EXPLICIT RELATIVE TRACE FORMULAS FOR HILBERT MODULAR FORMS

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Contents

Introduction

0.1. In number theory, it is important to study L-functions and their special values. In this thesis, we focus on central values of automorphic L-functions for $\mathrm{GL}(2)$. Such central values have been studied by many mathematicians. As one of remarkable results, Waldspurger [48] gave a beautiful formula which relates the central L-value associated with a cuspidal automorphic representation of $\mathrm{GL}(2)$ to period integrals of cusp forms belonging the cuspidal representation along an elliptic torus of $\mathrm{GL}(2)$. Later, Jacquet [16], [17] gave another proof of Waldspurger's result introducing relative trace formulas. Inspired by Jacquet's result, Ramakrishnan and Rogawski [35] computed Jacquet's relative trace formula explicitly. In the case of holomorphic elliptic modular forms with a fixed weight and prime level, they gave an asymptotic formula of an average of the product of central L-values and their twists by an odd quadratic Dirichlet character, when the level goes to infinity. The detail is as follows. Let $k \geqslant 4$ be an even integer. For a prime number N, let $S_k^{\mathrm{new}}(N)$ be the space of all elliptic cuspidal new forms of weight k and level N (for $\Gamma_0(N)$). The space $S_k^{\mathrm{new}}(N)$ has an orthogonal basis $\mathcal{F}_k^{\mathrm{new}}(N)$ consisting of normalized Hecke eigen forms. For $\varphi \in S_k^{\mathrm{new}}(N)$, we denote by $L(s,\varphi)$ the completed automorphic L-function for φ whose center at a symmetry of a functional equation is $1/2$. Let η be a quadratic Dirichlet character of conductor D with $\eta(-1) = -1$. The Dirichlet L-series associated with η is denoted by $L_{\mathrm{fin}}(s,\eta)$. For a fixed prime $p \nmid D$, $\mathcal{I}_{p,\eta}^{+}$ denotes the set of all primes N satisfying both $\gcd(p,N) = \gcd(D,N) = 1$ and $\eta(N) = -1$. We define the p-th Fourier coefficient $a_p(\varphi)$ of φ by the Fourier expansion $f(z) = \sum_{n=1}^{\infty} n^{(k-1)/2} a_n(\varphi) e^{2\pi i n z}$. In this setting, Ramakrishnan and Rogawski proved the following theorem.

Theorem 0.1. [35, Theorem A] *For any interval $J \subset [-2,2]$, we have*

$$\lim_{\substack{N\to\infty \\ N\in\mathcal{I}_{p,\eta}^{+}}} \sum_{\substack{\varphi\in\mathcal{F}_k^{\mathrm{new}}(N), \\ a_p(\varphi)\in J}} \frac{L(1/2,\varphi)L(1/2,\varphi\otimes\eta)}{||\varphi||^2} = 2^{k-1}\frac{\{(k/2-1)!\}^2}{\pi(k-2)!} L_{\mathrm{fin}}(1,\eta)\mu_{p,\eta}(J),$$

where $||\varphi||$ denotes the Petersson norm of φ and $\mu_{p,\eta}$ denotes the probability measure on $[-2,2]$ defined by

$$\mu_{p,\eta}(x) = \begin{cases} \dfrac{p-1}{(p^{1/2}+p^{-1/2}-x)^2}\ \mu_{\mathrm{ST}}(x) & (\eta(p)=1), \\[2ex] \dfrac{p+1}{(p^{1/2}+p^{-1/2})^2-x^2}\ \mu_{\mathrm{ST}}(x) & (\eta(p)=-1). \end{cases}$$

Here, $\mu_{\mathrm{ST}}(x)$ is the Sato-Tate measure $(2\pi)^{-1}\sqrt{4-x^2}dx$.

Feigon and Whitehouse [6] generalized their result to the case of holomorphic Hilbert modular forms, imposing that the level is square-free and that a quadratic Hecke character concerned with twisted L-values is non-trivial at all archimedean places, by using a refined Waldspurger's formula [25] and the relative trace formula given by [18]. For non-holomorphic Hilbert modular forms, Tsuzuki [47] gave an analogous result for even Hilbert Maass forms by using automorphic Green functions on $\mathrm{GL}(2)$. Even in his result, the condition that the squre-freeness of levels was not able to be removed from assumptions.

In this thesis, we generalize results [35], [6] and [47] to several directions, without the square-freeness of level and the oddness condition on a quadratic Hecke character. Essential ingredients are automorphic Green functions on $\mathrm{GL}(2)$, which were introduced by Tsuzuki and explicit relative trace formulas for $\mathrm{GL}(2)$ resulting from automorphic Green functions. This thesis is based on [42], [43] and [44], which were given during the author's doctoral program. Here, [43] and [44] are joint works with Masao Tsuzuki.

To state our results in this thesis, we prepare some notation. Let F be a totally real algebraic number field of finite degree and $\mathfrak{o}$ its integer ring. The adele ring of F is denoted by $\mathbb{A}$. Let D_F be the absolute value of the discriminant of $F/\mathbb{Q}$. We denote by Σ_∞ and Σ_{fin} the set of all infinite places and all finite places of F, respectively. For each $v \in \Sigma_F = \Sigma_\infty \cup \Sigma_{\mathrm{fin}}$, we denote by $|\cdot|_v$ the modulas of the completion

F_v of F at v and fix a uniformizer ϖ_v of the integer ring $\mathfrak{o}_v$ of F_v if $v \in \Sigma_{\rm fin}$. Then, $q_v = |\varpi_v|_v^{-1}$ is the order of $\mathfrak{o}_v/\varpi_v\mathfrak{o}_v$. For an ideal $\mathfrak{a}$ of $\mathfrak{o}$, let $S(\mathfrak{a})$ denote the set of all $v \in \Sigma_{\rm fin}$ such that ${\rm ord}_v(\mathfrak{a}) \geqslant 1$. The absolute norm of $\mathfrak{a}$ is denoted by ${\rm N}(\mathfrak{a})$.

Fix a real valued character $\eta = \prod_{v\in\Sigma_F} \eta_v$ of $F^\times\backslash\mathbb{A}^\times$ of conductor $\mathfrak{f}_\eta$ and fix a finite subset S of Σ_F such that $S \cap S(\mathfrak{f}_\eta) = \emptyset$. Then, $\eta(t) \in \{\pm 1\}$ for all $t \in \mathbb{A}^\times$. Let $\mathcal{I}_{S,\eta}^{+}$ (resp. $\mathcal{I}_{S,\eta}^{-}$) be the set of all ideals $\mathfrak{n}$ of $\mathfrak{o}$ satisfying the following three conditions:

(1) $S(\mathfrak{n}) \cap S(\mathfrak{f}_\eta) = \emptyset$ and $S(\mathfrak{n}) \cap S = \emptyset$,
(2) $\eta_v(\varpi_v) = -1$ for all $v \in S(\mathfrak{n})$,
(3) $\{\prod_{v\in\Sigma_\infty} \eta_v(-1)\}\tilde{\eta}(\mathfrak{n}) = +1 \qquad$ (resp. $\{\prod_{v\in\Sigma_\infty} \eta_v(-1)\}\tilde{\eta}(\mathfrak{n}) = -1$).

Here we put $\tilde{\eta}(\mathfrak{n}) = \prod_{v\in\Sigma_{\rm fin}} \eta_v(\varpi_v^{{\rm ord}_v(\mathfrak{n})})$. The completed Hecke L-function for η is denoted by $L(s, \eta)$.

Let π be an irreducible cuspidal automorphic representation of ${\rm GL}(2, \mathbb{A})$ with trivial central character. We denote by $L(s, \pi)$ the completed standard autumorphic L-function associated with π. The adjoint L-function $L(s, \pi, {\rm Ad})$ of π is the completed standard automorphic L-function of the adjoint lift of π, which is a cuspidal automorphic representation of ${\rm GL}(3, \mathbb{A})$ (cf. [7]). Let $\mathfrak{f}_\pi$ be the conductor of π and S_π the set of all finite places of F such that ${\rm ord}_v\, \mathfrak{f}_\pi \geqslant 2$. Then, $L^{S_\pi}(s, \pi, {\rm Ad})$ is defined as the product of v-th Euler factors $L(s, \pi_v, {\rm Ad})$ over all $v \in \Sigma_F - S_\pi$.

Throughout this thesis, automorphic L-functions are normalized so that their functional equations have $1/2$ as a center of symmetry, and we use the subscript "fin" to represent non-completed L-functions, such as $L_{\rm fin}(s, \eta)$, $L_{\rm fin}(s, \pi)$. For any self-dual irreducible cuspidal automorphic representation π of ${\rm GL}(2, \mathbb{A})$, the epsilon factor of π is denoted by $\epsilon(s, \pi)$, which has an explicit form $\epsilon(s, \pi) = \epsilon(1/2, \pi)\{{\rm N}(\mathfrak{f}_\pi)D_F^2\}^{1/2-s}$ with $\epsilon(1/2, \pi) \in \{\pm 1\}$. Then, the functional equation for $L(s, \pi)$ is of the form $L(s, \pi) = \epsilon(s, \pi)L(1-s, \pi)$.

From Part 1 to Part 3, we consider asymptotic formulas of

$$\frac{1}{{\rm N}(\mathfrak{n})}\sum_{\pi} \frac{L(1/2, \pi)L(1/2, \pi\otimes\eta)}{L^{S_\pi}(1, \pi, {\rm Ad})}\alpha(\nu_S(\pi))$$

with a quantity $\alpha(\nu_S(\pi))$, where π runs over irreducible cuspidal automorphic representatons of ${\rm PGL}(2, \mathbb{A})$ with $\mathfrak{f}_\pi = \mathfrak{n}$, being imposed some conditions on its archimedean components.

0.2. Part 1 : Results for non-holomorphic even Hilbert Maass forms. In Part 1, we explain a generalization of [47] to the case of arbitrary levels. Let $\mathbf{K}_\infty$ be the standard maximal compact subgroup of ${\rm GL}(2, F \otimes_{\mathbb{Q}} \mathbb{R})$. For an ideal $\mathfrak{n}$ of $\mathfrak{o}$, let $\Pi_{\rm cus}(\mathfrak{n})$ denote the set of all irreducible cuspidal automorphic representations of ${\rm PGL}(2, \mathbb{A})$ with $\mathfrak{f}_\pi$ dividing $\mathfrak{n}$. We denote by $\Pi_{\rm cus}^*(\mathfrak{n})$ the set of all $\pi \in \Pi_{\rm cus}(\mathfrak{n})$ with $\mathfrak{f}_\pi = \mathfrak{n}$. We assume that $S \supset \Sigma_\infty$ and that all archimedean components of η are trivial.

For an ideal $\mathfrak{n}$ of $\mathfrak{o}$ relatively prime to S and $\pi = \otimes_v\pi_v \in \Pi_{\rm cus}(\mathfrak{n})$, π_v is isomorphic to a unitarizable spherical principal series representation $I_v(\nu_v) = {\rm Ind}_{B(F_v)}^{{\rm GL}(2,F_v)}(|\cdot|_v^{\nu_v/2} \boxtimes |\cdot|_v^{-\nu_v/2})$ of ${\rm GL}(2, F_v)$ for all $v \in S$, where B is the Borel subgroup of ${\rm GL}(2)$ consisting of all upper triangular matrices. We can take ν_v so that $\nu_v \in \mathfrak{X}_v^{0+}$, where $\mathfrak{X}_v^{0+} = i\mathbb{R}_{\geqslant 0} \cup (0, 1)$ for $v \in \Sigma_\infty$ and $\mathfrak{X}_v^{0+} = i[0, 2\pi(\log q_v)^{-1}] \cup \{x + iy \mid x \in (0, 1),\ y \in \{0, 2\pi(\log q_v)^{-1}\}\ \}$ for $v \in S_{\rm fin} = S \cap \Sigma_{\rm fin}$, respectively. Such $\nu_v \in \mathfrak{X}_v^{0+}$ is denoted by $\nu_v(\pi)$. The spectral parameter $\nu_S(\pi)$ of π at S is defined as $\nu_S(\pi) = (\nu_v(\pi))_{v\in S} \in \mathfrak{X}_S^{0+} = \prod_{v\in S} \mathfrak{X}_v^{0+}$.

Set $\mathfrak{X}_v^0 = \mathfrak{X}_v^{0+} \cap i\mathbb{R}$ for any $v \in S$ and $\mathfrak{X}_S^0 = \prod_{v\in S} \mathfrak{X}_v^0$. We define a positive Radon measure λ_S^η on $\mathfrak{X}_S^0$ by $4D_F^{3/2}L(1, \eta)\bigotimes_{v\in S} \lambda_v^{\eta_v}$, and for each $v \in S$, the measure $\lambda_v^{\eta_v}$ on $\mathfrak{X}_v^0$ is given by

$$d\lambda_v^{\eta_v}(iy) = \frac{L(1/2, I_v(iy))L(1/2, I_v(iy)\otimes\eta_v)}{L(1, \eta_v)} \times \begin{cases} \dfrac{1}{4\pi}|\Gamma(iy/2)|^{-2}dy & (v \in \Sigma_\infty), \\ \dfrac{\log q_v}{4\pi}|1 - q_v^{-iy}|^2dy & (v \in S_{\rm fin}). \end{cases}$$

Then, we remark that for $v \in \Sigma_{\mathrm{fin}}$,

$$d\lambda_v^{\eta_v}(iy_v) = \begin{cases} \dfrac{q_v - 1}{(q_v^{1/2} + q_v^{-1/2} - x_v)^2}\, d\mu_{\mathrm{ST}}(x_v) & (\eta_v(\varpi_v) = +1), \\ \dfrac{q_v + 1}{(q_v^{1/2} + q_v^{-1/2})^2 - x_v^2}\, d\mu_{\mathrm{ST}}(x_v) & (\eta_v(\varpi_v) = -1) \end{cases}$$

by the variable change $x_v = q_v^{iy_v/2} + q_v^{-iy_v/2}$. When $F = \mathbb{Q}$ and $v = p < \infty$, $\lambda_v^{\eta_v}(iy)$ is exactly equal to $\mu_{p,\eta}(x)$.

For any ideal $\mathfrak{n}$ of $\mathfrak{o}$, put

$$\nu(\mathfrak{n}) = \prod_{v \in S_2(\mathfrak{n})} \{1 - (q_v^2 - q_v)^{-1}\} \prod_{v \in S(\mathfrak{n}) - (S_1(\mathfrak{n}) \cup S_2(\mathfrak{n}))} (1 - q_v^{-2}),$$

where $S_1(\mathfrak{n})$ (resp. $S_2(\mathfrak{n})$) denotes the set of all $v \in S(\mathfrak{n})$ such that $\mathrm{ord}_v(\mathfrak{n}) = 1$ (resp. $\mathrm{ord}_v(\mathfrak{n}) = 2$).

Theorem 0.2. *Suppose that η is non-trivial and that $\eta_v(-1) = 1$ for all $v \in \Sigma_\infty$. Let Λ be an infinite subset of $\mathcal{I}_{S,\eta}^+$. For any $f \in C_c(\mathfrak{X}_S^{0+})$, we have*

$$\frac{1}{\mathrm{N}(\mathfrak{n})\nu(\mathfrak{n})} \sum_{\pi \in \Pi_{\mathrm{cus}}^*(\mathfrak{n})} \frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{L^{S_\pi}(1,\pi,\mathrm{Ad})} f(\nu_S(\pi)) \to \int_{\mathfrak{X}_S^0} f(\mathbf{s})\, d\lambda_S^\eta(\mathbf{s})$$

as $\mathrm{N}(\mathfrak{n}) \to \infty$ in $\mathfrak{n} \in \Lambda$. In particular, for any non-empty bounded Borel set $\mathbf{J}$ of $\mathfrak{X}_S^{0+}$ such that its boundary is a null set with respect to λ_S^η, the following formula holds:

$$\lim_{\substack{\mathrm{N}(\mathfrak{n})\to\infty \\ \mathfrak{n}\in\Lambda}} \frac{1}{\mathrm{N}(\mathfrak{n})\nu(\mathfrak{n})} \sum_{\substack{\pi \in \Pi_{\mathrm{cus}}^*(\mathfrak{n}), \\ \nu_S(\pi) \in \mathbf{J}}} \frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{L^{S_\pi}(1,\pi,\mathrm{Ad})} = 4D_F^{3/2} L(1,\eta)\mathrm{vol}(\mathbf{J}, \otimes_{v\in S} d\lambda_v^{\eta_v}).$$

We remark that Theorem 0.2 is compatible with [47, Theorem 1.1] since $\nu(\mathfrak{n}) = 1$ holds if $\mathfrak{n}$ is square-free. This asymptotic formula gives the following counterpart of [35, Corollary B].

Corollary 0.3. *Let S and η be the same as in Theorem 0.2 and let $\{J_v\}_{v\in S}$ be a family of intervals such that J_v is contained in $[1/4, \infty)$ for each $v \in \Sigma_\infty$ and in $[-2, 2]$ for each $v \in S_{\mathrm{fin}}$. Then, for any sequence $\{\mathfrak{n}_k\}_{k\in\mathbb{N}}$ of $\mathcal{I}_{S,\eta}^+$ such that $\lim_{k\to\infty} \mathrm{N}(\mathfrak{n}_k) = +\infty$, there exists $k_0 > 0$ such that for any $k \geqslant k_0$, there exists $\pi \in \Pi_{\mathrm{cus}}^*(\mathfrak{n}_k)$ satisfying the following conditions:*

1. *Both $L(1/2,\pi) \neq 0$ and $L(1/2,\pi\otimes\eta) \neq 0$ hold.*
2. *The spectral parameter $\nu_S(\pi) = (\nu_v(\pi))_{v\in S}$ of π satisfies $(1 - \nu_v(\pi)^2)/4 \in J_v$ for all $v \in \Sigma_\infty$ and $q_v^{-\nu_v(\pi)/2} + q_v^{\nu_v(\pi)/2} \in J_v$ for all $v \in S_{\mathrm{fin}}$.*

We remark that $L(1/2,\pi)L(1/2,\pi\otimes\eta) > 0$ if $L(1/2,\pi)L(1/2,\pi\otimes\eta) \neq 0$ by Guo's result [10]. As for equidistribution results for Hecke eigenvalues of Maass forms without weighting central L-values, there is a work [21] by Knightly and Li when $F = \mathbb{Q}$.

Let $\{v_j\}_{j\in\mathbb{N}}$ be the set of all places $v \in \Sigma_{\mathrm{fin}} - (S \cup S(\mathfrak{f}_\eta))$ such that $\eta_v(\varpi_v) = -1$ and let $\{\mathfrak{p}_j\}_{j\in\mathbb{N}}$ be the set of all prime ideals of $\mathfrak{o}$ corresponding to $\{v_j\}_{j\in\mathbb{N}}$. Here are some examples of $\{\mathfrak{n}_k\}_{k\in\mathbb{N}}$ in Theorem 0.2 and Corollary 0.3:

1. $\{\mathfrak{n} = \mathfrak{p}_1 \cdots \mathfrak{p}_{2n} \mid n \in \mathbb{N}\}$,
2. $\{\mathfrak{n} = \mathfrak{p}_1^{2n} \mid n \in \mathbb{N}\}$,
3. $\{\mathfrak{n} = \mathfrak{p}_n^{2a} \mid n \in \mathbb{N}\}$ for a fixed $a \in \mathbb{N}$,
4. $\{\mathfrak{n} = \mathfrak{p}_1^{an}\mathfrak{p}_2^{bn} \mid n \in \mathbb{N}\}$ for fixed odd integers $a, b > 0$.

The case (1) was treated by Tsuzuki [47, Theorem 1.1 and Corollary 1.2].

Motohashi [30] studied the growth of the square mean of central values of automorphic L-functions attached to Maass forms with full level via Kuznetsov's trace formula. Tsuzuki [47, Theorem 1.3] considered a similar growth in the case where the level is square-free and the base field is totally real. We can generalize [47, Theorem 1.3] to the case of arbitrary levels.

Theorem 0.4. *Suppose $S = \Sigma_\infty$ and set $d_F = [F : \mathbb{Q}]$. Let $\mathfrak{n}$ be an arbitrary ideal of $\mathfrak{o}$ and η a real valued character of $F^\times \backslash \mathbb{A}^\times$ such that $\mathfrak{f}_\eta$ is relatively prime to $\mathfrak{n}$, $\tilde{\eta}(\mathfrak{n}) = 1$ and $\eta_v(-1) = 1$ for all $v \in \Sigma_\infty$. Let J be a compact subset of $\prod_{v\in\Sigma_\infty} i\mathbb{R}_{>0}$ with smooth boundary. Then, for any $\epsilon > 0$, we have*

$$\sum_{\substack{\pi\in\Pi_{\mathrm{cus}}(\mathfrak{n}),\\ \nu_{\Sigma_\infty}(\pi)\in tJ}} \frac{w^\eta_{\mathfrak{n}}(\pi)}{[\mathbf{K}_0(\mathfrak{f}_\pi) : \mathbf{K}_0(\mathfrak{n})]} \frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{\mathrm{N}(\mathfrak{f}_\pi)L^{S_\pi}(1,\pi,\mathrm{Ad})}$$
$$= \frac{4D_F^{3/2}}{(2\pi)^{d_F}}\{1+\delta(\mathfrak{n}=\mathfrak{o})\}\,\mathrm{vol}(J)\,t^{d_F}\left(d_F \operatorname{Res}_{s=1} L(s,\eta)\log t + \mathbf{C}^\eta(F,\mathfrak{n})\right)$$
$$+\mathcal{O}(t^{d_F-1}(\log t)^3) + \mathcal{O}(t^{d_F(1+4\theta)+\epsilon}),\ \ t\to\infty,$$

where $w^\eta_\pi(\mathfrak{n})$ is a constant explicitly defined in Lemma 3.6,

$$\delta(\mathfrak{n}=\mathfrak{o}) = \begin{cases} 1 & (\mathfrak{n}=\mathfrak{o}),\\ 0 & (\mathfrak{n}\neq\mathfrak{o}),\end{cases}$$

the values $\operatorname{Res}_{s=1} L(s,\eta)$ and $\mathrm{CT}_{s=1}L(s,\eta)$ denote the residue and the constant term of $L(s,\eta)$ at $s=1$, respectively,

$$\mathbf{C}^\eta(F,\mathfrak{n}) = \mathrm{CT}_{s=1}L(s,\eta) + \operatorname{Res}_{s=1} L(s,\eta)\left\{\frac{d_F}{2}(C_{\mathrm{Euler}} + 2\log 2 - \log\pi) + \log(D_F\mathrm{N}(\mathfrak{n})^{1/2})\right\},$$

the value C_{Euler} is the Euler constant, and $\theta\in\mathbb{R}$ is a constant such that

$$|L_{\mathrm{fin}}(1/2+it,\chi)| \ll \mathfrak{q}(\chi|\cdot|^{it}_{\mathbb{A}})^{1/4+\theta},\quad t\in\mathbb{R}$$

holds uniformly for any character χ of $F^\times\backslash\mathbb{A}^\times$. Here $\mathfrak{q}(\chi|\cdot|^{it}_{\mathbb{A}})$ is the analytic conductor of $\chi|\cdot|^{it}_{\mathbb{A}}$ (cf. §1.3).

Moreover, we obtain the following result on subconvexity bounds depending on $\theta < 0$.

Theorem 0.5. *Let $\mathfrak{n}$ be an arbitrary ideal of $\mathfrak{o}$ and let θ be as in Theorem 0.4. Let $J \subset \mathfrak{X}^{0+}_S$ be a closed cone such that $J - \{0\} \subset \prod_{v\in\Sigma_\infty} i\mathbb{R}_{>0}$. Then, for any $\epsilon > 0$, we have*

$$|L_{\mathrm{fin}}(1/2,\pi)| \ll_\epsilon (1+||\nu_{\Sigma_\infty}(\pi)||)^{d_F/2+\sup(2d_F\theta,-1/2)+\epsilon}$$

for $\pi\in\Pi_{\mathrm{cus}}(\mathfrak{n})_J = \{\pi\in\Pi_{\mathrm{cus}}(\mathfrak{n}) \mid \nu_{\Sigma_\infty}(\pi)\in J\}$. Here the implied constant may depend on $\mathfrak{n}$ and J, and $||\nu|| = (\sum_{v\in\Sigma_\infty}|\nu_v|^2)^{1/2}$ is the Euclidean norm of $\nu = (\nu_v)_{v\in\Sigma_\infty}$.

We remark that Theorem 0.5 was proved by Tsuzuki [47, Corollary 1.4] when $\mathfrak{n}$ is square-free. When $F = \mathbb{Q}$ and $\mathfrak{n} = \mathbb{Z}$, there are works [13], [12] and [19], by which we have $|L_{\mathrm{fin}}(1/2,\pi)| \ll_\epsilon (1+||\nu_{\Sigma_\infty}(\pi)||)^{1/3+\epsilon}$ uniformly for $\pi\in\Pi_{\mathrm{cus}}(\mathbb{Z})$. Recently, Michel and Venkatesh [27] gave subconvexity bounds for automorphic L-functions for GL(1) and GL(2) in a more general case. Their result asserts existence of a subconvexity estimate

$$|L_{\mathrm{fin}}(1/2,\pi)| \ll (1+||\nu_{\Sigma_\infty}(\pi)||)^{d_F/2-2d_F\delta}\mathrm{N}(\mathfrak{n})^{1/4-\delta},\quad \pi\in\Pi_{\mathrm{cus}}(\mathfrak{n})$$

with implicit $\delta > 0$. Since θ can be taken so that $\theta < 0$ by [27, Theorem 1.1], Theorem 0.5 gives an explicit subconvex exponent in the Laplacian eigenvalue aspect. In particular, if $d_F > 1/4|\theta|$, then we have explicitly a subconvex exponent $d_F/2 - 1/2 + \epsilon$ not depending on θ.

To prove results for even Hilbert Maass forms, we generalize Tsuzuki's method explained in [47, §1.3] to the case of non square-free levels. In §2, the number of Hecke characters is estimated, which is used in Lemma 10.3. In §3, we recall the notion of regularized (H,η)-periods $P^{\eta}_{\mathrm{reg}}(\varphi)$ for automorphic forms φ, which was introduced in [47]. Let H be the diagonal maximal torus of $\mathrm{GL}(2)$ and let Z be the center of $\mathrm{GL}(2)$. Then, for a real valued Hecke character η, the (H,η)-period integral for cusp forms φ on $\mathrm{PGL}(2,\mathbb{A})$ is originaly defined by

$$P^{\eta}(\varphi)=\int_{Z_{\mathbb{A}}H_F\backslash H_{\mathbb{A}}}\varphi(h)\,\eta(\det(h))dh=\int_{Z_{\mathbb{A}}H_F\backslash H_{\mathbb{A}}}\varphi(h)\,\eta(\det(h))|\det(h)|_{\mathbb{A}}^{s-1/2}dh\bigg|_{s=1/2}.$$

Since the regularization procedure compensates the divergence arising from $\mathrm{vol}(F^{\times}\backslash\mathbb{A}^{\times})=\infty$, the value $P^{\eta}_{\mathrm{reg}}(\varphi)$ makes sense even if φ is not cuspidal.

Further, in §3.2 and §3.5, we recall explicit formulas of reguralized (H,η)-periods of cusp forms, Eisenstein series, and of the residues and the constant terms of Eisenstein series at $\nu=1$, which were proved in a previous paper [41]. Although some constant terms of Eisenstein series do not have the regularized (H,η)-periods, computation of the spectral side works. Adelic Green functions $\Psi^{(z)}(\mathfrak{n}|\mathbf{s};g)$ defined in §6 are described by using Green functions on $\mathrm{GL}(2,F_v)$. These Green functions on local groups are revised in §4 and §5.

In §7, we regularize the automorphic Poincaré series

$$\boldsymbol{\Psi}^{(0)}(\mathfrak{n}|\mathbf{s};g)=\sum_{\gamma\in H_F\backslash\mathrm{GL}(2,F)}\Psi^{(0)}(\mathfrak{n}|\mathbf{s};\gamma g),\qquad g\in\mathrm{GL}(2,\mathbb{A}) \tag{0.1}$$

and the multidimensional contour integral of the Poincaré series

$$\hat{\boldsymbol{\Psi}}^{(0)}(\mathfrak{n}|\alpha;g)=\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathfrak{X}_S^0}\boldsymbol{\Psi}^{(0)}(\mathfrak{n}|\mathbf{s};g)\alpha(\mathbf{s})d\mu_S(\mathbf{s}),\qquad g\in\mathrm{GL}(2,\mathbb{A})$$

with respect to a measure μ_S for special test functions α on the complex manifold $\mathfrak{X}_S=\prod_{v\Sigma_\infty}\mathbb{C}\times\prod_{v\in S_{\mathrm{fin}}}\mathbb{C}/4\pi i(\log q_v)^{-1}\mathbb{Z}$, in which spectral parameters range. The series (0.1) would link Green functions with $(H,\mathbf{1})$-period integrals if we could ignore the divergence arizing from $\mathrm{vol}(F^{\times}\backslash\mathbb{A}^{\times})=\infty$. For this reason, we study a regularization of the series (0.1) in the same way as the regularization of $(H,\mathbf{1})$-period integrals. First, by using $\lambda\in\mathbb{C}$ and an even holomorphic function $\beta(z)$ with rapid decay as $|\mathrm{Im}(z)|\to\infty$, we define $\Psi_{\beta,\lambda}(\mathfrak{n}|\mathbf{s};g)$ with a parameter (β,λ) as

$$\Psi_{\beta,\lambda}(\mathfrak{n}|\mathbf{s};g)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\{\Psi^{(z)}(\mathfrak{n}|\mathbf{s};g)+\Psi^{(-z)}(\mathfrak{n}|\mathbf{s};g)\}\frac{\beta(z)}{z+\lambda}dz,$$

provided with $\sigma\gg1$. This function is an object to be studied by the relation $\Psi_{\beta,0}(\mathfrak{n}|\mathbf{s};g)=\Psi^{(0)}(\mathfrak{n}|\mathbf{s};g)\beta(0)$. Next, for an even holomorphic function α on $\mathfrak{X}_S$ with rapid decay as $|\mathrm{Im}(z)|\to\infty$, set

$$\hat{\Psi}_{\beta,\lambda}(\mathfrak{n}|\alpha;g)=\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathrm{Re}(\mathbf{s})=\mathbf{c}}\Psi_{\beta,\lambda}(\mathfrak{n}|\mathbf{s};g)\alpha(\mathbf{s})d\mu_S(\mathbf{s}),\qquad g\in\mathrm{GL}(2,\mathbb{A}),$$

where $\mathbf{c}\in\mathbb{R}^S$ is sufficiently large, and

$$\hat{\boldsymbol{\Psi}}_{\beta,\lambda}(\mathfrak{n}|\alpha;g)=\sum_{\gamma\in H_F\backslash\mathrm{GL}(2,F)}\hat{\Psi}_{\beta,\lambda}(\mathfrak{n}|\alpha;\gamma g),\qquad g\in\mathrm{GL}(2,\mathbb{A}).$$

The absolute convergence of the integral and series as above is guaranteed for $\mathrm{Re}(\lambda)\gg1$. Instead of substituting $\lambda=0$, we continue the function $\lambda\mapsto\hat{\boldsymbol{\Psi}}_{\beta,\lambda}(\mathfrak{n}|\alpha;g)$ meromorphically to a right half plane $\mathrm{Re}(\lambda)>-\epsilon$ for some $\epsilon>0$. In this case, we can define the automorphic Green function $\hat{\boldsymbol{\Psi}}_{\mathrm{reg}}(\mathfrak{n}|\alpha;g)$ by the relation $\hat{\boldsymbol{\Psi}}_{\beta,\lambda}(\mathfrak{n}|\alpha;g)=\hat{\boldsymbol{\Psi}}_{\mathrm{reg}}(\mathfrak{n}|\alpha;g)\beta(0)$, where $\mathrm{CT}_{\lambda=0}\hat{\boldsymbol{\Psi}}_{\beta,\lambda}(\mathfrak{n}|\alpha;g)$ is the constant term of the Laurent expansion of $\hat{\boldsymbol{\Psi}}_{\beta,\lambda}(\mathfrak{n}|\alpha;g)$ at $\lambda=0$. The main tool, what we call *an explicit relative trace* *formula* for even Hilbert Maass forms, is obtained by computing $P^{\eta}_{\rm reg}(\hat{\boldsymbol{\Psi}}_{\rm reg}(\mathfrak{n}|\alpha))$ in different two ways. In §8, we describe explicitly one side of the formula in terms of (H,η)-periods of automorphic forms. We call this *the spectral side*. To complete the continuous part $\mathbb{I}^{\eta}_{\rm eis}(\mathfrak{n}|\alpha)$ and the residual part $\mathbb{D}^{\eta}(\mathfrak{n}|\alpha)$ in the spectral side, hard calculation is executed in Lemmas 7.3, 7.4, 8.3 and 8.4. A generalized Siegel's theorem for Hecke L-functions is needed in Lemma 7.5 in order to prove the moderate growth condition of $\hat{\boldsymbol{\Psi}}_{\rm reg}(\mathfrak{n}|\alpha)$. The phenomenon requiring hard calculation as above did not occur in the case of square-free levels ([47]). As the other aspect, $P^{\eta}_{\rm reg}(\hat{\boldsymbol{\Psi}}_{\rm reg}(\mathfrak{n}|\alpha))$ is decomposed into a sum over the double coset space $H_F\backslash{\rm GL}(2,F)/H_F$ in §9. We call this *the geometric side*. In §10, following Tsuzuki's method developed in [47], we estimate terms in the relative trace formula and deduce Theorems 0.2, 0.3, 0.4 and 0.5.

0.3. Part 2 : Results for holomorphic Hilbert modular forms. Part 2 is based on a joint work [43] with Masao Tsuzuki. In Part 2, we study an analogous asymptotic formula to Theorem 0.2 in the case where modular forms are holomorphic. The formula is a generalization of [35] and of [6], and is given by the method developed in [47] and in Part 1. For a family $l=(l_v)_{v\in\Sigma_\infty}\in(2\mathbb{N})^{\Sigma_\infty}$ and an ideal $\mathfrak{n}$ of $\mathfrak{o}$, let $\Pi_{\rm cus}(l,\mathfrak{n})$ be the set of all irreducible cuspidal automorphic representations π of ${\rm PGL}(2,\mathbb{A})$ such that its local component π_v for each $v\in\Sigma_\infty$ is isomorphic to the discrete series representation of ${\rm PGL}(2,\mathbb{R})$ of weight l_v and $\mathfrak{f}_\pi$ divides $\mathfrak{n}$. We denote by $\Pi^*_{\rm cus}(l,\mathfrak{n})$ the set of all $\pi\in\Pi_{\rm cus}(l,\mathfrak{n})$ with $\mathfrak{f}_\pi=\mathfrak{n}$. We assume that S is a finite subset of $\Sigma_{\rm fin}$.

In [6], it was indispensable to assume that $\mathfrak{n}$ is square-free and that $\eta_v(-1)=-1$ for all $v\in\Sigma_\infty$. By a relative trace formula developed in Part 2, we work with a more general sign condition on η than [6] allowing the level $\mathfrak{n}$ to be a general ideal not necessarily square-free. It is known that, for any $\pi\in\Pi_{\rm cus}(l,\mathfrak{n})$, the spectral parameter $\nu_S(\pi)$ of π at S is contained in $\mathfrak{X}^0_S=\prod_{v\in S}i[0,2\pi(\log q_v)^{-1}]$ (cf. [1]). In this setting, we have the following asymptotic formula.

Theorem 0.6. *Assume that $l=(l_v)_{v\in\Sigma_\infty}\in(2\mathbb{N})^{\Sigma_\infty}$ satisfies $\underline{l}=\inf_{v\in\Sigma_\infty}l_v\geqslant 6$. Let η be a quadratic character of $F^\times\backslash\mathbb{A}^\times$ and S a finite set of finite places relatively prime to $\mathfrak{f}_\eta$. Then, for $\mathfrak{n}\in\mathcal{I}^+_{S,\eta}$ and for any even holomorphic function $\alpha(\mathbf{s})$ on the complex manifold $\mathfrak{X}_S=\prod_{v\in S}(\mathbb{C}/\frac{4\pi i}{\log q_v}\mathbb{Z})$, we have the asymptotic formula*

$$(0.2)\qquad {\rm AL}^*(\mathfrak{n};\alpha)=\ \{\prod_{v\in\Sigma_\infty}\frac{2\pi\,(l_v-2)!}{\{(l_v/2-1)!\}^2}\}\times\frac{1}{{\rm N}(\mathfrak{n})}\sum_{\pi\in\Pi^*_{\rm cus}(l,\mathfrak{n})}\frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{L^{S_\pi}(1,\pi,{\rm Ad})}\alpha(\nu_S(\pi))$$

$$=4D_F^{3/2}\nu(\mathfrak{n})L_{\rm fin}(1,\eta)\int_{\mathfrak{X}^0_S}\alpha(\mathbf{s})(\otimes_{v\in S}d\lambda^{\eta_v}_v(s_v))+\mathcal{O}({\rm N}(\mathfrak{n})^{-1+\epsilon}).$$

for any $\epsilon>0$ with the implied constant independent of $\mathfrak{n}$. Here, the error term can be replaced with $\mathcal{O}({\rm N}(\mathfrak{n})^{-\underline{l}/2+1+\epsilon})$ when $\mathfrak{n}$ varies in the set of square-free ideals.

Sarnak and Iwaniec [15] announced results on certain densities of holomorphic cusp forms whose central L-values are non-zero when $F=\mathbb{Q}$. Trotabas [46] also gave estimates of a density of holomorphic Hilbert cusp forms whose central L-values are non-zero. For a density of Hecke eigenvalues, there are equidistribution results for Hecke eigenvalues of holomorphic Hilbert modular forms without weighting central L-values by Li [22] and by Knightly and Li [20]. In our setting, Theorem 0.6 provides us some nonvanishing results of central L-values and the density of spectral parameters simultaneously as follows (cf. [35, Corollary B] and [47, Corollary 1.2]).

Corollary 0.7. *Assume that $l=(l_v)_{v\in\Sigma_\infty}\in(2\mathbb{N})^{\Sigma_\infty}$ satisfies $\inf_{v\in\Sigma_\infty}l_v\geqslant 6$. Let η be a quadratic character of $F^\times\backslash\mathbb{A}^\times$ with conductor $\mathfrak{f}_\eta$. Let S be a finite set of finte places relatively prime to $\mathfrak{f}_\eta$ and $\{J_v\}_{v\in S}$ a collection of subintervals of $[-2,2]$. Given a sequence of ideals $\{\mathfrak{n}_k\}_{k\in\mathbb{N}}$ in $\mathcal{I}^+_{S,\eta}$ such that $\lim_{k\to\infty}{\rm N}(\mathfrak{n}_k)=+\infty$, there exists k_0 satisfying the following property: For any $k\geqslant k_0$, there exists $\pi\in\Pi^*_{\rm cus}(l,\mathfrak{n}_k)$ such that $L_{\rm fin}(1/2,\pi)L_{\rm fin}(1/2,\pi\otimes\eta)\neq 0$ and $q_v^{-\nu_v(\pi)/2}+q_v^{\nu_v(\pi)/2}\in J_v$ for all $v\in S$.*

The estimate of the form

$$|L_{\mathrm{fin}}(1/2,\pi)| \ll_\epsilon \{\mathrm{N}(\mathfrak{n}) \prod_{v\in\Sigma_\infty} l_v^2\}^{1/4+\epsilon}, \qquad \pi \in \Pi_{\mathrm{cus}}(l,\mathfrak{n})$$

for any $\epsilon > 0$ is called the convexity bound. When $F = \mathbb{Q}$, the bound $|L_{\mathrm{fin}}(1/2,\pi)| \ll_\epsilon l^{1/3+\epsilon}$ for $\pi \in \Pi_{\mathrm{cus}}(l,\mathbb{Z})$, which breaks the trivial bound in the weight aspect, has long been known (cf. [33], [19]). Recently, Michel and Venkatesh [27] gave a subconvexity bound for $L_{\mathrm{fin}}(1/2,\pi)$ in a more general setting. By applying our relative trace formula for a general totally real field F, we give an explicit subconvex exponent in the weight aspect for the L-function $L_{\mathrm{fin}}(s,\pi)\,L_{\mathrm{fin}}(s,\pi\otimes\eta)$ with η an quadratic character of $F^\times\backslash\mathbb{A}^\times$ which is odd at all archimedean places.

Theorem 0.8. *Assume that $l = (l_v)_{v\in\Sigma_\infty} \in (2\mathbb{N})^{\Sigma_\infty}$ satisfies $\inf_{v\in\Sigma_\infty} l_v \geqslant 6$. Let $\mathfrak{n}$ be an arbitrary ideal of $\mathfrak{o}$ and η a real valued character of $F^\times\backslash\mathbb{A}^\times$ such that $\eta_v(-1) = -1$ for all $v \in \Sigma_\infty$. Suppose that the conductor $\mathfrak{f}_\eta$ of η is relatively prime to $\mathfrak{n}$. Then, for any $\epsilon > 0$,*

$$|L_{\mathrm{fin}}(1/2,\pi)\,L_{\mathrm{fin}}(1/2,\pi\otimes\eta)| \ll_\epsilon \mathrm{N}(\mathfrak{f}_\eta)^{3/4+\epsilon}\mathrm{N}(\mathfrak{n})^{1+\epsilon}\{\prod_{v\in\Sigma_\infty} l_v\}^{7/8+\epsilon}$$

with the implied constant independent of l, $\mathfrak{n}$, η and $\pi \in \Pi_{\mathrm{cus}}(l,\mathfrak{n})$.

The function $L(s,\pi)L(s,\pi\otimes\eta)$ is the L-function associated with the quadratic base change lift of π for E/F, where E is the quadratic extension of F corresponding to η by class field theory. Thus Theorem 0.8 also gives a subconvexity estimate of automorphic L-functions for $\mathrm{Res}_{E/F}\,\mathrm{GL}(2)$.

In the frame work of [6], the authors of [6] used the Jacquet-Langlands correspondence and the compactness of an anisotropic inner form of GL(2) which is easier to treat period integrals than GL(2). Instead of the Jacquet-Langlands correspondence, we consider the regularized period integral and holomorphic automorphic Green functions in the same way as Part 1. Holomorphic Shintani functions on $\mathrm{GL}(2,\mathbb{R})$ explained in §11 are used to construct adelic Green functions $\Psi_l^{(z)}(\mathfrak{n}|\mathbf{s};g)$ on $\mathrm{GL}(2,\mathbb{A})$. By the same procedure as in Part 1, the automorphic Green function $\hat{\mathbf{\Psi}}_{\mathrm{reg}}^l(\mathfrak{n}|\alpha;g)$ of weight $l = (l_v)_{v\in\Sigma_\infty}$ is defined in §12 when $\inf_{v\in\Sigma_\infty} l_v \geqslant 4$ is satisfied. The holomorphic condition deduces the cuspidality of $\hat{\mathbf{\Psi}}_{\mathrm{reg}}^l(\mathfrak{n}|\alpha)$. Thus it is much easier to treat than automorphic Green functions given in Part 1. The geometric side is calculated in §14 and §15 under the condition $\inf_{v\in\Sigma_\infty} l_v \geqslant 6$. Here, although $\hat{\mathbf{\Psi}}_{\mathrm{reg}}^l(\mathfrak{n}|\alpha)$ is cuspidal, the regularization of the period P^η is indispensable since in the geometric side the idempotent term $\mathrm{CT}_{\lambda=0}\mathbb{J}_{\mathrm{id}}^\eta(l;\beta,\lambda;\alpha) = 0$ compensates $\mathrm{vol}(F^\times\backslash\mathbb{A}^\times) = \infty$ successfully (see Lemma 15.1). The condition $\inf_{v\in\Sigma_\infty} l_v \geqslant 6$ is needed in the estimation of the hyperbolic term in §14.1. Finally, we obtain an explicit relative trace formula in §16. The formula is computable not only in the case where $\mathfrak{n}$ is not necessarily square-free but also in the case where $\eta_v(-1) = 1$ for some $v \in \Sigma_\infty$; the case $\eta = \mathbf{1}$ is also contained. From §17 to §19, we consider a special test function α called Iwaniec's amplifier as an application to subconvexity estimates. In §17, we give an explicit formula of $\mathbb{J}_{\mathrm{hyp}}^\eta(l,\mathfrak{n}|\alpha)$ as computable as possible. An explicit formula of $\mathbb{J}_{\mathrm{u}}^\eta(l,\mathfrak{n}|\alpha)$ is given in §18.

0.4. Part 3 : Results for holomorphic Hilbert modular forms : derivatives of L-series. Part 3 is based on a joint work [44] with Masao Tsuzuki. In Part 3, we give a refined asymptotic formula of the previous result and its derivative version for $L(1/2,\pi)L'(1/2,\pi\otimes\eta)$ for special test functions. As an application, by using Royer's method in [37], we prove existence of $\pi \in \Pi_{\mathrm{cus}}^*(l,\mathfrak{n})$ such that $L(1/2,\pi)L(1/2,\pi\otimes\eta) \neq 0$ or $L(1/2,\pi)L'(1/2,\pi\otimes\eta) \neq 0$, and the Hecke field of π has a sufficiently large degree over $\mathbb{Q}$. As in the previous subsection, we consider a finite subset S of places such that $S \subset \Sigma_{\mathrm{fin}}$.

Let $\mathfrak{n}$ be an ideal of $\mathfrak{o}$, $l = (l_v)_{v\in\Sigma_\infty} \in (2\mathbb{N})^{\Sigma_\infty}$ and η a quadratic character of $F^\times\backslash\mathbb{A}^\times$ as in the beginning of the introduction. Let $\mathfrak{a} \subset \mathfrak{o}$ be an ideal relatively prime to $\mathfrak{f}_\eta\mathfrak{n}$ such that $S(\mathfrak{a}) \subset S$. In Part

3, under the condition $\mathfrak{n} \in \mathcal{I}^-_{S,\eta}$ instead of $\mathcal{I}^+_{S,\eta}$, we investigate the asymptotic behavior of the following average

$$\mathrm{ADL}^*_-(\mathfrak{n};\alpha) = \frac{C_l}{\mathrm{N}(\mathfrak{n})} \sum_{\substack{\pi \in \Pi^*_{\mathrm{cus}}(l,\mathfrak{n}) \\ \epsilon(1/2,\pi\otimes\eta)=-1}} \frac{L(1/2,\pi)\,L'(1/2,\pi\otimes\eta)}{L^{S_\pi}(1,\pi,\mathrm{Ad})}\,\alpha(\nu_S(\pi)) \tag{0.3}$$

with

$$C_l = \prod_{v\in\Sigma_\infty} \frac{2\pi\,(l_v-2)!}{\{(l_v/2-1)!\}^2}. \tag{0.4}$$

For $n \in \mathbb{N}$, let $X_n(x)$ be the Tchebyshev polynomial $X_n(x)$ defined by the relation

$$X_n(x) = \sin((n+1)\theta)/\sin\theta \quad \text{for } x = 2\cos\theta \tag{0.5}$$

and set

$$\alpha_{\mathfrak{a}}(\nu) = \prod_{v\in S} X_{n_v}(q_v^{-\nu_v/2} + q_v^{\nu_v/2}), \quad \nu = (\nu_v)_{v\in S} \in \mathfrak{X}_S \tag{0.6}$$

if $\mathfrak{a} = \prod_{v\in S} \mathfrak{p}_v^{n_v}$, where $\mathfrak{p}_v$ is the prime ideal of $\mathfrak{o}$. For such $\mathfrak{a}$, define $\mathfrak{a}_\eta^{\pm} = \prod_{\substack{v\in S(\mathfrak{a}) \\ \tilde\eta(\mathfrak{p}_v)=\pm1}} \mathfrak{p}_v^{n_v}$, $d_1(\mathfrak{a}) = \prod_{v\in S(\mathfrak{a})}(n_v+1)$ and $\delta_\square(\mathfrak{a}) = \prod_{v\in S(\mathfrak{a})} \delta(n_v \in 2\mathbb{N})$, where $\delta(\mathrm{P})$ for a condition P is 1 (resp. 0) if P is true (resp. false). We have the following theorem for $\mathrm{ADL}^*_-(\mathfrak{n};\alpha_{\mathfrak{a}})$ and for $\mathrm{AL}^*(\mathfrak{n};\alpha_{\mathfrak{a}})$.

Theorem 0.9. *Suppose $\underline{l} = \inf_{v\in\Sigma_\infty} l_v \geqslant 6$. Set $c = [F:\mathbb{Q}]^{-1}(\underline{l}/2-1)$. For any $\epsilon > 0$, we have*

$$\begin{aligned}\mathrm{AL}^*(\mathfrak{n};\alpha_{\mathfrak{a}}) &= 4D_F^{3/2}\,L_{\mathrm{fin}}(1,\eta)\,\nu(\mathfrak{n})\,\mathrm{N}(\mathfrak{a})^{-1/2}\delta_\square(\mathfrak{a}_\eta^-)d_1(\mathfrak{a}_\eta^+) \\ &\quad + \mathcal{O}_{\epsilon,l,\eta}\left(\mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}\right), \qquad \mathfrak{n} \in \mathcal{I}^+_{S,\eta},\end{aligned} \tag{0.7}$$

$$\begin{aligned}&\mathrm{ADL}^*_-(\mathfrak{n};\alpha_{\mathfrak{a}}) \\ &= 4D_F^{3/2}\,L_{\mathrm{fin}}(1,\eta)\,\nu(\mathfrak{n})\,\mathrm{N}(\mathfrak{a})^{-1/2}d_1(\mathfrak{a}_\eta^+)\Bigg\{\delta_\square(\mathfrak{a}_\eta^-)\Bigg(\log(\sqrt{\mathrm{N}(\mathfrak{n})\mathrm{N}(\mathfrak{a})^{-1}}\mathrm{N}(\mathfrak{f}_\eta)D_F) \\ &\quad + \sum_{v\in S(\mathfrak{n})-(S_1(\mathfrak{n})\cup S_2(\mathfrak{n}))} \frac{\log q_v}{q_v^2-1} + \sum_{v\in S_2(\mathfrak{n})} \frac{\log q_v}{q_v^2-q_v-1} + \frac{L'}{L}(1,\eta) + \mathfrak{C}(l)\Bigg) \\ &\quad + \sum_{v\in S(\mathfrak{a}_\eta^-)} \delta_\square(\mathfrak{a}_\eta^-\mathfrak{p}_v^{-1})\log(q_v^{n_v+\frac12}) \\ &\quad + \mathcal{O}_{\epsilon,l,\eta}\left(\mathrm{N}(\mathfrak{a})^{-1/2}d_1(\mathfrak{a}_\eta^+)\delta_\square(\mathfrak{a}_\eta^-)\,X(\mathfrak{n}) + \mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\mathrm{N}(\mathfrak{n})^{-\inf(1,c)+\epsilon}\right), \quad \mathfrak{n} \in \mathcal{I}^-_{S,\eta},\end{aligned} \tag{0.8}$$

where

$$\mathfrak{C}(l) = \sum_{v\in\Sigma_\infty}\Bigg(\sum_{k=1}^{l_v/2-1}\frac{1}{k} - \frac12\log\pi - \frac12 C_{\mathrm{Euler}} - \delta(\eta_v(-1)=-1)\,\log 2\Bigg),$$

$$X(\mathfrak{n}) = \sum_{u\in S(\mathfrak{n})}\frac{\log q_u}{q_u} + \sum_{u\in S(\mathfrak{n})}\frac{\log q_u}{(q_u-1)^2}$$

and C_{Euler} is the Euler constant. The constants implicit in the $\mathcal{O}$-symbols in both formulas are independent of $\mathfrak{n}$ and $\mathfrak{a}$.

For $N \in \mathbb{N}$, let $J_0^{\mathrm{new}}(N)$ be the new part of the Jacobian variety of the modular curve $X_0(N)$ of level N. Serre [39] showed that the largest dimension of $\mathbb{Q}$-simple factors of $J_0^{\mathrm{new}}(N)$ tends to infinity as $N \to \infty$ (cf. [39, Theorem 7]). This result was refined in several ways by Royer [37]. He obtained a quantitative version of Serre's theorem giving a lower bound of the largest dimension of $\mathbb{Q}$-simple factors A of $J_0^{\mathrm{new}}(N)$ with or without rank conditions for the Model-Weil group of A. By the correspondence between the $\mathbb{Q}$-simple factors A of $J_0^{\mathrm{new}}(N)$ and the normalized Hecke eigen newforms f of level N and weight two, and by invoking the progress toward the Birch and Swinnerton-Dyer conjecture, the lower bound for the largest $\dim A$ is obtained from a lower bound of the maximal value of the absolute degree of the Hecke field $\mathbb{Q}(f) = \mathbb{Q}(\{n^{1/2}a_n(f) \mid n \in \mathbb{N}\})$ with or without conditions on $\mathrm{ord}_{s=1/2} L_{\mathrm{fin}}(s,f)$. Thus, one of Royer's results in [37] can be stated in the language of modular forms as follows.

Theorem 0.10. [37, Theorems 1.2 and 1.3] *Let p be a prime. There exist constants $C_p > 0$ and $N_p > 0$ satisfying the following properties:*

(1) *For any $N > N_p$ relatively prime to p, there exists a normalized Hecke eigen newform f of level N and weight two satisfying the conditions:*
  (i) $L_{\mathrm{fin}}(1/2, f) \neq 0$,
  (ii) $[\mathbb{Q}(f) : \mathbb{Q}] \geqslant C_p \sqrt{\log\log N}$.

(2) *For any $N > N_p$ relatively prime to p, there exists a normalized Hecke eigen newform f_1 of level N and weight two satisfying the conditions:*
  (i) *The sign of the functional equation of $L_{\mathrm{fin}}(s, f_1)$ is -1.*
  (ii) $L'_{\mathrm{fin}}(1/2, f_1) \neq 0$.
  (iii) $[\mathbb{Q}(f_1) : \mathbb{Q}] \geqslant C_p \sqrt{\log\log N}$.

We obtain an analogue of this theorem for higher parallel weight Hilbert cusp forms by using Theorem 0.9. For a cuspidal representation $\pi \in \Pi^*_{\mathrm{cus}}(l, \mathfrak{n})$, we denote by $\mathbb{Q}(\pi)$ the field of rationality of π (for definition, see §25.1).

Theorem 0.11. *Assume that $l = (l_v)_{v\in\Sigma_\infty} \in (2\mathbb{N})^{\Sigma_\infty}$ satisfies $l_v = k$ for all $v \in \Sigma_\infty$ with $k \geqslant 6$, and η a quadratic character of $F^\times\backslash\mathbb{A}^\times$. Let S be a finite subset of $\Sigma_{\mathrm{fin}} - S(\mathfrak{f}_\eta)$ and $\mathbf{J} = \{J_v\}_{v\in S}$ a family of closed subintervals of $(-2,2)$. Given a prime ideal $\mathfrak{q}$ prime to $S \cup S(\mathfrak{f}_\eta)$, there exist constants $C_\mathfrak{q} > 0$ and $N_{\mathfrak{q},S,l,\eta,\mathbf{J}} > 0$ satisfying the following property: For any ideal $\mathfrak{n} \in \mathcal{I}^+_{S\cup S(\mathfrak{q}),\eta}$ with $\mathrm{N}(\mathfrak{n}) > N_{\mathfrak{q},S,l,\eta,\mathbf{J}}$, there exists $\pi \in \Pi^*_{\mathrm{cus}}(l,\mathfrak{n})$ such that*
  (i) $L(1/2, \pi) \neq 0$ *and* $L(1/2, \pi\otimes\eta) \neq 0$,
  (ii) $[\mathbb{Q}(\pi) : \mathbb{Q}] \geqslant C_\mathfrak{q} \sqrt{\log\log \mathrm{N}(\mathfrak{n})}$ *and*
  (iii) $q_v^{-\nu_v(\pi)/2} + q_v^{\nu_v(\pi)/2} \in J_v$ *for all* $v \in S$.

We should note that this can be regarded as a refinement of Corollary 0.7.

As for central derivatives, Trotabas [46] estimated a density of holomorphic Hilbert cusp forms whose central derivatives of L-functions are non-zero. As a corollary of Theorem 0.9, we have a conditional result.

Theorem 0.12. *Let $l = (l_v)_{v\in\Sigma_\infty}$ and η be the same as in Theorem 0.11. Suppose that*

$$\frac{d}{ds}\Big|_{s=1/2}(L(s,\pi)L(s,\pi\otimes\eta)) \geqslant 0 \tag{0.9}$$

is satisfied for all $\pi \in \Pi^*_{\mathrm{cus}}(l,\mathfrak{n})$ and any integral ideal $\mathfrak{n}$ such that $\mathfrak{n}$ is prime to $\mathfrak{f}_\eta$ and $\{\prod_{v\in\Sigma_\infty}\eta_v(-1)\}\,\tilde\eta(\mathfrak{n}) = -1$. Let S be a finite subset of $\Sigma_{\mathrm{fin}} - S(\mathfrak{f}_\eta)$ and $\mathbf{J} = \{J_v\}_{v\in S}$ a family of closed subintervals of $(-2,2)$. Given a prime ideal $\mathfrak{q}$ prime to $S \cup S(\mathfrak{f}_\eta)$ and a constant $M > 1$, there exist constants $C_\mathfrak{q} > 0$ and $N_{\mathfrak{q},S,l,\eta,\mathbf{J},M} > 0$ satisfying the following property: For any ideal $\mathfrak{n} \in \mathcal{I}^-_{S\cup S(\mathfrak{q}),\eta}$ with $\mathrm{N}(\mathfrak{n}) > N_{\mathfrak{q},S,l,\eta,\mathbf{J},M}$ and $\sum_{v\in S(\mathfrak{n})}\frac{\log q_v}{q_v} \leqslant M$, there exists $\pi \in \Pi^*_{\mathrm{cus}}(l,\mathfrak{n})$ such that
  (i) $\epsilon(1/2, \pi\otimes\eta) = -1$,

(ii) $L(1/2,\pi) \neq 0$ *and* $L'(1/2,\pi\otimes\eta) \neq 0$,
(iii) $[\mathbb{Q}(\pi):\mathbb{Q}] \geqslant C_{\mathfrak{q}}\sqrt{\log\log \mathrm{N}(\mathfrak{n})}$ *and*
(iv) $q_v^{-\nu_v(\pi)/2} + q_v^{\nu_v(\pi)/2} \in J_v$ *for all* $v \in S$.

We should note that the assumption (0.9) is a consequence of the generalized Riemann hypothesis for the L-function $L(s,\pi)L(s,\pi\otimes\eta)$. Furthermore, there are some works [9], [51], [52], [53] and [50] in the view point of arithmetic geometry of modular varieties via the Gross-Zagier formula for Hilbert cusp forms. From these works, (0.9) holds in the parallel weight two case. As is seen from this, we can expect that (0.9) holds in the higher weight case (cf. [51, Corollary 0.3.6]).

Theorem 0.11 (Theorem 0.12) yields a Hilbert cusp form of arbitrarily large level with arbitrarily large degree of the Hecke field, such that the central value of the L-function and the central value (derivative) of its prescribed quadratic twist are nonzero simultaneously. Although we can expect a counterpart of the parallel weight two case, our method does not work as it is for such low weight cases. In order to treat these interesting cases, the technique of Green's functions as in [47] and in Part 1 may be useful.

Part 3 strongly depends on results of Part 2. We use the automorphic Green function $\hat{\mathbf{\Psi}}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha;g)$ constructed in Part 2 to consider a derivative version of the relative trace formula obtained in Part 2. By the integral $\partial P^\eta_{\beta,\lambda}(\hat{\mathbf{\Psi}}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha))$, which regularizes

$$\partial P^\eta(\varphi) = \int_{Z_{\mathbb{A}}H_F\backslash H_{\mathbb{A}}} \varphi(h)\eta(\det(h))\log|\det(h)|_{\mathbb{A}}dh = \frac{d}{ds}\int_{Z_{\mathbb{A}}H_F\backslash H_{\mathbb{A}}} \varphi(h)\eta(\det(h))|\det(h)|_{\mathbb{A}}^{s-1/2}dh\bigg|_{s=1/2}$$

for cusp forms φ on $\mathrm{PGL}(2,\mathbb{A})$ and a fixed quadratic Hecke character η, we obtain *a derivative relative trace formula* (see Theorem 21.9). A spectral expansion and a geometric expansion of $\partial P^\eta_{\beta,\lambda}(\hat{\mathbf{\Psi}}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha))$ are given in §20 and §21, respectively. Some lemmas on the $\mathcal{N}$-transform defined in Part 1 are prepared in §22. In §23, we recall the relative trace formula explicated in Part 2. Then, the first formula (0.7) on $\mathrm{AL}^*(\mathfrak{n};\alpha_{\mathfrak{a}})$ is established for the special test functions $\alpha_{\mathfrak{a}}$ determined by ideals $\mathfrak{a}$ with $S=S(\mathfrak{a})$. The average $\mathrm{ADL}^*_-(\mathfrak{n};\alpha_{\mathfrak{a}})$ is investigated in §24. In computation in §24, all terms except for $\mathrm{ADL}^*_-(\mathfrak{n};\alpha_{\mathfrak{a}})$ in the spectral side are included in the error term with the aid of the asymptotic formula of $\mathrm{AL}^*(\mathfrak{n};\alpha_{\mathfrak{a}})$ given in §23. In §24, the second formula (0.8) in Theorem 0.9 is proved. Royer's method in [37] is generalized to our case in §25. The hyperbolic term $\mathbb{W}^\eta_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha)$ and the unipotent term $\tilde{\mathbb{W}}^\eta_{\mathrm{u}}(l,\mathfrak{n}|\alpha)$ are explicitly described in §26. In §27, Theorem 27.1 on a estimate $\Theta(\Lambda)$ of a sum over a $\mathbb{Z}$-lattice Λ is prepared. The estimate given there is used to control the hyperbolic term $\mathbb{J}^\eta_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha)$ and the derivative one $\mathbb{W}^\eta_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha)$.

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1. Preliminaries

We prepare notation, which is used from Part 1 to Part 3.

1.1. We write $\mathbb{N}$ for the set of positive integers and put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For sets A and B, the set $\mathrm{Map}(A, B)$ denotes the set of mappings from A to B. For $f, g \in \mathrm{Map}(A, \mathbb{R}_{\geqslant 0})$ and a given data P, let us denote by $f(x) \ll_P g(x), x \in A$ an inequality $f(x) \leqslant C\, g(x)$ for all $x \in A$ with some constant $C > 0$ depending on P. We write $f(x) \asymp g(x), x \in A$ if both $f(x) \ll g(x), x \in A$ and $g(x) \ll f(x), x \in A$ hold. For a given condition P, $\delta(P) \in \{0, 1\}$ is defined by $\delta(P) = 1$ (resp. $\delta(P) = 0$) if P is true (resp. false). For a set X and its subset A, the characteristic function of A is denoted by ch_A.

For any $z \in \mathbb{C}^\times$ and $\alpha \in \mathbb{C}$, we define $\log z$ and z^α by the formula

$$\log z = \log r + i\theta, \qquad z^\alpha = \exp(\alpha \log z)$$

with $z = re^{i\theta}\,(r > 0, \theta \in (-\pi, \pi])$. For a complex function $f(z)$ in $z \in \mathbb{C}$ and for $\sigma \in \mathbb{R}$, the contour integral $\int_{\sigma - i\infty}^{\sigma + i\infty} f(z)dz$ along the vertical line $\mathrm{Re}(z) = \sigma$ is sometimes denoted by $\int_{L_\sigma} f(z)dz$. If f is a meromorphic function on a domain $D \subset \mathbb{C}$, we denote by $\mathrm{Res}_{z=a}\, f(z)$ and by $\mathrm{CT}_{z=a} f(z)$ the residue and the constant term of $f(z)$ in the Laurent expansion at $a \in D$, respectively. We set $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$.

1.2. Let F be a totally real algebraic number field with its degree d_F and $\mathfrak{o}$ its integer ring. Let $\mathbb{A}$ and $\mathbb{A}_{\mathrm{fin}}$ be the adele ring and the finite adele ring of F, respectively. The symbols Σ_∞ and Σ_{fin} denote the set of all infinite places and the set of all finite places of F, respectively. For a place $v \in \Sigma_F = \Sigma_\infty \cup \Sigma_{\mathrm{fin}}$, let $|\cdot|_v$ denote the normalized valuation of the completion F_v of F at v. For each $v \in \Sigma_{\mathrm{fin}}$, let ϖ_v be a uniformizer of F_v. Then, $\mathfrak{p}_v = \varpi_v \mathfrak{o}_v$ is a maximal ideal of the integer ring $\mathfrak{o}_v$ of F_v and we have $|\varpi_v|_v = q_v^{-1}$, where q_v is the cardinality of the residue field $\mathfrak{o}_v/\mathfrak{p}_v$. For an ideal $\mathfrak{a}$ of $\mathfrak{o}$, let $S(\mathfrak{a})$ denote the set of all $v \in \Sigma_{\mathrm{fin}}$ such that v divides $\mathfrak{a}$. For any $k \in \mathbb{N}$, we write $S_k(\mathfrak{a})$ for the set of all $v \in S(\mathfrak{a})$ with $\mathrm{ord}_v(\mathfrak{a}) = k$, where $\mathrm{ord}_v(\mathfrak{a})$ is the order of $\mathfrak{a}$ at v. Let $\mathrm{N}(\mathfrak{a})$ denote the absolute norm of $\mathfrak{a}$.

Let G be the algebraic group $\mathrm{GL}(2)$ with unit element $e = 1_2$. For any F-algebraic subgroup M of G, we set $M_F = M(F)$, $M_v = M(F_v)$ (for $v \in \Sigma_F$), $M_{\mathbb{A}} = M(\mathbb{A})$ and $M_{\mathrm{fin}} = M(\mathbb{A}_{\mathrm{fin}})$, respectively. The diagonal maximal split torus of G is denoted by H. Then, the Borel subgroup $B = HN$ of G consists of all upper triangular matrices, where N is the subgroup of G consisting of all unipotent matrices. The center of G is denoted by Z. We put $\mathbf{K}_v = \mathrm{GL}(2, \mathfrak{o}_v)$ for $v \in \Sigma_{\mathrm{fin}}$ and

$$\mathbf{K}_0(\mathfrak{p}_v^n) = \left\{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathbf{K}_v \mid c \equiv 0 \pmod{\mathfrak{p}_v^n} \right\}$$

for any $n \in \mathbb{N}_0$. For an ideal $\mathfrak{a}$ of $\mathfrak{o}$, we put $\mathbf{K}_0(\mathfrak{a}) = \prod_{v \in \Sigma_{\mathrm{fin}}} \mathbf{K}_0(\mathfrak{a}\mathfrak{o}_v)$, which is an open compact subgroup of $\mathbf{K}_{\mathrm{fin}} = \prod_{v \in \Sigma_{\mathrm{fin}}} \mathbf{K}_v$. For each $v \in \Sigma_\infty$, let $\mathbf{K}_v$ be the image of $\mathrm{O}(2, \mathbb{R})$ by the isomorphism $\mathrm{GL}(2, \mathbb{R}) \cong G_v$. Note that $\mathbf{K}_v^0$ is isomorphic to the rotation group $\mathrm{SO}(2, \mathbb{R})$. Set $\mathbf{K}_\infty = \prod_{v \in \Sigma_\infty} \mathbf{K}_v$ and $\mathbf{K} = \mathbf{K}_\infty\, \mathbf{K}_{\mathrm{fin}}$.

For θ, $r \in \mathbb{R}$, set

$$k_\theta = \left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right], \qquad a_r = \left[\begin{smallmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{smallmatrix}\right].$$

They are elements of $\mathrm{GL}(2, \mathbb{R})$ and we have $\mathrm{SO}(2, \mathbb{R}) = \{k_\theta | \, \theta \in \mathbb{R}\,\}$.

1.3. Let $\mathbb{A}_{\mathbb{Q}}$ be the adele ring of $\mathbb{Q}$ and $\psi_{\mathbb{Q}} = \prod_p \psi_p$ the additive character of $\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}$ with archimedean component $\psi_\infty(x) = \exp(2\pi i x)$ for $x \in \mathbb{R}$. Then, $\psi_F = \psi \circ \mathrm{tr}_{F/\mathbb{Q}} = \prod_{v \in \Sigma_F} \psi_{F_v}$ is a non-trivial additive character of $F\backslash\mathbb{A}$. Let $\mathfrak{D}_{F/\mathbb{Q}}$ be the global different of $F/\mathbb{Q}$ and set $d_v = \mathrm{ord}_v\, \mathfrak{D}_{F/\mathbb{Q}}$ for any $v \in \Sigma_{\mathrm{fin}}$. Put $D_F = \mathrm{N}(\mathfrak{D}_{F/\mathbb{Q}}) = \prod_{v \in \Sigma_{\mathrm{fin}}} q_v^{d_v}$. Then, D_F coincides with the absolute value of the discriminant of $F/\mathbb{Q}$. The completed Dedekind zeta function is denoted by $\zeta_F(s)$, i.e., $\zeta_F(s) = \prod_{v \in \Sigma_\infty} \Gamma_{\mathbb{R}}(s) \prod_{v \in \Sigma_{\mathrm{fin}}} (1 - q_v^{-s})^{-1}$.

For $v \in \Sigma_F$, let dx_v be the self-dual Haar measure of F_v with respect to ψ_{F_v}. We set $d^\times x_v = (1 - q_v^{-1})^{-1} dx_v/|x_v|_v$ for $v \in \Sigma_{\mathrm{fin}}$ and $d^\times x_v = d^\times x_v/|x_v|_v$ for $v \in \Sigma_\infty$, respectively. Then, $d^\times x_v$ is a Haar measure of $F_v^\times$ and the product measure $d^\times x = \prod_{v \in \Sigma_F} d^\times x_v$ gives a Haar measure on $\mathbb{A}^\times$. For each $v \in \Sigma_F$, we take a Haar measure dk_v on $\mathbf{K}_v$ such that total volume is one, and take a Haar measure dg_v on G_v in the following way. Let dh_v (resp. dn_v) denotes the Haar measure on H_v (resp. N_v) induced via the isomorphism $H_v \cong F_v^\times \times F_v^\times$ (resp. $N_v \cong F_v$). Then, $dg_v = dh_v dn_v dk_v$ gives a Haar measure on G_v via the Iwasawa decomposition $g_v = h_v n_v k_v \in H_v N_v \mathbf{K}_v$. We remark that the volume $\mathrm{vol}(\mathbf{K}_v, dg_v)$ equals $q_v^{-3d_v/2}$ for any $v \in \Sigma_{\mathrm{fin}}$. We denote the Haar measure $\prod_{v\in\Sigma_F} dk_v$ of $\mathbf{K}$ by dk. We fix a Haar measure dg on $G_{\mathbb{A}}$ by taking the product of Haar measures dg_v on G_v over all $v \in \Sigma_F$.

