifications. In order to achieve these design specifications, control system design methods by which output responses are specified quite exactly are needed. In view of this demand, we first focus on systems with norm bounded structured uncertainties of one-block for which a quadratically stabilizing control law is derived analytically and it can be parameterized by new design parameters. The parameterization of feedback gains and positive definite solutions of the Riccati equations are well known in the inverse problem of linear quadratic design problem. Then using this parameterization, we can construct the control system which is able to achieve decoupled desirable responses asymptotically in the configuration of one degree of freedom. Next the solutions of the quadratic stabilization problem already obtained are applied to the design problems of servo systems in order to guarantee robust stability. This design problem is first considered for the state feedback case, and then discussed also for the observer-based output feedback case since state variables are not often available in practical control systems. Finally robust stability conditions with respect to new design parameters are derived and then design algorithms of robust servo systems are proposed based on these conditions. Using these algorithms, both robust stability and output response specification can be achieved almost independently in the configuration of one degree of freedom.

1.2 Overview of the Previous and Related Research

As pointed in [Soro84] etc., if we do not take uncertainties into consideration in control system design, the resulting control system can be fragile under uncertainties. Therefore, robust control theory is one of the most important research area in control engineering and a lot of researches about it have been done recently. In the course of these researches, robust control problems have been attacked in several framework. Especially, robust stabilization problem is tackled using interpolation theory in [Kim84] etc., H_∞ control theory in [Doy89] etc., quadratic stability theory in [Peter87a, Peter87b, Peter88], [Khar90], [Wei90], μ theory in [Doy82] etc., L_1 control theory in [Vid86] etc., Kharitonov theorem in [Khari78]. Furthermore, robust control problems focusing on systems structures [Wei90, Wei94], [Ame94b, Ame96a, Ame97a],[Maye81],[Hu96, Hu97], [Jing96] are also considered.

Focusing on the structure of systems clarifies such basic properties as controllability, stability etc. and it makes analysis of systems easy. Controllability of systems is one of the most important system properties in connection with control system design as well as analysis of system dynamics. In addition, in control system design we may construct a simpler robust controller by paying attention to system structure. Hence, qualitative analysis problems focusing on the structure of systems have been investigated so far [Itom81] for the structured system which consists of fixed zero entries, independently varying entries or sign-invariant varying entries of arbitrary sizes. It is worth mentioning some of them. First, for a kind of the continuous-time structured system with fixed zero entries and independently varying entries alone, Lin [Lin74] defined *structural controllability*, meaning that a system is controllable for almost all parameters of system matrices. He then derived a necessary and sufficient condition for structural controllability using graph theory. For a discrete-time system structural controllability was discussed in [Muro92]. Next, in [Maye79] Mayeda and Tanaka derived a necessary and sufficient condition for another strong structural controllability, meaning that a system is controllable as long as some independently varying entries do not become zeros. Ishida *et al.* tackled the problem of sign stability and sign observability for the system having three kinds of entries, such as, fixed zero entries, positive entries and negative entries in [Ishi81]. In [Maye81] a necessary and sufficient condition that the servo problem for structured systems has a solution was derived and a design algorithm of a robust servo compensator for structured systems was proposed.

Next similar researches were developed on another kind of structured system, which consists of fixed zero entries, sign-invariant varying entries and independently varying entries. There have been some important researches on quadratic stabilizability in connection with this thesis. For example, Wei [Wei90] showed a necessary and sufficient condition of quadratic stabilizability for a kind of a single input structured system in terms of a geometric pattern with respect to the location of uncertain parameters, which is called antisymmetric stepwise configuration. This result was then extended for a slightly larger class of interval systems by Hu *et al.* [Hu96]. For the multi-input case only a sufficient condition was obtained in [Wei89a] and those for a different class of systems in [Dai96],[Hu97]. Furthermore, Su and Fu treated nonlinear systems and derived a design method for a class of nonlinear uncertain systems with an up-triangular structure in [Su98]. The antisymmetric stepwise configuration includes up-triangular structure.

With regard to stabilizability, meaning that all the roots of a system belonging to some pattern are in the strict left half of the complex plane under uncertainties, some studies have been made. For example, based on the preliminary results in [Wei85, Wei89b], a necessary and sufficient condition of stabilizability in the sense of making the roots of a perturbed characteristic equation stable was obtained in [Wei92, Wei94] for a kind of a single input structured system, referred to as interval system. The condition is given in terms of a geometric pattern as in [Wei94]. Jing *et al.* [Jing96] extended slightly the class of single input stabilizable interval systems. For a multi-input system a sufficient condition was derived in [Wei89c].

Stabilization problems for linear delay systems were considered and conditions for delay-independent stabilization were derived in terms of a geometric pattern with respect to the location of uncertain parameters as in the above researches. The conditions were obtained in terms of a geometric pattern without uncertainties in the system matrices except a coefficient matrix of time delay in [Ame83, Akaz87, Ame88]. In [Ame94a, Ame94b, Ame96a, Ame96b, Ame97a] robust stabilization problems for linear delay system were considered and conditions in terms of a geometric pattern were obtained. In [Ame97b], those conditions obtained in [Wei90],[Ame96b] were explained by singular perturbation approach. These conditions for linear delay systems are based on the idea of designing feedback such that the minus of the state matrix of the closed-loop system becomes an M-matrix.

Furthermore, the conditions of adaptive stabilizability were also given in terms of a geometric pattern with respect to the location of uncertain parameters in [Koko91], [Kane91].

Robust stabilization problem for systems with arbitrarily large parameter variations except the above structured systems has been considered using bound invariant Lyapunov function in [Holl87],[Zhou88b]. Furthermore, in [Peter87a] Petersen considered a notion of completely stabilizability with an arbitrary degree of stability, which implies that an uncertain system is quadratically stabilizable with an arbitrary degree of stability for arbitrarily large parameter variations. He then derived a necessary and sufficient condition for it and in addition considered the relation between the stability and controllability invariance, which implies that a system is controllable for all uncertainties. However, the complete clarification of the connection between these two notions was not made. We consider, therefore, the relation between controllability invariance and robust stabilizability for a class of structured systems in this thesis.

Quadratic stability theory treats stability of uncertain systems using a Lyapunov function in the quadratic form with a positive definite matrix invariant for uncertainties and provides effective design methods for systems with structured uncertainties. In [Leit79] this theory was applied to an uncertain system in order to guarantee uniformly asymptotic stability of the closed-loop system and then *quadratic stabilizability* was first defined in [Holl80]. Barmish derived a necessary and sufficient condition of quadratic stabilizability condition for uncertain systems by a continuous state feedback control mapping [Bar85]. In [Stein85] quadratic stabilization problem using output feedback was considered for a system having uncertainties only in the input matrix under matching condition. Schmittendorf constructed a quadratically stabilizing state feedback control law under matching condition when there exist uncertainties both in the state and input matrices [Schm87]. Next he derived a sufficient condition in the state feedback case without matching condition when there exist uncertainties both in the state and input matrices [Schm88]. Petersen derived a necessary and sufficient condition of quadratic stabilizability using state feedback in the case [Peter87b] where there exist uncertainties only in the state matrix and in the case [Peter88] that there exist uncertainties only in the input matrix. In [Khar90] a necessary and sufficient condition in the state feedback case was derived when there exist uncertainties both in the state and input matrices. Asai *et al.* discussed the quadratic stabilization problem for a descriptor system [Asai95]. In the observer-based output feedback case quadratic stabilization problem was considered in [Bar86, Gali86, Holl86, Jabb97, Peter85]. In [Osuk89] quadratic stabilization problem by output feedback was considered. Furthermore quadratic stabilization problem with disturbance attenuation was considered in [Xie92, Xie96], etc.

One of the most important issue in control system design is the servo problem whose objective is to design a control system such that output responses track step reference inputs, which is a quantitative synthesis problem. Furthermore it is essential to track output responses to step reference inputs under parameter variations and hence numerous researches have been made recently. In [Davi76] desired tracking is guaranteed essentially for small parameter variations, whereas in [Schm86] it is guaranteed for large parameter variations but under a strong assumption of matching condition. In addition numerous researches have been made based on the above-mentioned results concerning quadratic stability. Tsuchida and Suda considered robust servo problem for systems with time invariant uncertainties by state feedback without the assumption of matching condition and introduced tuning parameters into a feedback control law [Tsuch91]. When the above-mentioned results of quadratic stabilization problem are applied to robust servo systems, the relations between design parameters and design specifications are not clarified. In order to overcome this difficulty, ILQ design method has been developed for systems with no uncertainties in [Fujii87a, Fujii87b, Fujii87c, Fujii88, Shimo93, Kuroe96, Kuroe98]. Another ILQ design method was developed by a polynomial approach in [Sugi95]. In this thesis we apply the parameterization of a feedback control law well known in the inverse problem of linear quadratic control problem to this robust servo problem and propose some practical design algorithms in the cases of state feedback and observer-based output feedback. These algorithms achieve asymptotic specification of desirable output responses. From a similar viewpoint to the one in this thesis, this servo problem was considered in [Fujii93a],[Sugi97, Sugi98]. Using the same parameterization as the one in this thesis, the regulator problem for systems with structured uncertainties was treated in [Shimo98].

Furthermore, in [Hoz97] robust servo problem with H_∞ norm constraint was considered for system with time invariant uncertainties by output feedback. For systems with time varying uncertainties, robust servo problem was also considered in [Itoh98],[Yama93, Yama94]. Other approach to robust servo system is to apply the results in [Chil96, Garc96a, Garc96b, Masu95, Sche97] with pole assignment constraints, H_∞ norm constraints etc. to the augmented system including an integrator.

1.3 Thesis Outline

In Chapter 2 we introduce some kinds of linear time invariant uncertain systems treated in this thesis. Then features of those systems and comparison to other types of systems are described. Moreover, the important notions treated in this thesis are introduced. They are controllability invariance and some kinds of robust stabilizability which are important from a system theoretic point of view.

In Chapter 3 we consider the relation between uncertainty structure and robust stabilizability for interval systems. Interval system is a kind of structured systems and consists of fixed zero entries, sign invariant entries and sign varying entries. In this chapter a standard system of interval system is considered and has a special structure necessary for robust stabilization. First, the results in [Wei90, Wei92, Wei94] are introduced, in which conditions for robust stabilization are given in terms of a geometric pattern with respect to the location of uncertain parameters. Next we derive the main result which shows the equivalence between controllability invariance and stabilizability for interval systems. Finally we show the proofs of the lemma and propositions which play key roles in the proof of the main theorem.

Chapter 4 addresses robust servo problems for systems with norm bounded structured uncertainties of one-block. First, control objectives are stated and the augmented system used in the design of servo systems for a step reference input is introduced. Next the solutions of quadratic stabilization problem are introduced, some transformations are applied to the system and then some preliminary results are derived. The above transformations are necessary in order to apply the result on the parameterization of a feedback control law to the robust servo problem. Then we introduce the result on the parameterization which leads to the special structure of the closed-loop system; this structure is important so as to achieve desirable output responses asymptotically. Next we introduce a necessary and sufficient condition of quadratic stability and state main results for design methods of robust servo systems. The results for the case of state feedback are stated when there exists uncertainties entering into the state matrix or both into the state and input matrix. The results for the case of observer-based output feedback are also stated. After stating main results, design algorithms are proposed based on the main results in the above-mentioned cases. The proofs of the main results are described and the effectiveness of the proposed design algorithms are explained by design examples.

In Chapter 5 we summarize the conclusions and point out some future research issues.

Finally, some useful lemma is stated in Appendix A. In Appendix B the proof of the preliminary result is shown and Appendix C discusses the derivation of the augmented system for the design of servo systems. The calculation of the feedback gains of ILQ design method is described in Appendix D. In Appendix E illustrative examples of generalized antisymmetric stepwise configurations as stabilizability condition of interval systems and antisymmetric stepwise configurations as quadratic stabilizability condition are listed.

Chapter 2

Systems and Definitions

2.1 Systems

We introduce linear systems with time invariant structured uncertainties treated in this thesis.

Remark 2.1 *There are examples of time invariant uncertainties in the following.*

One example of time invariant uncertainties is the uncertainty generated by the differences among each characteristic of a group of a kind of plant. Each characteristic of plants is usually distributed in a specified range, the range is determined by an upper bound and a lower bound and the characteristic is fixed for each plant. Therefore, the above-mentioned uncertainty is time invariant.

The other is the uncertainty in the following. The characteristic of a plant changes slowly as time passes. Therefore, the change is regarded as the other type of time invariant unceratintties.

2.1.1 Interval Systems

We consider a single input time invariant interval system (or an interval system for short) (A, b), which is a set of systems described by the state equation

$$\dot{x} = Ax + bu \tag{2.1}$$

where $x \in \boldsymbol{R}^n$ is the state; $u \in \boldsymbol{R}$ is the control; the entries of A and b are unknown but bounded in given compact sets; i.e., $A = \{a_{ij}\}$ and $\overline{a}_{ij} \geq a_{ij} \geq \underline{a}_{ij}$; $b = \{b_i\}$ and $\overline{b}_i \geq b_i \geq \underline{b}_i$. Note that the entries of A, b vary independently. We will write $a_{ij} \equiv 0\,(b_i \equiv 0)$ and it is called a **fixed zero entry** if $\overline{a}_{ij} = \underline{a}_{ij} = 0$ (resp. $\overline{b}_i = \underline{b}_i = 0$).

The entry a_{ij} or b_i is called a **sign-invariant entry** if $\overline{a}_{ij} \times \underline{a}_{ij} > 0$ or $\overline{b}_i \times \underline{b}_i > 0$ and a **sign-varying entry** if $\overline{a}_{ij} \times \underline{a}_{ij} < 0$ or $\overline{b}_i \times \underline{b}_i < 0$.

An example of interval systems is shown below.

$$\dot{x} = \begin{bmatrix} 0 & 0 & * \\ * & 0 & \theta \\ * & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ * \\ 0 \end{bmatrix} u$$

0 : fixed zero entry, θ : sign-invariant entry, $*$: sign-varying entry

Noting in Theorem 3.1 in [Wei90] and Theorem 2.6 in [Wei94] that a robustly stabilizable system must have at least the same number of sign-invariant entries in the system matrices as the system order, we restrict our attention to a class of interval systems which is called a standard system as defined below.

Definition 2.1 *[Wei92],[Wei94] An $n \times (n+1)$ interval matrix M is called the* **associated matrix** *of the interval system (A, b) if*

$$M = \begin{bmatrix} A & b \end{bmatrix} \tag{2.2}$$

Furthermore an interval system (A, b) is **standard** *if the associated matrix $M = \{m_{ij}\}$ has the property that m_{ii+1} is a sign-invariant entry for each i, $1 \le i \le n$.*

Remark 2.2 *In the definition of the interval system we focus on the location of structured uncertainties and do not restrict size of uncertainties. Therefore, we can investigate the relation between uncertainty structure and robust stabilizability.*

The following is an example of standard systems.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & \theta & * \\ 0 & * & \theta \\ 0 & * & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} * \\ 0 \\ \theta \end{bmatrix} u, \quad M = \begin{bmatrix} 0 & \theta & * & * \\ 0 & * & \theta & 0 \\ 0 & * & 0 & \theta \end{bmatrix}$$

0 : fixed zero entry, θ : sign-invariant entry, $*$: sign-varying entry

We can find it easily that m_{ii+1} $(i = 1, \ldots, 3)$ are sign-invariant entries.

Remark 2.3 *The result in Chapter 3 holds for the system whose states are permuted because permutation guarantees independence of parameter variations.*

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} * & 0 & \theta \\ \theta & 0 & * \\ * & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ * \\ \theta \end{bmatrix} u$$

Remark 2.4 *At least n sign-invariant entries in a system are necessary in order to stabilize the system robustly by [Wei90, Wei94]. Therefore, there are n sign-invariant entries in a standard system. However, there may be more than n sign-invariant entries in a practical system. Therefore, dealing with a system which has more than n sign-invariant entries is left as a future research.*

Remark 2.5 *Theorem 3.1 in [Wei90] and Theorem 2.6 in [Wei94] do not show where sign-invariant entries are located. In [Wei90, Wei94] a standard system is defined as a kind of interval systems which meet the condition in Theorem 3.1 in [Wei90] and Theorem 2.6 in [Wei94], and the location of sign-invariant entries in the system is the same as that of* 1 *in the canonical form. In a standard system the directions of the inputs from* x_2 *to* $\dot{x}_1$*, from* x_3 *to* $\dot{x}_2$*,* $\ldots$*, from* x_n *to* $\dot{x}_{n-1}$*, and from* u *to* $\dot{x}_n$ *are invariant, i.e., there exists at least one route where the directions of the inputs between each states are invariant.*

However, we have to consider the location and the number of sign-invariant entries in order to make a class of standard systems large. The location of sign-invariant entries in [Wei89c],[Dai96] and [Hu97] is different from that of 1 *in the canonical form in the multi-input case. In [Wei89a] a quasi stable uncertain matrix was defined. Therefore, it may be better that in the multi-input case the location of sign-invariant entries in a system is similar to that of* 1 *in the canonical form.*

2.1.2 Systems with norm bounded structured uncertainties

Consider a linear system with time invariant norm bounded structured uncertainty of one-block described by

$$\begin{aligned} \dot{x}(t) &= [A + \Delta A]x(t) + [B + \Delta B]u(t) \\ y(t) &= Cx(t) \\ \Delta A &= DFE_a, \Delta B = DFE_b. \end{aligned} \tag{2.3}$$

Here $x(t) \in \boldsymbol{R}^n$ is the state, $u(t) \in \boldsymbol{R}^m$ is the control input and $y(t) \in \boldsymbol{R}^m$ is the output; A, B, and C are the nominal system matrices with rank $B = m$; D, E_a and E_b are known real matrices characterizing the structure of the uncertainties. In addition, F is a matrix of uncertain parameters with its maximum singular value bounded by unity, i.e.,

$$F \in \boldsymbol{F} = \{F : \|F\| \leq 1\}. \tag{2.4}$$

Remark 2.6 *There are two kinds of norm bounded structured uncertainties other than the above-mentioned uncertainties as follows.*
Parametric structured uncertainties :

$$\Delta A(\boldsymbol{r}) = \sum_{i=1}^{p} r_i A_i, \quad |r_i| \leq 1$$
$$\Delta B(\boldsymbol{s}) = \sum_{i=1}^{q} s_i B_i, \quad |s_i| \leq 1$$

Block structured uncertainties :

$$\Delta A(\boldsymbol{r}) = \sum_{i=1}^{h} D_{ai}\Delta_{ai}(\boldsymbol{r})E_{ai}, \quad \|\Delta_{ai}\| \leq 1$$
$$\Delta B(\boldsymbol{s}) = \sum_{i=1}^{k} D_{bi}\Delta_{bi}(\boldsymbol{s})E_{bi}, \quad \|\Delta_{bi}\| \leq 1$$

Here $\boldsymbol{r}$, $\boldsymbol{s}$ *are vectors representing uncertainties.*

These kinds of structured uncertainties have more degree of freedom in representing uncertainties than one-block one in this thesis. However, only sufficient conditions of robust stabilizability have been obtained for the former systems in the analytical expression, while necessary and sufficient conditions will be obtained for the system treated in this thesis in the analytical expression. Therefore, we can parameterize the feedback gain in the analytical expression for the system with one-block uncertainties and propose practical design algorithms of constructing robust servo systems.

Remark 2.7 *Here we state the treatment of nonlinear systems using the above type of systems. In Fig. 2.1 the idea of the treatment is shown. When we treat nonlinear systems in the framework of the systems with norm bounded structured uncertainties, linear approximate models at the operating points* $(i = 1, \ldots, n)$ *in Fig. 2.1 have to be derived and then the mathematical model (2.3) which includes the linear approximate models be constructed.*

However, the effective range of this approach using constant linear feedback in this thesis is probably narrower than that of the approaches using nonlinear feedback.

2.2 Systems Properties

In this section an important notion of controllability invariance with respect to qualitative system property is introduced as follows.

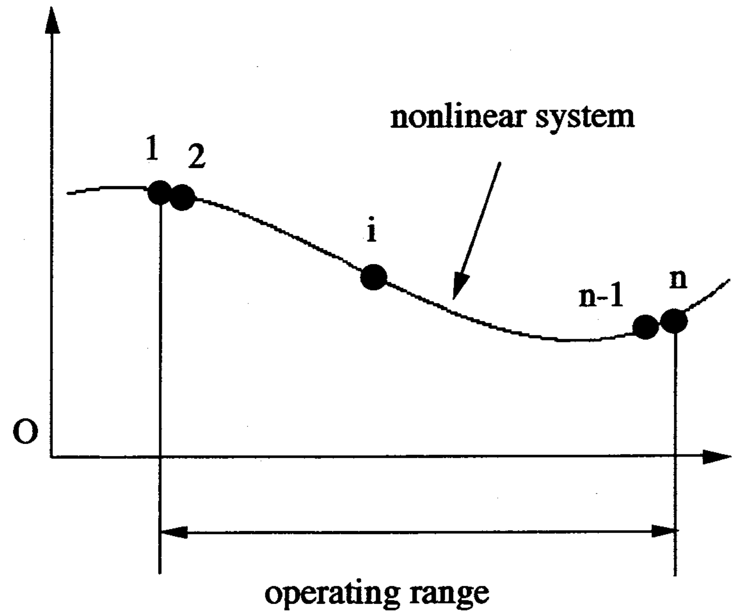

Figure 2.1: Treatment of a nonlinear system

Definition 2.2 *Consider an interval system (A, b) and assume that the associated matrix M is a standard system. Then the interval system (A, b) is said to be* **controllability invariant** *if the pair (A, b) is controllable in a usual sense for any fixed value of uncertain parameters, that is,*

$$\text{rank}\left[\begin{array}{cc} A - sI & b \end{array}\right] = n \tag{2.5}$$

for every $s \in \boldsymbol{C}$ and every $a_{ij} \in [\underline{a}_{ij}, \overline{a}_{ij}], b_i \in [\underline{b}_i, \overline{b}_i]$.

Remark 2.8 *The notion of "controllability invariance" defined above, which is essentially the same as that defined in [Peter87a], is a natural extension of the familiar "controllability" in the linear system theory.*

Remark 2.9 *In a decentralized control system a* **fixed mode** *is defined as a pole which cannot be moved for any feedback control. The necessary and sufficient condition for pole-assignability of a system by a decentralized control is that there exists no fixed mode. A controllability invariant interval system has no fixed mode with respect to* *state feedback control for each uncertainty and therefore the system may be stabilizable by some decentralized control. Then the problem of stabilization by a decentralized state feedback control is equivalen to the problem of stabilization of multi-input system.*

The condition the a system is controllable and observable is a necessary condition and not a sufficient condition for pole-assignability of a system by a decentralized control. However, controllability invariance and observability invariance are stronger condition than controllability and observability. Therefore, the system which is controllability invariant and observability invariant is pole-assignable by a decentralized control. It is left as a future study.

An example of controllability invariant interval systems is given as follows.

$$\dot{x} = \begin{bmatrix} 0 & \theta_1 & 0 \\ 0 & a_{22} & \theta_2 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} b_1 \\ 0 \\ \theta_3 \end{bmatrix} u = Ax + bu$$

$$\theta_i \ (i = 1, 2, 3) \ : \text{sign-invariant entries}, \ a_{22}, b_1 \ : \text{sign-varying entries}$$

The equation (2.5) can be replaced with the following equation.

$$\text{rank} \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix} = n$$

This condition is equivalent to the following condition.

$$\det \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix} \neq 0$$

Therefore, we investigate this condition ($n = 3$) in order to show that the above system is controllability invariant.

$$\det \begin{bmatrix} b & Ab & A^2b \end{bmatrix} = \begin{bmatrix} b_3 & 0 & \theta_1\theta_2\theta_3 \\ 0 & \theta_2\theta_3 & a_{22}\theta_2\theta_3 \\ \theta_3 & 0 & 0 \end{bmatrix} = -\theta_1\theta_2^2\theta_3^3 \neq 0$$

The above system is controllability invariant from this equation.

Thus the condition of controllability invariance can be checked by symbolic computation.

Remark 2.10 *In the system theory stabilizability is important together with controllability. Therefore, the notion of stabilizability invariance seems to be important together with controllability invariance. Here the notion of stabilizability invariance is defined in the following.*

***Definition 2.3** Consider an interval system (A, b) and assume that the associated matrix M is a standard system. Then the interval system (A, b) is said to be* **stabilizability invariant** *if the pair (A, b) is stabilizable in a usual sense for any fixed value of uncertain parameters, that is,*

$$\operatorname{rank}\begin{bmatrix} A - sI & b \end{bmatrix} = n \tag{2.6}$$

for every $s \in \{C^+ \cup C^0\}$ and every $a_{ij} \in [\underline{a}_{ij}, \overline{a}_{ij}], b_i \in [\underline{b}_i, \overline{b}_i]$.

For example, we consider the following system.

$$\dot{x} = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

$$\theta_1,\ \theta_2\ :\ \text{sign-invariant entries}$$

When θ_1 and θ_2 are minus, this system is stabilizability invariant. However, this system is not a standard system and does not become a standard system even by permutation of states. Hence, the above system is not in a class of systems which are dealt with in this thesis. However, this system is robustly stabilizable clearly.

According to this discussion, we would better treat a wider class of systems than standard systems by changing the position of sign-invariant entries and discuss the relations between robust stabilizability, controllability invariance etc.

2.3 Notions of Robust Stabilizability

Several notions of stabilizability for uncertain systems treated in this paper are defined in this subsection. First *quadratic stability* is defined in the following.

Definition 2.4 *[Bar85] The unforced system (2.3) with $u = 0$ is said to be* **quadratically stable** *if there exists an $n \times n$ real symmetric matrix $P > 0$ and a constant $\alpha > 0$ such that for any admissible uncertainty F, the Lyapunov function $V(x) = x^T P x$ satisfies*

$$L(x) := \dot{V} = 2x^T P \left[A + DFE_a \right] x \leq -\alpha \left\| x \right\|^2 \tag{2.7}$$

for all pairs $(x, t) \in R^n \times R$.

Remark 2.11 *"Quadratic stability" is a kind of stability for an uncertain system, to which usual Lyapunov stability for a deterministic system is extended for an uncertain system. When $F = 0$, quadratic stability in Definition 2.4 means Lyapunov stability for a deterministic system.*

Next we introduce one of the notion of robust stabilizability, that is, quadratic stabilizability. It is defined for both systems with norm bounded structured uncertainties and interval systems.

Definition 2.5 *[Holl80],[Bar85] The system (2.3) is said to be* **quadratically stabilizable via linear control** *if there exists a linear static state feedback control $u = -Kx$ with $K \in \boldsymbol{R}^{m \times n}$, $P > 0$ and $\alpha > 0$ such that the following condition holds : for any admissible uncertainty F,*

$$\boldsymbol{L}(x) = 2x^T P \left[(A + DFE_a) - (B + DFE_b)K \right] x \leq -\alpha \|x\|^2 \tag{2.8}$$

for all pairs $(x, t) \in \boldsymbol{R}^n \times \boldsymbol{R}$. where $\boldsymbol{L}(x)$ is the Lyapunov derivative for the quadratic Lyapunov function $V(x) = x^T Px$ along the trajectories of the closed-loop system.

