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UNIPOTENT ELEMENTS AND CHARACTERS OF FINITE CHEVALLEY GROUPS

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Introduction

Let $\mathfrak{G}$ be a connected semisimple linear algebraic group defined over an algebraically closed field K of characteristic $p>0$, and σ a surjective endomorphism of $\mathfrak{G}$ such that the group $\mathfrak{G}_\sigma$ of elements fixed by σ is finite. The finite groups $\mathfrak{G}_\sigma$ obtained in this manner can be classified as follows (Steinberg [20]): If $\mathfrak{G}$ is simple, $\mathfrak{G}_\sigma$ is either the group of rational points of a F-form of $\mathfrak{G}$ for an appropriate finite field F or one of the groups defined by M. Suzuki and R. Ree. If $\mathfrak{G}$ is not simple, $\mathfrak{G}_\sigma$ is essentially a direct product of the groups mentioned above.

In this paper, a finite group G is called a finite Chevalley group[1] if it can be realized as $\mathfrak{G}_\sigma$ for some $\mathfrak{G}$ and σ. Let (G, B, N, S) be a Tits system (or BN-pair) associated to a finite Chevalley group G. We denote by W its Weyl group. Let G^1 be the set of unipotent elements (or p-elements) of G and U the p-Sylow subgroup of G contained in B. The main purpose of this paper is to establish the following two results:

(I) *Let w be an arbitrary element of W, and w_S the element of W of maximal length. Then the number of unipotent elements contained in the double coset BwB is $|BwB\cap w_S U w_S^{-1}|\,|U|$, which can be written explicitly as a polynomial in $q_s=|BsB/B|$ $(s\in S)$*[2]. (As a corollary, we obtain $|G^1|=|U|$, a result of Steinberg [20].)

(II) *Assume that the characteristic p is good* (see Definition 6.2) *for $\mathfrak{G}$. Let g be an element of $G=\mathfrak{G}_\sigma$, and C a regular unipotent conjugacy class of G. Then the number $|Bg\cap C|$ depends neither on g nor C.*

As far as the author knows, these results are new even for $G=SL_n(F)$ with F a finite field. In this case an arbitrary prime is good and a unipotent element

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1) This definition is slightly different from the one given, for example, in [19]. But such difference is not essential for our purpose.

2) For a finite set A, $|A|$ denotes the number of its elements.

is regular if and only if its Jordan normal form consists of a single block.

The proof of (I) is given in §4. The main tool is the construction of the Steinberg character of G due to C.W. Curtis. In §5, combining (I) with an elementary lemma 2.4 we show

(III) *Let X be an irreducible complex character of G contained in the character $i[1_B|B\to G]$inducedfrom the trivial character 1_B of B. Then*

$$\sum_{u\in G^1}\chi(u) = |U|\hat{\chi}(1),$$

where $\hat{X}$ is the "dual character" (see Definition 2.8) *of* χ.

In particulr, if % is trivial, $\hat{\%}$ is the Steinberg character, whose degree is known to be $|U|$. Hence we obtain the equality $|G^1| - |U|^2$ again.

It may be remarked that all the properties of G required for the proofs of (I) and (III) are formal consequences of the following two facts:
(1) (B, N) is a split BN-pair (see [2; part B]).
(2) The commutator relations (Proposion 1.3(c)) are satisfied.

In §6 after recalling some known facts on regular unipotent elements, we prove a key lemma 6.10. As the first application of this, we obtain

(IV) *Assume tht $\mathfrak{G}$ is adjoint and p is goodfor ®. Let % be an irreducible cuspidal character of $G=\mathfrak{G}_\sigma$, and u a regular unipotent element of G. Then $\chi(u)=\pm 1$ if χ is contained in the character inducedfrom a linear character of U in "general position" in the sense of Gel'fand and Graev* [10], *and $\chi(u)=0$ otherwise.*

The proof of (II) is given in §7. We first prove the following result.

(V) *Assume thatp isgoodfor $\mathfrak{G}$. Let χ be a non-trivial irredubcile character of $G=\mathfrak{G}_\sigma$ contained in $i[1_B|B\to G]$. Then % vanishes on the set of regular unipotent elements of G.*

The main tool for the proof of (V) is Lemma 6.10 again. We also use a result (Theorem 3.4) in §3. It allows us to assume that © is adjoint, in which case the set of regular unipotent elements of $\mathfrak{G}_\sigma$ forms a single conjugacy class. Combining (V) with Lemma 2.4 we obtain (II).

It is quite likely that the main results (I) (II) reflect interesting relations between the variety of unipotent elements and the Bruhat decomposition of $\mathfrak{G}$.

Notations. Let A be a set. If σ is a transformation of A, A_σ denotes the set of fixed points of σ. If f is a mapping from A into another set and B is a subset of A, $f|B$ denotes the restriction of / to B. Let G be a group and H a subset of G. Then $C_G(H)$ denotes the conjugacy class of H. Let G be finite. The inner product for complex valued functions /, g on G is defined by $(/, g)_G =$ I $G|^{-1}\sum_{x\in G}f(x)\overline{g(x)}$. Let H be a subgroupf G and % a character of H. The character of G induced from % is denoted by $i[\chi | H \rightarrow G]$.

1. Finite Che valley groups $\mathfrak{G}_\sigma$

Let $G=\mathfrak{G}_\sigma$ be as in Introduction. In this section we recall some known facts about G and establish some notations frequently used in the paper. References are Steinberg [19], [20] and Bourbaki [3].

Let S3 be a Borel subgroup of @, and $\mathfrak{T}$ a maximal torus of © contained in S3. We can choose S3 and £ to be fixed by σ. Then the unipotent radical $\mathfrak{U}$ of S3 and the normalizer $\mathfrak{N}$ of $\mathfrak{T}$ in © are also fixed by σ. We shall write B, T, U and N for the groups $\mathfrak{B}_\sigma$, $\mathfrak{T}_\sigma$, $\mathfrak{U}_\sigma$ and $\mathfrak{N}_\sigma$ respectively. Let $\mathfrak{W}=\mathfrak{N}/\mathfrak{T}$, the Weyl group of © with respect to $\mathfrak{T}$. Then σ acts naturally on $\mathfrak{W}$ and the group $W=\mathfrak{W}_\sigma$ of fixed points is called the Weyl group of G (with respect to T). It is known that W is canonically isomorphic to N/T.

Let $X(\mathfrak{T})$be the character module of $\mathfrak{T}$, and Σ C$X(\mathfrak{T})$ the root system of © with respect to $\mathfrak{T}$. For $\alpha \in \Sigma$ there is an isomorphism x_α of the additive group (of K) onto a closed subgroup $\mathfrak{U}_\alpha$ of © such that

(1.1) $$t x_\alpha(k) t^{-1} = x_\alpha(\alpha(t) k) \qquad (t \in \mathfrak{T}, k \in \text{tf})\ .$$

Choose an order on Σ so that $\mathfrak{U}=\prod_{\alpha>0}\mathfrak{U}_\alpha$ Let Σ^+ and Π be the set of positive and simple roots respectively. We denote by σ* the dual action of σ$|$ T on the real vector space $V=X(\mathfrak{T})\otimes_Z \boldsymbol{R}$.

Lemma 1.1. ([20; §11]) *Let the notations be as above.*

(a) *There exists a permutation p of* Σ *andfor each* $\alpha \in \Sigma$ *a power* $q(\alpha)$ *ofp such that* $\sigma^* \rho \alpha = q(\alpha) \alpha$.

(b) $\sigma x_\alpha(k) = x_{\rho\alpha}(c_\alpha k^{q(\alpha)})$*for some* $c_\alpha \in K^*$ *and all* $k \in K$.

(c) Σ^+ *and* Π *are stable under p.*

(d) *Let* π *be a p-orbít of* Π. *Then* $\prod_{\alpha\in\pi} q(\alpha) > 1$.

For each p-orbit π of Π, let Σ_π^+ be the set of positive roots which are linear combinations of the elements of π. Then $R' = \{w \Sigma_\pi^+ \,|\, w \in W, \pi$ is a ρ-orbit of $\Pi\}$ forms a partition of Σ.

We fix a W-invariant positive definite inner product on $V=X(\mathfrak{T})\otimes_Z \boldsymbol{R}$. Then $\mathfrak{W}$ can be identified with the Weyl group $W(\Sigma)$ of the root system Σ.

Proposition 1.2. ([19; §11)]) *Consider the projections a of roots a on the subsapce* V_{σ^*} *of V.*

(a) *Let* α_1 *and* α_2 *be roots. Then* $\bar{\alpha}_1$ *is a positive multiple of* α_2 *if and only if there exists an element* $a \in R'$ *containing* α_1 *and* α_2.

(b) *For each* $a \in R'$, *let a be the shortest vector in* $\{\bar{\alpha} \,|\, \alpha \in a\}$. *Then* $R = \{\bar{a} \,|\, a \in R'\}$ *is a reduced root system in* V_{σ^*}.

(c) *The Weylgroup W of G is canonically isomorphic to the Weylgroup $W(R)$ of the root system R.*

By Proposition 1.2 we can identify R with R and W with $W(R)$. For $a\in R$ we write w_a for the corresponding reflection. Put $R(S)=\{\Sigma_\pi^+ \mid \pi$ is a p-orbit of $\Pi\}$. This is a set of fundamental roots (or base) of R. We denote by R^+ the set of positive roots with respect to $R(S)$. Let $S=\{w_a \mid a\in R(S)\}$. Then (W, S) is a Coxeter system. Hence a reduced decomposition $s(w)=(s_1, s_2, \cdots, s_l)(s_i\in S)$ and the length $l(w)$ of $w\in W$ can be defined (see [3]). The element of W of maximal length is denoted by w_S.

Proposition 1.3. ([19; §11]) *For $a\in R$ $(=R')$, let $\mathfrak{U}_a=\prod_{\alpha\in a}\mathfrak{U}_\alpha$.*

(a) *$\mathfrak{U}_a$ is σ-stable.*

(b) *Let $U_a=(\mathfrak{U}_a)_\sigma$ and $q_a=|U_a|$. Then $q_a=\prod_{\alpha\in a}q(\alpha)$, where $q(\alpha)$'s are defined by Lemma 1.1 (a).*

(c) *If a, $b\in R$ and $a\neq\pm b$, the commutator (U_a, U_b) is contained in $\prod U_{ia+jb}$, where the product is taken over all roots $ia+jb$ $(i, j>0)$ arranged in some fixed order.*

(d) *$wU_aw^{-1}=U_{wa}$ for any $w\in W$ and $a\in R$.*

For $w\in W$, let $R_w^+=\{a\in R^+ \mid wa>0\}$, $R_w^-=\{a\in R^+ \mid wa<0\}$, $U_w^+=U\cap w^{-1}Uw$ and $U_w^-=U\cap w^{-1}U^-w$, where $U^-=w_SUw_S^{-1}$.

Lemma 1.4. ([19; §11]) *Let w be any element of W.*

(a) $U=U_w^+U_w^-$ and $U_w^+\cap U_w^-=\{1\}$.

(b) *$U_w^+=\prod_a U_a (a\in R_w^+)$ with the product taken in some fixed order and there is uniqueness of expression on the right.*

The quadruplet (G, B, N, S) is a Tits system. In particular G has the Bruhat decomposition:

$$G=\bigcup_{w\in W}BwB \qquad \text{(disjoint union)}.$$

Lemma 1.5. ([20; 11.1]) *If $\{n_w \mid w\in W\}$ is a system of representatives for $W=N/T$, $BwB=Bn_wU_w^-$ and each element $x\in G$ can be expressed uniquely in the form $x=bn_wu$ with $b\in B$, $w\in W$ and $u\in U_w^-$.*

A subgroup of G which is conjugate to B is called a Borel subgroup. A subgroup of G which contains a Borel subgroup is called a parabolic subgroup.

Proposition 1.6. ([3; Ch. 4, §2]) *For each subset X of S, Let W_X be the subgroup of W generated by X. Then*

(a) *$P_X=BW_XB$ is a parabolic subgroup of G containing B. Conversely, any parabolic subgroup containing B may be obtained in this manner.*

(b) *Two parabolic subgroups* P_X *and* $P_Y(X, Y \subset S)$ *are conjugate to each other if and only if* $X= Y$.

(c) *For* $X, Y \subset S$, *there is a bijection between* $W_X \backslash W / W_Y$ *and* $P_X \backslash G / P_Y$ *given by* $\Xi \to B\Xi B$, $\Xi \in W_X \backslash W / W_Y$.

Lemma 1.7. *Let X be an arbitrary subset of S, and* w_X *the element of* W_X *of maximal length. Then the normalizer of* $U^+_{w_X}$ *in G is the parabolic subgroup* P_X.

Proof. Let 57 be the normalizer of $U^+_{w_X}$ in G. By Lemma 1.4 (b) and Proposition 1.3(c), it sufficiesto show that n_w $(w \in W)$ is contained in 57 if and only if w is contained in W_X. By Lemma 1.4 (b), $U^+_{w_X} = \prod_a U_a$ $(a \in R^+_{w_X})$. Hence $n_w \in \mathfrak{N}$ if and only if $R^+_{w_X} \subset R^+_w$, i.e. $R^-_{w_X} \supset R^-_w$. By [3; p. 158, Cor. 2], this is the case if and only if $w \in W_X$. This proves the lemma.