Let $|\cdot|_{\mathbb{A}} = \prod_{v\in\Sigma_F} |\cdot|_v$ be the idele norm of $\mathbb{A}^\times$ and set $\mathbb{A}^1 = \{x \in \mathbb{A}^\times \mid |x|_{\mathbb{A}} = 1\}$. For $y \in \mathbb{R}_{>0}$, $\underline{y}$ denotes the idele such that the v-th component of $\underline{y}$ satisfies $\underline{y}_v = y^{1/d_F}$ (resp. $\underline{y}_v = 1$) for $v \in \Sigma_\infty$ (resp. $v \in \Sigma_{\mathrm{fin}}$). We take a Haar measure du on $\mathbb{A}^1$ which satisfies $d^\times x = du d^\times y$ via $x = u\underline{y} \in \mathbb{A}^\times$ with $u \in \mathbb{A}^1$ and $y > 0$.

For $v \in \Sigma_{\mathrm{fin}}$ and a quasi-character χ_v of $F_v^\times$, $\mathfrak{p}_v^{f(\chi_v)}$ denotes the conductor of χ_v. We define the Gauss sum associated with χ_v by

$$\mathcal{G}(\chi_v) = \int_{\mathfrak{o}_v^\times} \chi_v(u\varpi_v^{-d_v - f(\chi_v)})\psi_{F_v}(u\varpi_v^{-d_v-f(\chi_v)})d^\times u.$$

If χ_v is unramified, $\mathcal{G}(\chi_v)$ equals $\chi_v(\varpi_v^{-d_v})q_v^{-d_v/2}$. If χ_v is ramified, then $|\mathcal{G}(\chi_v)|$ equals $q_v^{-f(\chi_v)/2-d_v/2}(1-q_v^{-1})^{-1}$. For any quasi-character $\chi = \prod_{v\in\Sigma_F}\chi_v$ of $F^\times\backslash\mathbb{A}^\times$, the conductor of χ is denoted by $\mathfrak{f}_\chi$. The Gauss sum $\mathcal{G}(\chi)$ associated with χ is defined by the product of $\mathcal{G}(\chi_v)$ over all $v \in \Sigma_{\mathrm{fin}}$. We set $\tilde{\chi}(\mathfrak{a}) = \prod_{v\in\Sigma_{\mathrm{fin}}}\chi_v(\varpi_v^{\mathrm{ord}_v(\mathfrak{a})})$ for any ideal $\mathfrak{a}$ of $\mathfrak{o}$. For $v \in \Sigma_F$, we denote the trivial character of $F_v^\times$ by $\mathbf{1}_v$, and the trivial character of $\mathbb{A}^\times$ by $\mathbf{1}$. Throughout this thesis, any quasi-character χ of $F^\times\backslash\mathbb{A}^\times$ is assumed to satisfy $\chi(\underline{y}) = 1$ for all $y \in \mathbb{R}_{>0}$. Such a quasi-character is a character. For any $v \in \Sigma_\infty$ (resp. $v \in \Sigma_{\mathrm{fin}}$) and any character χ_v of $F_v^\times$, let $b(\chi_v)$ denote $b_v \in \mathbb{R}$ (resp. $b_v \in [0, 2\pi(\log q_v)^{-1})$) such that the restriction of χ_v to $\mathbb{R}_{>0}$ (resp. $\varpi_v^{\mathbb{Z}}$) is of the form $|\cdot|_v^{ib_v}$. For any character χ of $F^\times\backslash\mathbb{A}^\times$, the analytic conductor $\mathfrak{q}(\chi)$ of χ is defined to be

$$\mathfrak{q}(\chi) = \{\prod_{v\in\Sigma_\infty}(3 + |b(\chi_v)|)\}\mathrm{N}(\mathfrak{f}_\chi).$$

1.4. Let η be a real valued character of $F^\times\backslash\mathbb{A}^\times$ with conductor $\mathfrak{f}_\eta$. Such η is quadratic or trivial. For any $v \in \Sigma_\infty$, there exists $\epsilon_v \in \{0, 1\}$ such that $\eta_v(x) = (x/|x|_v)^{\epsilon_v}$. We call ϵ_v the sign of η at v, and set $\epsilon(\eta) = \sum_{v\in\Sigma_\infty}\epsilon_v$. Let $I(\mathfrak{f}_\eta)$ be the group of fractional ideals relatively prime to $\mathfrak{f}_\eta$. Then we define a character $\tilde{\eta} : I(\mathfrak{f}_\eta) \to \{\pm 1\}$ by setting $\tilde{\eta}(\mathfrak{p}_v \cap \mathfrak{o}) = \eta_v(\varpi_v)$ for any $v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{f}_\eta)$. We define the adele $x_\eta \in \mathbb{A}$ by $x_{\eta,v} = 0$ for $v \in \Sigma_\infty$ and $x_{\eta,v} = \varpi_v^{-f(\eta_v)}$ for $v \in \Sigma_{\mathrm{fin}}$, respectively. It determines the idele x_η^* such that all its archimedean components are equal to one and the projection of x_η^* to $\mathbb{A}_{\mathrm{fin}}$ coincides with that of x_η.

1.5. Let φ be a smooth function on $G_{\mathbb{A}}$. The right translation of φ by $g \in G_{\mathbb{A}}$ is denoted by $R(g)\varphi$, i.e., $[R(g)\varphi](h) = \varphi(hg)$. For any compactly supported smooth function f on the product $\prod_{v\in S} G_v$ for a finite subset $S \subset \Sigma_F$, the right translation of φ by f is defined by the convolution $R(f)\varphi(x) = \int_{\prod_{v\in S} G_v}\varphi(xg_S)\, f(g_S)\, dg_S$ for $x \in G_{\mathbb{A}}$ with respect to the product measure $dg_S = \otimes_{v\in S} dg_v$. The derived action of the universal enveloping algebra of the complexified Lie algebra $\mathfrak{g}_\infty$ of G_∞ on smooth functions on $G_{\mathbb{A}}$ is also denoted by R.

Let W and $\overline{W}$ be the element $\frac{1}{2}\left[\begin{smallmatrix} 1 & -i \\ -i & -1\end{smallmatrix}\right]$ of $\mathfrak{sl}_2(\mathbb{C})$ and its complex conjugate, respectively. Let Ω denote the Casimir element of $\mathrm{GL}(2, \mathbb{R})$ defined by

$$\Omega = \frac{1}{2}\{\left[\begin{smallmatrix} 0 & -i \\ i & 0\end{smallmatrix}\right]^2 + 2\overline{W}W + 2W\overline{W}\} = \frac{1}{2}\{\left[\begin{smallmatrix} 0 & -i \\ i & 0\end{smallmatrix}\right]^2 + 2\left[\begin{smallmatrix} 0 & -i \\ i & 0\end{smallmatrix}\right] + 4W\overline{W}\}.$$

Then, Ω corresponds to the differential operator $(-2) \times (-y^2)(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ on the Poincaré upper half plane. For any $v \in \Sigma_\infty$, the elements of $\mathrm{Lie}(G_v)_{\mathbb{C}}$ corresponding to Ω, W and $\overline{W}$ are denoted by Ω_v, W_v and $\overline{W}_v$, respectively.

1.6. Fix a relatively compact subset ω_B of $B^1_{\mathbb{A}} = \{\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] \mid a,\, d \in \mathbb{A}^1,\, b \in \mathbb{A}\}$ such that $B^1_{\mathbb{A}} = B_F\,\omega_B$. Let $\mathfrak{S}^1 = \omega_B \left\{\left[\begin{smallmatrix} \underline{t} & 0 \\ 0 & \underline{t}^{-1} \end{smallmatrix}\right] \mid t > 0,\, t^2 > c\right\}\mathbf{K}$ with some $c > 0$ be a Siegel domain such that $G_{\mathbb{A}} = Z_{\mathbb{A}} G_F\, \mathfrak{S}^1$. Define $y : G_{\mathbb{A}} \to \mathbb{R}_{>0}$ by setting $y\left(\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] k\right) = |a/d|_{\mathbb{A}}$ for any $\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] \in B_{\mathbb{A}}$ and $k \in \mathbf{K}$.

1.7. In this subsection, we give a result on weighted equidistributions. The following assertion is needed to prove non-vanishing results of L-values in Part 1, Part 2 and Part 3. It is a generalization of [39, Proposition 1].

Proposition 1.1. *Let X be a locally compact Hausdorff space and μ a positive Radon measure on X. Let $\{\mu_\lambda\}_{\lambda \in \Lambda}$ be a directed sequence of positive Radon measures on X such that μ_λ converges weakly to μ on $C_c(X)$. Suppose that A is a μ-measurable set of X satisfying*

(1) *Its boundary ∂A is μ-null,*

(2) *There exists a compact subset K of X containing A such that its boundary ∂K is μ-null.*

Then, we have

$$\lim_{\overrightarrow{\lambda \in \Lambda}} \mu_\lambda(A) = \mu(A).$$

Proof. The restriction of μ_λ and μ to K gives Radon measures $\mu_\lambda|_K$ and $\mu|_K$ on K, respectively. By [3, Chap. IV, §5, n°12, Proposition 22], it is sufficient to prove $\lim_\lambda \mu_\lambda|_K(K) = \mu|_K(K)$ for any relatively compact subset K of X such that ∂K is μ-null. The proof is given in the following way, which was suggested by Tsuzuki.

Let K° and $\bar{K}$ be the interior and closure of K, respectively. By $\mu(\bar{K}) - \mu(K^\circ) = \mu(\bar{K} - K^\circ) = \mu(\partial K) = 0$, we have $\mu(K^\circ) = \mu(K) = \mu(\bar{K})$. Let $\epsilon > 0$ be a positive number. By inner regularity of μ (and Urysohn's lemma), there exists $f_\epsilon \in C_c(X)$ such that $0 \leqslant f_\epsilon \leqslant \mathrm{ch}_{K^\circ}$ and $\mu(K^\circ) - \epsilon/2 < \mu(f_\epsilon)$. In a similar way, by outer regularity of μ, there exists $g_\epsilon \in C_c(X)$ such that $\mathrm{ch}_{\bar{K}} \leqslant g_\epsilon$ and $\mu(g_\epsilon) < \mu(\bar{K}) + \epsilon/2$.

For ϵ, f_ϵ and g_ϵ, there exists $\lambda_\epsilon \in \Lambda$ such that we have $|\mu_\lambda(f_\epsilon) - \mu(f_\epsilon)| < \epsilon/2$ and $|\mu_\lambda(g_\epsilon) - \mu(g_\epsilon)| < \epsilon/2$ for any $\lambda \geqslant \lambda_\epsilon$. Then, we obtain

$$\mu(K) = \mu(K^\circ) < \mu(f_\epsilon) + \epsilon/2 < \mu_\lambda(f_\epsilon) + \epsilon \leqslant \mu_\lambda(K) + \epsilon$$

and

$$\mu_\lambda(K) \leqslant \mu_\lambda(g_\epsilon) < \mu(g_\epsilon) + \epsilon/2 < \mu(\bar{K}) + \epsilon = \mu(K) + \epsilon$$

for any $\lambda \geqslant \lambda_\epsilon$. This completes the proof. $\square$

Part 1. Relative trace formulas for even Hilbert Maass forms

2. Estimates of number of characters

Before introducing a relative trace formula, we prepare Lemma 2.1, which is used to estimate a continuous spectrum of the relative trace formula in Lemma 10.3.

Let $\mathfrak{n}$ be an ideal of $\mathfrak{o}$. For an ideal $\mathfrak{c}$ of $\mathfrak{o}$, let $\Xi_0(\mathfrak{c})$ be the set of all characters χ of $F^\times\backslash\mathbb{A}^\times$ such that $\mathfrak{f}_\chi = \mathfrak{c}$ and $\chi_v(-1) = 1$ for all $v \in \Sigma_\infty$. We write $\Xi(\mathfrak{n})$ for $\bigcup_{\mathfrak{c}^2|\mathfrak{n}} \Xi_0(\mathfrak{c})$. Let U_F^+ be the set of all totally positive units of $\mathfrak{o}$ and set

$$\log U_F^+ = \{(\log u_v)_{v\in\Sigma_\infty} \mid (u_v)_{v\in\Sigma_\infty} \in U_F^+\}.$$

Then, $\log U_F^+$ is a lattice of $\mathbb{Z}$-rank $d_F - 1$ in V, where $V = \{(x_v)_{v\in\Sigma_\infty} \in \mathbb{R}^{d_F} \mid \sum_{v\in\Sigma_\infty} x_v = 0\}$. Set

$$L_0 = \{(b_v)_{v\in\Sigma_\infty} \in V \mid \sum_{v\in\Sigma_\infty} b_v l_v \in \mathbb{Z} \textit{ for all } (l_v)_{v\in\Sigma_\infty} \in \log U_F^+\}.$$

Then, L_0 is also a $\mathbb{Z}$-lattice in V. Let χ be a character of $F^\times\backslash\mathbb{A}^\times$. Since $\chi(\underline{y}) = 1$ for any $y \in \mathbb{R}_{>0}$, we have $\sum_{v\in\Sigma_\infty} b(\chi_v) = 0$. Thus, if we denote by $b(\chi)$ the element $(b(\chi_v))_{v\in\Sigma_\infty}$ of $\mathbb{R}^{d_F}$, then $b(\chi) \in L_0$ holds. Therefore the mapping $\chi \mapsto b(\chi)$ is a surjection from $\Xi(\mathfrak{n})$ onto L_0 and the kernel $\Xi_{\ker}(\mathfrak{n})$ of this mapping is a finite abelian group.

Lemma 2.1. *Let $X(\mathfrak{n})$ be the order of $\Xi_{\ker}(\mathfrak{n})$. Then, for any $\epsilon > 0$, the estimate*

$$X(\mathfrak{n}) \ll \mathrm{N}(\mathfrak{n})^{1/2+\epsilon}$$

holds with the implied constant independent of $\mathfrak{n}$.

Proof. For any ideal $\mathfrak{a}$ of $\mathfrak{o}$, we set $I_F(\mathfrak{a}) = \prod_{v\in\Sigma_\infty} \mathbb{R}^\times \times \prod_{v\in\Sigma_{\mathrm{fin}}} (1+\mathfrak{a}\mathfrak{o}_v)$. Then, the ray class group $C_F(\mathfrak{a})$ modulo $\mathfrak{a}$ is defined by $C_F(\mathfrak{a}) = F^\times\backslash F^\times I_F(\mathfrak{a})$. For any fixed $\mathfrak{c}$ satisfying $\mathfrak{c}^2|\mathfrak{n}$, the group $\Xi_0(\mathfrak{c}) \cap \Xi_{\ker}(\mathfrak{n})$ is equal to the set of all characters of $F^\times\backslash\mathbb{A}^\times$ of finite order contained in $\Xi_0(\mathfrak{c})$. Hence

$$\#(\Xi_0(\mathfrak{c}) \cap \Xi_{\ker}(\mathfrak{n})) \leqslant \#(F^\times\backslash\mathbb{A}^\times/I_F(\mathfrak{c})) = h_F\#(C_F(\mathfrak{o})/C_F(\mathfrak{c})) \leqslant h_F \mathrm{N}(\mathfrak{c}) \leqslant h_F \mathrm{N}(\mathfrak{n})^{1/2}$$

holds, where h_F is the class number of F. Noting $\sum_{\mathfrak{c}^2|\mathfrak{n}} 1 \ll \mathrm{N}(\mathfrak{n})^\epsilon$ for any $\epsilon > 0$, we obtain the assertion. □

3. Regularized periods of automorphic forms

In this section, we recall explicit formulas in [41] of the regularized periods of automorphic forms on $G_\mathbb{A}$.

3.1. Zeta integrals of cusp forms.

Let π be a $\mathbf{K}_\infty$-spherical irreducible cuspidal automorphic representation of $G_\mathbb{A}$ with trivial central character, where the representation space V_π is realized in $L^2(Z_\mathbb{A}G_F\backslash G_\mathbb{A})$. For any quasi-character η of $F^\times\backslash\mathbb{A}^\times$ and $\varphi \in V_\pi$, we define the global zeta integral by

$$Z(s,\eta,\varphi) = \int_{F^\times\backslash\mathbb{A}^\times} \varphi\left(\left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right]\right) \eta(t)|t|_\mathbb{A}^{s-1/2} d^\times t, \qquad s \in \mathbb{C}.$$

The defining integral converges absolutely for any $s \in \mathbb{C}$, and hence $Z(s,\eta,\varphi)$ is an entire function in s.

We fix a family $\{\pi_v\}_{v\in\Sigma_F}$ consisting of irreducible admissible representations such that $\pi \cong \bigotimes_{v\in\Sigma_F} \pi_v$. The conductor of π is denoted by $\mathfrak{f}_\pi$, which is the ideal of $\mathfrak{o}$ defined by $\mathfrak{f}_\pi\mathfrak{o}_v = \mathfrak{p}_v^{c(\pi_v)}$ for all $v \in \Sigma_{\mathrm{fin}}$, where $\mathfrak{p}_v^{c(\pi_v)}$ is the conductor of π_v. Let $\mathfrak{n}$ be an ideal of $\mathfrak{o}$ which is divided by $\mathfrak{f}_\pi$.

Let n be the maximal non-negative integer m such that $S_m(\mathfrak{n}\mathfrak{f}_\pi^{-1}) \neq \emptyset$. For $\rho = (\rho_k)_{1 \leqslant k \leqslant n} \in \Lambda_\pi^0(\mathfrak{n}) = \prod_{k=1}^n \mathrm{Map}\left(S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1}), \{0, \dots, k\}\right)$, let $\varphi_{\pi,\rho}$ denote the cusp form in $V_\pi^{\mathbf{K}_\infty \mathbf{K}_0(\mathfrak{n})}$ corresponding to

$$\bigotimes_{v \in \Sigma_\infty} \phi_{0,v} \otimes \bigotimes_{k=1}^{n} \bigotimes_{v \in S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1})} \phi_{\rho_k(v),v} \otimes \bigotimes_{v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{n}\mathfrak{f}_\pi^{-1})} \phi_{0,v}$$

by the isomorphism $V_\pi \cong \bigotimes_{v \in \Sigma_F} V_{\pi_v}$. Here, V_{π_v} denotes the Whittaker model of π_v with respect to ψ_{F_v}, $\phi_{0,v}$ is the spherical vector in V_{π_v} given in [41, §1.4] for $v \in \Sigma_\infty$, and the function $\phi_{k,v}$ is the $\mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v)$-invariant vector constructed in [41, §2 and §3] for $v \in \Sigma_{\mathrm{fin}}$. Then, the finite set $\{\varphi_{\pi,\rho}\}_{\rho \in \Lambda_\pi^0(\mathfrak{n})}$ is an orthogonal basis of $V_\pi^{\mathbf{K}_\infty \mathbf{K}_0(\mathfrak{n})}$. Here $V_\pi \subset L^2(Z_\mathbb{A} G_F \backslash G_\mathbb{A})$ is equipped with the L^2-inner product (cf. [41, Proposition 17]).

We consider a character η of $F^\times \backslash \mathbb{A}^\times$ satisfying

$$\begin{cases} \eta^2 = \mathbf{1}, \\ v \in \Sigma_\infty \Rightarrow \eta_v = \mathbf{1}_v, \\ \mathfrak{f}_\eta \text{ is relatively prime to } \mathfrak{n} \text{ and } \tilde{\eta}(\mathfrak{n}) = 1. \end{cases} \tag{3.1}$$

For such a character η and $\varphi \in V_\pi^{\mathbf{K}_\infty \mathbf{K}_0(\mathfrak{n})}$, we define the modified global zeta integral by

$$Z^*(s, \eta, \varphi) = \eta(x_\eta^*) Z\left(s, \eta, \pi(\left[\begin{smallmatrix} 1 & x_\eta \\ 0 & 1 \end{smallmatrix}\right])\varphi\right), \qquad s \in \mathbb{C}.$$

Here x_η (resp. x_η^*) is the adele (resp. idele) determined by η (see §1.4).

3.2. Regularized periods of cusp forms. We recall a definition of regularized periods of automorphic forms on $G_\mathbb{A}$ defined in [47, §7]. Let $\mathcal{B}$ be the space of all holomorphic even functions β on $\mathbb{C}$ satisfying that there exist $A > 0$ and $B \in \mathbb{R}$ such that

$$|\beta(\sigma + it)| \ll e^{-A(|t|+B)^2}, \ \sigma \in [a, b], t \in \mathbb{R}$$

holds for any interval $[a, b] \subset \mathbb{R}$. We note that $\mathcal{B}$ as above is a proper subspace of $\mathcal{B}$ defined in [47, §6.1]. The growth condition of β is essentially used in Part 2.

For $\beta \in \mathcal{B}$ and $\lambda \in \mathbb{C}$, we define a function $\hat{\beta}_\lambda$ on $\mathbb{R}_{>0}$ by

$$\hat{\beta}_\lambda(t) = \frac{1}{2\pi i} \int_{L_\sigma} \frac{\beta(z)}{z + \lambda} t^z dz, \quad (\sigma > -\mathrm{Re}(\lambda)),$$

where L_σ is as in the notation. The estimate $|\hat{\beta}_\lambda(t)| \ll \inf\{t^\sigma, \ t^{-\mathrm{Re}(\lambda)}\}, \ t > 0$ is given in [47, Lemma 7.1].

For $\beta \in \mathcal{B}$, $\lambda \in \mathbb{C}$, a character η of $F^\times \backslash \mathbb{A}^\times$ satisfying (3.1) and a function $\varphi : Z_\mathbb{A} G_F \backslash G_\mathbb{A} \to \mathbb{C}$, we consider

$$P_{\beta,\lambda}^\eta(\varphi) = \int_{F^\times \backslash \mathbb{A}^\times} \{\hat{\beta}_\lambda(|t|_\mathbb{A}) + \hat{\beta}_\lambda(|t|_\mathbb{A}^{-1})\} \varphi\left(\left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & x_\eta \\ 0 & 1 \end{smallmatrix}\right]\right) \eta(t x_\eta^*) d^\times t.$$

Now we assume that for any $\beta \in \mathcal{B}$, there exists a constant $C \in \mathbb{R}$ such that the integral $P_{\beta,\lambda}^\eta(\varphi)$ converges if $\mathrm{Re}(\lambda) > C$ and the function $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > C\} \ni \lambda \mapsto P_{\beta,\lambda}^\eta(\varphi)$ is continued meromorphically to a neighborhood of $\lambda = 0$. Then a constant $P_{\mathrm{reg}}^\eta(\varphi)$ is called *the regularized* (H, η)*-period of* φ if $\mathrm{CT}_{\lambda=0} P_{\beta,\lambda}^\eta(\varphi) = P_{\mathrm{reg}}^\eta(\varphi)\beta(0)$ for all $\beta \in \mathcal{B}$. Then the following was proved in [41].

Proposition 3.1. [41, Main Theorem A] *For any $\rho = (\rho_k)_{1 \leqslant k \leqslant n} \in \Lambda_\pi^0(\mathfrak{n})$ and η satisfying* (3.1)*, the period $P_{\mathrm{reg}}^\eta(\varphi_{\pi,\rho})$ can be defined and we have*

$$P_{\mathrm{reg}}^\eta(\varphi_{\pi,\rho}) = Z^*(1/2, \eta, \varphi_{\pi,\rho}) = \mathcal{G}(\eta) \{\prod_{k=1}^n \prod_{v \in S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1})} Q_{\rho_k(v),v}^{\pi_v}(\eta_v, 1)\} L(1/2, \pi \otimes \eta),$$

where the constants $Q_{\rho_k(v),v}^{\pi_v}(\eta_v, 1)$ are given as follows:

- *If* $c(\pi_v)=0$ *and* (α_v,α_v^{-1}) *is the Satake parameter of* π_v, *then*

$$Q_{k,v}^{\pi_v}(\eta_v,1)=\begin{cases}1 & (k=0),\\ \eta_v(\varpi_v)-\dfrac{\alpha_v+\alpha_v^{-1}}{q_v^{1/2}+q_v^{-1/2}} & (k=1),\\ q_v^{-1}\eta_v(\varpi_v)^{k-2}(\alpha_v q_v^{1/2}\eta_v(\varpi_v)-1)(\alpha_v^{-1}q_v^{1/2}\eta_v(\varpi_v)-1) & (k\geqslant 2).\end{cases}$$

- *If* $c(\pi_v)=1$, *then* π_v *is isomorphic to a special representation* $\sigma(\chi_v|\cdot|_v^{1/2},\chi_v|\cdot|_v^{-1/2})$ *for some unramified character* χ_v *of* $F_v^\times$ *and*

$$Q_{k,v}^{\pi_v}(\eta_v,1)=\begin{cases}1 & (k=0),\\ \eta_v(\varpi_v)^{k-1}(\eta_v(\varpi_v)-q_v^{-1}\chi_v(\varpi_v)^{-1}) & (k\geqslant 1).\end{cases}$$

- *If* $c(\pi_v)\geqslant 2$, *then* $Q_{k,v}^{\pi_v}(\eta_v,1)=\eta_v(\varpi_v)^k$ *for any* $k\in\mathbb{N}_0$.

3.3. Preliminaries for regularized periods of Eisenstein series. We fix a character $\chi=\prod_{v\in\Sigma_F}\chi_v$ of $F^\times\backslash\mathbb{A}^\times$. For $\nu\in\mathbb{C}$, we denote by $I(\chi|\cdot|_{\mathbb{A}}^{\nu/2})$ the space of all smooth $\mathbb{C}$-valued right $\mathbf{K}$-finite functions f on $G_{\mathbb{A}}$ with the $B_{\mathbb{A}}$-equivariance

$$f\left(\left[\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right]g\right)=\chi(a/d)|a/d|_{\mathbb{A}}^{(\nu+1)/2}f(g)$$

for all $\left[\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right]\in B_{\mathbb{A}}$ and $g\in G_{\mathbb{A}}$. If $\nu\in i\mathbb{R}$, then the space $I(\chi|\cdot|_{\mathbb{A}}^{\nu/2})$ is unitarizable and a $G_{\mathbb{A}}$-invariant hermitian inner product is given by

$$(f_1|f_2)=\int_{\mathbf{K}}f_1(k)\overline{f_2(k)}dk$$

for any $f_1,f_2\in I(\chi|\cdot|_{\mathbb{A}}^{\nu/2})$.

For $\nu\in\mathbb{C}$ and $f^{(\nu)}\in I(\chi|\cdot|_{\mathbb{A}}^{\nu/2})$, the family $\{f^{(\nu)}\}_{\nu\in\mathbb{C}}$ is called a flat section if the restriction of $f^{(\nu)}$ to $\mathbf{K}$ is independent of $\nu\in\mathbb{C}$. We define the Eisenstein series for $f^{(\nu)}\in I(\chi|\cdot|_{\mathbb{A}}^{\nu/2})$ by

$$E(f^{(\nu)},g)=\sum_{\gamma\in B_F\backslash G_F}f^{(\nu)}(\gamma g),\quad g\in G_{\mathbb{A}}.$$

The defining series converges absolutely if $\mathrm{Re}(\nu)>1$. If $\{f^{(\nu)}\}_{\nu\in\mathbb{C}}$ is a flat section, then $E(f^{(\nu)},g)$ is continued meromorphically to $\mathbb{C}$ as a function in ν. We remark that the function $E(f^{(\nu)},g)$ is holomorphic on $i\mathbb{R}$. On the half plane $\mathrm{Re}(\nu)>0$, $E(f^{(\nu)},g)$ is holomorphic except for $\nu=1$, and $\nu=1$ is a pole of $E(f^{(\nu)},g)$ if and only if $\chi^2=\mathbf{1}$.

Let $\mathfrak{n}$ be an ideal of $\mathfrak{o}$. Throughout §3, we assume that a character χ of $F^\times\backslash\mathbb{A}^\times$ is contained in $\Xi(\mathfrak{n})$.

3.4. Zeta integrals of Eisenstein series. We consider Eisenstein series for $f\in I(\chi|\cdot|_{\mathbb{A}}^{\nu/2}))^{\mathbf{K}_\infty\mathbf{K}_0(\mathfrak{n})}$. Let n be the maximal non-negative integer m such that $S_m(\mathfrak{n}\mathfrak{f}_\chi^{-2})\neq\emptyset$. For each $v\in\Sigma_F$, the space $I(\chi_v|\cdot|_v^{\nu/2})$ is defined in the same way as the global case. Set

$$\Lambda_\chi(\mathfrak{n})=\prod_{k=1}^{n}\mathrm{Map}(S_k(\mathfrak{n}\mathfrak{f}_\chi^{-2}),\{0,\dots,k\}).$$

For $\rho=(\rho_k)_{1\leqslant k\leqslant n}\in\Lambda_\chi(\mathfrak{n})$, let $f_{\chi,\rho}^{(\nu)}$ denote the vector in $I(\chi|\cdot|_{\mathbb{A}}^{\nu/2})$ corresponding to

$$\bigotimes_{v\in\Sigma_\infty}f_{0,\chi_v}^{(\nu)}\otimes\bigotimes_{k=1}^{n}\bigotimes_{v\in S_k(\mathfrak{n}\mathfrak{f}_\chi^{-2})}\tilde{f}_{\rho_k(v),\chi_v}^{(\nu)}\otimes\bigotimes_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{n}\mathfrak{f}_\chi^{-2})}\tilde{f}_{0,\chi_v}^{(\nu)}$$

by the isomorphism $I(\chi|\cdot|_{\mathbb{A}}^{\nu/2}) \cong \bigotimes_{v\in\Sigma_F} I(\chi_v|\cdot|_v^{\nu/2})$, where $f_{0,\chi_v}^{(\nu)}$ is the spherical vector in $I(\chi_v|\cdot|_v^{\nu/2})$ normalized so that $f_{0,\chi_v}^{(\nu)}(1_2)$ equals one for $v\in\Sigma_\infty$ and $\tilde{f}_{k,\chi_v}^{(\nu)}$ is the $\mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v)$-invariant vector constructed in [41, §7 and §8] for $v \in \Sigma_{\mathrm{fin}}$. Then, for any $\rho = (\rho_k)_{1\leqslant k\leqslant n} \in \Lambda_\chi(\mathfrak{n})$, the family $\{f_{\chi,\rho}^{(\nu)}\}_{\nu\in\mathbb{C}}$ is a flat section. Moreover, if $\nu \in i\mathbb{R}$, the finite set $\{f_{\chi,\rho}^{(\nu)}\}_{\rho\in\Lambda_\chi(\mathfrak{n})}$ is an orthonormal basis of $I(\chi|\cdot|_{\mathbb{A}}^{\nu/2})^{\mathbf{K}_\infty\mathbf{K}_0(\mathfrak{n})}$ (cf. [41, Proposition 33]).

Let $\rho\in\Lambda_\chi(\mathfrak{n})$ and set $E_{\chi,\rho}(\nu,g) = E(f_{\chi,\rho}^{(\nu)},g)$. The constant term of $E(f_{\chi,\rho}^{(\nu)},g)$ is defined by

$$E_{\chi,\rho}^{\circ}(\nu,g) = \int_{F\backslash\mathbb{A}} E_{\chi,\rho}\left(\nu, \left[\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right] g\right) dx.$$

For $k\in\{1,\dots,n\}$, the sets $U_k(\rho)$, $R_k(\rho)$ and $R_0(\rho)$ are defined as follows:

$$U_k(\rho) = \bigcup_{m=k}^{n} \rho_m^{-1}(k) - S(\mathfrak{f}_\chi), \qquad R_k(\rho) = \bigcup_{m=k}^{n} \rho_m^{-1}(k) \cap S(\mathfrak{f}_\chi),$$

$$R_0(\rho) = \left(\bigcup_{m=0}^{n} \rho_m^{-1}(0)\cap S(\mathfrak{f}_\chi)\right) \bigcup (S(\mathfrak{f}_\chi) - S(\mathfrak{n}\mathfrak{f}_\chi^{-2})).$$

Furthermore, for any $k\in\mathbb{N}_0$, set

$$S_k(\rho) = \begin{cases} R_0(\rho) & (k=0),\\ U_k(\rho)\cup R_k(\rho) & (k\geqslant 1),\end{cases}$$

$R(\rho) = \bigcup_{k=0}^n R_k(\rho)$ and $S(\rho) = \bigcup_{k=0}^n S_k(\rho)$. Then, by [41, Proposition 34] we have

$$E_{\chi,\rho}^{\circ}(\nu,g) = f_{\chi,\rho}^{(\nu)}(g) + D_F^{-1/2} A_{\chi,\rho}(\nu) \frac{L(\nu,\chi^2)}{L(1+\nu,\chi^2)} f_{\chi^{-1},\rho}^{(-\nu)}(g),$$

where

$$A_{\chi,\rho}(\nu) = \mathrm{N}(\mathfrak{f}_\chi)^{-\nu} \prod_{k=0}^{n}\prod_{v\in S_k(\rho)} \left\{ q_v^{d_v/2} q_v^{-k\nu} \frac{\epsilon(1-\nu,\chi_v^{-2},\psi_{F_v})\epsilon(1+\nu/2,\chi_v,\psi_{F_v})}{\epsilon(1-\nu/2,\chi_v^{-1},\psi_{F_v})} \frac{L(1+\nu,\chi_v^2)}{L(1-\nu,\chi_v^{-2})}\right\}.$$

We fix a character η of $F^\times\backslash\mathbb{A}^\times$ satisfying (3.1). 3.1. For any $v\in\Sigma_{\mathrm{fin}} - S(\mathfrak{f}_\eta)$ and $k\in\mathbb{N}_0$, let $Q_{k,\chi}^{(\nu)}(\eta_v, X)$ be the polynomial defined in [41, §9] as follows:

- For $v\in\Sigma_{\mathrm{fin}} - S(\mathfrak{f}_\chi)$, set

$$Q_{k,\chi_v}^{(\nu)}(\eta_v,X) := \begin{cases} 1 & (k=0),\\ \eta_v(\varpi_v)X - \dfrac{\chi_v(\varpi_v)q_v^{-\nu/2} + \chi_v(\varpi_v)^{-1}q_v^{\nu/2}}{q_v^{1/2}+q_v^{-1/2}} & (k=1),\\ q_v^{-1}\eta_v(\varpi_v)^{k-2}X^{k-2} & \\ \times(\chi_v(\varpi_v)q_v^{(1-\nu)/2}\eta_v(\varpi_v)X-1)(\chi_v(\varpi_v)^{-1}q_v^{(1+\nu)/2}\eta_v(\varpi_v)X-1) & (k\geqslant 2).\end{cases}$$

- For $v\in S(\mathfrak{f}_\chi)$, set

$$Q_{k,\chi_v}^{(\nu)}(\eta_v,X) := \eta_v(\varpi_v)^k X^k.$$

Then, we have the following.

Proposition 3.2. [41, Proposition 35] *We set* $E^{\natural}_{\chi,\rho}(\nu,g) = E_{\chi,\rho}(\nu,g) - E^{\circ}_{\chi,\rho}(\nu,g)$. *Then* $E^{\natural}_{\chi,\rho}(\nu,-)$ *is left* B_F*-invariant and we have*

$$Z^*(s,\eta,E^{\natural}_{\chi,\rho}(\nu,-)) = \mathcal{G}(\eta)D_F^{-\nu/2}\mathrm{N}(\mathfrak{f}_\chi)^{1/2-\nu}B^{\eta}_{\chi,\rho}(s,\nu)\frac{L(s+\nu/2,\chi\eta)L(s-\nu/2,\chi^{-1}\eta)}{L(1+\nu,\chi^2)},$$

where

$$\begin{aligned} B^{\eta}_{\chi,\rho}(s,\nu) =& D_F^{s-1/2}\left\{\prod_{k=0}^{n}\prod_{v\in S_k(\rho)} Q^{(\nu)}_{k,\chi_v}(\eta_v,q_v^{1/2-s})L(1+\nu,\chi_v^2)\right\} \\ &\times \prod_{v\in U_1(\rho)}(1+q_v^{-1})q_v^{-\nu/2}\prod_{k=2}^{n}\prod_{v\in U_k(\rho)}\left(\frac{q_v+1}{q_v-1}\right)^{1/2}q_v^{-k\nu/2} \\ &\times \left\{\prod_{k=0}^{n}\prod_{v\in R_k(\rho)} q_v^{d_v/2-k\nu/2}(1-q_v^{-1})^{1/2}\overline{\mathcal{G}(\chi_v)}\right\}\prod_{v\in\Sigma_{\mathrm{fin}}-R(\rho)}\chi_v(\varpi_v)^{d_v}. \end{aligned}$$

3.5. Regularized periods of Eisenstein series. For any characters χ_1 and χ_2 of $F^\times\backslash\mathbb{A}^\times$, we put $\delta_{\chi_1,\chi_2} = \delta(\chi_1=\chi_2)$. The regularized period $P^{\eta}_{\mathrm{reg}}(E_{\chi,\rho}(\nu,-))$ was computed as follows in [41].

Proposition 3.3. [41, Main Theorem B] *Assume* $\nu\in i\mathbb{R}$. *Then the integral* $P^{\eta}_{\beta,\lambda}(E_{\chi,\rho}(\nu,-))$ *converges absolutely for any* $(\beta,\lambda)\in\mathcal{B}\times\mathbb{C}$ *such that* $\mathrm{Re}(\lambda)>1$. *Moreover* $P^{\eta}_{\mathrm{reg}}(E_{\chi,\rho}(\nu,-))$ *can be defined, and we have*

$$P^{\eta}_{\mathrm{reg}}(E_{\chi,\rho}(\nu,-)) = \mathcal{G}(\eta)D_F^{-\nu/2}\mathrm{N}(\mathfrak{f}_\chi)^{1/2-\nu}B^{\eta}_{\chi,\rho}(1/2,\nu)\frac{L((1+\nu)/2,\chi\eta)L((1-\nu)/2,\chi^{-1}\eta)}{L(1+\nu,\chi^2)}.$$

We define two functions $\mathfrak{e}_{\chi,\rho,-1}$ and $\mathfrak{e}_{\chi,\rho,0}$ on $G_{\mathbb{A}}$ by the Laurent expansion

$$E_{\chi,\rho}(\nu,g) = \frac{\mathfrak{e}_{\chi,\rho,-1}(g)}{\nu-1} + \mathfrak{e}_{\chi,\rho,0}(g) + \mathcal{O}(\nu-1), \qquad (\nu\to1).$$

We explain the regularized periods of $\mathfrak{e}_{\chi,\rho,-1}$ and that of $\mathfrak{e}_{\chi,\rho,0}$. Set $R_F = \mathrm{Res}_{s=1}\zeta_F(s) = \mathrm{vol}(F^\times\backslash\mathbb{A}^1)$.

Proposition 3.4. [41, Lemma 38 and Theorem 39] *We have*

$$\mathfrak{e}_{\chi,\rho,-1}(g) = \delta\left(\chi^2=\mathbf{1},\mathfrak{f}_\chi=\mathfrak{o},S(\rho)=\emptyset\right)\frac{D_F^{-1/2}R_F}{\zeta_F(2)}\chi(\det g)$$

for any $g\in G_{\mathbb{A}}$. *Moreover, for* $\lambda\in\mathbb{C}$ *such that* $\mathrm{Re}(\lambda)>0$, *we have*

$$P^{\eta}_{\beta,\lambda}(\mathfrak{e}_{\chi,\rho,-1}) = \delta\left(\chi=\eta,\mathfrak{f}_\chi=\mathfrak{o},S(\rho)=\emptyset\right)\frac{2D_F^{-1/2}R_F^2}{\zeta_F(2)}\frac{\beta(0)}{\lambda}$$

and $P^{\eta}_{\mathrm{reg}}(\mathfrak{e}_{\chi,\rho,-1})=0$.

For any character ξ of $F^\times\backslash\mathbb{A}^\times$, we define $R(\xi)$, $C_0(\xi)$ and $C_1(\xi)$ by the Laurent expansion

$$L(s,\xi) = \frac{R(\xi)}{s-1} + C_0(\xi) + C_1(\xi)(s-1) + \mathcal{O}((s-1)^2), \qquad (s\to1).$$

We note $R(\xi) = \delta_{\xi,\mathbf{1}}R_F$ for any character ξ of $F^\times\backslash\mathbb{A}^\times$.

Proposition 3.5. [41, Theorem 40 and Corollary 41] *Let* η *be a character of* $F^\times\backslash\mathbb{A}^\times$ *satisfying (3.1). The integral* $P^{\eta}_{\beta,\lambda}(\mathfrak{e}_{\chi,\rho,0})$ *converges absolutely for any* $(\beta,\lambda)\in\mathcal{B}\times\mathbb{C}$ *such that* $\mathrm{Re}(\lambda)>1$. *There exists* *an entire function* $f(\lambda)$ *on* $\mathbb{C}$ *such that*

$$\begin{aligned} P^{\eta}_{\beta,\lambda}(\mathfrak{e}_{\chi,\rho,0}) &= \delta_{\chi,\eta}R_F f^{(1)}_{\chi,\rho}(1_2)\left\{\frac{1}{\lambda-1}+\frac{1}{\lambda+1}\right\}\beta(1) \\ &+2\delta_{\chi,\eta}R_F\frac{D_F^{-1/2}f^{(1)}_{\chi,\rho}(1_2)}{\zeta_F(2)}\left\{R_F\left(-\frac{\zeta_F'(2)}{\zeta_F(2)}A_{\chi,\rho}(1)+A'_{\chi,\rho}(1)\right)+C_0(\mathbf{1})A_{\chi,\rho}(1)\right\}\frac{\beta(0)}{\lambda} \\ &+f(\lambda)-\mathcal{G}(\eta)D_F^{-1/2}R_F\delta_{\chi,\eta}\left\{-\frac{\tilde{B}^{\eta}_{\chi,\rho}(1)}{\lambda+1}+\frac{\tilde{B}^{\eta}_{\chi,\rho}(-1)}{\lambda-1}\right\}\beta(1) \\ &-\frac{\mathcal{G}(\eta)D_F^{-1/2}}{\zeta_F(2)}\delta_{\chi,\eta}\left\{-(\tilde{B}^{\eta}_{\chi,\rho})'(0)R_F^2\frac{\beta(0)}{\lambda}+\tilde{B}^{\eta}_{\chi,\rho}(0)R_F^2\frac{\beta(0)}{\lambda^2}\right\}, \end{aligned}$$

where $\tilde{B}^{\eta}_{\chi,\rho}(z)=\epsilon(-z,\chi^{-1}\eta)B^{\eta}_{\chi,\rho}(-z+1/2,1)$*. Moreover we have*

$$\mathrm{CT}_{\lambda=0}P^{\eta}_{\beta,\lambda}(\mathfrak{e}_{\chi,\rho,0})=\frac{\mathcal{G}(\eta)D_F^{-1/2}\mathrm{N}(\mathfrak{f}_\chi)^{-1/2}}{L(2,\chi^2)}\left\{-\frac{1}{2}\delta_{\chi,\eta}\tilde{B}^{\eta}_{\chi,\rho}(0)R_F^2\beta''(0)+a^{\eta}_{\chi,\rho}(0)\beta(0)\right\},$$

where

$$a^{\eta}_{\chi,\rho}(0)=-\frac{1}{2}\delta_{\chi,\eta}(\tilde{B}^{\eta}_{\chi,\rho})''(0)R_F^2-2\delta_{\chi,\eta}\tilde{B}^{\eta}_{\chi,\rho}(0)R_FC_1(\mathbf{1})+\tilde{B}^{\eta}_{\chi,\rho}(0)C_0(\chi\eta)^2.$$

3.6. An orthonormal basis of $V_\pi^{\mathbf{K}_\infty\mathbf{K}_0(\mathfrak{n})}$. Let π be a cuspidal automorphic representation of $G_{\mathbb{A}}$ such that $\pi\in\Pi_{\rm cus}(\mathfrak{n})$. We put

$$\mathbb{P}^{\eta}(\pi;\mathbf{K}_0(\mathfrak{n}))=\sum_{\varphi\in\mathcal{B}(\pi;\mathfrak{n})}\overline{Z^*(1/2,\mathbf{1},\varphi)}Z^*(1/2,\eta,\varphi),$$

where $\mathcal{B}(\pi;\mathfrak{n})$ is an orthonormal basis of $V_\pi^{\mathbf{K}_\infty\mathbf{K}_0(\mathfrak{n})}$. In this subsection, we examine $\mathbb{P}^{\eta}(\pi;\mathbf{K}_0(\mathfrak{n}))$. Set $\varphi_\pi^{\rm new}=\varphi_{\pi,\rho_\pi}$, where ρ_π is a unique element of $\Lambda_\pi^0(\mathfrak{f}_\pi)$.

Lemma 3.6. *The value* $\mathbb{P}^{\eta}(\pi;\mathbf{K}_0(\mathfrak{n}))$ *is independent of the choice of* $\mathcal{B}(\pi;\mathfrak{n})$ *and we have*

$$\mathbb{P}^{\eta}(\pi;\mathbf{K}_0(\mathfrak{n}))=D_F^{-1/2}\mathcal{G}(\eta)w^{\eta}_{\mathfrak{n}}(\pi)\frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{||\varphi_\pi^{\rm new}||^2}.$$

Here $w^{\eta}_{\mathfrak{n}}(\pi)$ *is an explicit non-negative constant defined as*

$$w^{\eta}_{\mathfrak{n}}(\pi)=\prod_{k=1}^{n}\prod_{v\in S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1})}r(\pi_v,\eta_v,k)=\prod_{v\in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})}r(\pi_v,\eta_v,\mathrm{ord}_v(\mathfrak{n}\mathfrak{f}_\pi^{-1}))$$

where $r(\pi_v,\eta_v,k)$ *is defined as follows:*

- *If* $c(\pi_v)\geqslant 2$*, then* $r(\pi_v,\eta_v,k)=\begin{cases}k+1 & (\eta_v(\varpi_v)=1),\\ 2^{-1}(1+(-1)^k) & (\eta_v(\varpi_v)=-1).\end{cases}$
- *If* $c(\pi_v)=1$*, then* π_v *is isomorphic to* $\sigma(\chi_v|\cdot|_v^{1/2},\chi_v|\cdot|_v^{-1/2})$ *for some unramified character* χ_v *of* $F_v^\times$*. Then*

$$r(\pi_v,\eta_v,k)=\begin{cases}1+\dfrac{1-\chi_v(\varpi_v)q_v^{-1}}{1+\chi_v(\varpi_v)q_v^{-1}}\,k & (\eta_v(\varpi_v)=1),\\ \\ 2^{-1}(1+(-1)^k) & (\eta_v(\varpi_v)=-1).\end{cases}$$

- *If $c(\pi_v)=0$ and $(\alpha_v, \alpha_v^{-1})$ is the Satake parameter of π_v, then*
$$r(\pi_v,\eta_v,k)=\begin{cases}\dfrac{2}{1+Q(\pi_v)}+\dfrac{1-Q(\pi_v)}{1+Q(\pi_v)}\dfrac{q_v+1}{q_v-1}(k-1) & (\eta_v(\varpi_v)=1),\\[2ex] \dfrac{q_v+1}{q_v-1}\dfrac{1+(-1)^k}{2} & (\eta_v(\varpi_v)=-1),\end{cases}$$
where $Q(\pi_v)=(\alpha_v+\alpha_v^{-1})(q_v^{1/2}+q_v^{-1/2})^{-1}$.

Moreover, $\mathcal{G}(\eta)^{-1}\mathbb{P}^\eta(\pi;\mathbf{K}_0(\mathfrak{n}))$ is non-negative.

Proof. The first assertion is obvious. Thus we may take $\{||\varphi_{\pi,\rho}||^{-1}\varphi_{\pi,\rho}\}_{\rho\in\Lambda^0_\pi(\mathfrak{n})}$ as $\mathcal{B}(\pi;\mathfrak{n})$. By virtue of Proposition 3.1, we have
$$\mathbb{P}^\eta(\pi;\mathbf{K}_0(\mathfrak{n}))=\sum_{\rho\in\Lambda^0_\pi(\mathfrak{n})}\frac{1}{||\varphi_{\pi,\rho}||^2}Z^*(1/2,\mathbf{1},\varphi)Z^*(1/2,\eta,\varphi)$$
$$=\sum_{\rho\in\Lambda^0_\pi(\mathfrak{n})}\prod_{k=1}^{n}\prod_{v\in S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1})}\left\{\frac{\overline{Q^{\pi_v}_{\rho_k(v),v}(\mathbf{1}_v,1)}Q^{\pi_v}_{\rho_k(v),v}(\eta_v,1)}{\tau_{\pi_v}(\rho_k(v),\rho_k(v))}\right\}\frac{\mathcal{G}(\mathbf{1})\mathcal{G}(\eta)L(1/2,\pi)L(1/2,\pi\otimes\eta)}{||\varphi_\pi^{\rm new}||^2}.$$
Then, we obtain the second assertion by setting
$$w_{\mathfrak{n}}^\eta(\pi)=\sum_{\rho\in\Lambda^0_\pi(\mathfrak{n})}\prod_{k=1}^{n}\prod_{v\in S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1})}\left\{\frac{\overline{Q^{\pi_v}_{\rho_k(v),v}(\mathbf{1}_v,1)}Q^{\pi_v}_{\rho_k(v),v}(\eta_v,1)}{\tau_{\pi_v}(\rho_k(v),\rho_k(v))}\right\}.$$
Here $\tau_{\pi_v}(j,j)=||\phi_{j,v}||_v^2$ for $j\in\mathbb{N}$, and $||\cdot||_v$ is the norm on V_{π_v} defined by the G_v-invariant inner product normalized so that $||\phi_{0,v}||_v=1$. We remark that an explicit formula of $\tau_{\pi_v}(j,j)$ was given in [41, Corollaries 12, 16 and Lemma 3] (see (20.6)). By definition and a direct computation, we have
$$w_{\mathfrak{n}}^\eta(\pi)=\prod_{k=1}^{n}\left\{\sum_{(j_v)_v\in\{0,\dots,k\}^{S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1})}}\prod_{v\in S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1})}r_{v,j_v}\right\}=\prod_{k=1}^{n}\prod_{v\in S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1})}\sum_{j=0}^{k}r_{v,j}$$
and $\sum_{j=0}^k r_{v,j}=r(\pi_v,\eta_v,k)$, where $r_{v,j}=\overline{Q^{\pi_v}_{j,v}(\mathbf{1}_v,1)}Q^{\pi_v}_{j,v}(\eta_v,1)\tau_{\pi_v}(j,j)^{-1}$.

Then, one can check $w_{\mathfrak{n}}^\eta(\pi)\in\mathbb{R}_{\geqslant 0}$ easily by noting $|Q(\pi_v)|<1$ when $c(\pi_v)=0$. The estimate $L(1/2,\pi)L(1/2,\pi\otimes\eta)\geqslant 0$ by [10] gives $\mathcal{G}(\eta)^{-1}\mathbb{P}^\eta(\pi;\mathbf{K}_0(\mathfrak{n}))\geqslant 0$. □

Since $\eta^2=\mathbf{1}$, we have $\tilde\eta(\mathfrak{n})=\pm1$. We consider only the case of $\tilde\eta(\mathfrak{n})=1$ because of the following reason.

Lemma 3.7. *Let π be a $\mathbf{K}_\infty$-spherical irreducible cuspidal automorphic representation of $G_{\mathbb{A}}$ with trivial central character. Let η be a character of $F^\times\backslash\mathbb{A}^\times$ such that $\eta^2=\mathbf{1}$ and $\mathfrak{f}_\eta$ is relatively prime to $\mathfrak{f}_\pi$. Suppose that $\eta_v(-1)=1$ for all $v\in\Sigma_\infty$. Then, $L(1/2,\pi)L(1/2,\pi\otimes\eta)=0$ unless $\tilde\eta(\mathfrak{f}_\pi)=1$.*

Proof. By the argument in the proof of [47, Lemma 2.3], it is enough to show $\epsilon(1/2,\pi_v,\psi_{F_v})\epsilon(1/2,\pi_v\otimes\eta_v,\psi_{F_v})=\eta_v(\varpi_v^{c(\pi_v)})$ for any $v\in\bigcup_{k\geqslant2}S_k(\mathfrak{f}_\pi)$. It follows immediately from fundamental properties of ϵ-factors (cf. [38, 1.1]). We note that η_v is unramified if $v\in S(\mathfrak{f}_\pi)$. □

3.7. Adjoint L-functions. Let π be a cuspidal automorphic representation of $G_{\mathbb{A}}$ contained in $\Pi_{\rm cus}(\mathfrak{n})$. To examine an explicit description of $||\varphi_\pi^{\rm new}||^2$ in terms of the adjoint L-function of π, we compute the Rankin-Selberg convolution of $\varphi_\pi^{\rm new}$ and the $\mathbf{K}$-spherical Eisenstein series
$$E_{\mathbf{1},\rho_0}(\nu,g)=\sum_{\gamma\in B_F\backslash G_F}y(\gamma g)^{(\nu+1)/2},\qquad g\in G_{\mathbb{A}},\ {\rm Re}(\nu)>1,$$

where ρ_0 denotes a unique element of $\Lambda_{\mathbf{1}}(\mathfrak{o})$. For any $v \in \Sigma_F$, $Z_v(s)$ denotes the local Rankin-Selberg integral

$$\int_{\mathbf{K}_v}\int_{F_v^\times} \phi_{0,v}\left(\left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right] k_v\right) \overline{\phi_{0,v}\left(\left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right] k_v\right)} |t_v|_v^{s-1} d^\times t_v dk_v.$$

Lemma 3.8. *Set $S_\pi = \{v \in \Sigma_{\mathrm{fin}} \mid \mathrm{ord}_v(\mathfrak{f}_\pi) \geqslant 2\}$. We have*

$$\begin{aligned}&\int_{Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}} \varphi_\pi^{\mathrm{new}}(g)\overline{\varphi_\pi^{\mathrm{new}}(g)} E_{\mathbf{1},\rho_0}(2s-1,g)dg \\ =&[\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{f}_\pi)]^{-1}\mathrm{N}(\mathfrak{f}_\pi)^s D_F^{s-3/2}\zeta_F(2s)^{-1}\zeta_F(s)L(s,\pi,\mathrm{Ad}) \prod_{v\in S_\pi} \frac{q_v^{d_v(3/2-s)} q_v^{c(\pi_v)(1-s)} Z_v(s)}{L(s,\pi_v,\mathrm{Ad})} \frac{1+q_v^{-1}}{1+q_v^{-s}}\end{aligned}$$

for $\mathrm{Re}(s) \gg 1$. *Moreover, we have* $||\varphi_\pi^{\mathrm{new}}||^2 = 2\mathrm{N}(\mathfrak{f}_\pi)[\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{f}_\pi)]^{-1} L^{S_\pi}(1,\pi,\mathrm{Ad})$.

Proof. If $v \in \Sigma_F - S_\pi$, then $Z_v(s)$ is computed in [47, Lemma 2.14]. Hence, it suffices to examine $Z_v(s)$ when $v \in S_\pi$. By $[\mathbf{K}_v : \mathbf{K}_0(\mathfrak{p}_v^{c(\pi_v)})] = q_v^{c(\pi_v)}(1+q_v^{-1})$, we obtain the first assertion.

We note $Z_v(1) = q_v^{-d_v/2}$ for $v \in S_\pi$. Then we obtain the second assertion by taking the residue at $s = 1$ since $\mathrm{Res}_{s=1} E_{\mathbf{1},\rho_{\mathbf{0}}}(s,g) = D_F^{-1/2} R_F \zeta_F(2)^{-1}$ holds by Proposition 3.4. □

4. Green's functions on $\mathrm{GL}(2,\mathbb{R})$

In this section, we review the definition of Green functions on $\mathrm{GL}(2,\mathbb{R})$, which was introduced in [47, §4]. For $s, z \in \mathbb{C}$ such that $\mathrm{Re}(s) > 2|\mathrm{Re}(z)|$, set

$$\begin{aligned}(4.1)\quad &\Psi^{(z)}(s;g) \\ =&|\det g|^{(s+1)/2} \frac{-1}{8\sqrt{\pi}} \frac{\Gamma((s+2z+1)/4)\Gamma((s-2z+1)/4)}{\Gamma(s/2+1)} (a^2+b^2)^{-(s-2z+1)/4}(c^2+d^2)^{-(s+2z+1)/4} \\ &\times {}_2F_1\left(\frac{s+2z+1}{4}, \frac{s-2z+1}{4}; \frac{s}{2}+1; \frac{(\det g)^2}{(a^2+b^2)(c^2+d^2)}\right)\end{aligned}$$

for any $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}(2,\mathbb{R})$. We call this the Green function on $\mathrm{GL}(2,\mathbb{R})$.

Lemma 4.1. [47, §4] *Set $T = \{\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] \mid t_1, t_2 \in \mathbb{R}^\times\}$.*

(1) *For any $\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] \in T$ and $k \in \mathrm{O}(2,\mathbb{R})$, we have*

$$\Psi^{(z)}(s; \left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] gk) = |t_1/t_2|^z \Psi^{(z)}(s;g), \quad g \in \mathrm{GL}(2,\mathbb{R}).$$

(2) *For any $a_r = \left[\begin{smallmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{smallmatrix}\right]$ with $r \in \mathbb{R}$, we have*

$$\begin{aligned}\Psi^{(z)}(s;a_r) =& \frac{-1}{8\sqrt{\pi}} \frac{\Gamma((s+2z+1)/4)\Gamma((s-2z+1)/4)}{\Gamma(s/2+1)} (\cosh 2r)^{-(s+1)/2} \\ &\times {}_2F_1\left(\frac{s+2z+1}{4}, \frac{s-2z+1}{4}; \frac{s}{2}+1; \frac{1}{\cosh^2 2r}\right).\end{aligned}$$

(3) *The function $\Psi^{(z)}(s)$ is smooth on $\mathrm{GL}(2,\mathbb{R}) - T\,\mathrm{O}(2,\mathbb{R})$ and a Casimir eigenfunction with eigenvalue $(s^2-1)/2$, i.e.,*

$$[R(\Omega)\Psi^{(z)}(s)](a_r) = \frac{s^2-1}{2}\Psi^{(z)}(s;a_r), \quad r \in \mathbb{R} - \{0\}.$$

(4) *We have*

$$\lim_{r\to+0} \frac{d}{dr}\Psi^{(z)}(s;a_r) - \lim_{r\to-0}\frac{d}{dr}\Psi^{(z)}(s;a_r) = 1.$$

In particular, $\Psi^{(z)}(s)$ is continuous on $\mathrm{GL}(2,\mathbb{R})$ but not smooth on $T\,\mathrm{O}(2,\mathbb{R})$.

Remark 4.2. *Tsuzuki considered the differential equation in (3) under the conditions (1) and (4). He solved it and gave an explicit formula (2), which suffices to obtain Lemma 4.1 by using the decomposition* $\mathrm{GL}(2,\mathbb{R}) = T\{a_r \mid r \in \mathbb{R}\}\mathrm{SO}(2,\mathbb{R})$ *and the equivariance (1).*

Proposition 4.3. [47, §4] *Assume that* $s \in \mathbb{C}$ *and* $z \in \mathbb{C}$ *satisfy* $\mathrm{Re}(s) > |2\,\mathrm{Re}(z)|+1$. *Let* $f : \mathrm{GL}(2,\mathbb{R}) \to \mathbb{C}$ *be a smooth function such that* $f\left(\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] gk\right) = |t_1/t_2|_v^{-z} f(g)$ *for all* $\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] \in T$, $k \in \mathrm{O}(2,\mathbb{R})$ *and* $g \in \mathrm{GL}(2,\mathbb{R})$. *Suppose*

$$\sum_{m=0}^{2} \left| \frac{d^m}{dr^m} f(a_r) \right| \ll (\cosh 2r)^{|\,\mathrm{Re}(z)|}, \qquad r \in \mathbb{R}.$$

Then the equality

$$\int_{T\backslash \mathrm{GL}(2,\mathbb{R})} \Psi^{(z)}(s,g)[R(\Omega - (s^2-1)/2)f](g)dg = f(1_2)$$

holds with the integral being convergent absolutely.

5. Green's functions on GL(2) over non-archimedean local fields

This section is a review of results in [47, §5]. We fix a place $v \in \Sigma_{\mathrm{fin}}$. For $z \in \mathbb{C}$, there exists a unique function $\Phi_{0,v}^{(z)} : G_v \to \mathbb{C}$ such that

$$\Phi_{0,v}^{(z)}(\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right] k) = |t_1/t_2|_v^{z}\,\delta(x \in \mathfrak{o}_v), \qquad \left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] \in H_v,\ \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right] \in N_v,\ k \in \mathbf{K}_v. \tag{5.1}$$

Given $z \in \mathbb{C}$ and $s \in \mathbb{C}/4\pi i(\log q_v)^{-1}\mathbb{Z}$, we consider the following inhomogeneous equation

$$R\left(\mathbb{T}_v - (q_v^{(1-s)/2} + q_v^{(1+s)/2})\,1_{\mathbf{K}_v}\right)\Psi = \Phi_{0,v}^{(z)} \tag{5.2}$$

where $\Psi : G_v \to \mathbb{C}$ satisfies the $(H_v, \mathbf{K}_v)$-equivariance

$$\Psi\left(\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] gk\right) = |t_1/t_2|_v^{z}\,\Psi(g), \qquad \left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] \in H_v,\ k \in \mathbf{K}_v. \tag{5.3}$$

Here $\mathbb{T}_v$ and $1_{\mathbf{K}_v}$ are elements of the spherical Hecke algebra $\mathcal{H}(G_v, \mathbf{K}_v)$ defined by

$$\mathbb{T}_v = \frac{1}{\mathrm{vol}(\mathbf{K}_v; dg)}\,\mathrm{ch}_{\mathbf{K}_v\left[\begin{smallmatrix} \varpi_v & 0 \\ 0 & 1 \end{smallmatrix}\right]\mathbf{K}_v}, \qquad 1_{\mathbf{K}_v} = \frac{1}{\mathrm{vol}(\mathbf{K}_v; dg)}\,\mathrm{ch}_{\mathbf{K}_v}.$$

The function $\mathbb{T}_v$ is called the v-th Hecke operator.

Lemma 5.1. [47, Lemma 5.2] *Suppose* $\mathrm{Re}(s) > |2\,\mathrm{Re}(z) - 1|$. *Then, there exists a unique bounded function* $\Psi_v^{(z)}(s; -) : G_v \to \mathbb{C}$ *satisfying* (5.2) *and* (5.3), *whose values on* N_v *are given by*

(5.4)

$$\Psi_v^{(z)}(s; \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right]) = -q_v^{-(s+1)/2}(1 - q_v^{-(s-2z+1)/2})^{-1}(1 - q_v^{-(s+2z+1)/2})^{-1}\sup(1, |x|_v)^{-(s-2z+1)/2}, \qquad x \in F_v.$$

Proof. We note the decomposition $G_v = \coprod_{m \geqslant 0} H_v \mathrm{n}_m \mathbf{K}_v$ with $\mathrm{n}_m = \left[\begin{smallmatrix} 1 & \varpi_v^{-m} \\ 0 & 1 \end{smallmatrix}\right]$. Hence, the condition (5.3) implies that a function Ψ satisfying (5.2) and (5.3) is determined by all values $a(m) = \Psi(\mathrm{n}_m)$, $m \geqslant 0$. By (5.2), we have a relation on $a(m-1)$, $a(m)$ and $a(m+1)$. By solving the recurrence equation and noting the boundedness of $\{a(m)\}_{m \geqslant 0}$, we determine $\{a(m)\}_{m \geqslant 0}$ uniquely. We can refer to [47, Lemmas 5.1 and 5.2] for details. □

The following lemma is used in the proof of Propositions 6.1 and 12.2.

Lemma 5.2. [47, Lemma 5.4] *Let* $f : G_v \to \mathbb{C}$ *be a smooth function such that* $f\left(\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] gk\right) = |t_1/t_2|_v^{-z} f(g)$ *for any* $t_1, t_2 \in F_v^\times$ *and for any* $k \in \mathbf{K}_v$. *Then, the equality*

$$\int_{H_v\backslash G_v} \Psi_v^{(z)}(s; g)\,[R(\mathbb{T}_v - (q_v^{(1+s)/2} + q_v^{(1-s)/2})\,1_{\mathbf{K}_v})f](g)\,dg = \mathrm{vol}(H_v\backslash H_v\mathbf{K}_v)\,f(1_2) \tag{5.5}$$

holds as long as the integral on the left-hand side converges absolutely.

6. Even non-holomorphic adelic Green functions

We define the adelic Green function on $G_{\mathbb{A}}$ associated with an arbitrary ideal $\mathfrak{n}$ of $\mathfrak{o}$. This was introduced in [47] in the case where $\mathfrak{n}$ is square-free.

Assume that $s \in \mathbb{C}$ and $z \in \mathbb{C}$ satisfy $\mathrm{Re}(s) > 2|\mathrm{Re}(z)|$. For each $v \in \Sigma_\infty$, we denote by $\Psi_v^{(z)}(s, -)$ the Green function on $G_v \cong \mathrm{GL}(2, \mathbb{R})$ defined in §4. For each $v \in S(\mathfrak{n})$, we set

$$\Phi_{\mathfrak{n},v}^{(z)}\left(\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right] k\right) = |t_1/t_2|_v^z \delta(x \in \mathfrak{o}_v)\delta(k \in \mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v))$$

for any $t_1, t_2 \in F_v^\times$, $x \in F_v$ and $k \in \mathbf{K}_v$ and put $\Phi_{0,v}^{(z)} = \Phi_{\mathfrak{n},v}^{(z)}$ for $v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{n})$.

Fix a finite subset S of Σ_F such that $\Sigma_\infty \subset S$ and set $S_{\mathrm{fin}} = S \cap \Sigma_{\mathrm{fin}}$. Set $\mathfrak{X}_S = \prod_{v \in \Sigma_\infty} \mathbb{C} \times \prod_{v \in S_{\mathrm{fin}}} (\mathbb{C}/4\pi i (\log q_v)^{-1}\mathbb{Z})$ and $q(\mathbf{s}) = \inf_{v \in S}(\mathrm{Re}(s_v) + 1)/4$ for any $\mathbf{s} \in \mathfrak{X}_S$. For any $\mathbf{s} \in \mathfrak{X}_S$ and $z \in \mathbb{C}$ such that $q(\mathbf{s}) > |\mathrm{Re}(z)| + 1$, *the adelic Green function* is defined by

$$\Psi^{(z)}(\mathfrak{n}|\mathbf{s}; g) = \prod_{v \in \Sigma_\infty} \Psi_v^{(z)}(s_v; g_v) \prod_{v \in S_{\mathrm{fin}}} \Psi_v^{(z)}(s_v; g_v) \prod_{v \in S(\mathfrak{n})} \Phi_{\mathfrak{n},v}^{(z)}(g_v) \prod_{v \notin S \cup S(\mathfrak{n})} \Phi_{0,v}^{(z)}(g_v)$$

for any $g = (g_v)_{v \in \Sigma_F} \in G_{\mathbb{A}}$. Note that the function $\Psi^{(z)}(\mathfrak{n}|\mathbf{s})$ on $G_{\mathbb{A}}$ is right $\mathbf{K}_\infty \mathbf{K}_0(\mathfrak{n})$-invariant and continuous on $G_{\mathbb{A}}$. Moreover, we have $\Psi^{(z)}\left(\mathfrak{n}|\mathbf{s}; \left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] g\right) = |t_1/t_2|_{\mathbb{A}}^z \Psi^{(z)}(\mathfrak{n}|\mathbf{s}; g)$ for all $\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] \in H_{\mathbb{A}}$ and $g \in G_{\mathbb{A}}$.

To state a main property of adelic Green functions, we consider the integral

$$\varphi^{H,(z)}(g) = \int_{Z_{\mathbb{A}} H_F \backslash H_{\mathbb{A}}} \varphi(hg)\chi_z(h)dh$$

for any $\varphi \in C_c^\infty(Z_{\mathbb{A}} G_F \backslash G_{\mathbb{A}})$, where $\chi_z : H_F \backslash H_{\mathbb{A}} \to \mathbb{C}^\times$ is the quasi-character defined by

$$\chi_z(\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right]) = |t_1/t_2|_{\mathbb{A}}^z$$

for any $t_1, t_2 \in \mathbb{A}^\times$. The integral $\varphi^{H,(z)}(g)$ converges absolutely and $\varphi^{H,(z)}(hg) = \chi_z(h)^{-1}\varphi^{H,(z)}(g)$ holds for any $h \in H_{\mathbb{A}}$ (cf. [47, §6.2]).

Let $\mathfrak{Z}(\mathfrak{g}_\infty)$ be the center of the universal enveloping algebra of the complexification of $\mathfrak{g}_\infty$. For $\mathbf{s} \in \mathfrak{X}_S$, the element $\mathbf{\Omega}_S(\mathbf{s})$ of the algebra $\mathfrak{Z}(\mathfrak{g}_\infty) \otimes \{\bigotimes_{v \in S_{\mathrm{fin}}} \mathcal{H}(G_v, \mathbf{K}_v)\}$ is defined as

$$\mathbf{\Omega}_S(\mathbf{s}) = \bigotimes_{v \in \Sigma_\infty} \left(\Omega_v - \frac{s_v^2 - 1}{2}\right) \bigotimes_{v \in S_{\mathrm{fin}}} \left(\mathbb{T}_v - (q_v^{(1-s_v)/2} + q_v^{(1+s_v)/2})1_{\mathbf{K}_v}\right).$$

The following proposition is proved in the same way as [47, Lemma 6.3].