Definition 2.6 *[Holl80],[Bar85] An interval system (A, b) is said to be* **quadratically stabilizable via linear control** *if there exists a linear static state feedback control $u = -kx$ with $k^T \in \boldsymbol{R}^n$, $P > 0$ and $\alpha > 0$ such that the following condition holds : for all (A, b)*

$$\boldsymbol{L}(x) = x^T \left[A^T P + PA \right] x - 2x^T Pbkx \leq -\alpha \|x\|^2 \tag{2.9}$$

where $\boldsymbol{L}(x)$ is the Lyapunov derivative for the quadratic Lyapunov function $V(x) = x^T Px$ along the trajectories of the closed-loop system.

Here stabilizability of interval systems is defined.

Definition 2.7 *[Wei92],[Wei94] An interval system (A, b) is said to be* **stabilizable** *if there exists a linear static state feedback control law $u = -kx$ with $k^T \in \boldsymbol{R}^n$ such that the characteristic polynomial of the closed-loop system*

$$f(s) = \det(sI - A + bk) \tag{2.10}$$

is a Hurwitz invariant polynomial; i.e., all the roots of the uncertain polynomial $f(s)$ are in the strict left half of the complex plane.

Remark 2.12 *The stabilizability of interval systems is an extended notion of stabilizability for a deterministic system which means that all the roots of the closed-loop system are in the strict left half of the complex plane. The notion of* **stabilizability** *in Definition 2.7 is often called* **Hurwitz stabilizability**.

2.4 Summary

In this chapter we introduce some kinds of linear time invariant uncertain systems treated in this thesis. Then features of those systems are described and comparison to other types of systems are pointed out. Moreover, the important notions treated in this thesis are introduced. They are controllability invariance and some kinds of robust stabilizability which are important from a system theoretic point of view.

Chapter 3

Uncertainty Structure and Robust Stabilizability

3.1 Introduction

In recent years, the robust stabilization problem has attracted a considerable amount of interest in the field of robust control. Various kinds of necessary and sufficient conditions have been derived so far for the existence of robustly stabilizing controllers. One example is a condition given in terms of the Pick matrix using interpolation theory [Kim84]. Another is a type of condition given in terms of the solutions to Riccati equations both in H_∞ control problem [Doy89] and in quadratic stabilization problem (see [Khar90] and the references therein).

For a certain class of linear interval systems, conditions for robust stabilization are given in terms of a geometric pattern with respect to the location of uncertain parameters both in the quadratic stabilization problem [Wei90] and in the robust stabilization problem [Wei92, Wei94]. In the former the pattern is called ***antisymmetric stepwise configuration*** and in the latter ***generalized antisymmetric stepwise configuration.*** These conditions are easy to check, but not necessarily easy to understand from a system theoretic point of view. In particular, the connection of these conditions with more familiar notions in the linear system theory, for instance, controllability and other notions, are not so clear. Along this line in [Peter87a] Petersen defined the notion of controllability invariance in that a linear uncertain system is controllability invariant if it is controllable in usual sense for each fixed value of uncertain parameters and discussed its connection with complete quadratic stabilizability with an arbitrary degree of stability. However, he did not succeed in making a complete clarification of the connection between these two notions.

The purpose of this chapter is to investigate the robust stabilization problem for interval systems along the direction in [Peter87a]. Based on the results in [Wei90], [Wei92] and[Wei94], we restrict attention to uncertain systems having a specific structure which is described in greater detail in Section 2.1.1. It is called standard system. Along this line we show, with the help of the results in [Wei90, Wei92, Wei94], that the notion of controllability invariance plays an important role in this problem. We establish that controllability invariance is necessary and sufficient for robust stabilizability, and is necessary but not sufficient for quadratic stabilizability. Thus, the main contribution of this chapter is to give an intuitively appealing interpretation for the condition for robust stabilizability given in [Wei92, Wei94].

3.2 Preliminary Results

In this chapter we consider standard system among a single input time invariant interval system defined in Section 2.1.1. The system is a set of systems described by the state equation and rewritten in the following for convenience.

$$\dot{x} = Ax + bu \tag{2.1}$$

where $x \in \boldsymbol{R}^n$ is the state; $u \in \boldsymbol{R}$ is the control; the entries of A and b are unknown but bounded in given compact sets; i.e., $A = \{a_{ij}\}$ and $\bar{a}_{ij} \geq a_{ij} \geq \underline{a}_{ij}$; $b = \{b_i\}$ and $\bar{b}_i \geq b_i \geq \underline{b}_i$. Note that the entries of A, b vary independently. We will write $a_{ij} \equiv 0\,(b_i \equiv 0)$ and they are called **fixed zero entries** if $\bar{a}_{ij} = \underline{a}_{ij} = 0$ (resp. $\bar{b}_i = \underline{b}_i = 0$). The entry a_{ij} or b_i is called a **sign-invariant entry** if $\bar{a}_{ij} \times \underline{a}_{ij} > 0$ or $\bar{b}_i \times \underline{b}_i > 0$ and a **sign-varying entry** if $\bar{a}_{ij} \times \underline{a}_{ij} < 0$ or $\bar{b}_i \times \underline{b}_i < 0$.

In this chapter we treat the standard system. Hence, the associated matrix $M = \{m_{ij}\} = \begin{bmatrix} A & B \end{bmatrix}$ has the property that m_{ii+1} is a sign-invariant entry for each i, $1 \leq i \leq n-1$.

In this section we first state the definition of the notation mentioned in Section 3.1, which is given in terms of a geometric pattern.

Definition 3.1 *[Wei92],[Wei94] An $n \times (n+1)$ matrix $P = \{p_{ij}\}$ is said to be a* **pattern matrix** *if every entry p_{ij} of the matrix is either 0 or 1. Let Σ denote the set of all standard systems (A, b) as in Definition 2.1. For a given pattern matrix P, we define Σ_p as a subset of Σ determined by the following rule: A standard interval system $(A, b) \in \Sigma_p$ if $p_{ij} = 0$ implies $m_{ij} \equiv 0$ for any i, j.*

According to the above definition, in order to check if an interval system $(A, b) \in \Sigma_p$ we only need to check if it is a standard system and in addition $m_{ij} \equiv 0$ when $p_{ij} = 0$.

Definition 3.2 *[Wei92],[Wei94] An $n \times (n+1)$ pattern matrix $P = \{p_{ij}\}$ is said to have a* **generalized antisymmetric stepwise configuration** *if the following conditions hold:*

1. *$p_{ii+1} = 1$ for all $i = 1, 2, \cdots, n$.*
2. *If $p \geq h+2$ and $p_{hp} = 1$, then $p_{uv} = 0$ for all $u \geq v$, $u \leq p-1$ and $v \leq h$.*
3. *$\det(P^r) \equiv p_{12}p_{23}\cdots p_{nn+1}$, where P^r is the right submatrix of P defined by*

$$P^r := \begin{bmatrix} p_{12} & p_{13} & \cdots & p_{1n+1} \\ p_{22} & p_{23} & \cdots & p_{2n+1} \\ \vdots & \vdots & \vdots & \vdots \\ p_{n2} & p_{n3} & \cdots & p_{nn+1} \end{bmatrix}. \tag{3.1}$$

The following are all examples of the third order system whose pattern matrix has GAS configuration.

0 : fixed zero entry, θ : sign-invariant entry, $*$: sign-varying entry

$$A = \begin{bmatrix} a_{11} & \theta_1 & 0 \\ a_{21} & a_{22} & \theta_2 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ \theta_3 \end{bmatrix}; \quad A = \begin{bmatrix} 0 & \theta_1 & a_{13} \\ 0 & 0 & \theta_2 \\ 0 & 0 & 0 \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \theta_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & \theta_1 & a_{13} \\ 0 & 0 & \theta_2 \\ 0 & 0 & a_{33} \end{bmatrix}, b = \begin{bmatrix} b_1 \\ 0 \\ \theta_3 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & \theta_1 & a_{13} \\ 0 & 0 & \theta_2 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ \theta_3 \end{bmatrix}$$

$$A_* = \begin{bmatrix} 0 & \theta_1 & 0 \\ 0 & a_{22} & \theta_2 \\ 0 & 0 & 0 \end{bmatrix}, b_* = \begin{bmatrix} b_1 \\ 0 \\ \theta_3 \end{bmatrix}$$

Next we show by confirming three conditions in Definition 3.2 that the pattern matrix of (A_*, b_*) has GAS configuration.

1.

$$P = \begin{bmatrix} 0 & p_{12} & 0 & p_{14} \\ 0 & p_{22} & p_{23} & 0 \\ 0 & 0 & 0 & p_{34} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Longrightarrow \quad p_{12} = p_{23} = p_{34} = 1$$

2.

$$\begin{aligned} & p_{14}=1, \quad h=1, \quad p=4 \\ \Rightarrow\ & p_{uv}=0 \quad u \geq v, \quad u \leq p-1=3, \quad v \leq h=1 \\ \Rightarrow\ & p_{11}=p_{21}=p_{31}=0 \end{aligned}$$

3.

$$\begin{aligned} P^r &= \begin{bmatrix} p_{12} & 0 & p_{14} \\ p_{22} & p_{23} & 0 \\ 0 & 0 & p_{34} \end{bmatrix} \\ \det P^r &= p_{12}p_{23}p_{34} + p_{22}\cdot 0(p_{33})\cdot p_{14} + 0(p_{32})\cdot 0(p_{24})\cdot 0(p_{13}) \\ &\quad -p_{14}p_{23}\cdot 0(p_{32}) - 0(p_{13})\cdot p_{22}\cdot p_{34} - p_{12}\cdot 0(p_{33})\cdot 0(p_{24}) = p_{12}p_{23}p_{34} \end{aligned}$$

The entry corresponding to fixed zero entry is written in () in order to show which entry each fixed zero entry corresponds to. The relation $p_{32}=p_{33}=0$ has to hold in order to satisfy $\det P^r = p_{12}p_{23}p_{34}$.

The following lemma shows a necessary condition for controllability invariance and will be used later for proving one of the two key propositions, that is, Proposition 3.1, (see Section 3.4 for its proof).

Lemma 3.1 *If every interval system (A,b) in Σ_p is controllability invariant, then the following conditions hold;*

1. *If $b_k \not\equiv 0$, then $a_{ij} \equiv 0 (i \geq j, 1 \leq j \leq k,\ 1 \leq i \leq n)$ and $a_{nk+1} \equiv 0$.*
2. *If $v \geq u+2$ and $a_{uv} \not\equiv 0$, then $a_{ij} \equiv 0 (i \geq j, 1 \leq j \leq u,\ 1 \leq i \leq v-1)$ and $a_{v-1u+1} \equiv 0$.*

The following are two key propositions for deriving one of the main results, that is, Theorem 3.1.

Proposition 3.1 *Let P be a given pattern matrix. Every interval system (A,b) in Σ_p is controllability invariant if and only if the matrix P has a generalized antisymmetric stepwise configuration.*

The proof of this proposition is given in Section 3.4.

Proposition 3.2 *[Wei92],[Wei94] Let P be a given pattern matrix. Every interval system (A,b) in Σ_p is stabilizable if and only if the matrix P has a generalized antisymmetric stepwise configuration.*

3.3 Main Result

This first main result stated below is a direct consequence of Propositions 3.1 and 3.2.

Theorem 3.1 *Every interval system (A, b) in Σ_p is stabilizable if and only if every system (A, b) in Σ_p is controllability invariant.*

Remark 3.1 *This result has the following interpretation. Suppose a standard interval system is controllability invariant, then for each fixed values of the uncertain parameters, there exists a stabilizing feedback control law which may depend on the uncertain parameter. However, the theorem guarantees that the feedback control law can be chosen to be independent of the uncertain parameters, in other words, it depends only on the upper and lower bounds of the uncertain parameters. Conversely, if every standard interval system in Σ_p is robustly stabilizable, then controllability invariance must hold.*

Remark 3.2 *For a deterministic system, controllability is not necessary but sufficient condition for stabilizability. However, in Theorem 3.1 controllability invariance is equivalent to stabilizability. The reason why the equivalence holds is probably that the relation between these properties is considered for any large parameter variations. Hence, there exists the gap between the results for a deterministic system and an uncertain system.*

For this sense of robust stabilizability, we can easily obtain a corresponding result to Theorem 3.1, by noting that quadratic stabilizability implies robust stabilizability.

Corollary 3.1 *Every interval system (A, b) in Σ_p is quadratically stabilizable only if every system (A, b) in Σ_p is controllability invariant.*

Remark 3.3 *We can create an example showing that the converse statement is not always true. For example, it is easy to show that the following interval system (A, b) is controllability invariant, but not quadratically stabilizable according to the main result of [Wei90].*

Example

Here we consider the system with the system matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & a_{22} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ 0 \\ 1 \end{bmatrix} \tag{3.2}$$

where a_{22} and b_1 are sign-varying entries.

This system is controllability invariant, but not always quadratically stabilizable for a_{22} and b_1.

Proof of Example

First, we introduce a fact for quadratic stabilizability.

Fact 3.1 *[Bar85] A certain system (3.2) is quadratically stabilizable if and only if there exists an $n \times n$ positive definite matrix S such that*

$$x^T(AS + SA^T)x < 0 \tag{3.3}$$

for all $x \in \boldsymbol{N}$ with $x \neq 0$ and all A where $\boldsymbol{N} := \left\{x \in \boldsymbol{R}^n : Ker(b^T),\ i.e.,\ b^T x = 0 \text{ for some } b \in \operatorname{conv}\{b\}\right\}$.

In order to show that an interval system (3.2) is not quadratically stabilizable, we claim by the example that there does not exist a matrix S as in Fact 3.1. Suppose that the system (3.2) is quadratically stabilizable. Then there exists $S > 0$ as in Fact 3.1 described below.

$$S := \begin{bmatrix} 1 & s_2 & s_3 \\ s_2 & s_4 & s_5 \\ s_3 & s_5 & s_6 \end{bmatrix}$$

The vector $\beta_1^T := [1 \quad 0 \quad -b_1]^T$ belongs to $\boldsymbol{N}$. Then the following relation must hold.

$$\begin{bmatrix} 1 & 0 & -b_1 \end{bmatrix} \left(AS + SA^T\right) \begin{bmatrix} 1 \\ 0 \\ -b_1 \end{bmatrix} = 2(s_2 - s_5 b_1) < 0 \tag{3.4}$$

When the bound of $|b_1|$ is greater than 1, the following condition holds.

$$|s_2| > |s_5| \tag{3.5}$$

The vector $\beta_2^T := [1 \quad x_2 \quad 0]$ where x_2 is arbitrary real number belongs to $\boldsymbol{N}$. Then

$$\begin{bmatrix} 1 & x_2 & 0 \end{bmatrix} \left(AS + SA^T\right) \begin{bmatrix} 1 \\ x_2 \\ 0 \end{bmatrix} \tag{3.6}$$

$$= (a_{22}s_4 + s_5)x_2^2 + (a_{22}s_2 + s_3 + s_4)x_2 + s_2 < 0 \tag{3.7}$$

Since this relation holds in the system (3.2), it follows that

$$(a_{22}s_2 + s_3 + s_4)^2 - 4s_2(a_{22}s_4 + s_5) < 0. \tag{3.8}$$

When a_{22} takes the value r_{22} and $-r_{22}$, respectively, we obtain

$$8s_2s_5 > 2r_{22}^2s_2^2 + 2(s_3 + s_4)^2 > 2r_{22}^2s_2^2$$

When the value of x_2 is more than 1, it follows that

$$|s_2| < |s_5|$$

which contradicts (3.5). Hence, there does not exist S as in Fact 3.1 for the system. This completes the proof.

Remark 3.4 *Illustrative examples are taken in Appendix E.1 with respect to a generalized antisymmetric stepwise configuration as stabilizability of interval systems and an antisymmetric stepwise configuration as quadratic stabilizability.*

3.4 Proofs of Preliminary Results

3.4.1 Proof of Lemma 3.1

Below we denote sign-invariant entries by θ_i or θ.

Suppose that the following system Σ_{cl} is controllability invariant.

$$\Sigma_{cl} \; : A := \begin{bmatrix} 0 & \theta_1 & \cdot & \cdot & 0 \\ \cdot & & \cdot & a_{hp} & \cdot \\ a_{uv} & & & \cdot & \cdot \\ \cdot & & & & \theta_{n-1} \\ 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} \quad b := \begin{bmatrix} 0 \\ \cdot \\ b_k \\ \cdot \\ \theta_n \end{bmatrix} \tag{3.9}$$

Then by Definition 2.2 we can assume that the rank condition (2.5) holds for the case where $\theta_i = 1\,(i = 1, \cdots, n)$, and A and b contain only two sign-varying entries, b_k and a_{uv}, or a_{hp} and a_{uv}, which vary sufficiently largely. Define the matrix

$$N := \begin{bmatrix} A - sI & b \end{bmatrix} \tag{3.10}$$

and denote N_k as the $(n-k+1) \times (n-k+1)$ down-right submatrix of N.

For the former part we suppose that $a_{hp} \equiv 0$, $b_k \not\equiv 0$. Then clearly by (3.9) $\text{rank}[A - sI] = n - 1$ for s taking the eigenvalues of A and hence N_1 is of full rank, or,

$\det N_1 \neq 0$. We also note that the characteristic polynomial of A for the system Σ_{cl} is described by

$$(-s)^{n-u+v-1}\{(-s)^{u-v+1} + a_{uv}\} = 0 \tag{3.11}$$

for $u \geq v$. Below we find conditions for the following two cases such that $\det N_1 \neq 0$ for all the roots s of the equation (3.11), or all eigenvalues of A.

Case 1: $k = 1$, meaning that b_1 is a sign-varying entry :

Furthermore, we separate the case depending on the location of a_{uv}.

1) The case with $v = 1$, meaning that there exists a sign-varying entry in the first column of A : For the matrix

$$N_1 := \begin{bmatrix} 1 & & & & b_1 \\ -s & \cdot & & \mathbf{O} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & 0 \\ \mathbf{O} & & & -s & 1 \end{bmatrix} \tag{3.12}$$

we have

$$\det N_1 = 1 + b_1(-s)^{n-1}. \tag{3.13}$$

Substituting the eigenvalues of A into s in (3.13) yields

$$\det N_1 = 1 \qquad (s = 0) \tag{3.14}$$

$$\det N_1 = 1 + b_1 \left\{(-1)^{u+1}a_{u1}\right\}^{\frac{n-1}{u}} (-1)^{n-1}$$

$$(s = \sqrt[u]{|(-1)^{u+1}a_{u1}|}). \tag{3.15}$$

The latter equation implies that for $a_{u1} \not\equiv 0$ and $b_1 \not\equiv 0$, there exists a_{u1} and b_1 such that $\det N_1 = 0$. Hence if $b_1 \not\equiv 0$, then $a_{u1} \equiv 0\,(u = 1, \cdots, n)$.

2) The case with $v > 1$, meaning that there does not exist a sign-varying entry in the first column of A : Define

$$N_1 := \begin{bmatrix} 1 & & & & & b_1 \\ -s & \cdot & & & \mathbf{O} & 0 \\ 0 & \cdot & \cdot & & & \cdot \\ \cdot & a_{uv} & \cdot & \cdot & & 0 \\ 0 & \cdot & 0 & -s & & 1 \end{bmatrix} \tag{3.16}$$

then the following equation is derived.

$$\det N_1 = 1 + (-1)^{n+1} b_1 (-s)^{n-u+v-2} \left\{ (-s)^{u-v+1} + a_{uv} \right\}$$

$$(u \neq n \text{ or } v \neq 2) \tag{3.17}$$

$$\det N_1 = 1 + (-1)^{n+1} b_1 \left\{ (-s)^{n-1} + a_{uv} \right\}$$

$$(u = n \text{ and } v = 2). \tag{3.18}$$

For $s = 0$, we have

$$\det N_1 = 1 \qquad (u \neq n \text{ or } v \neq 2) \tag{3.19}$$

$$\det N_1 = 1 + (-1)^{n+1} b_1 a_{uv} \ (u = n \text{ and } v = 2). \tag{3.20}$$

For $s = \sqrt[u-v+1]{(-1)^{u-v+2} a_{uv}}$, we obtain

$$\det N_1 = 1 \qquad (u \neq n \text{ or } v \neq 2) \tag{3.21}$$

$$\det N_1 = 1 \qquad (u = n \text{ and } v = 2). \tag{3.22}$$

Equation (3.20) implies that if $b_1 \not\equiv 0$, then $a_{n2} \equiv 0$, where a_{n2} is the lower left hand corner of N_1. Combining 1) and 2) as above, we see that this lemma is valid for $k = 1$.

Case 2 : $k > 1$, meaning that b_k is a sign-varying entry : Define

$$N_1 := \left[\begin{array}{ccccccccc} & & k-1 & k & & & n & \\ 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & \\ -s & \cdot & \cdot & & \mathbf{O} & & 0 & \\ 0 & \cdot & 1 & \cdot & & & 0 & k-1 \\ \cdot & \cdot & \cdot & 1 & & & b_k & k \\ \cdot & a_{uv} & \cdot & -s & \cdot & & 0 & \\ \cdot & & & \cdot & \cdot & \cdot & 0 & \\ 0 & & \cdot & \cdot & 0 & -s & 1 & n \end{array} \right] \tag{3.23}$$

then we get

$$\det N_1 = \det N_k. \tag{3.24}$$

1) When N_k does not contain a_{uv}, i.e., $v \leq k$, we have

$$\det N_k = 1 + b_k (-s)^{n-k}. \tag{3.25}$$

Since A has nonzero eigenvalues, $b_k \not\equiv 0$ implies $a_{uv} \equiv 0$ $(v = 1, \cdots, k)$ by (3.25).

2) When N_k contains a_{uv}, that is, $v > k$, it becomes

$$N_k := \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & b_k \\ -s & \cdot & \cdot & & \mathbf{O} & \cdot \\ \cdot & \cdot & \cdot & & & 0 \\ \cdot & a_{uv} & \cdot & \cdot & & 0 \\ \cdot & \cdot & & & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & -s & 1 \end{bmatrix}. \tag{3.26}$$

Like in Case 1, we can show that $b_k \not\equiv 0$ implies that the lower left hand corner of N_k must be zero, or $a_{nk+1} \equiv 0$.

For the latter part we consider the case of $a_{hp} \not\equiv 0$ and $b_k \equiv 0$ as described below:

$$A := \begin{bmatrix} 0 & 1 & \cdot & \cdot & 0 \\ \cdot & & \cdot & a_{hp} & \cdot \\ a_{uv} & & & \cdot & \cdot \\ \cdot & & & & 1 \\ 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix}, \quad b := \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \tag{3.27}$$

First we show that this case can be reduced to the case of $a_{hp} \equiv 0$ and $b_k \not\equiv 0$ as treated in the former part. For proving this, we need the next claim, which is easy to verify.

Claim 3.1 *An interval system (A, b) is controllability invariant if and only if an interval system (A^+, b^+) defined below is controllability invariant.*

$$A^+ := \begin{bmatrix} A & b \\ 0 & 0 \end{bmatrix}, \quad b^+ := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.28}$$

Considering the case of $u \le v$, $u \le p - 1$, and define (A_{hp}, b_{hp}) as follows.

$$\Sigma_{hp} : A_{hp} := \begin{bmatrix} 0 & \theta_1 & & & \\ \cdot & & \cdot & \mathbf{O} & \\ \cdot & & & \cdot & \\ \cdot & a_{uv} & & & \theta_{p-2} \\ 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} \quad b_{hp} := \begin{bmatrix} 0 \\ a_{hp} \\ \cdot \\ 0 \\ \theta_{p-1} \end{bmatrix} \tag{3.29}$$

By noting the similarity of (3.27) and (3.28), and successive use of claim 3.1 we see that controllability invariance of the system (3.27) is equivalent to controllability invariance of the system (3.29), which belongs to the class of systems as treated in the former case (i.e., $a_{hp} \equiv 0$, $b_k \not\equiv 0$). We, therefore, conclude from the discussion for the former part that that if $a_{hp} \not\equiv 0$, then $a_{uv} \equiv 0$ for all $1 \leq u \leq p-1$, $1 \leq v \leq h$ and $a_{p-1h+1} \equiv 0$. This completes the proof of Lemma.

3.4.2 Proof of Proposition 3.1

(Necessity)

Suppose that a standard system described below is controllability invariant.

$$A := \begin{bmatrix} a_{11} & \theta_1 & a_{13} & \cdot & a_{1n} \\ a_{21} & \cdot & \theta_2 & \cdot & \cdot \\ \cdot & & \cdot & \cdot & a_{n-2n} \\ \cdot & & & \cdot & \theta_{n-1} \\ a_{n1} & \cdot & \cdot & \cdot & a_{nn} \end{bmatrix}, \ b := \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ b_{n-1} \\ \theta_n \end{bmatrix} \tag{3.30}$$

This implies the two conditions of Lemma 3.1, which include Condition 2 of Definition 3.2 as a subset. It thus remains to show Condition 3 of Definition 3.2 under the assumption, that is,

$$\text{rank} \begin{bmatrix} A - sI & b \end{bmatrix} = n \tag{3.31}$$

for every $s \in \boldsymbol{C}$ and every a_{ij}, $b_i \in \boldsymbol{R}$. Taking $s = 0$ in (3.31) yields

$$\text{rank} \begin{bmatrix} A & b \end{bmatrix} = n \tag{3.32}$$

for any a_{ij}, b_i. By considering the case where the first column of A take a value of 0, we see

$$\det \begin{bmatrix} \theta_1 & a_{13} & \cdot & a_{1n} & b_1 \\ a_{22} & \theta_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & a_{n-2n} & \cdot \\ \cdot & \cdot & & \cdot & b_{n-1} \\ a_{n2} & \cdot & \cdot & \cdot & \theta_n \end{bmatrix} := \det \widetilde{M} \neq 0 \tag{3.33}$$

for any a_{ij}, b_i. Expanding $\widetilde{M}$ with respect to the first column of $\widetilde{M}$ yields

$$\det \widetilde{M} = \theta_1 \Delta(\theta_1) + a_{22}\Delta(a_{22}) + \cdots + a_{n2}\Delta(a_{n2}) \neq 0 \tag{3.34}$$

where $\Delta(*)$ is the cofactor of $*$. All terms of the above equation except for the first term contain uncertain entries, a_{ij}, varying independently, hence these terms must be equal to 0. Repeating the same discussion for the remaining first term leads to the following equation.

$$\det \widetilde{M} = \theta_1\theta_2\cdots\theta_n \tag{3.35}$$

Therefore, the conditions of Definition 3.2 hold for the system (A, b), namely this system has a generalized antisymmetric stepwise configuration.