DEFINITION 1.8. Let X be an arbitrary subset of S. Put $V_X = U^+_{w_X}$. We call V_X the *unipotent radical* of the parabolic subgroup P_X. In general, the *unipotent radical* V_P of a parabolic subgroup $P = gP_Xg^{-1} (g \in G)$ is defined by $V_P = gV_Xg^{-1}$. (The well-definedness of V_P follows from Lemma 1.7 and the fact that P is its own normalizer in G.)

Lemma 1.9. ([3; p. 37, Ex. 3]) *Let* $X, Y \subset S$. *There exists a unique element of minimal length in each* (W_X, W_Y)*-coset in* W. *Moreover, the following conditions for an element w of W are equivalent*:

(1) *w is the element of minimal length in* $W_X w W_Y$.

(2) $l(w_1w) = l(w_1) + l(w)$ *and* $l(ww_2) = l(w) + l(w_2)$ *for all* $w_1 \in W_X$ *and* $w_2 \in W_Y$.

(3) $l(xw) > l(w)$ *and* $l(wy) > l(w)$ *for all* $x \in X$ *and* $y \in Y$.

DEFINITION 1.10. An element w of W satisfying the conditions in Lemma 1.9 is called a *(X, Y)-reduced element.*

Lemma 1.11. *Let* $X \subset S$. *There exists a unique element of maximal length in each* W_X*-coset in* W. *Moreover, the following conditions for an element v of W are equivalent*:

(1) *v is the element of maximal length in* vW_X.

(2) $v = ww_X$ *for some* (ϕ, X)*-reduced element w of W.*

(3) $l(vw_1) = l(v) - l(w_1)$ *for all* $w_1 \in W_X$.

(4) $l(vx) < l(v)$ *foall* $x \in X$.

Proof. Let w be an arbitary (ϕ, X)-reduced element of W. Any element v' of wW_X can be written in the form $v' = ww_1'$ with $w_1' \in W_X$. By Lemma 1.9 (2), $l(v') = l(w) + l(w_1')$. Thus the length of v' is maximal among the elements in wW_X if and only if $w_1' = w_X$. This proves the first assertion in the lemma and the equivalence of the conditions (1) and (2). If (2) holds and $w_1 \in W_X$, then $l(vw_1)=l(ww_Xw_1)=l(w)+l(w_Xw_1)$ by Lemma 1.9 (2). On the other hand, we have $l(w_Xw_1)=l(w_X)-l(w_1)$ from [3; p. 43, Ex. 22]. Hence

$$l(vw_1) = l(w)+l(w_X)-l(w_1) = l(v)-l(w)\,,$$

which is (3). It is trivial that (3) implies (4). Assume that (4) holds. Let v_1 be the element of maximal length in vW_X. Then $v=v_1w_1'$ with $w_1' \in W_X$. If $w_1' \neq 1$, there is an element x of X such that $l(w_1'x)<l(w_1')$. We have $l(vx)= l(v_1w_1'x) = l(v_1) - l(w_1'x) = l(v_1) - l(w_1')+l(x) = l(v_1w_1')+l(x) = l(v)+l(x)>l(v)$, a contradiction. Therefore $w_1'=1$. Hence $v=v_1$, which is (1). This completes the proof of the lemma.

Lemma 1.12. *Any parabolic subgroup P of G can be uniquely written in the form $P=uwP_Xw^{-1}u^{-1}$, where X is a subset of S, w is a (φ, X)-reduced element of W and u is an element of $U^-_{w^{-1}}$.*

Proof. By Proposition 1.6 (b), X is uniquely determined by P. Assume that $u_1w_1P_Xw_1^{-1}u_1^{-1}=u_2w_2P_Xw_2^{-1}u_2^{-1}$ for two distinct (ϕ, X)-reduced elements w_1, w_2 of W and some elements u_1, u_2 of U. Then $w_1^{-1}u_1^{-1}u_2w_2 \in P_X$ because P_X is its own normalizer in G. Thus $w_2 \in Bw_1P_X$. On the other hand, we have $w_1W_X \neq w_2W_X$ from Lemma 1.9. Hence $Bw_1P_X \cap Bw_2P_Y=\phi$ by Proposition 1.6 (c), a contradiction. This proves the uniqueness of w in the lemma. Next, assume that $u_1wP_Xw^{-1}u_1^{-1}=u_2wP_Xw^{-1}u_2^{-1}$ for a (ϕ, X)-reduced element w of W *and* elements u_1, u_2 of $U^-_{w^{-1}}$. Then $w^{-1}u_2^{-1}u_1w \in P_X$. Thus we have

$$(1.2) \qquad u_1u \in u_2w\big(\textstyle\bigcup_{v\in W_X} BvB\big)\,.$$

By Lemma 1.9, $l(wv)=l(w)+l(v)$ for all $v \in W_X$. Hence $u_2wBvB \subset BwvB$ by [3; p. 26, Cor. 1]. Therefore (1.2) implies that $u_1w \in u_2wB$. Hence we have $u_1=u_2$ from Lemma 1.5. This completes the proof of the uniqueness part of the lemma. Let P be an arbitrary parabolic subgroup. By Proposition 1.6 and Lemma 1.9 there exist a subset X of S and a (ϕ, X)-reduced element w of W and an element u of U such that $P=uwP_Xw^{-1}u^{-1}$. By Lemma 1.4 (a), $u=u_1u_2$ for some $u_1 \in U^-_{w^{-1}}$ and $u_2 \in U^+_{w^{-1}}$. Thus $P=u_1wP_Xw^{-1}u_1^{-1}$. The proof of Lemma 1.12 is now complete.

2. Hecke algebra $H_C(G, B)$ and characters in $i[1_B | B \to G]$

Let G be an arbitrary finite group, and B a subgroup of G. We denote by C[G] the group algebra of G over a complex number field C. Then $e_1=|B|^{-1}\sum_{x\in B} x$ is an idempotent in C[G] and the left $C[G]$-module $C[G]e_1$ affords the character $i[1_B | B\to G]$ induced from the trivial character 1_B of B.

DEFINITION 2.1. The *Heche algebra* $H_C(G, B)$ is defined to be the subalgebra $e_1\boldsymbol{C}[G]e_1$ of the group algebra $\boldsymbol{C}[G]$.

Lemma 2.2. (Curtis-Fossum [8]) *Let % be an irreducible complex character of G* ***contained in*** $i[1_B|B\to G]$. *We also denote by* X *the corresponding* ***irreduicble*** *character of* $\boldsymbol{C}[G]$.

(a) *The restriction* $\chi|H_C(G, B)$ *of* χ*to* $H_C(G, B)$ *is an irreducible character of* $H_C(G, B)$. *Conversely, each* ***irreducible*** *character of* $H_C(G, B)$ *is the* ***restriction*** *to* $H_C(G, B)$ *of a* ***unique*** *irreducible character of* $\boldsymbol{C}[G]$.

(b) *Let* a_χ *be a* ***primitive idempotentin*** $H_C(G, B)$ *corresponding to* % $|H_C(G, B)$. *Then* a_χ *is also a* ***primitive*** *ídempotent in C[G] and the left* $\boldsymbol{C}[G]$***-module*** $\boldsymbol{C}[G]a_\chi$ *affords* χ.

Lemma 2.3. (Littlewood [15; §4.4]) *Let* $\sum_{x\in G}\lambda_x x$ *be a* ***primitive*** *ídempotent in C[G] corresponding to an* ***irreducible*** *character* χ. *Then*

$$\chi(g)=|G||C_G(g^{-1})|^{-1}\{\textstyle\sum_x \lambda_x\} \qquad (x\in C_G(g^{-1}))\,.$$

Let A be a set of representatives for the (B, B)-double cosets decomposition of G. For $a\in A$, let

$$(2.1)\qquad e_a=|B|^{-1}\textstyle\sum_{x\in BaB} x\,.$$

Then $\{e_a|a\in A\}$ forms a $\boldsymbol{C}$-basis for $H_C(G, B)$. For an arbitrary element $h=\sum_{a\in A}h_a e_a\,(h_a\in \boldsymbol{C})$ of $H_C(G, B)$, define the following complex valued class functions on G:

$$(2.2)\qquad f_h(g)=\textstyle\sum_{a\in A}h_a|B|^{-1}|G||C_G(g^{-1})|^{-1}|BaB\cap C_G(g^{-1})|\,.$$

Later, we shall often require the following lemma.

Lemma 2.4. (a) *Let % be an* ***irreducible*** *character of G contained* ***in*** $i[1_B|B\to G]$*,and* a_χ *a* ***primitive*** *idempotent in* $H_C(G,B)$ *corresponding to % in the sense of Lemma* 2.2. *Then*

$$\chi(g)=f_{a_\chi}(g) \qquad (g\in G)\,,$$

where f_{a_χ} *is the class* ***function*** *on G* ***defined*** *by* (2.2).

(b) ***Each function*** $f_h\,(h\in H_C(G,B))$ *can be* ***written*** *as a* ***linear combination*** *of* ***irreducible*** *characters contained* ***in*** *í* $[1_B|B\to G]$.

Proof. (a) This is a consequence of Lemma 2.3, (2.1) and (2.2).

(b) Let $X(G, B)$ be the set of all irreducible characters of G contained in $i[1_B|B\to G]$. It suffices to show that each $f_{e_a}(a\in A)$ can be written as a linear combination of elements of $X(G, B)$. Since $H_C(G, B)$ is a semisimple algebra, it is isomorphic to a direct sum of full matrix algebras $M(n_i, C)$ $(i=1, 2, \cdots, m)$. It is easy to see that each algebra $M(n_i, C)$ has a basis which consists of primitive idempotents. Hence $H_C(G,B)$ also has such a basis $\{v_j \mid 1 \leqq j \leqq N\}$ $(N=|A|)$. Let $v_j=\sum_{a\in A} c_{j,a}e_a$ $(c_{j,} \in C, 1\leqq j\leqq N)$ Then, from part (a) we have

$$(2.3) \qquad \sum_{a\in A} c_{j,a} f_{a} = \chi_j \quad (1\leqq j\leqq N)$$

for some $\chi_j \in X(G, B)$. Solving (2.3) in $f_{e_a}(a\in A)$, we get the required result.

We shall also need the following

Lemma 2.5. *$f_{hk}=f_{kh}$ foall elements h, k of $H_C(G, B)$.*

Proof. Let x and y be any elements of G. Then xy and yx belong to the same conjugacy class of G. The assertion follows from this fact and (2.2).

Assume henceforth that G is a finite Chevalley group and B is a Borel subgroup of G. We shall also use other notations given in §1. By the Bruhat decomposition of G, $\{e_w \mid w\in W\}$ (see (2.1)) forms a basis for $H_C(G,B)$. Hence for any element h of $H_C(G, B)$ there exist unique complex numbers $[h : e_w]$ (we W) such that $h=\sum_{w\in W} [A : e_w] e_w$.

Theorem 2.6. (Iwahori [2], Matsumoto [16]) *Let ind $e_w=|BwB/B|$ for $w\in W$, and $q_s=\text{ind}\, e_s$ for $s\in S$. Then*

(a) $q_s = q_a = \prod_{\alpha\in a} q(\alpha)$ *if* $s = w_a$ *with* $a\in R(S)$.

(b) $[e_w e_{w'} : e_{w''}] = BwB \cap n_{w''} U^-_{w'} n_{w'^{-1}}|$ *for all elements* w , w' ', w'' *of* W.

(c) *For* $s\in S$ *and* $w\in W$,

$$e_1 e_w = e_w e_1 = e_w\,,$$

$$e_s e_w = e_{sw} \qquad \textit{if} \quad l(sw)>l(w)$$

and $$e_s e_w = q_s e_{sw} + (q_s - 1) e_w \qquad \textit{if} \quad l(sw)<l(w)\,.$$

(d) *If* $s(w) = (s_1, s_2, \cdots, s_l)(s_i\in S)$ *is a reduced decomposition of* $w\in W$, *then*

$$e_w - e_{s_1} e_{s_2} \quad e_{s_l}\,.$$

(e) *Let the notations be as in (d). Then*

$$\textit{ind}\, e_w = |U^-_w| = q_{s_1} q_{s_2} \cdots q_{s_l}\,.$$

Proof. (a) This is a consequence of Lemma 1.5 and Proposition 1.3 (b).

(b) By [12; (3)], the left hand side is equal to $|(BwB \cap w'' U w'^{-1} B)/B|$, which is equal to $|BwB \cap n_{w''} U^-_{w'} n_{w'^{-1}}|$by Lemma 1.5.

(c) and (d) are proved in [12] and [16].
(e) is a consequence of (d) and [12; Lemma 1.2].

By Theorem 2.6 (c) we have

$$(2.4)\qquad (e_s)^{-1}=q_s^{-1}\{e_s+(1-q_s)e_1\}\qquad (s\in S).$$

By (2.4) and Theorem 2.6 (d), e_w is invertible for an arbitrary $w\in W$. The following theorem was obtained by O. Goldman (see [12]) for untwisted Chevalley groups G and by R. W. Kilmoyer [14] for general G.