Proposition 6.1. *Suppose $q(\mathbf{s}) > 2|\mathrm{Re}(z)|+1$. Then, for any $\varphi \in C_c^\infty(Z_{\mathbb{A}} G_F \backslash G_{\mathbb{A}})^{\mathbf{K}_\infty \mathbf{K}_0(\mathfrak{n})}$, the function $g \mapsto \Psi^{(z)}(\mathfrak{n}|\mathbf{s}; g)\varphi^{H,(z)}(g)$ is integrable on $H_{\mathbb{A}} \backslash G_{\mathbb{A}}$ and we have*

$$\int_{H_{\mathbb{A}} \backslash G_{\mathbb{A}}} \Psi^{(z)}(\mathfrak{n}|\mathbf{s}, g)[R(\Omega_S(\mathbf{s}))\varphi^{H,(z)}](g)dg = \mathrm{vol}(H_{\mathrm{fin}} \backslash H_{\mathrm{fin}} \mathbf{K}_0(\mathfrak{n}))\varphi^{H,(z)}(1_2).$$

7. Spectral expansions of renormalized Green functions

The set $\mathfrak{X}_S = \prod_{v \in \Sigma_\infty} \mathbb{C} \times \prod_{v \in S_{\mathrm{fin}}} (\mathbb{C}/4\pi i (\log q_v)^{-1}\mathbb{Z})$ is considered as a complex manifold with respect to a usual complex structure. Let $\mathcal{A}_S$ be the space of holomorphic functions $\alpha(\mathbf{s})$ on $\mathfrak{X}_S$ such that for any $v \in S$ and $\mathbf{s}' \in \mathfrak{X}_{S - \{v\}}$, the function $s_v \mapsto \alpha(\mathbf{s}', s_v)$ is contained in $\mathcal{B}$.

For $\mathbf{c} \in \mathbb{R}^S$, we put $\mathbb{L}_S(\mathbf{c}) = \{\mathbf{s} \in \mathfrak{X}_S \mid \mathrm{Re}(\mathbf{s}) = \mathbf{c}\}$. A multidimensional contour integral of a holomorphic function $f(\mathbf{s})$ on $\mathfrak{X}_S$ along $\mathbb{L}_S(\mathbf{c})$ is defined inductively as

$$\int_{\mathbb{L}_S(\mathbf{c})} f(\mathbf{s})d\mu_S(\mathbf{s}) = \int_{L_v(c_v)} \left\{ \int_{\mathbb{L}_{S-\{v\}}(\mathbf{c}')} f(\mathbf{s}', s_v)d\mu_{S-\{v\}}(\mathbf{s}') \right\} d\mu_v(s_v)$$

for $\mathbf{c} = (\mathbf{c}', c_v) \in \mathbb{R}^S$, where

$$d\mu_v(s) = \begin{cases} sds & (v \in \Sigma_\infty), \\ \frac{1}{2}(\log q_v)(q_v^{(1+s)/2} - q_v^{(1-s)/2})ds & (v \in \Sigma_{\mathrm{fin}}) \end{cases}$$

and $L(c_v)$ stands for $c_v + i\mathbb{R}$ and $c_v + \mathbb{C}/4\pi i(\log q_v)^{-1}\mathbb{Z}$ for $v \in \Sigma_\infty$ and $v \in \Sigma_{\mathrm{fin}}$, respectively. Then, for $\mathbf{c} \in \mathbb{R}^S$ and $z \in \mathbb{C}$ such that $q(\mathbf{c}) > |\operatorname{Re}(z)| + 1$, the integral

$$\hat{\Psi}^{(z)}(\mathfrak{n}|\alpha; g) = \left(\frac{1}{2\pi i}\right)^{\#S} \int_{\mathbb{L}_S(\mathbf{c})} \Psi^{(z)}(\mathfrak{n}|\mathbf{s}; g)\alpha(\mathbf{s})d\mu_S(\mathbf{s})$$

is absolutely convergent and is independent of the choice of $\mathbf{c}$, and the function $z \mapsto \hat{\Psi}^{(z)}(\mathfrak{n}|\alpha; g)$ is entire. Furthermore, for $\beta \in \mathcal{B}$, $\lambda \in \mathbb{C}$ and $g \in G_{\mathbb{A}}$, we consider the integral

$$\hat{\Psi}_{\beta,\lambda}(\mathfrak{n}|\alpha; g) = \frac{1}{2\pi i}\int_{L_\sigma} \frac{\beta(z)}{z+\lambda}\{\hat{\Psi}^{(z)}(\mathfrak{n}|\alpha; g) + \hat{\Psi}^{(-z)}(\mathfrak{n}|\alpha; g)\}dz$$

for $\sigma \in \mathbb{R}$ such that $-\inf(q(\mathbf{s})-1, \operatorname{Re}(\lambda)) < \sigma < q(\mathbf{s})-1$. The integral of the right-hand side is absolutely convergent and is independent of the choice of σ. Moreover, for $\alpha \in \mathcal{A}_S$, $\beta \in \mathcal{B}$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) > 0$, the Poincaré series of $\hat{\Psi}_{\beta,\lambda}(\mathfrak{n}|\alpha; g)$ is defined to be

$$\hat{\mathbf{\Psi}}_{\beta,\lambda}(\mathfrak{n}|\alpha; g) = \sum_{\gamma \in H_F\backslash G_F} \hat{\Psi}_{\beta,\lambda}(\mathfrak{n}|\alpha; \gamma g)$$

for $g \in G_{\mathbb{A}}$. In the same way as [47, Proposition 9.1], we obtain

(1) The series $\hat{\mathbf{\Psi}}_{\beta,\lambda}(\mathfrak{n}|\alpha; g)$ is absolutely convergent locally uniformly in $\{\operatorname{Re}(\lambda) > 0\} \times G_{\mathbb{A}}$. Moreover, the function $\lambda \mapsto \hat{\mathbf{\Psi}}_{\beta,\lambda}(\mathfrak{n}|\alpha; g)$ on $\operatorname{Re}(\lambda) > 0$ is holomorphic and the function $g \mapsto \hat{\mathbf{\Psi}}_{\beta,\lambda}(\mathfrak{n}|\alpha; g)$ on $G_{\mathbb{A}}$ is continuous, left G_F-invariant and right $\mathbf{K}_\infty\mathbf{K}_0(\mathfrak{n})$-invariant.
(2) For $\operatorname{Re}(\lambda) > 0$, we have $\hat{\mathbf{\Psi}}_{\beta,\lambda}(\mathfrak{n}|\alpha) \in L^m(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})$ for any $m > 0$ such that $m(1 - \operatorname{Re}(\lambda)) < 1$.

Let us compute the spectral expansion of $\hat{\mathbf{\Psi}}_{\beta,\lambda}(\mathfrak{n}|\alpha)$ explicitly. Recall spectral parameters at S of automorphic forms (cf. [47, 9.1.3]). For a given automorphic form φ on $G_{\mathbb{A}}$, if there exists $\nu_{\varphi,S} = (\nu_{\varphi,v})_{v\in S} \in \mathfrak{X}_S$ such that

$$R(\Omega_v)\varphi = \frac{\nu_{\varphi,v}^2 - 1}{2}\varphi$$

and

$$R(\mathbb{T}_v)\varphi = (q_v^{(1-\nu_{\varphi,v})/2} + q_v^{(1+\nu_{\varphi,v})/2})\varphi$$

hold for all $v \in \Sigma_\infty$ and all $v \in S_{\mathrm{fin}}$, respectively, then we call $\nu_{\varphi,S}$ the spectral parameter at S of φ. Set

$$C(\mathfrak{n}, S) = (-1)^{\#S}\mathrm{vol}(H_{\mathrm{fin}}\backslash H_{\mathrm{fin}}\mathbf{K}_0(\mathfrak{n})) = (-1)^{\#S}D_F^{-1/2}[\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{n})]^{-1}.$$

By using Proposition 6.1 and the argument similar to [47, Lemma 9.4], we have the following.

Lemma 7.1. *Assume* $\operatorname{Re}(\lambda) > 1$. *Then, for any automorphic form φ on $G_{\mathbb{A}}$ with spectral parameter $\nu_{\varphi,S}$, we have*

$$\langle\hat{\mathbf{\Psi}}_{\beta,\lambda}(\mathfrak{n}|\alpha)|\varphi\rangle_{L^2} = C(\mathfrak{n}, S)\alpha(\nu_{\varphi,S})P^{\mathbf{1}}_{\beta,\lambda}(\overline{\varphi}),$$

where $\langle\cdot|\cdot\rangle_{L^2}$ is the L^2-inner product on $L^2(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})$.

For any character χ of $F^\times\backslash\mathbb{A}^\times$ and $\alpha \in \mathcal{A}_S$, we define the function $\tilde{\alpha}_\chi$ on $\mathbb{C}$ by

$$\tilde{\alpha}_\chi(\nu) = \alpha((\nu + 2ib(\chi_v))_{v\in S})$$

and write $\tilde{\alpha}(\nu)$ for $\tilde{\alpha}_{\mathbf{1}}(\nu)$.

Fix an orthonormal basis $\mathcal{B}_{\mathrm{cus}}(\mathfrak{n})$ of $\sum_{\pi\in\Pi_{\mathrm{cus}}(\mathfrak{n})} V_\pi^{\mathbf{K}_\infty\mathbf{K}_0(\mathfrak{n})}$ and let $\mathcal{B}_{\mathrm{res}}(\mathfrak{n})$ be the orthonormal system consisting of all functions $\varphi_\chi = \mathrm{vol}(Z_\mathbb{A}G_F\backslash G_\mathbb{A})^{-1/2}\,\chi\circ\det$ on $G_\mathbb{A}$ for any $\chi\in\Xi_0(\mathfrak{o})$ such that $\chi^2=\mathbf{1}$. We write $\Lambda(\mathfrak{n})$ for $\Lambda_{\mathbf{1}}(\mathfrak{n})$. Lemma 7.1 and the same method as [47, Lemma 9.6] give the following.

Lemma 7.2. *Assume* $\mathrm{Re}(\lambda)>1$. *Then we have the expression*

$$\begin{aligned}\hat{\mathbf{\Psi}}_{\beta,\lambda}(\mathfrak{n}|\alpha;g) =&C(\mathfrak{n},S)\Bigg\{\sum_{\varphi\in\mathcal{B}_{\mathrm{cus}}(\mathfrak{n})}\alpha(\nu_{\varphi,S})P^{\mathbf{1}}_{\beta,\lambda}(\overline{\varphi})\varphi(g)+\sum_{\varphi\in\mathcal{B}_{\mathrm{res}}(\mathfrak{n})}\alpha(\nu_{\varphi,S})P^{\mathbf{1}}_{\beta,\lambda}(\overline{\varphi})\varphi(g)\\ &+\sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\frac{R_F^{-1}}{8\pi i}\int_{i\mathbb{R}}\tilde{\alpha}_\chi(\nu)P^{\mathbf{1}}_{\beta,\lambda}(\overline{E_{\chi,\rho}(\nu,-)})E_{\chi,\rho}(\nu,g)d\nu\Bigg\}.\end{aligned}$$

The series and integrals in the right-hand side converge absolutely and locally uniformly on $Z_\mathbb{A}G_F\backslash G_\mathbb{A}$.

Lemma 7.3. *For any* $g\in G_\mathbb{A}$, *the function* $\lambda\mapsto\hat{\mathbf{\Psi}}_{\beta,\lambda}(\mathfrak{n}|\alpha;g)$ *on* $\mathrm{Re}(\lambda)>1$ *is continued to a meromorphic function on* $\mathrm{Re}(\lambda)>-1/2$.

Proof. Let $\Psi_{\mathrm{cus}}(\lambda)=\Psi_{\mathrm{cus}}(\lambda,\alpha,g)$, $\Psi_{\mathrm{res}}(\lambda)=\Psi_{\mathrm{res}}(\lambda,\alpha,g)$ and $\Psi_{\mathrm{ct}}(\lambda)=\Psi_{\mathrm{ct}}(\lambda,\alpha,g)$ be the cuspidal part, residual part and Eisenstein part divided by $C(\mathfrak{n},S)$ in the spectral expansion of $\hat{\mathbf{\Psi}}_{\beta,\lambda}(\mathfrak{n}|\alpha;g)$ given in Lemma 7.2, respectively.

First we examine $\Psi_{\mathrm{res}}(\lambda)$. For $\mathrm{Re}(\lambda)>0$, by applying Proposition 3.4, the function $\Psi_{\mathrm{res}}(\lambda)$ is written as

$$\Psi_{\mathrm{res}}(\lambda)=\sum_{\chi\in\Xi_0(\mathfrak{o}),\chi^2=\mathbf{1}}\alpha(\nu_{\varphi_\chi,S})P^{\mathbf{1}}_{\beta,\lambda}(\overline{\varphi_\chi})\varphi_\chi(g)=2\tilde{\alpha}(1)\frac{R_F}{\mathrm{vol}(Z_\mathbb{A}G_F\backslash G_\mathbb{A})}\frac{\beta(0)}{\lambda}$$

and has a meromorphic continuation to $\mathbb{C}$. From this, $\mathrm{CT}_{\lambda=0}\Psi_{\mathrm{res}}(\lambda)=0$ holds.

Next we examine $\Psi_{\mathrm{cus}}(\lambda)$. By the same computation as in the proof of [47, Lemma 9.8], the series $\Psi_{\mathrm{cus}}(\lambda)$ converges absolutely and the estimate

$$|\Psi_{\mathrm{cus}}(\lambda,\alpha,g)|\ll y(g)^{-m},\quad g\in\mathfrak{S}^1$$

holds. Moreover, $\Psi_{\mathrm{cus}}(\lambda)$ is analytically continued to an entire function and we have

$$\mathrm{CT}_{\lambda=0}\Psi_{\mathrm{cus}}(\lambda)=\sum_{\varphi\in\mathcal{B}_{\mathrm{cus}}(\mathfrak{n})}\alpha(\nu_{\varphi,S})\overline{P^{\mathbf{1}}_{\mathrm{reg}}(\varphi)}\varphi(g).$$

Therefore, it is enough to examine $\Psi_{\mathrm{ct}}(\lambda)$. Assume $\mathrm{Re}(\lambda)>1$ and $\nu\in i\mathbb{R}$. By the proof of [41, Theorem 37], the integral $P^{\mathbf{1}}_{\lambda,\beta}(E_{\chi^{-1},\rho}(-\nu,-))$ can be expressed as

$$\begin{aligned}P^{\mathbf{1}}_{\lambda,\beta}(E_{\chi^{-1},\rho}(-\nu,-))=&P_{\chi^{-1}}(\mathbf{1},\lambda,-\nu)+D_F^{-1/2}A_{\chi^{-1},\rho}(-\nu)\frac{L(-\nu,\chi^{-2})}{L(1-\nu,\chi^{-2})}P_\chi(\mathbf{1},\lambda,\nu)\\ &+Q^+_{\chi^{-1},\rho}(\mathbf{1},\lambda,-\nu)+Q^-_{\chi^{-1},\rho}(\mathbf{1},\lambda,-\nu),\end{aligned}$$

where

$$P_{\chi^{\pm1}}(\eta,\lambda,\pm\nu)=f^{(\pm\nu)}_{\chi^{\pm1},\rho}(1_2)\delta_{\chi,\eta}R_F\left\{\frac{\beta((\mp\nu-1)/2)}{\lambda-(\pm\nu+1)/2}+\frac{\beta((\pm\nu+1)/2)}{\lambda+(\pm\nu+1)/2}\right\}$$

and

$$Q^{\pm}_{\chi^{-1},\rho}(\eta,\lambda,-\nu)=\frac{1}{2\pi i}\int_{L_{\pm\sigma}}Z^*(\pm z+1/2,\eta,E^\natural_{\chi^{-1},\rho}(-\nu,-))\frac{\beta(z)}{\lambda+z}dz.$$

We remark $\overline{E_{\chi,\rho}(\nu,-)}=E_{\chi^{-1},\rho}(-\nu,-)$. Furthermore, by the residue theorem, we have

$$\begin{aligned}&P^{\mathbf{1}}_{\lambda,\beta}(E_{\chi^{-1},\rho}(-\nu,-))\\ =&P_{\chi^{-1}}(\mathbf{1},\lambda,-\nu)+D_F^{-1/2}A_{\chi^{-1},\rho}(-\nu)\frac{\zeta_F(-\nu)}{\zeta_F(1-\nu)}P_\chi(\mathbf{1},\lambda,\nu)+Q^0_{\chi^{-1},\rho}(\mathbf{1},\lambda,-\nu)\end{aligned}$$

$$
\begin{aligned}
&-\left\{\frac{\beta((-\nu+1)/2)}{\lambda+(-\nu+1)/2}\operatorname{Res}_{z=(-\nu+1)/2}+\frac{\beta((\nu+1)/2)}{\lambda+(\nu+1)/2}\operatorname{Res}_{z=(\nu+1)/2}\right.\\
&\left.+\frac{\beta((-\nu-1)/2)}{\lambda+(-\nu-1)/2}\operatorname{Res}_{z=(-\nu-1)/2}+\frac{\beta((\nu-1)/2)}{\lambda+(\nu-1)/2}\operatorname{Res}_{z=(\nu-1)/2}\right\}f^{\mathbf{1}}_{\chi^{-1},\rho}(-z,-\nu),
\end{aligned}
$$

where we put

$$f^{\eta}_{\chi,\rho}(z,\nu)=Z^*(z+1/2,\eta,E^{\natural}_{\chi,\rho}(\nu,-))$$

and

$$Q^0_{\chi,\rho}(\eta,\lambda,\nu)=\frac{1}{2\pi i}\int_{L_\sigma}\{f^{\eta}_{\chi,\rho}(z,\nu)+f^{\eta}_{\chi,\rho}(-z,\nu)\}\frac{\beta(z)}{\lambda+z}dz$$

for $\mathrm{Re}(\lambda)>-\sigma$. Thus we express $\Psi_{\mathrm{ct}}(\lambda)$ as the sum of the following four terms:

$$
\begin{aligned}
\Phi_1(\lambda)=&\frac{1}{8\pi i}\sum_{\rho\in\Lambda(\mathfrak{n})}f^{(-\nu)}_{\mathbf{1},\rho}(1_2)\int_{i\mathbb{R}}\tilde{\alpha}(\nu)\beta((\nu-1)/2)\left\{\frac{1}{\lambda-(-\nu+1)/2}+\frac{1}{\lambda+(-\nu+1)/2}\right\}E_{\mathbf{1},\rho}(\nu,g)d\nu,\\
\Phi_2(\lambda)=&\frac{1}{8\pi i}\sum_{\rho\in\Lambda(\mathfrak{n})}f^{(\nu)}_{\mathbf{1},\rho}(1_2)\int_{i\mathbb{R}}\tilde{\alpha}(\nu)D_F^{-1/2}A_{\mathbf{1},\rho}(-\nu)\frac{\zeta_F(-\nu)}{\zeta_F(1-\nu)}\beta((\nu+1)/2)\\
&\times\left\{\frac{1}{\lambda-(\nu+1)/2}+\frac{1}{\lambda+(\nu+1)/2}\right\}E_{\mathbf{1},\rho}(\nu,g)d\nu,\\
\Phi_3(\lambda)=&\frac{1}{8\pi i}\sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\int_{i\mathbb{R}}\tilde{\alpha}_\chi(\nu)Q^0_{\chi^{-1},\rho}(\mathbf{1},\lambda,-\nu)E_{\chi,\rho}(\nu,g)d\nu,\\
\Phi_4(\lambda)=&-\sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\frac{R_F^{-1}}{8\pi i}\int_{i\mathbb{R}}\left\{\frac{\beta((\nu+1)/2)}{\lambda+(\nu+1)/2}\operatorname{Res}_{z=(\nu+1)/2}\right.\\
&+\frac{\beta((-\nu+1)/2)}{\lambda+(-\nu+1)/2}\operatorname{Res}_{z=(-\nu+1)/2}+\frac{\beta((\nu-1)/2)}{\lambda+(\nu-1)/2}\operatorname{Res}_{z=(\nu-1)/2}\\
&\left.+\frac{\beta((-\nu-1)/2)}{\lambda+(-\nu-1)/2}\operatorname{Res}_{z=(-\nu-1)/2}\right\}\{f^{\mathbf{1}}_{\chi^{-1},\rho}(-z,-\nu)\}\tilde{\alpha}_\chi(\nu)E_{\chi,\rho}(\nu,g)d\nu.
\end{aligned}
$$

By the functional equation

$$D_F^{-1/2}A_{\mathbf{1},\rho}(-\nu)\frac{\zeta_F(-\nu)}{\zeta_F(1-\nu)}E_{\mathbf{1},\rho}(\nu,g)=E_{\mathbf{1},\rho}(-\nu,g)$$

of the Eisenstein series, the following equalities hold:

$$
\begin{aligned}
\Phi_2(\lambda)=&\frac{1}{8\pi i}\sum_{\rho\in\Lambda(\mathfrak{n})}f^{(\nu)}_{\mathbf{1},\rho}(1_2)\int_{i\mathbb{R}}\tilde{\alpha}(\nu)D_F^{-1/2}A_{\mathbf{1},\rho}(-\nu)\frac{\zeta_F(-\nu)}{\zeta_F(1-\nu)}E_{\mathbf{1},\rho}(\nu,g)\beta((\nu+1)/2)\\
&\times\left\{\frac{1}{\lambda-(\nu+1)/2}+\frac{1}{\lambda+(\nu+1)/2}\right\}d\nu\\
=&\frac{1}{8\pi i}\sum_{\rho\in\Lambda(\mathfrak{n})}f^{(\nu)}_{\mathbf{1},\rho}(1_2)\int_{i\mathbb{R}}\tilde{\alpha}(\nu)E_{\mathbf{1},\rho}(\nu,g)\beta((-\nu+1)/2)\left\{\frac{1}{\lambda-(-\nu+1)/2}+\frac{1}{\lambda+(-\nu+1)/2}\right\}d\nu\\
=&\Phi_1(\lambda).
\end{aligned}
$$

Thus we have to consider only $\Phi_1(\lambda)$, $\Phi_3(\lambda)$ and $\Phi_4(\lambda)$.

We take $c>1$. Then $\Phi_1(\lambda)$ is expressed as

$$\Phi_1(\lambda)=\frac{1}{8\pi i}\sum_{\rho\in\Lambda(\mathfrak{n})}f^{(-\nu)}_{\mathbf{1},\rho}(1_2)\int_{i\mathbb{R}}\tilde{\alpha}(\nu)\beta((\nu-1)/2)\left\{\frac{1}{\lambda-(-\nu+1)/2}+\frac{1}{\lambda+(-\nu+1)/2}\right\}E_{\mathbf{1},\rho}(\nu,g)d\nu$$

$$=\frac{1}{8\pi i}\sum_{\rho\in\Lambda(\mathfrak{n})} f_{\mathbf{1},\rho}^{(-\nu)}(1_2)\int_{i\mathbb{R}}\tilde{\alpha}(\nu)\beta((\nu-1)/2)\frac{1}{\lambda+(-\nu+1)/2}E_{\mathbf{1},\rho}(\nu,g)d\nu$$
$$+\frac{1}{8\pi i}\sum_{\rho\in\Lambda(\mathfrak{n})} f_{\mathbf{1},\rho}^{(-\nu)}(1_2)\Big\{\int_{L_c}\tilde{\alpha}(\nu)\beta((\nu-1)/2)\frac{1}{\lambda-(-\nu+1)/2}E_{\mathbf{1},\rho}(\nu,g)d\nu$$
$$-2\pi i\tilde{\alpha}(1)\beta(0)\frac{2}{\lambda}\mathfrak{e}_{\mathbf{1},\rho,-1}(g)\Big\}.$$

The first term is holomorphic on $\mathrm{Re}(\lambda)>-1/2$, the second term is holomorphic on $\mathrm{Re}(\lambda)>(-c+1)/2$ and the third term is holomorphic on $\mathbb{C}-\{0\}$. Hence $\Phi_1(\lambda)=\Phi_2(\lambda)$ has a meromorphic continuation to $\mathrm{Re}(\lambda)>-1/2$. Since $\Phi_3(\lambda)$ is described as an absolutely convergent double integral, $\Phi_3(\lambda)$ has an analytic continuation to $\mathbb{C}$. We note that the integral $Q^0_{\chi^{-1},\rho}(\mathbf{1},\lambda,-\nu)$ is absolutely convergent and is entire as a function in λ. In order to examine $\Phi_4(\lambda)$, we consider the following residues:

$$\mathrm{Res}_{z=(\nu+1)/2}\, f^{\mathbf{1}}_{\chi^{-1},\rho}(-z,-\nu)=D_F^{-1/2+\nu/2}\mathrm{N}(\mathfrak{f}_\chi)^{1/2+\nu}B^{\mathbf{1}}_{\chi^{-1},\rho}(-\nu/2,-\nu)\frac{L(-\nu,\chi^{-1})}{L(1-\nu,\chi^{-2})}\delta_{\chi,\mathbf{1}}D_F^{1/2}R_F,$$

$$\mathrm{Res}_{z=(-\nu+1)/2}\, f^{\mathbf{1}}_{\chi^{-1},\rho}(-z,-\nu)=D_F^{-1/2+\nu/2}\mathrm{N}(\mathfrak{f}_\chi)^{1/2+\nu}B^{\mathbf{1}}_{\chi^{-1},\rho}(\nu/2,-\nu)\frac{L(\nu,\chi^{-1})}{L(1-\nu,\chi^{-2})}\delta_{\chi,\mathbf{1}}D_F^{1/2}R_F,$$

$$\mathrm{Res}_{z=(\nu-1)/2}\, f^{\mathbf{1}}_{\chi^{-1},\rho}(-z,-\nu)=D_F^{-1/2+\nu/2}\mathrm{N}(\mathfrak{f}_\chi)^{1/2+\nu}B^{\mathbf{1}}_{\chi^{-1},\rho}(1-\nu/2,-\nu)\frac{L(1-\nu,\chi^{-1})}{L(1-\nu,\chi^{-2})}(-\delta_{\chi,\mathbf{1}}R_F),$$

$$\mathrm{Res}_{z=(-\nu-1)/2}\, f^{\mathbf{1}}_{\chi^{-1},\rho}(-z,-\nu)=D_F^{-1/2+\nu/2}\mathrm{N}(\mathfrak{f}_\chi)^{1/2+\nu}B^{\mathbf{1}}_{\chi^{-1},\rho}(1+\nu/2,-\nu)\frac{L(1+\nu,\chi)}{L(1-\nu,\chi^{-2})}(-\delta_{\chi,\mathbf{1}}R_F).$$

The functions $\mathrm{Res}_{z=(\pm\nu\pm1)/2}\, f^{\mathbf{1}}_{\chi^{-1},\rho}(-z,-\nu)$ are holomorphic on $i\mathbb{R}$ as functions in ν and vanish unless $\chi=\mathbf{1}$. Therefore, the integral

$$\int_{i\mathbb{R}}\left\{\frac{\beta((\nu+1)/2)}{\lambda+(\nu+1)/2}\mathrm{Res}_{z=(\nu+1)/2}+\frac{\beta((-\nu+1)/2)}{\lambda+(-\nu+1)/2}\mathrm{Res}_{z=(-\nu+1)/2}\right\} f^{\mathbf{1}}_{\chi^{-1},\rho}(-z,-\nu)\tilde{\alpha}_\chi(\nu)E_{\chi,\rho}(\nu,g)d\nu$$

is holomorphic on $\mathrm{Re}(\lambda)>-1/2$.

Consider the integral

$$\int_{i\mathbb{R}}\left\{\frac{\beta((\nu-1)/2)}{\lambda+(\nu-1)/2}\mathrm{Res}_{z=(\nu-1)/2}+\frac{\beta((-\nu-1)/2)}{\lambda+(-\nu-1)/2}\mathrm{Res}_{z=(-\nu-1)/2}\right\} f^{\mathbf{1}}_{\chi^{-1},\rho}(-z,-\nu)\tilde{\alpha}_\chi(\nu)E_{\chi,\rho}(\nu,g)d\nu.$$

Set $F^+_\rho(\nu)=\mathrm{Res}_{z=(\nu-1)/2}\, f^{\mathbf{1}}_{\mathbf{1},\rho}(-z,-\nu)$. We note that $F^+_\rho(\nu)$ is entire. By taking $c>1$, we obtain

$$\int_{i\mathbb{R}}\frac{\beta((\nu-1)/2)}{\lambda+(\nu-1)/2}\mathrm{Res}_{z=(\nu-1)/2}\, f^{\mathbf{1}}_{\mathbf{1},\rho}(-z,-\nu)\tilde{\alpha}(\nu)E_{\mathbf{1},\rho}(\nu,g)d\nu$$
$$=\int_{L_c}\frac{\beta((\nu-1)/2)}{\lambda+(\nu-1)/2}F^+_\rho(\nu)\tilde{\alpha}(\nu)E_{\mathbf{1},\rho}(\nu,g)d\nu-2\pi i\frac{\beta(0)}{\lambda}F^+_\rho(1)\tilde{\alpha}(1)\mathfrak{e}_{\mathbf{1},\rho,-1}(g).$$

The first term of the right-hand side is holomorphic on $\mathrm{Re}(\lambda)>(-c+1)/2$ and the second term is meromorphic on $\mathbb{C}$. Set $F^-_\rho(\nu)=\mathrm{Res}_{z=(-\nu-1)/2}\, f^{\mathbf{1}}_{\mathbf{1},\rho}(-z,-\nu)$. By the relation $B^{\mathbf{1}}_{\mathbf{1},\rho}(1-\nu/2,-\nu)=B^{\mathbf{1}}_{\mathbf{1},\rho}(1-\nu/2,\nu)A_{\mathbf{1},\rho}(-\nu)$, we have

$$F^-_\rho(-\nu)D_F^{-1/2}A_{\mathbf{1},\rho}(-\nu)\frac{\zeta_F(-\nu)}{\zeta_F(1-\nu)}=B^{\mathbf{1}}_{\mathbf{1},\rho}(1-\nu/2,\nu)D_F^{\nu/2}(-R_F)D_F^{-1/2}A_{\mathbf{1},\rho}(-\nu)=F^+_\rho(\nu),$$

and hence, we obtain

$$\int_{i\mathbb{R}} \frac{\beta((-\nu-1)/2)}{\lambda+(-\nu-1)/2} F_\rho^-(\nu)\tilde{\alpha}(\nu)E_{\mathbf{1},\rho}(\nu,g)d\nu = \int_{i\mathbb{R}} \frac{\beta((\nu-1)/2)}{\lambda+(\nu-1)/2} F_\rho^-(-\nu)\tilde{\alpha}(\nu)E_{\mathbf{1},\rho}(-\nu,g)d\nu$$
$$= \int_{i\mathbb{R}} \frac{\beta((\nu-1)/2)}{\lambda+(\nu-1)/2} F_\rho^-(-\nu)D_F^{-1/2}A_{\mathbf{1},\rho}(-\nu)\frac{\zeta_F(-\nu)}{\zeta_F(1-\nu)}\tilde{\alpha}(\nu)E_{\mathbf{1},\rho}(\nu,g)d\nu$$
$$= \int_{i\mathbb{R}} \frac{\beta((\nu-1)/2)}{\lambda+(\nu-1)/2} F_\rho^+(\nu)\tilde{\alpha}(\nu)E_{\mathbf{1},\rho}(\nu,g)d\nu$$
$$= \int_{L_c} \frac{\beta((\nu-1)/2)}{\lambda+(\nu-1)/2} F_\rho^+(\nu)\tilde{\alpha}(\nu)E_{\mathbf{1},\rho}(\nu,g)d\nu - 2\pi i\frac{\beta(0)}{\lambda}F_\rho^+(1)\tilde{\alpha}(1)\mathfrak{e}_{\mathbf{1},\rho,-1}(g).$$

Then, in the last line of the equalities above, the first term is holomorphic on $\mathrm{Re}(\lambda) > (-c+1)/2$ and the second term is meromorphic on $\mathbb{C}$. Hence $\Phi_4(\lambda)$ has a meromorphic continuation to $\mathrm{Re}(\lambda) > -1/2$. This gives us a meromorphic continuation of $\Psi_{\mathrm{ct}}(\lambda)$ to $\mathrm{Re}(\lambda) > -1/2$. $\square$

Lemma 7.4. *We have*

$$\begin{aligned}\mathrm{CT}_{\lambda=0}\hat{\mathbf{\Psi}}_{\beta,\lambda}(\mathfrak{n}|\alpha;g) =& C(\mathfrak{n},S)\Bigg\{\sum_{\varphi\in\mathcal{B}_{\mathrm{cus}}(\mathfrak{n})} \alpha(\nu_{\varphi,S})\overline{P^{\mathbf{1}}_{\mathrm{reg}}(\varphi)}\varphi(g) \\ &+ \sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})} \frac{R_F^{-1}}{8\pi i}\int_{i\mathbb{R}} \tilde{\alpha}_\chi(\nu)P^{\mathbf{1}}_{\mathrm{reg}}(E_{\chi^{-1},\rho}(-\nu,-))E_{\chi,\rho}(\nu,g)d\nu \\ &+ \sum_{\rho\in\Lambda(\mathfrak{n})}\{f^{(0)}_{\mathbf{1},\rho}(1_2)+D_{\mathbf{1}}(\rho)\}\{\tilde{\alpha}'(1)\mathfrak{e}_{\mathbf{1},\rho,-1}(g)+\tilde{\alpha}(1)\mathfrak{e}_{\mathbf{1},\rho,0}(g)\}\Bigg\}\beta(0),\end{aligned}$$

where we put $D_\eta(\rho) = \delta(\cup_{k=2}^n S_k(\rho)=\emptyset)\prod_{v\in S_1(\rho)}\{-\eta_v(\varpi_v)q_v^{-1/2}\}$ *for* $\eta\in\Xi_0(\mathfrak{o})$.

Proof. In the proof of Lemma 7.3, we gave the constant terms of the cuspidal and residual parts at $\lambda=0$. Therefore, it is enough to evaluate the constant term of the Eisenstein part $\Psi_{\mathrm{ct}}(\lambda) = 2\Phi_1(\lambda)+\Phi_3(\lambda)+\Phi_4(\lambda)$. By the residue theorem, we have

$$\begin{aligned}\mathrm{CT}_{\lambda=0}\Phi_1(\lambda) =& \frac{1}{8\pi i}\sum_{\rho\in\Lambda(\mathfrak{n})} f^{(-\nu)}_{\mathbf{1},\rho}(1_2)\int_{i\mathbb{R}}\tilde{\alpha}(\nu)\beta((\nu-1)/2)\frac{-1}{(\nu-1)/2}E_{\mathbf{1},\rho}(\nu,g)d\nu \\ &+ \frac{1}{8\pi i}\sum_{\rho\in\Lambda(\mathfrak{n})} f^{(-\nu)}_{\mathbf{1},\rho}(1_2)\int_{L_c}\tilde{\alpha}(\nu)\beta((\nu-1)/2)\frac{1}{(\nu-1)/2}E_{\mathbf{1},\rho}(\nu,g)d\nu \\ =& \frac{1}{8\pi i}\sum_{\rho\in\Lambda(\mathfrak{n})} f^{(-\nu)}_{\mathbf{1},\rho}(1_2)2\pi i\,\mathrm{Res}_{\nu=1}\{\tilde{\alpha}(\nu)\beta((\nu-1)/2)\frac{1}{(\nu-1)/2}E_{\mathbf{1},\rho}(\nu,g)\} \\ =& \frac{1}{2}\sum_{\rho\in\Lambda(\mathfrak{n})} f^{(-\nu)}_{\mathbf{1},\rho}(1_2)\{\tilde{\alpha}'(1)\mathfrak{e}_{\mathbf{1},\rho,-1}(g)+\tilde{\alpha}(1)\mathfrak{e}_{\mathbf{1},\rho,0}(g)\}\beta(0)\end{aligned}$$

and the integral $Q^0_{\chi^{-1},\rho}(\mathbf{1},0,-\nu)$ is written as

$$\begin{aligned}Q^0_{\chi^{-1},\rho}(\mathbf{1},0,-\nu) =& \frac{1}{2\pi i}\int_{L_\sigma}\{f^{\mathbf{1}}_{\chi^{-1},\rho}(z,-\nu)+f^{\mathbf{1}}_{\chi^{-1},\rho}(-z,-\nu)\}\frac{\beta(z)}{z}dz \\ &= f^{\mathbf{1}}_{\chi^{-1},\rho}(0,-\nu)\beta(0) + \{\mathrm{Res}_{z=(1+\nu)/2}+\mathrm{Res}_{z=(1-\nu)/2}+\mathrm{Res}_{z=(-1+\nu)/2} \\ &\quad + \mathrm{Res}_{z=(-1-\nu)/2}\}\left\{f^{\mathbf{1}}_{\chi^{-1},\rho}(z,-\nu)\frac{\beta(z)}{z}\right\}.\end{aligned}$$

Thus the constant term of $\Phi_3(\lambda)$ is evaluated as

$$\begin{aligned}
&\mathrm{CT}_{\lambda=0}\Phi_3(\lambda)\\
=&\frac{R_F^{-1}}{8\pi i}\sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\int_{i\mathbb{R}}\tilde{\alpha}_\chi(\nu)Q^0_{\chi^{-1},\rho}(\mathbf{1},0,-\nu)E_{\chi,\rho}(\nu,g)d\nu\\
=&\frac{R_F^{-1}}{8\pi i}\sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\bigg\{\int_{i\mathbb{R}}\tilde{\alpha}_\chi(\nu)f^{\mathbf{1}}_{\chi^{-1},\rho}(0,-\nu)E_{\chi,\rho}(\nu,g)d\nu\beta(0)+\int_{i\mathbb{R}}\{\mathrm{Res}_{z=(1+\nu)/2}\\
&+\mathrm{Res}_{z=(1-\nu)/2}+\mathrm{Res}_{z=(-1+\nu)/2}+\mathrm{Res}_{z=(-1-\nu)/2}\}\bigg\{f^{\mathbf{1}}_{\chi^{-1},\rho}(z,-\nu)\frac{\beta(z)}{z}\bigg\}\tilde{\alpha}_\chi(\nu)E_{\chi,\rho}(\nu,g)d\nu\bigg\}.
\end{aligned}$$

We examine the constant term of $\Phi_4(\lambda)$. By the expression of $\Phi_4(\lambda)$ given in the proof of Lemma 7.3, we have

$$\begin{aligned}
&\mathrm{CT}_{\lambda=0}\Phi_4(\lambda)\\
=&-\frac{R_F^{-1}}{8\pi i}\sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\int_{i\mathbb{R}}\bigg\{\frac{\beta((\nu+1)/2)}{(\nu+1)/2}\mathrm{Res}_{z=(\nu+1)/2}+\frac{\beta((-\nu+1)/2)}{(-\nu+1)/2}\mathrm{Res}_{z=(-\nu+1)/2}\bigg\}f^{\mathbf{1}}_{\chi^{-1},\rho}(-z,-\nu)\\
&\times\tilde{\alpha}_\chi(\nu)E_{\chi,\rho}(\nu,g)d\nu\\
&-2\times\frac{R_F^{-1}}{8\pi i}\sum_{\rho\in\Lambda(\mathfrak{n})}\bigg\{\int_{L_c}\frac{\beta((\nu-1)/2)}{(\nu-1)/2}F^+_\rho(\nu)\tilde{\alpha}(\nu)E_{\mathbf{1},\rho}(\nu,g)d\nu\bigg\}.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\mathrm{CT}_{\lambda=0}\{\Phi_3(\lambda)+\Phi_4(\lambda)\}\\
=&\frac{R_F^{-1}}{8\pi i}\sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\bigg\{\int_{i\mathbb{R}}\tilde{\alpha}_\chi(\nu)f^{\mathbf{1}}_{\chi^{-1},\rho}(0,-\nu)E_{\chi,\rho}(\nu,g)d\nu\beta(0)\\
&+\int_{i\mathbb{R}}\{\mathrm{Res}_{z=(-1+\nu)/2}+\mathrm{Res}_{z=(-1-\nu)/2}\}\bigg\{f^{\mathbf{1}}_{\chi^{-1},\rho}(z,-\nu)\frac{\beta(z)}{z}\bigg\}\tilde{\alpha}_\chi(\nu)E_{\chi,\rho}(\nu,g)d\nu\bigg\}\\
&-2\times\frac{R_F^{-1}}{8\pi i}\sum_{\rho\in\Lambda(\mathfrak{n})}\bigg\{\int_{L_c}\frac{\beta((\nu-1)/2)}{(\nu-1)/2}F^+_\rho(\nu)\tilde{\alpha}(\nu)E_{\mathbf{1},\rho}(\nu,g)d\nu\bigg\}.
\end{aligned}$$

By noting the relation

$$\int_{i\mathbb{R}}F^+_\rho(\nu)\frac{\beta((\nu-1)/2)}{(\nu-1)/2}\tilde{\alpha}(\nu)E_{\mathbf{1},\rho}(\nu,g)d\nu=\int_{i\mathbb{R}}F^-_\rho(\nu)\frac{\beta((-\nu-1)/2)}{(-\nu-1)/2}\tilde{\alpha}(\nu)E_{\mathbf{1},\rho}(\nu,g)d\nu,$$

we have

$$\begin{aligned}
&\frac{R_F^{-1}}{8\pi i}\sum_{\rho\in\Lambda(\mathfrak{n})}\int_{i\mathbb{R}}\{\mathrm{Res}_{z=(-1+\nu)/2}+\mathrm{Res}_{z=(-1-\nu)/2}\}\bigg\{f^{\mathbf{1}}_{\mathbf{1},\rho}(z,-\nu)\frac{\beta(z)}{z}\bigg\}\tilde{\alpha}(\nu)E_{\mathbf{1},\rho}(\nu,g)d\nu\\
&-2\times\frac{R_F^{-1}}{8\pi i}\sum_{\rho\in\Lambda(\mathfrak{n})}\int_{L_c}\frac{\beta((\nu-1)/2)}{(\nu-1)/2}F^+_\rho(\nu)\tilde{\alpha}(\nu)E_{\mathbf{1},\rho}(\nu,g)d\nu\\
=&\frac{R_F^{-1}}{8\pi i}\sum_{\rho\in\Lambda(\mathfrak{n})}\int_{i\mathbb{R}}\bigg\{F^+_\rho(\nu)\frac{\beta((\nu-1)/2)}{(\nu-1)/2}+F^-_\rho(\nu)\frac{\beta((-\nu-1)/2)}{(-\nu-1)/2}\bigg\}\tilde{\alpha}(\nu)E_{\mathbf{1},\rho}(\nu,g)d\nu\\
&-2\times\frac{R_F^{-1}}{8\pi i}\sum_{\rho\in\Lambda(\mathfrak{n})}\int_{L_c}\frac{\beta((\nu-1)/2)}{(\nu-1)/2}F^+_\rho(\nu)\tilde{\alpha}(\nu)E_{\mathbf{1},\rho}(\nu,g)d\nu
\end{aligned}$$

$$
\begin{aligned}
=&2\times\frac{R_F^{-1}}{8\pi i}\sum_{\rho\in\Lambda(\mathfrak{n})}\int_{i\mathbb{R}}F_\rho^{+}(\nu)\frac{\beta((\nu-1)/2)}{(\nu-1)/2}\tilde\alpha(\nu)E_{\mathbf{1},\rho}(\nu,g)d\nu\\
&-2\times\frac{R_F^{-1}}{8\pi i}\sum_{\rho\in\Lambda(\mathfrak{n})}\int_{L_c}\frac{\beta((\nu-1)/2)}{(\nu-1)/2}F_\rho^{+}(\nu)\tilde\alpha(\nu)E_{\mathbf{1},\rho}(\nu,g)d\nu\\
=&\frac{R_F^{-1}}{4\pi i}\sum_{\rho\in\Lambda(\mathfrak{n})}(-2\pi i)\,\mathrm{Res}_{\nu=1}\left\{\frac{\beta((\nu-1)/2)}{(\nu-1)/2}F_\rho^{+}(\nu)\tilde\alpha(\nu)E_{\mathbf{1},\rho}(\nu,g)\right\}.
\end{aligned}
$$

Here the residue is expressed as

$$
\begin{aligned}
&\mathrm{Res}_{\nu=1}\left\{\frac{\beta((\nu-1)/2)}{(\nu-1)/2}F_\rho^{+}(\nu)\tilde\alpha(\nu)E_{\mathbf{1},\rho}(\nu,g)\right\}\\
=&\,\mathrm{Res}_{\nu=1}\left\{\frac{\beta((\nu-1)/2)}{(\nu-1)/2}\tilde\alpha(\nu)E_{\mathbf{1},\rho}(\nu,g)D_F^{-1/2+\nu/2}B_{\mathbf{1},\rho}^{\mathbf{1}}(1-\nu/2,-\nu)(-R_F)\right\}\\
=&\{2\tilde\alpha'(1)\mathfrak{e}_{\mathbf{1},\rho,-1}(g)+2\tilde\alpha(1)\mathfrak{e}_{\mathbf{1},\rho,0}(g)\}D_{\mathbf{1}}(\rho)(-R_F)\beta(0).
\end{aligned}
$$

We note $D_F^{(\nu-1)/2}B^{\eta}_{\eta,\rho}(1-\nu/2,-\nu)=\tilde\eta(\mathfrak{D}_{F/\mathbb{Q}})D_\eta(\rho)$ for any $\eta\in\Xi_0(\mathfrak{o})$ satisfying $\eta^2=\mathbf{1}$. Therefore we obtain

$$
\begin{aligned}
&\frac{R_F^{-1}}{8\pi i}\sum_{\rho\in\Lambda(\mathfrak{n})}\int_{i\mathbb{R}}\{\mathrm{Res}_{z=(-1+\nu)/2}+\mathrm{Res}_{z=(-1-\nu)/2}\}\left\{f_{\mathbf{1},\rho}^{\mathbf{1}}(z,-\nu)\frac{\beta(z)}{z}\right\}\tilde\alpha(\nu)E_{\mathbf{1},\rho}(\nu,g)d\nu\\
&-2\times\frac{R_F^{-1}}{8\pi i}\sum_{\rho\in\Lambda(\mathfrak{n})}\int_{L_c}\frac{\beta((\nu-1)/2)}{(\nu-1)/2}F_\rho^{+}(\nu)\tilde\alpha(\nu)E_{\mathbf{1},\rho}(\nu,g)d\nu\\
=&\frac{R_F^{-1}}{4\pi i}\sum_{\rho\in\Lambda(\mathfrak{n})}2\pi i\{2\tilde\alpha'(1)\mathfrak{e}_{\mathbf{1},\rho,-1}(g)+2\tilde\alpha(1)\mathfrak{e}_{\mathbf{1},\rho,0}(g)\}D_{\mathbf{1}}(\rho)R_F\beta(0)\\
=&\sum_{\rho\in\Lambda(\mathfrak{n})}D_{\mathbf{1}}(\rho)\{\tilde\alpha'(1)\mathfrak{e}_{\mathbf{1},\rho,-1}(g)+\tilde\alpha(1)\mathfrak{e}_{\mathbf{1},\rho,0}(g)\}\beta(0),
\end{aligned}
$$

and hence

$$
\begin{aligned}
\mathrm{CT}_{\lambda=0}\Psi_{\mathrm{ct}}(\lambda)=&\frac{R_F^{-1}}{8\pi i}\sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\int_{i\mathbb{R}}\tilde\alpha_\chi(\nu)f^{\mathbf{1}}_{\chi^{-1},\rho}(0,-\nu)E_{\chi,\rho}(\nu,g)d\nu\beta(0)\\
&+\sum_{\rho\in\Lambda(\mathfrak{n})}\{f^{(0)}_{\mathbf{1},\rho}(1_2)+D_{\mathbf{1}}(\rho)\}\{\tilde\alpha'(1)\mathfrak{e}_{\mathbf{1},\rho,-1}(g)+\tilde\alpha(1)\mathfrak{e}_{\mathbf{1},\rho,0}(g)\}\beta(0).
\end{aligned}
$$

This gives the expression of $\mathrm{CT}_{\lambda=0}\hat{\mathbf{\Psi}}_{\beta,\lambda}(\mathfrak{n}|\alpha;g)$. □

We define the regularized smoothed kernel $\hat{\mathbf{\Psi}}_{\mathrm{reg}}(\mathfrak{n}|\alpha;g)$ by the relation

$$
\mathrm{CT}_{\lambda=0}\hat{\mathbf{\Psi}}_{\beta,\lambda}(\mathfrak{n}|\alpha;g)=\hat{\mathbf{\Psi}}_{\mathrm{reg}}(\mathfrak{n}|\alpha;g)\beta(0),\quad \beta\in\mathcal{B}.
$$

Lemma 7.5. *The following estimates hold for any $g\in\mathfrak{S}^1$ uniformly.*

(1) *For any $m>0$, we have $\sum_{\varphi\in\mathcal{B}_{\mathrm{cus}}(\mathfrak{n})}|\alpha(\nu_{\varphi,S})\overline{P^{\mathbf{1}}_{\mathrm{reg}}(\varphi)}\varphi(g)|\ll_m y(g)^{-m}$.*

(2) *There exists $N\in\mathbb{N}$ such that we have the estimate*

$$
\sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\frac{R_F^{-1}}{8\pi i}\int_{i\mathbb{R}}|\tilde\alpha_\chi(\nu)P^{\mathbf{1}}_{\mathrm{reg}}(E_{\chi^{-1},\rho}(-\nu,-))E_{\chi,\rho}(\nu,g)||d\nu|\ll_N y(g)^N.
$$

(3) *For any $\rho\in\Lambda(\mathfrak{n})$, we have $|\mathfrak{e}_{\mathbf{1},\rho,0}(g)|+|\mathfrak{e}_{\mathbf{1},\rho,-1}(g)|\ll y(g)$.*

(4) *For N as in (2), we have*

$$|\hat{\mathbf{\Psi}}_{\mathrm{reg}}(\mathfrak{n}|\alpha;g)| \ll_N y(g)^N.$$

Proof. There exists a positive constant C such that $L_{\mathrm{fin}}(s,\chi)$ does not vanish for any non-quadratic character χ of $F^\times\backslash\mathbb{A}^\times$ if $\mathrm{Re}(s) \geqslant 1 - C/\log\{\mathfrak{q}(\chi)(3+|\mathrm{Im}(s)|)\}$ (cf. [14, Theorem 5.10]). Hence, by virtue of the proof of [45, Theorem 3.11], the estimate

$$\frac{1}{|L_{\mathrm{fin}}(1,\chi)|} \ll \log \mathfrak{q}(\chi)$$

holds uniformly for non-quadratic characters χ of $F^\times\backslash\mathbb{A}^\times$. Next we give a generalized Siegel's theorem for quadratic characters of $F^\times\backslash\mathbb{A}^\times$. By [28, Theorem 2.3.1], for any $\epsilon > 0$, the estimate

$$|L_{\mathrm{fin}}(1,\chi)| \gg \mathfrak{q}(\chi)^{-\epsilon}$$

holds uniformly for quadratic characters χ of $F^\times\backslash\mathbb{A}^\times$. Indeed, [28, Theorem 2.3.1] works for general L-functions over F in the sense of [4].

As a consequence, we have the estimate

$$\frac{1}{|L_{\mathrm{fin}}(1+\nu,\chi^2)|} \ll \mathfrak{q}(\chi^2|\cdot|_{\mathbb{A}}^{\nu})^{\epsilon}, \ \nu \in i\mathbb{R}$$

with the implied constant independent of $\chi \in \Xi(\mathfrak{n})$ and $\mathfrak{n}$. Combining this with the argument of the proof of [47, Lemma 9.9], we have the assertions. $\square$

8. Periods of regularized automorphic smoothed kernels: the spectral side

By (4) in Lemma 7.5, the integral $P^{\eta}_{\beta,\lambda}(\hat{\mathbf{\Psi}}_{\mathrm{reg}}(\mathfrak{n}|\alpha))$ converges absolutely for $\mathrm{Re}(\lambda) > N$ and is holomorphic on $\mathrm{Re}(\lambda) > N$. The following is given in the same way as [47, Lemma 10.1].

Lemma 8.1. *For $\mathrm{Re}(\lambda) > N$, we have the expression*

$$P^{\eta}_{\beta,\lambda}(\hat{\mathbf{\Psi}}_{\mathrm{reg}}(\mathfrak{n}|\alpha)) = C(\mathfrak{n},S)\{\mathbb{P}^{\eta}_{\mathrm{cus}}(\beta,\lambda,\alpha) + \mathbb{P}^{\eta}_{\mathrm{eis}}(\beta,\lambda,\alpha) + \mathbb{P}^{\eta}_{\mathrm{res}}(\beta,\lambda,\alpha)\},$$

where

$$\mathbb{P}^{\eta}_{\mathrm{cus}}(\beta,\lambda,\alpha) = \sum_{\varphi\in\mathcal{B}_{\mathrm{cus}}(\mathfrak{n})} \alpha(\nu_{\varphi,S})\overline{P^{\mathbf{1}}_{\mathrm{reg}}(\varphi)}P^{\eta}_{\beta,\lambda}(\varphi),$$

$$\mathbb{P}^{\eta}_{\mathrm{eis}}(\beta,\lambda,\alpha) = \sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})} \frac{R_F^{-1}}{8\pi i}\int_{i\mathbb{R}} \tilde{\alpha}_\chi(\nu)P^{\mathbf{1}}_{\mathrm{reg}}(E_{\chi^{-1},\rho}(-\nu,-))P^{\eta}_{\beta,\lambda}(E_{\chi,\rho}(\nu,-))d\nu$$

and

$$\mathbb{P}^{\eta}_{\mathrm{res}}(\beta,\lambda,\alpha) = \sum_{\rho\in\Lambda(\mathfrak{n})} \{f^{(0)}_{\mathbf{1},\rho}(1_2) + D_{\mathbf{1}}(\rho)\}(\tilde{\alpha}'(1)P^{\eta}_{\beta,\lambda}(\mathfrak{e}_{\mathbf{1},\rho,-1}) + \tilde{\alpha}(1)P^{\eta}_{\beta,\lambda}(\mathfrak{e}_{\mathbf{1},\rho,0})).$$

Here the series converge absolutely and locally uniformly on $\mathrm{Re}(\lambda) > N$.

By Propositions 3.4 and 3.5, we have the following.

Lemma 8.2. *The function $\lambda \mapsto \mathbb{P}^{\eta}_{\mathrm{res}}(\beta,\lambda,\alpha)$ on $\mathrm{Re}(\lambda) > N$ is analytically continued to a meromorphic function on $\mathbb{C}$. Its constant term at $\lambda = 0$ is given by*

$$\mathrm{CT}_{\lambda=0}\mathbb{P}^{\eta}_{\mathrm{res}}(\beta,\lambda,\alpha) = \sum_{\rho\in\Lambda(\mathfrak{n})} \{f^{(0)}_{\mathbf{1},\rho}(1_2) + D_{\mathbf{1}}(\rho)\}\tilde{\alpha}(1)\frac{\mathcal{G}(\eta)D_F^{-1/2}}{\zeta_F(2)} \times\left\{-\frac{1}{2}\delta_{\eta,\mathbf{1}}\tilde{B}^{\mathbf{1}}_{\mathbf{1},\rho}(0)R_F^2\beta''(0) + a^{\eta}_{\mathbf{1},\rho}(0)\beta(0)\right\}.$$

Here $\tilde{B}^{\eta}_{\chi,\rho}(z) = \epsilon(-z,\chi^{-1}\eta)B^{\eta}_{\chi,\rho}(-z+1/2,1)$ *and*

$$a^{\eta}_{\mathbf{1},\rho}(0) = -\frac{1}{2}(\tilde{B}^{\mathbf{1}}_{\mathbf{1},\rho})''(0)\delta_{\eta,\mathbf{1}}R_F^2 - 2\tilde{B}^{\mathbf{1}}_{\mathbf{1},\rho}(0)R_F C_1(\mathbf{1})\delta_{\eta,\mathbf{1}} + \tilde{B}^{\eta}_{\mathbf{1},\rho}(0)C_0(\eta)^2.$$

Lemma 8.3. *The function* $\lambda \mapsto \mathbb{P}^{\eta}_{\mathrm{eis}}(\beta,\lambda,\alpha)$ *on* $\mathrm{Re}(\lambda) > N$ *is analytically continued to a meromorphic function on* $\mathrm{Re}(\lambda) > -1/2$.

Proof. By Proposition 3.2, we have

$$Z^*(s,\eta,E^{\natural}_{\chi,\rho}(\nu,-)) = \mathcal{G}(\eta)D_F^{-\nu/2}\mathrm{N}(\mathfrak{f}_\chi)^{1/2-\nu}B^{\eta}_{\chi,\rho}(s,\nu)\frac{L(s+\nu/2,\chi\eta)L(s-\nu/2,\chi^{-1}\eta)}{L(1+\nu,\chi^2)}.$$

Set

$$\mathfrak{L}^{\eta}_{\chi,\rho}(\nu) = D_F^{\nu/2}\mathrm{N}(\mathfrak{f}_\chi)^{1/2+\nu}B^{\eta}_{\chi^{-1},\rho}(1/2,-\nu)\frac{L((1+\nu)/2,\chi\eta)L((1-\nu)/2,\chi^{-1}\eta)}{L(1-\nu,\chi^{-2})}$$

and recall the expression

$$P^{\eta}_{\beta,\lambda}(E_{\chi,\rho}(\nu,-)) = P_\chi(\eta,\lambda,\nu) + D_F^{-1/2}A_{\chi,\rho}(\nu)\frac{L(\nu,\chi^2)}{L(1+\nu,\chi^2)}P_{\chi^{-1}}(\eta,\lambda,-\nu) + Q^{+}_{\chi,\rho}(\eta,\lambda,\nu) + Q^{-}_{\chi,\rho}(\eta,\lambda,\nu).$$

We remark

$$\mathbb{P}^{\eta}_{\mathrm{eis}}(\beta,\lambda,\alpha) = \sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\frac{R_F^{-1}}{8\pi i}\int_{i\mathbb{R}}\tilde{\alpha}_\chi(\nu)\mathcal{G}(\mathbf{1})\mathfrak{L}^{\mathbf{1}}_{\chi,\rho}(\nu)\Big\{P_\chi(\eta,\lambda,\nu)$$
$$+ D_F^{-1/2}A_{\chi,\rho}(\nu)\frac{L(\nu,\chi^2)}{L(1+\nu,\chi^2)}P_{\chi^{-1}}(\eta,\lambda,-\nu) + Q^{0}_{\chi,\rho}(\eta,\lambda,\nu)$$
$$- \sum_{a=(\pm\nu\pm1)/2}\frac{\beta(a)}{\lambda+a}\mathrm{Res}_{z=a}\{f^{\eta}_{\chi,\rho}(-z,\nu)\}\Big\}d\nu.$$

In order to examine $\mathbb{P}^{\eta}_{\mathrm{eis}}(\beta,\lambda,\alpha)$, we decompose this into the following four terms:

$$\Phi_1^{+}(\lambda) = \sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\frac{R_F^{-1}}{8\pi i}\int_{i\mathbb{R}}\tilde{\alpha}_\chi(\nu)D_F^{-1/2}\mathfrak{L}^{\mathbf{1}}_{\chi,\rho}(\nu)f^{(0)}_{\chi,\rho}(1_2)$$
$$\times\,\delta_{\chi,\eta}R_F\left\{\frac{1}{\lambda-(\nu+1)/2}+\frac{1}{\lambda+(\nu+1)/2}\right\}\beta((\nu+1)/2)d\nu,$$

$$\Phi_1^{-}(\lambda) = \sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\frac{R_F^{-1}}{8\pi i}\int_{i\mathbb{R}}\tilde{\alpha}_\chi(\nu)D_F^{-1/2}\mathfrak{L}^{\mathbf{1}}_{\chi,\rho}(\nu)D_F^{-1/2}A_{\chi,\rho}(\nu)\frac{L(\nu,\chi^2)}{L(1+\nu,\chi^2)}f^{(0)}_{\chi^{-1},\rho}(1_2)$$
$$\times\,\delta_{\chi,\eta}R_F\left\{\frac{1}{\lambda-(-\nu+1)/2}+\frac{1}{\lambda+(-\nu+1)/2}\right\}\beta((-\nu+1)/2)d\nu,$$

$$\Phi_2(\lambda) = \sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\frac{R_F^{-1}}{8\pi i}\int_{i\mathbb{R}}\tilde{\alpha}_\chi(\nu)D_F^{-1/2}\mathfrak{L}^{\mathbf{1}}_{\chi,\rho}(\nu)Q^{0}_{\chi,\rho}(\eta,\lambda,\nu)d\nu,$$

$$\Phi_3(\lambda) = -\sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\frac{R_F^{-1}}{8\pi i}\int_{i\mathbb{R}}\tilde{\alpha}_\chi(\nu)D_F^{-1/2}\mathfrak{L}^{\mathbf{1}}_{\chi,\rho}(\nu)\sum_{a=(\pm\nu\pm1)/2}\frac{\beta(a)}{\lambda+a}\mathrm{Res}_{z=a}\{f^{\eta}_{\chi,\rho}(-z,\nu)\}d\nu.$$

When $\chi=\eta$, then $\mathfrak{f}_\eta=\mathfrak{o}$ holds and by using the functional equations

$$\mathfrak{L}^{\mathbf{1}}_{\eta,\rho}(\nu)D_F^{-1/2}A_{\eta,\rho}(\nu)\frac{\zeta_F(\nu)}{\zeta_F(1+\nu)} = \mathfrak{L}^{\mathbf{1}}_{\eta,\rho}(-\nu)$$

and $B^{\mathbf{1}}_{\eta,\rho}(1/2,\nu) = B^{\mathbf{1}}_{\eta,\rho}(1/2,-\nu)A_{\eta,\rho}(\nu)$, we obtain $\Phi_1^+(\lambda) = \Phi_1^-(\lambda)$. The term $\Phi_1^+(\lambda)$ is expressed as

$$\Phi_1^+(\lambda) = \sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\Bigg\{\frac{R_F^{-1}}{8\pi i}\int_{i\mathbb{R}}\tilde{\alpha}_\chi(\nu)D_F^{-1/2}\mathfrak{L}^{\mathbf{1}}_{\chi,\rho}(\nu)f^{(0)}_{\chi,\rho}(1_2)\delta_{\chi,\eta}R_F\frac{1}{\lambda+(\nu+1)/2}\beta((\nu+1)/2)d\nu$$
$$+\frac{R_F^{-1}}{8\pi i}\int_{i\mathbb{R}}\tilde{\alpha}_\chi(\nu)D_F^{-1/2}\mathfrak{L}^{\mathbf{1}}_{\chi,\rho}(\nu)f^{(0)}_{\chi,\rho}(1_2)\delta_{\chi,\eta}R_F\frac{1}{\lambda-(\nu+1)/2}\beta((\nu+1)/2)d\nu\Bigg\}.$$

Then the first term in the summation is holomorphic on $\mathrm{Re}(\lambda) > -1/2$. For any fixed $\sigma > 1$, the second term in the summation is transformed into

$$\frac{R_F^{-1}}{8\pi i}D_F^{-1/2}\Bigg\{\int_{L_{-\sigma}}\tilde{\alpha}_\chi(\nu)\mathfrak{L}^{\mathbf{1}}_{\chi,\rho}(\nu)f^{(0)}_{\chi,\rho}(1_2)\delta_{\chi,\eta}R_F\frac{1}{\lambda-(\nu+1)/2}\beta((\nu+1)/2)d\nu$$
$$+\delta_{\chi,\eta}\,2\pi i\,\mathrm{Res}_{\nu=-1}\left(\frac{\beta((\nu+1)/2)}{\lambda-(\nu+1)/2}\tilde{\alpha}_\eta(\nu)\mathfrak{L}^{\mathbf{1}}_{\eta,\rho}(\nu)\right)f^{(0)}_{\chi,\rho}(1_2)R_F\Bigg\}.$$

The first term in the expression above is meromorphic on $\mathrm{Re}(\lambda) > (-\sigma+1)/2$. In order to prove the meromorphicity of the second term in the expression above, we put

$$D_F^{\nu/2}\frac{L((1+\nu)/2,\eta)L((1-\nu)/2,\eta)}{\zeta_F(1-\nu)} = \frac{D^\eta_{-2}}{(\nu+1)^2}+\frac{D^\eta_{-1}}{\nu+1}+D^\eta_0+\mathcal{O}((\nu+1)),\ (\nu\to-1),$$

$$B^{\mathbf{1}}_{\eta,\rho}(1/2,-\nu) = p^\eta_0(\rho)+p^\eta_1(\rho)(\nu+1)+p^\eta_2(\rho)(\nu+1)^2+\mathcal{O}((\nu+1)^3),\ (\nu\to-1)$$

and

$$\frac{\beta((\nu+1)/2)}{\lambda-(\nu+1)/2}\tilde{\alpha}_\eta(\nu) = q^\eta_0(\lambda)+q^\eta_1(\lambda)(\nu+1)+\mathcal{O}((\nu+1)^2),\ (\nu\to-1).$$

Then these give the following expressions:

$$\mathrm{Res}_{\nu=-1}\left\{\frac{\beta((\nu+1)/2)}{\lambda-(\nu+1)/2}\tilde{\alpha}_\eta(\nu)\mathfrak{L}^{\mathbf{1}}_{\eta,\rho}(\nu)\right\} = p^\eta_0(\rho)q^\eta_1(\lambda)D^\eta_{-2}+p^\eta_0(\rho)q^\eta_0(\lambda)D^\eta_{-1}+p^\eta_1(\rho)q^\eta_0(\lambda)D^\eta_{-2},$$

$$q^\eta_0(\lambda) = \frac{\tilde{\alpha}_\eta(1)\beta(0)}{\lambda},\quad q^\eta_1(\lambda) = \left(\frac{\tilde{\alpha}'_\eta(1)}{\lambda}+\frac{\tilde{\alpha}_\eta(1)}{2\lambda^2}\right)\beta(0).$$

Therefore $\Phi_1^+(\lambda) = \Phi_1^-(\lambda)$ has a meromorphic continuation to $\mathrm{Re}(\lambda) > -1/2$. Since $\Phi_2(\lambda)$ is described as an absolutely convergent double integral, $\Phi_2(\lambda)$ is entire.