(Sufficiency)

Suppose that M has a generalized antisymmetric stepwise configuration. Then by Condition 2 of Definition 3.2 the matrix

$$N := \left[\begin{array}{cc} A - sI & b \end{array}\right] \tag{3.36}$$

has the form :

$$N = \left[\begin{array}{cccccccccccccccc}
-s & \theta & * & * & \cdot & \cdot & \cdot & \cdot & \cdot & * & * & * & 0 & \cdot & \cdot & 0 \\
0 & -s & \theta & * & & & & & & & & * & 0 & & & 0 \\
\cdot & & \cdot & \cdot & \cdot & & & & & & & \cdot & \cdot & & & \cdot \\
\cdot & & & \cdot & \cdot & \cdot & & & & & & \cdot & \cdot & & & \cdot \\
0 & & & & -s & \theta & * & * & \cdot & \cdot & * & * & 0 & & & 0 \\
0 & & & & 0 & *-s & \theta & 0 & 0 & 0 & 0 & 0 & 0 & & & 0 \\
\cdot & & & & \cdot & \cdot & \cdot & \theta & 0 & & & & & & & 0 \\
\cdot & & & & \cdot & \cdot & & \cdot & \cdot & \cdot & & & & & & \cdot \\
\cdot & & & & \cdot & \cdot & & & \cdot & \cdot & \cdot & & & & & \cdot \\
0 & & & & 0 & * & & & & *-s & \theta & 0 & & & & 0 \\
0 & 0 & \cdot & \cdot & 0 & * & & & & & *-s & \theta & 0 & & & 0 \\
* & * & \cdot & \cdot & * & * & & & & & & *-s & \theta & \cdot & & 0 \\
* & * & \cdot & \cdot & * & * & & & & & & & *-s & \cdot & \cdot & 0 \\
* & & & & & & & & & & & & & \cdot & \cdot & 0 \\
* & * & \cdot & \cdot & * & * & \cdot & \cdot & \cdot & \cdot & * & * & * & * & *-s & \theta
\end{array}\right] \tag{3.37}$$

θ : a sign-invariant entry, $*$: a sign-varying entry

Let the (h, p) entry of N be the lowest and rightest sign-varying entry in the upper part of sign-invariant entries (we set $h = 0$ if there is no sign-varying entry in the upper part of sign-invariant entries). In the sequel, we prove that N has a full rank both for $s = 0$ and for $s \neq 0$, by taking Condition 3 of Definition 3.2 into consideration and by elementary transformation respectively.

1) The case with $s = 0$

Taking $s = 0$ in (3.37) yields

$$
N = \left[\begin{array}{ccccccccccccccccc}
 & & & & h & h+1 & & & & & p-1 & p & p+1 & & & & \\
0 & \theta & * & * & \cdot & \cdot & \cdot & \cdot & \cdot & * & * & * & 0 & \cdot & \cdot & 0 & \\
0 & 0 & \theta & * & & & & & & & & * & 0 & & & 0 & \\
\cdot & & \cdot & \cdot & \cdot & & & & & & & \cdot & \cdot & & & \cdot & \\
\cdot & & & \cdot & \cdot & \cdot & & & & & & \cdot & \cdot & & & \cdot & \\
0 & & & & 0 & \theta & * & * & \cdot & \cdot & * & * & 0 & & & 0 & h \\
0 & & & & 0 & * & \theta & 0 & 0 & 0 & 0 & 0 & 0 & & & 0 & h+1 \\
\cdot & & & & \cdot & \cdot & \cdot & \theta & 0 & & & & & & & 0 & \\
\cdot & & & & \cdot & \cdot & & \cdot & \cdot & \cdot & & & & & & \cdot & \\
\cdot & & & & \cdot & \cdot & & & \cdot & \cdot & \cdot & & & & & \cdot & \\
0 & & & & 0 & * & & & & * & \theta & 0 & & & & 0 & \\
0 & 0 & \cdot & \cdot & 0 & * & & & & & * & \theta & 0 & & & 0 & p-1 \\
* & * & \cdot & \cdot & * & * & & & & & & * & \theta & \cdot & & 0 & p \\
* & * & \cdot & \cdot & * & * & & & & & & & * & \cdot & \cdot & 0 & \\
* & & & & & & & & & & & & & \cdot & \cdot & 0 & \\
* & * & \cdot & \cdot & * & * & \cdot & \cdot & \cdot & \cdot & * & * & * & * & * & \theta &
\end{array}\right]. \tag{3.38}
$$

Eliminating the first column of N above yields the right matrix $\widetilde{M}$ of the associated matrix M, whose determinant is equal to the product of all diagonal elements $\theta\theta \cdots \theta$ by Condition 3 of Definition 3.2. So N has a full rank.

2)The case with $s \neq 0$

By shifting the first h columns of N to the right by one column and at the same time moving the $(h + 1)$-th column to the first column, we obtain

$$N = \left[\begin{array}{ccccccccccccccccc}
 & & & h & h+1 & & & & & & p-1 & p & p+1 & & & & \\
 & -s & \theta & * & * & \cdot & \cdot & \cdot & \cdot & * & * & * & 0 & \cdot & \cdot & 0 & \\
\cdot & 0 & -s & \theta & * & & & & & & & * & 0 & & & 0 & \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & & & & & \cdot & \cdot & & & \cdot & \\
* & \cdot & & & \cdot & \cdot & & & & & & \cdot & \cdot & & & \cdot & \\
\theta & 0 & & & & -s & * & * & \cdot & \cdot & * & * & 0 & & & 0 & h \\
*-s & 0 & & & & 0 & \theta & 0 & 0 & 0 & 0 & 0 & 0 & & & 0 & h+1 \\
\cdot & \cdot & & & & \cdot & *-s & \theta & 0 & & & & & & & 0 & \\
\cdot & \cdot & & & & \cdot & & \cdot & \cdot & \cdot & & & & & & \cdot & \\
\cdot & \cdot & & & & \cdot & & & \cdot & \cdot & \cdot & & & & & \cdot & \\
* & 0 & & & & 0 & & & & *-s & \theta & 0 & & & & 0 & \\
* & 0 & 0 & \cdot & \cdot & 0 & & & & & *-s & \theta & 0 & & & 0 & p-1 \\
* & * & * & \cdot & \cdot & * & & & & & & *-s & \theta & \cdot & & 0 & p \\
\cdot & & & & & & & & & & & & *-s & \cdot & \cdot & 0 & \\
\cdot & & & & & & & & & & & & & \cdot & \cdot & 0 & \\
* & * & \cdot & \cdot & * & * & \cdot & \cdot & \cdot & \cdot & * & * & * & * & *-s & \theta &
\end{array}\right]. \tag{3.39}$$

It is then easy to see that the determinant of $(n \times n)$ right submatrix is equal to a nonzero value $\theta^{n-h} \times (-s)^h$. By 1) and 2) as above, it follows

$$\text{rank}\left[\begin{array}{cc} A - sI & b \end{array}\right] = n \tag{3.40}$$

for every $s \in \boldsymbol{C}$ and every $a_{ij}, b_i \in \boldsymbol{R}$. Hence (A, b) is controllability invariant. This completes the proof.

3.5 Robust Stabilizability Conditions with respect to Uncertainty Structure

In this chapter we have treated a special kind of structured systems called interval systems and derived the relation between system properties in terms of a geometric pattern with respect to the location of uncertain parameters. Several results are also given in terms of a geometric pattern except those introduced in the previous sections. In this section we discuss the relation between system properties and a geometric pattern based on those results.

A stabilizability condition for interval systems is given by the *generalized antisymmetric stepwise configuration* [Wei92, Wei94] and a quadratic stabilizability condition by the *antisymmetric stepwise configuration* [Wei90]. The former configuration includes the latter configuration.

In [Koko91] the geometric pattern condition called *pure feedback form* was obtained as a sufficient condition and the condition called *extended matching condition struc-*

ture was obtained as a sufficient condition in [Kane91] for adaptive stabilizability. In [Naha95] the relations between these conditions and conditions by Wei were discussed. The antisymmetric stepwise configuration includes pure feedback form which includes extended matching condition structure. Therefore, the following corollary holds.

Corollary 3.2 *If the pattern matrix P is in the pure feedback form or extended matching condition structure, every interval system (A, b) in Σ_p is controllability invariant.*

Stabilization problems for linear delay systems were considered and conditions of delay-independent stabilization for the system were derived in terms of a geometric pattern with respect to the location of uncertain parameters as in the above researches. The sufficient conditions were obtained in terms of a geometric pattern with no uncertainties in the system matrices except a coefficient matrix of time delay in [Ame83, Akaz87, Ame88]. In [Ame94a, Ame94b, Ame96a, Ame96b, Ame97a] robust stabilization problems for linear delay systems were considered and sufficient conditions in terms of a geometric pattern were obtained. In [Ame97b], conditions obtained in [Wei90],[Ame96b] were explained by singular perturbation approach. These conditions for linear delay systems are based on the idea to make minus of the state matrix of the closed-loop system M-matrix. A class of systems satisfying the condition obtained by Wei is larger than one satisfying the conditions in terms of a geometric pattern by Amemiya, etc.

The relation of the above-mentioned conditions is shown in Appendix E.2.

3.6 Summary

Theorem 3.1 and Corollary 3.1 lead to the fact that as far as a certain class of interval systems is concerned, the notion of controllability invariance defined here is necessary and sufficient for stabilizability of the interval systems and is necessary but not sufficient for quadratic stabilizability. By this fact we have connected robust stabilizability with a natural extension of the familiar notion of controllability in the linear system theory. Thus, we have found that this notion plays an important role in this robust stabilization problem. The future research is to clarify the meaning of controllability invariance defined here in the context of robust stabilization problem and also to investigate its connection with other notions of controllability, for example, the feedback controllability as defined in [Peter90]. Furthermore we make comparisons between classes of systems satisfying the robust stabilizability condition in terms of a geometric pattern.

Chapter 4

Design of Servo Systems with Structured Uncertainties

4.1 Introduction

In this chapter we tackle the design problem of servo systems for systems with time invariant structured uncertainties. In [Davi76] desired tracking is guaranteed essentially for small parameter variations, whereas in [Schm86] it is guaranteed for large parameter variations but under a strong assumption of a matching condition, meaning that uncertainties are included in the range space of the input matrix, i.e., $\Delta A \in \text{Range}(B)$ and $\Delta B \in \text{Range}(B)$. Recently numerous researches have been made concerning quadratic stability, and various design methods for quadratically stabilizing controllers have been proposed in a similar form as in the LQ design [Peter87b, Peter88],[Khar90]. Tsuchida *et al.* [Tsuch91] considered a robust servo problem for systems with time invariant structured uncertainties by state feedback with no assumption of the matching condition and introduced tuning parameters into a feedback control law. Although these methods can be applied directly to the robust servo problem, the resulting design method involves inherent practical difficulties in the choice of design parameters similar to those well known in LQ design. In order to overcome this difficulty, we apply the parameterization of a feedback control law well known in the inverse problem of a linear quadratic control problem to this problem and propose some practical design algorithms in the cases of state feedback and observer-based output feedback in this chapter. From the same viewpoint as the one in this chapter, this problem was considered in [Fujii93a],[Sugi97, Sugi98] and a different parameterization was used in the state feedback case but not in the observer-based feedback case in [Sugi97, Sugi98].

Several researches have been done recently in the output feedback case. For example, [Gali86, Bar86, Peter85] dealt with the observer-based output feedback case, and [Hoz97, Chil96, Masu95, Yama94] in other types. To be more specific, in [Gali86, Bar86] a quadratic stabilizability condition was derived in terms of the observer gain for a given state feedback control achieving robust tracking. In [Peter85] quadratic stabilizability conditions were established for a state feedback gain as well as an observer gain. However, the relation between weighting matrices in the Riccati equations and the resulting closed-loop responses is not so clear. In [Yama93, Yama94] a design method of two degree of freedom control systems achieving robustness of responses was proposed. However, they treated only the single input case and the resulting system turned out to be complicated. In [Hoz97] they provided H_∞ norm conditions based on linear matrix inequalities (LMI). Other approach to robust servo system is to apply those results in [Chil96, Garc96a, Garc96b, Masu95, Sche97] using the linear matrix inequality with pole assignment constraints, H_∞ norm constraints etc. to the augmented system including an integrator.

The advantages of our design method over the above-mentioned ones are as follows. With our proposed method we can provide a guideline of choosing state feedback gains as well as observer gains unlike the methods proposed in [Gali86, Bar86]. Next we can achieve desired responses more explicitly using a new parameterization of a quadratically stabilizing control law unlike [Peter85]. Then unlike [Yama93, Yama94] we adopt one degree of freedom configuration, which nevertheless achieves desirable responses as well as robust stability.

Furthermore, in [Hoz97],[Chil96, Garc96a, Garc96b, Masu95, Sche97], etc. the LMI based methods of an H_∞ control problem into which the quadratically stabilizing servo problem can be transformed were proposed; this methods provide a necessary and sufficient condition for the existence of a controller, whereas our proposed method is more conservative. However, with our method, we can construct a one degree of freedom control system in which decoupled desirable output responses can be achieved asymptotically. Such output responses would be difficult to achieve by the LMI based method with pole assignment constraints and in addition the resulting controllers tend to have higher orders.

4.2 Systems

In this chapter we consider the system with time invariant norm bounded structured uncertainties introduced in Chapter 2, which is rewritten in the following for convenience.

$$\begin{aligned}\dot{x}(t) &= [A+\Delta A]x(t) + [B+\Delta B]u(t) \\ y(t) &= Cx(t) \\ \Delta A = DFE_a,& \quad \Delta B = DFE_b\end{aligned} \tag{2.3}$$

Here $x(t) \in \boldsymbol{R}^n$ is the state, $u(t) \in \boldsymbol{R}^m$ is the control input and $y(t) \in \boldsymbol{R}^m$ is the output; A, B, and C are the nominal system matrices with rank $B = m$; D, E_a and E_b are known real matrices characterizing the structure of the uncertainties. In addition, F is a matrix of uncertain parameters with its maximum singular value bounded by unity, i.e.,

$$F \in \boldsymbol{F} = \{F : \|F\| \leq 1\} \tag{2.4}$$

For this uncertain system (2.3) we consider a design problem of robust servo systems tracking a step reference input $r(t)$ such that

1) the closed-loop system is quadratically stable under the parameter uncertainty described above (***robust stability***),

2) its output $y(t)$ approaches $r(t)$ asymptotically as $t \to \infty$ for all allowable F (***robust tracking***),

by use of state feedback or observer-based output feedback controllers.

To solve this servo problem, we first consider a familiar augmented system used in a design problem of servo systems for a step reference input $r(t)$:

$$\begin{aligned}\dot{\xi}_e &= \begin{bmatrix} A+DFE_a & 0 \\ C & 0 \end{bmatrix} \xi_e + \begin{bmatrix} B+DFE_b \\ 0 \end{bmatrix} u \\ &= [A_\xi + D_\xi F E_{a\xi}]\xi_e + [B_\xi + D_\xi F E_b]u \\ y &= \begin{bmatrix} C & 0 \end{bmatrix} \xi_e,\end{aligned} \tag{4.1}$$

where

$$A_\xi = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, B_\xi = \begin{bmatrix} B \\ 0 \end{bmatrix}, D_\xi = \begin{bmatrix} D \\ 0 \end{bmatrix}, E_{a\xi} = \begin{bmatrix} E_a & 0 \end{bmatrix}.$$

This augmented system is derived in Appendix C.

Remark 4.1 *When the augmented system is constructed, it is assumed that there exist constant steady states as $t \to \infty$, i.e. the outputs arrive at the step reference inputs. For systems with time invariant uncertainties, there exist the constant steady states. However, there does not always exist the constant steady states for systems with time varying uncertainties. Therefore, the approach using this augmented system in this chapter is effective only for systems with time invariant uncertainties and not for systems with time varying uncertainties. The robust servo problems for systems with time varying uncertainties were treated in [Yama93, Yama94],[Itoh98].*

Here we make the following assumption as is usual in this type of servo problems.

$$\det \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \neq 0 \tag{4.2}$$

Then the above servo problem can be reduced to the design problem of quadratically stabilizing controllers for the augmented system (4.1). We can construct a desired robust servo system by the following state feedback plus integral control:

$$u(t) = -K_F x(t) + K_I \int_0^t (r(\tau) - y(\tau))d\tau \tag{4.3}$$

or by the following type of an observer-based output feedback control:

$$\begin{aligned} \dot{\xi}(t) &= (A - LC)\xi(t) + Ly(t) + Bu(t) &(4.4)\\ u(t) &= -K_F\xi(t) + K_I \int_0^t (r(\tau) - y(\tau))d\tau. &(4.5) \end{aligned}$$

Figs. 4.1 and 4.2 show the configurations of control systems with these controls.

Here we describe a quadratic stabilization problem (**QSP**) for the system (4.1) as follows.

Definition 4.1 *(**QSP**) Determine first whether the system (4.1) is quadratically stabilizable or not, and if so, construct a quadratically stabilizing control (**QSC**).*

In this chapter we consider this problem and its inverse problem, and derive design algorithms for constructing quadratically stabilizing control laws.

Remark 4.2 *In this chapter we treat systems in which uncertainties enter into the input matrix and those in which uncertainties enter into both the state and input matrices. However, systems with uncertainties in the output matrix are not dealt with because there exists only a sufficient condition for quadratic stabilizability for the systems [Osuk89] and the result based on the condition becomes conservative.*

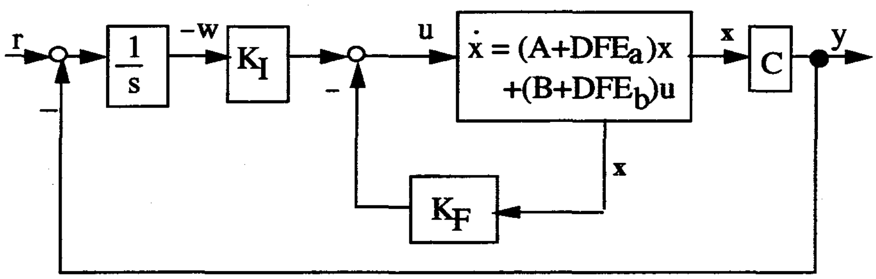

Figure 4.1: Configuration of Control System 1 (State feedback)

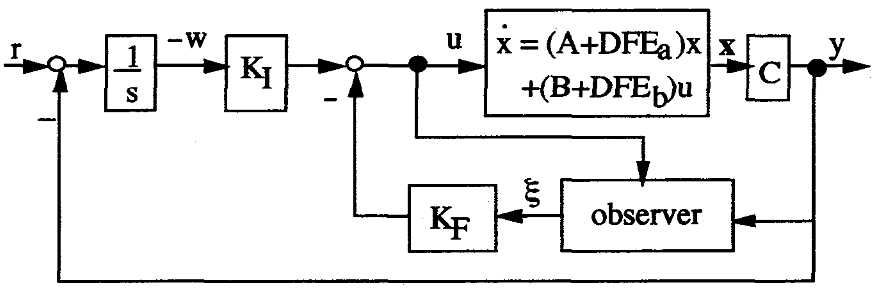

Figure 4.2: Configuration of Control System 1 (Observer-based output feedback)

4.3 Preliminaries

In this section the results of quadratic stabilization problems are introduced and some preliminary lemmas are shown.

4.3.1 The case in which uncertainties enter into the state matrix

The solution to QSP in the case that only ΔA exists is stated below.

Fact 4.1 *[Peter87b] The system (4.1) is quadratically stabilizable via state feedback if* *and only if there exists $\epsilon > 0$ such that the Riccati equation*

$$A_\xi^T P + P A_\xi^T + P[D_\xi D_\xi^T - B_\xi R^{-1} B_\xi^T] P + E_{a\xi}^T E_{a\xi} + Q = 0 \tag{4.6}$$

with $R = \epsilon R_0$, $Q = \epsilon Q_0$ has a real symmetric solution $P > 0$. In addition, a QSC is given by

$$u = -R^{-1} B_\xi^T P \xi_e \tag{4.7}$$

Then this system can be transformed by $\xi_e = \Gamma x_e$ to

$$\begin{aligned} \dot{x}_e &= (A_e + D_e F E_{ea}) x_e + B_e u \\ y &= C_e x_e \end{aligned} \tag{4.8}$$

where

$$\Gamma = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \quad (\det \Gamma \neq 0) \tag{4.9}$$

$$A_e = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, B_e = \begin{bmatrix} 0 \\ I \end{bmatrix}, \ C_e = \begin{bmatrix} C & 0 \end{bmatrix} \tag{4.10}$$

$$D_e = \Gamma^{-1} \begin{bmatrix} D \\ 0 \end{bmatrix}, E_{ea} = \begin{bmatrix} E_a & 0 \end{bmatrix} \Gamma. \tag{4.11}$$

Then the above servo problem can be reduced to *a design problem of quadratically stabilizing controllers for the augmented system (4.8).* According to the design theory of servo systems, if this stabilizing controller is given in the form of state feedback:

$$u = -K_e x_e \tag{4.12}$$

a desired robust servo system can be constructed by state feedback control (4.3).

Lemma 4.1 *The system (4.1) is quadratically stabilizable via state feedback if and only if there exists $\epsilon > 0$ such that the Riccati equation*

$$A_e^T P_e + P_e A_e^T + P[D_e D_e^T - B_e R^{-1} B_e^T] P + E_{ea}^T E_{ea} + \Gamma^T Q \Gamma = 0 \tag{4.13}$$

with $R = \epsilon R_0$, $Q = \epsilon Q_0$ has a real symmetric solution $P > 0$. In addition, a QSC is given by

$$\begin{aligned} u &= -K_e x_e \\ K_e &= R^{-1} B_e^T P_e \end{aligned} \tag{4.14}$$

It is called "a central QSC" like a central solution of H_∞ control for convenience.

We then state below some preliminary results.

Fact 4.2 *[Fujii91] A feedback control $v = -K_e x_e$ is a central quadratically stabilizing control for the system (4.8) if and only if there exist $R > 0$ and $P_e > 0$ that satisfy the following relations:*

$$
\begin{aligned}
RK_e &= B_e^T P_e && (4.15)\\
P_e\left(\frac{1}{2}B_eK_e - A_e\right) &+ \left(\frac{1}{2}B_eK_e - A_e\right)^T P_e \\
&- P_eD_eD_e^TP_e - E_{ea}^TE_{ea} > 0. && (4.16)
\end{aligned}
$$

The proof of this fact is shown in Appendix B. Furthermore we get the following result using Schur complement in Lemma A.1 in Appendix A.

Lemma 4.2 *The Riccati inequality (4.16) is equivalent to the following linear matrix inequality.*

$$
\begin{bmatrix} P_e\Psi_e + \Psi_e^TP_e - E_e^TE_e & P_eD_e \\ D_e^TP_e & I \end{bmatrix} > 0 \qquad (4.17)
$$

$$
\Psi_e := \frac{1}{2}B_eK_e - A_e \qquad (4.18)
$$

4.3.2 The case in which uncertainties enter into the state and input matrix

The solution to QSP in the case that ΔA and ΔB exist is stated below.

Fact 4.3 *[Khar90] The system (4.1) is quadratically stabilizable via state feedback if and only if there exist $\epsilon > 0$ and $Q > 0$ such that the Riccati equation*

$$
\begin{aligned}
[A_\xi - B_\xi\Xi E_b^TE_{a\xi}]^TP + P[A_\xi - B_\xi\Xi E_b^TE_{a\xi}] &+ P[D_\xi D_\xi^T - B_\xi R^{-1}B_\xi^T]P \\
+E_{a\xi}^T\{I - E_b\Xi E_b^T\}E_{a\xi} &+ Q = 0 && (4.19)
\end{aligned}
$$

has a real symmetric solution $P > 0$. In addition, a QSC is given by

$$
u = -(R^{-1}B_\xi^TP + \Xi E_b^TE_{a\xi})\xi_e \qquad (4.20)
$$

where R and Ξ are defined by

$$
R = (V_1J^{-2}V_1 + \frac{1}{2\epsilon}V_2V_2^T)^{-1},\ \Xi = V_1J^{-2}V_1^T \qquad (4.21)
$$

based on the singular value decomposition of E_b:

$$E_b = [U_1 \quad U_2]\begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}. \tag{4.22}$$

Remark 4.3 *In [Khar90] Q in (4.19) is set to ϵI. However, necessity and sufficiency in Fact 4.3 hold also in case of Q.*

Now we need two transformations in order to apply a new parameterization of QSC derived in Fact 4.5.

In view of Fact 4.3, we transform the system (4.1) first by a feedback transformation

$$u = v - \Xi E_b^T E_{a\xi}\xi_e \tag{4.23}$$

into

$$\begin{aligned} \dot{\xi}_e &= \begin{bmatrix} A_F + DFU_2U_2^TE_a & 0 \\ C & 0 \end{bmatrix}\xi_e + \begin{bmatrix} B + DFE_b \\ 0 \end{bmatrix} v \\ y &= \begin{bmatrix} C & 0 \end{bmatrix}\xi_e \end{aligned} \tag{4.24}$$

where

$$A_F = A - B\Xi E_b^T E_a$$

and then by a coordinate transformation $\xi_e = \Gamma_F x_e$ into

$$\begin{aligned} \dot{x}_e &= (A_e + D_eFE_{eab})x_e + (B_e + D_eFE_b)v \\ y &= C_e x_e \end{aligned} \tag{4.25}$$

where

$$\Gamma_F = \begin{bmatrix} A_F & B \\ C & 0 \end{bmatrix} \quad (\det \Gamma_F \neq 0) \tag{4.26}$$

$$\begin{aligned} A_{eF} &= \begin{bmatrix} A_F & B \\ 0 & 0 \end{bmatrix}, D_{eF} = \Gamma_F^{-1}\begin{bmatrix} D \\ 0 \end{bmatrix} \\ E_{eab} &= \begin{bmatrix} U_2U_2^TE_a & 0 \end{bmatrix}\Gamma_F. \end{aligned} \tag{4.27}$$

As a result of these transformations, the QSP for the system (4.1) can be reduced to that for the system (4.25), namely, we obtain the following lemma directly from Fact 4.3.

Lemma 4.3 *The system (4.1) is quadratically stabilizable if and only if there exist* $\epsilon > 0$ *and* $Q > 0$ *such that the Riccati equation*

$$P_eA_{eF}+A_{eF}^TP_e-P_e(D_{eF}D_{eF}^T-B_eR^{-1}B_e^T)P_e+E_{eab}^T(I-E_b\Xi E_b^T)E_{eab}+\Gamma_F^TQ\Gamma_F=0 \quad (4.28)$$

has a real symmetric solution $P_e > 0$*. Furthermore a QSC is given by*

$$\begin{aligned} v &= -K_ex_e \quad (4.29) \\ K_e &= R^{-1}B_e^TP_e \end{aligned}$$

where R *and* Ξ *are defined by (4.21). This control law is called "a central QSC" like a central solution of* H_∞ *control for convenience.*

The term of $\Xi E_b^T E_{a\xi}$ is deleted by the former feedback transformation and the matrix $B_\xi = [B \quad 0]^T$ is transformed to the matrix $B_e = [0 \quad I]^T$ by the latter transformation. We are, therefore, now ready to apply Fact 4.5 to this QSC because we get B_e and K_e in the form of (4.10) and (4.29) by the above transformations.

Fact 4.4 *[Fujii91] A feedback control* $v = -K_ex_e$ *is a central quadratically stabilizing control for the system (4.25) if and only if there exist* $R > 0$ *of the form (4.21) and* $P_e > 0$ *that satisfy the following relations:*

$$\begin{aligned} RK_e &= B_e^TP_e \quad (4.30) \\ P_e\left(\frac{1}{2}B_eK_e - A_{eF}\right) &+ \left(\frac{1}{2}B_eK_e - A_{eF}\right)^T P_e \\ &- P_eD_{eF}D_{eF}^TP_e - (U_2^TE_{eab})^T(U_2^TE_{eab}) > 0. \quad (4.31) \end{aligned}$$

Furthermore we get the following result using Schur complement in Lemma A.1 in Appendix A.

Lemma 4.4 *The Riccati inequality (4.31) is equivalent to the following linear matrix inequality.*

$$\begin{bmatrix} P_e\Psi_{eF} + \Psi_{eF}^TP_e - (U_2^TE_{eab})^T(U_2^TE_{eab}) & P_eD_{eF} \\ D_{eF}^TP_e & I \end{bmatrix} > 0 \quad (4.32)$$

$$\Psi_{eF} := \frac{1}{2}B_eK_e - A_{eF} \quad (4.33)$$

4.3.3 Parameterization of a Control Law

In order to parameterize a QSC we consider the following two types of inverse quadratic stabilization problems.