Theorem 2.7. *The Hecke algebra $H_C(G, B)$ has an* ***involutoryautomorphism*** *defined by*

$$\hat{e}_w=(-1)^{l(w)}\,|\,U_w^-\,|\,(e_{w^{-1}})^{-1}\,.$$

Let χ be an irreducible character of G contained in $i[1_B\,|\,B\to G]$, and a_χ a primitive idempotent in $H_C(G, B)$ corresponding to χ (see Lemma 2.2). Then a_χ is also a primitive idempotent in $H_C(G, B)$. It is easy to see that the irreducible character of G corresponding to a_χ is independent of the choice of a_χ.

DEFINITION 2.8. Let the notations be as above. The *dual* $\hat{\chi}$ of χ is defined as the irreducible character of G corresponding to a_χ.

REMARK 2.9. In [14], R. Kilmoyer defined $\hat{\chi}$ by $(\hat{\chi}\,|\,H_C(G,B))(h)=(\chi\,|\,H_C(G, B))(\hat{h})$. It is easy to see that these two definitions are in fact identical.

Notation 2.10. Let w, w' and w'' be any elements of W, and $s(w)=(s_l, s_{l-1}, \cdots, s_1)$ a reduced decomposition of w. We denote by $\mathcal{J}(s(w), w', w'')$ the set of all integer sequences $J=(j_k, j_{k-1}, \cdots, j_0)$ satisfying the following conditions (cf. [2; (3.19)]):

(a) $l(w)=l\geqq j_k>j_{k-1}>\cdots>j_1>j_0=0$
(b) $s_{j_k}s_{j_{k-1}}\cdots s_{j_0}w'=w''$ where we put $s_{j_0}=s_0=1$ for convention.
(c) $(s_p s_{j_h}s_{j_{h-1}}\cdots s_{j_0}w')<l(s_{j_h}s_{j_{h-}}\cdots s_{j_0}w')$ for $j_h<p<j_{h+1}$ if $0\leqq h\leqq k-1$ and $j_k<p\leqq l$ if $h=k$.

For each $J\in\mathcal{J}(s(w), w', w'')$ we put $J_1=\{j_h\in J\,|\,l(s_{j_h}s_{j_{h-1}}\cdots s_{j_0}w')<l(s_{j_{h-1}}\cdots s_{j_0}w')\}$ and $J_2=\{j_h\in J\,|\,j_h\neq 0, j_h\notin J_1\}$.

Notation 2.11. Let $S_i (i\in I)$ be the equivalence classes for the relation "s is conjugate to r in W" between elements s, r of S. Let $\boldsymbol{t}=(t_i)_{i\in I}$ be a family of indeterminates indexed by I, and for each $s\in S$ define t_s to be t_i if $s\in S_i$.

DEFINITION 2.12. Let w, w' and w'' be any elements of W. Using the above notations we define the following polynomial in $t=(t_i)_{i\in I}$:

$$\begin{aligned} &F(w, w', w'')(t)= F(w, w', w'')((t_i)_{i\in I}) \\ &= \sum_J \{\Pi_{j\in J_1} \vee \Pi_{i\notin J}(t_{s_j}-1)\} \qquad (J\in \mathcal{J}(s(w), w', w'')) \end{aligned}$$

where $s(w)=(s_l, s_{l-1}, \cdots, s_1)$ is a reduced decomposition of w.

REMARK 2.13. The polynomial $F(w, w', w'')(t)$ does not depend on the choice of a reduced decomposition $s(w)$ of w. This fact follows from Lemma 2.14 (b) given below.

Lemma 2.14. (a) *Let s, r be elements of S. Then $q_s=q_r$ if s and r belong to the same equivalence class S_i defined in Notation* 2.11.

(b) *Put $q_i=q_s$ if $s\in S_i$. Then $[e_we_{w'}: e_{w''}]=F(w, w', w'')((q_i)_{i\in I})$ for all elements w, w' and w'' of W.*

(c) $[e_we_{w'}: e_{w'}]=[\hat{e}_we_{w'w_S}: e_{w'w_S}]$ *for all elements w and w' of W.*

Proof. (a) This is a consequence of Theorem 2.6 (a) and Proposition 1.3 (d).

(b) We will prove this by the induction on the length $l(w)$ of w. It is easy to see that (b) is true if $w=1$. Let $s(w)=(s_l, s_{l-1}, \cdots, s_1)$ be a reduced decomposition of $w\neq 1$. Then $l(s_lw)<l(w)$. By the induction assumption,

$$e_{s_lw}e_{w'} = \sum_{v\in W} F(s_lw, w', v)((q_i)_{i\in I})e_v\,.$$

Multiplying e_{s_l} from the left, we have

$$e_we_{w'} = \sum_{v\in W} F(s_lw, w', v)((q_i)_{i\in I})e_{s_l}e_v$$

by Theorem 2.6 (c).

Comparing this formula with (b) and using Theorem 2.6 (c), we see that it suffices to prove the following:

(1) $F(w, w', w'')(t)=F(s_lww', s_lw'')(t)+(t_{s_l}-1)F(s_lww', w'')(t)$ if $l(s_lw'')<l(w'')$.

(2) $F(w, rf, w'')(t)=t_{s_l}F(s_lw,w', s_lw'')(t)$ if $l(s_lw'')>l(w'')$.

We first prove (1). Let $J'=(j_m', \cdots, j_0')$ be an element of $\mathcal{J}(s(s_lw), w', s_lw'')$, where $s(s_lw)=(s_{l-1}, s_{l-2}, \cdots, s_1)$. Then $J=(l, j_m', \cdots, j_0')$ is an element of $\mathcal{J}(s(w), w', w'')$. Next, let $J''=(j_n'', \cdots, j_0'')$ be an element of $\mathcal{J}(s(s_lw), w', w'')$. Then $J=J''$ is an element of $\mathcal{J}(s(w), w', w'')$. In fact the condition in Notation 2.10 (c) is satisfied by the assumption that $l(s_lw'')<l(w'')$. Moreover, every element $J\in J(s(w), w', w'')$ can be obtained from $J'\in \mathcal{J}(s(s_lw), s', s_lw'')$ or $J''\in \mathcal{J}(s(s_lw)w', w'')$ in this manner. Hence, from the definition of

$F(w, w', w'')(\boldsymbol{t})$, we obtain (1). To prove (2), we note that for any element $J=(j_k, j_{k-1}, \cdots, j_0)$ of $\mathcal{J}(s(w), w', w'')$, j_k must be l and $J'=(j_{k-1}, j_{k-2}, \cdots, j_0)$ is an element of $\mathcal{J}(s(s_l w), w'', s_l w'')$. This follows from the assumption that $l(s_l w'')>l(w'')$. The rest of the proof is similar to that of (1). The proof of (b) is over.

(c) By Theorem 2.6 (c) and (2.4), it is easy to see that for $s\in S$ and $w\in W$,

$$e_s e_{wwS} = -q_s e_{swwS} \text{ if } l(sw)>l(w)$$
$$\text{and } e_s e_{wwS} = -e_{swwS}+(q_s-1)\, e_{wwS} \text{ if } l(sw)<l(w)\,.$$

By the almost same argument as in (a), we get

$$[\hat{e}_w e_{w'wS} \; e_{w''wS}] = E(w, w', w'')((q_i)_{i\in I})\,,$$

where $E(w, w'', w'')(\boldsymbol{t})$ is a polynomial in $\boldsymbol{t}=(t_i)_{i\in I}$ defined as follows:

$$E(w, w', w'')(\boldsymbol{t})$$
$$=\textstyle\sum_J \{(-1)^{|J_1|}\prod_{j\in J_2}(-t_{s_j})\prod_{j\notin J}(t_{s_j}-1) \qquad (J\in\mathcal{J}(s(w), w', w''))$$

where $\boldsymbol{s}(w)=(s_l, s_{l-1}, \cdots, s_1)$ is a reduced decomposition of w.

Therefore, for the proof of (c) it sufficies to show

$$\text{(2.5)} \qquad \textstyle\prod_{j\in J_1} t_{s_j} = \prod_{j\in J_2} t_{s_j}$$

for all $J\in\mathcal{J}(\boldsymbol{s}(w), w', w'')$ $(w, w'\in W)$. Let $J=(j_k, j_{k-1}, \cdots, j_0)$. Then $s_{j_k}s_{j_{k-1}}\cdots s_{j_1}w'=w''$. Hence (2.5) is a special case of Lemma 2.15 (b) below.

Lemma 2.15. (a) *Let w be an element of W, and $\boldsymbol{s}(w)=(s_1, s_2, \cdots, s_l)$ a reduced decomposition of W. Then the monomial $\boldsymbol{t}_w=t_{s_1}t_{s_2}\;\;t_{s_l}$ is independent of the choice of reduced decomposition $\boldsymbol{s}(w)$.*

(b) *Let $(s_1, s_2, \quad, s_m)$ be a sequence of elements of S. Put $w=s_1s_2\cdots s_m$. Then*

$$\boldsymbol{t}_w = t_{s_1}^{a_1}t_{s_m}^{a_2}\cdots t_{s_m}^{a_m}\,,$$

where $a_i=1$ or -1 according as $l(s_1s_2\cdots s_{i-1})<l(s_1s_2\cdots s_i)$ or $l(s_1s_2\;\; s_{i-1})>l(s_1s_2\;\; s_i)$ respectively.

Proof. (a) This follows from [3: p. 16, Proposition 5.].

(b) We prove this by induction on m. If $m=l$ the assertion is trivially true. Put $w'=s_1s_2\;\; s_{m-1}$. Then by the induction assumption we have $\boldsymbol{t}_{w'}=t_{s_1}^{a_1}t_{s_2}^{a_2}\cdots t_{s_{m-1}}^{a_{m-1}}$ If $l(w)=l(w's_m)>l(w')$, $\boldsymbol{t}_w=\boldsymbol{t}_{w'}t_{s_m}$ by part (a). If $l(w)=l(w's_m)<l(w')$, $\boldsymbol{t}_{w'}=\boldsymbol{t}_w t_{s_m}$ by part (a). In any case we have $\boldsymbol{t}_w=t_{s_1}^{a_1}t_{s_2}^{a_2}\cdots t_{s_m}^{a_m}$, as required.

3. Central isogenies and characters in $i[1_B|B\to G]$

The main purpose in this section is to prove Theorem 3.4. This result, which is of independent interest, will be used in §7.

First we recall some facts on central isogenies. References are [5] and [2; §2]. Let © and ©' be connected semisimple linear algebraic groups defined over an algebraically closed field K, 33 a Borel subgroup of ©, and $\mathfrak{T}$ a maximal torus of © contained in 33. We also use other notations in §1. Assume that there exists a central isogeny $\psi: \mathfrak{G}\to\mathfrak{G}'$. Then the following statements are valid:

(3.1) $\mathfrak{B}'=\psi(\mathfrak{B})$ is a Borel subgroup of ©'.

(3.2) $\mathfrak{T}'=\psi(\mathfrak{T})$is a maximal torus of ©' contained in $\mathfrak{B}'$.

(3.3) Let $X(\mathfrak{T})$ and $X(\mathfrak{T}')$ be character modules of X and $\mathfrak{T}'$ respectively. Let ψ^*: $X(\mathfrak{T}')\otimes_Z \boldsymbol{Q}\to X(\mathfrak{T})\otimes_Z$ ©be the $\boldsymbol{Q}$-linearextention of the transpose of $\psi\,|\mathfrak{T}$. Then ψ^* is injective and $\psi^*(X(\mathfrak{T}'))\subset X(\mathfrak{T})$and $\psi^*(\Sigma')=\Sigma$, where Σ' is the root system of ©' with respect to $\mathfrak{T}'$.

(3.4) $\mathfrak{U}'=\psi(\mathfrak{U})$ is the unipotent radical of $\mathfrak{B}'$ and $\psi|\mathfrak{U}$: $\mathfrak{U}\to\mathfrak{U}'$ is an isomorpism.

(3.5) For $\alpha\in\Sigma$, tt£/= $\psi(\mathfrak{U}_\alpha)$ is the root subgroup of ©' with respect to *a!* correspondingto $\alpha'=\psi^{*-1}(\alpha)$.

(3.6) $\mathfrak{N}'=\psi(\mathfrak{N})$ is the normalizer of 37 and ψ induces an isomorphism ψ from the Weyl group $\mathfrak{W}=\mathfrak{N}/\mathfrak{T}$ of © onto the one $\mathfrak{W}'=\mathfrak{N}'/\mathfrak{T}'$ of $\mathfrak{G}'$. If $w_\alpha(\alpha\in\Sigma)$ is a reflection in W, then $\bar{\psi}(w_\alpha)=w_{\alpha'}$, where $\alpha'=\psi^{*-1}(\alpha)$.

In the following, let ©, 33, σ, G, B, etc. be as in §1.

Theorem 3.1. (a) *Let* $\tilde{©}$ *be the simply connected group of the same type as* $\mathfrak{G}$, *and let* ψ: $\tilde{\mathfrak{G}}$– ›$\mathfrak{G}$ *be a central isogeny. Then there exists a unique surjective endomorphism* σ *of* $\tilde{\mathfrak{G}}$*such that* (1) $\psi\circ\tilde{\sigma}=\sigma\circ\psi$ *and (2)* $\tilde{\mathfrak{G}}_{\tilde{\sigma}}$ $<\infty$.