We examine $\Phi_3(\lambda)$. This is written as

$$\Phi_3(\lambda) = -\sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\frac{R_F^{-1}}{8\pi i}\Bigg\{\int_{i\mathbb{R}}\tilde{\alpha}_\chi(\nu)D_F^{-1/2}\mathfrak{L}^{\mathbf{1}}_{\chi,\rho}(\nu)\sum_{a=(\pm\nu+1)/2}\frac{\beta(a)}{\lambda+a}\mathrm{Res}_{z=a}\,f^\eta_{\chi,\rho}(-z,\nu)d\nu$$
$$+\int_{i\mathbb{R}}\tilde{\alpha}_\chi(\nu)D_F^{-1/2}\mathfrak{L}^{\mathbf{1}}_{\chi,\rho}(\nu)\sum_{a=(\pm\nu-1)/2}\frac{\beta(a)}{\lambda+a}\mathrm{Res}_{z=a}\,f^\eta_{\chi,\rho}(-z,\nu)d\nu\Bigg\}.$$

In the bracket of the right-hand side, the first term is holomorphic on $\mathrm{Re}(\lambda) > -1/2$ and the part of $a = (-\nu-1)/2$ in the second term is transposed into

$$\int_{i\mathbb{R}}\tilde{\alpha}_\chi(\nu)D_F^{-1/2}\mathfrak{L}^{\mathbf{1}}_{\chi,\rho}(\nu)\frac{\beta((-\nu-1)/2)}{\lambda+(-\nu-1)/2}\mathrm{Res}_{z=(-\nu-1)/2}\,f^\eta_{\chi,\rho}(-z,\nu)d\nu$$
$$=\int_{L_{-\sigma}}\tilde{\alpha}_\chi(\nu)D_F^{-1/2}\mathfrak{L}^{\mathbf{1}}_{\chi,\rho}(\nu)\frac{\beta((-\nu-1)/2)}{\lambda+(-\nu-1)/2}\mathrm{Res}_{z=(-\nu-1)/2}\,f^\eta_{\chi,\rho}(-z,\nu)d\nu$$
$$+2\pi i\ \delta_{\chi,\eta}\,\mathrm{Res}_{\nu=-1}(\mathfrak{L}^{\mathbf{1}}_{\eta,\rho}(\nu))\tilde{\alpha}_\eta(1)D_F^{-1/2}\frac{\beta(0)}{\lambda}\mathcal{G}(\eta)D_F^{1/2}B^\eta_{\eta,\rho}(1/2,-1)(-R_F)$$

for any fixed $\sigma > 1$. We note $\mathrm{Res}_{\nu=-1}\,\mathfrak{L}^{\mathbf{1}}_{\eta,\rho}(\nu) = p_0^\eta(\rho)D^\eta_{-1} + p_1^\eta(\rho)D^\eta_{-2}$. Thus the part of $a = (-\nu-1)/2$ is meromorphic on $\mathrm{Re}(\lambda) > -1/2$. Noting that the part of $a = (\nu-1)/2$ equals that of $a = (-\nu-1)/2$, the function $\Phi_3(\lambda)$ has a meromorphic continuation to $\mathrm{Re}(\lambda) > -1/2$. This completes the proof. □

Lemma 8.4. *We have*

$$\begin{aligned}
&\mathrm{CT}_{\lambda=0}\mathbb{P}^\eta_{\mathrm{eis}}(\beta,\lambda,\alpha) + \mathrm{CT}_{\lambda=0}\mathbb{P}^\eta_{\mathrm{res}}(\beta,\lambda,\alpha)\\
=&\Bigg\{\mathcal{G}(\eta)D_F^{-1/2}R_F^{-1}\sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\frac{1}{8\pi i}\int_{i\mathbb{R}}\tilde{\alpha}_\chi(\nu)\mathfrak{L}^{\mathbf{1}}_{\chi,\rho}(\nu)\mathfrak{L}^\eta_{\chi^{-1},\rho}(-\nu)d\nu\\
&+\delta(\mathfrak{f}_\eta=\mathfrak{o})\{Y_2^\eta(\mathfrak{n})\tilde{\alpha}''_\eta(1) + Y_1^\eta(\mathfrak{n})\tilde{\alpha}'_\eta(1) + Y_0^\eta(\mathfrak{n})\tilde{\alpha}_\eta(1)\} + Y^\eta_{-1}(\mathfrak{n})\tilde{\alpha}(1)\Bigg\}\beta(0),
\end{aligned}$$

where we put

$$\begin{aligned}
Y_2^\eta(\mathfrak{n}) &= \sum_{\rho\in\Lambda(\mathfrak{n})} D_F^{-1/2}\{f^{(0)}_{\eta,\rho}(1_2) + D_\eta(\rho)\}\frac{1}{2}p_0^\eta(\rho)D^\eta_{-2},\\
Y_1^\eta(\mathfrak{n}) &= \sum_{\rho\in\Lambda(\mathfrak{n})} D_F^{-1/2}\{f^{(0)}_{\eta,\rho}(1_2) + D_\eta(\rho)\}\{D^\eta_{-1}p_0^\eta(\rho) + D^\eta_{-2}p_1^\eta(\rho)\},\\
Y_0^\eta(\mathfrak{n}) &= \sum_{\rho\in\Lambda(\mathfrak{n})} D_F^{-1/2}\{f^{(0)}_{\eta,\rho}(1_2) + D_\eta(\rho)\}\{D^\eta_{-2}p_2^\eta(\rho) + D^\eta_{-1}p_1^\eta(\rho) + D^\eta_0 p_0^\eta(\rho)\}
\end{aligned}$$

and

$$Y^\eta_{-1}(\mathfrak{n}) = \sum_{\rho\in\Lambda(\mathfrak{n})}\frac{\mathcal{G}(\eta)D_F^{-1/2}}{\zeta_F(2)}\{f^{(0)}_{\mathbf{1},\rho}(1_2) + D_{\mathbf{1}}(\rho)\}a^\eta_{\mathbf{1},\rho}(0).$$

Proof. Let Φ_1^+, Φ_2 and Φ_3 be the functions defined in the proof of Lemma 8.3. Then, we obtain $\mathrm{CT}_{\lambda=0}\mathbb{P}^\eta_{\mathrm{eis}}(\beta,\lambda,\alpha) = \mathrm{CT}_{\lambda=0}(2\Phi_1^+(\lambda) + \Phi_2(\lambda) + \Phi_3(\lambda))$. A direct computation gives us

$$\begin{aligned}
\mathrm{CT}_{\lambda=0}\Phi_1^+(\lambda) =&\delta(\mathfrak{f}_\eta=\mathfrak{o})\sum_{\rho\in\Lambda(\mathfrak{n})}\frac{1}{8\pi i}\int_{i\mathbb{R}}\tilde{\alpha}_\eta(\nu)D_F^{-1/2}\mathfrak{L}^{\mathbf{1}}_{\eta,\rho}(\nu)f^{(0)}_{\eta,\rho}(1_2)\frac{\beta(0)}{(\nu+1)/2}d\nu\\
&+\delta(\mathfrak{f}_\eta=\mathfrak{o})\sum_{\rho\in\Lambda(\mathfrak{n})}\frac{1}{8\pi i}\int_{L_{-\sigma}}\tilde{\alpha}_\eta(\nu)D_F^{-1/2}\mathfrak{L}^{\mathbf{1}}_{\eta,\rho}(\nu)f^{(0)}_{\eta,\rho}(1_2)\frac{\beta(0)}{-(\nu+1)/2}d\nu
\end{aligned}$$

and

$$\mathrm{CT}_{\lambda=0}\Phi_2(\lambda) = \sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\frac{R_F^{-1}}{8\pi i}\int_{i\mathbb{R}}\tilde{\alpha}_\chi(\nu)D_F^{-1/2}\mathfrak{L}^{\mathbf{1}}_{\chi,\rho}(\nu)Q^0_{\chi,\rho}(\eta,0,\nu)d\nu,$$

where

$$Q^0_{\chi,\rho}(\eta,0,\nu) = f^\eta_{\chi,\rho}(0,\nu)\beta(0) + \sum_{a=(\pm\nu\pm1)/2}\mathrm{Res}_{z=a}\left\{f^\eta_{\chi,\rho}(-z,\nu)\frac{\beta(z)}{z}\right\}.$$

The constant term of $\Phi_3(\lambda)$ at $\lambda=0$ is evaluated as

$$\begin{aligned}
&\mathrm{CT}_{\lambda=0}\Phi_3(\lambda)\\
=&-\sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\frac{R_F^{-1}}{8\pi i}\left\{\int_{i\mathbb{R}}\tilde{\alpha}_\chi(\nu)D_F^{-1/2}\mathfrak{L}^{\mathbf{1}}_{\chi,\rho}(\nu)\sum_{a=(\pm\nu+1)/2}\frac{\beta(a)}{a}\mathrm{Res}_{z=a}\,f^\eta_{\chi,\rho}(-z,\nu)d\nu\right\}\\
&-2\delta(\mathfrak{f}_\eta=\mathfrak{o})\sum_{\rho\in\Lambda(\mathfrak{n})}\frac{R_F^{-1}}{8\pi i}\left\{\int_{L_{-\sigma}}\tilde{\alpha}_\eta(\nu)D_F^{-1/2}\mathfrak{L}^{\mathbf{1}}_{\eta,\rho}(\nu)\frac{\beta((-\nu-1)/2)}{(-\nu-1)/2}\mathrm{Res}_{z=(-\nu-1)/2}\,f^\eta_{\eta,\rho}(-z,\nu)d\nu\right\}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\mathrm{CT}_{\lambda=0}\mathbb{P}^{\eta}_{\mathrm{eis}}(\beta,\lambda,\alpha)\\
&=2\delta(\mathfrak{f}_\eta=\mathfrak{o})\sum_{\rho\in\Lambda(\mathfrak{n})}\frac{1}{8\pi i}D_F^{-1/2}f^{(0)}_{\eta,\rho}(1_2)\left(\int_{i\mathbb{R}}-\int_{L_{-\sigma}}\right)\tilde{\alpha}_\eta(\nu)\mathfrak{L}^{\mathbf{1}}_{\eta,\rho}(\nu)\frac{\beta((\nu+1)/2)}{(\nu+1)/2}d\nu\\
&\quad+\sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\frac{R_F^{-1}}{8\pi i}\int_{i\mathbb{R}}\tilde{\alpha}_\chi(\nu)D_F^{-1/2}\mathfrak{L}^{\mathbf{1}}_{\chi,\rho}(\nu)f^{\eta}_{\chi,\rho}(0,\nu)\beta(0)d\nu\\
&\quad+2\delta(\mathfrak{f}_\eta=\mathfrak{o})\sum_{\rho\in\Lambda(\mathfrak{n})}\frac{R_F^{-1}}{8\pi i}\Bigg\{\left(\int_{i\mathbb{R}}-\int_{L_{-\sigma}}\right)\tilde{\alpha}_\eta(\nu)D_F^{-1/2}\mathfrak{L}^{\mathbf{1}}_{\eta,\rho}(\nu)\frac{\beta((-\nu-1)/2)}{(-\nu-1)/2}\\
&\quad\times\mathrm{Res}_{z=(-\nu-1)/2}\,f^{\eta}_{\eta,\rho}(-z,\nu)d\nu\Bigg\}\\
&=\delta(\mathfrak{f}_\eta=\mathfrak{o})\sum_{\rho\in\Lambda(\mathfrak{n})}\frac{1}{2}D_F^{-1/2}f^{(0)}_{\eta,\rho}(1_2)\,\mathrm{Res}_{\nu=-1}\left\{\tilde{\alpha}_\eta(\nu)\mathfrak{L}^{\mathbf{1}}_{\eta,\rho}(\nu)\frac{\beta((\nu+1)/2)}{(\nu+1)/2}\right\}\\
&\quad+\sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\frac{R_F^{-1}}{8\pi i}\int_{i\mathbb{R}}\tilde{\alpha}_\chi(\nu)D_F^{-1/2}\mathfrak{L}^{\mathbf{1}}_{\chi,\rho}(\nu)\mathcal{G}(\eta)\mathfrak{L}^{\eta}_{\chi^{-1},\rho}(-\nu)\beta(0)d\nu\\
&\quad+\delta(\mathfrak{f}_\eta=\mathfrak{o})\sum_{\rho\in\Lambda(\mathfrak{n})}\frac{R_F^{-1}}{2}\,\mathrm{Res}_{\nu=-1}\left\{\tilde{\alpha}_\eta(\nu)D_F^{-1/2}\mathfrak{L}^{\mathbf{1}}_{\eta,\rho}(\nu)\frac{\beta((-\nu-1)/2)}{(-\nu-1)/2}\,\mathrm{Res}_{z=(-\nu-1)/2}\,f^{\eta}_{\eta,\rho}(-z,\nu)\right\}.
\end{aligned}$$

We remark

$$\mathrm{Res}_{z=(-\nu-1)/2}\,f^{\eta}_{\eta,\rho}(-z,\nu)=\mathcal{G}(\eta)(-R_F)D_F^{-\nu/2}B^{\eta}_{\eta,\rho}(\nu/2+1,\nu)=-R_F\,D_\eta(\rho)$$

and compute the residues as follows:

$$\begin{aligned}
&\mathrm{Res}_{\nu=-1}\left\{\tilde{\alpha}_\eta(\nu)\mathfrak{L}^{\mathbf{1}}_{\eta,\rho}(\nu)\frac{\beta((\nu+1)/2)}{(\nu+1)/2}\right\}=-\,\mathrm{Res}_{\nu=-1}\left\{\tilde{\alpha}_\eta(\nu)\mathfrak{L}^{\mathbf{1}}_{\eta,\rho}(\nu)\frac{\beta((-\nu-1)/2)}{(-\nu-1)/2}\right\}\\
&=\tilde{\alpha}''_\eta(1)p^{\eta}_0(\rho)D^{\eta}_{-2}\beta(0)+2\tilde{\alpha}'_\eta(1)\{p^{\eta}_0(\rho)D^{\eta}_{-1}+p^{\eta}_1(\rho)D^{\eta}_{-2}\}\beta(0)\\
&\quad+\tilde{\alpha}_\eta(1)\left\{D^{\eta}_{-2}\left(2p^{\eta}_2(\rho)\beta(0)+\frac{1}{4}p^{\eta}_0(\rho)\beta''(0)\right)+2D^{\eta}_{-1}p^{\eta}_1(\rho)\beta(0)+2D^{\eta}_0p^{\eta}_0(\rho)\beta(0)\right\}.
\end{aligned}$$

One can check that the sum of all terms containing $\beta''(0)$ in $\mathrm{CT}_{\lambda=0}\mathbb{P}^{\eta}_{\mathrm{eis}}(\beta,\lambda,\alpha)+\mathrm{CT}_{\lambda=0}\mathbb{P}^{\eta}_{\mathrm{res}}(\beta,\lambda,\alpha)$ vanishes with the aid of Lemma 8.2. As a consequence, we obtain the assertion. □

Lemma 8.5. *Suppose $\mathfrak{f}_\eta=\mathfrak{o}$. For any $\epsilon>0$, we have the following estimates*

$$|Y^{\eta}_j(\mathfrak{n})|\ll \mathrm{N}(\mathfrak{n})^{\epsilon},\quad j\in\{-1,0,1,2\},$$

where the implied constant is independent of $\mathfrak{n}$.

Proof. The proof is given by describing $Y^{\eta}_j(\mathfrak{n})$ for $j\in\{-1,0,1,2\}$ explicitly. Since η is unramified, we have

$$f^{(0)}_{\eta,\rho}(1_2)=\prod_{v\in S_1(\rho)}\eta_v(\varpi_v)q_v^{1/2}\prod_{k=2}^{n}\prod_{v\in S_k(\rho)}(1-q_v^{-1})\eta_v(\varpi_v)^k\left(\frac{q_v+1}{q_v-1}\right)^{1/2}q_v^{k/2}$$

and

$$p^{\eta}_0(\rho)=\tilde{\eta}(\mathfrak{D}_{F/\mathbb{Q}})\prod_{v\in S_1(\rho)}(1-\eta_v(\varpi_v))\frac{q_v}{q_v-1}q_v^{-1/2}$$

$$\times \prod_{k=2}^{n} \prod_{v\in S_k(\rho)} \left\{ \frac{(\eta_v(\varpi_v)-1)(\eta_v(\varpi_v)q_v-1)}{q_v-q_v^{-1}} \left(\frac{q_v+1}{q_v-1}\right)^{1/2} q_v^{-k/2}\right\}.$$

Moreover, we obtain expressions

$$p_1^\eta(\rho)=\tilde{\eta}(\mathfrak{D}_{F/\mathbb{Q}})\sum_{w\in S(\rho)}\big\{\prod_{v\in S(\rho)-\{w\}} Y_v^\eta(-1)\big\}(Y_w^\eta)'(-1)$$

and

$$\begin{aligned} p_2^\eta(\rho)=&\frac{\tilde{\eta}(\mathfrak{D}_{F/\mathbb{Q}})}{2}\sum_{w\in S(\rho)}\Big[\sum_{x\in S(\rho)-\{w\}}\Big\{\prod_{v\in S(\rho)-\{w,x\}} Y_v^\eta(-1)\Big\}(Y_w^\eta)'(-1)(Y_x^\eta)'(-1)\\ &+\Big\{\prod_{v\in S(\rho)-\{w\}} Y_v^\eta(-1)\Big\}(Y_w^\eta)''(-1)\Big], \end{aligned}$$

where we set

$$C_v=\delta(v\in S_1(\rho))+\delta\left(v\in\coprod_{k=2}^{n} S_k(\rho)\right)\left(\frac{q_v+1}{q_v-1}\right)^{1/2}$$

and

$$Y_v^\eta(\nu)=C_v\{q_v+1+\eta_v(\varpi_v)(q_v^{(1+\nu)/2}+q_v^{(1-\nu)/2})\}\frac{q_v^{k\nu/2}}{q_v-q_v^{\nu}}.$$

Further we have

$$(Y_v^\eta)'(-1)=C_v(\log q_v^k)q_v^{-k/2}\frac{-\eta_v(\varpi_v)q_v(q_v-1)^2+k(1+\eta_v(\varpi_v))q_v(q_v^2-1)+2(1+\eta_v(\varpi_v))q_v}{2k(q_v^2-1)(q_v-1)}$$

and

$$\begin{aligned} &(Y_v^\eta)''(-1)\\ =&C_v\Bigg[\eta_v(\varpi_v)(\log q_v^k)^2q_v^{-k/2}\frac{(1+q_v)(q_v-q_v^{-1})+(1-q_v)\{k(q_v-q_v^{-1})+2q_v^{-1}\}}{4k^2(q_v-q_v^{-1})^2}\\ &+(\log q_v^k)^2q_v^{-k/2}\frac{\eta_v(\varpi_v)\{k(q_v^3-q_v)+2q_v\}}{4k^2(1+q_v)(1-q_v^2)}\\ &+(\log q_v^k)^2q_v^{-k/2}\frac{(1+\eta_v(\varpi_v))q_v}{k^2(q_v^2-1)^3(q_v-1)}\Bigg\{\left(\frac{k^2}{4}(q_v^2-1)+1\right)(q_v^2-1)^2+(k(q_v^2-1)+2)(q_v^2-1)\Bigg\}\Bigg]. \end{aligned}$$

Thus, by noting $\#\Lambda(\mathfrak{n})\ll \mathrm{N}(\mathfrak{n})^\epsilon$, we obtain the estimates of $Y_j^\eta(\mathfrak{n})$ for $j\in\{0,1,2\}$.

Next let us examine $Y_{-1}^\eta(\mathfrak{n})$. We have the following expressions:

$$\tilde{B}_{\mathbf{1},\rho}^\eta(0)=\epsilon(0,\eta)B_{\mathbf{1},\rho}^\eta(1/2,1),$$

$$\begin{aligned} B_{\mathbf{1},\rho}^\eta(1/2,1)=&\prod_{v\in S_1(\rho)}\frac{(\eta_v(\varpi_v)-1)q_v^{-1/2}}{(1-q_v^{-1})}\\ &\times\prod_{k=2}^{n}\prod_{v\in S_k(\rho)}\left\{\frac{\eta_v(\varpi_v)^k(\eta_v(\varpi_v)-1)(\eta_v(\varpi_v)-q_v^{-1})}{1-q_v^{-2}}\left(\frac{q_v+1}{q_v-1}\right)^{1/2}q_v^{-k/2}\right\}, \end{aligned}$$

$$(\tilde{B}_{\mathbf{1},\rho}^{\mathbf{1}})''(0)=\epsilon''(0,\mathbf{1})B_\rho(0)+2\epsilon'(0,\mathbf{1})B_\rho'(0)+\epsilon(0,\mathbf{1})B_\rho''(0).$$

Here we set $B_\rho(z) = B^{\mathbf{1}}_{\mathbf{1},\rho}(-z+1/2,1) = D_F^{-z}\prod_{v\in S(\rho)} B_v(z)$ and

$$\begin{aligned}B_v(z) =&\delta(v\in S_1(\rho))(q_v^z-1)\frac{q_v^{-1/2}}{1-q_v^{-1}}\\ &+\sum_{k=2}^{n}\delta(v\in S_k(\rho))(q_v^{kz}-q_v^{(k-1)z-1}-q_v^{(k-1)z}+q_v^{(k-2)z-1})\left(\frac{q_v+1}{q_v-1}\right)^{1/2}\frac{q_v^{-k/2}}{1-q_v^{-2}}.\end{aligned}$$

A direct computation gives us

$$B_\rho'(0) = (\log D_F^{-1})\prod_{v\in S(\rho)} B_v(0) + \sum_{w\in S(\rho)}\Big\{\prod_{v\in S(\rho)-\{w\}} B_v(0)\Big\}B_v'(0),$$

$$\begin{aligned}B_\rho''(0) =&(\log D_F^{-1})^2\prod_{v\in S(\rho)} B_v(0) + 2(\log D_F^{-1})\sum_{w\in S(\rho)}\Big\{\prod_{v\in S(\rho)-\{w\}} B_v(0)\Big\}B_v'(0)\\ &+\sum_{w\in S(\rho)}\left\{\sum_{x\in S(\rho)-\{w\}}\Big\{\prod_{v\in S(\rho)-\{w,x\}} B_v(0)\Big\}B_w'(0)B_x'(0)+\Big\{\prod_{v\in S(\rho)-\{w\}} B_v(0)\Big\}B_w''(0)\right\},\end{aligned}$$

$$B_v'(0) = \delta(v\in S_1(\rho))(\log q_v)\frac{q_v^{-1/2}}{1-q_v^{-1}}+\sum_{k=2}^{n}\delta(v\in S_k(\rho))(\log q_v)\left(\frac{q_v+1}{q_v-1}\right)^{1/2}\frac{q_v^{-k/2}}{1+q_v^{-1}}$$

and

$$\begin{aligned}B_v''(0) =&\delta(v\in S_1(\rho))(\log q_v)^2\frac{q_v^{-1/2}}{1-q_v^{-1}}\\ &+\sum_{k=2}^{n}\delta(v\in S_k(\rho))(\log q_v^k)^2\frac{2k-1-(2k-3)q_v^{-1}}{k^2}\left(\frac{q_v+1}{q_v-1}\right)^{1/2}\frac{q_v^{-k/2}}{1-q_v^{-2}}.\end{aligned}$$

This completes the proof of the estimate of $Y^\eta_{-1}(\mathfrak{n})$. □

With the aid of Lemmas 8.1 and 8.4, we obtain the expression of the spectral side of $P^\eta_{\rm reg}(\hat{\mathbf{\Psi}}_{\rm reg}(\mathfrak{n}|\alpha))$.

Theorem 8.6. *The value $P^\eta_{\rm reg}(\hat{\mathbf{\Psi}}_{\rm reg}(\mathfrak{n}|\alpha))$ can be defined and we have*

$$P^\eta_{\rm reg}(\hat{\mathbf{\Psi}}_{\rm reg}(\mathfrak{n}|\alpha)) = C(\mathfrak{n},S)\{\mathbb{I}^\eta_{\rm cus}(\mathfrak{n}|\alpha)+\mathbb{I}^\eta_{\rm eis}(\mathfrak{n}|\alpha)+\mathbb{D}^\eta(\mathfrak{n}|\alpha)\}.$$

Here we put

$$\mathbb{I}^\eta_{\rm cus}(\mathfrak{n}|\alpha) = \sum_{\varphi\in\mathcal{B}_{\rm cus}(\mathfrak{n})}\alpha(\nu_{\varphi,S})\overline{P^{\mathbf{1}}_{\rm reg}(\varphi)}P^\eta_{\rm reg}(\varphi),$$

$$\mathbb{I}^\eta_{\rm eis}(\mathfrak{n}|\alpha) = \sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\frac{R_F^{-1}}{8\pi i}\int_{i\mathbb{R}}\tilde{\alpha}_\chi(\nu)P^{\mathbf{1}}_{\rm reg}(E_{\chi^{-1},\rho}(-\nu,-))P^\eta_{\rm reg}(E_{\chi,\rho}(\nu,-))d\nu$$

and

$$\mathbb{D}^\eta(\mathfrak{n}|\alpha) = \delta(\mathfrak{f}_\eta=\mathfrak{o})\{Y^\eta_2(\mathfrak{n})\tilde{\alpha}''_\eta(1)+Y^\eta_1(\mathfrak{n})\tilde{\alpha}'_\eta(1)+Y^\eta_0(\mathfrak{n})\tilde{\alpha}_\eta(1)\}+Y^\eta_{-1}(\mathfrak{n})\tilde{\alpha}(1).$$

9. Periods of regularized automorphic smoothed kernels: the geometric side

In this section, we describe the geometric expression of $\hat{\mathbf{\Psi}}_{\mathrm{reg}}(\mathfrak{n}|\alpha)$ and its regularized period $P^{\eta}_{\mathrm{reg}}(\hat{\mathbf{\Psi}}_{\mathrm{reg}}(\mathfrak{n}|\alpha))$. For $\delta \in G_F$, we put $\mathrm{St}(\delta) = H_F \cap \delta^{-1} H_F \delta$. By [47, Lemma 11.1], the following elements of G_F form a complete system of representatives of the double coset space $H_F \backslash G_F / H_F$:

$$e = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right],\ w_0 = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right],$$
$$u = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right],\ \overline{u} = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right],\ uw_0 = \left[\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right],\ \overline{u}w_0 = \left[\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right],$$
$$\delta_b = \left[\begin{smallmatrix} 1+b^{-1} & 1 \\ 1 & 1 \end{smallmatrix}\right],\quad b \in F^\times - \{-1\}.$$

Moreover, we have $\mathrm{St}(e) = \mathrm{St}(w_0) = H_F$ and $\mathrm{St}(\delta) = Z_F$ for any $\delta \in \{u, \overline{u}, uw_0, \overline{u}w_0\} \cup \{\delta_b | b \in F^\times - \{-1\}\}$. We note

$$H_F \backslash G_F = \coprod_{\delta \in H_F \backslash G_F / H_F} H_F \backslash (H_F \delta H_F) \cong \coprod_{\delta \in H_F \backslash G_F / H_F} \mathrm{St}(\delta) \backslash H_F.$$

Thus we obtain the following expression for $\mathrm{Re}(\lambda) > 0$:

$$\hat{\mathbf{\Psi}}_{\beta,\lambda}\left(\mathfrak{n}|\alpha; \left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & x_\eta \\ 0 & 1 \end{smallmatrix}\right]\right) = \sum_{\delta \in H_F \backslash G_F / H_F} \sum_{\gamma \in \mathrm{St}(\delta) \backslash H_F} \hat{\Psi}_{\beta,\lambda}\left(\mathfrak{n}|\alpha; \delta\gamma \left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & x_\eta \\ 0 & 1 \end{smallmatrix}\right]\right).$$

Set

$$J_\delta(\beta, \lambda, \alpha; t) = \sum_{\gamma \in \mathrm{St}(\delta) \backslash H_F} \hat{\Psi}_{\beta,\lambda}\left(\mathfrak{n}|\alpha; \delta\gamma \left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & x_\eta \\ 0 & 1 \end{smallmatrix}\right]\right)$$

for any $\delta \in H_F \backslash G_F / H_F$.

The following three lemmas are proved in similar fashions to [47, Lemma 11.2], [47, Lemma 11.3] and [47, Lemma 11.21].

Lemma 9.1. *Both functions $\lambda \mapsto J_e(\beta, \lambda, \alpha; t)$ and $\lambda \mapsto J_{w_0}(\beta, \lambda, \alpha; t)$ are analytically continued to entire functions. The values of these functions at $\lambda = 0$ are equal to $J_{\mathrm{id}}(\alpha; t)\beta(0)$ and $\delta(\mathfrak{n} = \mathfrak{o})J_{\mathrm{id}}(\alpha; t)\beta(0)$, respectively, where*

$$J_{\mathrm{id}}(\alpha; t) = \delta(\mathfrak{f}_\eta = \mathfrak{o}) \left(\frac{1}{2\pi i}\right)^{\#S} \int_{\mathbb{L}_S(\mathbf{c})} \Upsilon_S^{\mathbf{1}}(\mathbf{s})\alpha(\mathbf{s}) d\mu_S(\mathbf{s})$$

with

$$\Upsilon_S^{\mathbf{1}}(\mathbf{s}) = \left\{\prod_{v \in \Sigma_\infty} \frac{-1}{8} \frac{\Gamma((s_v+1)/4)^2}{\Gamma((s_v+3)/4)^2}\right\} \left\{\prod_{v \in S_{\mathrm{fin}}} (1 - q_v^{-(s_v+1)/2})^{-1}(1 - q_v^{(s_v+1)/2})^{-1}\right\}.$$

We put

$$J_{\mathrm{u}}(\beta, \lambda, \alpha; t) = J_u(\beta, \lambda, \alpha, t) + J_{\overline{u}w_0}(\beta, \lambda, \alpha, t)$$

and

$$J_{\bar{\mathrm{u}}}(\beta, \lambda, \alpha; t) = J_{uw_0}(\beta, \lambda, \alpha, t) + J_{\bar{u}}(\beta, \lambda, \alpha, t).$$

Lemma 9.2. *For $* \in \{\mathrm{u}, \bar{\mathrm{u}}\}$, the function $\lambda \mapsto J_*(\beta, \lambda, \alpha, t)$ is analytically continued to an entire function and the value at $\lambda = 0$ is equal to $J_*(\alpha; t)\beta(0)$, where*

$$J_{\mathrm{u}}(\alpha; t) = \left(\frac{1}{2\pi i}\right)^{\#S} \sum_{a \in F^\times} \int_{\mathbb{L}_S(\mathbf{c})} \left\{\Psi^{(0)}\left(\mathfrak{n}|\mathbf{s}; \left[\begin{smallmatrix} 1 & at^{-1} \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & x_\eta \\ 0 & 1 \end{smallmatrix}\right]\right)\right.$$
$$\left. + \delta(\mathfrak{n} = \mathfrak{o})\Psi^{(0)}\left(\mathfrak{n}|\mathbf{s}; \left[\begin{smallmatrix} 1 & 0 \\ at^{-1} & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 0 \\ -x_\eta & 1 \end{smallmatrix}\right] w_0\right)\right\}\alpha(\mathbf{s}) d\mu_S(\mathbf{s})$$

and

$$J_{\bar{\mathrm{u}}}(\alpha; t) = \left(\frac{1}{2\pi i}\right)^{\#S} \sum_{a \in F^\times} \int_{\mathbb{L}_S(\mathbf{c})} \left\{\Psi^{(0)}\left(\mathfrak{n}|\mathbf{s}; \left[\begin{smallmatrix} 1 & 0 \\ at & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & x_\eta \\ 0 & 1 \end{smallmatrix}\right]\right)\right.$$

$$+ \delta(\mathfrak{n}=\mathfrak{o})\Psi^{(0)}\left(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1 & at\\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 0\\ -x_\eta & 1\end{smallmatrix}\right]w_0\right)\Big\}\alpha(\mathbf{s})d\mu_S(\mathbf{s}).$$

These series-integrals are absolutely convergent.

We put

$$J_{\mathrm{hyp}}(\beta,\lambda,\alpha;t)=\sum_{b\in F^\times-\{-1\}}J_{\delta_b}(\beta,\lambda,\alpha;t)=\sum_{b\in F^\times-\{-1\}}\sum_{a\in F^\times}\hat{\Psi}_{\beta,\lambda}\left(\mathfrak{n}|\alpha;\delta_b\left[\begin{smallmatrix}at & 0\\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & x_\eta\\ 0 & 1\end{smallmatrix}\right]\right).$$

Lemma 9.3. *The function $J_{\mathrm{hyp}}(\beta,\lambda,\alpha;t)$ on $\mathrm{Re}(\lambda)>1$ is analytically continued to an entire function and the value at $\lambda=0$ is $J_{\mathrm{hyp}}(\alpha;t)\beta(0)$, where*

$$J_{\mathrm{hyp}}(\alpha;t)=\sum_{b\in F^\times-\{-1\}}\sum_{a\in F^\times}\hat{\Psi}^{(0)}\left(\mathfrak{n}|\alpha;\delta_b\left[\begin{smallmatrix}at & 0\\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & x_\eta\\ 0 & 1\end{smallmatrix}\right]\right).$$

The series converges absolutely and locally uniformly in $t\in\mathbb{A}^\times$.

Lemmas 9.1, 9.2 and 9.3 give the geometric expression of $\hat{\mathbf{\Psi}}_{\mathrm{reg}}\left(\mathfrak{n}|\alpha;\left[\begin{smallmatrix}t & 0\\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & x_\eta\\ 0 & 1\end{smallmatrix}\right]\right)$.

Proposition 9.4. *Let $\mathfrak{n}$ be an ideal of $\mathfrak{o}$ and S a finite subset of Σ_F satisfying $\Sigma_\infty\subset S$ and $S\cap S(\mathfrak{n})=\emptyset$. Let η be a character satisfying (3.1). Then, for any $\alpha\in\mathcal{A}_S$, we have*

$$\begin{aligned}&\hat{\mathbf{\Psi}}_{\mathrm{reg}}\left(\mathfrak{n}|\alpha;\left[\begin{smallmatrix}t & 0\\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & x_\eta\\ 0 & 1\end{smallmatrix}\right]\right)\\ =&(1+\delta(\mathfrak{n}=\mathfrak{o}))J_{\mathrm{id}}(\alpha;t)+J_{\mathrm{u}}(\alpha;t)+J_{\bar{\mathrm{u}}}(\alpha;t)+J_{\mathrm{hyp}}(\alpha;t),\quad t\in\mathbb{A}^\times.\end{aligned}$$

Next let us compute $P^\eta_{\mathrm{reg}}(\hat{\mathbf{\Psi}}_{\mathrm{reg}}(\mathfrak{n}|\alpha))$ explicitly. Define

$$\mathbb{J}^\eta_\natural(\beta,\lambda;\alpha)=\int_{F^\times\backslash\mathbb{A}^\times}J_\natural(\alpha;t)\{\hat{\beta}_\lambda(|t|_\mathbb{A})+\hat{\beta}_\lambda(|t|_\mathbb{A}^{-1})\}\eta(tx^*_\eta)d^\times t$$

for $\natural\in\{\mathrm{id},\mathrm{u},\bar{\mathrm{u}},\mathrm{hyp}\}$ and

$$\Upsilon^\eta_S(\mathbf{s})=\left\{\prod_{v\in\Sigma_\infty}\frac{-1}{8}\frac{\Gamma((s_v+1)/4)^2}{\Gamma((s_v+3)/4)^2}\right\}\left\{\prod_{v\in S_{\mathrm{fin}}}(1-q_v^{(s_v+1)/2})^{-1}(1-\eta_v(\varpi_v)q_v^{-(s_v+1)/2})^{-1}\right\}.$$

For any ideal $\mathfrak{a}$ of $\mathfrak{o}$, we set

$$\begin{aligned}\mathfrak{C}^\eta_{S,\mathfrak{a}}(\mathbf{s})=&C_0(\eta)+R(\eta)\Bigg\{\log(D_F\mathrm{N}(\mathfrak{a}))+\frac{d_F}{2}(C_{\mathrm{Euler}}+2\log 2-\log\pi)\\ &+\sum_{v\in S_{\mathrm{fin}}}\frac{\log q_v}{1-q_v^{(s_v+1)/2}}+\frac{1}{2}\sum_{v\in\Sigma_\infty}\left(\psi\left(\frac{s_v+1}{4}\right)+\psi\left(\frac{s_v+3}{4}\right)\right)\Bigg\},\end{aligned}$$

where $\psi(z)=\Gamma'(s)/\Gamma(s)$ is the digamma function and C_{Euler} is the Euler constant. We note that if $\eta\neq\mathbf{1}$, then $\mathfrak{C}^\eta_{S,\mathfrak{a}}(\mathbf{s})$ is independent of the choice of $\mathfrak{a}$, and $\mathfrak{C}^\eta_{S,\mathfrak{a}}(\mathbf{s})=C_0(\eta)=L(1,\eta)$. Put

$$\mathfrak{K}_\eta(\mathfrak{n}|\mathbf{s})=\sum_{b\in F^\times-\{-1\}}\int_{\mathbb{A}^\times}\Psi^{(0)}\left(\mathfrak{n}|\mathbf{s};\delta_b\left[\begin{smallmatrix}t & 0\\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & x_\eta\\ 0 & 1\end{smallmatrix}\right]\right)\eta(tx^*_\eta)d^\times t.$$

The defining series-integral converges absolutely if we take $c\in\mathbb{R}$ such that $\mathrm{Re}(\mathbf{s})=\underline{c}=(c)_{v\in S}$ and $(c+1)/4>1$. By the expression of $\hat{\mathbf{\Psi}}_{\mathrm{reg}}(\mathfrak{n}|\alpha)$ in Proposition 9.4 and the same computation as in the proof of [47, Theorem 12.1], we can express the geometric side of $P^\eta_{\mathrm{reg}}(\hat{\mathbf{\Psi}}_{\mathrm{reg}}(\mathfrak{n}|\alpha))$ as follows.

Theorem 9.5. *For any* $\natural \in \{\mathrm{id}, \mathrm{u}, \bar{\mathrm{u}}, \mathrm{hyp}\}$*, the integral* $\mathbb{J}_{\natural}^{\eta}(\beta, \lambda; \alpha)$ *converges absolutely and locally uniformly in* $\{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) > 1\}$*. The function* $\lambda \mapsto \mathbb{J}_{\natural}^{\eta}(\beta, \lambda; \alpha)$ *is analytically continued to a meromorphic function on* $\{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) > -1\}$*. Moreover, the constant term* $\mathrm{CT}_{\lambda=0}\mathbb{J}_{\natural}^{\eta}(\beta, \lambda; \alpha)$ *is equal to* $\mathbb{J}_{\natural}^{\eta}(\mathfrak{n}|\alpha)\beta(0)$*, where*

$$\mathbb{J}_{\mathrm{id}}^{\eta}(\mathfrak{n}|\alpha) = 0,$$

$$\mathbb{J}_{\mathrm{u}}^{\eta}(\mathfrak{n}|\alpha) = (1 + \delta(\mathfrak{n} = \mathfrak{o}))D_F^{1/2}\mathcal{G}(\eta)\int_{\mathbb{L}_S(\mathbf{c})} \Upsilon_S^{\eta}(\mathbf{s})\mathfrak{C}_{S,\mathfrak{o}}^{\eta}(\mathbf{s})\alpha(\mathbf{s})d\mu_S(\mathbf{s}),$$

$$\mathbb{J}_{\bar{\mathrm{u}}}^{\eta}(\mathfrak{n}|\alpha) = (1 + \delta(\mathfrak{n} = \mathfrak{o}))D_F^{1/2}\mathcal{G}(\eta)\int_{\mathbb{L}_S(\mathbf{c})} \Upsilon_S^{\eta}(\mathbf{s})\mathfrak{C}_{S,\mathfrak{n}}^{\eta}(\mathbf{s})\alpha(\mathbf{s})d\mu_S(\mathbf{s})$$

and

$$\mathbb{J}_{\mathrm{hyp}}^{\eta}(\mathfrak{n}|\alpha) = \left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\underline{c})} \mathfrak{K}_{\eta}(\mathfrak{n}|\mathbf{s})\alpha(\mathbf{s})d\mu_S(\mathbf{s}).$$

In particular, we have

$$P_{\mathrm{reg}}^{\eta}(\hat{\mathbf{\Psi}}_{\mathrm{reg}}(\mathfrak{n}|\alpha)) = \mathbb{J}_{\mathrm{u}}^{\eta}(\mathfrak{n}|\alpha) + \mathbb{J}_{\bar{\mathrm{u}}}^{\eta}(\mathfrak{n}|\alpha) + \mathbb{J}_{\mathrm{hyp}}^{\eta}(\mathfrak{n}|\alpha).$$

10. Proofs of main theorems for even Hilbert Maass forms

Fix a character η of $F^{\times}\backslash\mathbb{A}^{\times}$ so that $\eta^2 = \mathbf{1}$ and $\eta_v(-1) = 1$ for all $v \in \Sigma_{\infty}$. Let S be a finite subset of Σ_F such that $S \supset \Sigma_{\infty}$ and $S_{\mathrm{fin}} \cap S(\mathfrak{f}_{\eta}) = \emptyset$. Let $\mathcal{J}_{S,\eta}^{+}$ be the set of all ideals $\mathfrak{n}$ of $\mathfrak{o}$ such that $S(\mathfrak{n}) \cap (S \cup S(\mathfrak{f}_{\eta})) = \emptyset$ and $\tilde{\eta}(\mathfrak{n}) = 1$. By Theorems 8.6 and 9.5, we obtain the relative trace formula

$$C(\mathfrak{n}, S)\{\mathbb{I}_{\mathrm{cus}}^{\eta}(\mathfrak{n}|\alpha) + \mathbb{I}_{\mathrm{eis}}^{\eta}(\mathfrak{n}|\alpha) + \mathbb{D}^{\eta}(\mathfrak{n}|\alpha)\} = \mathbb{J}_{\mathrm{u}}^{\eta}(\mathfrak{n}|\alpha) + \mathbb{J}_{\bar{\mathrm{u}}}^{\eta}(\mathfrak{n}|\alpha) + \mathbb{J}_{\mathrm{hyp}}^{\eta}(\mathfrak{n}|\alpha)$$

for any $\alpha \in \mathcal{A}_S$ and $\mathfrak{n} \in \mathcal{J}_{S,\eta}^{+}$.

10.1. Estimates of error terms. The following estimate of $\mathbb{J}_{\mathrm{hyp}}^{\eta}(\mathfrak{n}|\alpha)$ is given by the same argument as in the proof of [47, Lemma 12.9].

Lemma 10.1. *For any* $\alpha \in \mathcal{A}_S$ *and* $q > 0$*, we have* $|\mathbb{J}_{\mathrm{hyp}}^{\eta}(\mathfrak{n}|\alpha)| \ll \mathrm{N}(\mathfrak{n})^{-q}$ *with the implied constant independent of* $\mathfrak{n} \in \mathcal{J}_{S,\eta}^{+}$.

Lemma 10.2. *For any* $\epsilon > 0$*, we have*

$$|B_{\chi,\rho}^{\eta}(1/2, \nu)| \ll \mathrm{N}(\mathfrak{f}_{\chi})^{-1/2-\epsilon}\mathrm{N}(\mathfrak{n})^{\epsilon}, \qquad \nu \in i\mathbb{R},\ \rho \in \Lambda_{\chi}(\mathfrak{n}),\ \chi \in \Xi(\mathfrak{n})$$

with the implied constant independent of $\mathfrak{n} \in \mathcal{J}_{S,\eta}^{+}$.

Proof. Assume $\nu \in i\mathbb{R}$. Then, the following estimate holds for any $\epsilon > 0$:

$$\begin{aligned}
&|B_{\chi,\rho}^{\eta}(1/2, \nu)| \\
&= \prod_{k=0}^{n}\prod_{v \in S_k(\rho)} |Q_{k,\chi_v}^{(\nu)}(\eta_v, 1)||L(1+\nu, \chi_v^2)| \prod_{v \in U_1(\rho)}(1 + q_v^{-1}) \\
&\quad \times \prod_{k=2}^{n}\prod_{v \in U_k(\rho)}\left(\frac{q_v+1}{q_v-1}\right)^{1/2}\prod_{k=0}^{n}\prod_{v \in R_k(\rho)} q_v^{d_v/2}(1-q_v^{-1})^{1/2}|\overline{\mathcal{G}(\chi_v)}| \\
&\ll \prod_{v \in U_1(\rho)}(1+q_v^{-1})\left(1 + \frac{2}{q_v^{1/2}+q_v^{-1/2}}\right)\frac{1}{1-q_v^{-1}}\prod_{k=2}^{n}\prod_{v \in U_k(\rho)}\left(\frac{q_v+1}{q_v-1}\right)^{1/2} q_v^{-1}(q_v^{1/2}+1)^2\frac{1}{1-q_v^{-1}} \\
&\quad \times \prod_{k=0}^{n}\prod_{v \in R_k(\rho)} q_v^{d_v/2}(1-q_v^{-1})^{1/2}\frac{q_v^{-f(\chi_v)/2}q_v^{-d_v/2}}{1-q_v^{-1}}\frac{1}{1-q_v^{-1}}
\end{aligned}$$

$$\ll \mathrm{N}(\mathfrak{f}_\chi)^{-1/2+\epsilon}\mathrm{N}(\mathfrak{n}\mathfrak{f}_\chi^{-2})^{\epsilon}.$$

This completes the proof. □

Note that $[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{n})]=\mathrm{N}(\mathfrak{n})\prod_{v\in S(\mathfrak{n})}(1+q_v^{-1})$ holds by an easy computation.

Lemma 10.3. *For any $\alpha\in\mathcal{A}_S$, there exists $\delta>0$ such that $|C(\mathfrak{n},S)\mathbb{I}^{\eta}_{\mathrm{eis}}(\mathfrak{n}|\alpha)|\ll\mathrm{N}(\mathfrak{n})^{-\delta}$ with the implied constant independent of $\mathfrak{n}\in\mathcal{J}^+_{S,\eta}$.*

Proof. We recall that for any $\epsilon>0$, the estimate $|L_{\mathrm{fin}}(1+\nu,\chi^2)|^{-1}\ll\mathfrak{q}(\chi^2|\cdot|_{\mathbb{A}}^{\nu})^{\epsilon},\ \nu\in i\mathbb{R}$ holds with the implied constant independent of $\chi\in\Xi(\mathfrak{n})$ and $\mathfrak{n}$. This was given in the proof of Lemma 7.5. Let θ be a real number such that $|L_{\mathrm{fin}}(1/2+it,\chi)|\ll\mathfrak{q}(\chi|\cdot|_{\mathbb{A}}^{it})^{1/4+\theta},\ t\in\mathbb{R}$ uniformly for any $\chi\in\Xi(\mathfrak{n})$ and $\mathfrak{n}$. We can take such θ so that $-1/4<\theta<0$ by [27, Theorem 1.1]. Thus, with the aid of Lemma 10.2 and

$$\prod_{v\in\Sigma_\infty}\left|\frac{L((1+\nu)/2,\chi_v)L((1-\nu)/2,\chi_v^{-1})}{L(1-\nu,\chi_v^{-2})}\right|\asymp\prod_{v\in\Sigma_\infty}(1+\nu+2ib(\chi_v))^{-1/2},$$

which follows from Stirling's formula, the explicit description of $P^{\eta}_{\mathrm{reg}}(E_{\chi,\rho}(\nu,-))$ in Proposition 3.3 gives us the estimate

$$\begin{aligned}|P^{\eta}_{\mathrm{reg}}(E_{\chi,\rho}(\nu,-))|&\ll\mathrm{N}(\mathfrak{f}_\chi)^{1/2}\mathrm{N}(\mathfrak{f}_\chi)^{-1/2-\epsilon}\mathrm{N}(\mathfrak{n})^{\epsilon}(\mathrm{N}(\mathfrak{f}_\chi)^{1/4+\theta})^2\mathrm{N}(\mathfrak{f}_\chi)^{\epsilon}\prod_{v\in\Sigma_\infty}(1+|\nu+2ib(\chi_v)|)^{2\theta+\epsilon}\\&=\mathrm{N}(\mathfrak{f}_\chi)^{1/2+2\theta}\mathrm{N}(\mathfrak{n})^{\epsilon}\prod_{v\in\Sigma_\infty}(1+|\nu+2ib(\chi_v)|)^{2\theta+\epsilon}\\&\ll\mathrm{N}(\mathfrak{n})^{1/4+\theta+\epsilon}\prod_{v\in\Sigma_\infty}(1+|\nu+2ib(\chi_v)|)^{2\theta+\epsilon}\end{aligned}$$

for any $\epsilon>0$, where the implied constant is independent of $\nu\in i\mathbb{R},\chi\in\Xi(\mathfrak{n})$ and $\mathfrak{n}\in\mathcal{J}^+_{S,\eta}$. With the aid of Lemma 2.1, we have

$$\begin{aligned}|C(\mathfrak{n},S)\mathbb{I}^{\eta}_{\mathrm{eis}}(\mathfrak{n}|\alpha)|&\ll[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{n})]^{-1}\sum_{\chi\in\Xi(\mathfrak{n})}\sum_{\rho\in\Lambda_\chi(\mathfrak{n})}\int_{i\mathbb{R}}|P^{\mathbf{1}}_{\mathrm{reg}}(E_{\chi^{-1},\rho}(-\nu,-))||P^{\eta}_{\mathrm{reg}}(E_{\chi,\rho}(\nu,-))||\tilde{\alpha}_\chi(\nu)||d\nu|\\&\ll\mathrm{N}(\mathfrak{n})^{-1}\sum_{\chi\in\Xi(\mathfrak{n})}\left(\sum_{\mathfrak{a}|\mathfrak{n}}1\right)\int_{y\in\mathbb{R}}\mathrm{N}(\mathfrak{n})^{1/2+2\theta+2\epsilon}\{\prod_{v\in\Sigma_\infty}(1+|y+2b(\chi_v)|)^{4\theta+2\epsilon}\}|\tilde{\alpha}_\chi(iy)|dy\\&\ll\mathrm{N}(\mathfrak{n})^{-1/2+2\theta+3\epsilon}\sum_{\chi\in\Xi_{\ker}(\mathfrak{n})}\sum_{b\in L_0}\int_{y\in\mathbb{R}}\{\prod_{v\in\Sigma_\infty}(1+|y+2b_v|)^{4\theta+2\epsilon}\}|\alpha((iy+2ib_v)_{v\in\Sigma_\infty})|dy\\&\ll\mathrm{N}(\mathfrak{n})^{2\theta+4\epsilon}\int_{y\in\mathbb{R}^{d_F}}(1+||y||^2)^{4\theta+2\epsilon}|\alpha(iy)|dy.\end{aligned}$$

Note $\sum_{\mathfrak{a}|\mathfrak{n}}1\ll\mathrm{N}(\mathfrak{n})^{\epsilon}$. Since we can take $\epsilon>0$ so that $2\theta+4\epsilon<0$, we obtain the assertion. □

Lemma 10.4. *For any $\epsilon>0$ and $\alpha\in\mathcal{A}_S$, we have $|C(\mathfrak{n},S)\mathbb{D}^{\eta}(\mathfrak{n}|\alpha)|\ll\mathrm{N}(\mathfrak{n})^{-1+\epsilon}$ with the implied constant independent of $\mathfrak{n}\in\mathcal{J}^+_{S,\eta}$.*

Proof. This follows immediately from Lemma 8.5. □

For $\mathfrak{n}\in\mathcal{J}^+_{S,\eta}$, we set

$$\langle\lambda^{\eta}_S(\mathfrak{n}),f\rangle=2D_F^{1/2}\mathcal{G}(\eta)^{-1}[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{n})]^{-1}\sum_{\pi\in\Pi_{\mathrm{cus}}(\mathfrak{n})}\mathbb{P}^{\eta}(\pi;\mathbf{K}_0(\mathfrak{n}))f(\nu_S(\pi))$$

for any $f \in C_c(\mathfrak{X}_S^{0+})$ or for any $f = \alpha \in \mathcal{A}_S$. The defining series is convergent in the same way as [47, Lemma 13.16]. Combining Lemmas 3.6, 3.8, 10.1, 10.3 and 10.4 with the argument in [47, Lemma 13.18], we have the following.

Proposition 10.5. *For a fixed $\alpha \in \mathcal{A}_S$, there exists $\delta > 0$ such that*

$$\langle \lambda_S^\eta(\mathfrak{n}), \alpha \rangle = \sum_{\pi \in \Pi_{\mathrm{cus}}(\mathfrak{n})} \frac{[\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{f}_\pi)]}{\mathrm{N}(\mathfrak{f}_\pi)[\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{n})]} w_{\mathfrak{n}}^\eta(\pi) \frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{L^{S_\pi}(1,\pi,\mathrm{Ad})} \alpha(\nu_S(\pi)) = \langle \lambda_S^\eta, \alpha \rangle + \mathcal{O}(\mathrm{N}(\mathfrak{n})^{-\delta})$$

as $\mathrm{N}(\mathfrak{n}) \to \infty$ in $\mathfrak{n} \in \mathcal{J}_{S,\eta}^+$.

10.2. Schwartz spaces on $\overline{\mathfrak{X}_S^{0+}}$. The Schwartz space $\mathcal{S}(\mathfrak{X}_S^{0+})$ was introduced in [47, §13.2]. However, the definition is inaccuracy; indeed, the Weierstrass approximation theorem does not work in the proof of [47, Lemma 13.17]. In this subsection, we introduce another Schwartz space $\mathcal{S}(\overline{\mathfrak{X}_S^{0+}})$. Set

$$\overline{\mathfrak{X}_v^{0+}} = \begin{cases} i\mathbb{R}_{\geqslant 0} \cup (0,1] & (v \in \Sigma_\infty), \\ i[0, 2\pi(\log q_v)^{-1}] \cup (0,1] \cup \{(0,1] + 2\pi i (\log q_v)^{-1}\} & (v \in S_{\mathrm{fin}}) \end{cases}$$

and $\overline{\mathfrak{X}_S^{0+}} = \prod_{v \in S} \overline{\mathfrak{X}_v^{0+}}$. We note

$$\overline{\mathfrak{X}_v^{0+}} \cong \mathbb{R}_{\geqslant 0}$$

by the homeomorphism $s \mapsto (1-s^2)/4$ if $v \in \Sigma_\infty$, and

$$\overline{\mathfrak{X}_v^{0+}} \cong \ [-(q_v^{1/2} + q_v^{-1/2}),\ q_v^{1/2} + q_v^{-1/2}]$$

by the homeomorphism $s \mapsto q_v^{-s/2} + q_v^{s/2}$ if $v \in S_{\mathrm{fin}}$.

Definition 10.6. *We define $\mathcal{S}(\overline{\mathfrak{X}_{\Sigma_\infty}^{0+}})$ as the space of all functions f on $\overline{\mathfrak{X}_{\Sigma_\infty}^{0+}}$ such that f is of the form $\varphi((\frac{1-s_v^2}{4})_{v \in \Sigma_\infty})$ for some $\varphi \in \mathcal{S}((\mathbb{R}_{\geqslant 0})^{\Sigma_\infty})$. Here $\mathcal{S}((\mathbb{R}_{\geqslant 0})^{\Sigma_\infty})$ is the Schwartz space in the usual sense.*

We define the Schwartz space on $\overline{\mathfrak{X}_S^{0+}} = \prod_{v \in S} \overline{\mathfrak{X}_v^{0+}}$, which is denoted by $\mathcal{S}(\overline{\mathfrak{X}_S^{0+}})$, as

$$\mathcal{S}(\overline{\mathfrak{X}_S^{0+}}) = \mathcal{S}(\overline{\mathfrak{X}_{\Sigma_\infty}^{0+}}) \otimes C(\overline{\mathfrak{X}_{S_{\mathrm{fin}}}^{0+}}) \quad \text{(algebraic tensor)}.$$

Both measures $\lambda_S^\eta(\mathfrak{n})$ and λ_S^η on $\mathfrak{X}_S^{0+}$ are naturally extended as linear functionals on $\mathcal{S}(\overline{\mathfrak{X}_S^{0+}})$ (cf. [47, Lemmas 13.14 and 13.16]).

Lemma 10.7. *We have the following.*

1. *Let A_{fin} denote the $\mathbb{C}$-vector space of all functions on $\overline{\mathfrak{X}_{S_{\mathrm{fin}}}^{0+}}$ generated by $\prod_{v \in S} Q_v(q_v^{-s_v/2} + q_v^{s_v/2})$ for any polynomials $Q_v[X] \in \mathbb{C}[X]$, $(v \in S_{\mathrm{fin}})$. Then, A_{fin} is dense in $C(\overline{\mathfrak{X}_{S_{\mathrm{fin}}}^{0+}})$ with respect to the topology by supremum norm.*
2. *The symbol A_∞ denotes the $\mathbb{C}$-vector space of all functions on $\overline{\mathfrak{X}_{\Sigma_\infty}^{0+}}$ generated by the functions*

$$\overline{\mathfrak{X}_{\Sigma_\infty}^{0+}} \ni \mathbf{s} = (s_v)_{v \in \Sigma_\infty} \mapsto \prod_{v \in \Sigma_\infty} Q_v(s_v^2) \exp((s_v^2 - 1)/4)$$

for any polynomials $Q_v(X) \in \mathbb{C}[X]$, $(v \in \Sigma_\infty)$. Then, A_∞ is dense in $\mathcal{S}(\overline{\mathfrak{X}_{\Sigma_\infty}^{0+}})$ with respect to the Fréchet topology determined by the semi-norms

$$p_{m,\mathbf{n}}(f) = \sup_{\mathbf{s} \in \overline{\mathfrak{X}_{\Sigma_\infty}^{0+}}} |\partial^{\mathbf{n}} f(\mathbf{s})|(1 + ||\mathbf{s}||^2)^m$$

for all $m \in \mathbb{N}_0$ and all $\mathbf{n} \in (\mathbb{N}_0)^{\Sigma_\infty}$. Here $\partial^{\mathbf{n}}$ denotes the higher order partial derivative $\prod_{v \in \Sigma_\infty} \partial^{n_v}/\partial s_v^{n_v}$ for any multi-index $n = (n_v)_{v \in \Sigma_\infty} \in (\mathbb{N}_0)^{\Sigma_\infty}$, and we put $||\mathbf{s}|| = \sum_{v \in \Sigma_\infty} |s_v|^2$.

Proof. To prove (1), we only have to use the Stone-Weierstrass theorem for the compact Hausdorff space $\overline{\mathfrak{X}^{0+}_{S_{\rm fin}}}$. The assertion (2) follows from [36, Theorem V.13 (p.143)] and [5, Lemma 9.3]. $\square$

10.3. Extraction of the new part. Let $\mathcal{I}^+_{S,\eta}$ be the set defined in §0.1. Set

$$\mathrm{AL}^*_{\mathrm{M}}(\mathfrak{n}; f) = \frac{1}{\mathrm{N}(\mathfrak{n})} \sum_{\pi \in \Pi^*_{\rm cus}(\mathfrak{n})} \frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{L^{S_\pi}(1,\pi,\mathrm{Ad})} f(\nu_S(\pi))$$

for any $\mathfrak{n} \in \mathcal{J}^+_{S,\eta}$ and any $f \in C_c(\overline{\mathfrak{X}^{0+}_S})$ (or $f = \alpha \in \mathcal{A}_S$). The convergence of the defining series is proved as follows. We note that $L(1/2,\pi)L(1/2,\pi\otimes\eta) \geqslant 0$ by [10] and that $\mathcal{G}(\eta)^{-1}\mathbb{P}^{\eta}(\pi; \mathbf{K}_0(\mathfrak{n})) \geqslant 0$ by Lemma 3.6. Furthermore, Lemma 3.6 gives us $w^{\eta}_{\mathfrak{n}}(\pi) = 1$ if $\pi \in \Pi_{\rm cus}(\mathfrak{n})$ satisfies $\mathfrak{f}_\pi = \mathfrak{n}$. Hence we have $|\mathrm{AL}^*_{\mathrm{M}}(\mathfrak{n}; f)| \leqslant \langle \lambda^{\eta}_S(\mathfrak{n}), |f|\rangle$. Here we note $\langle \lambda^{\eta}_S(\mathfrak{n}), |f|\rangle < \infty$ (cf. [47, Lemma 13.16]).

We extract the new part $\mathrm{AL}^*_{\mathrm{M}}(\mathfrak{n}; f)$ from $\langle \lambda^{\eta}_S(\mathfrak{n}), f\rangle$.

Theorem 10.8. *There exists a sufficiently small $\delta > 0$ such that*

$$\mathrm{AL}^*_{\mathrm{M}}(\mathfrak{n}; \alpha) = \nu(\mathfrak{n})\langle \lambda^{\eta}_S, \alpha\rangle + \mathcal{O}(\mathrm{N}(\mathfrak{n})^{-\delta})$$

holds for any $\alpha \in \mathcal{A}_S$ as $\mathrm{N}(\mathfrak{n}) \to \infty$ in $\mathfrak{n} \in \mathcal{I}^+_{S,\eta}$. Moreover, for any $f \in \mathcal{S}(\overline{\mathfrak{X}^{0+}_S})$, we have

$$\frac{1}{\nu(\mathfrak{n})}\mathrm{AL}^*_{\mathrm{M}}(\mathfrak{n}; f) \to \langle \lambda^{\eta}_S, f\rangle$$

as $\mathrm{N}(\mathfrak{n}) \to \infty$ in $\mathfrak{n} \in \mathcal{I}^+_{S,\eta}$. The limit for any $f \in \mathcal{S}(\overline{\mathfrak{X}^{0+}_S})$ as above is valid for any $f \in C_c(\mathfrak{X}^{0+}_S)$.

The proof of this theorem is given in §10.4. As a corollary, by Proposition 1.1, we have the following.

Corollary 10.9. *For any bounded Borel measure J of $\mathfrak{X}^{0+}_S$ with boundary λ^{η}_S-null, we have*

$$\frac{1}{\nu(\mathfrak{n})}\mathrm{AL}^*_{\mathrm{M}}(\mathfrak{n}; \mathrm{ch}_J) \to \int_J d\lambda^{\eta}_S$$

as $\mathrm{N}(\mathfrak{n}) \to \infty$ in $\mathfrak{n} \in \mathcal{I}^+_{S,\eta}$.

10.3.1. *The $\mathcal{N}$-transform.* We introduce $\mathcal{N}$-transform, which will be used in Part 2 and Part 3.

For any ideal $\mathfrak{c}$ and a place $v \in \Sigma_{\rm fin}$, set

$$\omega_v(\mathfrak{c}) = \begin{cases} 1 & (v \in S(\mathfrak{c})), \\ \dfrac{q_v+1}{q_v-1} & (v \notin S(\mathfrak{c})). \end{cases}$$

For any pair of integral ideals $\mathfrak{m}$ and $\mathfrak{b}$, define

$$\omega(\mathfrak{m}, \mathfrak{b}) = \delta(\mathfrak{m} \subset \mathfrak{b}) \prod_{v \in S(\mathfrak{b})} \omega_v(\mathfrak{m}\mathfrak{b}^{-1}).$$

Given an ideal $\mathfrak{n}$, let $\mathfrak{n}_0$ denote the largest square-free integral ideal dividing $\mathfrak{n}$; thus, there exists the unique integral ideal $\mathfrak{n}_1$ such that

$$\mathfrak{n} = \mathfrak{n}_0\mathfrak{n}_1^2.$$

Let $\mathcal{I}$ be a set of integral ideals such that if $\mathfrak{n} \in \mathcal{I}$, then all integral ideals $\mathfrak{m}$ dividing $\mathfrak{n}$ are elements of $\mathcal{I}$.

Proposition 10.10. *Let $B(\mathfrak{m})$ and $A(\mathfrak{m})$ be two arithmetic functions defined for ideals $\mathfrak{m} \in \mathcal{I}$. Then, the following two conditions are equivalent each other:*

(i) *For any $\mathfrak{n} \in \mathcal{I}$,*

$$B(\mathfrak{n}) = \sum_{\mathfrak{b} | \mathfrak{n}_1} \omega(\mathfrak{n}, \mathfrak{b}^2)\, A(\mathfrak{n}\mathfrak{b}^{-2}).$$

(ii) *For any* $\mathfrak{n} \in \mathcal{I}$,

$$A(\mathfrak{n}) = \sum_{I \subset S(\mathfrak{n}_1)} (-1)^{\#I} \{ \prod_{v \in I \cap S_1(\mathfrak{n}_1)} \omega_v(\mathfrak{n}_0) \} \, B(\mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2}).$$

Proof. We show that (i) implies (ii). By substituting (i), the right-hand side of (ii) becomes

$$\sum_{I \subset S(\mathfrak{n}_1)} (-1)^{\#I} \{ \prod_{v \in I \cap S_1(\mathfrak{n}_1)} \omega_v(\mathfrak{n}_0) \} \{ \sum_{\mathfrak{b} | \mathfrak{n}_1 \prod_{v \in I} \mathfrak{p}_v^{-1}} \omega(\mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2}, \mathfrak{b}^2) \, A(\mathfrak{n} \mathfrak{b}^{-2} \prod_{v \in I} \mathfrak{p}_v^{-2}) \}$$

$$= \sum_{\mathfrak{b}_1 | \mathfrak{n}_1} \left\{ \sum_{I \subset S(\mathfrak{n}_1 \mathfrak{b}_1^{-1})} (-1)^{\#I} \omega \left(\mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2}, \mathfrak{n}_1^2 \mathfrak{b}_1^{-2} \prod_{v \in I} \mathfrak{p}_v^{-2} \right) \prod_{v \in I \cap S_1(\mathfrak{n}_1) \cap S(\mathfrak{n}_1 \mathfrak{b}_1^{-1})} \omega_v(\mathfrak{n}_0) \right\} A(\mathfrak{n}_0 \mathfrak{b}_1^2)$$

Here to have the equality, we made the substitution $\mathfrak{b}_1 = \mathfrak{n}_1 \mathfrak{b}^{-1} \prod_{v \in I} \mathfrak{p}_v^{-1}$. If $\mathfrak{b}_1 = \mathfrak{n}_1$, the term inside the bracket is 1 obviously; otherwise it equals

$$\sum_{I \subset S(\mathfrak{n}_1 \mathfrak{b}_1^{-1})} (-1)^{\#I} \prod_{v \in S(\mathfrak{n}_1 \mathfrak{b}_1^{-1} \prod_{v \in I} \mathfrak{p}_v^{-1}) - S(\mathfrak{n}_0 \mathfrak{b}_1^2)} \frac{q_v + 1}{q_v - 1} \prod_{v \in I \cap S(\mathfrak{n}_1 \mathfrak{b}_1^{-1}) \cap S_1(\mathfrak{n}_1) - S(\mathfrak{n}_0)} \frac{q_v + 1}{q_v - 1}$$

$$= \sum_{I \subset S(\mathfrak{n}_1 \mathfrak{b}_1^{-1})} (-1)^{\#I} \prod_{v \in [(I - S_1(\mathfrak{n}_1 \mathfrak{b}_1^{-1})) \cup (S(\mathfrak{n}_1 \mathfrak{b}_1^{-1}) - I)] - S(\mathfrak{n}_0 \mathfrak{b}_1^2)} \frac{q_v + 1}{q_v - 1} \prod_{v \in I \cap S_1(\mathfrak{n}_1 \mathfrak{b}_1^{-1}) - S(\mathfrak{n}_0 \mathfrak{b}_1^2)} \frac{q_v + 1}{q_v - 1}$$

$$= \sum_{I \subset S(\mathfrak{n}_1 \mathfrak{b}_1^{-1})} (-1)^{\#I} \prod_{v \in (S(\mathfrak{n}_1 \mathfrak{b}_1^{-1}) - I) - S(\mathfrak{n}_0 \mathfrak{b}_1^2)} \frac{q_v + 1}{q_v - 1} \prod_{v \in I - S(\mathfrak{n}_0 \mathfrak{b}_1^2)} \frac{q_v + 1}{q_v - 1}$$

$$= \prod_{v \in S(\mathfrak{n}_1 \mathfrak{b}_1^{-1})} (\omega_v(\mathfrak{n}_0 \mathfrak{b}_1^2) - \omega_v(\mathfrak{n}_0 \mathfrak{b}_1^2)),$$

which is zero by $S(\mathfrak{n}_1 \mathfrak{b}_1^{-1}) \neq \emptyset$. We can prove that (ii) implies (i) in a similar fashion. $\square$

Set $\iota(\mathfrak{m}) = [\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{m})] = \prod_{v \in S(\mathfrak{m})} (1 + q_v) q_v^{\mathrm{ord}_v(\mathfrak{m}) - 1}$ for any ideal $\mathfrak{m}$ of $\mathfrak{o}$.

Definition 10.11. *For an arithmetic function $B : \mathcal{I} \to \mathbb{C}$, we define its $\mathcal{N}$-transform $\mathcal{N}[B] : \mathcal{I} \to \mathbb{C}$ by the formula*

$$\mathcal{N}[B](\mathfrak{n}) = \sum_{I \subset S(\mathfrak{n}_1)} (-1)^{\#I} \{ \prod_{v \in I \cap S_1(\mathfrak{n}_1)} \omega_v(\mathfrak{n}_0) \} \, \frac{\iota(\mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2})}{\iota(\mathfrak{n})} \, B(\mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2}).$$

Lemma 10.12. *For $t \in \mathbb{C}$, let N^t be the arithmetic function $\mathfrak{n} \mapsto \mathrm{N}(\mathfrak{n})^t$ on $\mathcal{I}$. For any ideal $\mathfrak{n}$, we have*

$$\mathcal{N}[\mathrm{N}^t](\mathfrak{n}) = \mathrm{N}(\mathfrak{n})^t \, \{ \prod_{v \in S(\mathfrak{n}_1) - S_2(\mathfrak{n})} (1 - q_v^{-2(1+t)}) \} \, \{ \prod_{v \in S_2(\mathfrak{n})} (1 - (1 - q_v^{-1})^{-1} \, q_v^{-2(1+t)}) \}.$$

In particular, $\mathcal{N}[1]$ is equal to

$$\nu(\mathfrak{n}) = \{ \prod_{v \in S(\mathfrak{n}_1) - S_2(\mathfrak{n})} (1 - q_v^{-2}) \} \, \{ \prod_{v \in S_2(\mathfrak{n})} (1 - (q_v^2 - q_v)^{-1}) \}.$$

Proof. For any subset $I \subset S(\mathfrak{n})$, we have

$$\frac{\iota(\mathfrak{n} \prod_{v \in I} \mathfrak{p}_v^{-2})}{\iota(\mathfrak{n})} = \prod_{v \in I} q_v^{-2} \prod_{v \in I \cap S_2(\mathfrak{n})} (1 + q_v^{-1})^{-1}.$$

Therefore, we obtain

$$\begin{aligned}
&\sum_{I\subset S(\mathfrak{n}_1)}(-1)^{\#I}\{\prod_{v\in I\cap S_1(\mathfrak{n}_1)}\omega_v(\mathfrak{n}_0)\}\frac{\iota(\mathfrak{n}\prod_{v\in I}\mathfrak{p}_v^{-2})}{\iota(\mathfrak{n})}\mathrm{N}(\mathfrak{n}\prod_{v\in I}\mathfrak{p}_v^{-2})^t\\
&=\mathrm{N}(\mathfrak{n})^t\sum_{I\subset S(\mathfrak{n}_1)}(-1)^{\#I}\{\prod_{v\in I\cap S_2(\mathfrak{n})}\frac{q_v+1}{q_v-1}\}\prod_{v\in I\cap S_2(\mathfrak{n})}(1+q_v^{-1})^{-1}\{\prod_{v\in I}q_v^{-2(1+t)}\}\\
&=\mathrm{N}(\mathfrak{n})^t\sum_{I\subset S(\mathfrak{n}_1)}(-1)^{\#I}\prod_{v\in I\cap S_2(\mathfrak{n})}(1-q_v^{-1})^{-1}\{\prod_{v\in I}q_v^{-2t}\}\\
&=\mathrm{N}(\mathfrak{n})^t\{\prod_{v\in S(\mathfrak{n}_1)-S_2(\mathfrak{n})}(1-q_v^{-2(1+t)})\}\{\prod_{v\in S_2(\mathfrak{n})}(1-(1-q_v^{-1})^{-1}\,q_v^{-2(1+t)})\}.
\end{aligned}$$

□

For any arithmetic function $B:\mathcal{I}\to\mathbb{C}$, we define another function $\mathcal{N}^+[B]$ by setting

$$\mathcal{N}^+[B](\mathfrak{n})=\sum_{I\subset S(\mathfrak{n}_1)}\{\prod_{v\in I\cap S_1(\mathfrak{n}_1)}\omega_v(\mathfrak{n}_0)\}\frac{\iota(\mathfrak{n}\prod_{v\in I}\mathfrak{p}_v^{-2})}{\iota(\mathfrak{n})}B(\mathfrak{n}\prod_{v\in I}\mathfrak{p}_v^{-2})$$

for $\mathfrak{n}=\mathfrak{n}_0\mathfrak{n}_1^2\in\mathcal{I}$. In a similar way to Lemma 10.12, we have

$$\mathcal{N}^+[\mathrm{N}^t]=\mathrm{N}(\mathfrak{n})^t\{\prod_{v\in S(\mathfrak{n}_1)-S_2(\mathfrak{n})}(1+q_v^{-2(t+1)})\}\{\prod_{v\in S_2(\mathfrak{n})}(1+(1-q_v^{-1})^{-1}q_v^{-2(t+1)})\}\tag{10.1}$$

for any $t\in\mathbb{R}$.