Definition 4.2 *Given a linear state feedback control* $v(t) = -K_e x_e(t)$ *(or* $u(t) = -K_e x_e(t)$*) for the system (4.25), find conditions such that it is a central QSC.*

Definition 4.3 *Given a linear state feedback control* $v(t) = -K_e x_e(t)$ *(or* $u(t) = -K_e x_e(t)$*) for the system (4.25), find conditions such that it is a QSC.*

Later on we use the solution of the former to give a parameterization of feedback control laws including central QSCs. The solution of the latter will then be used to find conditions on the associated design parameters for a possible central QSC so parameterized to be really a QSC. The reason we adopt such a two stage design method is that a complete parameterization of all the QSCs has not been derived so far. The solution of the former problem is given as follows.

Fact 4.5 *[Fujii87a] Let* B_e *be given by (4.10), and* K_e *by (4.14) and (4.29) for some* $R > 0$ *and* $P_e > 0$*. Then*

1. K_e *can be expressed as*
$$K_e = V^{-1}\Sigma V[K \quad I] \tag{4.34}$$
for some nonsingular matrix $V \in \boldsymbol{R}^{m\times m}$*, some positive diagonal matrix* $\Sigma = \mathrm{diag}\{\sigma_i\} \in \boldsymbol{R}^{m\times m}$*, and some real matrix* $K \in \boldsymbol{R}^{m\times m}$*.*

2. *The matrices* R *and* P_e *in (4.14) and (4.29) are expressed by*
$$P_e = (VK_e)^T \Lambda \Sigma^{-1}(VK_e) + \text{block-diag}(Y, 0) \tag{4.35}$$
$$R = V^T \Lambda V \tag{4.36}$$
for the matrix V *in (4.34), some positive definite diagonal matrix* $\Lambda \in \boldsymbol{R}^{m\times m}$ *and some positive definite matrix* $Y \in \boldsymbol{R}^{n\times n}$*.* Λ *and* Σ *are commutative, i.e.,* $\Lambda\Sigma = \Sigma\Lambda$*.*

Remark 4.4 *This parameterization can be applied to the feedback gain* $K_e = -R^{-1}B_e^T P_e$ *where* $B_e = \begin{bmatrix} 0 \\ I \end{bmatrix}$*,* $R > 0$*,* $P_e > 0$*. Therefore, this parameterization can be applied to the design problems in which there exists the feedback gain in the above form, for example, decentralized control problem etc.*

Based on this result, K_F, K_I in (4.3) and (4.5) is given for some K_F^0, K_I^0 by

$$\begin{aligned}\left[\begin{array}{cc} K_F & K_I \end{array}\right] &:= V^{-1}\Sigma V\left[\begin{array}{cc} K_F^0 & K_I^0 \end{array}\right] + \Xi E_b^T E_{a\xi} \\ &:= V^{-1}\Sigma V\left[\begin{array}{cc} K & I \end{array}\right]\Gamma_F^{-1} + \Xi E_b^T E_{a\xi} \\ &:= K_e\Gamma_F^{-1} + \Xi E_b^T E_{a\xi}. \end{aligned} \tag{4.37}$$

The observer-based output feedback control (4.4) and (4.5) is then realized as in Fig. 4.4 which is the servo system of our concern here. The solution of the latter problem is given as follows.

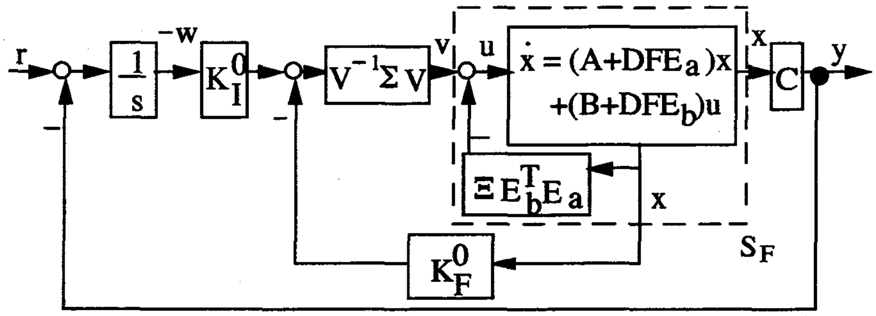

Figure 4.3: Configuration of Control System 2 (State feedback)

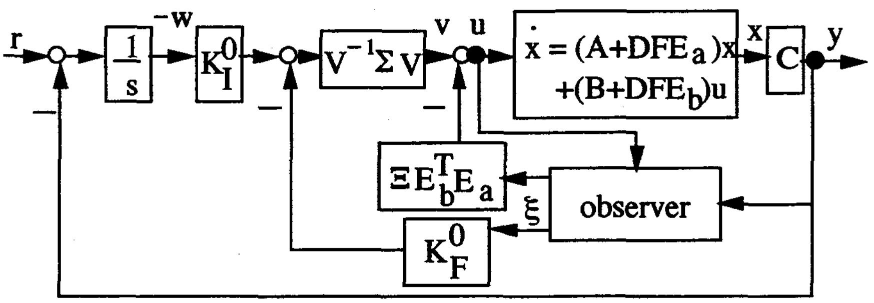

Figure 4.4: Configuration of Control System 2 (Observer-based output feedback)

Fact 4.6 *[Khar90] A feedback control $v(t) = -K_e x_e(t)$ is a QSC for the system (4.25) if and only if K_e satisfies the following conditions.*

1.

$$\Phi_e := A_e - B_e K_e \text{ is stable.} \tag{4.38}$$

2.

$$\|G_e(s)\|_\infty < 1, \tag{4.39}$$

where

$$G_e(s) := (E_e - E_b K_e)(sI - \Phi_e)^{-1} D_e.$$

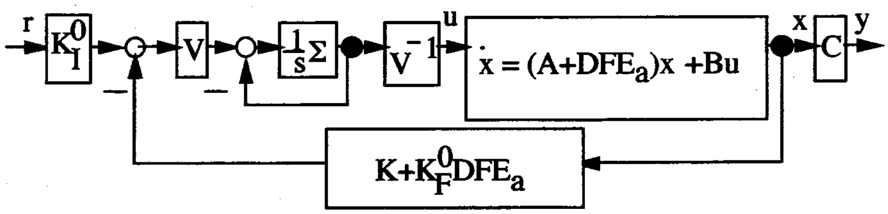

Figure 4.5: Equivalent Control System (State FB, ΔA)

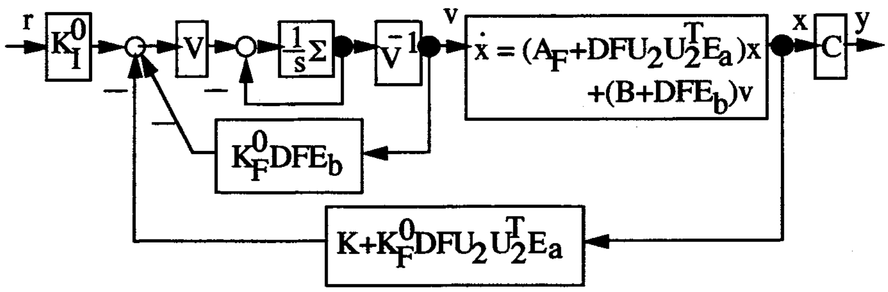

Figure 4.6: Equivalent Control System (State FB, ΔA, ΔB)

Now we state the effectiveness of this parameterization given in Fact 4.5. The structure of the closed-loop system using this parameterization was investigated in [Fujii93a, Fujii94]. According to the investigation, each system in Figs. 4.3 and 4.4 approaches each system in Figs. 4.8, 4.9 and 4.10 asymptotically as $\{\sigma_i\}$ increase

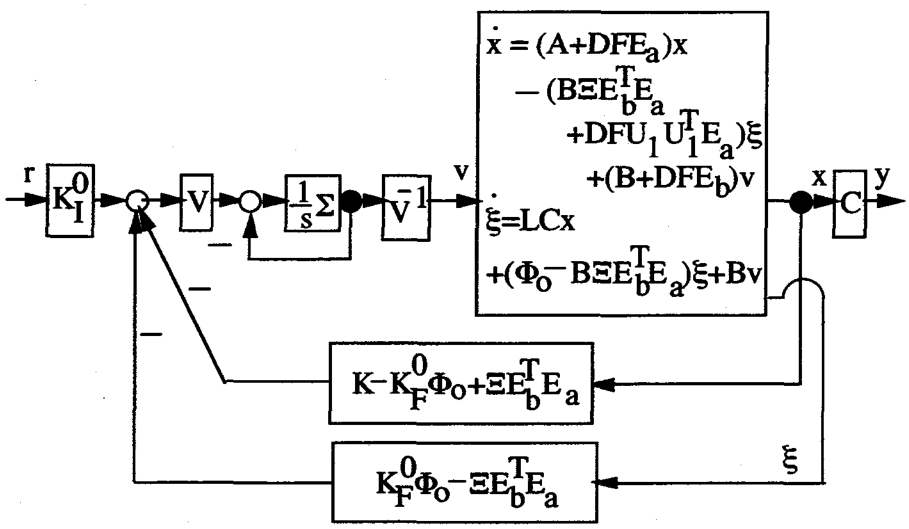

Figure 4.7: Equivalent Control System (Observer-based output FB, ΔA, ΔB)

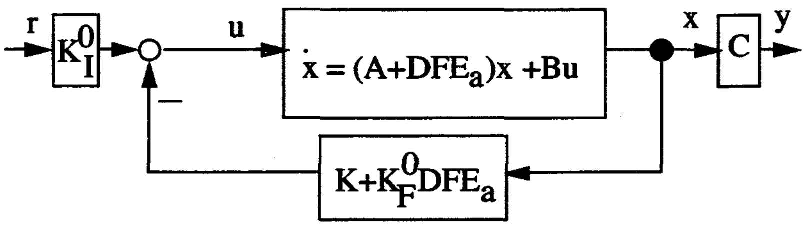

Figure 4.8: Asymptotic Control System (State FB, ΔA)

[Fujii93a, Fujii94], where K and K_I^0 are determined uniquely from (4.34) and (4.37) as follows.

$$K = K_F^0 A_F + K_I^0 C \tag{4.40}$$
$$K_F^0 = (CA_K^{-1}B)^{-1}CA_K^{-1} \tag{4.41}$$
$$K_I^0 = -(CA_K^{-1}B)^{-1} \tag{4.42}$$

where

$$A_K = A_F - BK.$$

The system in Figs. 4.8, 4.9 and 4.10 has no unknown parameters other than K. Hence, we can specify a desirable transfer function of the nominal system ($F = 0$) from r to

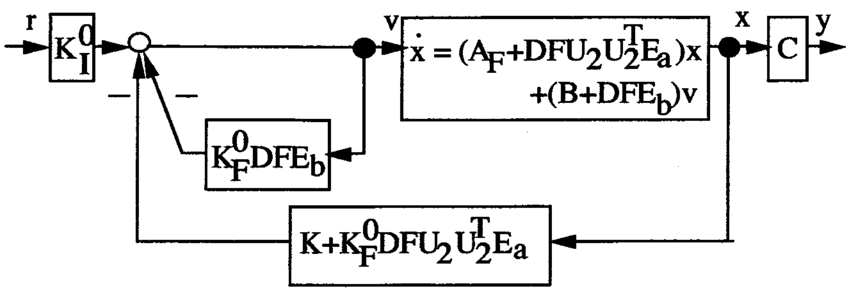

Figure 4.9: Asymptotic Control System (State FB, ΔA, ΔB)

y by proper choice of K alone.

In this way the above-mentioned system structure enables us to avoid difficulties in choosing design parameters similar to those well known in the usual LQ design.

Remark 4.5 *Figs. 4.5, 4.6, 4.8 and 4.9 are depicted in the case that both ΔA and ΔB exist. However, when only ΔA exists, we set $E_b = 0$ and $U_2 = I$ and then those figures hold in the case that only ΔA exists.*

When there do not exist uncertainties in the systems, the transfer characteristics from r to y in Figs. 4.8, 4.9 and 4.10 are equivalent each other in spite of the existence of an observer.

4.4 Main Results — State Feedback Case

In this section we show main results which are needed in order to propose design algorithms of robust servo systems in the case of a state feedback.

4.4.1 The case in which uncertainties enter into the state matrix

Here we treat the case in which uncertainties enter into the state matrix, i.e., $E_a \neq 0$, $E_b = 0$. To design a desired robust tracking controller, we first derive a parameterization of QSC from the Inverse LQ viewpoint, and then obtain conditions on the associated parameters for the state feedback control so parameterized to be a QSC,

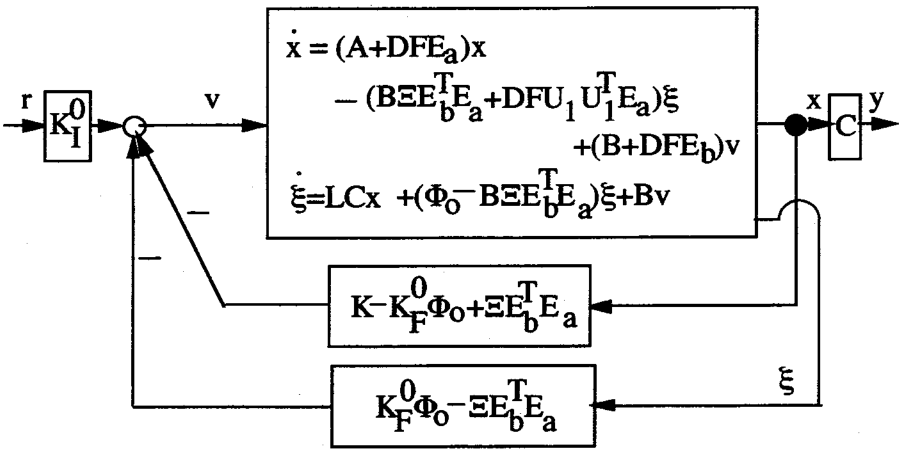

Figure 4.10: Asymptotic Control System (Observer-based output FB, ΔA, ΔB)

which leads finally to a desired design algorithm for QSC. The key idea of our design method of QSC is to parameterize a QSC law K_e in the form of (4.34) based on Lemma 4.1 and Fact 4.5, and then determine the associated parameter matrices V, K, and Σ based on Lemma 4.1 and Facts 4.5 and 4.6, in such a way that the K_e obtained is a QSC law. By Lemma 4.1 there is no restriction on the structure of R as in (4.36), and hence by 2. of Fact 4.5 we can choose any nonsingular matrix V. For simplifying the design procedure, we set $V = I$ in what follows.

We then show necessary and sufficient conditions for QSC of the state feedback control (4.3), and the observer-based output feedback control (4.4) and (4.5) associated with the parameterized gain K_e, which lead to determination of the remaining parameter matrices K and Σ.

Theorem 4.1 *[Fujii91] Let $V = I$ in (4.34). Then the state feedback control (4.3) associated with K_e given by (4.34) is a central QSC law for some Σ for (2.3) if and only if K satisfies the following conditions.*

1.

$$A_K \textit{ is stable.} \tag{4.43}$$

2.

$$\|G_a(s)\|_\infty < 1 \tag{4.44}$$

where

$$G_a(s) := E_a(sI - A_K)^{-1}(I - BB^-)D \quad (4.45)$$
$$B^- := (CA_K^{-1}B)^{-1}CA_K^{-1}. \quad (4.46)$$

Furthermore the matrix $G_e(s)$ has the following asymptotic property:

$$G_e(s) \to G_a(s) \quad (4.47)$$

as $\{\sigma_i\}$ increase.

The proof of this theorem is shown in Section 4.7.

Remark 4.6 *The condition (4.44) is interpreted as a small gain condition for the asymptoic system in Fig. 4.8 because the system in Fig. 4.8 is equivalent to the system in Fig. 4.11 by noting that $K_F^0 = B^-$. Moreover, Fig. 4.11 can be redrawn as Fig. 4.12, from which Fig. 4.12, $G_a(s)$ is derived directly.*

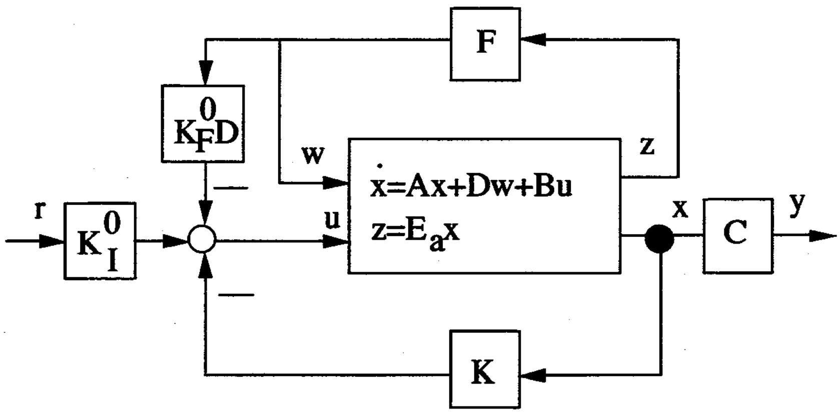

Figure 4.11: Small gain condition 1 for asymptotic system (State FB, ΔA)

4.4.2 The case in which uncertainties enter both into the state and input matrices

In this subsection we treat the case in which uncertainties enter into the state and input matrix, i.e., $E_a \neq 0$, $E_b \neq 0$. The key idea of our design method of QSC is to parameterize a QSC law K_e in the form of (4.34) based on Lemma 4.3 and Fact 4.5, and then determine the associated parameter matrices V, K, and Σ based on Lemma

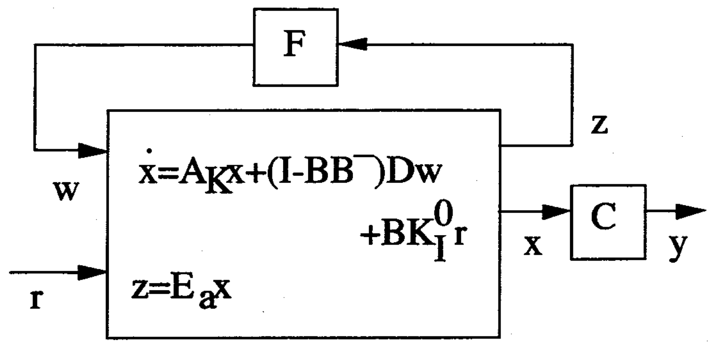

Figure 4.12: Small gain condition 2 for asymptotic system (State FB, ΔA)

4.3 and Facts 4.5 and 4.6. in such a way that the K_e so parameterized is a QSC law for the system (4.25). Note by Lemma 4.3 that the R as in (4.36) is restricted in the form of (4.21), or equivalently, by

$$R = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} J^2 & 0 \\ 0 & \epsilon I \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$$

and hence by 2 of Fact 4.5 we can determine V as

$$V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}^T. \tag{4.48}$$

We then show necessary and sufficient conditions for QSC of the state feedback control (4.3) associated with the parameterized gain K_e, which lead to determination of the remaining parameter matrices K and Σ.

Theorem 4.2 *[Fujii92b] Set V as in (4.48). Then the state feedback (4.3) associated with K_e given by (4.34) is a central QSC law for some Σ only if K satisfies the following conditions.*

1.

$$A_K \text{ is stable.} \tag{4.49}$$

2.

$$\left\| U_2^T G_a(s) \right\|_\infty < 1 \tag{4.50}$$

Furthermore the matrix $G_e(s)$ has the following asymptotic property:

$$G_e(s) \to U_2 U_2^T G_a(s) - G_b(s) \tag{4.51}$$

where

$$G_b(s) := E_b[K(sI - A_K)^{-1}(I - BB^-) + B^-]D \tag{4.52}$$

as $\{\sigma_i\}$ *increase.*

The proof of this theorem is shown in Section 4.7.

Remark 4.7 *The condition* $\|U_2U_2^TG_a(s) - G_b(s)\|_\infty < 1$ *is interpreted as a small gain condition for the asymptoic system in Fig. 4.9 because the system in Fig. 4.9 is equivalent to the system in Fig. 4.13.*

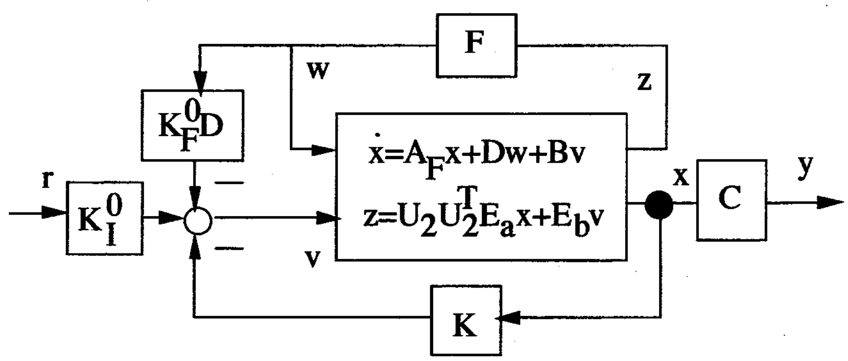

Figure 4.13: Small gain condition 1 for asymptotic system (State FB, ΔA, ΔB)

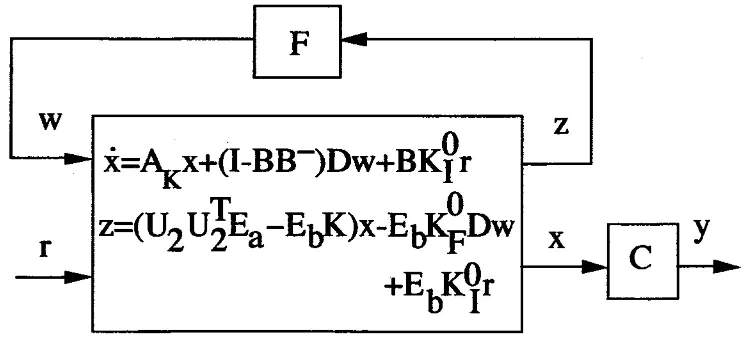

Figure 4.14: Small gain condition 2 for asymptotic system (State FB, ΔA, ΔB)

The system in Fig. 4.13 becomes that in Fig. 4.14. From Fig. 4.14, $U_2U_2^TG_a(s) - G_b(s)$ *is derived directly.*

4.5 Main Results — Observer-based Output Feedback Case

In this section we show main results which are needed in order to propose a design algorithm of robust servo systems in the case of an observer-based output feedback.

4.5.1 The case in which uncertainties enter into the state matrix

In this subsection we treat the case in which uncertainties enter only into the state matrix, i.e., $E_a \neq 0$, $E_b = 0$.

Theorem 4.3 *The closed-loop system (2.3),(4.4) and (4.5) is quadratically stable if and only if the two following conditions are satisfied.*

1.

$$\Phi_{oe} := \begin{bmatrix} \Phi_e & \begin{bmatrix} 0 \\ -K_F \end{bmatrix} \\ 0 & \Phi_o \end{bmatrix} \text{ is stable.} \tag{4.53}$$

2.

$$\|G_{oea}(s)\|_\infty < 1 \tag{4.54}$$

where

$$\begin{aligned} G_{oea}(s) &:= \begin{bmatrix} E_e & 0 \end{bmatrix} (sI - \Phi_{oe})^{-1} \begin{bmatrix} D_e \\ -D \end{bmatrix} \\ \Phi_o &:= A - LC. \end{aligned}$$

As $\{\sigma_i\}$ increases, $G_{oea}(s)$ approaches $\overline{G}_{oea}(s)$ defined below.

$$\overline{G}_{oea}(s) := G_a(s) + G_{ca}(s) \tag{4.55}$$

where

$$G_{ca}(s) := E_a(sI - A_K)^{-1}BsK_F^0(sI - \Phi_o)^{-1}D.$$

If $\|\overline{G}_{oea}(s)\|_\infty < 1$, the closed-loop system is quadratically stable for sufficiently large $\{\sigma_i\}$.

This result is derived directly from Theorem 4.4.

4.5.2 The case in which uncertainties enter both into the state and input matrix

In this subsection we treat the case in which uncertainties enter both into the state and input matrices, i.e., $E_a \neq 0$, $E_b \neq 0$. As we discussed in the section 4.4.2, by 2 of Fact 4.5 we can determine V as

$$V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}^T$$

where V_1, V_2 are given by (4.22), i.e., singular value decomposition of E_b.

Theorem 4.4 *The closed-loop system (2.3),(4.4) and (4.5) is quadratically stable if and only if the following two conditions are satisfied.*

1.

$$\Phi_{oe} := \begin{bmatrix} \Phi_e & \begin{bmatrix} 0 \\ -K_F \end{bmatrix} \\ 0 & \Phi_o \end{bmatrix} \text{ is stable.} \tag{4.56}$$

2.

$$\|G_{oe}(s)\|_\infty < 1 \tag{4.57}$$

$$G_{oe}(s) := \begin{bmatrix} E_e - E_b K_e & -E_b K_F \end{bmatrix} (sI - \Phi_{oe})^{-1} \begin{bmatrix} D_e \\ -D \end{bmatrix}$$

As $\{\sigma_i\}$ increases, $G_{oe}(s)$ approaches $\overline{G}_{oe}(s)$ defined below.

$$\overline{G}_{oe}(s) := U_2 U_2^T G_a(s) - G_b(s) + G_c(s) \tag{4.58}$$

where

$$\begin{aligned} G_c(s) &:= \left[E_b + (U_2 U_2^T E_a - E_b K)(sI - A_K)^{-1} B\right] \\ &\quad \times s K_F^0 (sI - \Phi_o)^{-1} D. \end{aligned} \tag{4.59}$$

If $\|\overline{G}_{oe}(s)\|_\infty < 1$, the closed-loop system is quadratically stable for sufficiently large $\{\sigma_i\}$.

The proof of this theorem is shown in Section 4.7.

Remark 4.8 *Here $\overline{G}_{oe}(s)$ is proven to be strictly proper. First, note that the direct transmission part of $G_c(s)$ is $E_b K_F^0 D$. Next expanding $G_c(s)$ yields*

$$\begin{aligned} G_c(s) &= E_b s K_F^0 (sI - \Phi_o)^{-1} D \\ &\quad + (U_2 U_2^T E_a - E_b K)(sI - A_K)^{-1} B s K_F^0 (sI - \Phi_o)^{-1} D \\ &:= G_{c1}(s) + G_{c2}(s) \end{aligned}$$

Clearly $G_{c2}(s)$ is strictly proper. Furthermore, calculating $G_{c1}(s)$ yields

$$G_{c1}(s) = E_b\left\{K_F^0 D + K_F^0\Phi_o(sI-\Phi_o)^{-1}D\right\}.$$

Hence, the direct transmission part of $G_c(s)$ is $E_bK_F^0D$. From (4.41) and (4.46), K_F^0 $= B^- = (CA_K^{-1}B)^{-1}\,CA_K^{-1}$. Therefore, $\overline{G}_{oe}(s)$ turns out to be strictly proper. We can also prove as in Theorem 2 of [Fujii94] that $\|G_{oe}(s)\|_\infty$ approaches $\|\overline{G}_{oe}(s)\|_\infty$ in the sense that $\|G_{oe}(s)-\overline{G}_{oe}(s)\|_\infty \to 0$ as $\{\sigma_i\}$ increases.