(b) *Let* ©' *be the adjointgroup of the sme type as* ©, *and let* ψ: $\mathfrak{G}\to\mathfrak{G}'$ *be a central isogeny. Then there exists a unique surjective endomorphism* σ' *of* ©' *such that* (1) $\psi\circ\sigma=\sigma'\circ\psi$ *and (2)* \ $\mathfrak{G}'_{\sigma'}|<\infty$.

Proof. (a) First we remark that σ: $\mathfrak{G}\to\mathfrak{G}$ is an isogeny because of the fact that ker σ is trivial ([20; 7.1]). The unique existence of an isogeny $\tilde{\sigma}$: $\tilde{\mathfrak{G}}\to\tilde{\mathfrak{G}}$ satisfying (1) follows from [20; 9.16]. Next, we prove that (1) implies (2). Let $x\in\tilde{\mathfrak{G}}_{\tilde{\sigma}}$. By (1), $\psi(x)\in\tilde{\mathfrak{G}}_{\tilde{\sigma}}$. Because ker ψ and $\mathfrak{G}_\sigma$ are finite, $\tilde{\mathfrak{G}}_{\tilde{\sigma}}$ is finite also.

(b) The uniqueness of σ follows from (1) and the surjectivity of ψ. We prove the existence of an isogeny σ' satisfying (1). Since $\mathfrak{G}'$ is adjoint, $X(\mathfrak{T}')$ is the $\boldsymbol{Z}$-module generated by Σ'. By (3.3) $\psi^*(X(T'))$ is the $\boldsymbol{Z}$-module $\Sigma_{\boldsymbol{Z}}$ generated by Σ. Since σ^* preserves $\Sigma_{\boldsymbol{Z}}$ by Lemma 1.1, we can define an automorphism γ of $X(\mathfrak{T}')$ by $\gamma=(\psi^*)^{-1}\circ\sigma^*\circ\psi^*$. From Lemma 1.1 we have $\gamma(\alpha')=q(\rho^{-1}\circ\psi^*(\alpha'))(\psi^*)^{-1}\circ\rho^{-1}\circ\psi^*(\alpha')$ for all $\alpha'\in\Sigma'$. Thus the existence of an isogeny σ' satisfying (1) follows from [5; 18-07, Proposition 5]. Next, we prove that (1) implies *(2)*. Assume that $\mathfrak{G}'_{\sigma'}$ is infinite. From the surjectivity of ψ and the finiteness of ker ψ, we see that the set $A— \{x\in G\mid\sigma'\circ\psi(x)=\psi(x)\}$ is infinite. Since $A=\{x\in G| x^{-1}\sigma(x)\in\ker\psi\}$ by (1) and ker ψ is finite, this fact implies that $A_c=\{x\in G\mid x^{-1}\sigma(x)=c\}$ is infinite for some $c\in\ker\psi$. If y and z are elements of A_c we have $y^{-1}\sigma(y)=z^{-1}\sigma(z)$,i.e. $yz^{-1}\in\mathfrak{G}_\sigma$. Hence $\mathfrak{G}_\sigma$ must be infinite, a contradiction. This proves (b).

Let ψ: $\mathfrak{G}\to\mathfrak{G}'$ be a central isogeny and σ' is a surjective endomorphism of $\mathfrak{G}'$ such that $\psi\circ\sigma=\sigma'\circ\psi$. By the proof of Theorem 3.1 (b), such σ is unique and $G'=\mathfrak{G}'_{\sigma'}$ is finite. In the following, the endomorphism σ' is denoted simply by σ.

Theorem *3.2. Let the notations be as above.*

(a) $\mathfrak{B}'=\psi(\mathfrak{B})$ *and* $\mathfrak{T}'=\psi(\mathfrak{T})$ *is fixed by* σ. (In the following, we write B' and T' for $\mathfrak{B}_\sigma'$ and $\mathfrak{T}_\sigma'$ respectively.)

(b) ψ^* *induces an isomorphism, which is also denoted by* ψ^*, *from the root system* R' *associated to* (G', T') (see Proposition 1.2) *onto the one* R *associated to* (G, T).

(c) ψ *induces an isomorphism between* U *and* $U'=\mathfrak{U}'_\sigma$.

(d) *If* $a\in R$,*then* $\psi(U_a)=U'_{a'}=(\prod_{\alpha\in a'}\mathfrak{U}'_\alpha)_\sigma$, *where* $a'=\psi^{*-1}(a)$.

(e) ψ *induces an isomorphism, which is also denoted by ty, from the Weyl group* $W=\mathfrak{N}_\sigma/\mathfrak{T}_\sigma$*of* G *onto the one* $W'=\mathfrak{N}_\sigma'/\mathfrak{T}_\sigma'$*of* G' '. *If* $a\in R$, *then* $\bar\psi(w_a)=w_{a'}$, *where* $a'=\psi^{*-1}(a)$.

Proof. These are easy consequences of the properties (3.1)~(3.6) of ψ, the assumption "$\psi\circ\sigma=\sigma\circ\psi$" and the definitions *(of* R, W, U, $\cdots$).

Corollary 3.3. *Let the notations be as in Theorem 3.2. The Hecke algebras* $H_C(G, B)$ *and* $H_C(G', B')$ *are isomophicby the natural mapping*: $e_w\to e_{\bar\psi(w)}(w\in W)$.

Proof. This follows from Theorem 3.2 and Theorem 2.6.

From Theorem 3.2 and Corollary 3.3, we may identify R, W, U, U_a $(a\in R)$ and $H_C(G, B)$ with R', W', U', $U'_{a'}$ $(a'=\psi^{*-1}(a))$ and $H_C(G',B')$ respectively. Put $H=H_C(G,B)=H_C(G',B')$. Let $X(G, B)$ (resp. $X(G',B')$) be the set of irreducible characters of G (resp. G') contained in $i[1_B\backslash B\to G]$ (resp. $i[1_{B'}\backslash B'\to G']$). Let % be an element of $X(G, 5)$, and h a primitive idempotent in H such that the left $\boldsymbol{C}[G]$-module $C[G]h$ affords χ' (see Lemma 2.2). Let χ' be an element of $X(G',B')$ afforded by the left $\boldsymbol{C}[G']$-module $C[G']h$. The correspondence: $\chi\to\chi'$ from $X(G, B)$ into $X(G', B')$ is clearly well defined and bijective.

Now, we can state the main result in this section:

Theorem 3.4. *Let the notations be as above. For $h\in H$, let f_h be the class function on G defined by* (2.2) *and f_h' the class function on G' defined in the same manner.*

(a) *For any $h\in H, f_h$ equals f_h' identically on $U\ (=U')$.*

(b) *// $\chi\in X(G, B)$ and $\chi'\in X(G',B')$ corresponds to each other in the sense mentioned above, % equals χ' identically on $U\ (=U')$.*

For the proof, we require some preliminary results.

Lemma 3.5. *Let $\mathcal{B}$ be the set of all Borel subgroups of G, and for $w\in W$, let $\mathcal{O}_w$ be the set of all couples $(B_1, B_2)\in\mathcal{B}$X $\mathcal{B}$ which are G-conjugate to the pair (B, wBw^{-1}). Then*

$$\mathcal{B}\times\mathcal{B}=\cup_{w\in W}\mathcal{O}_w \qquad (\textit{disjoint union})\ .$$

Proof. Let (B_1, B_2) be an arbitrary element of $\mathcal{B}\times\mathcal{B}$, and let g_1, g_2 be elements of G such that $B_i=g_iBg_i^{-1}$ $(i=1, 2)$. Then, the couple (B_1,B_2) is conjugate to $(B, g_1^{-1}g_2Bg_2^{-1}g_1)$. By the Bruhat decomposition of G, there exist $b, b'\in B$ and $w\in W$ such that $g_1^{-1}g_2=bwb'$. Hence (B_1, B_2) is conjugate to (B, wBw^{-1}), i.e. contained in $\mathcal{O}_w$. Thus we have $\mathcal{B}\times\mathcal{B}=\cup_{w\in W}\mathcal{O}_w$. Next, we prove the disjointness of this decomposition. Let w and w' be distinct elements of W. Assume that $\mathcal{O}_w \sqcap \mathcal{O}_{w'}\neq\phi$. Then (B, wBw^{-1}) and $(B, w'Bw'^{-1})$ are conjugate to each other. Hence there exist an element x of G such that $xBx^{-1}=B$ and $xwBw^{-1}x^{-1}=w'Bw'^{-1}$. Because B is its own normalizer in G, we have $x\in B$ and $w'^{-1}xw\in B$. Hence $BwB\ni w'$, a contradiction. Therefore, the decomposition is disjoint.

Lemma. 3.6. *Let the notations be as in Lemma* 3.5. *For $w\in W$ and $g\in G$, define the subset $F_{g\,w}$ of $\mathcal{B}$ by*

$$F_{g,w}=\{B_1\in\mathcal{B}\,|\,(gB_1g^{-1}, B_1)\in\mathcal{O}_w\}\ .$$

Then

(a) $\mathcal{B}=\cup_{w\in W}F_{g,w}$ *(disjoint union)* .

(b) $|F_{g,w}|=G\ \ |B|^{-1}|BwB\cap C_G(g^{-1})|\ \ C_G(g^{-1})\ \ ^{-1}=f_{e_w}(g)$ *for all $g\in G$.*

Proof. Part (a) follows from Lemma 3.5. We shall prove (b). Let g be a fixed element of G, and for $w\in W$, let A_w be the set of all $(B',g')\in\mathcal{B}\times C_G(g)$ such that $B'\in F_{g',w}$. Then we have

$$A_w = \cup_{B_1 \in \mathcal{B}} A_w(B_1)\,,$$

where $A_w(B_1)=A_w \cap (\{B_1\} \times C_G(g))$. Let $B_1=x_1Bx_1^{-1}$ with $x_1 \in G$. Assume that $A_w(B_1) \ni (B_1, g')$, i.e. $(g'x_1Bx_1^{-1}g'^{-1}x_1Bx_1^{-1}) \in \mathcal{O}_w$. This is the case if and only if $g'^{-1} \in x_1(BwB)x_1^{-1}$. Hence $A_w(B_1)|$ is equal to $|x_1(BwB)x_1^{-1} \cap C_G(g^{-1}) \doteqdot |BwB \cap C_G(g^{-1})|$. Therefore

$$(3.7) \qquad |A_w| = |G|\,|B|^{-1}|BwB \cap C_G(g^{-1})|\,.$$

On the other hand, we have

$$A_w = \cup_{g_1 \in C_G(g)}(F_{w,g_1} \times \{g_1\})$$

Clearly $|F_w(g_1)| = |F_w(g)|$ for $g_1 \in C_G(g)$. Hence

$$(3.8) \qquad |A_w| = |F_{w,g}|\,|C_G(g)| = |F_{w,G}|\,|C_G(g^{-1})|\,.$$

From (3.7) and (3.8), we have

$$|F_{w,G}| = |G|\,|B|^{-1}|BwB \cap C_G(g^{-1})|\,|C_G(g^{-1})|^{-1}\,,$$

as required.

REMARK 3.7. Let G be a (finite or infinite) group with a BN-pair (B, N), and W its Weyl group. Then, by the same arguments as above, we get a decomposition of $\mathcal{B}= C_G(B)$:

$$\mathcal{B} = \cup_{w \in W} F_{g,w}\,.$$

Proof of Theorem 3.4.

(a) It sufficies to prove the assertion for $h=e_w$ $(w \in W)$. Let B_1 be a Borel subgroup. By Lemma 1.12, B_1 can be written uniquely in the form $B_1=u_1w_1Bw_1^{-1}u_1^{-1}$ with $w_1 \in W$ and $u_1 \in U^-_{w_1^{-1}}$. Let u be an element of U and assume that B_1 is contained in $F_{u,w}$, i.e. $(uu_1w_1Bw_1^{-1}u_1^{-1}u^{-1}, u_1w_1Bw_1^{-1}u_1^{-1}) \in \mathcal{O}_w$. This is the case if and only if $w_1^{-1}u_1^{-1}u^{-1}u_1w_1 \in BwB$. Hence

$$|F_{u,w}| = \sum_{w_1 \in W} |\{u_1 \in U^-_{w_1^{-1}} \mid w_1^{-1}u_1^{-1}u^{-1}u_1w_1 \in BwB\}|\,.$$

By Theorem 3.2, the right hand side of this formula remains invariant when G, B, W etc. are replaced with G', B', W', etc. Since $f_{e_w}(u)=|F_{u,w}|$ by Lemma 3.6, the proof of part (a) is over.

(b) Let $h \in H(=H_C(G,B)=H_C(G',B'))$ be a primitive idempotent corresponding to χ and χ'. Then $\chi=f_h$ and $\chi'=f_h'$ by Lemma 2.4. This fact, together with part (a), implies (b).