Lemma 10.13. *Let $c>0$. For any sufficiently small $\epsilon>0$, we have*

$$\mathcal{N}^+[\mathrm{N}^{-c+\epsilon}](\mathfrak{n})\ll_\epsilon \mathrm{N}(\mathfrak{n})^{-\inf(c,1)+3\epsilon},\quad \mathfrak{n}\in\mathcal{I}.$$

Proof. From $\mathrm{N}(\mathfrak{n})^{-c+\epsilon}\leqslant\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}$, we have $\mathcal{N}^+[\mathrm{N}^{-c+\epsilon}](\mathfrak{n})\leqslant\mathcal{N}^+[\mathrm{N}^{-\inf(c,1)+\epsilon}](\mathfrak{n})$ obviously. Let us set $t=-\inf(c,1)+\epsilon$ and examine the right-hand side of the formula (10.1). We note that $t+1=1-\inf(c,1)+\epsilon\geqslant\epsilon>0$. The set $P(\epsilon)=\{v\in\Sigma_{\mathrm{fin}}\,|\,1-q_v^{-1}<q_v^{-\epsilon}\}$ is a finite set. For $v\in S_2(\mathfrak{n})-P(\epsilon)$, we have $(1-q_v^{-1})^{-1}\leqslant q_v^{\epsilon}$ and $q_v^{-2(t+1)}\leqslant q_v^{-2\epsilon}$; by these, the factor $1+(1-q_v^{-1})q_v^{-2(t+1)}$ is bounded by $1+q_v^{-\epsilon}$. For $v\in S(\mathfrak{n}_1)-S_2(\mathfrak{n})$ or $v\in S_2(\mathfrak{n})\cap P(\epsilon)$, we simply apply $q_v^{-2(t+1)}\leqslant q_v^{-2\epsilon}$. Thus,

$$\mathcal{N}^+[\mathrm{N}^t](\mathfrak{n})\leqslant\mathrm{N}(\mathfrak{n})^t\{\prod_{v\in S(\mathfrak{n}_1)-S_2(\mathfrak{n})}(1+q_v^{-2\epsilon})\}\{\prod_{v\in P(\epsilon)}(1+(1-q_v^{-1})^{-1}q_v^{-2\epsilon})\}\{\prod_{v\in S_2(\mathfrak{n})-P(\epsilon)}(1+q_v^{-\epsilon})\}.\tag{10.2}$$

In the right-hand side, the second factor is independent of $\mathfrak{n}$. The first and the last factors combined are estimated as

$$\begin{aligned}
\{\prod_{v\in S(\mathfrak{n}_1)-S_2(\mathfrak{n})}(1+q_v^{-2\epsilon})\}\{\prod_{v\in S_2(\mathfrak{n})-P(\epsilon)}(1+q_v^{-\epsilon})\}&\leqslant\{\prod_{v\in S(\mathfrak{n})}(1+q_v^{-\epsilon})\}^2\\
&\ll_\epsilon\{\prod_{v\in S(\mathfrak{n})}q_v^{\epsilon}\}^2\leqslant\mathrm{N}(\mathfrak{n})^{2\epsilon}.
\end{aligned}$$

Hence there exists a constant $C(\epsilon)>0$ dependent of ϵ such that (10.2) is less than $C(\epsilon)\,\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+3\epsilon}$ for any $\mathfrak{n}\in\mathcal{I}$. □

10.3.2. *The totally inert case over* $\mathfrak{n}$. Let $\mathcal{J}_{S,\eta}$ be the set of all ideals $\mathfrak{n}$ relatively prime to $S \cup S(\mathfrak{f}_\eta)$. Let $\mathcal{I}_{S,\eta}$ be the monoid of ideals generated by prime ideals $\mathfrak{p}_v \cap \mathfrak{o}$ such that $v \in \Sigma_{\text{fin}} - S \cup S(\mathfrak{f}_\eta)$ and $\tilde{\eta}(\mathfrak{p}_v \cap \mathfrak{o}) = -1$. Note that $\mathcal{I}_{S,\eta}$ is a submonoid of $\mathcal{J}_{S,\eta}$ and that $\mathcal{I}_{S,\eta} = \mathcal{I}_{S,\eta}^+ \cup \mathcal{I}_{S,\eta}^-$ (cf. §0.1). We relate $\mathrm{AL}^*_{\mathrm{M}}(\mathfrak{n};\alpha)$ to the $\mathcal{N}$-transforms of arithmetic functions $\langle \lambda_S^\eta(\cdot), \alpha\rangle$ on $\mathcal{I}_{S,\eta}$. We remark that an ideal $\mathfrak{n} \in \mathcal{I}_{S,\eta}$ satisfies the condition

$$\eta_v(\varpi_v) = -1, \qquad v \in S(\mathfrak{n}).$$

This means that the quadratic extension of F corresponding to η is inert over all places dividing $\mathfrak{n}$.

Lemma 10.14. *Let $\mathfrak{n} \in \mathcal{I}_{S,\eta}$. Then, for any $\pi \in \Pi_{\mathrm{cus}}(\mathfrak{n})$, we have $w_{\mathfrak{n}}^\eta(\pi) = 0$ unless $\mathfrak{n}\mathfrak{f}_\pi^{-1} = \mathfrak{b}^2$ for some integral ideal $\mathfrak{b}$, in which case*

$$w_{\mathfrak{n}}^\eta(\pi) = \omega(\mathfrak{n}, \mathfrak{n}\mathfrak{f}_\pi^{-1}).$$

Proof. Let $v \in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})$ and set $k_v = \mathrm{ord}_v(\mathfrak{n}\mathfrak{f}_\pi^{-1})$. From Lemma 3.6,

$$r(\pi_v, \eta_v, k_v) = \frac{1+(-1)^{k_v}}{2} \times \begin{cases} 1 & (c(\pi_v) \geqslant 1), \\ \dfrac{q_v+1}{q_v-1} & (c(\pi_v) = 0). \end{cases}$$

Thus $r(\pi_v, \eta_v, k_v) = 0$ unless $k_v = \mathrm{ord}_v(\mathfrak{n}\mathfrak{f}_\pi^{-1})$ is even. □

For any fixed $\alpha \in \mathcal{A}_S$ and $\mathfrak{n} \in \mathcal{I}_{S,\eta}$, set

$$\mathrm{AL}^*_{\mathrm{M}}(\mathfrak{n};\alpha) = \frac{1}{\mathrm{N}(\mathfrak{n})} \sum_{\pi \in \Pi^*_{\mathrm{cus}}(\mathfrak{n})} \frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{L^{S_\pi}(1,\pi,\mathrm{Ad})} \alpha(\nu_S(\pi)).$$

Lemma 10.15. *For any $\mathfrak{n} \in \mathcal{I}_{S,\eta}$,*

$$\langle \lambda_S^\eta(\mathfrak{n}), \alpha\rangle = \sum_{\mathfrak{b}} \omega(\mathfrak{n}, \mathfrak{b}^2)\, \frac{\iota(\mathfrak{n}\mathfrak{b}^{-2})}{\iota(\mathfrak{n})}\, \mathrm{AL}^*_{\mathrm{M}}(\mathfrak{n}\mathfrak{b}^{-2};\alpha),$$

where $\mathfrak{b}$ runs through all the integral ideals such that $\mathfrak{n} \subset \mathfrak{b}^2$.

Proof. This follows immediately from Lemma 10.14. □

Lemma 10.16. *For any $\mathfrak{n} \in \mathcal{I}_{S,\eta}$,*

$$\mathrm{AL}^*_{\mathrm{M}}(\mathfrak{n};\alpha) = \mathcal{N}[\langle \lambda_S^\eta(\cdot), \alpha\rangle](\mathfrak{n}).$$

Proof. By Lemma 10.15, we obtain the formula by applying Proposition 10.10 with $B(\mathfrak{m}) = \iota(\mathfrak{m})\,\langle \lambda_S^\eta(\mathfrak{m}), \alpha\rangle$ and $A(\mathfrak{m}) = \iota(\mathfrak{m})\,\mathrm{AL}^*_{\mathrm{M}}(\mathfrak{m};\alpha)$ both defined for $\mathfrak{m} \in \mathcal{I}_{S,\eta}$. □

10.4. Proof of Theorem 0.2. We prove Theorem 10.8, from which Theorem 0.2 follows immediately. For a fixed $\alpha \in \mathcal{A}_S$, by Proposition 10.5 and Lemmas 10.12, 10.13 and 10.16, we have

$$\begin{aligned}\mathrm{AL}^*_{\mathrm{M}}(\mathfrak{n};\alpha) =& \mathcal{N}[\langle \lambda_S^\eta(\cdot), \alpha\rangle](\mathfrak{n}) = \mathcal{N}[1](\mathfrak{n}) \times \langle \lambda_S^\eta, \alpha\rangle + \mathcal{O}(\mathcal{N}^+[\mathrm{N}^{-\delta+\epsilon}](\mathfrak{n})) \\ =& \nu(\mathfrak{n})\langle \lambda_S^\eta, \alpha\rangle + \mathcal{O}(\mathrm{N}^{-\delta+3\epsilon}(\mathfrak{n})).\end{aligned}$$

for $\mathfrak{n} \in \mathcal{I}_{S,\eta}^+$ with sufficiently small $\delta > 0$ and $\epsilon > 0$. Hence, we obtain the first assertion of Theorem 10.8.

With the aid of the proof of [47, Theorem 13.17] and the first assertion of Theorem 10.8, for any $f \in \mathcal{S}(\overline{\mathfrak{X}_S^{0+}})$ we have

$$\frac{1}{\nu(\mathfrak{n})}\mathrm{AL}^*_{\mathrm{M}}(\mathfrak{n};f) \to \langle \lambda_S^\eta, f\rangle$$

as $\mathrm{N}(\mathfrak{n}) \to \infty$ in $\mathfrak{n} \in \mathcal{I}_{S,\eta}^{+}$. Indeed, by Lemma 10.7, for any $f \in \mathcal{S}(\overline{\mathfrak{X}_S^{0+}})$, any $\epsilon > 0$ and any $m \in \mathbb{N}$, there exists a function α on $\overline{\mathfrak{X}_S^{0+}}$ of the form

$$\alpha(\mathbf{s}) = \sum_{j=1}^{N} \prod_{v \in \Sigma_\infty} Q_{v,j}(s_v^2) \exp((s_v^2 - 1)/4) \prod_{v \in S_{\mathrm{fin}}} Q_{v,j}(q_v^{-s_v/2} + q_v^{s_v/2}), \qquad \mathbf{s} \in \overline{\mathfrak{X}_S^{0+}}$$

for $N \in \mathbb{N}$ and a family of polynomials $Q_{v,j}(X) \in \mathbb{C}[X], (v \in S,\ 1 \leqslant j \leqslant N)$ such that

$$\sup_{\mathbf{s} \in \overline{\mathfrak{X}_S^{0+}}} |f(\mathbf{s}) - \alpha(\mathbf{s})|(1 + ||\mathbf{s}||^2)^m < \epsilon.$$

We regard naturally α as a function on $\mathfrak{X}_S$; then α is an element of $\mathcal{A}_S$. From this, the argument in the proof of [47, Lemma 13.17] is valid by using $\mathrm{AL}_{\mathrm{M}}^{*}(\mathfrak{n}; -)$ in place of $\lambda_S^{\eta}(\mathfrak{n})$. As a consequence, we obtain the second assertion of Theorem 10.8.

The assertion for $f \in \mathcal{S}(\overline{\mathfrak{X}_S^{0+}})$ deduces to that for $f \in C_c(\mathfrak{X}_S^{0+})$ in the following way. It suffices to prove the assertion for $f \in C_c(\overline{\mathfrak{X}_S^{0+}})$.

Take any $f \in C_c(\overline{\mathfrak{X}_S^{0+}})$ and any $\epsilon > 0$. Fix a locally compact bounded open subset U of $\overline{\mathfrak{X}_S^{0+}}$ such that $\mathrm{supp}(f) \subset U$. We may suppose $U = U_\infty \times U_{\mathrm{fin}}$ for some $U_\infty \subset \overline{\mathfrak{X}_\infty^{0+}}$ and $U_{\mathrm{fin}} \subset \overline{\mathfrak{X}_{S_{\mathrm{fin}}}^{0+}}$, where both U_∞ and U_{fin} are locally compact bounded open subsets. Then, we have $f|_U \in C_c(U)$. Let $C_c^\infty(U_\infty)$ be the space of all compactly supported functions h on U_∞ such that $h(s) = \varphi((\frac{1-s_v^2}{4})_{v \in \Sigma_\infty})$ for some C^∞-function φ on the set $\{x = (x_v)_{v \in \Sigma_\infty} \in (\mathbb{R}_{\geqslant 0})^{\Sigma_\infty} \mid (\sqrt{1 - 4x_v})_{v \in \Sigma_\infty} \in U_\infty\}$. By the Stone-Weierstrass theorem, $C_c^\infty(U_\infty) \otimes C(U_{\mathrm{fin}})$ is dense in $C_c(U)$ with respect to the topology by supremum norm. Thus there exists $g_\epsilon \in C_c^\infty(U_\infty) \otimes C(U_{\mathrm{fin}})$ satisfying

$$\sup_{\mathbf{s} \in U} |f(\mathbf{s}) - g_\epsilon(\mathbf{s})| < \epsilon.$$

By the extension by zero, the function g_ϵ is naturally extended as an element of $C_c^\infty(\overline{\mathfrak{X}_{\Sigma_\infty}^{0+}}) \otimes C(\overline{\mathfrak{X}_{S_{\mathrm{fin}}}^{0+}})$, which is also denoted by g_ϵ. Then, we have $g_\epsilon \in \mathcal{S}(\overline{\mathfrak{X}_S^{0+}})$.

From this and the second assertion of Theorem 10.8, there exists $M > 0$ such that for any $\mathfrak{n} \in \mathcal{I}_{S,\eta}^{+}$ with $\mathrm{N}(\mathfrak{n}) > M$, we have

$$|\nu(\mathfrak{n})^{-1} \mathrm{AL}_{\mathrm{M}}^{*}(\mathfrak{n}; g_\epsilon) - \langle \lambda_S^{\eta}, g_\epsilon \rangle| < \epsilon.$$

In the same way as [47, Lemmas 13.14 and 13.16], we have the estimates

$$|\nu(\mathfrak{n})^{-1} \mathrm{AL}_{\mathrm{M}}^{*}(\mathfrak{n}; f - g_\epsilon)| < C \sup_{\mathbf{s} \in U} (1 + ||\mathbf{s}||^2)^m \, \epsilon$$

and

$$|\langle \lambda_S^{\eta}, f - g_\epsilon \rangle| < C \sup_{\mathbf{s} \in U} (1 + ||\mathbf{s}||^2)^m \, \epsilon,$$

where $C > 0$ and $m \in \mathbb{N}$ are independent of $\mathfrak{n} \in \mathcal{I}_{S,\eta}^{+}$, the function f and $\epsilon > 0$. As a consequence, for any $\mathfrak{n} \in \mathcal{I}_{S,\eta}^{+}$ with $\mathrm{N}(\mathfrak{n}) > M$, we obtain

$$\begin{aligned} |\nu(\mathfrak{n})^{-1} \mathrm{AL}_{\mathrm{M}}^{*}(\mathfrak{n}; f) - \langle \lambda_S^{\eta}, f \rangle| &\leqslant |\nu(\mathfrak{n})^{-1} \mathrm{AL}_{\mathrm{M}}^{*}(\mathfrak{n}; f - g_\epsilon)| + |\nu(\mathfrak{n})^{-1} \mathrm{AL}_{\mathrm{M}}^{*}(\mathfrak{n}; g_\epsilon) - \langle \lambda_S^{\eta}, g_\epsilon \rangle| + |\langle \lambda_S^{\eta}, g_\epsilon - f \rangle| \\ &< \{1 + 2\, C \sup_{\mathbf{s} \in U} (1 + ||\mathbf{s}||^2)^m\} \, \epsilon. \end{aligned}$$

This completes the proof of the third assertion of Theorem 10.8. □

10.5. Proof of Theorem 0.3. We may assume that J_v for each $v \in \Sigma_\infty$ is bounded. Let $\mathbb{J}$ be the set of all $(\nu_v)_{v \in S} \in \mathfrak{X}_S^{0}$ such that $(1 - \nu_v^2)/4 \in J_v$ for all $v \in \Sigma_\infty$ and $q_v^{-\nu_v/2} + q_v^{\nu_v/2} \in J_v$ for all $v \in S_{\mathrm{fin}}$. Then, $\mathbb{J}$ is a bounded Borel set of $\mathfrak{X}_S^{0+}$ whose boundary is λ_S^{η}-null. Hence Theorem 0.3 follows from Corollary 10.9. □

10.6. Remarks on Theorems 0.4 and 0.5. Both theorems 0.4 and 0.5 are proved in the same way as [47, Theorem 1.3, Corollary 1.4] since we can generalize [47, Theorem 14.1] to the case of arbitrary levels by using the relative trace formula explained in §10. We remark that $|L_{\mathrm{fin}}^{S_\pi}(1,\pi,\mathrm{Ad})| \asymp |L_{\mathrm{fin}}(1,\pi,\mathrm{Ad})| \ll (1+||\nu_{\Sigma_\infty}(\pi)||)^\epsilon$ is due to [23]. Although [29] is refered to in [47], the Rankin-Selberg condition (A5) in [29] is valid only for general L-functions over $\mathbb{Q}$.

Part 2. Relative trace formulas for holomorphic Hilbert modular forms

11. Holomorphic Shintani functions on $\mathrm{GL}(2,\mathbb{R})$

11.1. Discrete series of $\mathrm{PGL}(2,\mathbb{R})$**.** For $n \in \mathbb{Z}$, let τ_n be the character of $\mathrm{SO}(2,\mathbb{R})$ defined by

$$\tau_n(k_\theta) = e^{in\theta}, \qquad \theta \in \mathbb{R}.$$

Let $l \geqslant 2$ be an even integer. Recall that there correspond discrete series representations D_l^+ and D_l^- of $\mathrm{SL}(2,\mathbb{R})$ such that $D_l^\pm|\mathrm{SO}(2,\mathbb{R})$ is a direct sum of characters τ_n for all $n \in \pm(l+2\mathbb{N}_0)$. We have a unitary representation D_l of $\mathrm{GL}(2,\mathbb{R})$ such that (a) D_l has the trivial central character and (b) $D_l|\mathrm{SL}(2,\mathbb{R}) = D_l^+ \oplus D_l^-$. We call D_l the discrete series representation of $\mathrm{PGL}(2,\mathbb{R})$ of minimal $\mathrm{SO}(2,\mathbb{R})$-type l.

11.2. Shintani functions. Let $f(\tau)$ be a cusp form on the upper half plane satisfying the modularity condition $f((a\tau+b)/(c\tau+d)) = (c\tau+d)^l f(\tau)$ for any matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ in a fixed congruence subgroup Γ of $\mathrm{PSL}(2,\mathbb{Z})$. Then it is lifted to a left Γ-invariant function $\tilde{f}$ on the group $\mathrm{GL}(2,\mathbb{R})$ by setting

$$\tilde{f}(g) = (\det g)^{l/2}(ci+d)^{-l} f\left(\tfrac{ai+b}{ci+d}\right) \times \delta(\det g > 0), \qquad g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}(2,\mathbb{R}).$$

Let $\tilde{f}_c$ be the complex conjugate of $\tilde{f}$. Then, $\tilde{f}_c$ satisfies the conditions

$$\tilde{f}_c(gk_\theta) = \tau_l(k_\theta)\,\tilde{f}_c(g), \ \ (\forall k_\theta \in \mathrm{SO}(2,\mathbb{R})), \qquad [R(\overline{W})\tilde{f}_c](g) = 0.$$

Since $\mathrm{Ad}(k_\theta)\overline{W} = e^{-2i\theta}\,\overline{W}$ in any $(\mathfrak{gl}_2(\mathbb{R}), \mathrm{O}(2,\mathbb{R}))$-module (π, V), we have $\pi(\overline{W})V[\tau_l] \subset V[\tau_{l-2}]$, where

$$V[\tau_l] = \{v \in V \,|\, \pi(k_\theta)v = e^{il\theta}\, v\ (\forall k_\theta \in \mathrm{SO}(2,\mathbb{R}))\,\}.$$

Let V be the $(\mathfrak{gl}_2(\mathbb{R}), \mathrm{O}(2,\mathbb{R}))$-submodule of the regular representation $L^2(\Gamma\backslash\mathrm{GL}(2,\mathbb{R}))$ generated by $\tilde{f}_c$. Then the condition above, or equivalently $\tilde{f}_c \in V[\tau_l]$ and $R(\overline{W})\tilde{f}_c = 0$, tells us that inside the module V (which is a finite sum of discrete series D_l) the vector $\tilde{f}_c$ is extremal. For $z \in \mathbb{C}$, let χ_z be the quasi-character of the diagonal split torus T defined by $\chi_z\left(\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right]\right) = |t_1/t_2|^z$. The integral

$$\phi(g) = \int_{\Gamma\cap T\backslash T} \tilde{f}_c(hg)\,\chi_{-z}(h)\,dh, \qquad g \in \mathrm{GL}(2,\mathbb{R}),$$

often called the (T,χ_z)-period integral of $\tilde{f}_c$, satisfies the following two conditions:

- $\phi\left(\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] g\,k_\theta\right) = |t_1/t_2|^z\,\tau_l(k_\theta)\,\phi(g)$ for all $\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] \in T$ and $\theta \in \mathbb{R}$,
- $R(\overline{W})\phi = 0$.

A function satisfying these conditions is called a holomorphic Shintani function of weight l. The next proposition tells that these conditions determine the function $\phi(g)$ uniquely up to a constant multiple.

Proposition 11.1. [11, Proposition 5.3] *Let $z \in \mathbb{C}$. For each even integer $l \geqslant 2$, there exists a unique $\mathbb{C}$-valued C^∞-function $\Psi^{(z)}(l;-)$ on $\mathrm{GL}(2,\mathbb{R})$ with the properties:*

(S-i) *It satisfies the equivariance condition*

$$\Psi^{(z)}\left(l; \left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] g\,k_\theta\right) = |t_1/t_2|^z\,\tau_l(k_\theta)\,\Psi^{(z)}(l;g) \quad \text{for all } \left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] \in T \text{ and } \theta \in \mathbb{R}.$$

(S-ii) *It satisfies the differential equation*

$$R(\overline{W})\,\Psi^{(z)}(l;-) = 0.$$

(S-iii) $\Psi^{(z)}(l;1_2) = 1$.

We have the explicit formula

$$\Psi^{(z)}(l;a_r) = 2^{-l/2}\,(-y)^{(2z-l)/4}\,(1-y)^{l/2} \qquad \text{with } y = \left(\frac{e^{2r}-i}{e^{2r}+i}\right)^2.$$

We remark that all values $\Psi^{(z)}(l;a_r), r \in \mathbb{R}$ characterize the function $\Psi^{(z)}(l;-)$ by (S-i) and the decomposition $\mathrm{GL}(2,\mathbb{R}) = T\,\{a_r | r \in \mathbb{R}\}\,\mathrm{SO}(2,\mathbb{R})$ (cf. [11, Lemma 3.1]).

Lemma 11.2. *Let $\Psi^{(z)}(l;-)$ be as in Proposition 11.1. Then,*

$$\Psi^{(z)}\left(l; \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right]\right) = (1+ix)^{z-l/2}, \qquad x \in \mathbb{R}.$$

Proof. By a direct computation, $\left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right] a_r\, k_\theta$ with

$$t = (1+x^2)^{1/4}, \qquad \cosh 2r = (1+x^2)^{1/2}, \qquad \sinh 2r = x,$$

$$e^{i\theta} = \frac{(\sqrt{1+x^2}+1)^{1/2}}{\sqrt{2}\,(1+x^2)^{1/4}} \left(1 - \frac{ix}{\sqrt{1+x^2}+1}\right),$$

and $y = \frac{x-i}{x+i}$, $1-y = \frac{2i}{x+i}$. Using these, we have the desired formula by a direct computation. $\square$

Lemma 11.3. *We have the estimate*

$$\left|\Psi^{(z)}\left(l; \left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] a_r k\right)\right| \leqslant 2^{-l/2}\,|t_1/t_2|^{\mathrm{Re}(z)} e^{\pi|\mathrm{Im}(z)|/2} (\cosh 2r)^{-l/2}$$

for any $t_1, t_2 \in \mathbb{R}^\times$, $r \in \mathbb{R}$ and $k \in \mathrm{SO}(2,\mathbb{R})$.

Proof. Set $y = \left(\frac{e^{2r}-i}{e^{2r}+i}\right)^2$. Then,

$$y = \left(\tanh 2r - \frac{i}{\cosh 2r}\right)^2 = 1 - \frac{2}{\cosh^2 2r} - \frac{2i\tanh 2r}{\cosh 2r}.$$

Hence, by a direct computation, we have $|1-y| = (\cosh 2r)^{-1}$. Furthermore, by $|y| = 1$, we have $|(-y)^{(2z-l)/4}| \leqslant e^{\pi|\mathrm{Im}(z)|/2}$. This completes the proof. $\square$

11.3. An inner product formula of Shintani functions. For an even integer $l \geqslant 2$ and $z \in \mathbb{C}$, let us consider the integral

$$C_l(z) = \int_1^\infty \left\{ \left(-\left(\frac{u-i}{u+i}\right)^2\right)^z + \left(-\left(\frac{u+i}{u-i}\right)^2\right)^z \right\} (1+u^2)^{1-l}\, u^{l-2}\, du.$$

Lemma 11.4. *The integral $C_l(z)$ converges absolutely. It has the following properties.*

(i) *The function $z \mapsto C_l(z)$ is entire and satisfies the functional equation*

$$C_l(-z) = C_l(z).$$

(ii) *The value at $z = 0$ is given by*

$$C_l(0) = 2^{-1}\Gamma((l-1)/2)^2\Gamma(l-1)^{-1} = 2^{3-2l}\pi\Gamma(l-1)\Gamma(l/2)^{-2}.$$

(iii) *The estimate*

$$|C_l(z)| \leqslant C_l(0)\, \exp(\pi|\mathrm{Im}(z)|), \qquad z \in \mathbb{C}$$

holds.

Proof. By the variable change $v^{-1} = 1+u^2$, we have

$$C_l(0) = 2\int_1^\infty (1+u^2)^{1-l} u^{l-2}\, du = 2^{-1}\int_0^1 (1-v)^{(l-3)/2} v^{(l-3)/2}\, dv = 2^{-1}\Gamma((l-1)/2)^2\Gamma(l-1)^{-1}$$

as desired in (ii). Remark that the second equality in (ii) is obtained by the duplication formula. Since $w = -((u-i)/(u+i))^2$ satisfies $|w| = 1$, by definition, we have $w^z = \exp(i\theta z)$ with $\theta \in (-\pi, \pi]$. Thus, $|w^z| = \exp(-\mathrm{Im}(z)\,\theta) \leqslant \exp(\pi|\mathrm{Im} z|)$, by which (iii) is immediate. From definition, we have the relation $w^{-z} = (w^{-1})^z$, which shows the functional equation in (i). $\square$

The inner product of Shintani functions $\Psi^{(z)}(l;-)$ and $\Psi^{(-\bar{z})}(l;-)$ is given as follows.

Proposition 11.5. *We have*

$$\int_{T\backslash \mathrm{GL}(2,\mathbb{R})} \Psi^{(z)}(l;g)\,\overline{\Psi^{(-\bar z)}(l;g)}\,dg = 2^{l-1}\,C_l(z).$$

Proof. Set $f(g) = \Psi^{(z)}(l;g)\,\overline{\Psi^{(-\bar z)}(l;g)}$. We have

$$\int_{T\backslash \mathrm{GL}(2,\mathbb{R})} f(g)\,dg = 2\int_{\mathbb{R}} f(a_r)\cosh 2r\,dr$$

by the formula [47, (3.3)]. From Proposition 11.1,

$$f(a_r) = 2^{-l}\,(-y)^{-l/2+z}\,(1-y)^l \quad \text{with } y = \left(\frac{e^{2r}-i}{e^{2r}+i}\right)^2.$$

By this, we compute

$$\begin{aligned}
2\int_0^{+\infty} f(a_r)\cosh 2r\,dr &= 2^{1-l}\int_0^{\infty}(-y)^{-l/2+z}\,(1-y)^l\cosh 2r\,dr\\
&= 2^{l-1}\int_1^{\infty}\left\{-\left(\frac{u-i}{u+i}\right)^2\right\}^z(1+u^2)^{1-l}\,u^{l-2}\,du,
\end{aligned}$$

setting $u = e^{2r}$. In the same way, we have

$$2\int_{-\infty}^{0} f(a_r)\cosh 2r\,dr = 2^{l-1}\int_1^{\infty}\left\{-\left(\frac{u+i}{u-i}\right)^2\right\}^z(1+u^2)^{1-l}\,u^{l-2}\,du.$$

□

11.4. Orbital integrals of Shintani functions. Set $w_0 = k_{\pi/2} = \left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right]$.

Lemma 11.6. *If $0 < \mathrm{Re}(z) < l/2$, then, for ϵ, $\epsilon' \in \{0,1\}$, we have*

$$\begin{aligned}
&\int_{\mathbb{R}^\times} \Psi^{(0)}\left(l; \left[\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right] w_0^{\epsilon'}\right)|x|^z \mathrm{sgn}^\epsilon(x) d^\times x = 2i^{l\epsilon'}\Gamma(z)\Gamma(l/2-z)\Gamma(l/2)^{-1} i^\epsilon \cos\left(\tfrac{\pi}{2}(z+\epsilon)\right),\\
&\int_{\mathbb{R}^\times} \Psi^{(0)}\left(l; \left[\begin{smallmatrix}1 & 0\\ x & 1\end{smallmatrix}\right] w_0^{\epsilon'}\right)|x|^z \mathrm{sgn}^\epsilon(x) d^\times x = 2i^{l\epsilon'}\Gamma(z)\Gamma(l/2-z)\Gamma(l/2)^{-1}(-i)^\epsilon \cos\left(\tfrac{\pi}{2}(z+\epsilon)\right).
\end{aligned}$$

Proof. Let $J_{l,\epsilon}(z)$ denote the first integral with $\epsilon' = 0$. From Lemma 11.2, we have $J_{l,\epsilon}(z) = J_l^+(z) + (-1)^\epsilon J_l^-(z)$ with

$$J_l^{\pm}(z) = \int_0^\infty (1 \pm ix)^{-l/2}\,x^z\,d^\times x.$$

By the formula [8, 3.194.3], we have

$$J_l^{\pm}(z) = (\pm i)^{-z}B(z, l/2-z) = (\pm i)^{-z}\Gamma(z)\Gamma(l/2-z)\Gamma(l/2)^{-1} \quad (l/2 > \mathrm{Re}(z) > 0).$$

Hence,

$$J_{l,\epsilon}(z) = \Gamma(z)\Gamma(l/2-z)\Gamma(l/2)^{-1}\{i^{-z} + (-1)^\epsilon(-i)^{-z}\}.$$

Since $i^{-z} + (-1)^\epsilon(-i)^{-z} = 2i^\epsilon\cos(\pi(z+\epsilon)/2)$, we are done. We have the Iwasawa decomposition

$$\left[\begin{smallmatrix}1 & 0\\ t & 1\end{smallmatrix}\right] = = \begin{bmatrix}\frac{1}{\sqrt{1+t^2}} & 0\\ 0 & \sqrt{1+t^2}\end{bmatrix}\left[\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right] k_\theta \qquad \text{with } e^{i\theta} = \frac{1+it}{\sqrt{1+t^2}}.$$

Hence, by Lemma 11.2, we obtain

$$\Psi^{(z)}\left(l;\left[\begin{smallmatrix}1&0\\t&1\end{smallmatrix}\right]\right)=\left(\frac{1}{1+t^2}\right)^{z}\times\left(\frac{1+it}{\sqrt{1+t^2}}\right)^{l}\times(1+it)^{z-l/2}=(1-it)^{-z-l/2}$$

Using this formula, in the same way as above, we can prove the second formula with $\epsilon'=0$. The remaining two formulas follow immediately from the proved ones by the relation $\Psi^{(0)}(l;gw_0)=i^l\,\Psi^{(0)}(l;g)$. $\square$

12. Holomorphic automorphic Green functions

Let $S\subset\Sigma_{\rm fin}$ be a finite subset. Put

$$\mathfrak{X}_S=\prod_{v\in S}\left(\mathbb{C}/4\pi i(\log q_v)^{-1}\mathbb{Z}\right),$$

which we regard as a complex manifold in the obvious way. Note that for any $\mathbf{c}\in\mathbb{R}^S$, the slice $\mathbb{L}_S(\mathbf{c})=\{\mathbf{s}\in\mathfrak{X}_S|\,\mathrm{Re}(\mathbf{s})=\mathbf{c}\}$ is a compact set homeomorphic to the torus $(\mathbb{S}^1)^S$.

Given $\mathbf{s}\in\mathfrak{X}_S$, $z\in\mathbb{C}$, an ideal $\mathfrak{n}\subset\mathfrak{o}$ such that $S(\mathfrak{n})\cap S=\emptyset$, and a family $l=(l_v)_{v\in\Sigma_\infty}\in(2\mathbb{Z}_{\geqslant 2})^{\Sigma_\infty}$, *the adelic Green function* $\Psi_l^{(z)}(\mathfrak{n}|\mathbf{s},-)$ is defined by

$$\Psi_l^{(z)}(\mathfrak{n}|\mathbf{s};g):=\prod_{v\in\Sigma_\infty}\Psi_v^{(z)}(l_v;g_v)\prod_{v\in S}\Psi_v^{(z)}(s_v;g_v)\prod_{v\in S(\mathfrak{n})}\Phi_{\mathfrak{n},v}^{(z)}(g_v)\prod_{v\in\Sigma_{\rm fin}-(S\cup S(\mathfrak{n}))}\Phi_{0,v}^{(z)}(g_v)$$

for any $g=(g_v)_{v\in\Sigma_F}\in G_{\mathbb{A}}$, where $\Psi_v^{(z)}(l_v;-)$ for $v\in\Sigma_\infty$ is the holomorphic Shintani function on $G_v\cong\mathrm{GL}(2,\mathbb{R})$ defined in Proposition 11.1, $\Psi_v^{(z)}(s;-)$ for $v\in S$ is the Green function recalled in §5, and for any $v\in\Sigma_{\rm fin}$, we set

$$\Phi_{\mathfrak{n},v}^{(z)}\left(\left[\begin{smallmatrix}t_1&0\\0&t_2\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right]k\right)=|t_1/t_2|_v^z\delta(x\in\mathfrak{o}_v)\delta(k\in\mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v)),\quad t_1,t_2\in F_v^\times,\,x\in F_v,\,k\in\mathbf{K}_v.$$

We remark that $\Phi_{\mathfrak{n},v}^{(z)}=\Phi_{0,v}^{(z)}$ if $v\in\Sigma_{\rm fin}-S(\mathfrak{n})$. The adelic Green function $\Psi_l^{(z)}(\mathfrak{n}|\mathbf{s};-)$ is a smooth function on $G_{\mathbb{A}}$ having the equivariance property

$$\Psi_l^{(z)}(\mathfrak{n}|\mathbf{s};hgk_\infty k_{\rm fin})=\{\prod_{v\in\Sigma_\infty}\tau_{l_v}(k_v)\}\chi_z(h)\Psi_l^{(z)}(\mathfrak{n}|\mathbf{s},g),\quad g\in G_{\mathbb{A}}$$

for any $h\in H_{\mathbb{A}}$, $k_\infty=(k_v)_{v\in\Sigma_\infty}\in\mathbf{K}_\infty^0$ and $k_{\rm fin}\in\mathbf{K}_0(\mathfrak{n})=\prod_{v\in\Sigma_{\rm fin}}\mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v)$, where $\chi_z:H_F\backslash H_{\mathbb{A}}\rightarrow\mathbb{C}^\times$ is the quasi-character defined by

$$\chi_z\left(\left[\begin{smallmatrix}t_1&0\\0&t_2\end{smallmatrix}\right]\right)=|t_1/t_2|_{\mathbb{A}}^z,\quad t_1,t_2\in\mathbb{A}^\times.$$

To state the most important property of the adelic Green functions, we introduce the (H,χ_z)-period integral of $\varphi\in C_c^\infty(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})$ by setting

$$\varphi^{H,(z)}(g)=\int_{Z_{\mathbb{A}}H_F\backslash H_{\mathbb{A}}}\varphi(hg)\chi_z(h)dh.$$

The integral $\varphi^{H,(z)}(g)$ converges absolutely and satisfies $\varphi^{H,(z)}(hg)=\chi_z(h)^{-1}\varphi^{H,(z)}(g)$ for any $h\in H_{\mathbb{A}}$. Let $C_c^\infty(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})[\tau_l]$ be the space of $\varphi\in C_c^\infty(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})$ such that

$$\varphi(gk_\infty)=\{\prod_{v\in\Sigma_\infty}\tau_{l_v}(k_v)\}\,\varphi(g)\qquad\text{for any }k_\infty=(k_v)_{v\in\Sigma_\infty}\in\mathbf{K}_\infty^0\text{ and }g\in G_{\mathbb{A}}.$$

Lemma 12.1. *Suppose $\varphi\in C_c^\infty(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})[\tau_l]$ and $R(\overline{W}_v)\,\varphi=0$ for all $v\in\Sigma_\infty$. Then we have*

$$\overline{\varphi}^{H,(z)}(g_{\rm fin}g_\infty)=\{\prod_{v\in\Sigma_\infty}\overline{\Psi_v^{(-\bar{z})}(l_v;g_v)}\}\overline{\varphi}^{H,(z)}(g_{\rm fin})$$

for $g_\infty=(g_v)_{v\in\Sigma_\infty}\in G_\infty$ and $g_{\rm fin}\in G_{\rm fin}$.

Proof. Let $g_{\rm fin} \in G_{\rm fin}$. For any $v \in \Sigma_\infty$, we can easily verify

$$\overline{\varphi}^{H,(z)}\left(g_{\rm fin}\left[\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right] g_\infty k\right) = |t_1/t_2|_v^{-z} \tau_{l_v}(k)^{-1} \overline{\varphi}^{H,(z)}(g_{\rm fin} g_\infty), \quad t_1, t_2 \in F_v^\times, k \in \mathbf{K}_v^0,\ g_\infty \in G_\infty.$$

Moreover we have $R(W_v)(\overline{\varphi}^{H,(z)}) = 0$ by the equality $R(W_v)(\overline{\varphi}^{H,(z)}) = (\overline{R(\bar{W}_v)\varphi})^{H,(z)}$. Thus the uniqueness of Shintani functions (Proposition 11.1) yields a constant C such that

$$\overline{\varphi}^{H,(z)}(g_{\rm fin} g_\infty) = C \prod_{v \in \Sigma_\infty} \overline{\Psi_v^{(-\bar{z})}(l_v; g_v)} \quad \text{for all } g_\infty \in G_\infty.$$

By setting $g_\infty = 1_2$, we have $C = \overline{\varphi}^{H,(z)}(g_{\rm fin})\{\prod_{v \in \Sigma_\infty} \overline{\Psi_v^{(-\bar{z})}(l_v; 1_2)}\}^{-1} = \overline{\varphi}^{H,(z)}(g_{\rm fin})$. This completes the proof. □

For $\mathbf{s} \in \mathfrak{X}_S$, we consider the element

$$\mathbf{T}_S(\mathbf{s}) = \bigotimes_{v \in S} \{\mathbb{T}_v - (q_v^{(1-s_v)/2} + q_v^{(1+s_v)/2})\, 1_{\mathbf{K}_v}\}$$

of the Hecke algebra $\bigotimes_{v \in S} \mathcal{H}(G_v, \mathbf{K}_v)$. We also set

$$q(\mathbf{s}) = \inf\{(\mathrm{Re}(s_v) + 1)/4 \,|\, v \in S\}.$$

Proposition 12.2. *Let $l = (l_v)_{v \in \Sigma_\infty}$ be a family of even positive integers, and suppose $q(\mathbf{s}) > 2|\mathrm{Re}(z)| + 1$. For $\varphi \in C_c^\infty(Z_{\mathbb{A}} G_F \backslash G_{\mathbb{A}})[\tau_l]^{\mathbf{K}_0(\mathfrak{n})}$ such that $R(\overline{W}_v)\varphi = 0$ for all $v \in \Sigma_\infty$, the function $g \mapsto \Psi_l^{(z)}(\mathfrak{n}|\mathbf{s}; g)\overline{\varphi}^{H,(z)}(g)$ is integrable on $H_{\mathbb{A}} \backslash G_{\mathbb{A}}$. Moreover, we have*

$$\int_{H_{\mathbb{A}} \backslash G_{\mathbb{A}}} \Psi_l^{(z)}(\mathfrak{n}|\mathbf{s}; g)[R(\mathbf{T}_S(\mathbf{s}))\overline{\varphi}^{H,(z)}](g) dg = \{\prod_{v \in \Sigma_\infty} 2^{l_v - 1} C_{l_v}(z)\} \mathrm{vol}(H_{\rm fin} \backslash H_{\rm fin} \mathbf{K}_0(\mathfrak{n})) \overline{\varphi}^{H,(z)}(1_2).$$

Proof. We follow the argument in the proof of [47, Lemma 6.3]. By Lemma 12.1, the integral in the left-hand side is

$$\int_{H_{\mathbb{A}} \backslash G_{\mathbb{A}}} \{\prod_{v \in \Sigma_\infty} \Psi_v^{(z)}(l_v; g_v) \overline{\Psi_v^{(-\overline{z})}(l_v; g_v)}\}\{\prod_{v \in S} \Psi_v^{(z)}(s_v; g_v) \prod_{v \in S(\mathfrak{n})} \Phi_{\mathfrak{n},v}^{(z)}(g_v) \prod_{v \in \Sigma_{\rm fin} - (S \cup S(\mathfrak{n}))} \Phi_{0,v}^{(z)}(g_v)\}$$
$$\times [R(\mathbf{T}_S(\mathbf{s}))\overline{\varphi}^{H,(z)}](g_{\rm fin})\, dg.$$

Hence, by Proposition 11.5 and Lemma 5.2, we obtain the assertion. □

12.1. Automorphic smoothed kernels. Set $\underline{l} = \inf_{v \in \Sigma_\infty} l_v$ for a family $l = (l_v)_{v \in \Sigma_\infty} \in (2\mathbb{N})^{\Sigma_\infty}$. In this subsection, we introduce the automorphic renormalized smoothed kernel function $\hat{\mathbf{\Psi}}_{\beta,\lambda}^l(\mathfrak{n}|\alpha; g)$ depending on a complex parameter λ and study its properties when $\underline{l} \geqslant 4$ and $1/2 < \mathrm{Re}(\lambda) < \underline{l}/2 - 1$. It is defined by the Poincaré series (12.1).

Let $\mathcal{B}$ denote the space of all the entire functions $\beta(z)$ on $\mathbb{C}$ satisfying $\beta(z) = \beta(-z)$ and that there exist $A > 0$ and $B \in \mathbb{R}$ such that the estimate

$$|\beta(\sigma + it)| \ll e^{-A(|t|+B)^2}, \qquad \sigma \in [a, b],\ t \in \mathbb{R}$$

holds for any interval $[a, b] \subset \mathbb{R}$. We have $C_l \mathcal{B} \subset \mathcal{B}$ by Lemma 11.4 (iii).

We define the renormalized Green function by

$$\Psi_{\beta,\lambda}^l(\mathfrak{n}|\mathbf{s}; g) = \frac{1}{2\pi i} \int_{L_\sigma} \frac{\beta(z)}{z + \lambda} \{\Psi_l^{(z)}(\mathfrak{n}|\mathbf{s}; g) + \Psi_l^{(-z)}(\mathfrak{n}|\mathbf{s}; g)\} dz$$

for $\sigma \in \mathbb{R}$ such that $-\inf(q(\mathbf{s}) - 1, \mathrm{Re}(\lambda)) < \sigma < q(\mathbf{s}) - 1$. The defining integral is absolutely convergent and independent of σ as above. The normalization is meaningful to link Green functions and regularized periods in Lemma 12.4.

For any $(\beta, \lambda, \mathbf{s}) \in \mathcal{B} \times \mathbb{C} \times \mathfrak{X}_S$ such that $\mathrm{Re}(\lambda) > 0$ and $q(\mathbf{s}) > 1$, we consider the average of the renormalized Green function $\Psi^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}; g)$ over the G_F-orbits:

$$\mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}; g) = \sum_{\gamma \in H_F\backslash G_F} \Psi^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}; \gamma g), \quad g \in G_{\mathbb{A}}.$$

Lemma 12.3. *Suppose $\underline{l} \geqslant 4$.*

(1) *The series $\mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}; g)$ converges absolutely and locally uniformly in $(\lambda, \mathbf{s}, g) \in \{\mathrm{Re}(\lambda) > 0\} \times \{q(\mathbf{s}) > 1\} \times G_{\mathbb{A}}$. For a fixed $(\lambda, \mathbf{s})$ in this region, $\mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}; g)$ is a continuous function in $g \in G_{\mathbb{A}}$, which is left $Z_{\mathbb{A}}G_F$-invariant and right $\mathbf{K}_0(\mathfrak{n})$-invariant, and satisfies*

$$\mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}; gk_v) = \tau_{l_v}(k_v)\mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}; g)$$

for all $v \in \Sigma_\infty$ and $k_v \in \mathbf{K}^0_v$.

(2) *Let $(\lambda, \mathbf{s})$ be an element of $\mathbb{C} \times \mathfrak{X}_S$ such that $2\,\mathrm{Re}(\lambda) > 1$, $q(\mathbf{s}) > 2\,\mathrm{Re}(\lambda) + 1$ and $\underline{l}/2 > \mathrm{Re}(\lambda) + 1$. Then, for any $\sigma \in (1/2, \mathrm{Re}(\lambda))$, we have the estimate*

$$|\mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}; g)| \ll y(g)^{1-\sigma}, \quad g \in \mathfrak{S}^1.$$

Proof. The same proof as in [47, Proposition 8.1] goes through with a minor modification; Lemma 11.3 is used in the course. The outline is as follows. For $p > 0$ and $q > 1$, set

$$\begin{aligned}\Xi_{l,p,q,S}([\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}](a_{r_v})_{v\in\Sigma_\infty}([\begin{smallmatrix} 1 & x_v \\ 0 & 1 \end{smallmatrix}])_{v\in\Sigma_{\mathrm{fin}}}k) = &\inf\{|t_1/t_2|^p_{\mathbb{A}}, |t_1/t_2|^{-p}_{\mathbb{A}}\} \prod_{v\in\Sigma_\infty} (\cosh 2r_v)^{-\underline{l}/2} \\ &\times \prod_{v\in S} \sup(1, |x_v|_v)^{-q} \prod_{v\in\Sigma_{\mathrm{fin}}-S} \delta(x_v \in \mathfrak{o}_v)\end{aligned}$$

for $t_1, t_2 \in \mathbb{A}^\times$, $(r_v)_{v\in\Sigma_\infty} \in \mathbb{R}^{\Sigma_\infty}$ and $(x_v)_{v\in\Sigma_{\mathrm{fin}}} \in \mathbb{A}_{\mathrm{fin}}$, and set

$$\mathbf{\Xi}_{l,p,q,S}(g) = \sum_{\gamma\in H_F\backslash G_F} \Xi_{l,p,q,S}(\gamma g), \qquad g \in G_{\mathbb{A}}.$$

Since $\underline{l}/2 > 1$, the series $\mathbf{\Xi}_{l,p,q,S}(g)$ is locally uniformly convergent in $G_{\mathbb{A}}$. Moreover, if $1 + 2q < p$ and $1 + p < \underline{l}/2$, we have

$$\mathbf{\Xi}_{l,p,q,S}(g) \ll y(g)^{1-p}, \quad g \in \mathfrak{S}^1.$$

Indeed, it is enough to replace q in the archimedean factors of $\Xi_{p,q,S}$ used in [47, Lemma 3.5] with $\underline{l}/2$. We also note that the condition $1 + p < \underline{l}/2$ is needed to guarantee $\int_{\mathbb{R}} \cosh{(2r_v)}^{p-\underline{l}/2+1}dr_v < \infty$. In this setting, $\Xi_{l,\sigma,q(\mathbf{s}),S}$ with $0 < \sigma < \inf(\mathrm{Re}(\lambda), q(\mathbf{s}) - 1)$ gives a majorant of $\Psi^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s})$ in the same way as [47, Lemma 6.7]. Thus $\mathbf{\Xi}_{l,\sigma,q(\mathbf{s}),S}$ is also a mojorant of $\mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s})$. $\square$

For a fixed $(\lambda, \mathbf{s})$ such that $2\,\mathrm{Re}(\lambda) > 1$, $q(\mathbf{s}) > 2\,\mathrm{Re}(\lambda) + 1$ and $\underline{l}/2 > \mathrm{Re}(\lambda) + 1$, the function $\mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s})$ defines a distribution on $Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}$ by

$$\langle \mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}), \varphi \rangle = \int_{Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}} \mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}; g)\varphi(g)dg, \qquad \varphi \in C^\infty_c(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})^{\mathbf{K}_0(\mathfrak{n})}.$$

We remark that the absolute convergence of the integral is valid for any rapidly decreasing function φ by Lemma 12.3 (2).

12.2. Regularized periods. (For details, see [47, §7] and [41].) We recall the regularization of period integrals along H explained in §3.2. Although such a regularization is not needed in the spectral side of our relative trace formula since (12.1) is cuspidal, the regularization as below plays a role in the geometric side in §15.

For a real valued character η of $F^\times\backslash\mathbb{A}^\times$, let x_η and x_η^* be as in §1.4. A continuous function φ on $Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}$ is said to have the regularized (H,η)-period $P^\eta_{\rm reg}(\varphi)\in\mathbb{C}$ if, for any $\beta\in\mathcal{B}$, the integral

$$P^\eta_{\beta,\lambda}(\varphi)=\int_{F^\times\backslash\mathbb{A}^\times}\varphi\left(\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\right)\eta(tx_\eta^*)\,\{\hat\beta_\lambda(|t|_{\mathbb{A}})+\hat\beta_\lambda(|t|_{\mathbb{A}}^{-1})\}\,d^\times t$$

converges absolutely when $\mathrm{Re}(\lambda)\gg 1$ and is continued meromorphically in a neighborhood of $\lambda=0$ with the constant term $\mathrm{CT}_{\lambda=0}P^\eta_{\beta,\lambda}(\varphi)=P^\eta_{\rm reg}(\varphi)\beta(0)$ in its Laurent expansion at $\lambda=0$. We note that if $\varphi\in C^\infty(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})$ is rapidly decreasing on $\mathfrak{S}^1$, then by [47, Lemma 7.3], the regularized period $P_{\rm reg}(\varphi)$ coincides with the $(H,\mathbf{1})$-period.

Lemma 12.4. *Assume $\underline{l}\geqslant 4$. Let $(\lambda,\mathbf{s})$ be an element of $\mathbb{C}\times\mathfrak{X}_S$ such that $2\,\mathrm{Re}(\lambda)>1$, $q(\mathbf{s})>2\,\mathrm{Re}(\lambda)+1$ and $\underline{l}/2>\mathrm{Re}(\lambda)+1$. Then, for any rapidly decreasing function $\varphi\in C^\infty(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})[\tau_l]^{\mathbf{K}_0(\mathfrak{n})}$ such that $R(\overline{W}_v)\varphi=0$ for all $v\in\Sigma_\infty$, we have*

$$\langle\mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}),R(\mathbf{T}_S(\mathbf{s}))\overline{\varphi}\rangle=\{\prod_{v\in\Sigma_\infty}2^{l_v-1}\}\mathrm{vol}(H_{\rm fin}\backslash H_{\rm fin}\mathbf{K}_0(\mathfrak{n}))P^{\mathbf{1}}_{\beta C_l,\lambda}(\overline{\varphi}),$$

where $C_l(z)=\prod_{v\in\Sigma_\infty}C_{l_v}(z)$.

Proof. The proof is given in the same way as [47, Lemma 8.2] with the aid of Lemma 11.3 and Proposition 12.2. We note that $P^{\mathbf{1}}_{\beta C_l,\lambda}(\overline{\varphi})$ is well-defined because βC_l belongs to $\mathcal{B}$. □

Assume $\underline{l}\geqslant 4$. Given a holomorphic function $\alpha(\mathbf{s})$ on $\mathfrak{X}_S$ such that $\alpha(\varepsilon\mathbf{s})=\alpha(\mathbf{s})$ for all $\varepsilon\in\{\pm1\}^S$, we define the renormalized smoothed kernel

$$\hat\Psi^l_{\beta,\lambda}(\mathfrak{n}|\alpha;g)=\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\Psi^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s};g)\,\alpha(\mathbf{s})d\mu_S(\mathbf{s})$$

for $\mathrm{Re}(\lambda)>0$ and $\mathbf{c}\in\mathbb{R}^S$ such that $q(\mathbf{c})>\sup(\mathrm{Re}(\lambda)+1,2)$, where $\int_{\mathbb{L}_S(\mathbf{c})}f(\mathbf{s})\,d\mu_S(\mathbf{s})$ means the multidimensional contour integral along the slice $\mathbb{L}_S(\mathbf{c})$ oriented naturally, which is as in the beginning of §7, with respect to the form $d\mu_S(\mathbf{s})=\prod_{v\in S}d\mu_v(s_v)$ with

$$d\mu_v(s_v)=2^{-1}\log q_v\,(q_v^{(1+s)/2}-q_v^{(1-s)/2})\,ds_v.$$

For $\mathrm{Re}(\lambda)>0$, let us consider the Poincaré series

$$\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha;g)=\sum_{\gamma\in H_F\backslash G_F}\hat\Psi^l_{\beta,\lambda}(\mathfrak{n}|\alpha;\gamma g),\quad g\in G_{\mathbb{A}}.\tag{12.1}$$

In the same way as [47], we analyze this series and obtain the following.

Lemma 12.5. (1) *The series $\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha;g)$ converges absolutely and locally uniformly in $(\lambda,g)\in\{\mathrm{Re}(\lambda)>0\}\times G_{\mathbb{A}}$. The function $g\mapsto\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha;g)$ is continuous on $G_{\mathbb{A}}$, left $Z_{\mathbb{A}}G_F$-invariant, and right $\mathbf{K}_0(\mathfrak{n})$-invariant; moreover it satisfies*

$$\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha;gk_v)=\tau_{l_v}(k_v)\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha;g)\tag{12.2}$$

for all $v\in\Sigma_\infty$ and $k_v\in\mathbf{K}^0_v$.

(2) *For $\mathrm{Re}(\lambda)>0$, the function $\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha;g)$ belongs to $L^m(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})$ for any $m>0$ such that $m(1-\mathrm{Re}(\lambda))<1$.*

Proof. The argument in the proof of [47, Proposition 9.1] works with a minor modification; We use $\Xi_{l,p,q,S}$ and $\boldsymbol{\Xi}_{l,p,q,S}$ given in the proof of Lemma 12.3. $\square$

Proposition 12.6. *For $1/2 < \mathrm{Re}(\lambda) < \underline{l}/2 - 1$, the function $\hat{\boldsymbol{\Psi}}^{l}_{\beta,\lambda}(\mathfrak{n}|\alpha; g)$ is cuspidal.*

Proof. From Proposition 11.1 and Lemma 12.5, we have the equations

$$R(\overline{W}_v)\, \hat{\boldsymbol{\Psi}}^{l}_{\beta,\lambda}(\mathfrak{n}|\alpha; g) = 0, \qquad g \in G_{\mathbb{A}}, \tag{12.3}$$

$$R(\Omega_v)\hat{\boldsymbol{\Psi}}^{l}_{\beta,\lambda}(\mathfrak{n}|\alpha; g) = 2^{-1}(l_v^2 - 2l_v)\hat{\boldsymbol{\Psi}}^{l}_{\beta,\lambda}(\mathfrak{n}|\alpha; g), \qquad g \in G_{\mathbb{A}} \tag{12.4}$$

for all $v \in \Sigma_\infty$. Hence, by (12.4), there exists $f \in C_c^\infty(G_{\mathbb{A}})$ such that $\hat{\boldsymbol{\Psi}}^{l}_{\beta,\lambda}(\mathfrak{n}|\alpha) * f = \hat{\boldsymbol{\Psi}}^{l}_{\beta,\lambda}(\mathfrak{n}|\alpha)$ by [2, Theorem 2.14]. By Lemma 12.5 (2), $\hat{\boldsymbol{\Psi}}^{l}_{\beta,\lambda}(\mathfrak{n}|\alpha; g)$ belongs to $L^2(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})^{\mathbf{K}_0(\mathfrak{n})}$. Thus, for any $X \in \mathfrak{g}_\infty$, the derivative $R(X)\hat{\boldsymbol{\Psi}}^{l}_{\beta,\lambda}(\mathfrak{n}|\alpha) = \hat{\boldsymbol{\Psi}}^{l}_{\beta,\lambda}(\mathfrak{n}|\alpha) * R(-X)f$ also belongs to $L^2(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})^{\mathbf{K}_0(\mathfrak{n})}$. Let V be the $(\mathfrak{g}_\infty, \mathbf{K}_\infty)$-submodule of $L^2(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})^{\mathbf{K}_0(\mathfrak{n})}$ generated by $\hat{\boldsymbol{\Psi}}^{l}_{\beta,\lambda}(\mathfrak{n}|\alpha)$. From (12.2) and (12.3), V is decomposed into a finite sum of the discrete series representation $\boxtimes_{v\in\Sigma_\infty} D_{l_v}$ of $\mathrm{PGL}(2, F \otimes_{\mathbb{Q}} \mathbb{R})$ of weight $(l_v)_{v\in\Sigma_\infty}$. By Wallach's criterion [49, Theorem 4.3], the space V is contained in the cuspidal part of $L^2(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})$. $\square$

By Proposition 12.6, for $1/2 < \mathrm{Re}(\lambda) < \underline{l}/2 - 1$, the function $\hat{\boldsymbol{\Psi}}^{l}_{\beta,\lambda}(\mathfrak{n}|\alpha; g)$ has the spectral expansion

$$\hat{\boldsymbol{\Psi}}^{l}_{\beta,\lambda}(\mathfrak{n}|\alpha; g) = \sum_{\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})} \sum_{\varphi\in\mathcal{B}(\pi;l,\mathfrak{n})} \langle \hat{\boldsymbol{\Psi}}^{l}_{\beta,\lambda}(\mathfrak{n}|\alpha)|\varphi\rangle_{L^2}\varphi(g). \tag{12.5}$$

Here $\langle\cdot|\cdot\rangle_{L^2}$ is the L^2-inner product on $L^2(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})$ and $\mathcal{B}(\pi; l, \mathfrak{n})$ is an orthonormal basis of $V_\pi[\tau_l]^{\mathbf{K}_0(\mathfrak{n})}$. From the finite dimensionality of

$$\{\varphi \in L^2(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})[\tau_l]^{\mathbf{K}_0(\mathfrak{n})} \mid R(\overline{W}_v)\varphi = 0\, (\forall v \in \Sigma_\infty)\},$$

the sum in (12.5) is finite and the equality holds pointwisely for all g.

13. Spectral expansions

From this section until §18, we fix a family $l = (l_v)_{v\in\Sigma_\infty} \in (2\mathbb{N})^{\Sigma_\infty}$, an ideal $\mathfrak{n} \subset \mathfrak{o}$, a character η of $F^\times\backslash\mathbb{A}^\times$ such that $\eta^2 = \mathbf{1}$ whose conductor $\mathfrak{f}_\eta$ is relatively prime to $\mathfrak{n}$, and a finite subset $S \subset \Sigma_{\mathrm{fin}} - S(\mathfrak{n}\mathfrak{f}_\eta)$. Using the spectral expansion (12.5), we show that $\hat{\boldsymbol{\Psi}}^{l}_{\beta,\lambda}(\mathfrak{n}|\alpha; g)$ has an entire extension to the whole λ-plane. As the value at $\lambda = 0$ of the entire extension, we define the regularized kernel $\boldsymbol{\Psi}^{l}_{\mathrm{reg}}(\mathfrak{n}|\alpha; g)$ and obtain its spectral expression. The upshot of this section is Proposition 13.6, which gives the period integral of the regularized kernel.

13.1. Extremal Whittaker vectors of discrete series. For $v \in \Sigma_\infty$, let π_v be the discrete series representation of $\mathrm{PGL}(2, \mathbb{R})$ of minimal $\mathbf{K}_v^0$-type $l_v \geqslant 2$. Let V_{π_v} denote the Whittaker model of π_v with respect to the character ψ_{F_v} (see §1.3). It is known that $V_{\pi_v}[\tau_{l_v}]$ contains a unique vector $\phi^{l_v}_{0,v}$ characterized by the formula

$$\phi^{l_v}_{0,v}\left(\left[\begin{smallmatrix} y & 0 \\ 0 & 1\end{smallmatrix}\right]\right) = 2|y|_v^{l_v/2}e^{2\pi y}\delta(y < 0) \tag{13.1}$$

for any $y \in \mathbb{R}^\times$. We remark that $\phi^{l_v}_{0,v}$ is extremal, i.e., $\pi_v(\overline{W})\phi^{l_v}_{0,v} = 0$, and $V_{\pi_v}[\tau_{l_v}] = \mathbb{C}\phi^{l_v}_{0,v}$. The local standard L-factor of π_v is given by $L(s, \pi_v) = \Gamma_{\mathbb{C}}(s + (l_v - 1)/2)$, and the local epsilon factor of π_v is given as $\epsilon(s, \pi_v \otimes \mathrm{sgn}^m, \psi_{F_v}) = i^{l_v}$ for $m \in \{0, 1\}$.

13.2. Construction of a basis. Let (π, V_π) be an irreducible cuspidal automorphic representation of $G_{\mathbb{A}}$ with trivial central character such that $V_\pi \subset L^2(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})$. We fix a family $\{(\pi_v, V_{\pi_v})\}_{v\in\Sigma_F}$ of unitarizable irreducible admissible representations of G_v with V_{π_v} being contained in the ψ_{F_v}-Whittaker functions on G_v such that $\pi \cong \bigotimes_{v\in\Sigma_F}\pi_v$. Given an ideal $\mathfrak{n}$ of $\mathfrak{o}$ and $l=(l_v)_{v\in\Sigma_\infty}\in(2\mathbb{N})^{\Sigma_\infty}$, let $\Pi_{\rm cus}(l,\mathfrak{n})$ denote the set of all those cuspidal representations π such that $\pi_v\cong D_{l_v}$ for any $v\in\Sigma_\infty$ and the conductor $\mathfrak{f}_\pi$ of π divides $\mathfrak{n}$.

For a fixed $\pi\in\Pi_{\rm cus}(l,\mathfrak{n})$, we write $\Lambda_\pi(\mathfrak{n})$ for $\Lambda_\pi^0(\mathfrak{n})=\prod_{k=1}^{n}{\rm Map}(S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1}),\{0,\dots,k\})$, where n is the maximal non-negative integer m such that $S_m(\mathfrak{n}\mathfrak{f}_\pi^{-1})\neq\emptyset$. By the same procedure as in §3.1, corresponding to each $\rho=(\rho_k)_{k=1}^n\in\Lambda_\pi(\mathfrak{n})$, we have a cusp form $\varphi_{l,\pi,\rho}\in V_\pi[\tau_l]^{\mathbf{K}_0(\mathfrak{n})}$ as the image of the decomposable tensor

$$\bigotimes_{v\in\Sigma_\infty}\phi_{0,v}^{l_v}\otimes\bigotimes_{k=1}^{n}\bigotimes_{v\in S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1})}\phi_{\rho_k(v),v}\otimes\bigotimes_{v\in\Sigma_{\rm fin}-S(\mathfrak{n}\mathfrak{f}_\pi^{-1})}\phi_{0,v}$$

by the isomorphism $V_\pi\cong\bigotimes_{v\in\Sigma_F}V_{\pi_v}$, where for each $v\in\Sigma_{\rm fin}$, the system $\{\phi_{k,v}\}$ is the basis of $V_{\pi_v}^{\mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v)}$ constructed in [41]. In this way, we have an orthogonal basis $\{\varphi_{l,\pi,\rho}\mid\rho\in\Lambda_\pi(\mathfrak{n})\}$ of the finite dimensional space $V_\pi[\tau_l]^{\mathbf{K}_0(\mathfrak{n})}$ equipped with the L^2-inner product on $Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}$ (cf. [41, Propdosition 17]). The vector φ_{l,π,ρ_0} with $\rho_0(v)=0$ for all $v\in\Sigma_{\rm fin}$ is denoted by $\varphi_{l,\pi}^{\rm new}$.

We note that if $\varphi\in C^\infty(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})$ is rapidly decreasing on $\mathfrak{S}^1$, then by [47, Lemma 7.3], the period $P_{\rm reg}^\eta(\varphi)$ coincides with the global zeta integral $Z^*(1/2,\eta,\varphi)=\int_{F^\times\backslash\mathbb{A}^\times}\varphi\left(\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\right)\eta(tx_\eta^*)\,d^\times t$, which is absolutely convergent. The following proposition is obtained by computing the global zeta integral; the proof is a minor modification of that of [41, Main Theorem A].

Proposition 13.1. *For any $\rho\in\Lambda_\pi(\mathfrak{n})$, $\varphi_{l,\pi,\rho}$ has the regularized (H,η)-period given by*

$$P_{\rm reg}^\eta(\varphi_{l,\pi,\rho})=Z^*(1/2,\eta,\varphi_{l,\pi,\rho})=(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)\{\prod_{v\in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})}Q_{\rho(v),v}^{\pi_v}(\eta_v,1)\}L(1/2,\pi\otimes\eta).$$

Here we set $\rho(v)=\rho_k(v)$ for each $v\in S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1})$ and $Q_{\rho(v),v}^{\pi_v}(\eta_v,1)$ is the constant given in Proposition 3.1 (cf. [41, Main Theorem A]*).*

Remark: Here we note that, throughout [41], it is assumed that $\eta_v(-1)=1$ for all $v\in\Sigma_\infty$, and hence $(-1)^{\epsilon(\eta)}$ does not appear in [41, Main Theorem A].

Set

$$\mathbb{P}^\eta(\pi;l,\mathfrak{n})=\sum_{\varphi\in\mathcal{B}(\pi;l,\mathfrak{n})}\overline{P_{\rm reg}^{\mathbf{1}}(\varphi)}P_{\rm reg}^\eta(\varphi),$$

where $\mathcal{B}(\pi;l,\mathfrak{n})$ is an orthonormal basis of $V_\pi[\tau_l]^{\mathbf{K}_0(\mathfrak{n})}$.

Lemma 13.2. *The sum $\mathbb{P}^\eta(\pi;l,\mathfrak{n})$ is independent of the choice of $\mathcal{B}(\pi;l,\mathfrak{n})$. We have*

$$\mathbb{P}^\eta(\pi;l,\mathfrak{n})=D_F^{-1/2}(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)w_{\mathfrak{n}}^\eta(\pi)\frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{\|\varphi_{l,\pi}^{\rm new}\|^2},$$

and that the value $(-1)^{-\epsilon(\eta)}\mathcal{G}(\eta)^{-1}\mathbb{P}^\eta(\pi;l,\mathfrak{n})$ is non-negative. Here $w_{\mathfrak{n}}^\eta(\pi)$ is the explicit non-negative constant given in Lemma 3.6. In particular, if η satisfies $\eta_v(\varpi_v)=-1$ for all $v\in S(\mathfrak{n})$, then $w_{\mathfrak{n}}^\eta(\pi)=0$ unless $\mathfrak{n}\mathfrak{f}_\pi^{-1}$ is a square of integral ideal.

Proof. With the aid of Proposition 13.1, we obtain the assertion in the same way as Lemma 3.6. The non-negativity of $(-1)^{-\epsilon(\eta)}\mathcal{G}(\eta)^{-1}\mathbb{P}^\eta(\pi;l,\mathfrak{n})$ follows from $w_{\mathfrak{n}}^\eta(\pi)\geqslant 0$ combined with the non-negativity of $L(1/2,\pi)L(1/2,\pi\otimes\eta)$ proved in [18]. □

The sign of the functional equation of the L-function $L(s,\pi)L(s,\pi\otimes\eta)$ is given as follows.

Lemma 13.3. *We have $\epsilon(1/2,\pi)\epsilon(1/2,\pi\otimes\eta)=(-1)^{\epsilon(\eta)}\tilde\eta(\mathfrak{f}_\pi)$. In particular, $L(1/2,\pi)L(1/2,\pi\otimes\eta)=0$ unless $(-1)^{\epsilon(\eta)}\tilde\eta(\mathfrak{f}_\pi)=1$.*

Proof. Since l_v is even for all $v\in\Sigma_\infty$, by virtue of Lemma 3.7, we have

$$\epsilon(1/2,\pi)\epsilon(1/2,\pi\otimes\eta)=\prod_{v\in\Sigma_\infty}i^{2l_v}\prod_{v\in S(\mathfrak{f}_\eta)}\eta_v(-1)\prod_{v\in S(\mathfrak{f}_\pi)}\eta_v(\varpi_v^{c(\pi_v)})=\{\prod_{v\in\Sigma_{\rm fin}}\eta_v(-1)\}\tilde\eta(\mathfrak{f}_\pi)=(-1)^{\epsilon(\eta)}\tilde\eta(\mathfrak{f}_\pi).$$

By the functional equation, we are done. $\square$

13.3. Adjoint L-functions : holomorphic case. Let $E(\nu,g)=E_{\mathbf{1},\rho_0}(\nu,g)=\sum_{\gamma\in B_F\backslash G_F}y(\gamma g)^{(\nu+1)/2}$ for $\mathrm{Re}(\nu)>1$ be the $\mathbf{K}$-spherical Eisenstein series on $G_{\mathbb{A}}$ (see §3.7).

Lemma 13.4. *For any $\pi\in\Pi_{\rm cus}(l,\mathfrak{n})$,*

$$\int_{Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}}\varphi_{l,\pi}^{\rm new}(g)\overline{\varphi_{l,\pi}^{\rm new}}(g)E(2s-1,g)dg \tag{13.2}$$

$$=\{\prod_{v\in\Sigma_\infty}2^{1-l_v}\}\frac{\mathrm{N}(\mathfrak{f}_\pi)^sD_F^{s-3/2}}{[\mathbf{K}_{\rm fin}:\mathbf{K}_0(\mathfrak{f}_\pi)]}\frac{\zeta_F(s)L(s,\pi,\mathrm{Ad})}{\zeta_F(2s)}\prod_{v\in S_\pi}\frac{q_v^{d_v(3/2-s)}Z_v(s)}{q_v^{c(\pi_v)(s-1)}L(s,\pi_v,\mathrm{Ad})}\frac{1+q_v^{-1}}{1+q_v^{-s}}$$

for $\mathrm{Re}(s)\gg0$ and $\|\varphi_{l,\pi}^{\rm new}\|^2=2\{\prod_{v\in\Sigma_\infty}2^{1-l_v}\}\mathrm{N}(\mathfrak{f}_\pi)[\mathbf{K}_{\rm fin}:\mathbf{K}_0(\mathfrak{f}_\pi)]^{-1}L^{S_\pi}(1,\pi,\mathrm{Ad})$. Here we set $S_\pi:=\{v\in\Sigma_{\rm fin}|\,\mathrm{ord}_v(\mathfrak{f}_\pi)\geqslant2\}$ and $Z_v(s):=\int_{\mathbf{K}_v}\int_{F_v^\times}\phi_{0,v}([\begin{smallmatrix}t&0\\0&1\end{smallmatrix}]k)\overline{\phi_{0,v}([\begin{smallmatrix}t&0\\0&1\end{smallmatrix}]k)}|t|_v^{s-1}d^\times tdk$ for $v\in\Sigma_{\rm fin}$.

Proof. By the standard procedure, we see that the left-hand side of (13.2) is a product of the integrals $Z_v(s)$ over all $v\in\Sigma_F$, where $Z_v(s)$ for each $v\in\Sigma_\infty$ is defined for $\phi_{0,v}^{l_v}$ in the same way as the non-archimedean case. If $v\in\Sigma_\infty$, using (13.1), we easily have $Z_v(s)=2^{1-l_v}\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(2s)^{-1}L(s,\pi_v,\mathrm{Ad})$. Together with the computations at finite places (cf. [47, Lemma 2.14 and Corollary 2.15] and Lemma 3.8) , this completes the proof. $\square$

Remark : Nelson, Pitale and Saha [32] also considered the integrals $Z_v(s)$ and gave explicit formulas of $Z_v(s)$. However, as already remarked in [32, 1.3], it seems difficult to give a simple formula of $Z_v(s)$ for $v\in S_\pi$.