Remark 4.9 *The condition $\|U_2U_2^TG_a(s) - G_b(s) + G_c(s)\|_\infty < 1$ is interpreted as a small gain condition for the asymptoic system in Fig. 4.10 because the system in Fig. 4.10 is equivalent to the system in Fig. 4.15. Moreover, Fig. 4.15 can be redrawn as Fig. 4.16.*

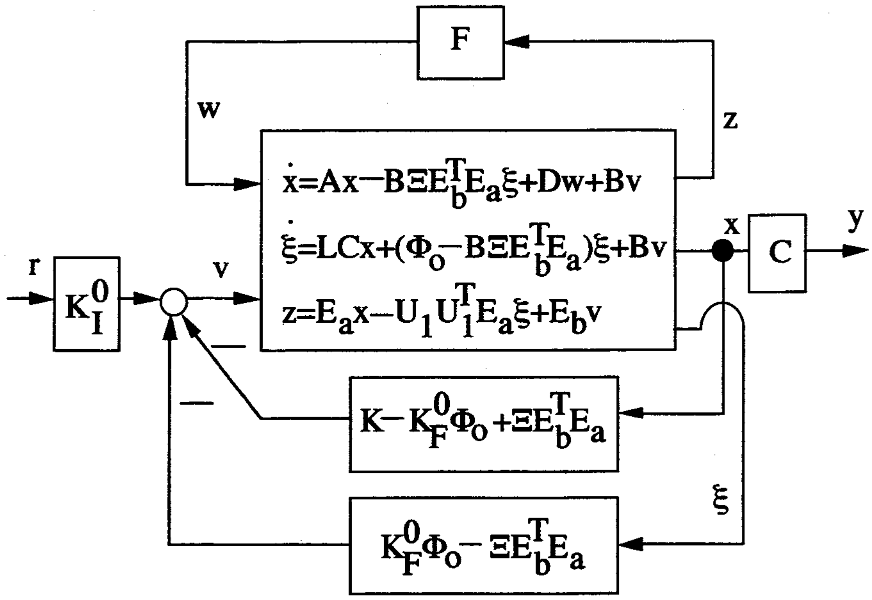

Figure 4.15: Small gain condition 1 for asymptotic system (Observer-based output FB, ΔA, ΔB)

Next it is shown that $G_{oe}(s)$ in 4.58 is equivalent to $G_{zw}(s)$ in Fig. 4.16. Here we have

$$\begin{aligned}
& U_2U_2^TG_a(s) - G_b(s) - G_c(s) \\
= \;& (U_2U_2^TE_a - E_bK)(sI-A_K)^{-1}D + E_bK_F^0\Phi_o(sI-\Phi_o)^{-1}D \\
& (U_2U_2^TE_a - E_bK)(sI-A_K)^{-1}BK_F^0\Phi_o(sI-\Phi_o)^{-1}D.
\end{aligned}$$

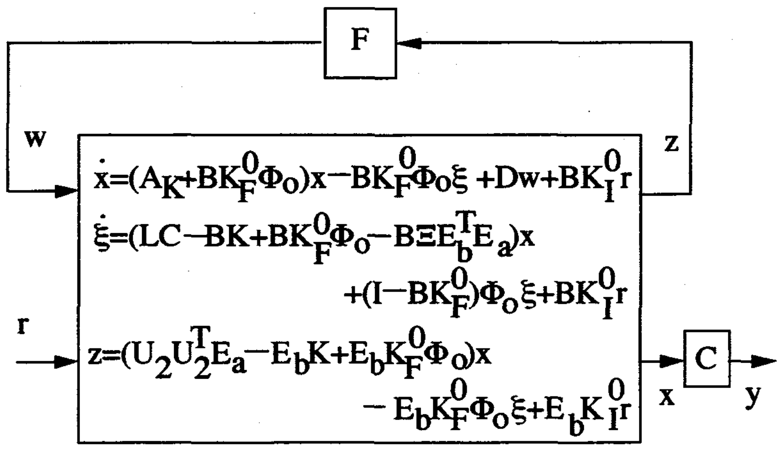

Figure 4.16: Small gain condition 2 for asymptotic system (Observer-based output FB, ΔA, ΔB)

From Fig. 4.16, $G_{zw}(s)$: the transfer function from w to z ($r = 0$) is derived as follows.

$$G_{zw}(s) = \left[\begin{array}{cc|c} A_K + BK_F^0\Phi_o & -BK_F^0\Phi_o & D \\ LC - BK + BK_F^0\Phi_o - B\Xi E_b^T E_a & (I - BK_F^0)\Phi_o & 0 \\ \hline U_2U_2^T E_a - E_bK + E_bK_F^0\Phi_o & -E_bK_F^0\Phi_o & 0 \end{array}\right]$$

This state space representation can be transformed by the coordinate transformation $\bar{x} = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} x \\ \xi - x \end{bmatrix}$ *into the following representation.*

$$G_{zw}(s) = \left[\begin{array}{cc|c} A_K & -BK_F^0\Phi_o & D \\ 0 & \Phi_o & -D \\ \hline U_2U_2^T E_a - E_bK & -E_bK_F^0\Phi_o & 0 \end{array}\right]$$

This representation is easily proven to be equivalent to $U_2U_2^T G_a(s) - G_b(s) - G_c(s)$.

4.6 Design Algorithms

4.6.1 State feedback case

This result leads to the following basic design algorithm of QSC.

Design algorithm 1 (ΔA exists)

Step 1 Set $V = I$.

Step 2 Determine K so as to satisfy the condition (4.44).

Step 3 Choose $\sigma_1 \sim \sigma_m$ large enough so as to satisfy the condition (4.39) for the control law K_e with K specified above.

Step 4 With the parameter matrices V,K, and Σ obtained above, the QSC law K_e can be obtained by (4.34), from which the desired gain matrices K_F,K_I in (4.3) of the servo system can be obtained by

$$[K_F \quad K_I] = \Sigma[K \quad I]\Gamma^{-1}. \tag{4.60}$$

Remark 4.10 *In practical control system design a decoupling property of closed-loop systems as well as specifying output responses are often required as design specifications. One of the effective methods of choosing K in Step 2 so as to meet the above two specifications is ILQ design method [Fujii87a], [Fujii87b], [Fujii93a], [Fujii88]. The features of the method are introduced in the next subsection and the calculation method of K is introduced in Appendix D.*

Remark 4.11 *In this design method we use the parameterization of the feedback gains in the form of $K_e = V^{-1}\Sigma V \begin{bmatrix} K & I \end{bmatrix}$. In this design algorithm K is selected as a decoupling gain in order to achieve a decoupling property as a design specification. This results in making a class of gains small and there may not be sometimes quadratically stabilizing gains even if a system is quadratically stabilizable. Therefore, this design algorithm yields a more conservative result than the design methods based on linear matrix inequalities or H_∞ control problems with respect to the existence of a quadratically stabilizing controller. However, this design algorithm can construct a control system in which decoupling characteristics and desirable output responses are achieved more easily that the latter design methods.*

Design algorithm 2 (ΔA and ΔB exist)

Step 1 Set $V = [V_1 \quad V_2]^T$.

Step 2 First, obtain a matrix K satisfying (4.49) by a pole assignment method. Then check whether it satisfies the condition $\|\overline{G}_e(s)\|_\infty < 1$ or not (see Theorem 4.2). If it satisfies this condition, proceed to Step 3; otherwise check whether it satisfies the condition (4.50). If it satisfies this condition, then proceed to Step 3; otherwise obtain a different K similarly and repeat this step.

Step 3 First, choose some positive values of $\sigma_1 \sim \sigma_m$ and then check whether the resulting control law K_e given by (4.34) satisfies the conditions (4.49) and (4.50). If it satisfies this condition, then proceed to Step 4; otherwise choose different $\{\sigma_i\}$ again and repeat this step.

Step 4 With the parameter matrices V,K, and Σ obtained above, the QSC law K_e can be obtained by (4.34), from which the desired gain matrices K_F and K_I in (4.5) of the servo system can be obtained by (4.37).

Remark 4.12 *Like Remark 4.10, one of the effective methods of choosing K in Step 2 so as to meet the specification is ILQ design method. The features of the method is introduced in the next subsection and the calculation method of K is introduced in Appendix D.*

4.6.2 Observer-based output feedback case

The Guideline of Choosing K

In this section we consider the guideline of choosing design parameters in observer-based output feedback case and propose a new design algorithm based on the Theorem 4.4.

In this section we propose a guideline of choosing K. For this purpose, we first clarify the structure of a closed-loop system using new design parameters. The closed-loop system of Fig. 4.4 can be transformed into a system of Figs. 4.5 and 4.6. When design parameters $\{\sigma_i\}$ increases, the resulting closed-loop system is known to approach the asymptotic system as shown in Figs. 4.8 and 4.9 [Fujii93a], [Fujii94], whose transfer function from r to y can be specified by suitable choice of K. In view of this observation we first have to choose K in order to make this asymptotic system stable, that is, $A_F - BK$ stable. In addition we want to choose K which achieve desirable closed-loop responses.

One of practical design methods of such a feedback gain K would be ILQ design method [Fujii87a], [Fujii87b], [Fujii93a], [Fujii88]. This design method for servo systems tracking a step reference input has been derived from the viewpoint of inverse LQ problem on the basis of a parameterization of the form (4.34) of LQ gains, and has several attractive features as shown below from the practical viewpoint.

1. The primary design computation is the determination of K in (4.34) by the pole assignment method, which is obviously much simpler than solving Riccati equations in the usual LQ design.
2. The decoupled step output response can be specified by suitable choice of specified poles $\{s_i\}$ for the pole assignment method above.
3. The design parameters $\{\sigma_i\}$ as in (4.34) can be used as the tradeoff parameters between the magnitude of control inputs and the tracking property of output responses.
4. The resulting feedback matrices can be expressed explicitly in terms of the system matrices and the design parameters selected.

In Theorem 4.4 no concrete guideline of choosing K is given except making $A_F - BK$ stable. However, a decoupling property of the closed-loop system is often required in practice as a design specification in multivariable feedback control. In this respect ILQ design method is more effective than other design methods for a feedback gain K.

The Guideline of Choosing L

The following upper bound of $\|\overline{G}_{oe}(s)\|$ can be obtained by using triangular inequality.

$$\|\overline{G}_{oe}(s)\|_\infty \leq \|U_2U_2^TG_a(s) - G_b(s)\|_\infty + \|G_c(s)\|_\infty \tag{4.61}$$

The first term of the right-hand side in (4.61) is related only to a feedback gain K and not to an observer gain L. An observer gain L is included only in the second term of the right-hand side in (4.61). Hence, an observer gain L is determined in order to make $\|G_c(s)\|_\infty$ small. Furthermore from the inequality a feedback gain K should be determined so as to not only achieve desirable responses but also make $\|G_a(s)-G_b(s)\|_\infty$ small.

One design method for L is to determine L by pole assignment method or to reduce the minimization of a norm with respect to a part of $\|G_c(s)\|_\infty$ to solvability of dual LMIs as shown in the following remark. The former method needs a little computation and the latter method much computation.

Remark 4.13 *The minimization of $\|sK_F^0(sI-\Phi_o)^{-1}D\|_\infty$, a latter part of $\|G_c(s)\|_\infty$, over L is reduced to dual LMIs. It, however, is difficult to solve.*

The following relation with respect to $\|\overline{G}_c(s)\|_\infty$ holds using triangular inequality.

$$\begin{aligned}\|G_c(s)\|_\infty &= \left\|\left[E_b+(U_2U_2^TE_a-E_bK)(sI-A_K)^{-1}B\right]sK_F^0(sI-\Phi_o)^{-1}D\right\|_\infty \\ &\leq \|E_b+(U_2U_2^TE_a-E_bK)(sI-A_K)^{-1}B\|_\infty\|sK_F^0(sI-\Phi_o)^{-1}D\|_\infty\end{aligned}$$

A latter part of a right-hand side in this equation is related only to an observer gain L. We, therefore, consider a minimization of $\|G_{cL}(s)\|_\infty$ by L.

$$G_{cL}(s) := sK_F^0(sI-\Phi_o)^{-1}D = K_F^0D+K_F^0(A-LC)\left\{sI-(A-LC)\right\}^{-1}D$$

A generalized plant for the minimization of $\|G_{cL}(s)\|_\infty$ by L is obtained as follows.

$$\begin{aligned}\dot{x} &= Ax+u+\frac{1}{\gamma}Dw \\ y &= Cx \\ z &= K_F^0Ax+K_F^0u+\frac{1}{\gamma}K_F^0Dw \\ u &= -Ly\end{aligned}$$

Applying Theorem 2 in [Iwasa94] to this H_∞ control problem, we find that this problem has a solution if and only if the following linear matrix inequalities, as called dual LMI, have a solution $P=Q^{-1}>0$.

$$\begin{bmatrix} -K_F^0 & I\end{bmatrix}\begin{bmatrix} AP+PA^T+\frac{1}{\gamma^2}DD^T & PA^TK_F^{0\,T}+\frac{1}{\gamma^2}DD^TK_F^{0\,T} \\ K_F^0AP+\frac{1}{\gamma^2}K_F^0DD^T & \frac{1}{\gamma^2}K_F^0DD^TK_F^{0\,T}-I\end{bmatrix}\begin{bmatrix} -K_F^{0\,T} \\ I\end{bmatrix}<0 \tag{4.62}$$

$$\begin{aligned}&\begin{bmatrix} (C^T)^\perp & 0\end{bmatrix} \\ &\times\begin{bmatrix} QA+A^TQ+A^TK_F^{0\,T}K_F^0A & \frac{1}{\gamma}(QD+A^TK_F^{0\,T}K_F^0D) \\ \frac{1}{\gamma}(D^TQ+D^TK_F^{0\,T}K_F^0A) & \frac{1}{\gamma^2}D^TK_F^{0\,T}K_F^0D-I\end{bmatrix}\begin{bmatrix} (C^T)^{\perp\,T} \\ 0\end{bmatrix}<0 \\ \Leftrightarrow\quad &(C^T)^\perp\left(P^{-1}A+A^TP^{-1}+A^TK_F^{0\,T}K_F^0A\right)(C^T)^{\perp\,T}<0\end{aligned} \tag{4.63}$$

However, this condition is not convex with respect to P. Therefore, it is difficult to solve and hence [Iwasa95] suggested an effective method to solve it. However, solving it needs much computation and hence reducing the minimization of an upper bound $\|G_c(s)\|_\infty$ to dual LMIs is not so practical.

Design Algorithm

According to the guidelines shown in the previous subsections we suggest the following design algorithm of robust servo systems achieving desirable responses.

Design algorithm (Observer-based output feedback)

Step 1 Set $V = [V_1 \quad V_2]^T$.

Step 2 Determine K such that $A_F - BK$ is stable and desirable responses are achieved, for example, using ILQ design method. Determine L such that $A - LC$ is stable. Then check whether $||\overline{G}_{oe}(s)||_\infty < 1$ is satisfied or not (Theorem 4.4). If this condition is satisfied, go to Step 3. Otherwise repeat this step after choosing different K and L.

Step 3 First, choose positive $\{\sigma_i\}$ and check whether $||G_{oe}(s)||_\infty < 1$ is satisfied or not for the resulting K_e. If this condition is satisfied, go to Step 4. Otherwise repeat this step after choosing different $\{\sigma_i\}$. If $||G_{oe}(s)||_\infty \geq 1$ even after this choice, go to Step 2 and choose different K and L.

Step 4 Using V, K and Σ obtained in the above step, K_e is obtained from (4.34), K_F and K_I by (4.37).

Remark 4.14 *The condition $||\overline{G}_{oe}(s)||_\infty < 1$ is only a sufficient condition for quadratic stability by Theorem 4.6. This sufficient condition is not far from a necessary condition because it is necessary to increase Σ to some extent in order to achieve desirable responses [Fujii93a],[Fujii88].*

Remark 4.15 *As indicated in Remark 4.11, selecting K as a decoupling gain give a conservative result with respect to the existence of a quadratically stabilizing controller. In the observer-based output feedback case the design method for L in Remark 4.13 give a conservative result because $||G_{cL}||_\infty$ is a part of an upper bound for $||\overline{G}_{oe}||_\infty$ which is a sufficient condition for quadratic stability. Therefore, the formulation of the minimization of $||G_{oe}||_\infty$ into matrix inequalities with respect to L is needed in order to diminish the conservativeness.*

4.7 Proofs of Theorems

4.7.1 Proof of Theorem 4.1

State feedback case (ΔA exists)

We note by Fact 4.2 and Lemma 4.2 that the control (4.3) is a central quadratically stabilizing control of for the system (4.8) if and only if there exists a $P_e > 0$ in the form of (4.35) that satisfies (4.17). Hence we show below that the existence of such a P_e implies the two conditions in Theorem 4.1. First, we substitute (4.35) into (4.17) and make the following equivalent transformation.

$$\begin{bmatrix} T_e^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P_e\Psi_e + \Psi_e^T P_e - E_{ea}^T E_{ea} & P_e D_e \\ D_e^T P_e & I \end{bmatrix} \begin{bmatrix} T_e & 0 \\ 0 & I \end{bmatrix} > 0$$

This inequality can be rewritten as

$$\begin{bmatrix} X_e H_e + H_e^T X_e - (E_e T_e)^T (E_e T_e) & X_e \widetilde{D}_e \\ \widetilde{D}_e^T P_e & I \end{bmatrix} > 0 \tag{4.64}$$

where

$$T_e := \begin{bmatrix} T & 0 \\ G & V^{-1} \end{bmatrix} \tag{4.65}$$

$$X_e := T_e^T P_e T_e = \begin{bmatrix} T^T Y T & 0 \\ 0 & \Sigma\Lambda \end{bmatrix} \tag{4.66}$$

$$H_e := T_e^{-1} \Psi_e T_e = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2}\Sigma \end{bmatrix} - \begin{bmatrix} S & T^{-1}BV^{-1} \\ -VGS & VFBV^{-1} \end{bmatrix} \tag{4.67}$$

$$\widetilde{D}_e := T_e^{-1} D_e = \begin{bmatrix} Z_{11} \\ Z_{21} \end{bmatrix} D \tag{4.68}$$

$$\begin{bmatrix} Z_{11} \\ Z_{21} \end{bmatrix} := (\Gamma T_e)^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{4.69}$$

and T, S and G are those matrices satisfying the following relations.

$$\begin{aligned} TST^{-1} &= A_F - BK = A_K, \ \det T \neq 0 \\ G &= -KT \end{aligned} \tag{4.70}$$

Furthermore (4.64) can be transformed equivalently as follows.

$$(4.64) \Leftrightarrow X_e H_e + H_e^T X_e^T - (E_e T_e)^T (E_e T_e)$$

$$-(X_e\widetilde{D}_e)(X_e\widetilde{D}_e)^T := \begin{bmatrix} L_{11} & L_{12} \\ L_{12}^T & L_{22} \end{bmatrix} > 0$$

$$\Leftrightarrow \begin{cases} L_{11} > 0 \\ L_{22} - L_{12}^T L_{11}^{-1} L_{12} > 0 \end{cases} \tag{4.71}$$

$$\begin{aligned} L_{11} &:= -T^TYTS - (T^TYTS) - (E_aTS)^T(E_aTS) \\ &\quad -(T^TYTZ_{11}D)(T^TYTZ_{11}D)^T && (4.72) \\ L_{12} &:= -T^TYBV^{-1} + (\Sigma\Lambda VGS)^T - (E_aTS)^T(E_aBV^{-1}) \\ &\quad -(T^TYTZ_{11}D)(\Sigma\Lambda Z_{21}D)^T && (4.73) \\ L_{22} &:= \left(\frac{1}{2}\Sigma\Lambda\Sigma - \Sigma\Lambda VFBV^{-1}\right) + \left(\frac{1}{2}\Sigma\Lambda\Sigma - \Sigma\Lambda VFBV^{-1}\right)^T \\ &\quad -(E_aBV^{-1})^T(E_aBV^{-1}) - (\Sigma\Lambda Z_{21}D)(\Sigma\Lambda Z_{21}D)^T && (4.74) \end{aligned}$$

In the following we show that the inequality $L_{11} > 0$ implies (4.43) and (4.44) in Theorem 4.1. We first note that $L_{11} > 0$ is equivalent to

$$\begin{aligned} &Y(TST^{-1}) + (TST^{-1})^TY + \{E_a(TST^{-1})\}^T \\ &\quad \times\{E_a(TST^{-1})\} + Y(TZ_{11}D)(TZ_{11}D)^TY < 0 \end{aligned}$$

which can be rewritten by (4.70) as

$$YA_K + A_K^TY + (E_aA_K)^T(E_aA_K) + Y(TZ_{11}D)(TZ_{11}D)^TY < 0 \tag{4.75}$$

or equivalently

$$Y_KA_K + A_K^TY_K + E_a^TE_a + Y_K(A_KTZ_{11}D)(A_KTZ_{11}D)^TY_K < 0 \tag{4.76}$$

$$Y_K := (A_K^{-1})^TYA_K^{-1}. \tag{4.77}$$

By Bounded Real Lemma, i.e., Lemma A.2 in Appendix A, the inequality (4.75) is equivalent to

$$A_K \text{ is stable} \tag{4.78}$$

$$\|E_a(sI - A_K)^{-1}A_KTZ_{11}D\|_\infty < 1. \tag{4.79}$$

To derive (4.44) from (4.79), we premultiply (4.69) by the following term.

$$\begin{bmatrix} T & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} N^{-1}T_e \tag{4.80}$$

$$N := \begin{bmatrix} I & 0 \\ -K & I \end{bmatrix} \tag{4.81}$$

Then we have

$$TZ_{11} = \begin{bmatrix} I & 0 \end{bmatrix} (\Gamma N)^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} = A_K^{-1} + A_K^{-1} B(-CA_K^{-1}B)^{-1} CA_K^{-1}. \tag{4.82}$$

Substituting this equation into the transfer function of (4.79) yields

$$\begin{aligned} & E_a(sI - A_K)^{-1} A_K T Z_{11} D \\ = \;& E_a(sI - A_K)^{-1} \{I - B(CA_K^{-1}B)^{-1} CA_K^{-1}\} D. \end{aligned} \tag{4.83}$$

Hence, from (4.78) and (4.79) we obtain

$$A_K \text{ is stable} \tag{4.84}$$

$$\|E_a(sI - A_K)^{-1}(I - BB^-)D\|_\infty < 1. \tag{4.85}$$

Next we prove the sufficiency. In (4.36) $R > 0$ has no constraint and therefore let $\Lambda = \Sigma^{-1}$. Then $L_{22} - L_{12}^T L_{11}^{-1} L_{12} > 0$ can be written as follows.

$$\begin{aligned} & L_{22} - L_{12}^T L_{11}^{-1} L_{12} > 0 \\ \Leftrightarrow \; & \left(\frac{1}{2}\Sigma\Lambda\Sigma - \Sigma\Lambda VFBV^{-1}\right) + \left(\frac{1}{2}\Sigma\Lambda\Sigma - \Sigma\Lambda VFBV^{-1}\right)^T \\ & -(E_aBV^{-1})^T(E_aBV^{-1}) - (\Sigma\Lambda Z_{21}D)(\Sigma\Lambda Z_{21}D)^T \\ & \quad - \left\{-T^TYBV^{-1} + (\Sigma\Lambda VGS)^T - (E_aTS)^T(E_aBV^{-1}) \right. \\ & \quad \left. -(T^TYTZ_{11}D)(\Sigma\Lambda Z_{21}D)^T\right\}^T L_{11}^{-1} L_{12} > 0 \\ \Leftrightarrow \; & \left(\frac{1}{2}\Sigma - VFBV^{-1}\right) + \left(\frac{1}{2}\Sigma - VFBV^{-1}\right)^T \\ & -(E_aBV^{-1})^T(E_aBV^{-1}) - (Z_{21}D)(Z_{21}D)^T - L_{12}^T L_{11} L_{12} > 0 \\ \Leftrightarrow \; & \Sigma > VFBV^{-1} + (VFBV^{-1})^T + (E_aBV^{-1})^T(E_aBV^{-1}) \\ & +(Z_{21}D)(Z_{21}D)^T + L_{12}^T L_{11} L_{12} > 0 \end{aligned} \tag{4.86}$$

We can find that there always exists Σ satisfying (4.86) i.e., sufficiently large Σ satisfies (4.86). Therefore, if K satisfies (4.43) and (4.44), K_e become a central quadratically stabilizing control law.

Next we show the asymptoic property (4.47). First, T_{BF} is defined as follows.

$$T_{BK} := I + BK(sI - A)^{-1}$$

Then it is straightforward to derive the following relations.

$$T_{BK}B = B\left[I + K(sI - A)^{-1}B\right] \tag{4.87}$$

$$sI - A = T_{BK}^{-1}(sI - A_K) \tag{4.88}$$

$$(I - BB^-)T_{BK} = I - BB^- \tag{4.89}$$

Using (4.8) and (4.34), $G_e(s)$ becomes

$$\begin{aligned} & G_e(s) \\ = & \begin{bmatrix} E_a & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -F & I \end{bmatrix} \begin{bmatrix} sI - A_K & -B \\ -sK & sI + V^{-1}\Sigma V \end{bmatrix}^{-1} \\ & \times \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} D \\ 0 \end{bmatrix} \\ = & E_a \begin{bmatrix} A_K & B \end{bmatrix} \begin{bmatrix} (sI - A)A_K & (sI - A)B + BV^{-1}\Sigma V \\ C(sI - A_K) & -CB \end{bmatrix}^{-1} \begin{bmatrix} D \\ 0 \end{bmatrix}. \end{aligned}$$

Substituting (4.88) into this equation yields

$$\begin{aligned} & G_e(s) \\ = & E_a \begin{bmatrix} A_K & B \end{bmatrix} \begin{bmatrix} (sI - A_K)A_K & (sI - A_K)B + T_{BK}BV^{-1}\Sigma V \\ C(sI - A_K) & -CB \end{bmatrix}^{-1} \\ & \times \begin{bmatrix} T_{BK} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} D \\ 0 \end{bmatrix} \\ = & E_a(sI - A_K)^{-1}H_1 \left\{ H + \begin{bmatrix} T_{BK}BV^{-1} \\ 0 \end{bmatrix} (\Sigma^{-1})^{-1} \begin{bmatrix} 0 & V \end{bmatrix} \right\}^{-1} \begin{bmatrix} T_{BK}D \\ 0 \end{bmatrix} \\ = & E_a(sI - A_K)^{-1}H_1 \\ & \times \left\{ H^{-1} - H^{-1} \begin{bmatrix} T_{BK}BV^{-1} \\ 0 \end{bmatrix} \left(\Sigma^{-1} + \begin{bmatrix} 0 & V \end{bmatrix} H^{-1} \begin{bmatrix} T_{BK}BV^{-1} \\ 0 \end{bmatrix} \right)^{-1} \right. \\ & \left. \times \begin{bmatrix} 0 & V \end{bmatrix} H^{-1} \right\} \begin{bmatrix} I \\ 0 \end{bmatrix} T_{BK}D \end{aligned} \tag{4.90}$$

where

$$\begin{aligned} H & := \begin{bmatrix} (sI - A_K)A_K & (sI - A_K)B \\ C(sI - A_K) & -CB \end{bmatrix} \\ H_1 & := \begin{bmatrix} (sI - A_K)A_K & (sI - A_K)B \end{bmatrix} \\ H^{-1} & := \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}. \end{aligned}$$

Approaching Σ to ∞ and using (4.87), $G_e(s)$ in (4.90) approaches $\overline{G}_{ea}(s)$ as follows.

$$\begin{aligned} & G_e(s) \\ \to & E_a(sI - A_K)^{-1} \end{aligned}$$

$$
\begin{aligned}
& \times \left\{ H_1 H^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} - H_1 H^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} T_{BK} B V^{-1} (V X_{21} T_{BK} B V^{-1})^{-1} V X_{21} \right\} T_{BK} D \\
= \; & E_a (sI - A_K)^{-1} \left\{ I - T_{BK} B (X_{21} T_{BK} B)^{-1} X_{21} \right\} T_{BK} D \\
= \; & E_a (sI - A_K)^{-1} \left[I - B (X_{21} B)^{-1} X_{21} \right] T_{BK} D \\
:= \; & \overline{G}_{ea}(s)
\end{aligned}
$$

From the above definition of X_{21}, X_{22}, the following equation is derived.

$$
\begin{aligned}
X_{21} &= -X_{22} C (sI - A_K) \left[(sI - A_K) A_K \right]^{-1} \\
&= -X_{22} C A_K^{-1}
\end{aligned}
$$

Substituting this equation into $\overline{G}_{ea}(s)$ and using (4.89), we have

$$
\begin{aligned}
\overline{G}_{ea}(s) &= E_a (sI - A_K)^{-1} \left[I - B (C A_K^{-1} B)^{-1} C A_K^{-1} \right] T_{BK} D \\
&= E_a (sI - A_K)^{-1} (I - B B^{-}) D. \qquad (4.91)
\end{aligned}
$$

4.7.2 Proof of Theorem 4.2 State feedback case (ΔA and ΔB exist)

We note by Fact 4.4 and Lemma 4.4 that the control (4.3) is a central quadratically stabilizing control of for the system (4.25) only if there exists a $P_e > 0$ in the form of (4.35) that satisfies (4.32). Hence we show below that the existence of such a P_e implies the two conditions in Theorem 4.2.