4. Unipotent elements in (B, B)-double cosets

Let P be a parabolic subgroup of a finite Chevalley group G. By Lemma 1.12, P can be written uniquely in the form $P=uw'P_Xw'^{-1}u^{-1}$, where X is a subset of 5, w' is a (ϕ, X)-reduced element of $W$and u is an element of $U^-_{w'^{-1}}$. Let G^1 be the set of all unipotent elements of G. The main purpose of this section is to prove the following

Theorem 4.1. *Let $P=uw'P_Xw'^{-1}u^{-1}$ as above. For an arbitrary element w of W, the following formula holds.*

$$|G^1 \cap BwB \cap P| = |BwB \cap w'w_X U^-_{w'w_X} w_X^{-1} w'^{-1}| \, |U_{w'}| \, .$$

Before proving the theorem we state some corollaries which can be deduced easily from it.

Corollary 4.2. $|G^1 \cap BwB| = |w_S U w_S^{-1} \cap BwB| \, |U|$.

Proof. Put $P=G$ in the above theorem. In this case, $X=S$ and $w'=1$. Hence we have the desired formula.

Corollary 4.3. *Let P be as in Theorem 4.1, and let $F(w_1, w_2, w_3)(t)(w_i \in W$, $i=1,2,3)$ be polynomials defined in § 2. Then $|G^1 \cap BwB \cap P| = F(w, w'w_X, w'w_X)$ $((q_i)_{i\in I}) \prod_{a\in R^+(w')} q_a$. In particular, $|G^1 \cap BwB| = F(w, w_S, w_S)((q_i)_{i\in I}) \prod_{a>0} q_a$.*

Proof. This follows from Theorem 4.1, Theorem 2.6(b), Lemma 2.14(b) and Lemma 1.4(b).

Corollary 4.4. $|G^1| = |U|^2$.

Proof. From the Bruhat decomposition of G we have $|G^1| = \sum_{w\in W} |G^1 \cap BwB|$. Applying Corollary 4.2 we get $|G^1| = \sum_{w\in W} |w_S U w_S^{-1} \cap BwB| \, |U|$, which equals $|U|^2$. This proves the corollary.

REMARK 4.5. The Collorary 4.4 was originally proved by R. Steinberg [20]. We shall give another, more direct, proof of Corollary 4.4 in Remark 4.9.

The proof of Theorem 4.1 requires several lemmas.

Lemma 4.6. *Let X be a subset of S, and ξ_X the character of the parabolic subgroup P_X of G defined by*

$$\xi_X = \sum_{Y\subset X} (-1)^{|Y|} i[1_{P_Y} | P_Y \to P_X],$$

where 1_{P_Y} is the trivial character of P_Y. Then the following statements are valid:

(a) *ξ_X is an irreducible character of the group P_X.*

(b) $\xi_X(x) = \begin{cases} |U^-_{w_X}| & \textit{if} \quad x \in V_X, \\ 0 & \textit{if} \quad x \in U - V_X. \end{cases}$

Proof. (a) Let N_X be the subgroup of P_X generated by T and $\{n_w \backslash w \in W_X\}$. Then (B, N_X) is a BN-pair in P_X. Hence (a) follows from [6; Theorem 2].

(b) Applying [6; (3.5)] to P_X we obtain $\xi_X(x) = i[1_{T \cdot V_X} T\ V_X \to B](x)$ for $x \in B$. Since $V_X = U^+_{w_X}$ is normal in B (see Lemma 1.7), $\xi_X(x) = 0$ if $x \in U - V_X$ and $\xi_X(x) = |B||T\ V_X|^{-1} = |U||U^+_{w_X}|^{-1} = |U^-_{w_X}$ for $x \in V_X$, where we used Lemma 1.4(a). This proves part (b).

DEFINITION 4.7. The irreducible character ξ_X of the group P_X is called the *Steinberg character* of P_X.

Lemma 4.8. *Let* $X \subset S$. *Then*

$$1_{P_X} = \sum_{Y \subset X} (-1)^{|Y|} i[\xi_Y | P_Y \to P_X]$$

Proof. By Lemma 4.6 and transitivity of induction, the right hand side of the above formula equals

$$\sum_{Y \subset X} (-1)^{|Y|} \{\sum_{Z \subset Y} (-1)^{|Z|} i[1_{P_Z} | P_Z \to P_X]\} .$$

The coefficients of $i\ [1_{P_Z} | P_Z \to P_X]$ in this expression is

$$\{\sum_{Z \subset Y \subset X} (-1)^{|Y|}\} (-1)^{|Z|} ,$$

which is equal to 0 if $Z \neq X$ and 1 if $Z = X$. This proves the lemma.

REMARK 4.9. Here we show that Corollary 4.4 follows easily from Lemma 4.6 and Lemma 4.8. As already remarked in [6], Lemma 4.6 with $X = S$ implies

$$(4.1) \qquad |U| = \sum_{Y \subset S} (-1)^{|Y|} |G|\, |P_Y|^{-1} .$$

Let θ be the class function on G which is defined to be 1 on G^1 and 0 outside of it. Then using Lemma 4.8 with $X - S$ and Frobenius reciprocity we obtain

$$(\theta, 1_G)_G = \sum_{X \subset S} (-1)^{|Y|} (\theta | P_Y, \xi_Y)_{P_Y} .$$

Hence, by Lemma 4.6 (b),

$$(4.2) \qquad |G|^{-1} |G^1| = \sum_{Y \subset S} (-1)^{|Y|} |P_Y|^{-1} |U| .$$

Combining (4.1) and (4.2) we obtain $|G^1| = |U|^2$, as required.

Lemma 4.10. *Let X and Y be subsets of S such that* $Y \subset X$. *For* $x \in G$, *let*

$$\mathcal{P}_{X,Y}(x) = \{P' \in C_{P_X}(P_Y) | V_{P'} \ni x\} .$$

Then

$$|\mathcal{P}_{X,Y}(x)| = \begin{cases} |U^-_{w_Y}|^{-1} i[\xi_Y | P_Y \to P_X](x) & \text{if } x \in G^1 \cap P_X \\ 0 & \text{if } x \in G - G^1 \cap P_X . \end{cases}$$

Proof. It is clear that $|\mathcal{P}_{X,Y}(x)|=0$ if $x\in G-G^1\cap P_X$ Let x be an element of $G^1\cap P_X$. Let $\mathcal{A}_{X,Y}(x)$ be the set of all couples $(P',x')\in C_{P_X}(P_Y)\times C_{P_X}(x)$ such that $V_{P'}\ni x'$. Then

$$\mathcal{A}_{X,Y}(x) = \cup_{P_1} \mathcal{A}_{X,Y}(x, P_1) \qquad (P_1 \in C_{P_X}(P_Y)),$$

where $\mathcal{A}_{X,Y}(x,P_1)=\mathcal{A}_{X,Y}(x)\cap(\{P_1\}\times C_{P_X}(x))$.

Clearly $|\mathcal{A}_{X,Y}(x,P_1)| = |\mathcal{A}_{X,Y}(x,P_Y)| = |V_Y\cap C_{P_X}(x)|$. Thus we have

$$(4.3) \qquad |\mathcal{A}_{X,Y}(x)| = |P_X| \, |P_Y|^{-1} |V_Y \cap C_{P_X}(x)| .$$

On the other hand, we have

$$\mathcal{A}_{X,Y}(x) = \cup_{x_1} (\mathcal{P}_{X,Y}(x_1) \times \{x_1\}) \qquad (x_1 \in C_{P_X}(x)) .$$

Hence

$$(4.4) \qquad |\mathcal{A}_{X,Y}(x)| = |\mathcal{P}_{X,Y}(x)| \, |C_{P_X}(x)| .$$

By (4.3) and (4.4),

$$|\mathcal{P}_{X,Y}(x)| = |P_X| \, |P_Y|^{-1} |V_Y \cap C_{P_X}(x)| \, |C_{P_X}(x)|^{-1} .$$

Hence, using Lemma 4.6 (b) we obtain

$$|\mathcal{P}_{X,Y}(x)| = |U^-_{w_Y}|^{-1} i[\xi_Y | P_Y \to P_X](x),$$

as required.

Proof of Theorem 4.1.

Since $G^1\cap BwB$ is invariant under conjugations by elements of B, we may assume that $u=1$. Then from Lemma 4.8 and Lemma 4.10 the left hand side of the equality stated in the theorem is

$$(4.5) \qquad \sum_{Y\subset X} (-1)^{|Y|} \sum_{P'} |w' V_{P'} w'^{-1} \cap BwB| \, |U^-_{w_Y}| ,$$

where the second sum is taken over the set $C_{P_X}(P_Y)$ for each Y. By Lemma 1.12 each element P' of $C_{P_X}(P_Y)$ can be written uniquely in the form $P'=u'w''P_Yw''^{-1}u'^{-1}$, where w'', is a (ϕ, Y)-reduced element of W_X and u' is an element of $U^-_{w''^{-1}}$. Thus the expression (4.5) is equal to

$$(4.6) \qquad \sum_{Y\subset X} (-1)^{|Y|} \sum_{w''} |w'w''V_Yw''^{-1}w'^{-1} \cap BwB| \, |U^-_{w''^{-1}}| \, |U^-_{w_Y}| ,$$

where the second sum is over the set of all (ϕ, Y)-reduced elements of W_X for each Y. The summand corresponding to Y and w'' is

$$(4.7) \qquad |w'w''(U\cap w_Y U w_Y^{-1})w''^{-1}w'^{-1} \cap BwB| \, |U^-_{w''^{-1}}| \, |U^-_{w_Y}| .$$

Using Lemma 1 .4 and the fact that $w'w''$ is (ϕ, Y)-reduced the first factor in (4.7) can be written as

$$\begin{aligned} &|(w'w''U^-_{w'w''}w''^{-1}w'^{-1})(w'w''w_Y U w_Y^{-1} w''^{-1} w'^{-1} \cap U) \cap BwB| \\ &= |w'w''U^-_{w'w''}w''^{-1}w'^{-1} \cap BwB|\,|w'w''w_Y U w_Y^{-1} w''^{-1} w'^{-1} \cap U| \,. \end{aligned}$$

We have also $|w'w''w_Y U w_Y^{-1} w''^{-1} w'^{-1} \cap U|\,|U^-_{w''^{-1}}|\,|U^-_{w_Y}| = |U^+_{w'}|$ from Lemma 1.5 and Theorem 2.6 (e). Hence (4.6) is equal to

$$(4.8) \qquad \sum_{Y \subset X} (-1)^{|Y|} \sum_{w''} |w'w''U^-_{w'w''}w''^{-1}w'^{-1} \cap BwB|\,|U^+_{w'}| \,.$$

For each w'', let $X(w'') = \{s \in X \mid l(w''s) > l(w'')\}$. Then the coefficient of $|w'w''U^-_{w'w''}w''^{-1}w'^{-1} \cap BwB|\,|U^+_{w'}|$ in (4.8) is

$$\sum_{Y \subset X(w'')} (-1)^{|Y|} \,,$$

which is 0 if $X(w'') \neq \phi$ and 1 if $X(w'') = \phi$, i.e. $w'' = w_X$. Hence (4.8) is equal to

$$|w'w_X U^-_{w'w_X} w_X^{-1} w'^{-1} \cap BwB|\,|U^+_{w'}| \,.$$

This proves Theorem 4.1.

5. Unipotent elements and characters in $i[1_B | B \rightarrow G]$

The purpose of this section is to prove the following

Theorem 5.1. *Let χ be an irreducible character of a finite Chevalley group G contained in the induced character $i[1_B | B \rightarrow G]$. Let X be a subset of S, and ξ_X the Steinberg character of P_X. Then*

$$\sum_{u \in G^1} \chi(u) i[1_{P_X} | P_X \rightarrow G](u) = \sum_{u \in G^1} \hat{\chi}(u) i[\xi_X | P_X \rightarrow G](u) \,,$$

where G^1 is the set of all unipotent elements in G and $\hat{\chi}$ is the dual (see Definition 2.8) *of χ.*

Consider the special case where $P = G$. Then using Lemma 4.6 we obtain

Corollary 5.2. *Let the notations be as in Theorem* 5.1. *Then*

$$\sum_{u \in G^1} \chi(u) = |U| \hat{\chi}(1) \,.$$

REMARK 5.3. Let t be a semisimple element of G. Denote by $Z^1(t)$ the set of unipotent elements of G which commute with t. It is likely that the following formula holds for any irreducible chatacters χ_1 and χ_2 of G which are contained in $i[1_B | B \rightarrow G]$:

$$(5.1) \qquad \sum_{u \in Z^1(t)} \chi_1(tu) \chi_2(tu) = \sum_{u \in Z^1(t)} \hat{\chi}_1(tu) \hat{\chi}_2(tu) \,.$$

We shall state some evidences for (5.1). (1) Theorem 5.1 follows from the formula (5.1) with $t=l$. (2) Let t be an element of T such that its centralizer coincides with Γ. From a result of C.W. Curtis [7] we have $\chi(t)=\zeta_\chi(1)$ for any % in $i[1_B|B\to G]$, where ζ_χ is the character of W corresponding to % by a fixed isomorphism between $\boldsymbol{C}[W]$ and $H_C(G, B)$ (see [19; Theorem 48]). Since $\zeta_{\hat{\chi}}(w)=(-1)^{l(w)}\zeta_\chi(w)$ for we W, (5.1) holds in this case. (3) When G is of type A_n and $G=\mathfrak{G}_\sigma$ is untwisted, (5.1) can be proved for an arbitrary semisimple element t using a result of J.A. Green [11] (see also [13]).