13.3.1. *Spectral parameters.* Let $\pi\in\Pi_{\rm cus}(l,\mathfrak{n})$. For any $v\in\Sigma_{\rm fin}-S(\mathfrak{f}_\pi)$, the v-th component π_v of π is isomorphic to the $\mathbf{K}_v$-spherical principal series representation

$$I_v(\nu_v)=\mathrm{Ind}_{B_v}^{G_v}(|\cdot|_v^{\nu_v/2}\boxtimes|\cdot|_v^{-\nu_v/2})$$

where ν_v belongs to $i[0,2\pi(\log q_v)^{-1}]\cup\{x+iy\mid x\in(0,1),\,y\in\{0,2\pi(\log q_v)^{-1}\}\}$. The point $\nu_S(\pi)=(\nu_v)_{v\in S}$ of $\mathfrak{X}_S$ is called the spectral parameter of π at S.

13.4. The spectral side. We can describe the coefficients of $\hat{\mathbf{\Psi}}_{\beta,\lambda}^l(\mathfrak{n}|\alpha)$ in the L^2-expansion (12.5) in terms of $(H,\mathbf{1})$-period integrals and the spectral parameters of representations in $\Pi_{\rm cus}(l,\mathfrak{n})$.

Lemma 13.5. *Let $\pi\in\Pi_{\rm cus}(l,\mathfrak{n})$ and $\nu_S(\pi)=(\nu_v(\pi))_{v\in S}$ the spectral parameter of π at S. Then, for any $\varphi\in V_\pi[\tau_l]^{\mathbf{K}_0(\mathfrak{n})}$ and for $1/2<\mathrm{Re}(\lambda)<\underline{l}/2-1$, we have*

$$\langle\hat{\mathbf{\Psi}}_{\beta,\lambda}^l(\mathfrak{n}|\alpha)|\varphi\rangle_{L^2}=(-1)^{\#S}\{\prod_{v\in\Sigma_\infty}2^{l_v-1}\}D_F^{-1/2}[\mathbf{K}_{\rm fin}:\mathbf{K}_0(\mathfrak{n})]^{-1}\alpha(\nu_S(\pi))P_{\beta C_l,\lambda}^{\mathbf{1}}(\overline{\varphi}).$$

Proof. In the same way as [47, Lemma 9.2] with the aid of the majorant $\Xi_{l,\mathrm{Re}(\lambda)-\epsilon,q(\mathbf{c}),S}$ for any sufficiently small $\epsilon>0$ (Note: in the proof of [47, Lemma 9.2], the majorant of the integral (9.3) should be $\Xi_{\mathrm{Re}(\lambda)-\epsilon,q(\mathbf{c}),S_{\rm fin}}$), we have

$$\langle\hat{\mathbf{\Psi}}_{\beta,\lambda}^l(\mathfrak{n}|\alpha)|\varphi\rangle_{L^2}=\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\langle\mathbf{\Psi}_{\beta,\lambda}^l(\mathfrak{n}|\mathbf{s}),\bar\varphi\rangle\alpha(\mathbf{s})d\mu_S(\mathbf{s})$$

for any rapidly decreasing function $\varphi \in C^\infty(Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}})[\tau_l]^{\mathbf{K}_0(\mathfrak{n})}$, where $q(\mathbf{c})$ is sufficiently large. Contrary to [47, Lemma 9.2], the condition $\mathrm{Re}(\lambda) > 1$ is not needed. Indeed, in the proof of [47, Lemma 9.2], the estimate $|\varphi(g)| \ll ||g||_{\mathbb{A}}^{1+\epsilon}$ is replaced with $|\varphi(g)| \ll ||g||_{\mathbb{A}}^{-m}$ for any $m > 0$, and moreover, $\int_1^\infty y^{-\mathrm{Re}(\lambda)+1+2\epsilon} d^\times y$ is replaced with $\int_1^\infty y^{-\mathrm{Re}(\lambda)-m+\epsilon} d^\times y$ (Note: in the proof of [47, Lemma 9.2], $\int_1^\infty y^{-\mathrm{Re}(\lambda)+1} d^\times y$ should be $\int_1^\infty y^{-\mathrm{Re}(\lambda)+1+2\epsilon} d^\times y$). Thus, $\langle \hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha)|\varphi\rangle_{L^2}$ is equal to

$$\left(\frac{1}{2\pi i}\right)^{\#S} \int_{\mathbb{L}_S(\mathbf{c})} \langle \mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}), R(\mathbb{T}_S(\mathbf{s}))\bar{\varphi}\rangle$$
$$\times \{\prod_{v\in S}(q_v^{(1+\mathfrak{o})/2} + q_v^{(1-\nu_v(\pi))/2} - q_v^{(1+s_v)/2} - q_v^{(1-s_v)/2})\}^{-1}\alpha(\mathbf{s})d\mu_S(\mathbf{s}).$$

Here we use $q_v^{(1+\nu_v(\pi))/2} + q_v^{(1-\nu_v(\pi))/2} \in \mathbb{R}$. By Lemma 12.4, $\langle \mathbf{\Psi}^l_{\beta,\lambda}(\mathfrak{n}|\mathbf{s}), R(\mathbb{T}_S(\mathbf{s}))\bar{\varphi}\rangle$ is independent of $\mathbf{s}$, and hence [47, Lemma 9.5] works. We also note $\mathrm{vol}(H_{\mathrm{fin}}\backslash H_{\mathrm{fin}}\mathbf{K}_0(\mathfrak{n})) = D_F^{-1/2}[\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{n})]$ (cf. [47, Lemma 8.3]). As a result, we obtain the desired formula. □

By this lemma and (12.5), we have

$$\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; g) =(-1)^{\#S}\{\prod_{v\in\Sigma_\infty} 2^{l_v-1}\}D_F^{-1/2}[\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{n})]^{-1}$$
$$\times \sum_{\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})} \sum_{\varphi\in\mathcal{B}(\pi;l,\mathfrak{n})} \alpha(\nu_S(\pi))P^{\mathbf{1}}_{\beta C_l,\lambda}(\overline{\varphi})\varphi(g), \quad g \in G_{\mathbb{A}}.$$

The integral $P^{\mathbf{1}}_{\beta C_l,\lambda}(\overline{\varphi})$ is continued to an entire function in λ for any cusp form φ by [47, Lemma 7.3]. As a finite linear combination of such, the function $\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; g)$ has a holomorphic analytic continuation to the whole λ-plane. Since $\mathrm{CT}_{\lambda=0}P^\eta_{\beta C_l,\lambda}(\bar{\varphi}) = C_l(0)\, P^\eta_{\mathrm{reg}}(\bar{\varphi})\, \beta(0)$, we can define the regularized automorphic smoothed kernel $\hat{\mathbf{\Psi}}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha; g)$ by the relation

$$\mathrm{CT}_{\lambda=0}\hat{\mathbf{\Psi}}^l_{\beta,\lambda}(\mathfrak{n}|\alpha; g) = \hat{\mathbf{\Psi}}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha; g)\beta(0)$$

for any $\beta \in \mathcal{B}$. Indeed, we have the expression

$$\hat{\mathbf{\Psi}}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha; g) = \frac{(-1)^{\#S}\{\prod_{v\in\Sigma_\infty} 2^{l_v-1}\}C_l(0)D_F^{-1/2}}{[\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{n})]} \sum_{\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})} \sum_{\varphi\in\mathcal{B}(\pi;l,\mathfrak{n})} \alpha(\nu_S(\pi))\overline{P^{\mathbf{1}}_{\mathrm{reg}}(\varphi)}\varphi(g),$$

which is valid pointwisely with the summation being finite. From this, the regularized (H, η)-period $P^\eta_{\mathrm{reg}}(\hat{\mathbf{\Psi}}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha))$ is explicitly described as follows.

Proposition 13.6. *Suppose $\underline{l} \geqslant 4$. The function $\hat{\mathbf{\Psi}}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha)$ has the regularized (H, η)-period given by*

$$P^\eta_{\mathrm{reg}}(\hat{\mathbf{\Psi}}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha)) =(-1)^{\#S}\{\prod_{v\in\Sigma_\infty} 2\pi\frac{\Gamma(l_v-1)}{\Gamma(l_v/2)^2}\}D_F^{-1}[\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{n})]^{-1} \times (-1)^{\epsilon(\eta)}\mathcal{G}(\eta)$$
$$\times \sum_{\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})} w^\eta_{\mathfrak{n}}(\pi)\frac{L(1/2, \pi)L(1/2, \pi\otimes\eta)}{2\mathrm{N}(\mathfrak{f}_\pi)[\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{f}_\pi)]^{-1}L^{S_\pi}(1, \pi, \mathrm{Ad})}\alpha(\nu_S(\pi)).$$

Proof. By cuspidality of $\varphi \in \mathcal{B}(\pi; l, \mathfrak{n})$, $P^\eta_{\mathrm{reg}}(\varphi)$ becomes the usual absolutely convergent integral

$$\int_{F^\times\backslash\mathbb{A}^\times} \varphi\left(\left[\begin{smallmatrix} t & 0\\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & x_\eta\\ 0 & 1\end{smallmatrix}\right]\right)\, \eta(tx^*_\eta)\, d^\times t.$$

Thus, by term wise integration, we have

$$
\begin{aligned}
P^{\eta}_{\rm reg}(\hat{\mathbf{\Psi}}^{l}_{\rm reg}(\mathfrak{n}|\alpha)) =&(-1)^{\#S}\{\prod_{v\in\Sigma_\infty}2^{l_v-1}\}C_l(0)D_F^{-1/2}[\mathbf{K}_{\rm fin}:\mathbf{K}_0(\mathfrak{n})]^{-1}\\
&\times\sum_{\pi\in\Pi_{\rm cus}(l,\mathfrak{n})}\sum_{\varphi\in\mathcal{B}(\pi;l,\mathfrak{n})}\overline{P^{\mathbf{1}}_{\rm reg}(\varphi)}P^{\eta}_{\rm reg}(\varphi)\alpha(\nu_S(\pi)).
\end{aligned}
$$

Then we obtain the assertion by Lemma 11.4 (ii), Proposition 13.1, Lemmas 13.2 and 13.4. $\square$

14. Geometric expansions

Suppose $\underline{l}=\inf_{v\in\Sigma_\infty}l_v\geqslant 4$. In this section and the next, we compute the quantity $P^{\eta}_{\rm reg}(\hat{\mathbf{\Psi}}^{l}_{\rm reg}(\mathfrak{n}|\alpha))$ by using the series expression (12.1) (cf. §9). The firt step is to break the sum in (12.1) over $H_F\backslash G_F$ to a sum of subseries according to double cosets $H_F\delta H_F$. For $\delta\in G_F$, we put $\mathrm{St}(\delta):=H_F\cap\delta^{-1}H_F\delta$. Then, the following elements of G_F form a complete set of representatives of the double coset space $H_F\backslash G_F/H_F$:

$$
\begin{gathered}
e=\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right],\ w_0=\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right],\\
u=\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right],\ \overline{u}=\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right],\ uw_0=\left[\begin{smallmatrix}1&-1\\1&0\end{smallmatrix}\right],\ \overline{u}w_0=\left[\begin{smallmatrix}0&-1\\1&-1\end{smallmatrix}\right],\\
\delta_b=\left[\begin{smallmatrix}1+b^{-1}&1\\1&1\end{smallmatrix}\right],\ \ b\in F^\times-\{-1\}.
\end{gathered}
$$

Moreover, we have $\mathrm{St}(e)=\mathrm{St}(w_0)=H_F$ and $\mathrm{St}(\delta)=Z_F$ for any $\delta\in\{u,\overline{u},uw_0,\overline{u}w_0\}\cup\{\delta_b|b\in F^\times-\{-1\}\}$. (See [35, Lemma 1] and [47, Lemma 11.1]). Thus we obtain the following expression for $\mathrm{Re}(\lambda)>0$:

$$
\hat{\mathbf{\Psi}}^{l}_{\beta,\lambda}\left(\mathfrak{n}|\alpha;\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\right)=\sum_{\delta}J^l_\delta(\beta,\lambda,\alpha;t),
$$

where δ runs through the double coset representatives listed above and, for each such δ, $J^l_\delta(\beta,\lambda,\alpha;t)$ is the sum of $\hat{\Psi}^{l}_{\beta,\lambda}\left(\mathfrak{n}|\alpha;\delta\gamma\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\right)$ for $\gamma\in\mathrm{St}(\delta)\backslash H_F$.

Lemma 14.1. *The function $\lambda\mapsto J^l_e(\beta,\lambda,\alpha;t)$ and $\lambda\mapsto J^l_{w_0}(\beta,\lambda,\alpha;t)$ are entire on $\mathbb{C}$. Moreover their values at $\lambda=0$ are $J^l_{\rm id}(\alpha;t)\beta(0)$ and $i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o})J^l_{\rm id}(\alpha;t)\beta(0)$, respectively, where*

$$
J^l_{\rm id}(\alpha;t)=\delta(\mathfrak{f}_\eta=\mathfrak{o})\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\Upsilon^{\mathbf{1}}_S(\mathbf{s})\alpha(\mathbf{s})d\mu_S(\mathbf{s})
$$

with $\tilde{l}=\sum_{v\in\Sigma_\infty}l_v$ and

$$
\Upsilon^{\mathbf{1}}_S(\mathbf{s})=\prod_{v\in S}(1-q_v^{-(s_v+1)/2})^{-1}(1-q_v^{(s_v+1)/2})^{-1}.
$$

Proof. Since $\Psi^{(0)}_v(l_v;1_2)=1$ for all $v\in\Sigma_\infty$, the assertion is proved in the same way as [47, Lemma 11.2]. $\square$

We put

$$
J^l_{\rm u}(\beta,\lambda,\alpha;t)=J^l_u(\beta,\lambda,\alpha,t)+J^l_{\overline{u}w_0}(\beta,\lambda,\alpha,t)
$$

and

$$
J^l_{\bar{\rm u}}(\beta,\lambda,\alpha;t)=J^l_{uw_0}(\beta,\lambda,\alpha,t)+J^l_{\bar{u}}(\beta,\lambda,\alpha,t).
$$

Lemma 14.2. *For $*\in\{\mathrm{u},\bar{\mathrm{u}}\}$, the function $\lambda\mapsto J^l_*(\beta,\lambda,\alpha;t)$ on $\mathrm{Re}(\lambda)>0$ has a holomorphic continuation to $\mathbb{C}$ whose value at $\lambda=0$ is equal to $J^l_*(\alpha;t)\beta(0)$, where*

$$
J^l_{\rm u}(\alpha;t)=\left(\frac{1}{2\pi i}\right)^{\#S}\sum_{a\in F^\times}\int_{\mathbb{L}_S(\mathbf{c})}\left\{\Psi^{(0)}_l\left(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&at^{-1}\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\right)+\Psi^{(0)}_l\left(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&0\\at^{-1}&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&0\\-x_\eta&1\end{smallmatrix}\right]w_0\right)\right\}\alpha(\mathbf{s})d\mu_S(\mathbf{s})
$$

and

$$J_{\bar{\mathrm{u}}}^{l}(\alpha;t)=\left(\frac{1}{2\pi i}\right)^{\#S}\sum_{a\in F^\times}\int_{\mathbb{L}_S(\mathbf{c})}\left\{\Psi_l^{(0)}\left(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&0\\at&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\right)+\Psi_l^{(0)}\left(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&at\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&0\\-x_\eta&1\end{smallmatrix}\right]w_0\right)\right\}\alpha(\mathbf{s})d\mu_S(\mathbf{s}).$$

Proof. We follow the proof of [47, Lemma 11.3]. Take $\sigma>0$ such that $\underline{l}/2>\sigma+1$. Let us examine $J_{\mathrm{u}}^{l}(\beta,\lambda,\alpha;t)$. First we consider the sum of the functions $\hat{\Psi}_{\beta,\lambda}^{l}\left(\mathfrak{n}|\alpha;u\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}a^{-1}&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\right)$ over all $a\in F^\times$. We have

$$\begin{aligned}&\hat{\Psi}_{\beta,\lambda}^{l}\left(\mathfrak{n}|\alpha;u\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}a^{-1}&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\right)\\&=\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\left\{\frac{1}{2\pi i}\int_{L_\sigma}\frac{\beta(z)}{z+\lambda}\{|t|_{\mathbb{A}}^{z}\Psi^{(z)}\left(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&at^{-1}\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\right)\right.\\&\quad\left.+|t|_{\mathbb{A}}^{-z}\Psi^{(-z)}(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&at^{-1}\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right])\}dz\right\}\alpha(\mathbf{s})d\mu_S(\mathbf{s}).\end{aligned}$$

Here $\mathbf{c}$ is taken so that $q(\mathbf{c})$ is sufficiently large. There exists an ideal $\mathfrak{a}$ of F such that the estimate

$$\left|\Psi_l^{(\pm z)}\left(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&at^{-1}\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\right)\right|\ll f(a),\qquad a\in F^\times,(\mathbf{s},z)\in\mathbb{L}_S(\mathbf{c})\times L_\sigma$$

holds, where

$$f(a)=\prod_{v\in\Sigma_\infty}|1+ia_vt_v^{-1}|_v^{\sigma-l_v/2}\prod_{v\in S}\sup(1,|a_vt_v^{-1}|_v)^{-(2q(\mathbf{c})-\sigma)}\prod_{\Sigma_{\mathrm{fin}}-S}\delta(a_v\in\mathfrak{a}\mathfrak{o}_v),\quad a\in\mathbb{A}.$$

Thus to establish the absolute convergence of the sum of $\hat{\Psi}_{\beta,\lambda}^{l}\left(\mathfrak{n}|\alpha;u\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}a^{-1}&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\right)$ over $a\in F^\times$, it is enough to show $\sum_{a\in F^\times}f(a)<+\infty$. The convergence of the latter sum in turn follows from the convergence of the integral $\int_{\mathbb{A}}f(a)da$, which is a product of the archimedean integrals for all $v\in\Sigma_\infty$ convergent when $l_v/2-\sigma>1$ and the non-archimedean ones convergent for sufficiently large $q(\mathbf{c})$.

The sum of the functions $\hat{\Psi}_{\beta,\lambda}^{l}\left(\mathfrak{n}|\alpha;\bar{u}w_0\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}a^{-1}&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\right)$ over $a\in F^\times$ is analyzed similarly. By the estimate

$$\left|\Psi_l^{(\pm z)}\left(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&0\\at^{-1}&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&0\\-x_\eta&1\end{smallmatrix}\right]w_0\right)\right|\ll f(a),\qquad a\in F^\times,(\mathbf{s},z)\in\mathbb{L}_S(\mathbf{c})\times L_\sigma,$$

the problem is reduced to the convergence of the same series $\sum_{a\in F^\times}f(a)$ as above. Hence the assertion on $J_{\mathrm{u}}^{l}(\beta,\lambda,\alpha;t)$ is obtained. The integral $J_{\bar{\mathrm{u}}}^{l}(\beta,\lambda,\alpha;t)$ is examined in the same way. This completes the proof. $\square$

14.1. Hyperbolic terms. We consider the convergence of

$$J_{\mathrm{hyp}}^{l}(\beta,\lambda,\alpha;t)=\sum_{b\in F^\times-\{-1\}}J_{\delta_b}^{l}(\beta,\lambda,\alpha;t).$$

Let $v\in\Sigma_\infty$. For $t\in F_v^\times$, $b\in F_v^\times-\{-1\}$ and $\sigma,\rho\in\mathbb{R}$, set

$$f^{(\sigma)}(l_v;t,b)=\{(b+1)^2t^2+b^2\}^{\sigma/2-l_v/4}(1+t^{-2})^{-\sigma/2-l_v/4}|t|_v^{-2\sigma}$$

and

$$M_v(\sigma,\rho,l_v;b)=|b+1|_v^{-(\sigma-\rho)_-}|b|_v^{l_v/4-\sigma/2}\times\int_{F_v^\times}f^{(\sigma)}(l_v;t,b)|t|_v^{\sigma+\rho}d^\times t,$$

where $q_-=\inf(0,q)$ for $q\in\mathbb{R}$.

Lemma 14.3. *Let $v\in\Sigma_\infty$ and let $l_v\geqslant 2$ be an even integer. Then, for any $\sigma\in\mathbb{R}$ we have*

$$\left|\Psi_v^{(z)}\left(l_v;\left[\begin{smallmatrix}1+b^{-1}&1\\1&1\end{smallmatrix}\right]\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\right)\right|\leqslant|b|_v^{-\sigma}|t|_v^{\sigma}e^{\pi|\mathrm{Im}(z)|/2}f_v^{(\sigma)}(l_v;t,b),\qquad t\in F_v^\times,\ b\in F_v^\times-\{-1\},\ z\in L_\sigma.$$

Proof. By writing the Iwasawa decomposition $\left[\begin{smallmatrix}1+b^{-1}&1\\1&1\end{smallmatrix}\right]\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]=\left[\begin{smallmatrix}r&0\\0&r^{-1}\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right]k$ explicitly, we have $|1+ix|=(1+t^{-2})^{1/2}((b+1)^2t^2+b^2)^{1/2}$. Then the assertion follows from Proposition 11.1 and Lemma 11.2. $\square$

Lemma 14.4. *Let $v \in \Sigma_\infty$ and $l_v \in 2\mathbb{Z}_{\geqslant 2}$. Let $\sigma, \rho \in \mathbb{R}$. Then the estimate*

$$M_v(\sigma,\rho,l_v;b) \ll |b+1|_v^{-l_v/4+\sigma/2-(\sigma-\rho)_-}, \qquad b \in F_v^\times - \{-1\} \tag{14.1}$$

holds if $l_v/4 > |\rho| - \sigma/2$ and $l_v/4 > \sigma/2$. Moreover, for $\epsilon > 0$ and $c \in \mathbb{R}$, the function $|b(b+1)|_v^\epsilon\,|b|_v^{-l_v/4+(c+1)/4}\,M_v(\sigma,\rho,l_v;b)$ in $b \in F_v$ is locally bounded if

$$\big||\rho|-\sigma\big| + (\sigma-\rho)_- < \epsilon/3 < 1, \quad l_v/4 > \sigma/2 - (\sigma-\rho)_- + 1, \quad (c+1)/4 > \sigma/2 - (\sigma-\rho)_-. \tag{14.2}$$

Proof. The assertion is proved in a similar way to [47, Lemma 11.14]. By $b^2 + t^2(b+1)^2 \geqslant 2|b|\,|b+1|\,|t|$ and $\sigma/2 - l_v/4 < 0$, we estimate

$$\begin{aligned}M_v(\sigma,\rho,l_v;b) \ll& |b+1|^{-(\sigma-\rho)_-}|b|^{l_v/4-\sigma/2} \times \int_0^\infty \{|b||b+1||t|\}^{\sigma/2-l_v/4}(1+t^{-2})^{-\sigma/2-l_v/4}|t|^{-\sigma+\rho}d^\times t\\ =&|b+1|^{-l_v/4+\sigma/2-(\sigma-\rho)_-}\int_0^\infty |t|^{\rho+l_v/4+\sigma/2}(1+t^2)^{-\sigma/2-l_v/4}d^\times t.\end{aligned}$$

The integral converges absolutely if $l_v/4 > |\rho| - \sigma/2$. In the same way as in the proof of [47, Lemma 11.14], we have

$$\begin{aligned}&|b(b+1)|^\epsilon\,|b|^{-l_v/4+(c+1)/4}\,M_v(\sigma,\rho,l_v;b)\\ \ll&|b+1|^{\sigma-|\rho|-(\sigma-\rho)_-+\epsilon/3}|b|^{(c+1)/4+\sigma/2-|\rho|+\epsilon/3}\,|b(b+1)|^{\epsilon/3}\mathfrak{m}(r;b(b+1)),\end{aligned}$$

where $r = l_v + 2\sigma - 4|\rho| - 4\epsilon/3$ and $\mathfrak{m}(r;b(b+1)) = \int_0^\infty [(1+t^{-2})(b^2+t^2(b+1)^2\}]^{-r/4}\,d^\times t$. By [47, Lemma 15.5], the function $|b(b+1)|^\epsilon\,\mathfrak{m}(r;b(b+1))$ (with $r > 0$) is locally bounded on F_v. From this, $|b(b+1)|^\epsilon|b|^{-l_v/4+(c+1)/4}M_v(\sigma,\rho,l_v;b)$ is also locally bounded on F_v if

$$\sigma - |\rho| - (\sigma-\rho)_- + \epsilon/3 \geqslant 0, \quad r = l_v + 2\sigma - 4|\rho| - 4\epsilon/3 > 0, \quad (c+1)/4 + \sigma/2 - |\rho| + \epsilon/3 \geqslant 0.$$

This condition is satisfied by (14.2). Thus, under (14.2), the estimate (14.1) is extendable to F_v; from this, the last assertion is obvious. □

Let $\mathbf{c} = (c_v)_{v\in S} \in \mathbb{R}^S$, $l = (l_v)_{v\in\Sigma_\infty} \in (2\mathbb{Z}_{\geqslant 2})^{\Sigma_\infty}$, $t \in \mathbb{A}^\times$, $b \in F^\times - \{-1\}$ and $\sigma, \rho \in \mathbb{R}$. For $v \in S$, we put

$$\begin{aligned}f_v^{(\sigma)}(c_v;t_v,b) &= \inf(1,|t_v|_v^{-2})^\sigma \begin{cases}\sup(1,|t_v|_v^{-1}|b|_v)^{-(c_v+1)/2+\sigma} & (|t_v|_v \leqslant 1),\\ \sup(1,|t_v|_v|b+1|_v)^{-(c_v+1)/2+\sigma} & (|t_v|_v > 1),\end{cases}\\ M_v(\sigma,\rho,c;b) &= \sup(1,|b+1|_v)^{-(c+1)/4+\sigma/2+|\sigma-\rho|},\end{aligned}$$

and for $v \in \Sigma_{\rm fin} - S$, we put

$$f_v^{(\sigma)}(t_v,b) = \inf(1,|t_v|^{-2})^\sigma\,\delta(b \in \mathfrak{p}_v^{-f(\eta_v)}, q_v^{-2f(\eta_v)}|b|_v \leqslant |t_v|_v \leqslant |b+1|_v^{-1}).$$

Then, define

$$\begin{aligned}N(\mathfrak{n}|\sigma,l,\mathbf{c};t,b) =&|t|_{\mathbb{A}}^\sigma \prod_{v\in\Sigma_\infty} f_v^{(\sigma)}(l_v;t_v,b)\prod_{v\in S} f_v^{(\sigma)}(c_v;t_v,b)\\ &\times\prod_{v\in S(\mathfrak{n})}\delta(t_v\in\mathfrak{n}\mathfrak{o}_v)f_v^{(\sigma)}(t_v,b)\prod_{v\in\Sigma_{\rm fin}-(S\cup S(\mathfrak{n}))} f_v^{(\sigma)}(t_v,b),\end{aligned}$$

$$\begin{aligned}M(\mathfrak{n}|\sigma,\rho,l,\mathbf{c};b) =& \prod_{v\in\Sigma_\infty}|b|_v^{-l_v/4+\sigma/2}M_v(\sigma,\rho,l_v;b)\prod_{v\in S}|b|_v^{-(c_v+1)/4+\sigma/2}M_v(\sigma,\rho;c_v,b)\\ &\times\prod_{v\in\Sigma_{\rm fin}}\sup(1,|b|_v^{\sigma+\rho})\prod_{v\in\Sigma_{\rm fin}-S}\delta(b\in\mathfrak{f}_\eta^{-1}\mathfrak{n}\mathfrak{o}_v)\end{aligned}$$

and $M_\epsilon(\mathfrak{n}|\sigma,\rho,l,\mathbf{c};b)=\{\prod_{v\in\Sigma_\infty}|b(b+1)|_v^\epsilon\}\,M(\mathfrak{n}|\sigma,\rho,l,\mathbf{c};b)$ for $\epsilon\geqslant 0$. By closely following [47, §11.4], we have the following series of lemmas.

Lemma 14.5. *If $q(\mathbf{c})>|\sigma|+1$, then we have*

$$\left|\Psi_l^{(z)}\left(\mathfrak{n}|\mathbf{s};\delta_b[\begin{smallmatrix} t & 0\\ 0 & 1\end{smallmatrix}][\begin{smallmatrix} 1 & x_\eta\\ 0 & 1\end{smallmatrix}]\right)\right|\ll N(\mathfrak{n}|\sigma,l,\mathbf{c};t,b)e^{d_F\pi|\mathrm{Im}(z)|/2},\quad (z,\mathbf{s})\in L_\sigma\times\mathbb{L}_S(\mathbf{c}),\ b\in F^\times-\{-1\},\ t\in\mathbb{A}^\times$$

with the implied constant is independent of $\mathfrak{n}$.

Proof. This follows from Lemma 14.3, [47, Corollary 11.6, Lemma 11.10] and [47, Corollary 11.7], which can be generalized to the case of arbitrary ideals $\mathfrak{n}$; in the assertion of [47, Corollary 11.7], the factor $\delta(t\in\mathfrak{p}_v)$ is replaced with $\delta(t\in\mathfrak{n}\mathfrak{o}_v)$. □

Lemma 14.6. *If $q(\mathbf{c})>|\sigma|+|\rho|+1$, $\underline{l}/4>\sup(\sigma/2,|\rho|-\sigma/2)$ and $\sigma\neq\pm\rho$, then we have*

$$\int_{\mathbb{A}^\times}N(\mathfrak{n}|\sigma,l,\mathbf{c};t,b)|t|_{\mathbb{A}}^\rho d^\times t\ll_\epsilon M_\epsilon(\mathfrak{n}|\sigma,\rho,l,\mathbf{c};b)\mathrm{N}(\mathfrak{n})^\epsilon,\qquad b\in F^\times-\{-1\}$$

for any $\epsilon>0$, with the implied constant independent of the ideal $\mathfrak{n}$.

Proof. We can apply the same argument in [47, Lemma 11.16] by using l_ι in place of $c_\iota+1$ for all $\iota\in\Sigma_\infty$. □

Lemma 14.7. *Let U be a compact subset of $\mathbb{A}^\times$. If $q(\mathbf{c})>|\sigma|+|\rho|+1$, $\underline{l}/4>\sup(\sigma/2,|\rho|-\sigma/2)$ and $\sigma\neq\pm\rho$, then we have*

$$\sum_{t\in F^\times}N(\mathfrak{n}|\sigma,l,\mathbf{c};t,b)\ll_\epsilon M_\epsilon(\mathfrak{n}|\sigma,\rho,l,\mathbf{c};b)\,\mathrm{N}(\mathfrak{n})^\epsilon,\qquad b\in F^\times,\ t\in U$$

for any $\epsilon>0$, with the implied constant independent of the ideal $\mathfrak{n}$.

Proof. This follows from Lemma 14.6 and the argument in [47, Corollary 11.17]. □

Lemma 14.8. *If $\sigma+\rho>-1$, $\sigma\neq\pm\rho$, $(c+1)/4>5|\sigma|/2+2|\rho|+1$, $\underline{l}/4>|\sigma|+|\rho|+1$ and $\underline{l}/2>(c+1)/4+3|\sigma|/2+|\rho|+1$ hold, then, we have the estimate*

$$\sum_{b\in F^\times-\{-1\}}M_\epsilon(\mathfrak{n}|\sigma,\rho,l,\underline{c};b)\ll\mathrm{N}(\mathfrak{n})^{-(c+1)/4+\sigma/2+|\sigma+\rho|}$$

for some $\epsilon>0$ such that $\big||\rho|-\sigma\big|+(\sigma-\rho)_-<\epsilon/3<1$ and $\underline{l}/2>(c+1)/4+3|\sigma|/2+|\rho|+1+2\epsilon$, with the implied constant independent of $\mathfrak{n}$. Here $\underline{c}=(c_v)_{v\in S}$ with $c_v=c\,(\forall v\in S)$.

Proof. We give a proof in a similar way to [47, Lemma 11.19], replacing $c_\iota+1$ with l_ι for all $\iota\in\Sigma_\infty$. Under the assumption on l,σ,ρ,c in this lemma, the series

$$\sum_{b\in\mathfrak{o}(S)-\{-1\}}\{\prod_{v\in\Sigma_\infty}|b|_v^{-l_v/4+(c+1)/4}M_v(\sigma,\rho,l_v;b)\}\{\prod_{v\in S}\sup(1,|b|_v^{\sigma+\rho})M_v(\sigma,\rho,c_v;b)\}\,|\mathrm{N}(b(b+1))|^\epsilon,$$

which is denoted by $A_S(\sigma,\rho,l,c)$, converges for some $\epsilon>0$ such that $\big||\rho|-\sigma\big|+(\sigma-\rho)_-<\epsilon/3<1$ and $\underline{l}/2>(c+1)/4+3|\sigma|/2+|\rho|+1+2\epsilon$. Here $\mathfrak{o}(S)$ denotes the S-integer ring of F. Indeed, this follows from Lemma 14.4 and [47, Lemma 11.18]. By noting the Artin product formula $|b|_{\mathbb{A}}=1$ for $b\in F^\times$, we have

$$\begin{aligned}
&\sum_{b\in F^\times-\{-1\}} M_\epsilon(\mathfrak{n}|\sigma,\rho,l,\underline{c};b)\\
=&\sum_{b\in \mathfrak{f}_\eta^{-1}\mathfrak{n}\mathfrak{o}(S)-\{0,-1\}} \{\prod_{v\in\Sigma_{\rm fin}}\sup(1,|b|_v^{\sigma+\rho})\}\{\prod_{v\in S}|b|_v^{-(c+1)/4+\sigma/2}M_v(\sigma,\rho,c;b)\}\\
&\times\{\prod_{v\in\Sigma_\infty}|b|^{-l_v/4+\sigma/2}M_v(\sigma,\rho,l_v;b)\}|\mathrm{N}(b(b+1))|^\epsilon\\
=&\sum_{\substack{b\in \mathfrak{f}_\eta^{-1}\mathfrak{n}\mathfrak{o}(S)/\mathfrak{o}(S)^\times\\ b\neq 0,-1}} \{\prod_{v\in\Sigma_{\rm fin}-S}\sup(1,|b|_v^{\sigma+\rho})|b|_v^{(c+1)/4-\sigma/2}\}\\
&\times\sum_{u\in\mathfrak{o}(S)^\times}\{\prod_{v\in\Sigma_\infty}|ub|^{-l_v/4+(c+1)/4}M_v(\sigma,\rho,l_v;ub)\}\{\prod_{v\in S}\sup(1,|ub|_v^{\sigma+\rho})M_v(\sigma,\rho,c;ub)\}\\
&\times|\mathrm{N}(ub(ub+1))|^\epsilon\\
\ll&\sum_{\substack{b\in \mathfrak{f}_\eta^{-1}\mathfrak{n}\mathfrak{o}(S)/\mathfrak{o}(S)^\times\\ b\neq 0,-1}} \{\prod_{v\in\Sigma_{\rm fin}-S}\sup(1,|b|_v^{\sigma+\rho})|b|_v^{(c+1)/4-\sigma/2}\}\times A_S(\sigma,\rho,l,c).
\end{aligned}$$

We note that the series in the last as above is majorized by $\mathrm{N}(\mathfrak{n})^{-(c+1)/4+\sigma/2+|\sigma+\rho|}$ as in the proof of [47, Lemma 11.19]. $\square$

Lemma 14.9. *Let $l=(l_v)_{v\in\Sigma_\infty}\in(2\mathbb{Z}_{\geqslant 2})^{\Sigma_\infty}$ and $c,\sigma\in\mathbb{R}$. Assume the following conditions:*

- $\underline{l}\geqslant 6$,
- $\sigma>-1$,
- $(c+1)/4>9|\sigma|/2+1$,
- $\underline{l}/2>(c+1)/4+5|\sigma|/2+1$.

Then, for any compact subset U of $\mathbb{A}^\times$, the series

$$\sum_{b\in F^\times-\{-1\}}\sum_{a\in F^\times}\left|\Psi_l^{(z)}\left(\mathfrak{n}|\mathbf{s};\delta_b\left[\begin{smallmatrix}at&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\right)\right|$$

converges uniformly in $(t,z,\mathbf{s})\in U\times L_\sigma\times\mathbb{L}_S(\underline{c})$, and there exists $\epsilon>0$ such that, for any $\rho\in\mathbb{R}$ satisfying $0<||\rho|-\sigma|<\epsilon$ and $\sigma+\rho>-1$, the integral

$$\sum_{b\in F^\times-\{-1\}}\int_{t\in\mathbb{A}^\times}\left|\Psi_l^{(z)}\left(\mathfrak{n}|\mathbf{s};\delta_b\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\right)\right||t|_\mathbb{A}^\rho d^\times t$$

converges uniformly in $(z,\mathbf{s})\in L_\sigma\times\mathbb{L}_S(\underline{c})$.

Proof. By assumption, we can take $\rho\in\mathbb{R}$ such that $(c+1)/4>5|\sigma|/2+2|\rho|+1$, $\sigma+\rho>-1$, $\underline{l}/4>|\sigma|+|\rho|+1$ and $\underline{l}/2>(c+1)/4+3|\sigma|/2+|\rho|+1$ (we can take $\rho=0$ if $\sigma>-1$ and $\sigma\neq0$). Thus the assertion follows from Lemmas 14.5, 14.6, 14.7, and 14.8. We remark that the assumption $\underline{l}\geqslant 6$ is indispensable in our estimation of hyperbolic terms as above. Indeed, the third and the fourth inequalities in Lemma 14.9 imply $\underline{l}/2>7|\sigma|+2$, and hence $\underline{l}>4$. $\square$

Lemma 14.10. *Suppose $\underline{l}\geqslant 6$. The function $J_{\rm hyp}^l(\beta,\lambda,\alpha;t)$ on $\mathrm{Re}(\lambda)>1$ has a holomorphic continuation to $\mathbb{C}$ whose value at $\lambda=0$ equals $J^l_{\rm hyp}(\alpha;t)\beta(0)$, where*

$$J^l_{\rm hyp}(\alpha;t)=\sum_{b\in F^\times-\{-1\}}\sum_{a\in F^\times}\hat{\Psi}_l^{(0)}\left(\mathfrak{n}|\alpha;\delta_b\left[\begin{smallmatrix}at&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\right),$$

where we put

$$\hat{\Psi}_l^{(z)}(\mathfrak{n}|\alpha;g)=\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\Psi_l^{(z)}(\mathfrak{n}|\mathbf{s};g)\alpha(\mathbf{s})d\mu_S(\mathbf{s})$$

for $\mathbf{c}\in\mathbb{R}^S$ *such that* $q(\mathbf{c})>|\operatorname{Re}(z)|+1$ *(cf. §7). The series converges absolutely and uniformly in* $t\in\mathbb{A}^\times$.

Proof. This follows from Lemma 14.9 in the same way as [47, Lemma 11.21]. $\square$

From Lemmas 14.1, 14.2 and 14.10, we have

$$\hat{\mathbf{\Psi}}_{\mathrm{reg}}^l\left(\mathfrak{n}|\alpha;\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\right)=(1+i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))J_{\mathrm{id}}^l(\alpha;t)+J_{\mathrm{u}}^l(\alpha;t)+J_{\bar{\mathrm{u}}}^l(\alpha;t)+J_{\mathrm{hyp}}^l(\alpha;t)\tag{14.3}$$

for any $t\in\mathbb{A}^\times$.

15. Geometric side

Suppose $\underline{l}=\inf_{v\in\Sigma_\infty}l_v\geqslant 6$. We fix a holomorphic function $\alpha(\mathbf{s})$ on $\mathfrak{X}_S$ such that $\alpha(\varepsilon\mathbf{s})=\alpha(\mathbf{s})$ for any $\varepsilon\in\{\pm1\}^S$. Let $\beta\in\mathcal{B}$ as before. For $\natural\in\{\mathrm{id},\mathrm{u},\bar{\mathrm{u}},\mathrm{hyp}\}$, we set

$$\mathbb{J}_\natural^\eta(l;\beta,\lambda;\alpha)=\int_{F^\times\backslash\mathbb{A}^\times}J_\natural^l(\alpha;t)\{\hat{\beta}_\lambda(|t|_\mathbb{A})+\hat{\beta}_\lambda(|t|_\mathbb{A}^{-1})\}\eta(tx_\eta^*)d^\times t.$$

In this section, we shall show that this integral converges absolutely when $\operatorname{Re}(\lambda)\gg1$ and has a meromorphic continuation to a neighborhood of $\lambda=0$; at the same time, we determine the constant term in its Laurent expansion at $\lambda=0$. As a result, by the identity

$$P_{\beta,\lambda}^\eta(\hat{\mathbf{\Psi}}_{\mathrm{reg}}^l(\mathfrak{n}|\alpha))=\mathbb{J}_{\mathrm{id}}^\eta(l;\beta,\lambda;\alpha)+i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o})\mathbb{J}_{\mathrm{id}}^\eta(l;\beta,\lambda;\alpha)+\mathbb{J}_{\mathrm{u}}^\eta(l;\beta,\lambda;\alpha)+\mathbb{J}_{\bar{\mathrm{u}}}^\eta(l;\beta,\lambda;\alpha)+\mathbb{J}_{\mathrm{hyp}}^\eta(l;\beta,\lambda;\alpha)$$

obtained from (14.3), we have another expression of $P_{\mathrm{reg}}^\eta(\hat{\mathbf{\Psi}}_{\mathrm{reg}}^l(\mathfrak{n}|\alpha))$ already computed in Proposition 13.6 by means of the spectral expansion.

For the term $\mathbb{J}_{\mathrm{id}}^\eta(l;\beta,\lambda;\alpha)$, we have the following lemma, which is proved in the same way as [47, Lemma 11.2].

Lemma 15.1. *For* $\operatorname{Re}(\lambda)>0$, *the integral* $\mathbb{J}_{\mathrm{id}}^\eta(l;\beta,\lambda;\alpha)$ *converges absolutely and we have*

$$\mathbb{J}_{\mathrm{id}}^\eta(l;\beta,\lambda;\alpha)=\delta_{\eta,\mathbf{1}}\mathrm{vol}(F^\times\backslash\mathbb{A}^1)\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\Upsilon_S^{\mathbf{1}}(\mathbf{s})\alpha(\mathbf{s})d\mu_S(\mathbf{s})\frac{2\beta(0)}{\lambda}.$$

Moreover, the function $\mathbb{J}_{\mathrm{id}}^\eta(l;\beta,\lambda;\alpha)$ *in* λ *has a meromorphic continuation to* $\mathbb{C}$ *with* $\mathrm{CT}_{\lambda=0}\mathbb{J}_{\mathrm{id}}^\eta(l;\beta,\lambda;\alpha)=0$.

Let us examine the terms $\mathbb{J}_{\mathrm{u}}^\eta(l;\beta,\lambda;\alpha)$ and $\mathbb{J}_{\bar{\mathrm{u}}}^\eta(l;\beta,\lambda;\alpha)$. Assume that $q(\operatorname{Re}(\mathbf{s}))>\operatorname{Re}(\lambda)>\sigma$ and $1<\sigma<\underline{l}/2$ and set

$$U_{0,\eta}^{\pm}(\lambda;\mathbf{s})=\frac{1}{2\pi i}\int_{L_{\mp\sigma}}\frac{\beta(z)}{z+\lambda}\int_{\mathbb{A}^\times}\Psi_l^{(0)}\left(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&t^{-1}\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\right)\eta(tx_\eta^*)|t|_\mathbb{A}^{\pm z}d^\times t dz,$$

$$U_{1,\eta}^{\pm}(\lambda;\mathbf{s})=\frac{1}{2\pi i}\int_{L_{\mp\sigma}}\frac{\beta(z)}{z+\lambda}\int_{\mathbb{A}^\times}\Psi_l^{(0)}\left(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&0\\t^{-1}&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&0\\-x_\eta&1\end{smallmatrix}\right]w_0\right)\eta(tx_\eta^*)|t|_\mathbb{A}^{\pm z}d^\times t dz$$

and

$$\Upsilon_S^\eta(z;\mathbf{s})=\prod_{v\in S}(1-\eta_v(\varpi_v)q_v^{-(z+(s_v+1)/2)})^{-1}(1-q_v^{(s_v+1)/2})^{-1},$$

$$\Upsilon_{S,l}^\eta(z;\mathbf{s})=D_F^{-1/2}\{\#(\mathfrak{o}/\mathfrak{f}_\eta)^\times\}^{-1}\{\prod_{v\in\Sigma_\infty}\frac{2\Gamma(-z)\Gamma(l_v/2+z)}{\Gamma_\mathbb{R}(-z+\epsilon_v)\Gamma(l_v/2)}i^{\epsilon_v}\cos\left(\frac{\pi}{2}(-z+\epsilon_v)\right)\}\Upsilon_S^\eta(z;\mathbf{s}).$$

Here $\epsilon_v\in\{0,1\}$ is the sign of η_v for $v\in\Sigma_\infty$ (see §1.4).

Lemma 15.2. *The double integrals $U_{0,\eta}^{\pm}$ and $U_{1,\eta}^{\pm}$ converge absolutely and*

$$U_{0,\eta}^{\pm}(\lambda;\mathbf{s}) = \frac{1}{2\pi i}\int_{L_{\mp\sigma}} \frac{\beta(z)}{z+\lambda}\mathrm{N}(\mathfrak{f}_\eta)^{\mp z}L(\mp z,\eta)(-1)^{\epsilon(\eta)}\Upsilon_{S,l}^{\eta}(\pm z;\mathbf{s})dz,$$
$$U_{1,\eta}^{\pm}(\lambda;\mathbf{s}) = \frac{1}{2\pi i}\int_{L_{\mp\sigma}} \frac{\beta(z)}{z+\lambda}\mathrm{N}(\mathfrak{f}_\eta)^{\mp z}\mathrm{N}(\mathfrak{n})^{\pm z}\tilde{\eta}(\mathfrak{n})\delta(\mathfrak{n}=\mathfrak{o})L(\mp z,\eta)i^{\tilde{l}}\Upsilon_{S,l}^{\eta}(\pm z;\mathbf{s})dz,$$

where $\tilde{l} = \sum_{v\in\Sigma_\infty} l_v$ and $\epsilon(\eta) = \sum_{v\in\Sigma_\infty}\epsilon_v$.

Proof. This is proved in the same way as [47, Lemma 12.3]; to compute the archimedean integral, we use Lemma 11.6. □

By $1<\sigma<\underline{l}/2$, the possible poles of the integrand of $U_{0,\eta}^{+}(\lambda;\mathbf{s})$ in the region $-\sigma<\mathrm{Re}(z)<\sigma$ are $z=0,-1$. In fact, we observe that the integrand is holomorphic at $z=-1$. We shift the contour $L_{-\sigma}$ to L_σ; by the residue theorem,

$$U_{0,\eta}^{+}(\lambda;\mathbf{s}) = \frac{1}{2\pi i}\int_{L_\sigma}\frac{\beta(z)}{z+\lambda}\mathrm{N}(\mathfrak{f}_\eta)^{-z}L(-z,\eta)(-1)^{\epsilon(\eta)}\Upsilon_{S,l}^{\eta}(z;\mathbf{s})dz - \frac{\beta(0)}{\lambda}\delta_{\eta,\mathbf{1}}R_F(-1)^{\epsilon(\eta)}\Upsilon_S^{\mathbf{1}}(\mathbf{s}),$$

where $\delta_{\eta,\mathbf{1}} = \delta(\eta=\mathbf{1})$ and $R_F = \mathrm{Res}_{s=1}\zeta_F(s)$. In a similar manner,

$$U_{1,\eta}^{+}(\lambda;\mathbf{s}) = \frac{1}{2\pi i}\int_{L_\sigma}\frac{\beta(z)}{z+\lambda}\mathrm{N}(\mathfrak{f}_\eta)^{-z}\,L(-z,\eta)\,\delta(\mathfrak{n}=\mathfrak{o})\,i^{\tilde{l}}\,\Upsilon_{S,l}^{\eta}(z;\mathbf{s})dz - \frac{\beta(0)}{\lambda}\delta_{\eta,\mathbf{1}}\,R_F\,\delta(\mathfrak{n}=\mathfrak{o})\,i^{\tilde{l}}\,\Upsilon_S^{\mathbf{1}}(\mathbf{s}).$$

Define $C_0(\eta)$ and $R(\eta)$ by

$$L(s,\eta) = R(\eta)(s-1)^{-1} + C_0(\eta) + \mathcal{O}(s-1),\ \ (s\to 1).$$

We remark that $R_F = R(\eta)$ if η is trivial.

Lemma 15.3. *The function $\lambda\mapsto \mathbb{J}_{\mathrm{u}}^{\eta}(l;\beta,\lambda;\alpha)$ on $\mathrm{Re}(\lambda)>1$ has a meromorphic continuation to the region $\mathrm{Re}(\lambda)>-\underline{l}/2$. The constant term of $\mathbb{J}_{\mathrm{u}}^{\eta}(l;\beta,\lambda;\alpha)$ at $\lambda=0$ equals $\mathbb{J}_{\mathrm{u}}^{\eta}(l,\mathfrak{n}|\alpha)\beta(0)$. Here we put*

$$\mathbb{J}_{\mathrm{u}}^{\eta}(l,\mathfrak{n}|\alpha) = (-1)^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}(1+(-1)^{\epsilon(\eta)}\tilde{\eta}(\mathfrak{n})i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\Upsilon_S^{\eta}(\mathbf{s})\mathfrak{C}_{S,\mathrm{u}}^{\eta}(\mathbf{s})\alpha(\mathbf{s})d\mu_S(\mathbf{s})$$

with

$$\mathfrak{C}_{S,\mathrm{u}}^{\eta}(\mathbf{s}) = \pi^{\epsilon(\eta)}C_0(\eta) + R(\eta)\biggl\{-\frac{d_F}{2}(C_{\mathrm{Euler}}+\log\pi) + \sum_{v\in\Sigma_\infty}\sum_{k=1}^{l_v/2-1}\frac{1}{k} + \sum_{v\in S}\frac{\log q_v}{1-q_v^{(s_v+1)/2}} + \log D_F\biggr\}.$$

In particular, we have $\mathfrak{C}_{S,\mathrm{u}}^{\eta}(\mathbf{s}) = L_{\mathrm{fin}}(1,\eta)$ if η is non-trivial.

Proof. By definition,

$$\mathbb{J}_{\mathrm{u}}^{\eta}(l;\beta,\lambda;\alpha) = \left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}(U_{0,\eta}^{+}(\lambda;\mathbf{s}) + U_{0,\eta}^{-}(\lambda;\mathbf{s}) + U_{1,\eta}^{+}(\lambda;\mathbf{s}) + U_{1,\eta}^{-}(\lambda;\mathbf{s}))\alpha(\mathbf{s})d\mu_S(\mathbf{s}).$$

From Lemma 15.2 and the computation after it,

$$\begin{aligned}&\mathbb{J}_{\mathrm{u}}^{\eta}(l;\beta,\lambda;\alpha)\\ &=\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\frac{1}{2\pi i}\int_{L_\sigma}\frac{\beta(z)}{z+\lambda}((-1)^{\epsilon(\eta)}+i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))\{\mathrm{N}(\mathfrak{f}_\eta)^{-z}L(-z,\eta)\Upsilon_{S,l}^{\eta}(z;\mathbf{s})\\ &\quad+\mathrm{N}(\mathfrak{f}_\eta)^{z}L(z,\eta)\Upsilon_{S,l}^{\eta}(-z;\mathbf{s})\}dz\alpha(\mathbf{s})d\mu_S(\mathbf{s}) - \frac{(-1)^{\epsilon(\eta)}+i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o})}{2}\mathbb{J}_{\mathrm{id}}^{\eta}(l;\beta,\lambda;\alpha),\end{aligned}$$

with $1 < \sigma < \underline{l}/2$ and $\mathrm{Re}(\lambda) > -\sigma$. Since σ is arbitrary, this gives a meromorphic continuation of $\mathbb{J}_{\mathrm{u}}^{\eta}(l;\beta,\lambda;\alpha)$ to $\mathrm{Re}(\lambda) > -\underline{l}/2$. By the above expression,

$$\begin{aligned}
&\mathrm{CT}_{\lambda=0}\mathbb{J}_{\mathrm{u}}^{\eta}(l;\beta,\lambda;\alpha)\\
=&\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\frac{1}{2\pi i}\int_{L_\sigma}\frac{\beta(z)}{z}((-1)^{\epsilon(\eta)}+i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))\{\mathrm{N}(\mathfrak{f}_\eta)^{-z}L(-z,\eta)\Upsilon_{S,l}^{\eta}(z;\mathbf{s})\\
&+\mathrm{N}(\mathfrak{f}_\eta)^{z}L(z,\eta)\Upsilon_{S,l}^{\eta}(-z;\mathbf{s})\}dz\alpha(\mathbf{s})d\mu_S(\mathbf{s})\\
=&((-1)^{\epsilon(\eta)}+i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))\frac{1}{2\pi i}\int_{L_\sigma}\frac{\beta(z)}{z}\{f_{\mathrm{u}}(z)+f_{\mathrm{u}}(-z)\}dz\\
=&((-1)^{\epsilon(\eta)}+i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))\,\mathrm{Res}_{z=0}\left(\frac{\beta(z)}{z}f_{\mathrm{u}}(z)\right)=((-1)^{\epsilon(\eta)}+i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))\beta(0)\mathrm{CT}_{z=0}f_{\mathrm{u}}(z).
\end{aligned}$$

Here we put $f_{\mathrm{u}}(z)=\mathrm{N}(\mathfrak{f}_\eta)^{-z}L(-z,\eta)\Upsilon_{S,l}^{\eta}(z;\mathbf{s})$. By setting $\tilde{\Upsilon}_{S,l}^{\eta}(z;\mathbf{s})=D_F^{1/2}\{\#(\mathfrak{o}/\mathfrak{f}_\eta)^\times\}\Upsilon_{S,l}^{\eta}(z;\mathbf{s})$, the constant term is computed as follows:

$$\begin{aligned}
&\mathrm{CT}_{z=0}f_{\mathrm{u}}(z)=\frac{d}{dz}\mathrm{N}(\mathfrak{f}_\eta)^{-z}zL(-z,\eta)\Upsilon_{S,l}^{\eta}(z;\mathbf{s})\Big|_{z=0}\\
=&\frac{d}{dz}\Big\{\mathrm{N}(\mathfrak{f}_\eta)^{-z}\times zi^{\epsilon(\eta)}D_F^{1/2}\mathrm{N}(\mathfrak{f}_\eta)^{-1/2}\{\#(\mathfrak{o}/\mathfrak{f}_\eta)^\times\}\mathcal{G}(\eta)D_F^{1/2+z}\mathrm{N}(\mathfrak{f}_\eta)^{1/2+z}L(z+1,\eta)\\
&\times D_F^{-1/2}\{\#(\mathfrak{o}/\mathfrak{f}_\eta)^\times\}^{-1}\tilde{\Upsilon}_{S,l}^{\eta}(z;\mathbf{s})\Big\}\Big|_{z=0}\\
=&i^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}\times\frac{d}{dz}\Big\{D_F^{z}zL(z+1,\eta)\times\tilde{\Upsilon}_{S,l}^{\eta}(z;\mathbf{s})\Big\}\Big|_{z=0}\\
=&\mathcal{G}(\eta)D_F^{1/2}\pi^{\epsilon(\eta)}\tilde{\Upsilon}_{S}^{\eta}(\mathbf{s})\Big\{(\log D_F)R(\eta)+C_0(\eta)+R(\eta)\frac{\frac{d}{dz}\tilde{\Upsilon}_{S,l}^{\mathbf{1}}(z;\mathbf{s})|_{z=0}}{\tilde{\Upsilon}_{S,l}^{\mathbf{1}}(0;\mathbf{s})}\Big\}.
\end{aligned}$$

Here $\epsilon(\eta)=\sum_{v\in\Sigma_\infty}\epsilon_v$. We note that $\tilde{\Upsilon}_{S,l}^{\eta}(0;\mathbf{s})=(-i\pi)^{\epsilon(\eta)}\Upsilon_S^{\eta}(0;\mathbf{s})$ holds by

$$\frac{2\Gamma(-z)\Gamma(l_v/2+z)}{\Gamma_{\mathbb{R}}(-z+\epsilon_v)\Gamma(l_v/2)}i^{\epsilon_v}\cos\left(\frac{\pi}{2}(-z+\epsilon_v)\right)\Big|_{z=0}=(-i\pi)^{\epsilon_v}$$

for $v\in\Sigma_\infty$. The logarithmic derivative of $\tilde{\Upsilon}_{S,l}^{\mathbf{1}}(z;\mathbf{s})$ at $z=0$ is computed as

$$\begin{aligned}
&\sum_{v\in\Sigma_\infty}\frac{d}{dz}\log\left\{\frac{2\Gamma(-z)\Gamma(l_v/2+z)}{\Gamma_{\mathbb{R}}(-z)\Gamma(l_v/2)}\cos\left(\frac{\pi z}{2}\right)\right\}\Big|_{z=0}\\
&+\sum_{v\in S}\frac{d}{dz}\log(1-q_v^{-(z+(s_v+1)/2)})^{-1}(1-q_v^{(s_v+1)/2})^{-1}\Big|_{z=0}\\
=&\sum_{v\in\Sigma_\infty}\left\{\psi(l_v/2)-\frac{1}{2}\log\pi+\left(\frac{1}{2}\psi\left(\frac{-z}{2}\right)-\psi(-z)\right)\Big|_{z=0}\right\}+\sum_{v\in S}\frac{\log q_v}{1-q_v^{(s_v+1)/2}}.
\end{aligned}$$

By the formulas

$$\psi(l_v/2)=-C_{\mathrm{Euler}}+\sum_{k=1}^{l_v/2-1}\frac{1}{k},\qquad\left(\frac{1}{2}\psi\left(\frac{-z}{2}\right)-\psi(-z)\right)\Big|_{z=0}=\frac{1}{2}C_{\mathrm{Euler}},$$

we are done. $\square$

Assume that $q(\mathrm{Re}(\mathbf{s})) > \mathrm{Re}(\lambda) > \sigma$ and $1 < \sigma < \underline{l}/2$. Analyzing the integrals

$$\frac{1}{2\pi i}\int_{L_{\pm\sigma}}\int_{\mathbb{A}^\times}\Psi_l^{(0)}\left(\mathfrak{n}|\mathbf{s};[\begin{smallmatrix}1&0\\t&1\end{smallmatrix}][\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}]\right)\eta(tx_\eta^*)|t|_{\mathbb{A}}^{\pm z}d^\times t dz,$$
$$\frac{1}{2\pi i}\int_{L_{\pm\sigma}}\int_{\mathbb{A}^\times}\Psi_l^{(0)}\left(\mathfrak{n}|\mathbf{s};[\begin{smallmatrix}1&t\\0&1\end{smallmatrix}][\begin{smallmatrix}1&0\\-x_\eta&1\end{smallmatrix}]w_0\right)\eta(tx_\eta^*)|t|_{\mathbb{A}}^{\pm z}d^\times t dz$$

in the same way as $U_{\epsilon,\eta}^{\pm}(\lambda;\mathbf{s})$, we obtain the following lemma.

Lemma 15.4. *The function $\lambda \mapsto \mathbb{J}_{\bar{\mathrm{u}}}^{\eta}(l;\beta,\lambda;\alpha)$ on $\mathrm{Re}(\lambda) > 1$ has a meromorphic continuation to the region $\mathrm{Re}(\lambda) > -\underline{l}/2$. The constant term of $\mathbb{J}_{\bar{\mathrm{u}}}^{\eta}(l;\beta,\lambda;\alpha)$ at $\lambda = 0$ equals $\mathbb{J}_{\bar{\mathrm{u}}}^{\eta}(l,\mathfrak{n}|\alpha)\beta(0)$. Here we put*

$$\mathbb{J}_{\bar{\mathrm{u}}}^{\eta}(l,\mathfrak{n}|\alpha) = (-1)^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}((-1)^{\epsilon(\eta)}\tilde{\eta}(\mathfrak{n}) + i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\Upsilon_S^{\eta}(\mathbf{s})\mathfrak{C}_{S,\bar{\mathrm{u}}}^{\eta}(\mathbf{s})\alpha(\mathbf{s})d\mu_S(\mathbf{s})$$

with

$$\mathfrak{C}_{S,\bar{\mathrm{u}}}^{\eta}(\mathbf{s}) = \mathfrak{C}_{S,\mathrm{u}}^{\eta}(\mathbf{s}) + R(\eta)\log \mathrm{N}(\mathfrak{n}).$$

Let us consider the term $\mathbb{J}_{\mathrm{hyp}}^{\eta}(l;\beta,\lambda;\alpha)$, which is, by definition, equal to

$$\int_{\mathbb{A}^\times}\sum_{b\in F^\times-\{-1\}}\hat{\Psi}_l^{(0)}\left(\mathfrak{n}|\alpha;\delta_b\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\right)\{\hat{\beta}_\lambda(|t|_{\mathbb{A}}) + \hat{\beta}_\lambda(|t|_{\mathbb{A}}^{-1})\}\,\eta(tx_\eta^*)\,d^\times t.$$

Lemma 15.5. *Suppose $\underline{l} \geqslant 6$. The integral $\mathbb{J}_{\mathrm{hyp}}^{\eta}(l;\beta,\lambda;\alpha)$ converges absolutely and has an analytic continuation to the region $\mathrm{Re}(\lambda) > -\epsilon$ for some $\epsilon > 0$. Moreover, we have $\mathrm{CT}_{\lambda=0}\mathbb{J}_{\mathrm{hyp}}^{\eta}(l;\beta,\lambda;\alpha) = \mathbb{J}_{\mathrm{hyp}}^{\eta}(l,\mathfrak{n}|\alpha)\beta(0)$. Here $\mathbb{J}_{\mathrm{hyp}}^{\eta}(l,\mathfrak{n}|\alpha)$ is defined by*

$$\mathbb{J}_{\mathrm{hyp}}^{\eta}(l,\mathfrak{n}|\alpha) = \left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\mathfrak{K}_S^{\eta}(l,\mathfrak{n}|\mathbf{s})\,\alpha(\mathbf{s})\,d\mu_S(\mathbf{s})$$

with

$$\mathfrak{K}_S^{\eta}(l,\mathfrak{n}|\mathbf{s}) = \sum_{b\in F^\times-\{-1\}}\int_{\mathbb{A}^\times}\Psi_l^{(0)}\left(\mathfrak{n}|\mathbf{s};\delta_b\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\right)\,\eta(tx_\eta^*)\,d^\times t.$$

Proof. We take $c \in \mathbb{R}$ such that $\underline{l}/2 - 1 > (c+1)/4$. Then, from Lemma 14.9, there exists $\epsilon > 0$ such that, for $0 < |\rho| < \epsilon$ the integral

$$(15.1)\qquad \int_{\mathbb{L}_S(\underline{c})}|\alpha(\mathbf{s})||d\mu_S(\mathbf{s})|\int_{L_\rho}\frac{|\beta(z)|}{|z+\lambda|}\sum_{b\in F^\times-\{-1\}}\int_{\mathbb{A}^\times}\left|\Psi_l^{(0)}\left(\mathfrak{n}|\mathbf{s};\delta_b[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}][\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}]\right)\right|\{|t|_{\mathbb{A}}^{\rho}+|t|_{\mathbb{A}}^{-\rho}\}\,d^\times t|dz|,$$

which is majorized by

$$\int_{\mathbb{L}_S(\underline{c})}|\alpha(\mathbf{s})||d\mu_S(\mathbf{s})|\int_{L_\rho}\frac{|\beta(z)|e^{d_F\pi|\mathrm{Im}(z)|/2}}{|z+\lambda|}|dz|\sum_{b\in F^\times-\{-1\}}\{M_\epsilon(\mathfrak{n}|0,\rho,l,\underline{c};b) + M_\epsilon(\mathfrak{n}|0,-\rho,l,\underline{c};b)\},$$

is convergent. By $|t|_{\mathbb{A}}^{\rho} + |t|_{\mathbb{A}}^{-\rho} \geqslant 2$ ($t \in \mathbb{A}^\times$), the integral (15.1) is finite even for $\rho = 0$. Hence, we obtain an analytic continuation of the function

$$\begin{aligned}&\mathbb{J}_{\mathrm{hyp}}^{\eta}(l;\beta,\lambda;\alpha)\\&=\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\underline{c})}\left\{\frac{1}{2\pi i}\int_{L_\rho}\frac{\beta(z)}{z+\lambda}\left(\sum_{b\in F^\times-\{-1\}}\int_{\mathbb{A}^\times}\Psi_l^{(0)}\left(\mathfrak{n}|\mathbf{s};\delta_b[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}][\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}]\right)\right.\right.\\&\qquad\left.\left.\times\,(|t|_{\mathbb{A}}^{z}+|t|_{\mathbb{A}}^{-z})\,\eta(tx_\eta^*)\,d^\times t\right)dz\right\}\alpha(\mathbf{s})d\mu_S(\mathbf{s})\end{aligned}$$

in the variable λ to the region $\mathrm{Re}(\lambda) > -\epsilon$. $\square$

16. The relative trace formula for holomorphic Hilbert modular forms

Let $\mathfrak{n}$ be an integral ideal of F, $l = (l_v)_{v\in\Sigma_\infty} \in (2\mathbb{N})^{\Sigma_\infty}$ a family such that $l_v \geqslant 6$ for all $v \in \Sigma_\infty$, and η a real valued character of $F^\times\backslash\mathbb{A}^\times$ unramified at all $v \in S(\mathfrak{n})$. Let $\mathfrak{f}_\eta$ denote the conductor of η. We assume $(-1)^{\epsilon(\eta)}\tilde{\eta}(\mathfrak{n}) = 1$, where $\epsilon(\eta) = \sum_{v\in\Sigma_\infty} \epsilon_v$ is the sum of $\epsilon_v \in \{0,1\}$ such that $\eta_v = \mathrm{sgn}^{\epsilon_v}$ for any $v \in \Sigma_\infty$ (see §1.4). Put $\tilde{l} = \sum_{v\in\Sigma_\infty} l_v$. Let S be a finite subset of Σ_{fin} disjoint from $S(\mathfrak{n}) \cup S(\mathfrak{f}_\eta)$. For $v \in S$, let $\mathcal{A}_v$ be the space of all holomorphic functions $\alpha_v(s_v)$ in $s_v \in \mathbb{C}$ satisfying $\alpha_v(s_v) = \alpha_v(-s_v)$ and $\alpha_v(s_v + \frac{4\pi i}{\log q_v}) = \alpha_v(s_v)$. We denote by $\mathcal{A}_S$ the space of all holomorphic functions α on $\mathbb{C}^S$ such that for each $v_0 \in S$, the function $s_{v_0} \mapsto \alpha(s_{v_0}, \mathbf{s}')$ is contained in $\mathcal{A}_{v_0}$ for all $\mathbf{s}' \in \mathbb{C}^{S-\{v_0\}}$.

Theorem 16.1. *For any function $\alpha \in \mathcal{A}_S$, we have the identity*

$$C(l,\mathfrak{n},S)\sum_{\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})} \mathbb{I}^\eta_{\mathrm{cus}}(\pi; l, \mathfrak{n})\,\alpha(\nu_S(\pi)) = \tilde{\mathbb{J}}^\eta_{\mathrm{u}}(l,\mathfrak{n}|\alpha) + \mathbb{J}^\eta_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha) \tag{16.1}$$

Here $\nu_S(\pi) = \{\nu_v(\pi)\}_{v\in S}$ is the spectral parameter of π at S (see §13.3.1),

$$C(l,\mathfrak{n},S) = (-1)^{\#S}\{\prod_{v\in\Sigma_\infty} \frac{2\pi\,\Gamma(l_v-1)}{\Gamma(l_v/2)^2}\}\frac{D_F^{-1}}{2}[\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{n})]^{-1},$$

$$\mathbb{I}^\eta_{\mathrm{cus}}(\pi; l, \mathfrak{n}) = (-1)^{\epsilon(\eta)}\mathcal{G}(\eta)\ w^\eta_{\mathfrak{n}}(\pi)\,\frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{\mathrm{N}(\mathfrak{f}_\pi)\,[\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{f}_\pi)]^{-1}\,L^{S_\pi}(1,\pi;\mathrm{Ad})}$$

with $w^\eta_{\mathfrak{n}}(\pi)$ given in Lemma 3.6, and

$$\tilde{\mathbb{J}}^\eta_{\mathrm{u}}(l,\mathfrak{n}|\alpha) = 2(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}(1 + i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})} \mathfrak{U}^\eta_S(l,\mathfrak{n}|\mathbf{s})\alpha(\mathbf{s})d\mu_S(\mathbf{s}),$$

$$\mathbb{J}^\eta_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha) = \left(\frac{1}{2\pi i}\right)^{\#S}\int_{L_S(\mathbf{c})} \mathfrak{K}^\eta_S(l,\mathfrak{n}|\mathbf{s})\,\alpha(\mathbf{s})\,d\mu_S(\mathbf{s})$$

with

$$\mathfrak{U}^\eta_S(l,\mathfrak{n}|\mathbf{s}) = \prod_{v\in S}(1-\eta_v(\varpi_v)q_v^{-(s_v+1)/2})^{-1}(1-q_v^{(s_v+1)/2})^{-1}\left\{\mathbf{C}^\eta_F(l,\mathfrak{n}) + R(\eta)\sum_{v\in S}\frac{\log q_v}{1-q_v^{(s_v+1)/2}}\right\},$$

$$\mathfrak{K}^\eta_S(l,\mathfrak{n}|\mathbf{s}) = \sum_{b\in F^\times-\{-1\}}\int_{\mathbb{A}^\times}\Psi^{(0)}_l\left(\mathfrak{n}|\,\mathbf{s};\delta_b\left[\begin{smallmatrix} t & 0\\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & x_\eta\\ 0 & 1\end{smallmatrix}\right]\right)\eta(tx^*_\eta)\,d^\times t$$

and

$$\mathbf{C}^\eta_F(l,\mathfrak{n}) = \pi^{\epsilon(\eta)}C_0(\eta) + R(\eta)\left\{-\frac{d_F}{2}(C_{\mathrm{Euler}}+\log\pi) + \log(D_F\mathrm{N}(\mathfrak{n})^{1/2}) + \sum_{v\in\Sigma_\infty}\sum_{k=1}^{l_v/2-1}\frac{1}{k}\right\}.$$

We remark $\mathbf{C}^\eta_F(l,\mathfrak{n}) = L_{\mathrm{fin}}(1,\eta)$ if η is non-trivial.