This proof of conditions (4.49) and (4.50) is done by replacing E_{ea} in the proof of the previous subsection. by $U_2 U_2^T E_e$.

Next we prove the asymptotic property (4.51). First, $G_e(s)$ is represented as follows.

$$
\begin{aligned}
G_e(s) &= G_{ea}(s) - G_{eb}(s) \\
G_{ea}(s) &:= E_{eab} (sI - \Phi_e)^{-1} D_e \\
G_{eb}(s) &:= E_b K_e (sI - \Phi_e)^{-1} D_e
\end{aligned}
$$

The asymptotic property $G_{ea}(s) \to U_2 U_2^T G_a(s)$ is proven like that in Theorem 4.1. Here we prove the asymptotic property $G_{eb}(s) \to G_b(s)$. From the definition of $G_{eb}(s)$, $G_{eb}(s)$ becomes as follows.

$$
\begin{aligned}
& G_{eb}(s) \\
= \; & E_b K_e \begin{bmatrix} I & 0 \\ -K & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -K & I \end{bmatrix}^{-1} \begin{bmatrix} sI - A & -B \\ V^{-1} \Sigma V K & sI + V^{-1} \Sigma V \end{bmatrix}^{-1} D_e
\end{aligned}
$$

$$
\begin{aligned}
&= E_b V^{-1}\Sigma V \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} sI - A + BK & -B \\ -sK & sI + V^{-1}\Sigma V \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ -K & I \end{bmatrix} \\
&\quad \times \left(\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -K & I \end{bmatrix} \right)^{-1} \begin{bmatrix} D \\ 0 \end{bmatrix} \\
&= E_b V^{-1}\Sigma V \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} sI - A_K & -B \\ -KA_K & sI - KB + V^{-1}\sigma V \end{bmatrix}^{-1} \\
&\quad \times \begin{bmatrix} A_K^{-1}(I - BB^-) \\ B^- \end{bmatrix} D \qquad (4.92)
\end{aligned}
$$

Here W_{ij}, $(i, j = 1, 2)$ are defind as follows.

$$
\begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} := \begin{bmatrix} sI - A_K & -B \\ -KA_K & sI_K B + V^{-1}\sigma V \end{bmatrix}^{-1}
$$

$$
\begin{aligned}
W_{22} &= \left[(sI - KB + V^{-1}\Sigma V) - (-KA_K)(sI - A_K)^{-1}(-B) \right]^{-1} \\
&= \left[(sI - KB) - KA_K(sI - A_K)^{-1}B + V^{-1}\Sigma V \right]^{-1} \\
W_{21} &= -W_{22}(-KA_K)(sI - A_K)^{-1} \\
&= W_{22}(KA_K)(sI - A_K)^{-1}
\end{aligned}
$$

Substituting these equations into $G_{eb}(s)$ yields

$$
\begin{aligned}
G_{eb}(s) &= E_b \left[G_F(s) V^{-1}\Sigma^{-1} V + I \right]^{-1} \begin{bmatrix} KA_K(sI - A_K)^{-1} & I \end{bmatrix} \\
&\quad \times \begin{bmatrix} A_K^{-1}(I - BB^-) \\ B^- \end{bmatrix} D
\end{aligned}
$$

where

$$
G_F(s) := (sI - KB) - KA_K(sI - A_K)^{-1}B.
$$

As Σ approaches ∞, the following asymptoitc property is derived.

$$
G_{eb}(s) \to E_b \left[K(sI - A_K)^{-1}(I - BB^-) + B^- \right] D
$$

4.7.3 Proof of Theorem 4.4

Observer-based output feedback case (ΔA and ΔB exist)

First, a composite system which consists of the augmented system (4.25) and the observer (4.4) and (4.5) is transformed by the feedback transformation $u = v - \Xi B_b^T E_a \xi$

and we get the following system.

$$\begin{aligned}
\dot{x}_{oa} &= \left\{ \begin{bmatrix} A & 0 & -B\Xi E_b^T E_a \\ C & 0 & 0 \\ LC & 0 & A - LC - B\Xi E_b^T E_a \end{bmatrix} + \begin{bmatrix} D \\ 0 \\ 0 \end{bmatrix} F \begin{bmatrix} E_a & 0 & -U_1 U_1^T E_a \end{bmatrix} \right\} x_{oa} \\
&\quad + \left\{ \begin{bmatrix} B \\ 0 \\ B \end{bmatrix} + \begin{bmatrix} D \\ 0 \\ 0 \end{bmatrix} F E_b \right\} v \\
v &= -\begin{bmatrix} 0 & K_I & K_{F\xi} \end{bmatrix} x_{oa}
\end{aligned} \tag{4.93}$$

where

$$K_{F\xi} = K_F - \Xi E_b^T E_a.$$

The above equations are defined as

$$\dot{x}_{oa} = (A_{oe} + D_{oe} F E_{oa}) x_{oa} + (B_{oe} + D_{oe} F E_b) v, \ v = -K_{oe} x_{oa}.$$

Then the closed-loop system becomes as follows.

$$\dot{x}_{oa} = (A_{oe} - B_{oe} K_{oe}) x_{oa} + D_{oe} F (E_{oa} - E_b K_{oe}) x_{oa} \tag{4.94}$$

According to Fact 4.6, a necessary and sufficient condition for quadratic stability in the system (4.94) is that the following two conditions hold.

1. $(A_{oe} - B_{oe} K_{oe})$ is stable.
2. $\|G_{oe}(s)\|_\infty < 1$

   $G_{oe}(s) := (E_{oa} - E_b K_{oe}) \{sI - (A_{oe} - B_{oe} K_{oe})\}^{-1} D_{oe}$

This system is transformed by the following two coordinate transformation.

$$x_{oa} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ I & 0 & I \end{bmatrix} \begin{bmatrix} x \\ w \\ \xi - x \end{bmatrix} \tag{4.95}$$

$$\begin{bmatrix} x \\ w \\ \xi - x \end{bmatrix} = \begin{bmatrix} A_F & B & 0 \\ C & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x_e \\ \xi - x \end{bmatrix} \tag{4.96}$$

Then $G_{oe}(s)$ results in the following.

$$G_{oe}(s) = \begin{bmatrix} E_e - E_b K_e & -E_b K_F \end{bmatrix} \left\{ sI - \begin{bmatrix} \Phi_e & -\Gamma^{-1} \begin{bmatrix} BK_F \\ 0 \end{bmatrix} \\ 0 & \Phi_o \end{bmatrix} \right\}^{-1} \begin{bmatrix} D_e \\ -D \end{bmatrix} \tag{4.97}$$

Furthermore $G_{oe}(s)$ is expanded and arranged as follows.

$$\begin{aligned} G_{oe}(s) &= \begin{bmatrix} E_e - E_b K_e & -E_b K_F \end{bmatrix} \\ &\times \begin{bmatrix} (sI-\Phi_e)^{-1} & -(sI-\Phi_e)^{-1}\Gamma^{-1}\begin{bmatrix} BK_F \\ 0 \end{bmatrix}(sI-\Phi_o)^{-1} \\ 0 & (sI-\Phi_o)^{-1} \end{bmatrix} \begin{bmatrix} D_e \\ -D \end{bmatrix} \\ &= (E_e - E_b K_e)(sI-\Phi_e)^{-1} D_e \\ &\quad + (E_e - E_b K_e)(sI-\Phi_e)^{-1}\Gamma^{-1}\begin{bmatrix} BK_F \\ 0 \end{bmatrix}(sI-\Phi_o)^{-1} D \\ &\quad + E_b K_F (sI-\Phi_o)^{-1} D \end{aligned} \tag{4.98}$$

Here, $G_{ea}^F(s)$ and $G_{eb}^F(s)$ are defined as

$$\begin{aligned} G_{ea}^F(s) &= E_e(sI-\Phi_e)^{-1}\Gamma^{-1}\begin{bmatrix} I \\ 0 \end{bmatrix} \\ G_{eb}^F(s) &= E_b K_e (sI-\Phi_e)^{-1}\Gamma^{-1}\begin{bmatrix} I \\ 0 \end{bmatrix}. \end{aligned}$$

Then the definitions lead to

$$G_{oe}(s) = (G_{ea}^F - G_{eb}^F)D + G_{ea}^F B K_F (sI-\Phi_o)^{-1} D + (-G_{eb}^F B + E_b) K_F (sI-\Phi_o)^{-1} D. \tag{4.99}$$

Here, we consider $G_{ea}^F(s)$.

$$\begin{aligned} G_{ea}^F(s) &= \begin{bmatrix} U_2 U_2^T E_a & 0 \end{bmatrix} \begin{bmatrix} A_F & B \\ C & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -K & I \end{bmatrix} \\ &\times \begin{bmatrix} sI - A_K & -B \\ -sK & sI + V^{-1}\Sigma V \end{bmatrix}^{-1} \begin{bmatrix} A_F & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} \\ &= U_2 U_2^T E_a \begin{bmatrix} A_K & B \end{bmatrix} \begin{bmatrix} (sI-A_K)A_K & (sI-A_F)B + BV^{-1}\Sigma V \\ C(sI-A_K) & -CB \end{bmatrix}^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} \end{aligned}$$

Using the relation $sI - A_F = T_{BK}^{-1}(sI - A_K)$, the above equation becomes

$$\begin{aligned} G_{ea}^F(s) &= U_2 U_2^T E_a \begin{bmatrix} A_K & B \end{bmatrix} \\ &\times \begin{bmatrix} (sI-A_K)A_K & (sI-A_K)B + T_{BK}BV^{-1}\Sigma V \\ C(sI-A_K) & -CB \end{bmatrix}^{-1} \\ &\times \begin{bmatrix} T_{BK} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
&= U_2U_2^TE_a(sI-A_K)^{-1}H_1 \\
&\quad \times \left\{ H + \begin{bmatrix} T_{BK}BV^{-1} \\ 0 \end{bmatrix} (\Sigma^{-1})^{-1} [0\ V] \right\}^{-1} \begin{bmatrix} T_{BK} \\ 0 \end{bmatrix} \\
&= U_2U_2^TE_a(sI-A_K)^{-1}H_1 \left\{ H^{-1} - H^{-1} \begin{bmatrix} T_{BK}BV^{-1} \\ 0 \end{bmatrix} \right. \\
&\quad \times \left. \left(\Sigma^{-1} + [0\ V]\, H^{-1} \begin{bmatrix} T_{BK}BV^{-1} \\ 0 \end{bmatrix} \right)^{-1} [0\ V]\, H^{-1} \right\} \begin{bmatrix} I \\ 0 \end{bmatrix} T_{BK}
\end{aligned}$$

where

$$\begin{aligned}
H &:= \begin{bmatrix} (sI-A_K)A_K & (sI-A_K)B \\ C(sI-A_K) & -CB \end{bmatrix}, \quad H_1 := \begin{bmatrix} (sI-A_K)A_K & (sI-A_K)B \end{bmatrix} \\
H^{-1} &:= \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}, \quad T_{BK} := I + BK(sI-A_F)^{-1}.
\end{aligned}$$

Furthermore we compute this equation and get

$$\begin{aligned}
G_{ea}^F(s) &= U_2U_2^TE_a(sI-A_K)^{-1} \\
&\quad \left\{ I - T_{BK}BV^{-1}(\Sigma^{-1} + VX_{21}T_{BK}BV^{-1})^{-1}VX_{21} \right\} T_{BK}. \qquad (4.100)
\end{aligned}$$

Postmultiplying D by (4.100) yields

$$\begin{aligned}
G_{ea}^F(s)D &= U_2U_2^TE_a(sI-A_K)^{-1} \\
&\quad \times \left\{ I - T_{BK}BV^{-1}(\Sigma^{-1} + VX_{21}T_{BK}BV^{-1})^{-1}VX_{21} \right\} T_{BK}D.
\end{aligned}$$

As $\{\sigma_i\}$ approaches ∞, we get the following equation using the relation $T_{BK}B = B[I + K(sI-A_K)^{-1}B]$.

$$\begin{aligned}
G_{ea}^F(s)D &\to U_2U_2^TE_a(sI-A_K)^{-1} \left\{ I - T_{BK}B(X_{21}T_{BK}B)^{-1}X_{21} \right\} T_{BK}D \\
&= U_2U_2^TE_a(sI-A_K)^{-1} \left\{ I - B(X_{21}B)^{-1}X_{21} \right\} T_{BK}D
\end{aligned}$$

Furthermore $X_{21} = -X_{22}CA_K^{-1}$ holds from the definition of X_{21} and X_{22}. Using this relation, we have

$$G_{ea}^F(s)D \to U_2U_2^TE_a(sI-A_K)^{-1}(I - BB^-)D. \qquad (4.101)$$

We compute the second term of (4.99) using (4.100).

$$\begin{aligned}
&G_{ea}^F(s)BV^{-1}\Sigma VK_F^0(sI-\Phi_o)^{-1}D \\
&= U_2U_2^TE_a(sI-A_K)^{-1}
\end{aligned}$$

$$
\begin{aligned}
&\times\left\{I - T_{BK}BV^{-1}(\Sigma^{-1} + VX_{21}T_{BK}BV^{-1})^{-1}VX_{21}\right\} \\
&\times T_{BK}BV^{-1}\Sigma VK_F^0(sI - \Phi_o)^{-1}D \\
= \;& U_2U_2^TE_a(sI - A_K)^{-1}T_{BK}B \\
&\times\left\{I - V^{-1}(\Sigma^{-1} + VX_{21}T_{BK}BV^{-1})^{-1}VX_{21}T_{BK}B\right\} \\
&\times V^{-1}\Sigma VK_F^0(sI - \Phi_o)^{-1}D \\
= \;& U_2U_2^TE_a(sI - A_K)^{-1}T_{BK}B(V^{-1}\Sigma^{-1}V + X_{21}T_{BK}B)^{-1}K_F^0(sI - \Phi_o)^{-1}D
\end{aligned}
$$

Here, as $G_R(s) := I + K(sI - A_F)^{-1}B$ is defined, $X_{21}B = s^{-1}$ is substituted into the above equation and $\{\sigma_i\}$ approaches ∞, the above equation approaches

$$
\begin{aligned}
&G_{ea}^FBV^{-1}\Sigma VK_F^0(sI - \Phi_o)^{-1}D \\
\to\;& U_2U_2^TE_a(sI - A_K)^{-1}BG_R(s^{-1}G_R)^{-1}K_F^0(sI - \Phi_o)^{-1}D \\
=\;& sU_2U_2^TE_a(sI - A_K)^{-1}BK_F^0(sI - \Phi_o)^{-1}D. \qquad (4.102)
\end{aligned}
$$

Next we consider $G_{eb}^F(s)$.

$$
\begin{aligned}
G_{eb}^F(s) &= E_bK_e(sI - \Phi_e)^{-1}\Gamma^{-1}\begin{bmatrix} I \\ 0 \end{bmatrix} \\
&= E_bK_e\begin{bmatrix} I & 0 \\ -K & I \end{bmatrix}\begin{bmatrix} I & 0 \\ K & I \end{bmatrix}\begin{bmatrix} sI - A_F & -B \\ V^{-1}\Sigma VK & sI + V^{-1}\Sigma V \end{bmatrix}^{-1}\Gamma^{-1}\begin{bmatrix} I \\ 0 \end{bmatrix} \\
&= E_bV^{-1}\Sigma V\begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} sI - A_K & -B \\ -sK & sI + V^{-1}\Sigma V \end{bmatrix}^{-1} \\
&\quad\times\begin{bmatrix} I & 0 \\ -K & I \end{bmatrix}\left(\begin{bmatrix} A_F & B \\ C & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ -K & I \end{bmatrix}\right)^{-1}\begin{bmatrix} I \\ 0 \end{bmatrix} \\
&= E_bV^{-1}\Sigma V\begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} sI - A_K & -B \\ -KA_K & sI - KB + V^{-1}\Sigma V \end{bmatrix}^{-1}\begin{bmatrix} A_K^{-1}(I - BB^-) \\ B^- \end{bmatrix}
\end{aligned}
$$

The following definition is given.

$$
\begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} := \begin{bmatrix} sI - A_K & -B \\ -KA_K & sI - KB + V^{-1}\Sigma V \end{bmatrix}^{-1}
$$

W_{21} and W_{22} become

$$
\begin{aligned}
W_{22} &= \left[(sI - KB) - KA_K(sI - A_K)^{-1}B + V^{-1}\Sigma V\right]^{-1} := \left[G_F(s) + V^{-1}\Sigma V\right]^{-1} \\
W_{21} &= W_{22}(KA_K)(sI - A_K)^{-1}.
\end{aligned}
$$

Using these equations, $G_{eb}^F(s)$ is computed as follows.

$$\begin{aligned} G_{eb}^F(s) &= E_b V^{-1}\Sigma V(G_F(s)+V^{-1}\Sigma V)^{-1} \\ &\quad \times \begin{bmatrix} KA_K(sI-A_K)^{-1} & I \end{bmatrix} \begin{bmatrix} A_K^{-1}(I-BB^-) \\ B^- \end{bmatrix} \\ &= E_b V^{-1}\Sigma V(G_F(s)+V^{-1}\Sigma V)^{-1}\left\{KA_K(sI-A_K)^{-1}A_K^{-1}(I-BB^-)+B^-\right\} \\ &= E_b(G_F V^{-1}\Sigma^{-1}V+I)^{-1}\left\{K(sI-A_K)^{-1}(I-BB^-)+B^-\right\} \qquad (4.103) \end{aligned}$$

From this equation we have

$$\begin{aligned} G_{eb}^F(s)D &= E_b(G_F V^{-1}\Sigma^{-1}V+I)^{-1}\left\{K(sI-A_K)^{-1}(I-BB^-)+B^-\right\}D \\ &= E_b\left[G_F(V^{-1}\Sigma^{-1}V+G_F^{-1})\right]^{-1}\left\{K(sI-A_K)^{-1}(I-BB^-)+B^-\right\}D. \end{aligned}$$

As $\{\sigma_i\}$ approaches ∞, the following equation is derived.

$$G_{eb}^F(s)D \to E_b\left\{K(sI-A_K)^{-1}(I-BB^-)+B^-\right\}D \qquad (4.104)$$

Furthermore computing $(-G_{eb}^F B+E_b)\ K_F(sI-\Phi_o)^{-1}D$ based on (4.103) yields

$$\begin{aligned} (-G_{eb}^F B &+ E_b)K_F(sI-\Phi_o)^{-1}D \\ &= \left\{-E_b(G_F V^{-1}\Sigma^{-1}V+I)^{-1}+E_b\right\}V^{-1}\Sigma V K_F^0(sI-\Phi_o)^{-1}D \\ &= E_b(G_F V^{-1}\Sigma^{-1}V+G_F^{-1})^{-1}\Sigma K_F^0(sI-\Phi_o)^{-1}D \\ &= E_b(V^{-1}\Sigma^{-1}V+G_F^{-1})^{-1}K_F^0(sI-\Phi_o)^{-1}D. \end{aligned}$$

As $\{\sigma_i\}$ approaches ∞, we have

$$(-G_{eb}^F B+E_b)K_F(sI-\Phi_o)^{-1}D \to E_b G_F K_F^0(sI-\Phi_o)^{-1}D. \qquad (4.105)$$

From (4.101),(4.102),(4.104) and (4.105), $G_{oe}(s)$ approaches $\overline{G}_{oe}(s)$ as $\{\sigma_i\}$ approaches ∞.

$$\begin{aligned} G_{oe}(s) &\to U_2U_2^T E_a(sI-A_K)^{-1}(I-BB^-)D \\ &\qquad +E_b[K(sI-A_K)^{-1}(I-BB^-)+B^-]D \\ &\qquad\quad +[E_b+(U_2U_2^T E_a-E_bK)(sI-A_K)^{-1}B]sK_F^0(sI-\Phi_o)^{-1}D \\ &= \overline{G}_{oe}(s) \qquad (4.106) \end{aligned}$$

4.8 Design Examples

In this section we take some design examples to show the effectiveness of the proposed design algorithms in this chapter.

4.8.1 State feedback case (ΔA exists)

Here we deal with a design example of a movable mirror inside of of a Fourier Transform Infrared (FTIR) spectrometer [Fujii91],[Sakai90].

$$\begin{aligned} A &= \begin{bmatrix} 0 & 10 & 0 & 0 \\ -600 & -5 & 0.1 & 0 \\ 0 & 0 & -500 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 50 \end{bmatrix} \\ C &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \\ D &= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad E_a = \begin{bmatrix} 0 & 0 & 0.03 & 0 \end{bmatrix} \\ E_b &= 0 \end{aligned} \tag{4.107}$$

The above D, E_a and E_b denote that the $(2,3)$ entry of A matrix varies within a range of $\pm$ 30 % for a nominal value. Now we consider a design specification that an output response for a step reference input settles in a range of $\pm$ 1 % for the reference value under parameter variations by $t = 60$[ms]. This system has no zero and its relative degree is 4. Therefore, we can set a desired transfer characteristics from r to y to $\frac{1}{(1+Ts)^4}$.

Step 1 Set $V = I$.

Step 2 In this step K is determined using ILQ design method by specifying its design parameters s_1. From the above specification of settling time T have to be set to less than 5 [ms]. $\|G_a(s)\|_\infty$ is plotted for T in Fig. 4.17. From this Figure we choose $T = 4$[ms] in order to make $\|G_a(s)\|_\infty$ small.

Step 3 $\|G_e(s)\|_\infty$ is depicted for σ_1 in Fig. 4.18. Nominal stability is guaranteed for more than $\sigma_{\min} = 1367$ and $\|G_e(s)\|_\infty < 1$ holds for $\sigma > \sigma_{\min}$. In order to get good robust stability and achieve desirable responses, σ is set to 4000.

From the plot of $\|G_e(s)\|_\infty$ depicted in Fig. 4.18, $\|G_e(s)\|_\infty$ takes the values of 0.8662, 0.4655, 0.4580 for $\sigma_1 = \sigma_2 = 500, 4000, 10^4$ respectively. These values confirm the asymptotic property as stated in Theorem 4.1, i.e. $\|G_e(s)\|_\infty \to \|\overline{G}_e(s)\|_\infty = 0.4534$.

Step 4 From the above steps K_F and K_I are determined using (4.37) as follows.

$$K_F = \begin{bmatrix} 4.5224 \cdot 10^8 & 2.9122 \cdot 10^7 & 3.9600 \cdot 10^3 & 8.0000 \cdot 10^1 \end{bmatrix}$$
$$K_I = 3.1250 \cdot 10^{10}$$

Simulation result of step responses for $T = 4$[msec], $\sigma = 4000 > \sigma_{\min}$ is drawn in Fig. 4.19. When we choose $T = 26$[msec] for which $\|G_e(s)\|_\infty = 1.108 > 1$ ($\|G_a(s)\|_\infty = 1.143$) holds, unstable step responses generate for maximum parameter variation in Fig. 4.20. This result shows that $\|G_e(s)\|_\infty$ is useful to measure degree of robustness for the closed-loop system.

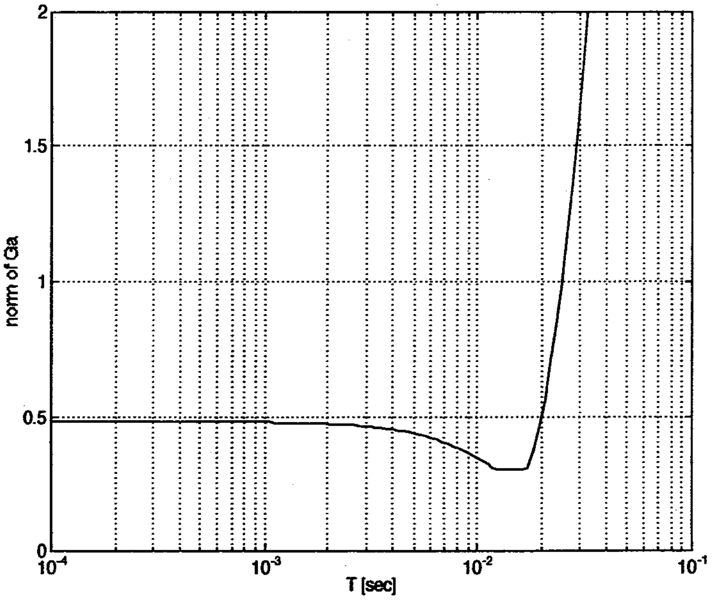

Figure 4.17: $\|G_a(s)\|_\infty$ for time constant T

In the above example the proposed method has several practical features as follows. Robustness of output responses is guaranteed under parameter variations and we can get degree of robustness quantitatively by $\|G_a(s)\|_\infty$ for K and $\|G_e(s)\|_\infty$ for Σ.