Now we turn to the proof of Theorem 5.1. First we prepare some lemmas.

Lemma 5.4. (a) *Let X be a subset of S. Then $X^* = \{x^*= w_S x w_S \setminus w\in X\}$ is also a subset of S.*

(b) *Let X and X^* be as above. An element w of W is* (φ, X)*-reduced if and only if ww_Xw_S is* (φ, X^*)*-reduced.*

Proof. (a) This appears in [3; p. 43, Ex. 22].

(b) Since $(X^*)^*=X$ and $w_Xw_Sw_{X^*}w_S=1$, it suffices to prove the if-part. Assume that ww_Xw_S is (φ, X^*)-reduced. Then, $l(ww_Xw_S\, w_{X^*}x^*)=l(ww_Xw_S)+l(w_{X^*}x^*)$ for all $x\in X$. Hence, for all $x\in X$, $l(wxw_S)=l(ww_Xw_S)+l(w_{X^*})-1=l(ww_Xw_Sw_*)-1=l(ww_S)-1$. Therefore $l(wx)=l(w)+1$ for all $x\in X$. Thus w is (φ, X)-reduced.

Lemma 5.5. *Let X and X^* be as in Lemma 5.4. Let w' be a* (φ, X)*-reduced element of W. Put $P=w'P_Xw'^{-1}$ and $P^*=w'w_Xw_SP_{X^*}w_S^{-1}w_X^{-1}w'^{-1}$. Then, for an arbitrary element w of W, the following formula holds.*

$$|G^1\cap BwB\cap P|\,|U^-_{w'}| = \textstyle\sum_{v\in W}[\hat{e}_w: e]\,|BvB\cap V_{P^*}|\,|U^+_{w'}|\,.$$

Proof. By Theorem 4.1, the left hand side of the above formula is equal to

$$\mathrm{I}\, BwB\, \Pi\, rfw_x\, U^-_{w'w_X}w_X^{-1}w'^{-1}\, \mathrm{I}\, U\,.$$

By Theorem 2.6 (b) and Lemma 2.14(c), this is equal to

$$\begin{aligned}
&[e_we_{w'w_X}:e_{w'w_X}]\,|U^-| = [\hat{e}_we_{ww_Xw_S'}\, e_{w'w_Xw_S}]\,|U^-|\\
&= \textstyle\sum_{v\in W}[\hat{e}_w: e_v][e_ve_{w'w_Xw_S}:e_{w'w_Xw_S}]\,|U^-|\\
&= \textstyle\sum_{v\in W}[\hat{e}_w: e_v]\,|BvB\cap w'w_Xw_SUw_S^{-1}w_X^{-1}w'^{-1}\cap U^-|\,|U|\,.
\end{aligned}$$

On the other hand

$$\begin{aligned}
V_{P^*} &= w'w_Xw_SV_{X^*}w_S^{-1}w_X^{-1}w'^{-1}\\
&= w'w_Xw_S(U\cap w_Sw_Xw_SUw_S^{-1}w_X^{-1}w_S^{-1})\,w_S^{-1}w_X^{-1}w'^{-1}\\
&= w'w_Xw_SUw_S^{-1}w_X^{-1}w'^{-1}\cap w'w_SUw_S^{-1}w'^{-1}\\
&= (w'w_Xw_SUw_S^{-1}w_X^{-1}w'^{-1}\cap U^-)(w'w_SUw_S^{-1}w'^{-1}\cap U)\,,
\end{aligned}$$

where we used Lemma 1.4. Hence we have

$$|BvB\cap V_{P^*}| = |BvB\cap w'w_Xw_SUw_S^{-1}w_X^{-1}w'^{-1}\cap U^-|\,|U^-_{w'}|\,.$$

Therefore

$$|G^1\cap BwB\cap P|\,|U^-_{w'}| = \textstyle\sum_{v\in W}[\hat{e}_w : e_v]\,|BvB\cap V_{P^*}|\,|U|\,|U^-_{w'}|^{-1}\,.$$

Since $|U|\,|U^-_{w'}|^{-1} = |U^+_{w'}|$ by Lemma 1.4 (a), the proof of Lemma 5.5 is over.

Lemma 5.6. (a) *For $w\in W$, let $w^*=w_Sww_S^{-1}$. Then $|BwB\cap C|=|Bw^*B\cap C|$ for any conjugacy class C of G.*

(b) *Let X and X^* be as in Lemma 5.4 (a). Then $i[\xi_X|P_X\to G]=i[\xi_{X^*}|P_{X^*}\to G]$.*

Proof. (a) By Theorem 2.6 (c), $e_{w_S}e_w(e_{w_S})^{-1}=e_{w^*}e_{w_Sw^{-1}}e_w(e_{w_Sw^{-1}}e_w)^{-1}=e_{w^*}$. Hence, by Lemma 2.5 we get $f_{e_w}=f_{e_{w^*}}$, from which (a) follows.

(b) By Lemma 4.6 and the definition of Y^* for $Y\subset S$,

$$i[\xi_X|P_X\to G]=\textstyle\sum_{Y\subset X}(-1)^{|Y|}i[1_{P_Y}|P_Y\to G]$$

and $$i[\xi_{X^*}|P_{X^*}\to G]=\textstyle\sum_{Y\subset X}(-1)^{|Y^*|}i[1_{P_{Y^*}}|P_{Y^*}\to G]\,.$$

Hence, for the proof of (b) it suffices to prove

$$(5.2)\qquad i[1_{P_Y}|P_Y\to G]=i[1_{P_{Y^*}}|P_{Y^*}\to G]$$

for $Y\subset S$. By the Bruhat decomposition of P_Y and P_{Y^*}, we have

$$|P_Y\cap C_G(x)| = \textstyle\sum_{w\in W_Y}|BwB\cap C_G(x)|$$

and $$|P_{Y^*}\cap C_G(x)| = \textstyle\sum_{w\in W_Y}|Bw^*B\cap C_G(x)|$$

for any $x\in G$. Hence $|P_Y\cap C_G(x)|=|P_{Y^*}\cap C_G(x)|$ by part (a). This fact, together with the definition of induced characters, implies (5.2). The proof of (b) is over.

Lemma 5.7. *Let X be a subset of S, w an element of W, and x an element of G. Then*

(a) $$f_{e_w}(x)i[\xi_X|P_X\to G](x)|C_G(x)| = |G|\,|B|^{-1}\textstyle\sum_{P'\in C_G(P_X)}|BwB\cap V_{P'}\cap C_G(x)|\,|U^-_{w_X}|$$

(b) $$f_{e_w}(x)i[1_{P_X}|P_X\to G](x)|C_G(x)| = |G|\,|B|^{-1}\textstyle\sum_{P'\in C_G(P_X)}|BwB\cap P'\cap C_G(x)|\,.$$

Proof. (a) Consider the set $\mathcal{H}=\mathcal{H}(x, w, X)$ of all triplets $(x', B', P')\in C_G(x)\times\mathcal{B}\times C_G(P_X)$ such that $(x'B'x'^{-1}B')\in\mathcal{O}_w$ and $x'\in V_{P'}$, where $\mathcal{B}$ and $\mathcal{O}_w$ ($w\in W$) are as in Lemma 3.5. From Lemma 3.6 (b) and Lemma 4.10, we have

$$(5.3)\qquad S_1 = f_{e_w}(x)\, i[\xi_X \mid P_X \to G](x) \mid C_G(x) \mid \mid U^-_{w_X} \mid^{-1}.$$

On the other hand, $\mathcal{H}$ can be decomposded into a disjoint union $\bigcup_{B_1 \in \mathcal{B}} \mathcal{H}(B_1)$, where $\mathcal{H}(B_1) = \mathcal{H} \cap (C_G(x) \times \{B_1\} \times C_G(P_X))$. Clearly $|\mathcal{H}(B_1)| = |\mathcal{H}(B)| = \sum_{P' \in C_G(P_X)} |BwB \cap V_{P'} \cap C_G(x)|$. Hence

$$(5.4)\qquad |\mathcal{H}| = |G|\,|B|^{-1} \sum_{P' \in C_G(P_X)} |BwB \cap V_{P'} \cap C_G(x)|.$$

The formula (a) follows from (5.3) and (5.4).

(b) Consider the set $Q(x, X)$ of all $P' \in C_G(P_X)$ such that $x \in P'$. Then, by a similar argument as in the proof of Lemma 4.10, we get

$$(5.5)\qquad |Q(x, X)| = i[1_{P_X} \mid P_X \to G](x).$$

Next, consider the set $\mathcal{L} = \mathcal{L}(x, w, X)$ of all triplets $(x', B', P') \in C_G(x) \times \mathcal{B} \times C_G(P_X)$ such that $(x'B'x'^{-1}, B') \in \mathcal{O}_w$ and $x' \in P'$. Using (5.5) instead of Lemma 4.10, we get

$$\begin{aligned} |\mathcal{L}| &= f_{e_w}(x)\, i[1_{P_X} \mid P_X \to G](x) \mid C_G(x) \mid \\ &= |G|\,|B|^{-1} \sum_{P' \in C_G(P_X)} |BwB \cap P' \cap C_G(x)| \end{aligned}$$

by a similar argument as in (a). This proves (b).

Proof of Theorem 5.1.

Let a_χ be a primitive idempotent of $H_C(G, B)$ corresponding to χ (see Lemma 2.2). Then $\chi = f_{a_\chi} = \sum_{w \in W} [a_\chi : e_w] f_{e_w}$ by Lemma 2.4 (a). Since $\hat{a}_\chi = \sum_{w \in W} [a_\chi : e_w] \hat{e}_w = \sum_{w \in W} [a_\chi : e_w] (\sum_{v \in W} [\hat{e}_w : e_v] e_v)$, $\hat{\chi} = f_{\hat{a}_\chi} = \sum_{w \in W} [a_\chi : e_w] (\sum_{v \in W} [\hat{e}_w : e_v] f_{e_v})$ Hence, for the proof of Theorem 5.1 it suffices to show

$$\begin{aligned} (5.6)\qquad & \sum_{u \in G^1} f_{e_w}(u)\, i[1_{P_X} \mid P_X \to G](u) \\ &= \sum_{v \in W} [\hat{e}_w : e_v] \left(\sum_{u \in G^1} f_{e_v}(u)\, i[\xi_X \mid P_X \to G](u)\right) \end{aligned}$$

for each $w \in W$ and $X \subset S$. By Lemma 5.7 (b) and Lemma 1.12, the left side of (5.6) is

$$|G|\,|B|^{-1} \sum_{w'} |G^1 \cap BwB \cap w' P_X w'^{-1}|\,|U^-_{w'^{-1}}|,$$

where the sum is over the set of all (ϕ, X)-reduced elements w' of W, By Lemma 5.5, this is equal to

$$|G|\,|B|^{-1} \sum_{w'} \sum_{v \in W} [\hat{e}_w : e_v] \mid BvB \cap w' w_X w_S V_{X^*} w_S^{-1} w_X^{-1} w'^{-1} \mid \mid U^+_{w'} \mid.$$

By Lemma 5.7 (a) and Lemma 1.12, this is

$$\sum_{v \in W} [\hat{e}_w : e_v] \left(\sum_{u \in G^1} f_{e_v}(u)\, i[\xi_{X^*} \mid P_{X^*} \to G](u)\right),$$

which is the right hand side of (5.6) by Lemma 5.6 (b). This completes the proof of Theorem 5.1.

6. Regular unipotent elements and induction from the subgroup U

The main results in this section are Lemma 6.10 and Theorem 6.12. We begin by recalling some known facts on regular unipotent elements.

An element x of a connected semisimple linear algebraic group © is called regular if the dimension of its centralizer $Z_{\mathfrak{G}}(x)$ is equal to the rank of ©. R. Steinberg [18] proved the following

Theorem 6.1. *Let $\mathfrak{G}$ be a connected semisimple linear algebraicgroup, $\mathfrak{B}$ a Borel subgroup, and $\mathfrak{T}$ a maximal torus contained in $\mathfrak{B}$. Let* U, Σ, Σ^+ *and* $\mathfrak{U}_\alpha$ $(\alpha\in\Sigma)$ *be as in* §1.

(a) *A unipotent element is regular if and only if it is contained in a unique Borel subgroup.*

(b) *An element* $x=\prod_{\alpha\in\Sigma^+}x_\alpha(x_\alpha\in\mathfrak{U}_\alpha)$ *of* $\mathfrak{G}$ *contained in* 11 *is regular if and only if* $x_\alpha\neq 1$ *for every simple root a.*

DEFINITION 6.2. Let $\Sigma=\Sigma_1\cup\Sigma_2\cup\cdots\cup\Sigma_k$ be the decmposition of the root system Σ of a connected semisimple group © into its irreducible components. Let $\tilde{\alpha}_i$ be the highest root of Σ_i in some order. Express α_i as an integral linear combination of the simple roots of Σ_i. If p is a prime number which does not devide any coefficient in such expressions for each $\imath$, p is called *good* for ©.