Proof. From Lemmas 15.1, 15.3, 15.4 and 15.5, $P^\eta_{\mathrm{reg}}(\hat{\mathbf{\Psi}}^l_{\mathrm{reg}}(\mathfrak{n}|\alpha))$ is given by the right-hand side of (16.1); the left-hand side is provided by Proposition 13.6. $\square$

We restrict our attention to the test functions of the form $\alpha(\mathbf{s}) = \prod_{v\in S}\alpha_v^{(m_v)}(s_v)$ with

$$\alpha_v^{(m)}(s_v) = q_v^{ms_v/2} + q_v^{-ms_v/2}, \qquad v \in S,\ m \in \mathbb{N}_0. \tag{16.2}$$

As is well known, these functions form a $\mathbb{C}$-basis of the image of the spherical Hecke algebra $\mathcal{H}(G_v, \mathbf{K}_v)$ by the spherical Fourier transform. Thus, by restricting our consideration to these functions, no generality is lost practically. The following two theorems are proved in §17 and §18.

Theorem 16.2. *For $\alpha = \otimes_{v\in S}\alpha_v$, we have*

$$\mathbb{J}^{\eta}_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha) = \sum_{b\in F^\times-\{-1\}} \{\prod_{v\in S} J_v^{\eta_v}(b;\alpha_v)\}\{\prod_{v\in\Sigma_\infty} J_v^{\eta_v}(l_v;b)\}\{\prod_{v\in\Sigma_{\mathrm{fin}}-S} J_v^{\eta_v}(b)\}.$$

Here $J_v^{\eta_v}(b;\alpha_v)$ is given by Lemma 17.2, $J_v^{\eta_v}(b)$ is given by Lemmas 17.4, 17.5 and 17.9, and $J_v^{\eta_v}(l_v;b)$ is given by Lemma 17.15.

Theorem 16.3. *For $\alpha = \otimes_{v\in S}\alpha_v$, we have*

$$\tilde{\mathbb{J}}^{\eta}_{\mathrm{u}}(l,\mathfrak{n}|\alpha) = 2(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)\, D_F^{1/2}(1+i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))$$
$$\times\left\{\mathbf{C}_F^{\eta}(l,\mathfrak{n})\prod_{v\in S}U_v^{\eta_v}(\alpha_v) + R(\eta)\sum_{v\in S}U'_v(\alpha_v)\prod_{w\in S-\{v\}}U_w^{\eta_w}(\alpha_w)\right\}.$$

Here $U_v^{\eta_v}(\alpha_v)$ and $U'_v(\alpha_v)$ are explicitly given in Proposition 18.1.

16.1. Proofs of Theorems 0.6 and 0.7. By the same procedure as in the proof of Theorems 0.2, with the aid of $\mathcal{N}$-transform (cf. §10.4), the estimation is reduced to that for the similar average over $\Pi_{\mathrm{cus}}(l,\mathfrak{n})$ (in place of $\Pi^*_{\mathrm{cus}}(l,\mathfrak{n})$). Since $\tilde{\mathbb{J}}^{\eta}_{\mathrm{u}}(l,\mathfrak{n}|\alpha)$ is evaluated by [47, Lemma 13.15], from Theorem 16.1, it suffices to show that $\mathbb{J}^{\eta}_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha) = \mathcal{O}_\epsilon(\mathrm{N}(\mathfrak{n})^{-\underline{l}/2+1+\epsilon})$ for any sufficiently small $\epsilon>0$, where $\underline{l} = \inf_{v\in\Sigma_\infty} l_v$. This follows from the proof of Lemma 15.5 and Lemmas 14.7 and 14.8 by taking $c\in\mathbb{R}$ and $\rho\neq 0$ such that $\underline{l}/2-1 > (c+1)/4 > (c+1)/4-|\rho| > \underline{l}/2-1-\epsilon > 1$ and $|\rho|$ is sufficiently small. By Lemma 10.13 with $c = \underline{l}/2-1$, the exponent of the error term is $-\inf(c,1)+\epsilon = -1+\epsilon$. □

Next we prove Theorem 0.7. Theorem 0.6 is also valid for any function

$$\alpha(\mathbf{s}) = \prod_{v\in S} P_v(q_v^{-s_v/2}+q_v^{s_v/2}),$$

where $P_v(X)\in\mathbb{C}[X]$ for each $v\in S$. Therefore, with the aid of the Stone-Weierstrass theorem, we have

$$\{\prod_{v\in\Sigma_\infty}\frac{2\pi\,(l_v-2)!}{\{(l_v/2-1)!\}^2}\}\times\frac{1}{\mathrm{N}(\mathfrak{n})\nu(\mathfrak{n})}\sum_{\pi\in\Pi^*_{\mathrm{cus}}(l,\mathfrak{n})}\frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{L^{S_\pi}(1,\pi,\mathrm{Ad})}f(\,(q_v^{-\nu_v(\pi)/2}+q_v^{\nu_v(\pi)/2})_{v\in S}\,)$$
$$\longrightarrow 4D_F^{3/2}L_{\mathrm{fin}}(1,\eta)\int_{[-2,2]^S} f(x)d\mu_{S,\eta}(x)$$

for any continuous function $f\in C([-2,2]^S)$ as $\mathrm{N}(\mathfrak{n})\to\infty$ in $\mathfrak{n}\in\mathcal{I}^+_{S,\eta}$. Here we set $d\mu_{S,\eta}(x) = \otimes_{v\in S}\,d\mu_{v,\eta_v}(x_v)$ and

$$d\mu_{v,\eta_v}(x_v) = \begin{cases} \dfrac{q_v-1}{(q_v^{1/2}+q_v^{-1/2}-x_v)^2}\,d\mu_{\mathrm{ST}}(x_v) & (\eta_v(\varpi_v)=+1), \\ \dfrac{q_v+1}{(q_v^{1/2}+q_v^{-1/2})^2-x_v^2}\,d\mu_{\mathrm{ST}}(x_v) & (\eta_v(\varpi_v)=-1). \end{cases}$$

Hence, by applying Proposition 1.1, we obtain Theorem 0.7. □

Remark : Suppose η is totally odd, i.e., $\eta_v(-1)=-1$ for all $v\in\Sigma_\infty$. If the level $\mathfrak{n}$ is sufficiently large compared with $\mathrm{N}(\mathfrak{f}_\eta)$ and the degree of α, the hyperbolic term $\mathbb{J}^{\eta}_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha)$ vanishes completely and the asymptotic formula in Theorem 0.6 is an exact formula without the remainder term $\mathcal{O}(\mathrm{N}(\mathfrak{n})^{-1+\epsilon})$; this kind of phenomenon, called the *stability*, was already observed in [26], [6] and [31].

17. Explicit formula of the hyperbolic term

In this section, we compute $\mathbb{J}^{\eta}_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha)$ further for particular test functions $\alpha=\otimes_{v\in S}\alpha_v$. By changing the order of integrals, we have

$$\mathbb{J}^{\eta}_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha)=\sum_{b\in F^{\times}-\{-1\}}\{\prod_{v\in S}J_v^{\eta_v}(b;\alpha_v)\}\{\prod_{v\in\Sigma_\infty}J_v^{\eta_v}(l_v;b)\}\{\prod_{v\in\Sigma_{\mathrm{fin}}-S}J_v^{\eta_v}(b)\},$$

where

$$J_v^{\eta_v}(b;\alpha_v)=\frac{1}{2\pi i}\int_{L_v(c)}\left\{\int_{F_v^\times}\Psi_v^{(0)}\left(s_v;\,\delta_b\,\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\right)\eta_v(t)\,d^\times t\right\}\alpha_v(s_v)\,d\mu_v(s_v)$$

for $v\in S$ with $\Psi_v(s_v;-)$ being the Green function on G_v (Lemma 5.1),

$$J_v^{\eta_v}(b)=\int_{F_v^\times}\Phi_{v,\mathfrak{n}}^{(0)}\left(\delta_b\,\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&\varpi_v^{-f(\eta_v)}\\0&1\end{smallmatrix}\right]\right)\eta_v(t\varpi_v^{-f(\eta_v)})\,d^\times t,\quad \text{if } v\in\Sigma_{\mathrm{fin}}-S,$$

$$J_v^{\eta_v}(l_v;b)=\int_{\mathbb{R}^\times}\Psi_v^{(0)}\left(l_v;\delta_b\,\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\right)\eta_v(t)\,d^\times t,\quad \text{if } v\in\Sigma_\infty$$

with $\Psi_v^{(0)}(l_v;-)$ being the Shintani function (Proposition 11.1).

17.1. An evaluation of non-archimedean integrals (for unramified η_v). In this paragraph, we explicitly compute the integrals $J_v^{\eta_v}(b;\alpha_v^{(m)})$ at $v\in S$ and the integrals $J_v^{\eta_v}(b)$ at $v\in\Sigma_{\mathrm{fin}}-S\cup S(\mathfrak{f}_\eta)$.

Lemma 17.1. *Let $v\in S$. Let $\alpha_v^{(m)}(s_v)=q_v^{ms_v/2}+q_v^{-ms_v/2}$ with $m\in\mathbb{N}_0$. Set*

$$\widehat{\Phi}_{vm}(g_v)=\frac{1}{2\pi i}\int_{L_v(c)}\Psi_v^{(0)}(s_v;\,g_v)\alpha_v^{(m)}(s_v)\,d\mu_v(s_v).$$

If $m>0$, then, for any $x\in F_v$ such that $\sup(|x|_v,1)=q_v^l$ *with $l\in\mathbb{N}_0$, we have*

$$\widehat{\Phi}_{vm}\left(\left[\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right]\right)=\begin{cases}0 & (l\geqslant m+1),\\ -q_v^{-m/2} & (l=m),\\ (m-l-1)q_v^{1-m/2}-(m-l+1)q_v^{-m/2} & (0\leqslant l<m).\end{cases}$$

If $m=0$, then for any $x\in F_v$ such that $\sup(|x|_v,1)=q_v^l$ *with $l\in\mathbb{N}_0$, we have*

$$\widehat{\Phi}_{v0}\left(\left[\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right]\right)=-2\,\delta(l=0)$$

Proof. From Lemma 5.1 and the formula $d\mu_v(s)=2^{-1}\log q_v\,(q_v^{(1+s)/2}-q_v^{(1-s)/2})\,ds$, we have

$$\begin{aligned}\widehat{\Phi}_{vm}\left(\left[\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right]\right)=\frac{1}{2\pi i}\int_{L_v(c)}&q_v^{-(s+1)l/2}\,(1-q_v^{-(s+1)/2})^{-1}(1-q_v^{(s+1)/2})^{-1}\\ &\times(q_v^{-ms/2}+q_v^{ms/2})\,2^{-1}\log q_v\,(q_v^{(1+s)/2}-q_v^{(1-s)/2})\,ds,\end{aligned}$$

where $L_v(c)$ is the contour $c+i\,[-\frac{2\pi}{\log q_v},\frac{2\pi}{\log q_v}]$. By the variable change $z=q_v^{s/2}$, this becomes

$$\frac{q_v^{(1-l)/2}}{2\pi i}\oint_{|z|=R}z^{-l}(1-q_v^{-1/2}z^{-1})^{-1}(1-q_v^{1/2}z)^{-1}(z^m+z^{-m})\,(z-z^{-1})\,\frac{dz}{z}$$

with $R=q_v^{c/2}\,(>1)$. Thus, by the residue theorem, we have the equality

$$\widehat{\Phi}_{vm}\left(\left[\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right]\right)=q_v^{(1-l)/2}\left(\mathrm{Res}_{z=q_v^{-1/2}}\phi(z)+\mathrm{Res}_{z=0}\phi(z)\right)\tag{17.1}$$

with $\phi(z)=\frac{(1-z^2)(z^m+z^{-m})}{(1-q_v^{1/2}z)^2}\frac{q_v^{1/2}}{z^{1+l}}$. By a direct computation, we have

$$\operatorname{Res}_{z=q_v^{-1/2}}\phi(z)=-\,q_v^{(1+l)/2}\Big(\{(m+l+1)(1-q_v^{-1})+2q_v^{-1}\}q_v^{m/2} + \{(-m+l+1)(1-q_v^{-1})+2q_v^{-1}\}q_v^{-m/2}\Big), \tag{17.2}$$

$$\begin{aligned}\operatorname{Res}_{z=0}\phi(z)=&\,\delta(l\geqslant m+1)\,\{(l-m+1)q_v^{(l-m+1)/2}-(l-m-1)q_v^{(l-m-1)/2}\}\\ &+\delta(l\geqslant 1-m)\,\{(l+m+1)q_v^{(l+m+1)/2}-(l+m-1)q_v^{(l+m-1)/2}\}\\ &+\{\delta(m=l)+\delta(m=-l)\}\,q_v^{1/2}.\end{aligned}\tag{17.3}$$

From (17.1), (17.2) and (17.3), we obtain the desired formula easily. □

Lemma 17.2. *Let $v\in S$. Let $\alpha_v^{(m)}(s_v)=q_v^{ms_v/2}+q_v^{-ms_v/2}$ with $m\in\mathbb{N}_0$. Then, for any $b\in F_v^\times-\{-1\}$,*

$$J_v^{\eta_v}(b;\alpha_v^{(m)})=I_v^+(m;\,b)+\eta_v(\varpi_v)I_v^+(m;\,\varpi_v^{-1}(b+1))$$

with

$$I_v^+(m;\,b)=\operatorname{vol}(\mathfrak{o}_v^\times)\,2^{\delta(m=0)}\Big(-q_v^{-m/2}\,\delta_m^{\eta_v}(b) + \sum_{l=\sup(0,-\operatorname{ord}_v(b))}^{m-1}\{(m-l-1)q_v^{1-m/2}-(m-l+1)q_v^{-m/2}\}\delta_l^{\eta_v}(b)\Big),$$

where for $n\in\mathbb{N}_0$,

$$\delta_n^{\eta_v}(b)=\delta(|b|_v\leqslant q_v^n)\,\eta_v(\varpi_v^n)\begin{cases}(\operatorname{ord}_v(b)+1)^{\delta(n=0)} & (\eta_v(\varpi_v)=1),\\ (2^{-1}(\eta_v(b)+1))^{\delta(n=0)} & (\eta_v(\varpi_v)=-1).\end{cases}$$

Proof. Let $m>0$. By definition, $J_v^{\eta_v}(b;\alpha_v)=I_v^+(m;b)+I_v^-(m;b)$ with $I_v^+(m;b)$ and $I_v^-(m;b)$ being the integrals of $\widehat{\Phi}_{vm}(\delta_b\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right])\,\eta_v(t)$ with respect to the measure $d^\times t$ over $|t|_v\leqslant 1$ and over $|t|_v>1$, respectively. From [47, Lemma 11.4],

$$\delta_b\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\in H_v\left[\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right]\mathbf{K}_v\quad\text{with}\quad |x|_v=\begin{cases}|t|_v^{-1}|b|_v & (|t|_v\leqslant 1),\\ |t|_v|b+1|_v & (|t|_v>1).\end{cases}\tag{17.4}$$

Hence, by Lemma 17.1, $I_v^+(m;b)$ becomes the sum of the integral

$$\int_{\substack{|t|_v\leqslant 1\\ \sup(1,|t|_v^{-1}|b|_v)=q_v^m}}(-q_v^{-m/2})\,\eta_v(t)\,d^\times t\tag{17.5}$$

and

$$\sum_{l=0}^{m-1}\int_{\substack{|t|_v\leqslant 1\\ \sup(1,|t|_v^{-1}|b|_v)=q_v^l}}\{(m-l-1)q_v^{1-m/2}-(m-l+1)q_v^{-m/2}\}\,\eta_v(t)\,d^\times t.\tag{17.6}$$

The condition $|t|_v\leqslant 1$, $\sup(1,|t|_v^{-1}|b|_v)=q_v^l$ is equivalent to $|b|_v\leqslant q_v^l$, $|t|_v=q_v^{-l}|b|_v$ if $l>0$ and to $|b|\leqslant|t|\leqslant 1$ if $l=0$. Hence, (17.5) is equal to $-q_v^{-m/2}\operatorname{vol}(\mathfrak{o}_v^\times)\,\delta_m^{\eta_v}(b)$, and (17.6) is equal to the following expression

$$\operatorname{vol}(\mathfrak{o}_v^\times)\sum_{l=\sup(0,-\operatorname{ord}_v(b))}^{m-1}\{(m-l-1)q_v^{1-m/2}-(m-l+1)q_v^{-m/2}\}\,\delta_l^{\eta_v}(b).$$

This completes the evaluation of the integral $I_v^+(m;b)$. In the same way as above, the other integral $I_v^-(m;b)$ is calculated in a similar form; from the resulting expression, $I_v^-(m;b)=\eta_v(\varpi_v)I_v^+(m;\varpi_v^{-1}(b+1))$ is observed. This settles our consideration when $m>0$. The other case $m=0$ is similar. $\square$

From Lemma 17.2, we have a useful estimate for the function $J_v(b,\alpha_v)$ in b.

Lemma 17.3. *Let $\alpha_v^{(m)}(s_v)=q_v^{ms_v/2}+q_v^{-ms_v/2}$ with $m\in\mathbb{N}_0$. If $m>0$, then*

$$|J_v^{\eta_v}(b,\alpha_v^{(m)})|\ll (m+1)^2\{\delta(|b|_v\leqslant q_v^{m-1})\,q_v^{1-m/2}+\delta(|b|_v=q_v^m)\,q_v^{-m/2}\},\qquad b\in F_v^\times-\{-1\}$$

with the implied constant independent of v and m. If $m=0$, then,

$$J_v^{\eta_v}(b,\alpha_v^{(0)})=-2\mathrm{vol}(\mathfrak{o}_v^\times)\,\Lambda_v^{\eta_v}(b),$$

where $\Lambda_v^{\eta_v}$ is a function on $F_v^\times-\{-1\}$ defined by

$$\Lambda_v^{\eta_v}(b)=\delta(b\in\mathfrak{o}_v)\delta_0^{\eta_v}(b(b+1)) \tag{17.7}$$

Proof. To infer the estimate from Lemma 17.2 in the case when $m>0$, it suffices to note that $I_m^+(\varpi_v^{-1}(b+1))=0$ if $|b|_v\geqslant q_v^m$, or equivalently if $|\varpi_v^{-1}(b+1)|_v\geqslant q_v^{m+1}$. The formula of $J_v(b,\alpha_v^{(0)})$ is obtained by noting the relation $\Lambda_v^{\eta_v}(b)=\delta_0^{\eta_v}(b)+\eta_v(\varpi_v)\,\delta_0^{\eta_v}(\varpi_v^{-1}(b+1))$. $\square$

Lemma 17.4. *Let $v\in\Sigma_{\mathrm{fin}}-S\cup S(\mathfrak{n}\mathfrak{f}_\eta)$. Then*

$$J_v^{\eta_v}(b)=\mathrm{vol}(\mathfrak{o}_v^\times)\,\Lambda_v^{\eta_v}(b)$$

with $\Lambda_v^{\eta_v}(b)$ being defined by (17.7).

Proof. This is proved in the same way as the case $m=0$ in Lemma 17.3. $\square$

Lemma 17.5. *Let $v\in S(\mathfrak{n})-S(\mathfrak{f}_\eta)$. If $\eta_v(\varpi_v)=1$, then*

$$J_v^{\eta_v}(b)=\mathrm{vol}(\mathfrak{o}_v^\times)\,\delta(b\in\mathfrak{n}\mathfrak{o}_v)\,\{\mathrm{ord}_v(b)-\mathrm{ord}_v(\mathfrak{n})+1\}.$$

If $\eta_v(\varpi_v)=-1$, then

$$J_v^{\eta_v}(b)=\mathrm{vol}(\mathfrak{o}_v^\times)\,\delta(b\in\mathfrak{n}\mathfrak{o}_v)\,2^{-1}(\eta_v(b)+(-1)^{\mathrm{ord}_v(\mathfrak{n})}).$$

Proof. This is proved in the same way as Lemma 17.4. We only have to remark that the assertion in the last sentence of [47, Lemma 11.4] is relevant here. $\square$

17.2. An evaluation of non-archimedean integrals (for ramified η_v). We shall calculate the integral $J_v^{\eta_v}(b)$ at finitely many places $v\in S(\mathfrak{f}_\eta)$. In what follows in this paragraph, we fix $v\in S(\mathfrak{f}_\eta)$ and set $f=f(\eta_v)$; thus f is a positive integer. For $l\in\mathbb{Z}$, consider the following subsets of $F_v^\times$ depending on $b\in F_v^\times-\{-1\}$.

$$D_l(b)=\{t\in F_v^\times|\,|t|_v=q_v^{-l},\ |1+t\varpi_v^{-f}|_v\,|b+t\varpi_v^{-f}(b+1)|_v\leqslant q_v^{-l}\,\},\quad (l\in\mathbb{Z}-\{f\}),$$
$$D_f(b)=\{t\in F_v^\times|\ -t\in\varpi_v^f(\mathfrak{o}_v^\times-U_v(f)),\ |1+t\varpi_v^{-f}|_v\,|b+t\varpi_v^{-f}(b+1)|_v\leqslant q_v^{-f}\,\},$$

where $U_v(m)=1+\mathfrak{p}_v^m$ for any positive integer m.

Lemma 17.6. *Let $l>f$. Then, $D_l(b)=\emptyset$ unless $l=f-\mathrm{ord}_v(b+1)+\mathrm{ord}_v(b)$, in which case, we have* $\mathrm{ord}_v(b)>0$, $\mathrm{ord}_v(b+1)=0$ *and*

$$\int_{t\in D_l(b)}\eta_v(t)\,d^\times t=\eta_v\left(\varpi_v^f\frac{-b}{b+1}\right)(1-q_v^{-1})^{-1}\,q_v^{-f-d_v/2}.$$

Proof. By the variable change $t = \varpi_v^l t'$, we have

$$\int_{t \in D_l(b)} \eta_v(t)\, d^\times t = \eta_v(\varpi_v^l) \int_{t' \in D'} \eta_v(t')\, d^\times t'$$

with $D' = \{t' \in \mathfrak{o}_v^\times \,|\, |1 + t'\varpi_v^{l-f}|_v\, |b + t'\varpi_v^{l-f}(b+1)|_v \leqslant q_v^{-l}\}$. Let $t' \in \mathfrak{o}_v^\times$. Then, the condition

$$|1 + t'\varpi_v^{l-f}|_v\, |b + t'\varpi_v^{l-f}(b+1)|_v \leqslant q_v^{-l}$$

is equivalent to

$$t' \in \varpi_v^{f-l} \tfrac{-b}{b+1}(1 + \varpi_v^l b^{-1}\mathfrak{o}_v). \tag{17.8}$$

If $|\varpi_v^l b^{-1}|_v > 1$, then $1 + \varpi_v^l b^{-1}\mathfrak{o}_v = \varpi_v^l b^{-1}\mathfrak{o}_v$. Hence, from (17.8),

$$1 = |t'|_v \leqslant \left|\varpi_v^{f-l} \tfrac{-b}{b+1} \cdot \varpi_v^l b^{-1}\right|_v = \left|\tfrac{\varpi_v^f}{b+1}\right|_v,$$

and $b + 1 \in \mathfrak{p}_v^f$ follows. Since $f > 0$, we obtain $|b|_v = 1$, which, combined with $|\varpi_v^l b^{-1}|_v > 1$, implies the inequality $|\varpi_v^l|_v > 1$ contradicting to $l > f > 0$.

If $|\varpi_v^l b^{-1}|_v = 1$, then $b \in \varpi_v^l \mathfrak{o}_v^\times$; thus, $|b + 1|_v = 1$ by $l > f > 0$. Hence, from (17.8), we have the inequality

$$1 = |t'|_v \leqslant \left|\varpi_v^{f-l} \tfrac{-b}{b+1}\right|_v = |\varpi_v^f|_v = q_v^{-f},$$

which is impossible due to $f > 0$. From the considerations so far, we have the inequality $|\varpi_v^l b^{-1}|_v < 1$, which yields $1 + \varpi_v^l b^{-1}\mathfrak{o}_v \subset \mathfrak{o}_v^\times$. Hence, from (17.8), we have the second equality of

$$1 = |t'|_v = \left|\varpi_v^{f-l} \tfrac{-b}{b+1}\right|_v,$$

which implies $l = f - \mathrm{ord}_v(b+1) + \mathrm{ord}_v(b)$. From this and $l > f$, we have $\mathrm{ord}_v(b+1) < \mathrm{ord}_v(b)$, which holds if and only if $\mathrm{ord}_v(b) > 0$ and $\mathrm{ord}_v(b+1) = 0$.

If we set $t' = \varpi_v^{f-l} \frac{-b}{b+1} r$, then (17.8) becomes $r \in 1 + \varpi_v^l b^{-1}\mathfrak{o}_v = 1 + \varpi_v^f \mathfrak{o}_v$; thus

$$\begin{aligned}\int_{t \in D_l(b)} \eta_v(t)\, d^\times t &= \eta_v(\varpi_v^l)\, \eta_v\left(\varpi_v^{f-l} \tfrac{-b}{b+1}\right) \int_{r \in 1 + \varpi_v^f \mathfrak{o}_v} \eta_v(r)\, d^\times r \\ &= \eta_v\left(\varpi_v^f \tfrac{-b}{b+1}\right) q_v^{-f-d_v/2}(1 - q_v^{-1})^{-1}.\end{aligned}$$

□

Lemma 17.7. *Let $l < f$. Then, $D_l(b) = \emptyset$ unless $l = f - \mathrm{ord}_v(b+1) + \mathrm{ord}_v(b)$, in which case, we have $\mathrm{ord}_v(b+1) > 0$, $\mathrm{ord}_v(b) = 0$ and*

$$\int_{t \in D_l(b)} \eta_v(t)\, d^\times t = \eta_v\left(\varpi_v^f \frac{-b}{b+1}\right) (1 - q_v^{-1})^{-1} q_v^{-f-d_v/2}.$$

Proof. This is proved in the same way as the previous lemma. □

Lemma 17.8. *The set $D_f(b)$ is empty unless $\mathrm{ord}_v(b) = \mathrm{ord}_v(b+1) \leqslant 0$, in which case*

$$\int_{t \in D_f(b)} \eta_v(t)\, d^\times t = \delta\left(\frac{b}{b+1} \in \mathfrak{o}_v^\times\right) \eta_v\left(\varpi_v^f \frac{-b}{b+1}\right) (1 - q_v^{-1})^{-1}\, q_v^{-f + \mathrm{ord}_v(b) - d_v/2}.$$

Proof. By $t = -\varpi_v^f t'$, the set $D_f(b)$ is mapped bijectively onto the set of t' such that

$$t' \in \mathfrak{o}_v^\times - U_v(f), \tag{17.9}$$

$$|1 - t'|_v\, |b - t'(b+1)|_v \leqslant q_v^{-f}. \tag{17.10}$$

We shall show that (17.9) and (17.10) are equivalent to the following conditions:

$$t' \in \tfrac{b}{b+1}(1+\varpi_v^f b^{-1}\mathfrak{o}_v), \tag{17.11}$$

$$\tfrac{b}{b+1} \in \mathfrak{o}_v^\times, \qquad b \notin \mathfrak{p}_v. \tag{17.12}$$

Noting that, under the condition (17.12), the sets $U_v(1)$ and $\frac{b}{b+1}(1+\varpi_v^f b^{-1}\mathfrak{o}_v)$ are disjoint, we see easily that (17.11) and (17.12) imply (17.9) and (17.10). To have the converse, we first observe that (17.9) is equivalent to $t' \in \mathfrak{o}_v^\times$ and $|t'-1|_v > q_v^{-f}$. Hence by (17.10),

$$|b - t'(b+1)|_v \leqslant q_v^{-f}|t'-1|_v^{-1} < 1,$$

or equivalently

$$b - t'(b+1) \in \mathfrak{p}_v. \tag{17.13}$$

If $b \in \mathfrak{p}_v$, then $b+1 \in \mathfrak{o}_v^\times$. From these and (17.13), $t' \in \frac{b}{b+1} + \mathfrak{p}_v = \mathfrak{p}_v$; this contradicts $t' \in \mathfrak{o}_v^\times$. Thus $b \notin \mathfrak{p}_v$ is obtained. From (17.13), we have $t'\frac{b+1}{b} \in 1+\mathfrak{p}_v \subset \mathfrak{o}_v^\times$. Since $t' \in \mathfrak{o}_v^\times$ by (17.9), $\frac{b}{b+1} \in \mathfrak{o}_v^\times$ is obtained. From (17.13),

$$t' \in \tfrac{b}{b+1} + \tfrac{1}{b+1}\mathfrak{p}_v = \tfrac{b}{b+1}(1+b^{-1}\mathfrak{p}_v).$$

Since $b^{-1} \in \mathfrak{o}_v$, we have $t' \in \frac{b}{b+1}\,U_v(1)$, which yields $t' \in \mathfrak{o}_v^\times - U_v(1)$ because $\frac{b}{b+1}U_v(1) \cap U_v(1) = \emptyset$. Thus $|t'-1|_v = 1$. Combining this with (17.10), we obtain (17.11). This settles the desired converse implication.

Consequently, we have

$$\begin{aligned}\int_{t \in D_f(b)} \eta_v(t)\,d^\times t &= \eta_v(-\varpi_v^f) \int \eta_v(t')\,d^\times t' \\ &= \delta\left(\tfrac{b}{b+1} \in \mathfrak{o}_v^\times,\, b \notin \mathfrak{p}_v\right) \eta_v(-\varpi_v^f)\,\eta_v\left(\tfrac{b}{b+1}\right) \int_{r \in 1+\varpi_v^f b^{-1}\mathfrak{o}_v} \eta_v(r)\,d^\times r \\ &= \delta\left(\tfrac{b}{b+1} \in \mathfrak{o}_v^\times\right) \eta_v\left(\varpi_v^f \tfrac{-b}{b+1}\right) \delta(b \in \mathfrak{o}_v^\times)\, q_v^{-f+\mathrm{ord}_v(b)-d_v/2}(1-q_v^{-1})^{-1}.\end{aligned}$$

□

Lemma 17.9. *Let η_v be a character of $F_v^\times$ of order 2 and of conductor $f>0$. Then, for $b \in F_v^\times - \{-1\}$, we have*

$$J_v^{\eta_v}(b) = \delta(b \in \mathfrak{p}_v^{-f})\left\{\eta_v(-1) + (\delta(b \in \mathfrak{o}_v) + \delta(b \notin \mathfrak{o}_v)q_v^{\mathrm{ord}_v(b)})\eta_v(-b(b+1))\right\} q_v^{-f-d_v/2}(1-q_v^{-1})^{-1}.$$

Proof. From [47, Lemmas 11.4 and 11.5],

$$J_v^{\eta_v}(b) = \delta(b \in \mathfrak{p}_v^{-f})(J_{v,1}^{\eta_v}(b) + J_{v,2}^{\eta_v}(b))$$

with

$$J_{v,1}^{\eta_v}(b) = \int_{\substack{-t \in \varpi_v^f U_v(f) \\ |t|_v|b+1|_v \leqslant 1}} \eta_v(-1)\,d^\times t, \qquad J_{v,2}^{\eta_v}(b) = \int_{\substack{-t \in F_v^\times - \varpi_v^f U_v(f) \\ |1+t\varpi_v^{-f}|_v\,|b+t\varpi_v^{-f}(b+1)|_v \leqslant |t|_v}} \eta_v(t\varpi_v^{-f})\,d^\times t.$$

If $b \in \mathfrak{p}_v^{-f}$, then $t \in -\varpi_v^f U_v(f)$ implies $|b+1|_v \leqslant q_v^f = |t|_v^{-1}$; thus,

$$J_{v,1}^{\eta_v}(b) = \eta_v(-1)\,\mathrm{vol}(-\varpi_v^f U_v(f); d^\times t) = \eta_v(-1)\mathrm{vol}(U_v(f); d^\times t) = \eta_v(-1)\,q_v^{-f-d_v/2}(1-q_v^{-1})^{-1}.$$

The integral domain of $J_{v,2}^{\eta_v}(b)$ is a disjoint union of the sets $D_l(b)\,(l \in \mathbb{Z})$. From Lemmas 17.6, 17.7 and 17.8, we have

$$J_{v,2}^{\eta_v}(b) = (\delta(b \in \mathfrak{o}_v) + \delta(b \notin \mathfrak{o}_v)q_v^{\mathrm{ord}_v(b)})\eta_v\left(\tfrac{-b}{b+1}\right) q_v^{-f-d_v/2}(1-q_v^{-1})^{-1}.$$

□

Lemma 17.10. *Let η be a character of $F^\times\backslash\mathbb{A}^\times$ with conductor $\mathfrak{f}_\eta$ such that $\eta^2=\mathbf{1}$. There exists a constant $C>1$ independent of η such that*

$$|J_v^{\eta_v}(b)|\leqslant C\,\delta(|b|_v\leqslant q_v^{f(\eta_v)})\,q_v^{-f(\eta_v)}$$

for any $v\in S(\mathfrak{f}_\eta)$ and for any $b\in F_v^\times-\{-1\}$.

Proof. This is obvious from the previous lemma. Indeed, $C=4$ is sufficient. □

Corollary 17.11. *For any $\epsilon>0$, we have*

$$|\prod_{v\in S(\mathfrak{f}_\eta)}J_v^{\eta_v}(b)|\ll_\epsilon\{\prod_{v\in S(\mathfrak{f}_\eta)}\delta(b\in\mathfrak{p}_v^{-f(\eta_v)})\}\,\mathrm{N}(\mathfrak{f}_\eta)^{-1+\epsilon},\quad b\in F^\times-\{-1\}$$

with the implied constant independent of η and $b\in F^\times-\{-1\}$.

Proof. Given $\epsilon>0$, let $P(\epsilon)$ be the set of $v\in\Sigma_{\rm fin}$ such that $q_v\leqslant C^{1/\epsilon}$, where $C>0$ is the constant in the previous lemma. Then, from the lemma,

$$|J_v^{\eta_v}(b)|\leqslant C\,\delta(|b|_v\leqslant q_v^{f(\eta_v)})\,q_v^{-f(\eta_v)+\epsilon}\quad\text{if } v\in S(\mathfrak{f}_\eta)\cap P(\epsilon)$$

and

$$|J_v^{\eta_v}(b)|\leqslant\delta(|b|_v\leqslant q_v^{f(\eta_v)})\,q_v^{-f(\eta_v)+\epsilon}\quad\text{if } v\in S(\mathfrak{f}_\eta)-P(\epsilon).$$

Taking the product of these inequalities, we have

$$\begin{aligned}|\prod_{v\in S(\mathfrak{f}_\eta)}J_v^{\eta_v}(b)|&=\{\prod_{v\in S(\mathfrak{f}_\eta)\cap P(\epsilon)}|J_v^{\eta_v}(b)|\}\{\prod_{v\in S(\mathfrak{f}_\eta)-P(\epsilon)}|J_v^{\eta_v}(b)|\}\\&\leqslant C^{\#(S(\mathfrak{f}_\eta)\cap P(\epsilon))}\{\prod_{v\in S(\mathfrak{f}_\eta)}\delta(b\in\mathfrak{p}_v^{-f(\eta_v)})\}\,\mathrm{N}(\mathfrak{f}_\eta)^{-1+\epsilon}\end{aligned}$$

□

17.3. An evaluation of archimedean integrals. In this subsection, we evaluate the integral

$$J^\eta(l;b)=\int_{\mathbb{R}^\times}\Psi^{(0)}\left(l;\delta_b\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\right)\eta(t)\,d^\times t,\quad b\in\mathbb{R}^\times-\{-1\}\tag{17.14}$$

explicitly, where $\eta:\mathbb{R}^\times\to\{\pm1\}$ is a character, and $\Psi^{(0)}(l;-)$ is the holomorphic Shintani function of weight $l\,(\geqslant 4)$.

Lemma 17.12. *We have*

$$J^\eta(l;b)=\int_{\mathbb{R}^\times}(1-it)^{-l/2}(1+b+t^{-1}bi)^{-l/2}\,\eta(t)\,d^\times t,\quad b\in\mathbb{R}^\times-\{-1\}.$$

Proof. From Lemma 11.2,

$$\Psi^{(0)}\left(l;\delta_b\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\right)=e^{il\theta}\,(1+ix)^{-l/2}\quad\text{if}\quad\delta_b\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\in T\left[\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right]k_\theta.$$

A direct computation yields $e^{i\theta}=\frac{1+it}{\sqrt{t^2+1}}$ and $x=bt^{-1}+t(b+1)$. Thus,

$$\begin{aligned}J^\eta(l;b)&=\int_{\mathbb{R}^\times}\left(\tfrac{1+it}{\sqrt{t^2+1}}\right)^l\{1+i(bt^{-1}+t(b+1))\}^{-l/2}\,\eta(t)\,d^\times t\\&=\int_{\mathbb{R}^\times}(1-it)^{-l/2}(1+b+t^{-1}bi)^{-l/2}\,\eta(t)\,d^\times t.\end{aligned}$$

□

Lemma 17.13. *Define*

$$J_+(l;b) = i^{l/2}\,(1+b)^{-l/2}\int_0^\infty (t+i)^{-l/2}\left(t+\tfrac{bi}{b+1}\right)^{-l/2} t^{l/2-1}\,dt.$$

Then

$$J^{\mathbf{1}}(l;b) = J_+(l;b) + \overline{J_+(l;b)}, \qquad J^{\mathrm{sgn}}(l;b) = J_+(l;b) - \overline{J_+(l;b)}.$$

Proof. By dividing the integral $J^\eta(l;b)$ to two parts according to $t>0$ and $t<0$, we obtain the assertions immediately. □

Lemma 17.14. *Suppose $b(b+1)>0$. Then*

$$\begin{aligned} J_+(l;b) &= (1+b)^{-l/2}\int_0^1 u^{l/2-1}(1-u)^{l/2-1}\left(\tfrac{-1}{b+1}u+1\right)^{-l/2} du \\ &= (1+b)^{-l/2}\,\Gamma(l/2)^2\Gamma(l)^{-1}\,{}_2F_1\left(l/2,l/2;l;(b+1)^{-1}\right) = 2\,Q_{l/2-1}(2b+1), \end{aligned}$$

where $Q_n(x)$ is the Legendre function of the 2nd kind.

Proof. If we set $f(z) = i^{l/2}(1+b)^{-l/2}\,(z+i)^{-l/2}\{z+bi/(b+1)\}^{-l/2}z^{l/2-1}$, then $f(z)$ is a meromorphic function on $\mathbb{C}$ with poles only at $z=-i$ and $\frac{-bi}{1+b}$, both of which are in the lower half plane $\mathrm{Im}(z)<0$. For $R>0$, let Q_R denote the rectangle $0\leqslant \mathrm{Im}(z)\leqslant R$, $0\leqslant \mathrm{Re}(z)\leqslant R$. Regarding ∂Q_R as a contour with counterclockwise orientation, by Cauchy's theorem, we have $\int_{\partial Q_R} f(z)\,dz = 0$. From this,

$$\begin{aligned} J_+(l;b) &= \int_{0i}^{i\infty} f(z)\,dz - \lim_{R\to\infty}\int_{\partial Q_R-[0,R]\cup i[0,R]} f(z)\,dz = \int_{0i}^{i\infty} f(z)\,dz \\ &= (1+b)^{-l/2}\int_0^{+\infty}(t+1)^{-l/2}\left(t+\tfrac{b}{b+1}\right)^{-l/2} t^{l/2-1}\,dt. \end{aligned}$$

By the variable change $t+1=u^{-1}$, this becomes

$$(1+b)^{-l/2}\int_0^1 u^{l/2-1}(1-u)^{l/2-1}\left(\tfrac{-1}{b+1}u+1\right)^{-l/2} du.$$

By using the integral representation of ${}_2F_1(a,b;c,z)$ in [24, p.54] here, we obtain

$$J_+(l;b) = (1+b)^{-l/2}\,\Gamma(l/2)^2\Gamma(l)^{-1}\,{}_2F_1\left(l/2,l/2;l;(b+1)^{-1}\right).$$

If we further apply the formula

$$2^{-n}(2n+1)!(n!)^{-2}\,Q_n(x) = (1+x)^{-(n+1)}{}_2F_1\left(n+1,n+1;2n+2;\tfrac{2}{x+1}\right)$$

([24, p.233]) with $n=l/2-1$ and $x=2b+1$, then $J_+(l;b) = 2\,Q_{l/2-1}(2b+1)$ as desired. □

Lemma 17.15. (1) *If $b(b+1)>0$, then*

$$J^{\mathbf{1}}(l;b) = (1+b)^{-l/2}\,2\,\Gamma(l/2)^2\Gamma(l)^{-1}\,{}_2F_1\left(l/2,l/2;l;(b+1)^{-1}\right), \qquad J^{\mathrm{sgn}}(l;b)=0.$$

(2) *If $b(b+1)<0$, then*

$$J^{\mathbf{1}}(l;b) = 2\log\left|\frac{b+1}{b}\right| P_{l/2-1}(2b+1) - \sum_{m=1}^{[l/4]}\frac{8(l-4m+1)}{(2m-1)(l-2m)}\,P_{l/2-2m}(2b+1), \tag{17.15}$$

$$J^{\mathrm{sgn}}(l;b) = 2\pi i\,P_{l/2-1}(2b+1), \tag{17.16}$$

where $P_n(z)$ denotes the Legendre polynomial of degree n.

Proof. First suppose $b \in \mathbb{R}$ and $b(b+1) > 0$. Then from the previous lemma, $J_+(l;b)$ is a real number. Thus, $J^{\mathbf{1}}(l;b) = 2J^+(l;b)$ and $J^{\rm sgn}(l;b) = 0$ by Lemma 17.13.

From $J_+(l;b) = 2Q_{l/2-1}(2b+1)$, applying the formula in [24, p.234], we obtain

$$J_+(l;b) = \log\left(\frac{b+1}{b}\right) P_{l/2-1}(2b+1) - \sum_{m=1}^{[l/4]} \frac{4(l-4m+1)}{(2m-1)(l-2m)} P_{l/2-2m}(2b+1) \tag{17.17}$$

for $b \in \mathbb{R}$ such that $b(b+1) > 0$. From the defining formula of $J_+(l;b)$, the function $b \mapsto J_+(l;b)$ on $\mathbb{R}^\times - \{-1\}$ has a holomorphic continuation to the whole complex b-plane away from the set $D = \{b \in \mathbb{C} |\, \frac{bi}{b+1} \in (-\infty, 0)\} \cup \{0, -1\}$, which is the upper half of the circle centered at $-1/2$ of radius $1/2$ with the edge points included. Thus, if we choose the branch of $\log(\frac{1+b}{b})$ on the domain $\mathbb{C} - D$ so that it is real for $b > 0$, then the formula (17.17) remains valid on $\mathbb{C} - D$ by analytic continuation. If $b \in \mathbb{R}$ satisfies $b(b+1) < 0$, then b is contained in $\mathbb{C} - D$. Hence, by taking the sum of (17.17) and its complex conjugate, we obtain the formula for $J^{\mathbf{1}}(l;b)$. As for $J^{\rm sgn}(l;b)$, we have

$$J^{\rm sgn}(l;b) = J_+(l;b) - \overline{J_+(l;b)} = \{\log(\tfrac{b+1}{b}) - \overline{\log(\tfrac{b+1}{b})}\}P_{l/2-1}(2b+1) = 2\pi i P_{l/2-1}(2b+1).$$

$\square$

18. Explicit formula of the unipotent term

Let $v \in S$. The aim of this section is to evaluate the integrals

$$U_v^{\eta_v}(\alpha_v) = \frac{1}{2\pi i}\int_{L_v(c)} (1 - \eta_v(\varpi_v) q_v^{-(s+1)/2})^{-1}(1 - q_v^{(s+1)/2})^{-1}\, \alpha_v(s)\, d\mu_v(s), \tag{18.1}$$

$$U_v'(\alpha_v) = \frac{\log q_v}{2\pi i}\int_{L_v(c)} (1 - q_v^{(s+1)/2})^{-2}(1 - q_v^{-(s+1)/2})^{-1}\, \alpha_v(s)\, d\mu_v(s). \tag{18.2}$$

for the test functions given by (16.2).

Proposition 18.1. *Let $\alpha_v(s) = q_v^{ms/2} + q_v^{-ms/2}$ with $m \in \mathbb{N}_0$. We have*

$$U_v^{\eta_v}(\alpha_v) = \begin{cases} \delta(m>0)\, q_v^{1-m/2}\, \{(m-1) - (m+1)\, q_v^{-1}\} - 2\,\delta(m=0) & (\eta_v(\varpi_v) = 1), \\ \delta(m \in 2\mathbb{N})\ q_v^{1-m/2}(1 - q_v^{-1}) - 2\ \delta(m=0) & (\eta_v(\varpi_v) = -1), \end{cases}$$

$$U_v'(\alpha_v) = -2^{-1}(\log q_v)\, q_v^{1-m/2}\, \delta(m>0)\, \{(m-1)(m-2) - m(m+1)\, q_v^{-1}\}.$$

Proof. We give an indication of proof when $\eta_v(\varpi_v) = -1$; the remaining cases are similar. By a variable change,

$$\begin{aligned} U_v^{\eta_v}(\alpha_v) =& \frac{1}{2\pi i}\oint_{|z|=q_v^{c/2}} (1 + q_v^{-1/2}z^{-1})^{-1}(1 - q_v^{1/2}z)^{-1}(z^m + z^{-m})q_v^{1/2}(z - z^{-1})\frac{dz}{z} \\ =& \{\mathrm{Res}_{z=q_v^{-1/2}} + \mathrm{Res}_{z=-q_v^{-1/2}} + \mathrm{Res}_{z=0}\}\phi(z), \end{aligned}$$

where $\phi(z) = \frac{(z^2-1)(z^m+z^{-m})}{1-q_v z^2}\frac{q_v}{z}$. By evaluating the residues, we are done. $\square$

19. Subconvexity estimates in the weight aspect

In this section we prove Theorem 0.8 by using the relative trace formula (Theorem 16.1); we take a particular test function $\alpha_S^\pi \in \mathcal{A}_S$ depending on a fixed cuspidal representation π with varying S. To have a good controle of the term $\mathbb{J}_{\rm hyp}^\eta(l, \mathfrak{n}|\alpha_S^\pi)$ explicating the dependence on S, our formula of local orbital integrals (Lemma 17.3) is indispensable. In this section, $\theta \in [0,1]$ denotes a real number such that the spectral radius of the Satake parameter $A_v(\pi)$ of $\pi \in \Pi_{\rm cus}(l, \mathfrak{n})$ at $v \in \Sigma_{\rm fin} - S(\mathfrak{f}_\pi)$ is no greater than $q_v^{\theta/2}$ for any $v \in \Sigma_{\rm fin} - S(\mathfrak{f}_\pi)$. Since the Ramanujan conjecture for the holomrphic Hilbert cusp forms is known ([1]), we can actually take $\theta = 0$; however, we let θ unspecified until the very end to be able to keep track of the dependence on the Ramanujan exponent θ in various estimations.

In this section, we abuse the symbol $\mathfrak{p}_v$ to denote the global ideal $\mathfrak{p}_v \cap \mathfrak{o}$.

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19.1. An auxiliary estimate of semilocal terms. Let S be a finite set of finite places v such that $\eta_v(\varpi_v) = -1$. For a decomposable function $\alpha_S(\mathbf{s}) = \prod_{v \in S} \alpha_v(s_v)$ in $\mathcal{A}_S$, we set

$$J_S(b; \alpha_S) = \prod_{v \in S} J_v(b; \alpha_v), \quad b \in F^\times - \{-1\},$$

where we simply write $J_v(b; \alpha_v)$ in place of $J_v^{\eta_v}(b; \alpha_v)$. Extending this linearly, we have a linear functional $\alpha_S \mapsto J_S(b; \alpha_S)$ on the space $\mathcal{A}_S$. Given $\pi \in \Pi_{\mathrm{cus}}(l, \mathfrak{n})$, set

$$\lambda_v(\pi) = \mathrm{tr}\, A_v(\pi), \qquad v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{f}_\pi)$$

with $A_v(\pi) \in \mathrm{GL}_2(\mathbb{C})$ the Satake parameter of π_v. Then, we define a function in $\mathcal{A}_S$ depending on the automorphic representation π as follows:

$$\alpha_S^\pi(\mathbf{s}) = \Big(\sum_{v \in S} \{\lambda_v(\pi)\,(z_v + z_v^{-1}) - (z_v^2 + z_v^{-2} + 1)\}\Big)^2,$$

where $z_v = q_v^{s_v/2}$ for each $v \in S$. We need an estimate of $J_S(b;\, \alpha_S^\pi)$ with varying b. For an integral ideal $\mathfrak{a}$ such that $S(\mathfrak{a}) \subset S$, let us define a function $D_S(\mathfrak{a}; -)$ on $F^\times - \{-1\}$ by

$$D_S(\mathfrak{a};\, b) = \{\prod_{w \in S - S(\mathfrak{a})} \Lambda_w(b)\}\,\{\prod_{v \in S(\mathfrak{a})} \delta(|b|_v \leqslant q_v^{\mathrm{ord}_v(\mathfrak{a})})\},$$

where $\Lambda_w(b) = \delta(b \in \mathfrak{o}_w)(\mathrm{ord}_w(b(b+1)) + 1)$.

Proposition 19.1. *Set $P = \{(v_1, v_2) \in S^2 |\, v_1 \neq v_2\,\}$. We have the estimate*

$$\begin{aligned}|J_S(b; \alpha_S^\pi)| \ll \sum_{v \in S} \Big\{ & D_S(\mathfrak{o}; b)\, q_v^{(\theta+1)/2} + D_S(\mathfrak{p}_v; b)\, q_v^{\theta} \\ & + D_S(\mathfrak{p}_v^2; b)\, q_v^{\theta-1} + D_S(\mathfrak{p}_v^3; b)\, q_v^{-1} + D_S(\mathfrak{p}_v^4; b)\, q_v^{-2}\Big\} \\ + \sum_{(v_1, v_2) \in P} \Big\{ & D_S(\mathfrak{o}; b)\, q_{v_1}^{(\theta+1)/2}\, q_{v_2}^{(\theta+1)/2} + D_S(\mathfrak{p}_{v_1}; b)\, q_{v_2}^{(\theta+1)/2} + D_S(\mathfrak{p}_{v_1}\mathfrak{p}_{v_2}; b) \\ & + D_S(\mathfrak{p}_{v_1}^2 \mathfrak{p}_{v_2}; b) q_{v_1}^{-1} + D_S(\mathfrak{p}_{v_1}^2; b)\, q_{v_1}^{-1}\, q_{v_2}^{(\theta+1)/2} + D_S(\mathfrak{p}_{v_1}^2 \mathfrak{p}_{v_2}^2; b) q_{v_1}^{-1} q_{v_2}^{-1}\Big\}\end{aligned}$$

for $b \in F^\times - \{-1\}$, where the implied constant is absolute.

Proof. Set $Z_v = \lambda_v(\pi)\,(z_v + z_v^{-1}) - (z_v^2 + z_v^{-2} + 1)$ for any $v \in S$. By expanding the square, we have

$$\alpha_S^\pi(\mathbf{s}) = \sum_{v \in S} Z_v^2 + \sum_{(v_1, v_2) \in P} Z_{v_1}\, Z_{v_2},$$

which, together with Lemma 17.3, gives us

(19.1)
$$\begin{aligned} J_S(b;\alpha_S^{\pi}) &= \sum_{v\in S}\{\prod_{w\in S-\{v\}} J_w(b;1)\}\, J_v(b;Z_v^2) + \sum_{(v_1,v_2)\in P}\{\prod_{w\in S-\{v_1,v_2\}} J_w(b;1)\}\, J_{v_1}(b;Z_{v_1})\, J_{v_2}(b;Z_{v_2}) \\ &= \sum_{v\in S}\{\prod_{w\in S-\{v\}} -2\mathrm{vol}(\mathfrak{o}_w^{\times})\,\Lambda_w(b)\}\, J_v(b;Z_v^2) \\ &\quad + \sum_{(v_1,v_2)\in P}\{\prod_{w\in S-\{v_1,v_2\}} -2\mathrm{vol}(\mathfrak{o}_w^{\times})\,\Lambda_w(b)\}\, J_{v_1}(b;Z_{v_1})\, J_{v_2}(b;Z_{v_2}). \end{aligned}$$

Let us estimate the integral $J_v(b;Z_v^2)$. By expanding the square,

$$\begin{aligned} Z_v^2 &= \lambda_v(\pi)^2\,(z_v^2+z_v^{-2}+2)+(z_v^4+z_v^{-4}+2)+2(z_v^2+z_v^{-2})+1-2\lambda_v(\pi)(z_v^3+z_v^{-3})-4\lambda_v(\pi)(z_v+z_v^{-1}) \\ &= \lambda_v(\pi)^2(\alpha_v^{(2)}+\alpha_v^{(0)})+\alpha_v^{(4)}+2\alpha_v^{(2)}+\frac{3}{2}\alpha_v^{(0)}-2\lambda_v(\pi)\alpha_v^{(3)}-4\lambda_v(\pi)\alpha_v^{(1)}. \end{aligned}$$

By this expression and by the estimates in Lemma 17.3, we obtain

(19.2)
$$\begin{aligned} &|J_v(b;Z_v^2)|\mathrm{vol}(\mathfrak{o}_v^{\times})^{-1} \\ &\ll \delta(|b|_v\leqslant 1)\,\{|\lambda_v(\pi)|q_v^{1/2}+\Lambda_v(b)(1+|\lambda_v(\pi)|^2)\}+\delta(|b|_v\leqslant q_v)\,\{|\lambda_v(\pi)|q_v^{-1/2}+1+|\lambda_v(\pi)|^2\} \\ &\quad+\delta(|b|_v\leqslant q_v^2)\,\{|\lambda_v(\pi)|q_v^{-1/2}+q_v^{-1}+|\lambda_v(\pi)|^2q_v^{-1}\}+\delta(|b|_v\leqslant q_v^3)\,\{|\lambda_v(\pi)|q_v^{-3/2}+q_v^{-1}\} \\ &\quad+\delta(|b|_v\leqslant q_v^4)\,q_v^{-2} \\ &\ll \delta(|b|_v\leqslant 1)\,\{q_v^{(\theta+1)/2}+\Lambda_v(b)\,q_v^{\theta}\} \\ &\quad+\delta(|b|_v\leqslant q_v)\,q_v^{\theta}+\delta(|b|_v\leqslant q_v^2)\,q_v^{\theta-1}+\delta(|b|_v\leqslant q_v^3)\,q_v^{-1}+\delta(|b|_v\leqslant q_v^4)\,q_v^{-2}, \end{aligned}$$

where to show the second inequality we use the estimate $|\lambda_v(\pi)|\leqslant 2q_v^{\theta/2}$ as well as the inequalities $-1\leqslant(\theta-1)/2\leqslant\theta\leqslant(\theta+1)/2$, $(\theta-3)/2\leqslant -1$. For $J_v(b;Z_v)$, directly from Lemma 17.3, we have

(19.3)
$$|J_v(b;Z_v)|\,\mathrm{vol}(\mathfrak{o}_v^{\times})^{-1}\ll\delta(|b|_v\leqslant 1)\,\{q_v^{(\theta+1)/2}+\Lambda_v(b)\}+\delta(|b|_v\leqslant q_v)+\delta(|b|_v\leqslant q_v^2)\,q_v^{-1}.$$

From (19.1), (19.2) and (19.3), we have the desired estimate immediately. □

19.2. A basic majorant for the hyperbolic term (odd case). For $b\in F^{\times}-\{-1\}$, viewing b as a real number, say b_v, by the mapping $F\hookrightarrow F_v\cong\mathbb{R}$ for each $v\in\Sigma_\infty$, we define

$$\mathfrak{m}_\infty(l;b)=\prod_{v\in\Sigma_\infty}|J^{\mathrm{sgn}}(l_v;b_v)|,$$

where $J^{\mathrm{sgn}}(l_v;b_v)$ is the integral (17.14). For relatively prime integral ideals $\mathfrak{n}$ and $\mathfrak{a}$ and for $l=(l_v)_{v\in\Sigma_\infty}\in(2\mathbb{Z}_{>2})^{\Sigma_\infty}$, we set

$$\mathfrak{I}(l,\mathfrak{n},\mathfrak{a}):=\sum_{b\in\mathfrak{n}\mathfrak{o}(S)-\{0,-1\}}\{\prod_{v\in\Sigma_{\mathrm{fin}}-S}\Lambda_v(b)\}\,D_S(\mathfrak{a};b)\,\mathfrak{m}_\infty(l;b),$$

where S is a finite set of places such that $S(\mathfrak{a})\subset S\subset\Sigma_{\mathrm{fin}}-S(\mathfrak{n})$, and $\mathfrak{o}(S)$ is the S-integer ring. As will be seen below, this is independent of the choice of such S.

Lemma 19.2. *Let $\mathfrak{a}$ and $\mathfrak{n}$ be relatively prime ideals. Then, for any $\epsilon>0$, the estimate*

$$\mathfrak{I}(l,\mathfrak{n},\mathfrak{a})\ll_\epsilon\{\prod_{v\in\Sigma_\infty}l_v\}^{-1/2}\,\mathrm{N}(\mathfrak{a})^{5/4+\epsilon}$$

holds with the implied constant depending on ϵ while independent of the data $(l,\mathfrak{n},\mathfrak{a})$.

If $\mathfrak{a}$ is trivial, the following vanishing holds.

Lemma 19.3. *For any $\mathfrak{n}$ and l, we have $\mathfrak{I}(l,\mathfrak{n},\mathfrak{o})=0$.*

19.3. Proofs of Lemmas 19.2 and 19.3. Set

$$\tau^{S(\mathfrak{a})}(b)=\prod_{v\in\Sigma_{\rm fin}-S(\mathfrak{a})}\Lambda_v(b)\prod_{v\in S(\mathfrak{a})}\delta(|b|_v\leqslant q_v^{\operatorname{ord}_v(\mathfrak{a})}).$$

Then we have

$$\mathfrak{I}(l,\mathfrak{n},\mathfrak{a})=\sum_{b\in\mathfrak{n}\mathfrak{a}^{-1}-\{0,-1\}}\tau^{S(\mathfrak{a})}(b)\,\mathfrak{m}_\infty(l;b). \tag{19.4}$$

Lemma 19.4. *For any $\epsilon>0$, we have*

$$\tau^{S(\mathfrak{a})}(b)\ll_\epsilon(\mathrm{N}(\mathfrak{a})^2\,|\mathrm{N}(b(b+1))|)^\epsilon,\qquad b\in\mathfrak{a}^{-1}-\{0,-1\}$$

with the implied constant independent of b.

Proof. Let $b\in\mathfrak{a}^{-1}-\{0\}$. Then $(b(b+1))\mathfrak{a}^2=\mathfrak{b}\prod_{j=1}^r\mathfrak{p}_j^{e_j}$, where e_j are positive integers, $\mathfrak{p}_j$ are prime ideals of $\mathfrak{o}$ relatively prime to $\mathfrak{a}$, and $\mathfrak{b}$ is an ideal of $\mathfrak{o}$ dividing $\mathfrak{a}$. For each j, there exist a prime number p_j and an integer $d_j\in\mathbb{N}$ such that $\mathrm{N}(\mathfrak{p}_j)=p_j^{d_j}$. By taking norms, we have

$$\mathrm{N}(\mathfrak{a})^2\,|\mathrm{N}(b(b+1))|=\mathrm{N}(\mathfrak{b})\prod_{j=1}^r\mathrm{N}(\mathfrak{p}_j)^{e_j}=\mathrm{N}(\mathfrak{b})\prod_{j=1}^r p_j^{d_je_j}.$$

Hence

$$d(\mathrm{N}(\mathfrak{a})^2|\mathrm{N}(b(b+1))|)=d(\mathrm{N}(\mathfrak{b}))\prod_{j=1}^r(e_jd_j+1)\geqslant\prod_{j=1}^r(e_j+1)=\tau^{S(\mathfrak{a})}(b),$$

where, for a natural number m, $d(m)$ denotes the number of positive divisors of m. Invoking the well known bound $d(m)\ll_\epsilon m^\epsilon$, we obtain the desired estimate. □

From Lemmas 19.4 and 17.15,

$$\mathfrak{I}(l,\mathfrak{n},\mathfrak{a})\ll_\epsilon\mathrm{N}(\mathfrak{a})^{2\epsilon}\sum_{b\in\mathfrak{n}\mathfrak{a}^{-1}\cap\mathcal{Q}_\infty}|\mathrm{N}(b(b+1))|^\epsilon\prod_{v\in\Sigma_\infty}|2\pi\,P_{l_v/2-1}(2b_v+1)|,$$

where $\mathcal{Q}_\infty$ denotes the cube $(-1,0)^{\Sigma_\infty}$ in $\prod_{v\in\Sigma_\infty}\mathbb{R}$. Invoking the inequality $|P_n(x)|\leqslant(1-x^2)^{-1/2}\,n^{-1/2}$ for $|x|<1$, $n\in\mathbb{N}$ ([24, p.237]), we have

$$\mathfrak{I}(l,\mathfrak{n},\mathfrak{a})\ll_\epsilon\pi^{d_F}\,\mathrm{N}(\mathfrak{a})^{2\epsilon}\sum_{b\in\mathfrak{n}\mathfrak{a}^{-1}\cap\mathcal{Q}_\infty}|\mathrm{N}(b(b+1))|^{-1/4+\epsilon}\prod_{v\in\Sigma_\infty}(l_v/2-1)^{-1/2}. \tag{19.5}$$

To estimate the sum $\sum_{b\in\mathfrak{n}\mathfrak{a}^{-1}\cap\mathcal{Q}_\infty}|\mathrm{N}(b(b+1))|^{-1/4+\epsilon}$, we need several lemmas.

Lemma 19.5. *For a positive integer c, let $\nu(c)$ be the number of integral ideals $\mathfrak{c}$ such that $\mathrm{N}(\mathfrak{c})=c$. Then, for any $\epsilon>0$, $\nu(c)\ll_\epsilon c^\epsilon$ with the implied constant independent of c.*

Proof. Suppose c is a prime power p^t. Then an ideal $\mathfrak{c}$ such that $\mathrm{N}(\mathfrak{c})=p^t$ must be a power of a prime ideal $\mathfrak{p}$ lying above p. The number of choices for such $\mathfrak{p}$ is at most $d_F=[F:\mathbb{Q}]$. If $\mathfrak{c}=\mathfrak{p}^e$, then $\mathrm{N}(\mathfrak{c})=p^t$ is equivalent to $p^{me}=p^t$, where $\mathrm{N}(\mathfrak{p})=p^m$. Hence $e=\frac{t}{m}\leqslant t\leqslant t\,\log_2 p\leqslant\log_2 p^t$. From this, we have the inequality $\nu(p^t)\leqslant d_F\,\log_2 p^t$. Given $\epsilon>0$, let $x(\epsilon)>1$ be a number such that $d_F\,\log_2 x\leqslant x^\epsilon$ for any $x\geqslant x(\epsilon)$. Let $Q(\epsilon)$ be the set of prime powers p^t such that $p^t\leqslant x(\epsilon)$. Noting that $Q(\epsilon)$ is a finite set, we set $C(\epsilon)=\prod_{q\in Q(\epsilon)}\nu(q)$, which is a constant depending only on ϵ. Let c' (resp. c'') be the product of the prime powers $p_i^{t_i}$ such that $p_i^{t_i}\in Q(\epsilon)$ (resp. $p_i^{t_i}\notin Q(\epsilon)$) in the prime factorization $c=\prod_i p_i^{t_i}$ of c. Since ν is multiplicative, we have

$$\nu(c)=\nu(c')\,\nu(c'')\leqslant\prod_{q\in Q(\epsilon)}\nu(q)\prod_{i;\,p_i|c''}d_F\,\log_2 p_i^{t_i}\leqslant C(\epsilon)\prod_{i;\,p_i|c''}p_i^{t_i\epsilon}\leqslant C(\epsilon)\,(\prod_i p_i^{t_i})^\epsilon\leqslant C(\epsilon)\,c^\epsilon.$$

This completes the proof. $\square$

Lemma 19.6. *Let $C=\{C_v\}_{v\in\Sigma_\infty}$ be a family of positive real numbers. For any $\epsilon>0$, we have*

$$\#\{u\in\mathfrak{o}^\times \,|\, |u_v|<C_v\ (\forall\, v\in\Sigma_\infty)\,\}\ll_\epsilon \left(\prod_{v\in\Sigma_\infty} C_v\right)^{\epsilon}$$

with the implied constant independent of C.

Proof. For simplicity, we set $d=d_F$. By the Dirichlet unit theorem, there exist fundamental units $\varepsilon_j\,(1\leqslant j\leqslant d-1)$ such that any $\gamma\in\mathfrak{o}^\times$ is written uniquely in the form $\gamma=\pm\varepsilon_1^{n_1}\cdots\varepsilon_{d-1}^{n_{d-1}}$ with integers $n_j\in\mathbb{Z}$. By this, the inequality $|\gamma_v|<C_v$ is written as

$$\sum_{j=1}^{d-1} n_j\,\log|(\varepsilon_j)_v|<\log C_v, \qquad (v\in\Sigma_\infty). \tag{19.6}$$

Let $\mathfrak{U}(C)$ be the set of $u\in\mathfrak{o}^\times$ such that $|u_v|<C_v$ for all $v\in\Sigma_\infty$. Thus, the number $\#\mathfrak{U}(C)$ is bounded from above by the number of integer points $(n_j)\in\mathbb{Z}^{d-1}$ lying on the Euclidean domain $D(C)$ in $\mathbb{R}^{d-1}$ defined by the system of linear inequalities (19.6). Fix an enumeration $\Sigma_\infty=\{v_1,\dots,v_d\}$ and let $E_i=(\log|(\varepsilon_j)_{v_i}|)_{1\leqslant j\leqslant d-1}\in\mathbb{R}^{d-1}$ for $1\leqslant i\leqslant d$. First $d-1$ vectors $E_i\,(1\leqslant i\leqslant d-1)$ form a basis of $\mathbb{R}^{d-1}$; let E_j^* $(1\leqslant j\leqslant d-1)$ be its dual basis. From the relation $|\mathrm{N}(\varepsilon_j)|=1$, we have $\sum_{i=1}^d E_i=0$. Hence, if we write a general point $y\in\mathbb{R}^{d-1}$ by $y=\sum_{i=1}^{d-1}(\log C_{v_i}-y_i)\,E_i^*$, then $y\in D(C)$ if and only if

$$y_i>0 \qquad (1\leqslant i\leqslant d-1), \quad \sum_{i=1}^{d-1} y_i<\sum_{j=1}^{d}\log C_{v_j}.$$

The volume of this region in the y-space with respect to the Euclidean measure is $\frac{1}{r_F\,(d-1)!}\,(\sum_{j=1}^d\log C_{v_j})^d$, where r_F is the regulator of F. Thus $\mathrm{vol}(D(C))\ll(\log\prod_v C_v)^d\ll_\epsilon(\prod_v C_v)^\epsilon$, and we are done. $\square$

Lemma 19.7. *Let $\mathfrak{a}$ be an integral ideal and c a positive integer. For any $\epsilon,\epsilon'>0$,*

$$\#\{b\in\mathfrak{a}^{-1}\cap\mathcal{Q}_\infty \,|\, \mathrm{N}((b)\mathfrak{a})=c\,\}\ll_{\epsilon,\epsilon'} c^{\epsilon'-\epsilon}\,\mathrm{N}(\mathfrak{a})^\epsilon$$

with the implied constant independent of $\mathfrak{a}$ and c.

Proof. Let $\mathfrak{c}$ be an integral ideal such that $\mathrm{N}(\mathfrak{c})=c$. From Lemma 19.5, the number of such $\mathfrak{c}$ is bounded by $c^{\epsilon'}$ for any $\epsilon'>0$. If $\mathfrak{c}\mathfrak{a}^{-1}$ is a principal ideal, say (ξ), then, using Lemma 19.6, we have

$$\begin{aligned}\#\{b\in\mathfrak{a}^{-1}\cap\mathcal{Q}_\infty \,|\, \mathfrak{c}=(b)\mathfrak{a}\,\}&=\#\{u\in\mathfrak{o}^\times \,|\, |u_v|<|\xi_v|^{-1}\ (\forall\, v\in\Sigma_\infty)\,\}\\ &\ll_\epsilon\big(\prod_{v\in\Sigma_\infty}|\xi_v|^{-1}\big)^\epsilon=(|\mathrm{N}(\xi)|^{-1})^\epsilon=(c^{-1}\,\mathrm{N}(\mathfrak{a}))^\epsilon.\end{aligned}$$

$\square$

19.3.1. *The completion of the proof of Lemma 19.2.* From (19.5), we have

$$\mathfrak{I}(l,\mathfrak{n},\mathfrak{a})\ll_\epsilon \mathrm{N}(\mathfrak{a})^{2\epsilon}\,\{\prod_{v\in\Sigma_\infty} l_v\}^{-1/2}\sum_{b\in\mathfrak{a}^{-1}\cap\mathcal{Q}_\infty}|\mathrm{N}(b(b+1))|^{-1/4+\epsilon} \tag{19.7}$$

with the implied constant independent of $(l,\mathfrak{n},\mathfrak{a})$. Setting $\mathrm{N}((b)\mathfrak{a})=c$, we rewrite the last summation in the following way.

$$\sum_{b\in\mathfrak{a}^{-1}\cap\mathcal{Q}_\infty}|\mathrm{N}(b(b+1))|^{-1/4+\epsilon}=\mathrm{N}(\mathfrak{a})^{1/4-\epsilon}\sum_{c=1}^{\infty}c^{-1/4+\epsilon}\sum_{\substack{b\in\mathfrak{a}^{-1}\cap\mathcal{Q}_\infty\\ |\mathrm{N}((b)\mathfrak{a})|=c}}|\mathrm{N}(b+1)|^{-1/4+\epsilon}.$$

The range of c is reduced to $1 \leqslant c \leqslant \mathrm{N}(\mathfrak{a})$ by the condition $b \in \mathcal{Q}_\infty$. Since $(0) \neq (b+1)\mathfrak{a} \subset \mathfrak{o}$, we have $\mathrm{N}((b+1)\mathfrak{a}) \geqslant 1$, by which the last summation in b is trivially bounded by $\mathrm{N}(\mathfrak{a})^{1/4-\epsilon}\,\#\{b \in \mathfrak{a}^{-1}\cap \mathcal{Q}_\infty \mid |\mathrm{N}((b)\mathfrak{a})| = c\}$ for any $\epsilon \in (0,1/4)$. Combining these considerations and by Lemma 19.7, we obtain the bound

$$\sum_{b \in \mathfrak{a}^{-1}\cap\mathcal{Q}_\infty} |\mathrm{N}(b(b+1))|^{-1/4+\epsilon} \ll_{\epsilon,\delta,\delta'} \mathrm{N}(\mathfrak{a})^{1/2-2\epsilon} \sum_{c=1}^{\mathrm{N}(\mathfrak{a})} c^{-1/4+\epsilon} c^{\delta'-\delta}\,\mathrm{N}(\mathfrak{a})^{\delta}$$

$$\ll_{\epsilon,\delta,\delta'} \mathrm{N}(\mathfrak{a})^{1/2-2\epsilon}\mathrm{N}(\mathfrak{a})^{\delta} \times \mathrm{N}(\mathfrak{a})^{3/4+\epsilon+\delta'-\delta} \log \mathrm{N}(\mathfrak{a}) \tag{19.8}$$

$$= \mathrm{N}(\mathfrak{a})^{5/4-\epsilon+\delta'} \log \mathrm{N}(\mathfrak{a}) \tag{19.9}$$

for any sufficiently small $\delta, \delta' > 0$. Consequently, we have the desired estimate from (19.7) and (19.9). $\square$

19.3.2. *Proof of Lemma 19.3.* Suppose $b \in \mathfrak{n} \cap \mathcal{Q}_\infty$. The integrality of b yields $\mathrm{N}(b) \in \mathbb{Z}$. From the condition $b \in \mathcal{Q}_\infty$, we have $0 < |b_v| < 1$ for all $v \in \Sigma_\infty$, from which $0 < |\mathrm{N}(b)| < 1$ is obtained. Thus, if $\mathfrak{a} = \mathfrak{o}$, then the summation in the right-hand side of (19.5) is empty. This completes the proof. $\square$

19.4. An estimate of the hyperbolic term. Given a quadratic character η of $F^\times\backslash\mathbb{A}^\times$ with conductor $\mathfrak{f}_\eta$ and an integral ideal $\mathfrak{n}$, for a large number $K \geqslant 2$, let $S = S_K^{\mathfrak{n};\eta} = \{v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{n}\mathfrak{f}_\eta) \mid \eta_v(\varpi_v) = -1,\ K \leqslant q_v \leqslant 2K\}$, and consider the test function $\alpha_S^\pi(\mathbf{s})$ depending on a cuspidal representation $\pi \in \Pi_{\mathrm{cus}}(l,\mathfrak{n})$.