4.8.2 State feedback case (ΔB exists)

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 1 \end{bmatrix} \tag{4.108}$$
$$D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad E_a = 0, \quad E_b = 0.2$$

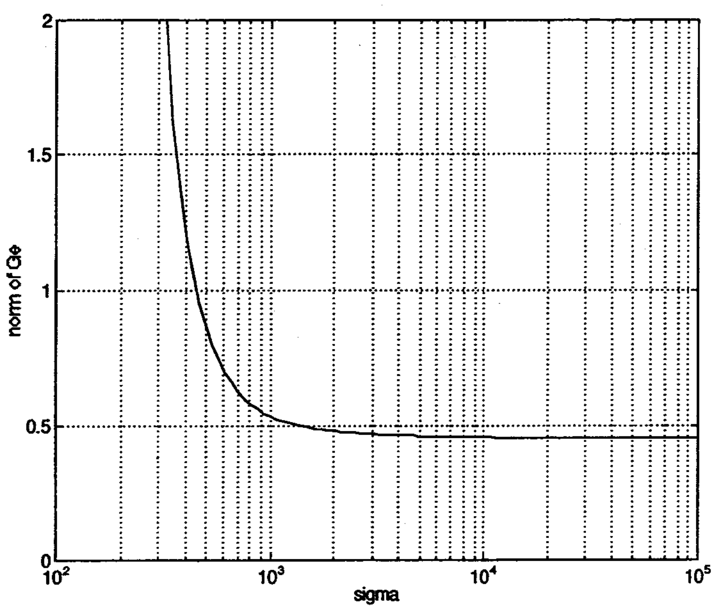

Figure 4.18: $\|G_e(s)\|_\infty$ for σ

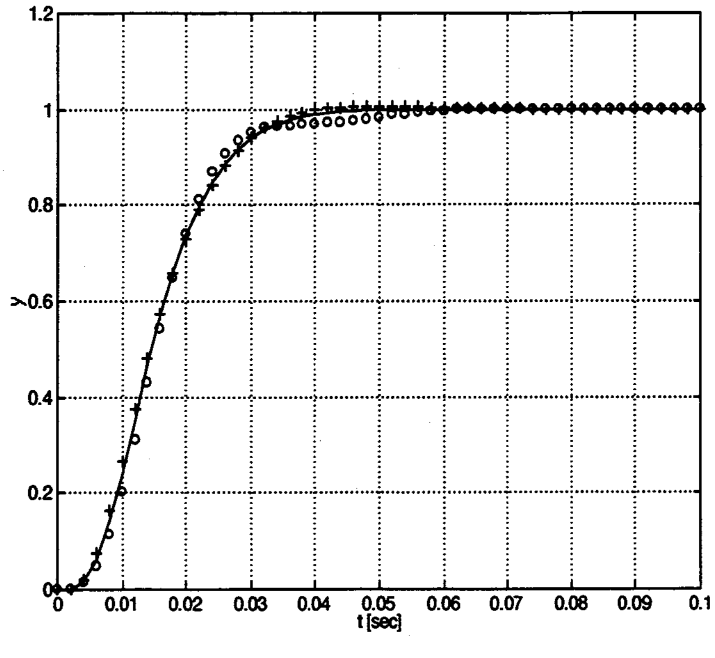

Figure 4.19: Step responses ($T = 4$ [msec]) $F = -1$:○, $F = 0$:—, $F = 1$:+

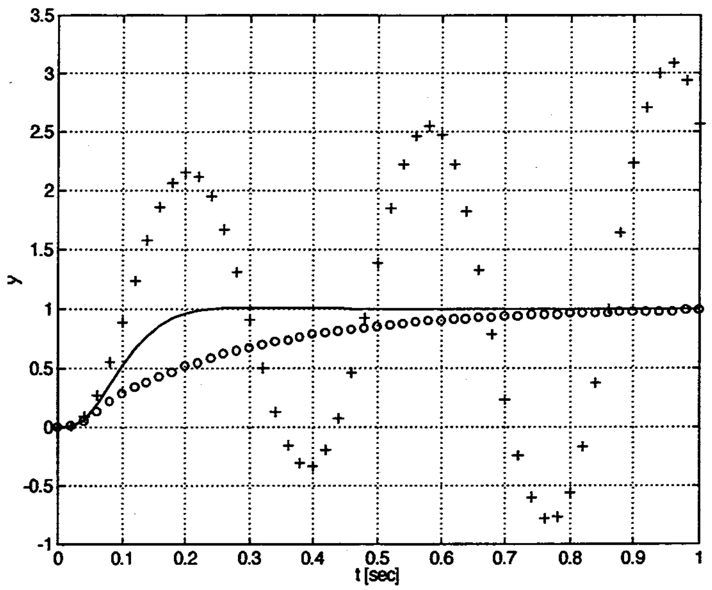

Figure 4.20: Step responses ($T = 26$ [msec]) $F = -1$:o, $F = 0$:—, $F = 1$:+

We consider a numerical example in [Soro84] in order to show the difficulty of designing robust controllers under uncertainty particularly in the input matrix.

The system has the following feature. The transfer function of this system is

$$G(s) = C(sI - A)^{-1}B = \frac{-1}{(s+1)(s+2)}.$$

The input matrix including the parameter variation is defined as follows.

$$B_\epsilon = \begin{bmatrix} 1+\epsilon \\ 1 \end{bmatrix}, \quad -0.2 \leq \epsilon \leq 0.2$$

Then the transfer function of the peturbed system $G_\Delta(s)$is

$$G_\Delta(s) = C(sI - A)^{-1}B_\epsilon = \frac{-(\epsilon s + 2\epsilon + 1)}{(s+1)(s+2)}.$$

When $\epsilon < 0$, $G_\Delta(s)$ has an unstable zero, that is, it becomes non-minimum phase system.

Step 1 Set $V = 1$.

Step 2 In this step K is determined using ILQ design method by specifying its design parameters $s_1 = -\frac{1}{8}$. $\|G_b(s)\|_\infty$ is plotted for T in Fig. 4.21. From this Figure we choose $T = 8$[s] ($\|G_b(s)\|_\infty = 1.2193$) in order to be able to choose σ_i which make $\|G_e(s)\|_\infty$ more than 1. We thus set desirable tracking characteristics of the nominal system to $\dfrac{1}{(1+8s)^2}$.

Step 3 From the plot of $\|G_e(s)\|_\infty$ depicted in Fig. 4.22, $\|G_e(s)\|_\infty$ takes the values of 0.68219, 1.1300, 1.2096 for $\sigma = 10, 100, 10^3$ respectively. These values confirm the asymptotic property as stated in Theorem 4.2, i.e., $\|G_e(s)\|_\infty \to \|G_b(s)\|_\infty = 1.2193$.

Step 4 Using V and K obtained above, K_F and K_I are calculated from (4.37).

$$K_F^0 = \begin{bmatrix} -7.5000 \cdot 10^{-1} & 1.7500 \end{bmatrix}, \quad K_I^0 = -1.5625 \cdot 10^{-2}$$

In Figs. 4.23 and 4.24 each simulation result for $\sigma = 10,\ 100$ is depicted. From Fig. 4.24 the closed-loop system turns out to become unstable under uncertainty. In the previous example $\|G_a(s)\|_\infty < 1$ is a necessary and sufficient condition. Therefore, the resulting closed-loop system becomes robustly stable as σ increases when $\|G_a(s)\|_\infty < 1$, and high gain leads to robust stabilization when the system is quadratically stabilizable. In this example $\|G_b(s)\|_\infty < 1$ is a sufficient condition. Hence there may exist σ which makes the resulting closed-loop system robustly stable when $\|G_b(s)\|_\infty > 1$. However, high gain does not lead to robust stabilization when the system is quadratically stabilizable. Thus this example shows the difficulty of designing robust controllers under uncertainty particularly in the input matrix.

4.8.3 State feedback case (ΔA and ΔB exist)

In this section we consider a design example of an engine test bed[Kawara90] for the system (4.25) in which coefficient matrices are given as follows.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2.500 & 0 \\ 50.00 & -50.00 & -0.1725 \end{bmatrix}, \tag{4.109}$$

$$B = \begin{bmatrix} 0.2490 & 0 \\ 0 & 0.01000 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, E_a = \begin{bmatrix} 0 & 0 & 0.2279 \\ 0 & 0 & 0 \end{bmatrix},$$

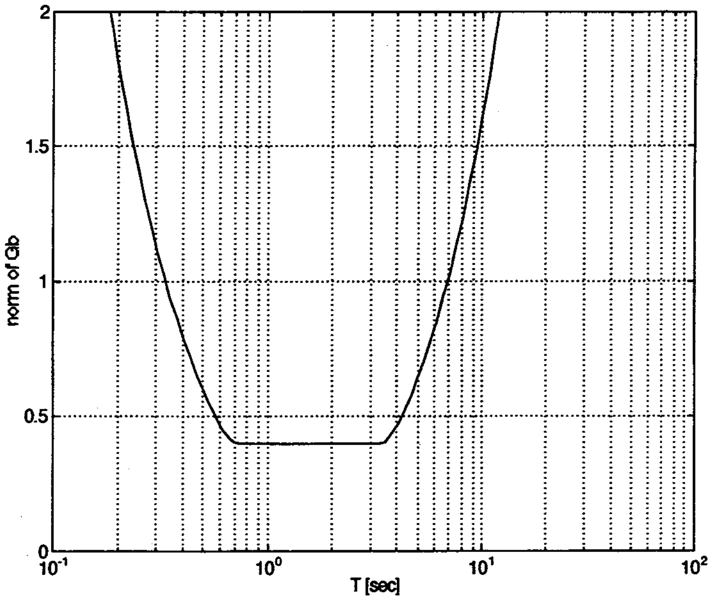

Figure 4.21: $\|G_b(s)\|_\infty$ for time constant T

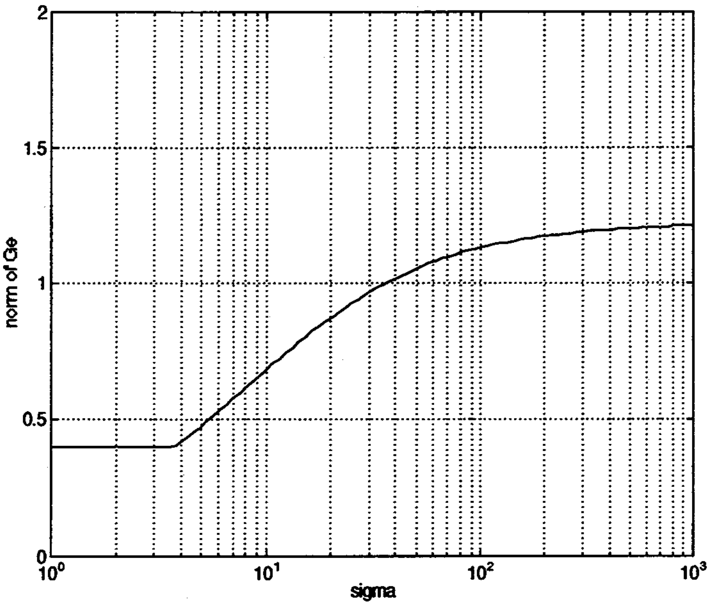

Figure 4.22: $\|G_e(s)\|_\infty$ for σ

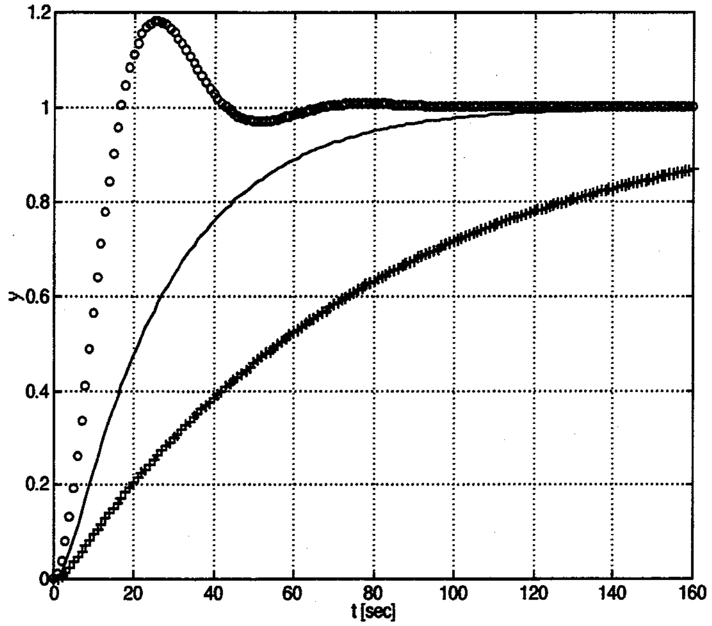

Figure 4.23: Step responses ($T = 8$ [sec]) $\sigma = 10$, $F = -1$:○, $F = 0$:—, $F = 1$:+

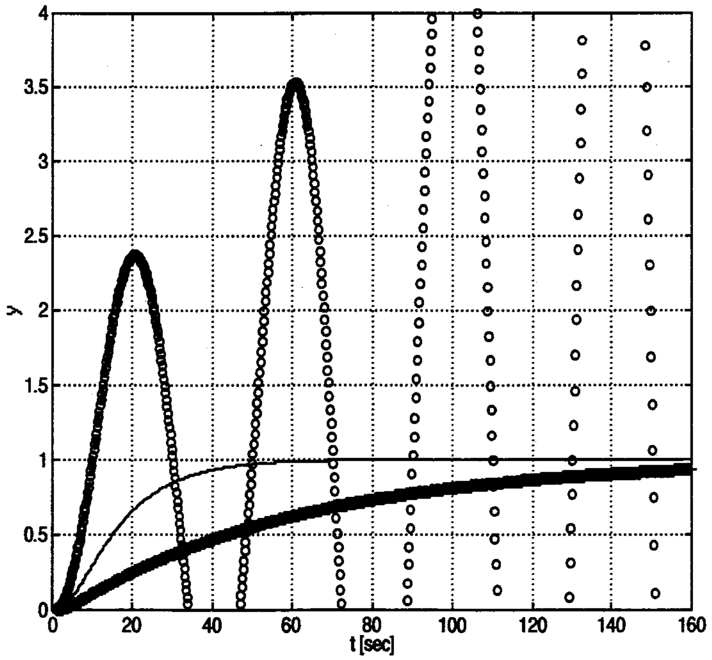

Figure 4.24: Step responses ($T = 8$ [sec]) $\sigma = 100$, $F = -1$:○, $F = 0$:—, $F = 1$:+

$$E_b = \begin{bmatrix} 0 & 0 \\ 0.001550 & 0 \end{bmatrix}$$

Step 1 Set $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Step 2 In this step K is determined using ILQ design method by specifying its design parameters s_1 and s_2. First, $\|U_2U_2^TG_a(s) - G_b(s)\|_\infty$ for $s_1 = s_2$ is depicted in Fig. 4.25, from which and a desirable response specification $s_1 = s_2 = -20$ yielding $\|U_2U_2^TG_a(s) - G_b(s)\|_\infty = 0.00799$ are chosen as specified poles. We thus set desirable tracking characteristics of the nominal system to diag $\left\{\frac{400}{(s+20)^2}, \frac{20}{s+20}\right\}$.

Step 3 From Fig. 4.26, the larger σ is, the robuster the resulting system is. $\sigma = 60$ is chosen as tuning parameters from step responses in Fig. 4.27. From the plot of $\|G_e(s)\|_\infty$ depicted in Fig. 4.26, $\|G_e(s)\|_\infty$ takes the values of 0.0593, 0.011, 0.00799 for $\sigma_1 = \sigma_2 = 20, 100, 10^5$ respectively. These values confirm the asymptotic property as stated in Theorem 4.2, i.e. $\|G_e(s)\|_\infty \to \|\overline{G}_e(s)\|_\infty = 0.00799$.

Step 4 Using V and K obtained above, K_F and K_I are calculated from (4.37).

$$K_F^0 = \begin{bmatrix} 2.0080 \cdot 10^{-1} & 0 & 1.5995 \cdot 10^{-1} \\ 0 & 5.0000 & 0 \end{bmatrix}$$

$$K_I^0 = \begin{bmatrix} 1.6064 & 4.0161 \\ 0 & 1.0000 \cdot 10^2 \end{bmatrix}$$

4.8.4 Observer-based output feedback case (ΔA and ΔB exist)

In this section we consider the same design example of an engine test bed[Kawara90] for the system (4.25) as that in the previous subsection.

Step 1 Set $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Step 2 In this step K is determined using ILQ design method by specifying its design parameters s_1 and s_2. First, $\|U_2U_2^TG_a(s) - G_b(s)\|_\infty$ for $s_1 = s_2$ is depicted in Fig. 4.28, from which and a desirable response specification $s_1 = s_2 = -20$ yielding $\|U_2U_2^TG_a(s) - G_b(s)\|_\infty = 0.00799$ are chosen as specified

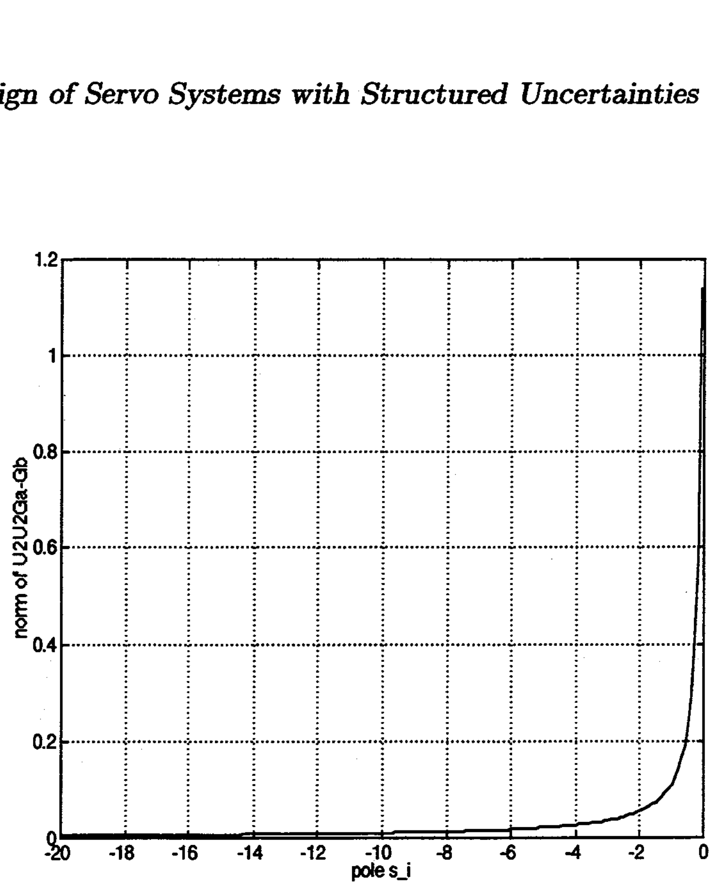

Figure 4.25: $\|U_2U_2^TG_a(s) - G_b(s)\|_\infty$ for specified poles

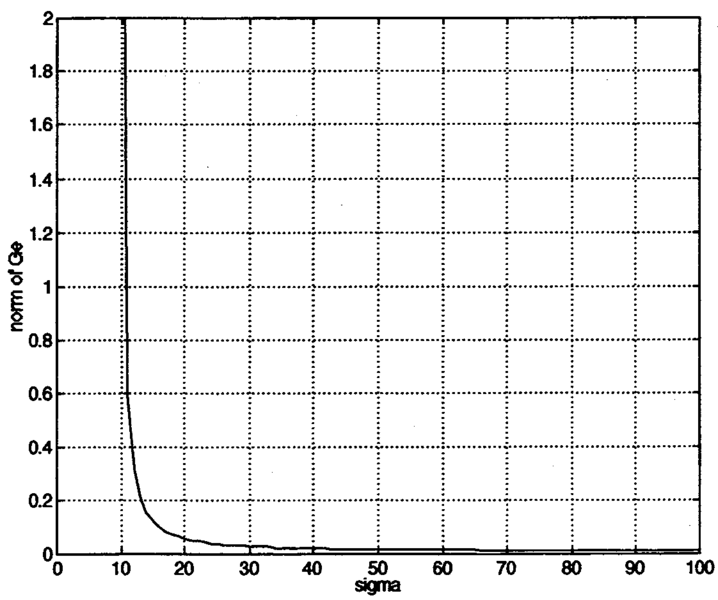

Figure 4.26: $\|G_e(s)\|_\infty$ for specified poles

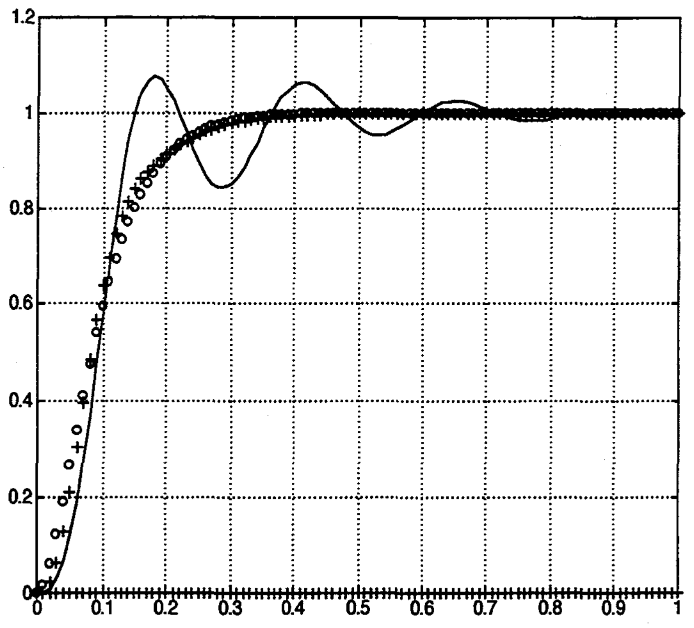

Figure 4.27: Step responses(solid($F = 1$,$\sigma_i = 20$),+($F = 1$,$\sigma_i = 60$) ,o(desired))

poles. We thus set desirable tracking characteristics of the nominal system to diag $\left\{\dfrac{400}{(s+20)^2}, \dfrac{20}{s+20}\right\}$. Next $\|G_c(s)\|_\infty$ is plotted in Fig. 4.29, from which we choose observer poles as, for example, -20 for which $\|G_c(s)\|_\infty = 0.03783$. In Fig. 4.30 $\|\overline{G}_{oe}(s)\|_\infty$ is drawn with $\|\overline{G}_{oe}(s)\|_\infty = 0.03945$ for $s_1 = s_2 = -20$. In Fig. 4.31 the plots of both sides of (4.61) is shown for various values of $\{s_i\}$, and we can see that the evaluation by triangular inequality we used here is not so conservative.

Step 3 From the plot of $\|G_{oe}(s)\|_\infty$ depicted in Fig. 4.32, $\|G_{oe}(s)\|_\infty$ takes the values of 0.10487, 0.044707, 0.039723 for $\sigma_1 = \sigma_2 = 20, 60, 1000$ respectively. These values confirm the asymptotic property as stated in Theorem 4.4, i.e. $\|G_{oe}(s)\|_\infty \to \|\overline{G}_{oe}(s)\|_\infty = 0.03945$.

Step 4 Using V and K obtained above, K_F and K_I are the same gains in Section 4.8.3 calculated from (4.37).

Finally we show in Fig. 4.33 some simulation results of step responses.

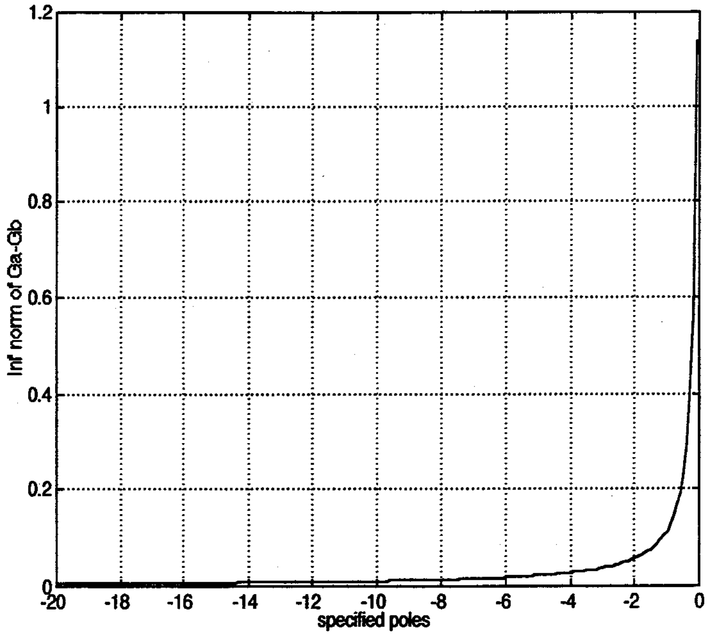

Figure 4.28: $||U_2U_2^TG_a(s) - G_b(s)||_\infty$ for specified poles

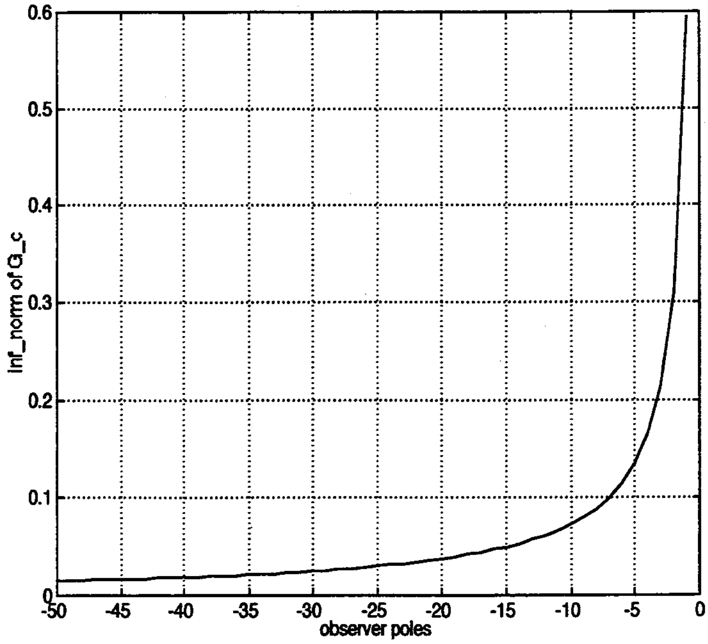

Figure 4.29: $||G_c(s)||_\infty$ for observer poles

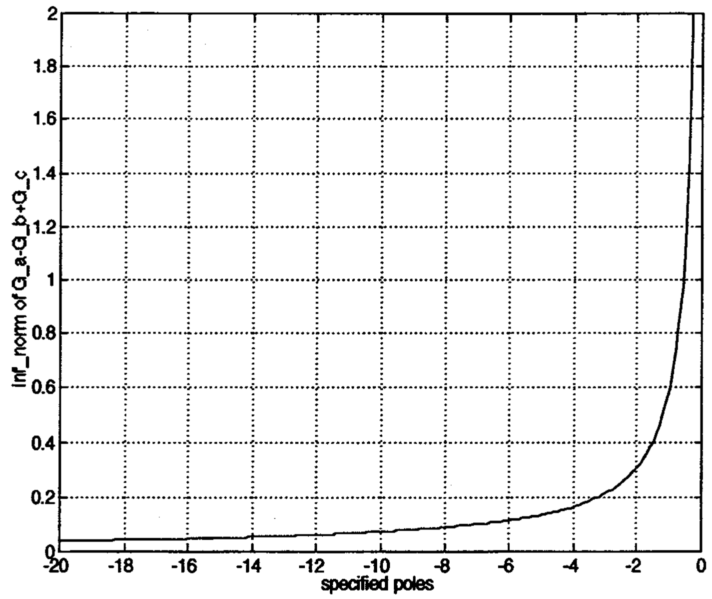

Figure 4.30: $\|U_2U_2^TG_a(s) - G_b(s) + G_c(s)\|_\infty$ for specified poles

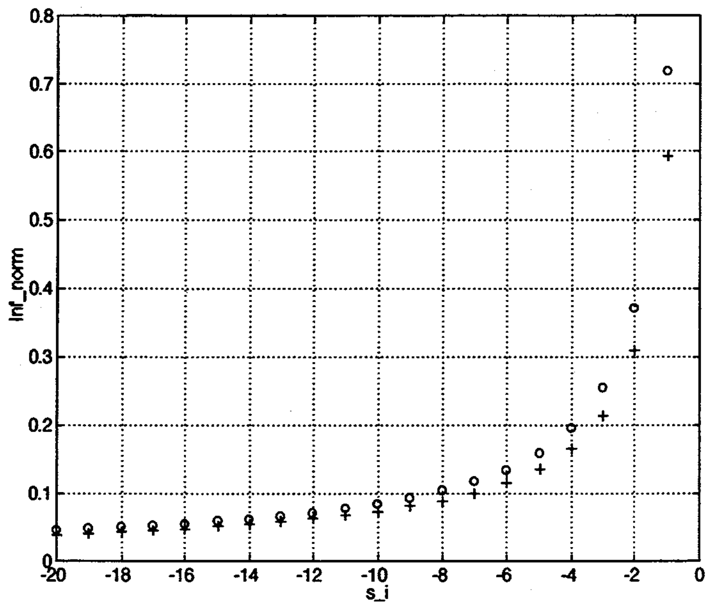

Figure 4.31: o : $\|U_2U_2^TG_a(s)-G_b(s)\|_\infty+\|G_c(s)\|_\infty$, + : $\|U_2U_2^TG_a(s)-G_b(s)+G_c(s)\|_\infty$