REMARK 6.3. For each simple type, good primes p can be directly defined as follows:

A_n: p arbitrary; B_n, C_n, D_n: $p\neq 2$; E_6, E_7, F_4, G_2: $p\neq 2$, 3; E_8: $p\neq 2$, 3, 5.

In the following, the notations in §1 will be used. We denote by G_r^1 the set of regular unipotent elements in a finite Chevalley group $G=\mathfrak{G}_\sigma$.

Lemma 6.4. (a) *Each element of G_r^1 is contained in a unique Borel subgroup.*

(b) *Let C be a regular unipotent conjugacy class of G. Then* $\backslash C\backslash=|G||B|^{-1}|B\cap C|$.

(c) $Z_G(u)=Z_B(u)=Z(G)Z_U(u)$ *fu*$\in B\cap G_r^1$, *where $Z(G)$ is the center of G, and $Z_G(u)$, $Z_B(u)$ and $Z_U(u)$ are centralízers of u in G, B and U respectively.*

Proof. Part (a) follows from Theorem 6.1 (a) and part (b) follows from part (a) and the fact that B is its own normalizer in G. Part (c) follows from [1 E-54, 1.14(a)].

Theorem 6.5. (Springer and Steinberg [1; E–55]) *Let $G=\mathfrak{G}_\sigma$, andp the characteristic of the field K over which ® is defined. Assume that © is adjoint and p is goodfor ©. Then the set G_r^1 of regular unipotent elements of G forms a single conjugacy class.*

Lemma 6.6. (Steinberg [19; p. 197 (2)]) *For* $a \in R(S)$, *let* $U_a = (\prod_{\alpha \in a} \mathfrak{U}_\alpha)_\sigma$ *as in* §1 *and* U_a^1 *the set of all elements* $u_a = \prod_{\alpha \in a} x_\alpha$ $(x_\alpha \in \mathfrak{U}_\alpha)$ *of* U_a *such that* $x_\alpha = 1$ *for every* $\alpha \in a \cap \Pi$.

(a) *The quotient group* U_a/U_a^1 *is isomorphic to the additive group of the Galois field* $\boldsymbol{F}_q$, *where* $q = \prod_{\alpha \in a \cap \Pi} q(\alpha)$.

(b) U_a^1 *is the derived group of* U_a.

Proof. (a) Let $\mathfrak{U}_a^1 = \prod_\alpha \mathfrak{U}_\alpha$ $(\alpha \in a - a \cap \Pi)$. This is σ-stable by Lemma 1.1. The quotient group $\mathfrak{U}_a/\mathfrak{U}_a^1$ is cannonically isomorphic to the direct product $\mathfrak{D}_a$ of the groups $\mathfrak{U}_\alpha$ for $\alpha \in a \cap \Pi$, and σ acts on the factors according to the formula in Lemma 1.1 (b). Let $a \cap \Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, where the suffixes are so chosen that

$$\sigma x_{\alpha_i}(k) = x_{\alpha_{i+1}}(c_i k^{q(i)}) \qquad (1 \leqq i \leqq n-1)$$

and

$$\sigma x_{\alpha_n}(k) = x_{\alpha_1}(c_n k^{q(n)})$$

for some $c_i \in K^*$ and all $k \in K$, where we wrote $q(i)$ for $q(\alpha_i)$. Let $(x_{\alpha_i}(k_i))_{1 \leqq i \leqq n}$ $(k_i \in K)$ be an element of $(\mathfrak{D}_a)_\sigma$. Then $c_1 k_1^{q(1)} = k_2$, $c_2 k_2^{q(2)} = k_3$, $\cdots$, $c_n k_n^{q(n)} = k_1$. From this fact and Lemma 1.1 (d), it follows that $(\mathfrak{D}_a)_\sigma$, hence $(\mathfrak{U}_a/\mathfrak{U}_a^1)_\sigma$, is isomorphic to $\boldsymbol{F}_q$. To complete the proof of (a) it suffices to notice that $U_a/U_a^1 \cong (\mathfrak{U}_a/\mathfrak{U}_a^1)_\sigma$, which follows from [20; 10.11].

(b) This can be checked directly using [19; Lemma 63].

Lemma 6.7. (a) $|B \cap G_r^1| = |B| \left(\prod_{\alpha \in \Pi} q(\alpha)\right)^{-1}$.
(b) $|G_r^1| = |G| \left(\prod_{\alpha \in \Pi} q(\alpha)\right)^{-1}$.

Proof. Part (b) follows from part (a) and Lemma 6.4 (b). We shall prove part (a). Let $\mathfrak{U}^1$ be the subgroup of $\mathfrak{U}$ generated by the group $\mathfrak{U}_\alpha$ for $\alpha \in \Sigma^+ - \Pi$. Since $\mathfrak{U}^1$ is normalized by $\mathfrak{T}$ and fixed by σ, $|\mathfrak{U}_\sigma^1| = \prod_{\alpha \in \Sigma^+ - \Pi} q(\alpha)$ by [20; 11.8]. It follows from [20; 10.11] that a coset $u\mathfrak{U}^1$ $(u \in \mathfrak{U})$ contains a σ-fixed elements if and only if $u\mathfrak{U}^1 \in (\mathfrak{U}/\mathfrak{U}^1)_\sigma$. In that case, the number of σ-fixed elements in $u\mathfrak{U}^1$ is clearly $|\mathfrak{U}_\sigma^1|$. On the other hand, it follows from Theorem 6.1 (b) that the set $\mathfrak{U}^r$ of regular unipotent elements is a union of $\mathfrak{U}^1$-cosets. The quotient set $\mathfrak{U}^r/\mathfrak{U}^1$ is canonically isomorphic to the direct product of the sets $\mathfrak{U}_\alpha - \{1\}$ for $\alpha \in \Pi$. Hence, by the proof of Lemma 6.6 (a) we get

$$|(\mathfrak{U}^r/\mathfrak{U}^1)_\sigma| = \prod_{a \in R(S)} \{(\prod_{\alpha \in a \cap \Pi} q(\alpha)) - 1\} .$$

Therefore, the number of regular unipotent elements in B is

$$\prod_{a \in R(S)} \{(\prod_{\alpha \in a \cap \Pi} q(\alpha)) - 1\} \prod_{\alpha \in \Sigma^+ - \Pi} q(\alpha) .$$

Since $|B| = \prod_{a \in R(S)} \{(\prod_{\alpha \in a \cap \Pi} (q(\alpha)) - 1\} \prod_{\alpha \in \Sigma^+} q(\alpha)$ ([20; 11.9, 10.10]), the proof of Lemma 6.7 is over.

For $a \in R(S)$, let L_a be the set of non-trivial linear character of U_a and L

the **direct** product of the sets L_a for $a \in R(S)$.

DEFINITION 6.8. For $X \subset S$ and $l=(l_a)_{a\in R(S)} \in L$, the linear character $\gamma_{X,l}$ of U is defined by $\gamma_{X,l}(u)=1$ if $\phi=X$ and by $\gamma_{X,l}(u)=\prod_a l_a(u_a)(a \in R(S)_X)$ if $X \neq \phi$, where $u=\prod_{a>0} u_a$ $(u_a \in U_a)$ and $R(S)_X=\{a \in R(S) \mid w_a \in X\}$.

Lemma 6.9. *Let $G=\mathfrak{G}_\sigma$, where © is adjoint. For $X \subset S$ and $l \in L$, let*

$$\Gamma_{X,l} = i[\gamma_{X,l} \mid U \to G] .$$

Then the character $\Gamma_{X,l}$ is independent of $l \in L$.

Proof. Let $\mathfrak{U}^1$ and $\mathfrak{U}^r$ be as in the proof of Lemma 6.7 (a). By (1.1) $\mathfrak{T}$ acts naturally on the set $\mathfrak{U}^r/\mathfrak{U}^1$. Hence T acts on $(\mathfrak{U}^r/\mathfrak{U}^1)_\sigma$. Let t be an arbitrary element of $\mathfrak{T}-\{1\}$. Since $\mathfrak{G}$ is adjoint, $\alpha(t) \neq 1$ for some $\alpha \in \Pi$. Hence the action of each $t \in T$ on $(\mathfrak{U}^r/\mathfrak{U}^1)_\sigma$ is non-trivial. On the other hand, $|T|=|(\mathfrak{U}^r/\mathfrak{U}^1)_\sigma|=\prod_{a\in R(S)}\{(\prod_{\alpha\in a\cap\Pi} q(\alpha))-1\}$ by [20; 11.2] and the proof of Lemma 6.7 (a). Hence the action of T on $(\mathfrak{U}^r/\mathfrak{U}^1)_\sigma$ is simply transitive. This fact, together with Lemma 6.6, implies that the action of T on L defined by

$$l_a^t(u_a) = l_a(tu_at^{-1})(t \in T,\ a \in R(S),\ l_a \in L_a,\ u_a \in U_a)$$

is simply transitive. Hence, for $u \in U$, we have

$$i[\gamma_{X,l} \mid U \to B](u) = \sum_{l' \in L} \gamma_{X,l'}(u) ,$$

which is independent of $l \in L$. The lemma follows from this fact and transitivity of induction.

In the following, if © is adjoint, we write Γ_X for the character $\Gamma_{X,l}$ $(X \subset S,\ l \in L)$ of $G=\mathfrak{G}_\sigma$.

We can now prove a key lemma:

Lemma 6.10. *For $X \subset S$ and $l \in L$, let $\Lambda_l=\sum_{X\subset S}(-1)^{|X|}\Gamma_{X,l}$.*

(a) *$\sum_{l\in L}\Lambda_l(x)$ is equal to $|L|\,|G|\,|G_r^1|^{-1}$ if $x \in G_r^1$ and 0 if $x \in G-G_r^1$.*

(b) *Assume that © is adjoint. Then $\Lambda(x)=\sum_{X\subset S}(-1)^{|X|}\Gamma_X(x)$ is equal to $|G|\,|G_r^1|^{-1}$ if $x \in G_r^1$ and 0 if $x \in G-G_r^1$.*

Proof. The proof depends on the following two results.

(1) For any $l \in L$, $\Lambda_l(x)=0$ if $x \in G-G_r^1$.

(2) For any $l \in L$, $(\Lambda_l, 1_G)_G=1$.

Let us deduce the lemma from (1) and (2).

(a) By Lemma 6.6 the function $\sum_{l\in L}\gamma_{X,l}$ on U takes the constant value $(-1)^{|X|}$ on the set of regular unipotent elements of U. Hence, by (1), we see that $\sum_{l\in L}\Lambda_l$ vainshes on $G-G_r^1$ and takes a constant value Q on G_r^1. Therefore $(\sum_{l\in L}\Lambda_l,\ 1_G)=|G|^{-1}\,|G_r^1|\ O$. On the other hand, we have

$(\sum_{l\in L}\Lambda_l, 1_G)_G = |L|$ from (2). Hence we obtain $Q= \backslash L\backslash\ \backslash G\ \backslash\ |G_r^1|^{-1}$, as required.

(b) This follows from part (a) and Lemma 6.9.

Next, we consider (1). Let $u=\prod_{a>0}u_a(u_a\in U_a)$be an element of U. Then

$$\sum_{X\subset S}(-1)^{|X|}\gamma_{X,l}(u) = \prod_a(1-l_a(u_a)) \qquad (a\in R(S)) .$$

By Lemma 6.6, this is 0 if u is not regular unipotent. Hence Λ_l vaishes on $G-G_r^1$. This proves (1).

It remains to prove (2). By Frobenius reciprocity,

$$(\Gamma_{X,l}, 1_G)_G = |U|^{-1}\sum_{u\in U}\gamma_{X,l}(u) ,$$

which is 0 if $X \neq \phi$ and 1 if $X=\phi$. The assertion (2) follows from this. The proof of Lemma 6.10 is now complete.

To state the first application of Lemma 6.10 we require the following notion due to Harish-Chandra.

DEFINITION 6.11. A complex valued function / on G is called a cusp form if

$$\sum_{u\in V_P} f(xu) = 0 ,$$

for all elements x of G and all parabolic subgroup $P\neq$ G. A character of G which is a cusp form is called a cuspidal character.

The importance of this notion is explained e.g. in [1; part *C]*. Some examples are given in [1; part D].

Theorem 6.12. *Let $G=\mathfrak{G}_\sigma$, and let % be an irreducible cuspidal character of G.*

(a) $|G_r^1|^{-1}\sum_{u\in G_r^1}\chi(u)=(-1)^{|S|}|L|^{-1}\{l\in L\,|\,\Gamma_{S,l}$ *contains* $\chi\}|$.

(b) *Assume that © is adjoint and the characteristic p of K is good for $\mathfrak{G}$. Then for any regular unipotent element $u\in G$, $\chi(u)$ equals $(-1)^{|S|}$ if Γ_S contains X and* 0 *if Γ_S does not contain χ.*

For the proof of Theorem 6.12 we require the following

Lemma 6.13. *Let % be a class function on G which is a cusp form. Then*

$$(\chi, \Gamma_{X,l})_G = 0$$

for any $X\subsetneq S$ and $l\in L$.