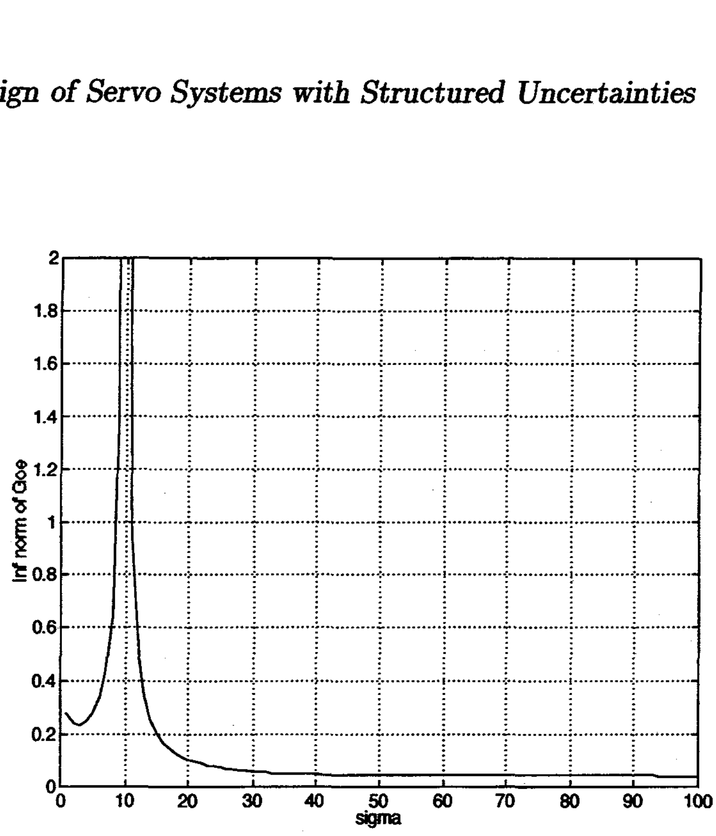

Figure 4.32: $\|G_{oe}(s)\|_\infty$ for σ_i

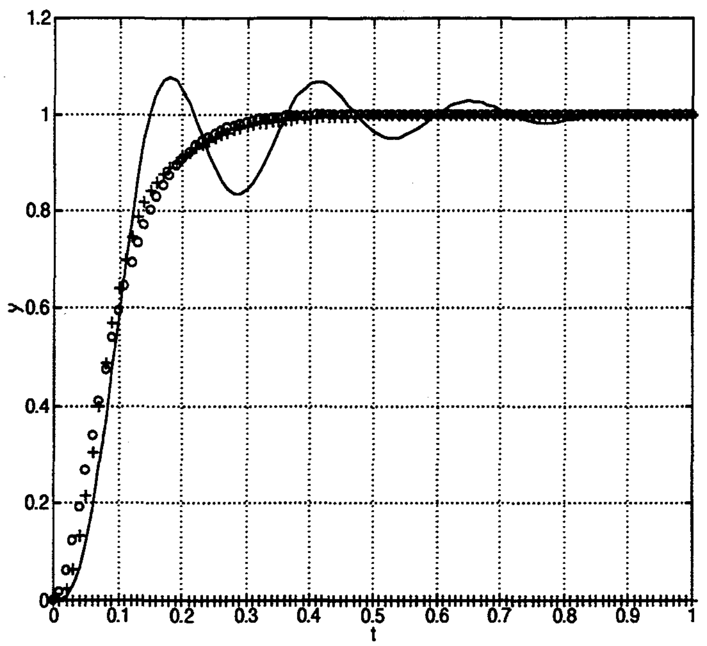

Figure 4.33: Step response (solid($F = 1$, $\sigma_i = 20$), + ($F = 1$, $\sigma_i = 60$), o (desired))

4.9 Summary

We have treated here a design problem of servo systems which achieve robust tracking of step reference inputs under uncertainties entering only in the state matrix of linear time invariant systems. Then the case where uncertainties enter into both the state and input matrices is considered by a similar approach. However, there exists the essential difference between these two cases in that for the latter case a high gain robust controller can be always designed, whenever a robust controller exists, but not for the former case as treated here. This may indicate the difficulty of designing robust tracking controllers under parameter variations particularly in the input matrix. Next we have developed a design method of observer-based robust tracking controllers for structured uncertainties. In this chapter we have proposed some design algorithms in which desirable responses and robust stability are both achieved by proper choice of design parameters to some extent independently. Finally the effectiveness of the proposed design methods are verified via design examples.

Chapter 5

Concluding Remarks

In this thesis we have first investigated the qualitative analysis problem related to robust control, where we discussed the relation between controllability invariance and several kinds of robust stabilizability, and then considered the quantitative synthesis problem related to robust control when we proposed the practical design methods of robust servo systems are proposed.

In this thesis by providing Theorem 3.1 and Corollary 3.1 we clarified the fact that as far as a certain class of interval systems is concerned, the notion of controllability invariance defined here is necessary and sufficient for stabilizability of the interval systems and is necessary but not sufficient for quadratic stabilizability. By this fact we have connected robust stabilizability with a natural extension of the familiar notion of controllability in the linear system theory. Thus, we have found that this notion plays an important role in this robust stabilization problem.

As far as a quantitative problem, we have first treated a design problem of servo systems which achieve robust tracking of step reference inputs under parameter variations entering only in the state matrix of linear time invariant systems. By a similar approach we then consider the case where uncertainties enter into both the state and input matrices is considered . However, there exists an essential difference between these two cases in that for the latter case a high gain robust controller can be always designed, whenever a robust controller exists, but not for the former case as treated here. This may indicate the difficulty of designing robust tracking controllers under parameter variations particularly existing in the input matrix. Next we have developed a design method of observer-based robust tracking controllers for structured uncertainties, and proposed a design algorithm in which desirable responses and robust stability are both achieved by proper choice of design parameters independently to some extent. Finally the effectiveness of the proposed design methods are shown via design examples.

Aside from the results reported above, there are several issues for future researches along the line of the research presented in this thesis which is stated in the following.

- The future research is to clarify the meaning of controllability invariance defined here in the context of the robust stabilization problem and also to investigate its connection with other notions of controllability, for example, the feedback controllability as defined in [Peter90].

- This thesis deals with a class of interval systems which have sign-invariant entries as super-diagonal entries in the state matrix, i.e., standard system. From this restriction, the class is not wide and there exists the stabilizable interval system as indicated in Remark 2.10. Therefore, the class have to be extended by locating sign-invariant entries other than in super-diagonal entries of the state matrix or increasing sign-invariant entries.

- There does not exist an algorithm by which we can judge whether an interval system can be transformed to a standard system or not. Furthermore, if it can be transformed to a standard system, we have to be able to transform it to a standard system. Therefore, an algorithm to achieve the judgment and the transformation stated above is needed. Moreover, deriving the algorithm results in clarifying a class of standard system.

- The condition the a system is controllable and observable is a necessary condition and not a sufficient condition for pole-assignability of a system by a decentralized control. However, controllability invariance and observability invariance are stronger condition than controllability and observability. Therefore, the system which is controllability invariant and observability invariant may be pole-assignable by a decentralized control. It is left as a future study.

- Wei *et al.* derived the robust stabilizability conditions in terms of a geometric pattern with respect to the location of uncertain parameters [Wei90, Hu96, Wei89a, Dai96, Hu97, Wei92, Wei94, Jing96, Wei89c]. They are based on the Lyapunov function with a constant positive definite matrix or making roots of the characteristic polynomial of the closed loop system stable. The other results given in terms of a geometric pattern are based on making the state matrix of the closed loop system stable by state or output feedback in [Akaz87, Ame83, Ame88, Ame94a, Ame94b, Ame96a, Ame97a] and adaptive control in [Naha95],[Koko91],[Kane91][Leit79]. These approaches give robust stability conditions in terms of a geometric pattern. However, the relations between these approaches have not been clarified. Hence, it is important to clarify them.

- In this thesis we have tackled the design problem of constructing a feedback control law analytically for uncertain systems using the parameterization of feedback gains. Recently numerical approaches for control system design using linear matrix inequality have been developed and applied to solve a control problem achieving multiple objectives. Therefore, by using such numerical approaches together with the parameterization of the solutions of the Riccati equations, we may obtained more practical design methods for multiple objectives.

- The result of obtained here on the design of robust servo systems using observer-based output feedback provides only a sufficient condition since the result is based on the previous result for the state feedback case. Hence, we would better consider the problem based on the result for the output feedback case in order to overcome the inherent difficulty due to the fact that the separation theorem as in linear quadratic gaussian control problem does not hold.

Appendix A

Preliminary Lemmas

Lemma A.1 *(Schur Complement) p.255 in [Skel97]*
The following three conditions are equivalent:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} > 0 \tag{A.1}$$

$$A_{22} > 0, \quad A_{11} - A_{12}A_{22}^{-1}A_{12}^T > 0 \tag{A.2}$$

$$A_{11} > 0, \quad A_{22} - A_{12}^T A_{11}^{-1} A_{12} > 0 \tag{A.3}$$

Lemma A.2 *(Bounded Real Lemma)[Zhou88a]*
Given a system $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$*, the following statements are equivalent:*

1. *A is a stability matrix and $\gamma > \|C(sI - A)^{-1}B + D\|_\infty$.*

2. *$\gamma^2 I - D^T D > 0$ and there exists a positive definite symmetric matrix $P > 0$ such that*

$$A^T P + PA + (PB + C^T D)(\gamma^2 I - D^T D)^{-1}(B^T P + D^T C) + C^T C < 0. \tag{A.4}$$

Appendix B

Proof of Fact

B.1 Proof of Fact 4.2

(Necessity)

If $u = -K_e x_e$ is a central quadratically stabilizing control law, there exists $P_e >$ and $R > 0$ such that the following equations holds

$$\begin{aligned} K_e &= R^{-1} B_e^T P_e & \text{(B.1)} \\ P_e A_e + A_e^T P_e &- P_e B_e R^{-1} B_e^T P_e + P_e D_e D_e^T P_e + E_{ea}^T E_{ea} + \Gamma^T Q \Gamma = 0 & \text{(B.2)} \end{aligned}$$

from Lemma 4.1. Premultiplying $P_e B_e$ and post-multiplying $B_e^T P_e$ by (B.1) yields

$$P_e B_e R^{-1} B_e^T P_e = -\frac{1}{2} P_e B_e K_e - \frac{1}{2} K_e^T B_e^T P_e. \qquad \text{(B.3)}$$

Substituting this relation into (B.2), we have

$$\begin{aligned} -P_e A_e - A_e^T P_e &+ \frac{1}{2} P_e B_e K_e + \frac{1}{2} K_e^T B_e^T P_e \\ &- P_e D_e D_e^T P_e - E_{ea}^T E_{ea} = Q > 0. \end{aligned}$$

From this inequality the following inequality is derived.

$$\begin{aligned} P_e \left(\frac{1}{2} B_e K_e - A_e \right) &+ \left(\frac{1}{2} B_e K_e - A_e \right)^T P_e \\ &- P_e D_e D_e^T P_e - E_{ea}^T E_{ea} > 0 \end{aligned}$$

(Sufficiency)

Suppose that there exists $R > 0$ and $P_e > 0$ which satisfy (4.15) and (4.16). Substituting (4.15) into (4.16) and defining $Q > 0$ in the following, we find that there exists

$R > 0$ and $P_e > 0$ such that

$$\begin{aligned} -Q &= P_e A_e + A_e^T P_e - P_e B_e R^{-1} B_e^T P_e \\ &\quad + P_e D_e D_e^T P_e + E_{ea}^T E_{ea}. \end{aligned}$$

The proof is complete.

Appendix C

Augmented System

The augmented system (4.1) is derived in this Appendix. In order to construct the control system in Fig. 4.2, we consider the augmented system in Fig. C.1.

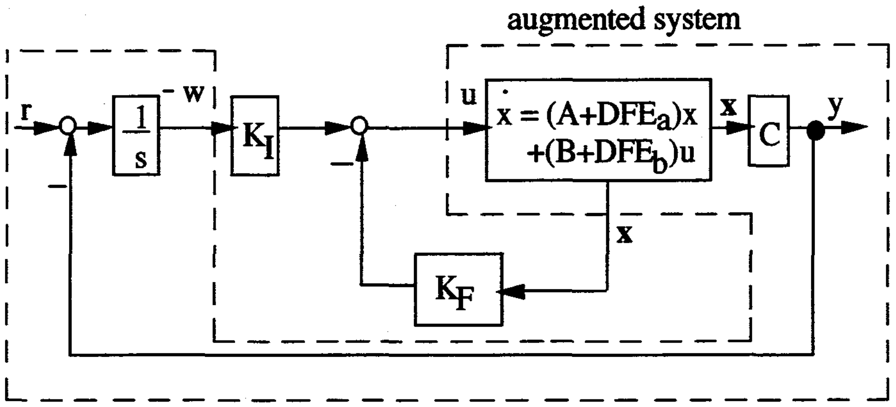

Figure C.1: Augmented system

In this Appendix we consider the system with norm bounded structured uncertainties introduced in Chapter 2, which is rewritten in the following for convenience.

$$\dot{x}(t) = [A + \Delta A]x(t) + [B + \Delta B]u(t) \tag{C.1}$$

$$y(t) = Cx(t) \tag{C.2}$$

$$\Delta A = DFE_a, \quad \Delta B = DFE_b$$

From Fig. C.1, the equation with respect to w is given as follows.

$$\dot{w} = y - r = Cx - r \tag{C.3}$$

From (C.1),(C.3), the state space representation of the augmented system in Fig. C.1 is written in the following.

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A + DFE_a & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} B + DFE_b \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -I \end{bmatrix} r \tag{C.4}$$

The following relation holds in steady state because uncertainties are time invariant.

$$\begin{bmatrix} 0 \\ r \end{bmatrix} = \begin{bmatrix} A + DFE_a & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} \tag{C.5}$$

where $\bar{x}$, $\bar{u}$ are the limiting state and input as $t \to \infty$. Arranging this equation yields

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A + DFE_a & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{w} \end{bmatrix} + \begin{bmatrix} B + DFE_b \\ 0 \end{bmatrix} \bar{u} + \begin{bmatrix} 0 \\ -I \end{bmatrix} r. \tag{C.6}$$

where $\bar{w}$ is the limiting state as $t \to \infty$. From (C.4) and (C.6), we have

$$\frac{d}{dt} \begin{bmatrix} x - \bar{x} \\ w - \bar{w} \end{bmatrix} = \begin{bmatrix} A + DFE_a & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} x - \bar{x} \\ w - \bar{w} \end{bmatrix} + \begin{bmatrix} B + DFE_b \\ 0 \end{bmatrix} (u - \bar{u}) \tag{C.7}$$

Defining $\xi_e = \begin{bmatrix} x - \bar{x} \\ w - \bar{w} \end{bmatrix}$, the above equation becomes

$$\dot{\xi}_e = \begin{bmatrix} A + DFE_a & 0 \\ C & 0 \end{bmatrix} \xi_e + \begin{bmatrix} B + DFE_b \\ 0 \end{bmatrix} (u - \bar{u}) \tag{C.8}$$

Then a state feedback control law is given by

$$u - \bar{u} = -K_F(x - \bar{x}) - K_I(w - \bar{w}) \tag{C.9}$$

Noting that $\bar{u} = -K_F\bar{x} - K_I\bar{w}$ holds with respect to the limiting states and inputs yields

$$u = -K_F x - K_I w \tag{C.10}$$

Thus it turns out that we can derive the same feedback gains K_F, K_I even if $\xi_e = \begin{bmatrix} x - \bar{x} \\ w - \bar{w} \end{bmatrix}$ and $u - \bar{u}$ is replaced with $\xi_e = \begin{bmatrix} x \\ w \end{bmatrix}$ and u. Therefore, we consider the augmented system (4.1) in Chapter 4.

Appendix D

Calculation of Gains of ILQ Design Method

Here K_F^0 and K_I^0 are determined as the ILQ principal gains for the system S_F indicated by the broken line in Fig.4.3 so that the closed loop output responses approach a specified responses as $\{\sigma_i\}$ increases, i.e., $\Sigma \to \infty$. Note that the definition in (4.37) yields

$$\begin{bmatrix} K_F^0 & K_I^0 \end{bmatrix} = \begin{bmatrix} K & I \end{bmatrix} \Gamma^{-1}. \tag{D.1}$$

Denote the i-th row of C by c_i $(1 \leq i \leq m)$ and define the following indices d_i, d and matrix M:

$$\begin{aligned} d_i &:= \min\{\, k \,|c_i A_F^k B \neq 0\} \\ &= \min\{\, k \,|c_i A^k B \neq 0\} \quad (1 \leq i \leq m) \end{aligned} \tag{D.2}$$

$$d := d_1 + d_2 + \cdots + d_m \tag{D.3}$$

$$M := \begin{bmatrix} c_1 A_F^{d_1} B \\ \vdots \\ c_m A_F^{d_m} B \end{bmatrix} = \begin{bmatrix} c_1 A^{d_1} B \\ \vdots \\ c_m A^{d_m} B \end{bmatrix} \tag{D.4}$$

and make the following assumption.

Assumption D.1 *The nominal system (2.3) is minimum phase with* $\det M \neq 0$.

Remark D.1 *ILQ design method was extended and can be applied to non-minimum phase [Shimo93]. However, the calculation only for minimum phase systems is introduced for simplicity because systems treated in design examples of this thesis are minimum phase.*

Let $\boldsymbol{K}_i$ be a set of integers with $d_i + 1$ elements such that

$$\{k\}_{k=1}^{d+m} = \boldsymbol{K}_1 \cup \boldsymbol{K}_2 \cup \cdots \cup \boldsymbol{K}_m \tag{D.5}$$

and define two polynomials for each $\boldsymbol{K}_i$,

$$\phi_i(s) := \prod_{k \in \boldsymbol{K}_i} (s - s_k) \quad 1 \le i \le m \tag{D.6}$$

$$\psi_i(s) := \{\phi_i(s) - \phi_i(0)\}/s \quad 1 \le i \le m \tag{D.7}$$

where $\{s_k\,, k \in \boldsymbol{K}_i\}_{i=1}^m$ are those stable poles specified freely in the pole assignment and the remaining poles should be specified by all the system zeros. With these assumption and definitions, we can state the analytical expression of the ILQ principal gains K_F^0 and K_I^0 as well as the pole assignment gain K as follows.

$$K = M^{-1}N \tag{D.8}$$

$$K_F^0 = M^{-1}N_0, \quad K_I^0 = M^{-1}M_0 \tag{D.9}$$

where

$$N := \begin{bmatrix} c_1\phi_1(A_F) \\ \vdots \\ c_m\phi_m(A_F) \end{bmatrix}$$

$$N_0 := \begin{bmatrix} c_1\psi_1(A_F) \\ \vdots \\ c_m\psi_m(A_F) \end{bmatrix} = \begin{bmatrix} c_1\psi_1(A) \\ \vdots \\ c_m\psi_m(A) \end{bmatrix} \tag{D.10}$$

$$M_0 := \mathrm{diag}\{\phi_1(0), \cdots, \phi_m(0)\}.$$

This expression yields the following result on which the second features of ILQ design method is based.

Theorem D.1 *Under the assumption, the step response of the nominal closed loop system shown in Fig. 4.3 approaches that of a system with the transfer function* $G_d(s)$ *as* $\{\sigma_i\} \to \infty$*, where*

$$G_d(s) := \mathrm{diag}\left\{\frac{\phi_i(0)}{\phi_i(s)}\right\}. \tag{D.11}$$

Furthermore, this property also holds even if we use the observer-based output feedback (4.4),(4.5) instead of the state feedback (4.3).

This result suggests us to use $\{s_k,\ k \in K_i\}$ in (D.6) as design parameters for specifying the i-th output response, and $\{\sigma_k\}$ in (4.34) as those tradeoff parameters mentioned earlier of an ILQ servo system shown in Fig. 4.3.

Appendix E

Supplement of Illustrative Examples

E.1 Antisymmetric stepwise configuration and generalized antisymmetric stepwise configuration

All 4-dimensional generalized antisymmetric stepwise configurations are shown as 1) and 2).

1) Systems which have a generalized antisymmetric stepwise configuration but do not have an antisymmetric stepwise configuration, i.e., are stabilizable but not quadratically stabilizable.

$$A = \begin{bmatrix} 0 & \theta_1 & 0 & a_{14} \\ 0 & a_{22} & \theta_2 & 0 \\ 0 & 0 & 0 & \theta_3 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \theta_4 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & \theta_1 & 0 & 0 \\ 0 & a_{22} & \theta_2 & 0 \\ 0 & a_{32} & a_{33} & \theta_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, b = \begin{bmatrix} b_1 \\ 0 \\ 0 \\ \theta_4 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & \theta_1 & a_{13} & 0 \\ 0 & 0 & \theta_2 & 0 \\ 0 & a_{32} & a_{33} & \theta_3 \\ 0 & 0 & a_{43} & 0 \end{bmatrix}, b = \begin{bmatrix} b_1 \\ 0 \\ 0 \\ \theta_4 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & \theta_1 & 0 & a_{14} \\ 0 & a_{22} & \theta_2 & 0 \\ 0 & 0 & 0 & \theta_3 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}, b = \begin{bmatrix} b_1 \\ 0 \\ 0 \\ \theta_4 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & \theta_1 & a_{13} & a_{14} \\ 0 & 0 & \theta_2 & 0 \\ 0 & 0 & a_{33} & \theta_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ 0 \\ \theta_4 \end{bmatrix}$$

As shown above, all the controllability invariant systems must have a part of zero entries according to Lemma 3.1.

2) Systems which have a generalized antisymmetric stepwise configuration and a antisymmetric stepwise configuration.

$$A=\begin{bmatrix} a_{11} & \theta_1 & 0 & 0 \\ a_{21} & a_{22} & \theta_2 & 0 \\ a_{31} & a_{32} & a_{33} & \theta_3 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, b=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \theta_4 \end{bmatrix}, \quad A=\begin{bmatrix} 0 & \theta_1 & a_{13} & 0 \\ 0 & 0 & \theta_2 & 0 \\ a_{31} & a_{32} & a_{33} & \theta_3 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, b=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \theta_4 \end{bmatrix}$$

$$A=\begin{bmatrix} 0 & \theta_1 & a_{13} & a_{14} \\ 0 & 0 & \theta_2 & 0 \\ 0 & 0 & a_{33} & \theta_3 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, b=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \theta_4 \end{bmatrix}, \quad A=\begin{bmatrix} 0 & \theta_1 & a_{13} & a_{14} \\ 0 & 0 & \theta_2 & 0 \\ 0 & 0 & a_{33} & \theta_3 \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}, b=\begin{bmatrix} b_1 \\ 0 \\ 0 \\ \theta_4 \end{bmatrix}$$

$$A=\begin{bmatrix} 0 & \theta_1 & a_{13} & a_{14} \\ 0 & 0 & \theta_2 & a_{24} \\ 0 & 0 & 0 & \theta_3 \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix}, b=\begin{bmatrix} b_1 \\ 0 \\ 0 \\ \theta_4 \end{bmatrix}, \quad A=\begin{bmatrix} 0 & \theta_1 & a_{13} & a_{14} \\ 0 & 0 & \theta_2 & a_{24} \\ 0 & 0 & 0 & \theta_3 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}, b=\begin{bmatrix} b_1 \\ b_2 \\ 0 \\ \theta_4 \end{bmatrix}$$

$$A=\begin{bmatrix} 0 & \theta_1 & a_{13} & a_{14} \\ 0 & 0 & \theta_2 & a_{24} \\ 0 & 0 & 0 & \theta_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, b=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \theta_4 \end{bmatrix}$$

These are all 4-dimensional systems that are both stabilizable and quadratically stabilizable.

E.2 Robust stabilizability conditions in terms of geometric pattern

In Chapter 4 some robust stabilizability conditions in terms of geometric pattern are introduced, for example, antisymmetric stepwise configuration(ASC) [Wei90], generalized antisymmetric stepwise configuration(GAS) [Wei92, Wei94], pure feedback form(PFF)

[Koko91], extended matching condition structure(EMC) [Kane91], Delay-independent stabilizability condition(DIS) [Ame96a]. Here we make comparison between them via examples.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & a_{33} \end{bmatrix}, b = \begin{bmatrix} b_1 \\ 0 \\ 1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & a_{22} & 1 \\ 0 & 0 & 0 \end{bmatrix}, b = \begin{bmatrix} b_1 \\ 0 \\ 1 \end{bmatrix}$$

ASC, GAS   not ASC, GAS

$$A = \begin{bmatrix} a_{11} & 1 & 0 \\ a_{21} & a_{22} & 1 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ a_{21} & a_{22} & 1 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

ASC, GAS, PFF, DIS   ASC, GAS, PFF, EMC, DIS

$$A = \begin{bmatrix} 0 & 1 & a_{13} \\ 0 & 0 & 1 \\ 0 & a_{32} & a_{33} \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & a_{13} \\ 0 & 0 & 1 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

ASC, GAS, DIS   ASC, GAS

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List of Publications

1. Takao Fujii, Taro Tsujino and Hideaki Uematsu, "The Design of Robust Servo Systems Based on the Quadratic Stability Theory", Trans. of the Institute of Systems, Control and Information Engineers, vol.4, no.11, pp.462-472, 1991 (in Japanese).

2. Takao Fujii and Taro Tsujino, "An Inverse LQ Based Approach to the Design of Robust Tracking controllers for Linear Uncertain Systems", Proc. of International Symposium on the Mathematical Theory of Networks and Systems, Kobe, Japan, June, 1991, Recent Advances in Mathematical Theory of Systems, Control, Networks and Signal processing I, Mita Press, pp.391-396, 1992.

3. Takao Fujii and Taro Tsujino, "An Inverse LQ Based Approach to the Design of Robust Tracking System with Quadratic Stability", Proc. of the Workshop on Robust Control, Tokyo, Japan, June, 1991, Robust Control, Lecture Notes on Control and Information Sciences, Springer Verlag, no.183, pp.106-113, 1992.

4. Taro Tsujino, Takao Fujii and Kehui Wei, "On the Connection between Controllability and Stabilizability of Linear Systems with Structural Uncertain parameters", Automatica, vol.29, no.1, pp.7-12, 1993.

5. Taro Tsujino and Takao Fujii, "The Design Method of Robust Servo Systems with Quadratic Stability using Output Feedback", Proc. of the IEEE Singapore International Symposium on Control Theory and Applications, pp. 52-56, Singapore, July 1997.

6. Taro Tsujino and Takao Fujii, "Design of Observer-based Robust Servo Systems from the Viewpoint of an Inverse Problem", Trans. of the Institute of Systems, Control and Information Engineers, accepted in 1998, to be published in vol.12, no.6, 1999.