Proof. Put $\Gamma'_{X,l}= i[\gamma_{X,l}\,|\,U\to P_X]$. Since the unipotent radical V_X of P_X is normal in $P_X$and $\gamma_{X,l}$istrivial on V_X, $\Gamma'_{X,l}$is constant on each V_X-coset in P_X. Hence, using transitivity of induction and Frobenius reciprocity we have

$$(\chi, \Gamma_{X,l})_G = (\chi, i[\Gamma'_{X,l} | P_X \to G])_G$$
$$= |P_X|^{-1} \textstyle\sum_x \Gamma'_{X,l}(x) (\sum_{u \in V_X} \chi(xu)),$$

where the first sum is taken over a set of representatives for V_X-cosets in P_X. Therefore, from the definition of cusp forms, we obtain the lemma.

Proof of Theorem 6.12.

Part (b) follows from part (a), Theorem 6.5 and Lemma 6.9. We shall prove part (a). By Lemma 6.10 (a),

$$|G_r^1|^{-1} \textstyle\sum_{u \in G_r^1} \chi(u) = |L|^{-1} (\chi, \sum_{l \in L} \Lambda_l)_G .$$

Hence, by Lemma 6.13,

$$|G_r^1|^{-1} \textstyle\sum_{u \in G_r^1} \chi(u) = (-1)^{|S|} |L|^{-1} \sum_{l \in L} (\chi, \Gamma_{S,l})_G .$$

Therefore, the proof of Theorem 6.12 is completed by the following theorem, which is proved by I.M. Gelfand and M.I. Graev [10] for SL_n, and by R. Steinberg [19; Theorem 49] for general G.

Theorem 6.14. *Let l be an element of L, and χ an irreducible character of G. Then*

$$(\chi, \Gamma_{S,l})_G = 1 \text{ or } 0 .$$

7. Regular unipotent elements in B-cosets and characters in $i[1_B | B \to G]$

Our main purpose in this section is to prove Theorem 7.1 and Theorem 7.2 below. Let $G = \mathfrak{G}_\sigma$ be a finite Chevalley group and p the characteristic of the field K over which $\mathfrak{G}$ is defined. We also use other notations in §1 and §6.

Theorem 7.1. *Assume that p is good for $\mathfrak{G}$ in the sense of Definition 6.2. Let χ be a nontrivial irreducible character of G contained in $i[1_B | B \to G]$. Then χ vanishes identically on the set G_r^1 of regular unipotent elements in G.*

Theorem 7.2. *Assume that p is good for $\mathfrak{G}$. Let g be an arbitrary element of G, and C an arbitrary regular unipotent class of G. Then the number $|Bg \cap C|$ depends neither on g nor C.*

REMARK 7.3. (a) The author believes, but can not prove, that theorems 7.1 and 7.2 hold without the assumption "p is good"[3]. Later we shall prove weaker results which hold in all characteristics $p > 0$.

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3) See "added in proof" at the end of the paper.

(b) Combining Theorem 7.2 with Lemma 6.7 (a) and [1; E-55], we see that the number $|Bg\cap C|$ in Theorem 7.2 is

$$|Z(\mathfrak{G})/(1-\sigma)Z(\mathfrak{G})\,\mathrm{I}^{-1}\,(\prod_{\alpha\in\Pi}q(\alpha))^{-1}|B|,$$

where $Z(\mathfrak{G})$ is the center of & and $(1-\sigma)Z(\mathfrak{G})=\{x(x^{-1})^{\sigma}\,|\,x\in Z(\mathfrak{G})\}$

(c) In §3, we showed that each element g of G gives rise to a decomposition of G/B into disjoint union:

$$G/B=\cup_{w\in W}F_{g,w}\,.$$

Assume that p is good. Let u be a regular unipotent element in G. Then, combining Lemma 3.6, Lemma 6.4 (b) and Theorem 7.2, we obtain

(7.1) $$|F_{u,w} = |U_{w}^{-}$$

Now let ® be a connected semisimple linear algebraic group defined over an algebraically closed field of arbitrary characteristic. Let g be an element of ®, $(\mathfrak{B},\mathfrak{N})$ an ordinary BN-pair in © and $\mathfrak{W}$ its Weyl group. Then $\mathfrak{G}/\mathfrak{B}$ has the decomposition

$$\mathfrak{G}/\mathfrak{B}=\cup_{w\in\mathfrak{W}}\mathfrak{F}_{g,w}$$

(see Remark 3.7). Each set $\mathfrak{F}_{g,w}$ has a natural structure of algebraic variety. In the special case that $w=1$, $\mathfrak{F}_{g,w}$ has been studied by several authors (see [3] and [21]). The formula (7.1) in the finite case suggests an interesting problem:

Let u be a regular unipotent element in $\mathfrak{G}$. Study the variety $\mathfrak{F}_{u,w}$. I[s] it the $l(w)$-dimensional affine space ?

The proofs of Theorem 7.1 and Theorem 7.2 depend on the following

Lemma 7.4. *Let X be a subset of S. Let $\Gamma_{X,l}$ $(l\in L)$ be a character of G defined in Lemma 6.9, and ξ_X the Steinberg character of P_X. Then*

$$(\Gamma_{X,l},\chi)_G=(i[\xi_X\backslash P_\chi\to\mathrm{G}],\ \chi)_G$$

for any irreducble character χ of G contained in $i[1_B\backslash B\to G]$.

Proof. From Frobenius reciprocity and Lemma 4.6 (b) we have

$$(i[\gamma_{X,l}\mathrm{I}\ U\to P_X],\ \xi_X)_{P_X}=(\gamma_{X,l},\ \xi_X\,|\,U)_U=1\,.$$

Hence the character $i[\gamma_{X,l}\backslash\ U\to P_X]$ of P_X contains the Steinberg character ξ_X. By transitivity of induction,

$$(\Gamma_{X,l},\ \chi)_G\geqq(i[\xi_X\,|\,P_X\to G],\ \chi)_G$$

for any irreducible character % of G contained in $i[1_B\,|\,B\to G]$. Therefore the proof of the lemma will be completed by the following formula.

(7.2) $\quad (\Gamma_{X,l},\ i[1_B|B\to G])_G = (i[\xi_X|P_X\to G],\ i[1_B|B\to G])_G.$

We shall prove (7.2). By a theorem of Mackey (see e.g. [9; p. 51]) and the Bruhat decomposition of G, the left hand side of (7.2) is equal to

$$\sum_{w\in W} |U_w^+|^{-1}\{\sum_{u\in U_w^+}\gamma_{Xl}(u)\} = |\{w\in W|\gamma_{X,l} \text{ is trivial on } U_w^+\}\backslash\ .$$

By the definition of $\gamma_{X,l}$,it is trivial on U_w^+ if and only if $l(wx)<l(w)$for all $x\in X$. The number of such $w\in W$ is $\backslash\ W/W_X|$ by Lemma 1.11. Hence the left hand side of (7.2) is $|W/W_X|$. On the other hand, by Proposition 1.6 (c) and the Mackey's theorem used above, the right hand side of (7.2) is

$$\sum_{Y\subset X}(-1)^{|Y|}|W/W_Y| = |W/W_X|\sum_{Y\subset X}(-1)^{|Y|}|W_X/W_Y|\ .$$

By a result of E. Witt (see e.g. [17; p. 378]), this is equal to $|W/W_X|$. This proves (7.2).

Proof of Theorem 7.1.

From Theorem 3.1 (b) and Theorem 3.4 we may assume that © is adjoint. Using Lemma 7.4 we get

(7.3) $\quad (\sum_{X\subset S}(-1)^{|X|}\Gamma_X,\ \chi)_G = (\sum_{X\subset S}(-1)^{|X|}i[\xi_X|P_X\to G],\ \chi)_G$

for any irreducible character % of G contained in $i[1_B|B\to G]$. By Theorem 6.5 and Lemma 6.10 (b), the left hand side of (7.3) equals $\chi(u)$ with $u\in G_r^1$. On the other hand, by Lemma 4.8, the right hand side of (7.3) equals $(1_G,\chi)_G$. Threfore

(7.4) $\quad \chi(u) = (1_G,\chi)_G \qquad (u\in G_r^1)\ .$

Hence, if % is non-trivial, % vanishes on G_r^1. This proves Theorem 7.1.

Proof of Theorem 7.2.

The proof depends on the following two results.

(1) Let C and C' be two regular unipotent conjugacy classes of G. Then $|B\cap C|=|B\cap C'|$

(2) Let C be a regular unipotent conjugacy class of G. Then $|C|^{-1}|BwB\cap C|=\backslash G|^{-1}|BwB|$ for any $w\in W$.

We will show that Theorem 7.1 is a consequence of (1) and (2). Let C be a regular unipotent conjugacy class of G. By (2) and Lemma 6.4 (b)

(7.5) $\quad |BwB\cap C||B\cap C|^{-1} = |BwB||B|^{-1}\ .$

It follows from Lemma 1.5 that $|BwB\backslash\backslash B\backslash^{-1} = \backslash U_w^-\backslash$ and that

$$\sum_u u(Bw\cap C)u^{-1} \qquad (u\in U_w^-)$$

is a decomposition of $BwB \cap C$ into a disjoint union. Hence, for $w \in W$, we have

$$(7.6) \qquad |Bw \cap C = |B \cap C|$$

from (7.5). Let g be an arbitrary element of G. Then, by the Bruhat decomposition of G, we can write $g=b'n_w b$ with b, $V \in B$ and $w \in W$. Hence

$$|Bg \cap C| = |b^{-1}(Bw \cap C)b| = |Bw \cap C| = |B \cap C|$$

by (7.6). Combining this formula with (1), we obtain Theorem 7.2.

Next we prove (1). Let u and u' be regular unipotent elements in B i.e. in U. Let ψ: $\mathfrak{G} \to \mathfrak{G}'$ be as in Theorem 3.1 (b). By (3.5), Theorem 3.2 and Theorem 6.1 (b), $\psi(u)$ and $\psi(u')$ are regular unipotent elements in $\mathfrak{G}_\sigma'$, This fact, togethr with Lemma 6.4 (c) and Theorem 6.5, implies that $\psi(u)$ and $\psi(u')$ are conjugate in $\psi(\mathfrak{B})_\sigma$. Hence, by Theorem 3.2 (c), there exists an automorphism of U which maps u to u' '. Therefore, $Z_U(u) \simeq Z_U(u')$.Combining this fact with Lemma 6.4 (c) we obtain $|Z_G(u)\backslash = \backslash Z_B(u)| = Z_B(u')| = \backslash Z_G(u')|$. Hence $|B \cap C_G(u)| = |B \cap C_B(u)| = B \cap C_B(u') = |B \cap C_G(u')$. This proves (1).

It remains to prove (2). From Theorem 3.1 (b) and Theorem 3.4 we may assume that © is adjoint. Then, it follows from (7.4) and Lemma 2.4 (b) that

$$f_{e_w}(u) = (1_G, f_{e_w})_G$$

for any $w \in W$ and $u \in G_r^1$. From this formula we obtain $C_G(u)|\ ^1 |BwB \cap C_G(u) \mathrm{I} - \mathrm{I}\, G|^{-1} \mathrm{I}\, BwB|$, as required. The proof of Theorem 7.2 is now comlete.

As mentioned already, the author does not know whether the theorems 7.1 and 7.2 hold in all charactristics $p>0$ or not. Here we content ourselves with the following weaker results.

Theorem 7.5. (a) *Let % be a non-trivial irreducible character of G contained in* $i[1_B \mathrm{I}\, B \to G]$. *Let* G_r^1 *be the set of regular unipotent elements in G. Then*

$$\textstyle\sum_{u \in G_r^1} \chi(u) = 0 \,.$$

(b) *Let g be an arbitrary element of G. Then the number* $\backslash Bg \Pi G_r^1|$ *is independent of g.*

Proof. (a) From Lemma 7.4 we have

$$(7.7) \qquad \textstyle\sum_{l \in L} (\sum_{X \subset S} (-1^{|X|} \Gamma_{X,l}, \chi)_G \\ = |L| \, (\sum_{X \subset S} (-1)^{|X|} i[\xi_X | P_X \to G], \chi)_G \,.$$

Hence

(7.8) $\quad |L||G_r^1|^{-1}\sum_{u\in G_r^1}\chi(u) = |L|(1_G, \chi)_G$

by Lemma 6.10 (a) and Lemma 4.8. Part (a) follows from this formula.

(b) Let w be any element of W. By (7.8) and Lemma 2.4 (b),

$$|G_r^1|^{-1}\sum_{u\in G_r^1} f_{e_w}(u) = (1_G, f_{e_w})_G\,.$$

Hence
$$|G_r^1|^{-1}|BwB\cap G_r^1| = |G|^{-1}|BwB|\,.$$

Combining this formula with Lemma 6.4 (b) we obtain

$$|BwB\cap G_r^1| = |B|^{-1}|BwB||B\cap G_r^1|\,.$$

Therefore, by the same method as in the proof of Theorem 7.2, we get $|Bg\cap G_r^1| = |B\cap G_r^1|$ for any element g of G. This proves (b).

Added in proof. Recently, the author received two preprints (Lehrer [22] and Green and Lehrer [23]), in which some of our results, in particular theorems 6.12 (b), 7.1 and 7.2, are proved independently. In [23], it is remarked that theorems 7.1 and 7.2 do *not* hold without the assumption "p is good for $\mathfrak{G}$". This can also be seen from [24].

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