Lemma 19.8. *There exists a constant $C > 1$ independent of $\mathfrak{n}$ and η such that $C^{-1}K(\log K)^{-1} < \#S < C\,K(\log K)^{-1}$ for all $K \geqslant 2$.*

Proof. This follows from an analogue of Dirichlet's theorem on arithmetic progression for number fields. $\square$

For $S = S_K^{\mathfrak{n},\eta}$ and for a given $\pi \in \Pi_{\mathrm{cus}}(l,\mathfrak{n})$, let $\alpha_S^\pi(\mathbf{s})$ be the function defined in §19.1.

Proposition 19.9. *For any $\epsilon > 0$, we have*

$$|\mathbb{J}_{\mathrm{hyp}}^{\eta}(l,\mathfrak{n}|\alpha_S^\pi)| \ll_\epsilon \{\prod_{v\in\Sigma_\infty} l_v\}^{-1/2}\,\mathrm{N}(\mathfrak{f}_\eta)^{1/4+\epsilon}\,K^{5+\epsilon}$$

with the implied constant independent of l, $\mathfrak{n}$, π, η and K.

Proof. Set $P = \{(v_1,v_2) \in S^2 \mid v_1 \neq v_2\}$. From Lemmas 17.4 and 17.5, we have the bound

$$|\mathbb{J}_{\mathrm{hyp}}^{\eta}(l,\mathfrak{n}|\alpha_S^\pi)| \ll \sum_{b\in F^\times-\{-1\}} |J_S(b;\alpha_S^\pi)|\,\{\prod_{v\in\Sigma_{\mathrm{fin}}-S\cup S(\mathfrak{f}_\eta)} \Lambda_v(b)\}\,\{\prod_{v\in S(\mathfrak{f}_\eta)} |J_v^{\eta_v}(b)|\}\,\mathfrak{m}_\infty(l;b).$$

Combining this with Corollary 17.11 and Proposition 19.1, we have that this is majorized by the $\mathrm{N}(\mathfrak{f}_\eta)^{-1+\epsilon}$ times the following expression.

$$\begin{aligned}&\{\sum_{v\in S} q_v^{(\theta+1)/2}\}\,\mathfrak{I}(l,\mathfrak{n},\mathfrak{f}_\eta) + \sum_{v\in S} q_v^{\theta}\,\mathfrak{I}(l,\mathfrak{n},\mathfrak{p}_v\mathfrak{f}_\eta) + \sum_{v\in S} q_v^{\theta-1}\,\mathfrak{I}(l,\mathfrak{n},\mathfrak{p}_v^2\mathfrak{f}_\eta)\\
&\quad + \sum_{v\in S} q_v^{-1}\,\mathfrak{I}(l,\mathfrak{n},\mathfrak{p}_v^3\mathfrak{f}_\eta) + \sum_{v\in S} q_v^{-2}\,\mathfrak{I}(l,\mathfrak{n},\mathfrak{p}_v^4\mathfrak{f}_\eta)\end{aligned}$$

$$+\{\sum_{(v_1,v_2)\in P} q_{v_1}^{(\theta+1)/2}\,q_{v_2}^{(\theta+1)/2}\}\,\mathfrak{I}(l,\mathfrak{n},\mathfrak{f}_\eta) \tag{19.10}$$

$$+\sum_{(v_1,v_2)\in P} q_{v_1}^{(\theta+1)/2}\,\mathfrak{I}(l,\mathfrak{n},\mathfrak{p}_{v_2}\mathfrak{f}_\eta) + \sum_{(v_1,v_2)\in P} \mathfrak{I}(l,\mathfrak{n},\mathfrak{p}_{v_1}\mathfrak{p}_{v_2}\mathfrak{f}_\eta) \tag{19.11}$$

$$+\sum_{(v_1,v_2)\in P} q_{v_1}^{-1}\,\mathfrak{I}(l,\mathfrak{n},\mathfrak{p}_{v_1}^2\mathfrak{p}_{v_2}\mathfrak{f}_\eta) + \sum_{(v_1,v_2)\in P} q_{v_1}^{-1}\,q_{v_2}^{(\theta+1)/2}\,\mathfrak{I}(l,\mathfrak{n},\mathfrak{p}_{v_1}^2\mathfrak{f}_\eta)$$

$$+ \sum_{(v_1,v_2)\in P} q_{v_1}^{-1} q_{v_2}^{-1}\, \mathfrak{I}(l,\mathfrak{n},\mathfrak{p}_{v_1}^2\mathfrak{p}_{v_2}^2\mathfrak{f}_\eta).$$

Invoking the bound $\sharp S \ll K$ obtained from Lemma 19.8 and applying Lemma 19.2 or Lemma 19.3, we estimate each term occurring above. Thus, after a power saving, we obtain $|\mathbb{J}^\eta_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha_S^\pi)| \ll_\epsilon \mathrm{N}(\mathfrak{f}_\eta)^{-1+\epsilon}\,\varphi(l,K)$ with $\varphi(l,K)$ being

$$\mathrm{N}(\mathfrak{f}_\eta)^{5/4+\epsilon}\,L^{-1/2}\,(K^{(\theta+3)/2} + K^{\theta+9/4+\epsilon} + K^{\theta+5/2+2\epsilon} + K^{15/4+3\epsilon} + K^{4+4\epsilon} + K^{3+\theta} + K^{(2\theta+15)/4+\epsilon}$$
$$+ K^{9/2+2\epsilon} + K^{19/4+3\epsilon} + K^{(\theta+8)/2+2\epsilon} + K^{5+4\epsilon}),$$

where $L=\prod_{v\in\Sigma_\infty} l_v$. Since $\theta\in[0,1]$, this is bounded by $\mathrm{N}(\mathfrak{f}_\eta)^{1/4+2\epsilon}\,L^{-1/2}\,K^{5+4\epsilon}$. □

19.5. An estimate of the unipotent term. Set $S=S_K^{\mathfrak{n},\eta}$ with $K\geqslant 2$.

Proposition 19.10. *Let $\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})$. For any $\epsilon>0$, we have*

$$|\tilde{\mathbb{J}}^\eta_{\mathrm{u}}(l,\mathfrak{n}|\alpha_S^\pi)| \ll_\epsilon |\mathcal{G}(\eta)|\mathrm{N}(\mathfrak{f}_\eta)^\epsilon\,K^{1+\theta},$$

with the implied constant independent of l, $\mathfrak{n}$, π, η and K.

Proof. We use the same notation as in the proof of Proposition 19.9. By substituting the expression $\alpha_S^\pi(\mathbf{s})=\sum_{v\in S} Z_v^2 + \sum_{(v_1,v_2)\in P} Z_{v_1}Z_{v_2}$, we obtain

$$\begin{aligned}
&|\mathcal{G}(\eta)^{-1}\,\tilde{\mathbb{J}}^\eta_{\mathrm{u}}(l,\mathfrak{n}|\alpha_S^\pi)|\\
\ll&|\mathbf{C}_F^\eta(l,\mathfrak{n})|\left(\sum_{v\in S}\{\prod_{w\in S-\{v\}}|U_w^{\eta_w}(1)|\}\,|U_v^{\eta_v}(Z_v^2)| + \sum_{(v_1,v_2)\in P}\{\prod_{w\in S-\{v_1,v_2\}}|U_w^{\eta_w}(1)|\}\,|U_{v_1}^{\eta_{v_1}}(Z_{v_1})||U_{v_2}^{\eta_{v_2}}(Z_{v_2})|\right)\\
\ll& L_{\mathrm{fin}}(1,\eta)\left(\sum_{v\in S}|U_v^{\eta_v}(Z_v^2)| + \sum_{(v_1,v_2)\in P}|U_{v_1}^{\eta_{v_1}}(Z_{v_1})||U_{v_2}^{\eta_{v_2}}(Z_{v_2})|\right),
\end{aligned}$$

where to simplify the terms, we use $U_w(1)=-1$ from Proposition 18.1. As in the proof of Proposition 19.1, using Proposition 18.1, we compute each term and estimate it as follows.

$$\begin{aligned}
U_v^{\eta_v}(Z_v^2) &= \lambda_v(\pi)^2\,\{U_v^{\eta_v}(\alpha_v^{(2)}) + U_v^{\eta_v}(\alpha_v^{(0)})\} + U_v^{\eta_v}(\alpha_v^{(4)}) + 2U_v^{\eta_v}(\alpha_v^{(2)}) + \frac{3}{2}U_v^{\eta_v}(\alpha_v^{(0)})\\
&= \lambda_v(\pi)^2\{(1-q_v^{-1})-2\} + q_v^{-1}(1-q_v^{-1}) + 2(1-q_v^{-1}) - 3.
\end{aligned}$$

By $|\lambda_v(\pi)|\ll q_v^{\theta/2}$ with $\theta\in[0,1]$, from this,

$$|U_v^{\eta_v}(Z_v^2)| \ll q_v^\theta(1+q_v^{-1}) + q_v^{-2} + q_v^{-1} + 1 \ll q_v^\theta.$$

In a similar way,

$$U_v^{\eta_v}(Z_v) = q_v^{-1}.$$

Applying these, we continue the estimate of $|\tilde{\mathbb{J}}^\eta_{\mathrm{u}}(l,\mathfrak{n}|\alpha_S^\pi)|$ as follows.

$$\begin{aligned}
&L_{\mathrm{fin}}(1,\eta)\,\big(\sum_{v\in S} q_v^\theta + \sum_{(v_1,v_2)\in P} q_v^{-1}q_v^{-1}\big)\\
&\ll \mathrm{N}(\mathfrak{f}_\eta)^\epsilon\,\{K/\log K\times K^\theta + (K/\log K)^2K^{-2}\} \ll \mathrm{N}(\mathfrak{f}_\eta)^\epsilon\,K^{\theta+1}.
\end{aligned}$$

We remark that $L_{\mathrm{fin}}(1,\eta)\ll_\epsilon \mathrm{N}(\mathfrak{f}_\eta)^\epsilon$ (cf. [4, Theorem 2]). This completes the proof. □

19.6. A subconvexity bound (odd case). Let $\mathfrak{n}$ be an ideal of $\mathfrak{o}$. For a family of positive even integers $l=(l_v)_{v\in\Sigma_\infty}$, let $\Pi_{\rm cus}^*(l,\mathfrak{n})$ denote the set of all cuspidal automorphic representations $\pi\cong\bigotimes_v\pi_v$ of $\mathrm{PGL}_2(\mathbb{A})$ such that $\mathfrak{f}_\pi=\mathfrak{n}$ and such that π_v is isomorphic to the discrete series representation D_{l_v} of minimal $\mathbf{K}_v^0$-type l_v for each $v\in\Sigma_\infty$.

Theorem 19.11. *Let η be a quadratic character of $F^\times\backslash\mathbb{A}^\times$ with conductor $\mathfrak{f}_\eta$ such that $\eta_v(-1)=-1$ for all $v\in\Sigma_\infty$. Let $\mathfrak{n}$ be an integral ideal relatively prime to $\mathfrak{f}_\eta$. Assume that $l_v\geqslant 6$ for all $v\in\Sigma_\infty$. Then, for any $\epsilon>0$ we have*

$$|L_{\rm fin}(1/2,\pi)\,L_{\rm fin}(1/2,\pi\otimes\eta)|\ll_\epsilon(\mathrm{N}(\mathfrak{n}\mathfrak{f}_\eta)KL)^\epsilon\,\mathrm{N}(\mathfrak{n})\;(LK^{\theta-1}+\mathrm{N}(\mathfrak{f}_\eta)^{3/4}L^{1/2}K^3),$$

where $L=\prod_{v\in\Sigma_\infty}l_v$ and with the implied constant independent of l, $\mathfrak{n}$, η, $K\geqslant 2$ and $\pi\in\Pi_{\rm cus}^*(l,\mathfrak{n})$.

Proof. Let $\pi\in\Pi_{\rm cus}^*(l,\mathfrak{n})$ and let $S=S_K^{\mathfrak{n};\eta}$. By applying Theorem 16.1 for the test function $\alpha_S^\pi(\mathbf{s})$, we have

$$|C(l,\mathfrak{n},S)|\,|\sum_{\pi'\in\Pi_{\rm cus}(l,\mathfrak{n})}\mathbb{I}_{\rm cus}^\eta(\pi';l,\mathfrak{n})\,\alpha_S^\pi(\nu_S(\pi'))|\leqslant|\tilde{\mathbb{J}}_{\rm u}^\eta(l,\mathfrak{n}|\alpha_S^\pi)|+|\mathbb{J}_{\rm hyp}^\eta(l,\mathfrak{n}|\alpha_S^\pi)|.$$

with $C(l,\mathfrak{n},S)=(-1)^{\#S}2^{-1}D_F^{-1}\,[\mathbf{K}_{\rm fin}:\mathbf{K}_0(\mathfrak{n})]^{-1}\prod_{v\in\Sigma_\infty}2\pi\Gamma(l_v-1)/\Gamma(l_v/2)^2$. From Proposition 13.6 and the non-negativity of $\mathbb{I}_{\rm cus}^\eta(\pi';l,\mathfrak{n})/(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)$ by Lemma 13.2, the left-hand side becomes

$$|C(l,\mathfrak{n},S)|\,|\mathcal{G}(\eta)|\sum_{\pi'\in\Pi_{\rm cus}(l,\mathfrak{n})}\frac{[\mathbf{K}_{\rm fin}:\mathbf{K}_0(\mathfrak{f}_{\pi'})]}{\mathrm{N}(\mathfrak{f}_{\pi'})}w_{\mathfrak{n}}^\eta(\pi')\,\frac{L(1/2,\pi')\,L(1/2,\pi'\otimes\eta)}{L^{S_\pi}(1,\pi',\mathrm{Ad})}\,\alpha_S^\pi(\nu_S(\pi')),$$

which is greater than the summand corresponding to π by the non-negativity again. Let us examine the π-term closely. First, from the explicit formula, $w_{\mathfrak{n}}^\eta(\pi)=1$ for $\mathfrak{f}_\pi=\mathfrak{n}$. Let $A_v(\pi)=\mathrm{diag}(z_v,z_v^{-1})$ be the Satake parameter of our π. Then, using Lemma 19.8, we obtain

$$\alpha_S^\pi(\nu_S(\pi))=\left(\sum_{v\in S}\{(z_v+z_v^{-1})^2-(z_v^2+z_v^{-2}+1)\}\right)^2=(\#S)^2\gg_\epsilon K^{2-\epsilon}.$$

Separating the gamma factors from the L-functions, we have

$$\begin{aligned}|C(l,\mathfrak{n},S)|\,\frac{L_\infty(1/2,\pi)\,L_\infty(1/2,\pi\otimes\eta)}{L_\infty(1,\pi;\mathrm{Ad})}&\asymp[\mathbf{K}_{\rm fin}:\mathbf{K}_0(\mathfrak{n})]^{-1}\prod_{v\in\Sigma_\infty}\frac{2\pi\,\Gamma(l_v-1)}{\Gamma(l_v/2)^2}\times\prod_{v\in\Sigma_\infty}\frac{\Gamma_{\mathbb{C}}(l_v/2)^2}{\Gamma_{\mathbb{C}}(l_v)}\\&\asymp[\mathbf{K}_{\rm fin}:\mathbf{K}_0(\mathfrak{n})]^{-1}\prod_{v\in\Sigma_\infty}(l_v-1)^{-1},\end{aligned}$$

where all the implied constants are only dependent on F. The remaining factors in the π-term are easily seen to be bounded from below by a constant independent of $(l,\mathfrak{n},\pi,\eta)$. Combining the considerations so far, we obtain the estimate

$$|\mathcal{G}(\eta)|K^{2-\epsilon}\,\mathrm{N}(\mathfrak{n})^{-1}\,L^{-1}\,\frac{L_{\rm fin}(1/2,\pi)\,L_{\rm fin}(1/2,\pi\otimes\eta)}{L_{\rm fin}^{S_\pi}(1,\pi,\mathrm{Ad})}\ll_\epsilon|\tilde{\mathbb{J}}_{\rm u}^\eta(l,\mathfrak{n}|\alpha_S^\pi)|+|\mathbb{J}_{\rm hyp}^\eta(l,\mathfrak{n}|\alpha_S^\pi)|.\tag{19.12}$$

From Propositions 19.9 and 19.10, the right-hand side is estimated by

$$\ll_\epsilon|\mathcal{G}(\eta)|\mathrm{N}(\mathfrak{f}_\eta)^\epsilon\,K^{1+\theta}+\mathrm{N}(\mathfrak{f}_\eta)^{1/4+\epsilon}\,L^{-1/2}K^{5+\epsilon}.$$

To complete the proof, we invoke the bound $L_{\rm fin}^{S_\pi}(1,\pi,\mathrm{Ad})\ll_\epsilon(\mathrm{N}(\mathfrak{n})L)^\epsilon$ which is known to hold for a general class of L-series (cf. [4, Theorem 2]). We remark that $|\mathcal{G}(\eta)|=D_F^{-1/2}\mathrm{N}(\mathfrak{f}_\eta)^{-1/2}\prod_{v\in S(\mathfrak{f}_\eta)}(1-q_v^{-1})^{-1}\geqslant D_F^{-1/2}\mathrm{N}(\mathfrak{f}_\eta)^{-1/2}$. $\square$

Theorem 19.12. *Let η be a quadratic character of $F^\times\backslash\mathbb{A}^\times$ such that $\eta_v(-1)=-1$ for all $v\in\Sigma_\infty$. Let $\mathfrak{n}$ be an integral ideal relatively prime to $\mathfrak{f}_\eta$. Assume that $l_v\geqslant 6$ for all $v\in\Sigma_\infty$. Then, for any $\epsilon>0$,*

$$|L_{\rm fin}(1/2,\pi)\,L_{\rm fin}(1/2,\pi\otimes\eta)|\ll_\epsilon \mathrm{N}(\mathfrak{f}_\eta)^{3/4+\epsilon}\,\mathrm{N}(\mathfrak{n})^{1+\epsilon}\,\{\prod_{v\in\Sigma_\infty} l_v\}^{(7-\theta)/(8-2\theta)+\epsilon}$$

with the implied constant independent of l, $\mathfrak{n}$, η and $\pi\in\Pi_{\rm cus}^*(l,\mathfrak{n})$.

Proof. We apply the estimate in Theorem 19.11 with taking K so that $LK^{\theta-1}\asymp L^{1/2}K^3$, or equivalently $K\asymp L^{1/(8-2\theta)}$. Then, we obtain the desired estimate. $\square$

If $\theta\in[0,1)$, the estimate in Theorem 19.12 breaks the convex bound $L_{\rm fin}(1/2,\pi)\,L_{\rm fin}(1/2,\pi\otimes\eta)\ll_\epsilon \{C(\pi)\,C(\pi\otimes\eta)\}^{1/4+\epsilon}\ll(\prod_{v\in\Sigma_\infty} l_v)^{1+\epsilon}$ in the weight aspect with a fixed level $\mathfrak{n}$ and a fixed character η. To have Theorem 0.8, we only have to invoke the Ramanujan bound $\theta=0$ (cf. [1]) in Theorem 19.12.

Part 3. Relative trace formulas for holomorphic Hilbert modular forms : derivatives of L-series

In this part, we establish an explicit relative trace formula for GL(2), which encodes central derivatives of automorphic L-functions. Thoroughout this part, assume that $l \in (2\mathbb{N})^{\Sigma_\infty}$ satisfies $\underline{l} = \inf_{v \in \Sigma_\infty} l_v \geqslant 6$ and that η is a quadratic character of $F^\times \backslash \mathbb{A}^\times$. Let $\mathfrak{n}$ be an ideal of $\mathfrak{o}$ relatively prime to $\mathfrak{f}_\eta$. We consider a finite subset S of $\Sigma_{\rm fin}$ relatively prime to $\mathfrak{n}\mathfrak{f}_\eta$ and the automorphic Green function $\hat{\mathbf{\Psi}}^l_{\rm reg}(\mathfrak{n}|\alpha; g)$ associated with l, $\mathfrak{n}$ and S. We abuse the symbol $\mathfrak{p}_v$ to represent the prime ideal $\mathfrak{p}_v \cap \mathfrak{o}$.

In §0.4, we fix S and consider ideals $\mathfrak{a} = \prod_{v \in S(\mathfrak{a})} \mathfrak{p}_v^{n_v}$ such that $S(\mathfrak{a}) \subset S$. From now on, we treat only the case $S(\mathfrak{a}) = S$. Although $v \notin S(\mathfrak{a})$ holds for places v such that $n_v = 0$, the case $S = S(\mathfrak{a})$ is sufficient to be considered by substituting 0 for some n_v formally.

20. Spectral average of derivatives of L-series : the spectral side

Let $\mathcal{B}$ be the space of functions defined in §3.2. Given $\beta \in \mathcal{B}$, $t > 0$ and $\lambda \in \mathbb{C}$, we set

$$\beta_\lambda^{(1)}(t) = \frac{1}{2\pi i} \int_{L_\sigma} \frac{\beta(z)}{(z+\lambda)^2} t^z dz,$$

where $L_\sigma = \{z \in \mathbb{C} \mid \mathrm{Re}(z) = \sigma\}$. The defining integral is independent of the choice of $\sigma > -\mathrm{Re}(\lambda)$. By the residue theorem,

$$\mathrm{CT}_{\lambda=0}\{\beta_\lambda^{(1)}(t) - \beta_\lambda^{(1)}(t^{-1})\} = \beta(0) \log t. \tag{20.1}$$

In the same way as [47, Lemma 7.1], we have the estimate

$$|\beta_\lambda^{(1)}(t)| \ll_\sigma \inf\{t^\sigma, t^{-\mathrm{Re}(\lambda)}\} \log t, \qquad t > 0, \quad \sigma > -\mathrm{Re}(\lambda). \tag{20.2}$$

Definition 20.1. *For a cusp form φ on* $\mathrm{PGL}(2, \mathbb{A})$*, set*

$$\partial P^\eta_{\beta,\lambda}(\varphi) = \int_{F^\times \backslash \mathbb{A}^\times} \varphi\left(\left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & x_\eta \\ 0 & 1 \end{smallmatrix}\right] \right) \eta(t x_\eta^*) \{\beta_\lambda^{(1)}(|t|_\mathbb{A}) - \beta_\lambda^{(1)}(|t|_\mathbb{A}^{-1})\}\, d^\times t, \quad \mathrm{Re}(\lambda) \gg 1.$$

By (20.2), the integral $\partial P^\eta_{\beta,\lambda}(\varphi)$ is absolutely convergent for $\lambda \in \mathbb{C}$ and the function $\lambda \mapsto \partial P^\eta_{\beta,\lambda}(\varphi)$ is entire on $\mathbb{C}$. Therefore, (20.1) gives us the formula

$$\mathrm{CT}_{\lambda=0} \partial P^\eta_{\beta,\lambda}(\varphi) = \int_{F^\times \backslash \mathbb{A}^\times} \varphi\left(\left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & x_\eta \\ 0 & 1 \end{smallmatrix}\right] \right) \eta(t x_\eta^*) \log |t|_\mathbb{A} d^\times t \; \beta(0) = \frac{d}{ds} Z^*(s, \eta, \varphi) \Big|_{s=1/2} \beta(0).$$

Here $Z^*(s, \eta, \varphi)$ is the modified global zeta integral considered in §3.1 (cf. [47, 2.6.2] and [41, §4]).

20.1. For $j \in \mathbb{N}_0$, a place $v \in \Sigma_{\rm fin}$, an irreducible admissible representation π_v of $\mathrm{PGL}(2, F_v)$ and for a character η_v of $F_v^\times$ such that $\eta_v^2 = 1$, we define a polynomial of X by setting $Q^{\pi_v}_{j,v}(\eta_v, X) =$

$$\begin{cases} 1 & (j = 0), \\ \eta_v(\varpi_v) X - Q(\pi_v) & (c(\pi_v) = 0,\ j = 1), \\ \eta_v(\varpi_v)^{j-1} X^{j-1} (\eta_v(\varpi_v) X - q_v^{-1} \chi_v(\varpi_v)^{-1}) & (c(\pi_v) = 1,\ j \geqslant 1), \\ q_v^{-1} \eta_v(\varpi_v)^{j-2} X^{j-2} (a_v q_v^{1/2} \eta_v(\varpi_v) X - 1)(a_v^{-1} q_v^{1/2} \eta_v(\varpi_v) X - 1) & (c(\pi_v) = 0,\ j \geqslant 2), \\ \eta_v(\varpi_v)^j X^j & (c(\pi_v) \geqslant 2,\ j \geqslant 1), \end{cases} \tag{20.3}$$

(cf. [41, Corollary 19]), where

$$Q(\pi_v) = (a_v + a_v^{-1})/(q_v^{1/2} + q_v^{-1/2}) \qquad \text{with } a_v^{\pm} \text{ the Satake parameter of } \pi_v \text{ if } c(\pi_v) = 0,$$

and χ_v is the unramified character of $F_v^\times$ such that $\pi_v \cong \sigma(\chi_v|\,|_v^{1/2}, \chi_v|\,|_v^{-1/2})$ if $c(\pi_v)=1$. For $\pi \in \Pi_{\rm cus}(l,\mathfrak{n})$, we set

$$Q_{\pi,\eta,\rho}(s) = \prod_{v\in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})} Q^{\pi_v}_{\rho(v),v}(\eta_v, q_v^{1/2-s}), \quad \rho\in\Lambda_\pi(\mathfrak{n}),$$

where $\Lambda_\pi(\mathfrak{n})$ denotes the set $\prod_{k=1}^n \mathrm{Map}(S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1}), \{0,\dots k\})$ with $n = \max_{v\in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})} \mathrm{ord}_v(\mathfrak{n}\mathfrak{f}_\pi^{-1})$ and we set $\rho(v)=\rho_k(v)$ for each $v\in S_k(\mathfrak{n}\mathfrak{f}_\pi^{-1})$. We recall here an explicit formula of the modified zeta integral $Z^*(s,\eta,\varphi_{l,\pi,\rho})$ for the basis $\{\varphi_{l,\pi,\rho}\}$ of $V_\pi[\tau_l]^{\mathbf{K}_0(\mathfrak{n})}$ ([41, Proposition 20] and Proposition 13.1):

$$Z^*(s,\eta,\varphi_{l,\pi,\rho}) = D_F^{s-1/2}(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)Q_{\pi,\eta,\rho}(s)L(s,\pi\otimes\eta) \tag{20.4}$$

for any $\pi\in\Pi_{\rm cus}(l,\mathfrak{n})$ and $\rho\in\Lambda_\pi(\mathfrak{n})$.

20.2. Let $\pi\in\Pi_{\rm cus}(l,\mathfrak{n})$ and $\rho\in\Lambda_\pi(\mathfrak{n})$. For a complex parameter z, we set

$$\begin{aligned} w_{\mathfrak{n}}^\eta(\pi;z) &= \sum_{\rho\in\Lambda_\pi(\mathfrak{n})}\prod_{v\in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})} \overline{Q^{\pi_v}_{\rho(v),v}(\mathbf{1},1)}\, Q^{\pi_v}_{\rho(v),v}(\eta_v, q_v^{1/2-z})/\tau_{\pi_v}(\rho(v),\rho(v)) \\ &= \prod_{v\in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})} r^{(z)}(\pi_v,\eta_v) \end{aligned} \tag{20.5}$$

with

$$r^{(z)}(\pi_v,\eta_v) = \sum_{j=0}^{\mathrm{ord}_v(\mathfrak{n}\mathfrak{f}_\pi^{-1})} \overline{Q^{\pi_v}_{j,v}(\mathbf{1},1)}\, Q^{\pi_v}_{j,v}(\eta_v, q_v^{1/2-z})/\tau_{\pi_v}(j,j)$$

Here $Q^{\pi_v}_{j,v}(\eta_v,X)$ is the polynomial defined by (20.3), and $\tau_{\pi_v}(j,j)$ is given by [41, Corollary 12, Corollary 16 and Lemma 3] as

$$\tau_{\pi_v}(j,j) = \begin{cases} 1 & (j=0 \text{ or } c(\pi_v)\geqslant 2), \\ 1-Q(\pi_v)^2 & (c(\pi_v)=0,\ j=1), \\ 1-q_v^{-2} & (c(\pi_v)=1,\ j\geqslant 1), \\ (1-Q(\pi_v)^2)(1-q_v^{-2}) & (c(\pi_v)=0,\ j\geqslant 2). \end{cases} \tag{20.6}$$

Here is the explicit determination of $r^{(z)}(\pi_v,\eta_v)$.

Lemma 20.2. *Let $v\in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})$ and set $k_v = \mathrm{ord}_v(\mathfrak{n}\mathfrak{f}_\pi^{-1})$ and $X = q_v^{1/2-z}$. Suppose $\eta_v(\varpi_v)=-1$. Then we have*

$$r^{(z)}(\pi_v,\eta_v) = \begin{cases} \frac{1-X}{1+Q(\pi_v)} + \frac{(1+a_vq_v^{1/2}X)(1+a_v^{-1}q_v^{1/2}X)}{(q_v-1)(1+Q(\pi_v))}\frac{1-(-X)^{k_v-1}}{1+X} & (c(\pi_v)=0), \\ 1 - \frac{X+q_v^{-1}\chi_v(\varpi_v)}{1+q_v^{-1}\chi_v(\varpi_v)}\frac{1-(-1)^{k_v}X^{k_v}}{1+X} & (c(\pi_v)=1), \\ \frac{1+(-1)^{k_v}X^{k_v+1}}{1+X} & (c(\pi_v)\geqslant 2). \end{cases}$$

Suppose $\eta_v(\varpi_v)=1$. Then we have

$$r^{(z)}(\pi_v,\eta_v) = \begin{cases} \frac{1+X}{1+Q(\pi_v)} + \frac{(1-a_vq_v^{1/2}X)(1-a_v^{-1}q_v^{1/2}X)}{(q_v-1)(1+Q(\pi_v))}(\sum_{j=2}^{k_v}X^{j-2}) & (c(\pi_v)=0), \\ 1 + \frac{X-q_v^{-1}\chi_v(\varpi_v)}{1+q_v^{-1}\chi_v(\varpi_v)}(\sum_{j=1}^{k_v}X^j) & (c(\pi_v)=1), \\ \sum_{j=0}^{k_v}X^j & (c(\pi_v)\geqslant 2). \end{cases}$$

Proof. From (20.3) and (20.6), we obtain the result by a direct computation. $\square$

We abbreviate $r^{(1/2)}(\pi_v,\eta_v)$ to $r(\pi_v,\eta_v)$. Define

$$w_{\mathfrak{n}}^{\eta}(\pi)=w_{\mathfrak{n}}^{\eta}(\pi;1/2),\qquad \partial w_{\mathfrak{n}}^{\eta}(\pi)=\left(\frac{d}{dz}\right)_{z=1/2}w_{\mathfrak{n}}^{\eta}(\pi;z).$$

Note that the first quantity $w_{\mathfrak{n}}^{\eta}(\pi)$ is the same one as in Lemma 3.6 and Lemma 13.2. From Lemma 20.2, the second quantity $\partial w_{\mathfrak{n}}^{\eta}(\pi)$ is also evaluated explicitly.

Corollary 20.3. *Set* $\partial r(\pi_v,\eta_v)=\frac{-1}{\log q_v}\left(\frac{d}{dz}\right)_{z=1/2}r^{(z)}(\pi_v,\eta_v)$. *When* $\eta_v(\varpi_v)=-1$,

$$\partial r(\pi_v,\eta_v)=\begin{cases}\frac{-1}{1+Q(\pi_v)}+\dfrac{1+(-1)^{k_v}}{2}\frac{2q_v+(q_v+1)Q(\pi_v)}{(q_v-1)(1+Q(\pi_v))}+\frac{(-1)^{k_v}(2k_v-3)-1}{4}\frac{q_v+1}{q_v-1} & (c(\pi_v)=0),\\ -\frac{1-(-1)^{k_v}}{2}\dfrac{1}{1+q_v^{-1}\chi_v(\varpi_v)}+\frac{1+(-1)^{k_v}(2k_v-1)}{4} & (c(\pi_v)=1),\\ \frac{(-1)^{k_v}(2k_v+1)-1}{4} & (c(\pi_v)\geqslant 2).\end{cases}$$

When $\eta_v(\varpi_v)=1$,

$$\partial r(\pi_v,\eta_v)=\begin{cases}\frac{1}{1+Q(\pi_v)}+(k_v-1)\frac{2q_v-(q_v+1)Q(\pi_v)}{(q_v-1)(1+Q(\pi_v))}+\frac{(k_v-2)(k_v-1)}{2}\frac{(q_v+1)(1-Q(\pi_v))}{(q_v-1)(1+Q(\pi_v))} & (c(\pi_v)=0),\\ \frac{k_v}{1+q_v^{-1}\chi_v(\varpi_v)}+\frac{1-q_v^{-1}\chi_v(\varpi_v)}{1+q_v^{-1}\chi_v(\varpi_v)}\frac{k_v(k_v+1)}{2} & (c(\pi_v)=1),\\ \frac{k_v(k_v+1)}{2} & (c(\pi_v)\geqslant 2).\end{cases}$$

20.3. Depending on a function $\alpha\in\mathcal{A}_S$, we have constructed a cusp form denoted by $\hat{\mathbf{\Psi}}_{\mathrm{reg}}^{l}(\mathfrak{n}|\alpha)$ in §13.4. Recall that it has the expression

(20.7)

$$\hat{\mathbf{\Psi}}_{\mathrm{reg}}^{l}(\mathfrak{n}|\alpha;g)=\frac{(-1)^{\#S}\{\prod_{v\in\Sigma_\infty}2^{l_v-1}\}C_l(0)D_F^{-1/2}}{[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{n})]}\sum_{\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})}\sum_{\rho\in\Lambda_\pi(\mathfrak{n})}\alpha(\nu_S(\pi))\frac{\overline{Z^*(1/2,\mathbf{1},\varphi_{l,\pi,\rho})}}{\|\varphi_{l,\pi,\rho}\|^2}\varphi_{l,\pi,\rho}(g).$$

Proposition 20.4. *We have*

$$\begin{aligned}\mathrm{CT}_{\lambda=0}\partial P_{\beta,\lambda}^{\eta}(\hat{\mathbf{\Psi}}_{\mathrm{reg}}^{l}(\mathfrak{n}|\alpha))=&(-1)^{\#S}\{\prod_{v\in\Sigma_\infty}2^{l_v-1}\}C_l(0)D_F^{-1}[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{n})]^{-1}(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)\\ &\times\Bigg[\sum_{\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})}(\log D_F)w_{\mathfrak{n}}^{\eta}(\pi)\frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{\|\varphi_{l,\pi}^{\mathrm{new}}\|^2}\alpha(\nu_S(\pi))\\ &+\sum_{\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})}\partial w_{\mathfrak{n}}^{\eta}(\pi)\,\frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{\|\varphi_{l,\pi}^{\mathrm{new}}\|^2}\alpha(\nu_S(\pi))\\ &+\sum_{\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})}w_{\mathfrak{n}}^{\eta}(\pi)\frac{L(1/2,\pi)L'(1/2,\pi\otimes\eta)}{\|\varphi_{l,\pi}^{\mathrm{new}}\|^2}\alpha(\nu_S(\pi))\Bigg]\beta(0).\end{aligned}$$

Proof. Since the spectral expansion (20.7) is a finite sum, we have

$$\begin{aligned}\mathrm{CT}_{\lambda=0}\partial P_{\beta,\lambda}^{\eta}(\hat{\mathbf{\Psi}}_{\mathrm{reg}}^{l}(\mathfrak{n}|\alpha))=&(-1)^{\#S}\prod 2^{l_v-1}C_l(0)D_F^{-1/2}[\mathbf{K}_{\mathrm{fin}}:\mathbf{K}_0(\mathfrak{n})]^{-1}\\ &\times\sum_{\pi\in\Pi_{\mathrm{cus}}(l,\mathfrak{n})}\sum_{\rho\in\Lambda_\pi(\mathfrak{n})}\alpha(\nu_S(\pi))\frac{\overline{Z^*(1/2,\mathbf{1},\varphi_{l,\pi,\rho})}}{\|\varphi_{l,\pi,\rho}\|^2}\,\frac{d}{ds}Z^*(s,\eta,\varphi_{l,\pi,\rho})\Big|_{s=1/2}\beta(0).\end{aligned}$$

By virtue of Proposition 13.1 and (20.4), we have

$$\sum_{\rho\in\Lambda_\pi(\mathfrak{n})}\frac{\overline{Z^*(1/2,\mathbf{1},\varphi_{l,\pi,\rho})}}{\|\varphi_{l,\pi,\rho}\|^2}\,\frac{d}{ds}Z^*(s,\eta,\varphi_{l,\pi,\rho})\Big|_{s=1/2}$$

$$
\begin{aligned}
=&\sum_{\rho\in\Lambda_\pi(\mathfrak{n})}\frac{1}{\|\varphi_{l,\pi,\rho}\|^2}D_F^{-1/2}Q_{\pi,\mathbf{1},\rho}(1/2)L(1/2,\pi)(\log D_F)\mathcal{G}(\eta)Q_{\pi,\eta,\rho}(1/2)L(1/2,\pi\otimes\eta)\\
&+\sum_{\rho\in\Lambda_\pi(\mathfrak{n})}\frac{1}{\|\varphi_{l,\pi,\rho}\|^2}D_F^{-1/2}Q_{\pi,\mathbf{1},\rho}(1/2)L(1/2,\pi)\mathcal{G}(\eta)(Q_{\pi,\eta,\rho})'(1/2)L(1/2,\pi\otimes\eta)\\
&+\sum_{\rho\in\Lambda_\pi(\mathfrak{n})}\frac{1}{\|\varphi_{l,\pi,\rho}\|^2}D_F^{-1/2}Q_{\pi,\mathbf{1},\rho}(1/2)L(1/2,\pi)\mathcal{G}(\eta)Q_{\pi,\eta,\rho}(1/2)L'(1/2,\pi\otimes\eta)\\
=&(\log D_F)D_F^{-1/2}\mathcal{G}(\eta)w_{\mathfrak{n}}^{\eta}(\pi)\frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{\|\varphi_{l,\pi}^{\mathrm{new}}\|^2}\\
&+\left\{\sum_{\rho\in\Lambda_\pi(\mathfrak{n})}\left(\prod_{v\in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})}\frac{Q_{\rho(v),v}^{\pi_v}(\mathbf{1}_v,1)}{\tau_{\pi_v}(\rho(v),\rho(v))}\right)(Q_{\pi,\eta,\rho})'(1/2)\right\}D_F^{-1/2}\mathcal{G}(\eta)\frac{L(1/2,\pi)L(1/2,\pi\otimes\eta)}{\|\varphi_{l,\pi}^{\mathrm{new}}\|^2}\\
&+D_F^{-1/2}\mathcal{G}(\eta)w_{\mathfrak{n}}^{\eta}(\pi)\frac{L(1/2,\pi)L'(1/2,\pi\otimes\eta)}{\|\varphi_{l,\pi}^{\mathrm{new}}\|^2}.
\end{aligned}
$$

By the definition (20.5) of $w_{\mathfrak{n}}^{\eta}(\pi,z)$, we have

$$\partial w_{\mathfrak{n}}^{\eta}(\pi)=\sum_{\rho\in\Lambda_\pi(\mathfrak{n})}\left\{\prod_{v\in S(\mathfrak{n}\mathfrak{f}_\pi^{-1})}\frac{\overline{Q_{\rho(v),v}^{\pi_v}(\mathbf{1},1)}}{\tau_{\pi_v}(\rho(v),\rho(v))}\right\}(Q_{\pi,\eta,\rho})'(1/2).$$

Thus we are done. □

21. Spectral average of derivatives of L-series: the geometric side

Recall that the function $\hat{\mathbf{\Psi}}_{\mathrm{reg}}^{l}(\mathfrak{n}|\alpha)$ has another expansion coming from the double coset space $H_F\backslash G_F/H_F$ in §14:

$$\hat{\mathbf{\Psi}}_{\mathrm{reg}}^{l}\left(\mathfrak{n}|\alpha;\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\right)=(1+i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))J_{\mathrm{id}}^{l}(\alpha;t)+J_{\mathrm{u}}^{l}(\alpha;t)+J_{\bar{\mathrm{u}}}^{l}(\alpha;t)+J_{\mathrm{hyp}}^{l}(\alpha;t),\quad t\in\mathbb{A}^\times, \tag{21.1}$$

where the terms in the right-hand side are defined in Lemmas 14.1, 14.2 and 14.10. For $\natural\in\{\mathrm{id},\mathrm{u},\bar{\mathrm{u}},\mathrm{hyp}\}$, we consider the "orbital integrals"

$$\mathbb{W}_{\natural}^{\eta}(\beta,\lambda;\alpha)=\int_{F^\times\backslash\mathbb{A}^\times}J_{\natural}^{l}(\alpha;t)\{\beta_\lambda^{(1)}(|t|_{\mathbb{A}})-\beta_\lambda^{(1)}(|t|_{\mathbb{A}}^{-1})\}\eta(tx_\eta^*)d^\times t$$

for $\alpha\in\mathcal{A}_S$, $\beta\in\mathcal{B}$ and $\lambda\in\mathbb{C}$ such that $\mathrm{Re}(\lambda)>1$. We shall show that these integrals converge absolutely individually when $\mathrm{Re}(\lambda)>1$ and admit an analytic continuation in a neighborhood of $\lambda=0$.

Lemma 21.1. *Let λ and w be complex numbers such that $\mathrm{Re}(w)<\mathrm{Re}(\lambda)$. Let ξ be a character of $F^\times\backslash\mathbb{A}^\times$. Then, we have*

$$\int_{F^\times\backslash\mathbb{A}^\times}\beta_\lambda^{(1)}(|t|_{\mathbb{A}})\xi(t)|t|_{\mathbb{A}}^{w}d^\times t=\delta_{\xi,\mathbf{1}}\mathrm{vol}(F^\times\backslash\mathbb{A}^1)\frac{\beta(-w)}{(\lambda-w)^2}.$$

Proof. The proof is given in the same way as [47, Lemma 7.6]. □

Lemma 21.2. *For $\mathrm{Re}(\lambda)>0$, the integral $\mathbb{W}_{\mathrm{id}}^{\eta}(\beta,\lambda;\alpha)$ converges absolutely and $\mathbb{W}_{\mathrm{id}}^{\eta}(\beta,\lambda;\alpha)=0$.*

Proof. This follows immediately from Lemma 21.1, since $J_{\mathrm{id}}(\alpha;t)$ is independent of the variable t (cf. Lemma 14.1). □

Assume that $q(\mathrm{Re}(\mathbf{s})) > \mathrm{Re}(\lambda) > \sigma$ and $1 < \sigma < \underline{l}/2$. Set

$$V_{0,\eta}^{\pm}(\lambda;\mathbf{s}) = \frac{1}{2\pi i}\int_{L_{\mp\sigma}} \frac{\beta(z)}{(z+\lambda)^2}\int_{\mathbb{A}^\times} \Psi_l^{(0)}\left(\mathfrak{n}|\mathbf{s}; [\begin{smallmatrix} 1 & t^{-1} \\ 0 & 1\end{smallmatrix}][\begin{smallmatrix} 1 & x_\eta \\ 0 & 1\end{smallmatrix}]\right)\eta(tx_\eta^*)|t|_{\mathbb{A}}^{\pm z}d^\times t dz,$$
$$V_{1,\eta}^{\pm}(\lambda;\mathbf{s}) = \frac{1}{2\pi i}\int_{L_{\mp\sigma}} \frac{\beta(z)}{(z+\lambda)^2}\int_{\mathbb{A}^\times} \Psi_l^{(0)}\left(\mathfrak{n}|\mathbf{s}; [\begin{smallmatrix} 1 & 0 \\ t^{-1} & 1\end{smallmatrix}][\begin{smallmatrix} 1 & 0 \\ -x_\eta & 1\end{smallmatrix}]w_0\right)\eta(tx_\eta^*)|t|_{\mathbb{A}}^{\pm z}d^\times t dz$$

and

$$\Upsilon_S^\eta(z;\mathbf{s}) = \prod_{v\in S}(1-\eta_v(\varpi_v)q_v^{-(z+(s_v+1)/2)})^{-1}(1-q_v^{(s_v+1)/2})^{-1},$$
$$\Upsilon_{S,l}^\eta(z;\mathbf{s}) = D_F^{-1/2}\{\#(\mathfrak{o}/\mathfrak{f}_\eta)^\times\}^{-1}\{\prod_{v\in\Sigma_\infty}\frac{2\Gamma(-z)\Gamma(l_v/2+z)}{\Gamma_{\mathbb{R}}(-z+\epsilon_v)\Gamma(l_v/2)}i^{\epsilon_v}\cos\left(\frac{\pi}{2}(-z+\epsilon_v)\right)\}\,\Upsilon_S^\eta(z;\mathbf{s}).$$

Lemma 21.3. *The double integrals $V_{j,\eta}^{\pm}(\lambda;\mathbf{s})$ converge absolutely and*

$$V_{0,\eta}^{\pm}(\lambda;\mathbf{s}) = \frac{1}{2\pi i}\int_{L_\sigma}\frac{\beta(z)}{(z+\lambda)^2}\mathrm{N}(\mathfrak{f}_\eta)^{\mp z}L(\mp z,\eta)(-1)^{\epsilon(\eta)}\Upsilon_{S,l}^\eta(\pm z;\mathbf{s})dz$$

and

$$V_{1,\eta}^{\pm}(\lambda;\mathbf{s}) = \frac{1}{2\pi i}\int_{L_\sigma}\frac{\beta(z)}{(z+\lambda)^2}\mathrm{N}(\mathfrak{f}_\eta)^{\mp z}\mathrm{N}(\mathfrak{n})^{\pm z}\tilde\eta(\mathfrak{n})\delta(\mathfrak{n}=\mathfrak{o})L(\mp z,\eta)i^{\tilde{l}}\Upsilon_{S,l}^\eta(\pm z;\mathbf{s})dz.$$

Proof. As in Lemma 15.2, we exchange the order of integrals and compute the t-integrals first. Since $\eta\neq 1$, the integrands in the remaining contour integrals in z are holomorphic on $|\mathrm{Re}(z)|<\sigma$; thus we can shift the contour $L_{-\sigma}$ to L_σ for $V_{0,\eta}^+$ and $V_{1,\eta}^+$. □

Lemma 21.4. *The integral $\mathbb{W}_{\mathrm{u}}^\eta(\beta,\lambda;\alpha)$ has an analytic continuation to the region $\mathrm{Re}(\lambda) > -\underline{l}/2$ as a function in λ. The constant term of $\mathbb{W}_{\mathrm{u}}^\eta(\beta,\lambda;\alpha)$ at $\lambda=0$ equals $\mathbb{W}_{\mathrm{u}}^\eta(l,\mathfrak{n}|\alpha)\beta(0)$ with*

$$\mathbb{W}_{\mathrm{u}}^\eta(l,\mathfrak{n}|\alpha) = (-1)^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}(1+(-1)^{\epsilon(\eta)}\tilde\eta(\mathfrak{n})i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\mathfrak{W}_S^\eta(l,\mathfrak{n}|\mathbf{s})\alpha(\mathbf{s})d\mu_S(\mathbf{s}),$$

where $\Upsilon_S^\eta(\mathbf{s}) = \Upsilon_S^\eta(0;\mathbf{s})$ and

$$\mathfrak{W}_S^\eta(\mathbf{s}) = \pi^{\epsilon(\eta)}\Upsilon_S^\eta(\mathbf{s})L(1,\eta)\Bigg\{\log D_F + \frac{L'(1,\eta)}{L(1,\eta)}$$
$$+\sum_{v\in\Sigma_\infty}\left(\sum_{k=1}^{l_v/2-1}\frac{1}{k} - \frac{1}{2}\log\pi - \frac{1}{2}C_{\mathrm{Euler}} - \delta_{\epsilon_v,1}\log 2\right) + \sum_{v\in S}\frac{\log q_v}{1-\eta_v(\varpi_v)q_v^{(s_v+1)/2}}\Bigg\}.$$

Proof. From Lemma 14.2, we have the expression

$$\mathbb{W}_{\mathrm{u}}^\eta(\beta,\lambda;\alpha) = \left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\{V_{0,\eta}^+(\lambda;\mathbf{s}) - V_{0,\eta}^-(\lambda;\mathbf{s}) + V_{1,\eta}^+(\lambda;\mathbf{s}) - V_{1,\eta}^-(\lambda;\mathbf{s})\}\alpha(\mathbf{s})d\mu_S(\mathbf{s}).$$

By Lemma 21.3, the right-hand side becomes

$$((-1)^{\epsilon(\eta)} + i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\frac{1}{2\pi i}\int_{L_\sigma}\frac{\beta(z)}{(z+\lambda)^2}\{\mathrm{N}(\mathfrak{f}_\eta)^{-z}L(-z,\eta)\Upsilon_{S,l}^\eta(z;\mathbf{s})$$
$$-\mathrm{N}(\mathfrak{f}_\eta)^zL(z,\eta)\Upsilon_{S,l}^\eta(-z;\mathbf{s})\}dz\alpha(\mathbf{s})d\mu_S(\mathbf{s}),$$

which is holomorphic on $\mathrm{Re}(\lambda) > -\sigma$. Since $1 < \sigma < \underline{l}/2$ is arbitrary, this gives an analytic continuation of $\mathbb{W}_{\mathrm{u}}^{\eta}(\beta, \lambda; \alpha)$ to the region $\mathrm{Re}(\lambda) > -\underline{l}/2$ and yields the equality

$$\begin{aligned}
&\mathrm{CT}_{\lambda=0}\mathbb{W}_{\mathrm{u}}^{\eta}(\beta, \lambda; \alpha)\\
&=\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\left(\frac{1}{2\pi i}\int_{L_\sigma}\frac{\beta(z)}{z^2}\{f_{\mathrm{u}}(z)-f_{\mathrm{u}}(-z)\}dz\right)\alpha(\mathbf{s})d\mu_S(\mathbf{s})\\
&=((-1)^{\epsilon(\eta)}+i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))\,\mathrm{Res}_{z=0}\left(\frac{\beta(z)}{z^2}f_{\mathrm{u}}(z)\right)\\
&=((-1)^{\epsilon(\eta)}+i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))\big\{\mathrm{CT}_{z=0}\frac{f_{\mathrm{u}}(z)}{z}\beta(0)+\frac{1}{2}\,\mathrm{Res}_{z=0}\,f_{\mathrm{u}}(z)\beta''(0)\big\},
\end{aligned}$$

where $f_{\mathrm{u}}(z) = \mathrm{N}(\mathfrak{f}_\eta)^{-z}L(-z, \eta)\Upsilon_{S,l}^{\eta}(z; \mathbf{s})$. Since η is non-trivial, by the functional equation

$$L(s,\eta) = i^{\epsilon(\eta)}D_F^{1-s}\mathrm{N}(\mathfrak{f}_\eta)^{-s}\,\#((\mathfrak{o}/\mathfrak{f}_\eta)^\times)\,\mathcal{G}(\eta)\,L(1-s,\eta),$$

$f_{\mathrm{u}}(z)$ is holomorphic at $z = 0$. Thus,

$$\begin{aligned}
&\mathrm{CT}_{z=0}\frac{f_{\mathrm{u}}(z)}{z}=\lim_{z\to 0}\frac{f_{\mathrm{u}}(z)-f_{\mathrm{u}}(0)}{z}\\
&=-(\log \mathrm{N}(\mathfrak{f}_\eta))L(0,\eta)\Upsilon_{S,l}^{\eta}(0;\mathbf{s})-L'(0,\eta)\Upsilon_{S,l}^{\eta}(0;\mathbf{s})+L(0,\eta)(\Upsilon_{S,l}^{\eta})'(0;\mathbf{s})\\
&=i^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}\tilde{\Upsilon}_{S,l}^{\eta}(0;\mathbf{s})\{-L(1,\eta)\log\mathrm{N}(\mathfrak{f}_\eta)+L(1,\eta)\log(D_F\mathrm{N}(\mathfrak{f}_\eta))+L'(1,\eta)+L(1,\eta)\frac{d}{dz}\log\tilde{\Upsilon}_{S,l}(z;\mathbf{s})|_{z=0}\}\\
&=\mathcal{G}(\eta)D_F^{1/2}\pi^{\epsilon(\eta)}\Upsilon_S^{\eta}(\mathbf{s})\{L(1,\eta)\log D_F+L'(1,\eta)+L(1,\eta)\frac{d}{dz}\log\tilde{\Upsilon}_{S,l}(z;\mathbf{s})|_{z=0}\},
\end{aligned}$$

where $\tilde{\Upsilon}_{S,l}^{\eta}(z; \mathbf{s}) = D_F^{1/2}\#((\mathfrak{o}/\mathfrak{f}_\eta)^\times)\Upsilon_{S,l}^{\eta}(z; \mathbf{s})$. Furthermore,

$$\begin{aligned}
\frac{d}{dz}\log\tilde{\Upsilon}_{S,l}(z;\mathbf{s})|_{z=0}=&\sum_{v\in\Sigma_\infty}\left(\psi(l_v/2)-\frac{1}{2}\log\pi+\frac{1}{2}\psi\left(\frac{-z+\epsilon_v}{2}\right)-\psi(-z)+\frac{\pi}{2}\tan\frac{\pi}{2}(-z+\epsilon_v)\right)\Bigg|_{z=0}\\
&+\sum_{v\in S}\frac{\log q_v}{1-\eta_v(\varpi_v)q_v^{(s_v+1)/2}}.
\end{aligned}$$

Here, by $\psi(1) = -C_{\mathrm{Euler}}$, $\psi(1/2) = -C_{\mathrm{Euler}} - 2\log 2$ and $\frac{d}{dt}\,(t\cot t)\,|_{t=0} = 0$, we have

$$\frac{1}{2}\psi\left(\frac{-z+\epsilon_v}{2}\right)-\psi(-z)+\frac{\pi}{2}\tan\frac{\pi}{2}(-z+\epsilon_v)\Bigg|_{z=0}=\begin{cases}\frac{1}{2}C_{\mathrm{Euler}} & (\epsilon_v=0),\\[2ex] \frac{1}{2}\psi\left(\frac{1}{2}\right)-\psi(1)=\frac{1}{2}C_{\mathrm{Euler}}-\log 2 & (\epsilon_v=1).\end{cases}$$

□

Assume that $q(\mathrm{Re}(\mathbf{s})) > \mathrm{Re}(\lambda) > \sigma$ and $1 < \sigma < \underline{l}/2$. Set

$$\begin{aligned}
\tilde{V}_{1,\eta}^{\pm}(\lambda;\mathbf{s})&=\frac{1}{2\pi i}\int_{L_{\pm\sigma}}\frac{\beta(z)}{(z+\lambda)^2}\int_{\mathbb{A}^\times}\Psi_l^{(0)}\left(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&0\\t&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right]\right)\eta(tx_\eta^*)|t|_{\mathbb{A}}^{\pm z}d^\times t dz,\\
\tilde{V}_{0,\eta}^{\pm}(\lambda;\mathbf{s})&=\frac{1}{2\pi i}\int_{L_{\pm\sigma}}\frac{\beta(z)}{(z+\lambda)^2}\int_{\mathbb{A}^\times}\Psi_l^{(0)}\left(\mathfrak{n}|\mathbf{s};\left[\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&0\\-x_\eta&1\end{smallmatrix}\right]w_0\right)\eta(tx_\eta^*)|t|_{\mathbb{A}}^{\pm z}d^\times t dz.
\end{aligned}$$

In the same way as Lemma 21.3, we obtain the following.

Lemma 21.5. *The double integrals $\tilde{V}^{\pm}_{j,\eta}(\lambda;\mathbf{s})$ converge absolutely and*

$$\tilde{V}^{\pm}_{1,\eta}(\lambda;\mathbf{s}) = \frac{1}{2\pi i}\int_{L_\sigma}\frac{\beta(z)}{(z+\lambda)^2}\mathrm{N}(\mathfrak{f}_\eta)^{\mp z}\mathrm{N}(\mathfrak{n})^{\mp z}\tilde{\eta}(\mathfrak{n})L(\pm z,\eta)\Upsilon^{\eta}_{S,l}(\mp z;\mathbf{s})dz,$$

$$\tilde{V}^{\pm}_{0,\eta}(\lambda;\mathbf{s}) = \frac{1}{2\pi i}\int_{L_\sigma}\frac{\beta(z)}{(z+\lambda)^2}\mathrm{N}(\mathfrak{f}_\eta)^{\mp z}\delta(\mathfrak{n}=\mathfrak{o})L(\pm z,\eta)(-1)^{\epsilon(\eta)}i^{\tilde{l}}\Upsilon^{\eta}_{S,l}(\mp z;\mathbf{s})dz.$$

Lemma 21.6. *The integral $\mathbb{W}^{\eta}_{\bar{\mathrm{u}}}(\beta,\lambda;\alpha)$ converges absolutely on $\mathrm{Re}(\lambda)>1$ and has an analytic continuation to the region $\mathrm{Re}(\lambda)>-\underline{l}/2$ as a function in λ. The constant term of $\mathbb{W}^{\eta}_{\bar{\mathrm{u}}}(\beta,\lambda;\alpha)$ at $\lambda=0$ equals $\mathbb{W}^{\eta}_{\bar{\mathrm{u}}}(l,\mathfrak{n}|\alpha)\beta(0)$ with*

$$\mathbb{W}^{\eta}_{\bar{\mathrm{u}}}(l,\mathfrak{n}|\alpha) = (-1)^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}((-1)^{\epsilon(\eta)}\tilde{\eta}(\mathfrak{n})+i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\mathfrak{W}^{\eta}_{S,\bar{\mathrm{u}}}(l,\mathfrak{n}|\mathbf{s})\alpha(\mathbf{s})d\mu_S(\mathbf{s}),$$

where

$$\mathfrak{W}^{\eta}_{S,\bar{\mathrm{u}}}(l,\mathfrak{n}|\mathbf{s}) = -\pi^{\epsilon(\eta)}\Upsilon^{\eta}_S(\mathbf{s})L(1,\eta)\log(\mathrm{N}(\mathfrak{n})\mathrm{N}(\mathfrak{f}_\eta)^2)-\mathfrak{W}^{\eta}_{S,\mathrm{u}}(l,\mathfrak{n}|\mathbf{s}).$$

Proof. From Lemma 14.2, we have the expression

$$\mathbb{W}^{\eta}_{\bar{\mathrm{u}}}(\beta,\lambda;\alpha) = \left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\{\tilde{V}^{+}_{0,\eta}(\lambda;\mathbf{s})-\tilde{V}^{-}_{0,\eta}(\lambda;\mathbf{s})+\tilde{V}^{+}_{1,\eta}(\lambda;\mathbf{s})-\tilde{V}^{-}_{1,\eta}(\lambda;\mathbf{s})\}\alpha(\mathbf{s})d\mu_S(\mathbf{s}).$$

By Lemma 21.5, the right-hand side becomes

$$\begin{aligned}&(\tilde{\eta}(\mathfrak{n})+(-1)^{\epsilon(\eta)}i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\frac{1}{2\pi i}\int_{L_\sigma}\frac{\beta(z)}{(z+\lambda)^2}\times\{\mathrm{N}(\mathfrak{f}_\eta)^{-z}\mathrm{N}(\mathfrak{n})^{-z}L(z,\eta)\Upsilon^{\eta}_{S,l}(-z;\mathbf{s})\\&-\mathrm{N}(\mathfrak{f}_\eta)^{z}\mathrm{N}(\mathfrak{n})^{z}L(-z,\eta)\Upsilon^{\eta}_{S,l}(z;\mathbf{s})\}dz\alpha(\mathbf{s})d\mu_S(\mathbf{s}).\end{aligned}$$

As before, this gives an analytic continuation of $\mathbb{W}^{\eta}_{\bar{\mathrm{u}}}(\beta,\lambda;\alpha)$ to the region $\mathrm{Re}(\lambda)>-\underline{l}/2$. We set $f_{\bar{\mathrm{u}}}(z)=-\mathrm{N}(\mathfrak{f}_\eta)^{2z}\mathrm{N}(\mathfrak{n})^{z}f_{\mathrm{u}}(z)$. Then,

$$\begin{aligned}&\mathrm{CT}_{\lambda=0}\mathbb{W}^{\eta}_{\bar{\mathrm{u}}}(\beta,\lambda;\alpha)\\=&(\tilde{\eta}(\mathfrak{n})+(-1)^{\epsilon(\eta)}i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o}))\big\{\mathrm{CT}_{z=0}\frac{f_{\bar{\mathrm{u}}}(z)}{z}\beta(0)+\frac{1}{2}\mathrm{Res}_{z=0}\,f_{\bar{\mathrm{u}}}(z)\beta''(0)\big\}.\end{aligned}$$

Since η is supposed to be non-trivial, $f_{\bar{\mathrm{u}}}(z)$ is holomorphic at $z=0$ and

$$\begin{aligned}\mathrm{CT}_{z=0}\frac{f_{\bar{\mathrm{u}}}(z)}{z}=&f'_{\bar{\mathrm{u}}}(0)=-\log(\mathrm{N}(\mathfrak{n})\mathrm{N}(\mathfrak{f}_\eta^2))f_{\mathrm{u}}(0)-f'_{\mathrm{u}}(0)\\=&\mathcal{G}(\eta)D_F^{1/2}\{-\pi^{\epsilon(\eta)}\Upsilon^{\eta}_S(\mathbf{s})L(1,\eta)\log(\mathrm{N}(\mathfrak{n})\mathrm{N}(\mathfrak{f}_\eta^2))-\mathfrak{W}^{\eta}_{S,\mathrm{u}}(l,\mathfrak{n}|\mathbf{s})\}.\end{aligned}$$

□

Lemma 21.7. *The integral $\mathbb{W}^{\eta}_{\mathrm{hyp}}(\beta,\lambda;\alpha)$ converges absolutely on $\mathrm{Re}(\lambda)>1$ and has an analytic continuation to the region $\mathrm{Re}(\lambda)>-\epsilon$ for some $\epsilon>0$. The constant term of $\mathbb{W}^{\eta}_{\mathrm{hyp}}(\beta,\lambda;\alpha)$ at $\lambda=0$ equals $\mathbb{W}^{\eta}_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha)\beta(0)$. Here*

$$\mathbb{W}^{\eta}_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha) = \left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\mathfrak{L}_\eta(l,\mathfrak{n}|\mathbf{s})\alpha(\mathbf{s})d\mu_S(\mathbf{s})$$

with

$$\mathfrak{L}_\eta(l,\mathfrak{n}|\mathbf{s}) = \sum_{b\in F-\{0,-1\}}\int_{\mathbb{A}^\times}\Psi^{(0)}_l(\mathfrak{n}|\mathbf{s},\delta_b\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right])\eta(tx^*_\eta)\log|t|_{\mathbb{A}}d^\times t.$$

Proof. The absolute convergence and analytic continuation of $\mathbb{W}^{\eta}_{\mathrm{hyp}}(\beta,\lambda;\alpha)$ are given in the same way as Lemma 15.5. We obtain the last assertion with the aid of (20.1). □

From the analysis so far, (21.1) yields the formula:

$$\partial P^{\eta}_{\beta,\lambda}(\hat{\mathbf{\Psi}}^{l}_{\mathrm{reg}}(\mathfrak{n}|\alpha)) = \mathbb{W}^{\eta}_{\mathrm{u}}(\beta,\lambda;\alpha) + \mathbb{W}^{\eta}_{\bar{\mathrm{u}}}(\beta,\lambda;\alpha) + \mathbb{W}^{\eta}_{\mathrm{hyp}}(\beta,\lambda;\alpha) \tag{21.2}$$

which is valid on a half plane $\mathrm{Re}(\lambda) > -\epsilon$ containing $\lambda = 0$.

21.1. The derivative relative trace formula. For any ideal $\mathfrak{m}$ of $\mathfrak{o}$, set

$$\iota(\mathfrak{m}) = [\mathbf{K}_{\mathrm{fin}} : \mathbf{K}_0(\mathfrak{m})] = \prod_{v \in S(\mathfrak{m})} (1+q_v) q_v^{\mathrm{ord}_v(\mathfrak{m})-1}.$$

Let $\mathcal{J}_{S,\eta}$ be the monoid of ideals generated by prime ideals $\mathfrak{p}_v$ for all $v \in \Sigma_{\mathrm{fin}} - S \cup S(\mathfrak{f}_\eta)$. We shall introduce several functionals in $\alpha \in \mathcal{A}_S$ depending on an ideal $\mathfrak{m} \in \mathcal{J}_{S,\eta}$:

$$\mathrm{AL}^{w}(\mathfrak{m};\alpha) = C_l \sum_{\pi \in \Pi_{\mathrm{cus}}(l,\mathfrak{m})} \frac{w^{\eta}_{\mathfrak{m}}(\pi)\,\iota(\mathfrak{f}_\pi)}{\mathrm{N}(\mathfrak{f}_\pi)\iota(\mathfrak{m})} \frac{L(1/2,\pi)\,L(1/2,\pi\otimes\eta)}{L^{S_\pi}(1,\pi,\mathrm{Ad})}\,\alpha(\nu_S(\pi)), \tag{21.3}$$

$$\mathrm{AL}^{\partial w}(\mathfrak{m};\alpha) = C_l \sum_{\pi \in \Pi_{\mathrm{cus}}(l,\mathfrak{m})} \frac{\partial w^{\eta}_{\mathfrak{m}}(\pi)\,\iota(\mathfrak{f}_\pi)}{\mathrm{N}(\mathfrak{f}_\pi)\,\iota(\mathfrak{m})} \frac{L(1/2,\pi)\,L(1/2,\pi\otimes\eta)}{L^{S_\pi}(1,\pi,\mathrm{Ad})}\,\alpha(\nu_S(\pi)), \tag{21.4}$$

$$\mathrm{ADL}^{w}_{\pm}(\mathfrak{m};\alpha) = C_l \sum_{\substack{\pi \in \Pi_{\mathrm{cus}}(l,\mathfrak{m}) \\ \epsilon(1/2,\pi\otimes\eta)=\pm 1}} \frac{w^{\eta}_{\mathfrak{m}}(\pi)\,\iota(\mathfrak{f}_\pi)}{\mathrm{N}(\mathfrak{f}_\pi)\,\iota(\mathfrak{m})} \frac{L(1/2,\pi)\,L'(1/2,\pi\otimes\eta)}{L^{S_\pi}(1,\pi,\mathrm{Ad})}\,\alpha(\nu_S(\pi)), \tag{21.5}$$

where C_l is the same as (0.4). The derivative of L-functions in ADL^{w}_{+} is eliminated by the functional equation.

Proposition 21.8. *We have*

$$\mathrm{ADL}^{w}_{+}(\mathfrak{m};\alpha) = C_l \sum_{\pi \in \Pi_{\mathrm{cus}}(l,\mathfrak{m})} \log\{\mathrm{N}(\mathfrak{f}_\pi\mathfrak{f}_\eta^2)D_F^2\}^{-1/2} \frac{w^{\eta}_{\mathfrak{m}}(\pi)\,\iota(\mathfrak{f}_\pi)}{\mathrm{N}(\mathfrak{f}_\pi)\,\iota(\mathfrak{m})} \frac{L(1/2,\pi)\,L(1/2,\pi\otimes\eta)}{L^{S_\pi}(1,\pi,\mathrm{Ad})}\,\alpha(\nu_S(\pi)).$$

Proof. By the functional equation,

$$L'(1/2,\pi\otimes\eta) = \frac{\epsilon'(1/2,\pi\otimes\eta)}{2} L(1/2,\pi\otimes\eta)$$

if $\epsilon(1/2,\pi\otimes\eta) = 1$. An explicit form of the ϵ-factor is given by $\epsilon(s,\pi\otimes\eta) = \epsilon(1/2,\pi\otimes\eta)\{\mathrm{N}(\mathfrak{f}_{\pi\otimes\eta})D_F^2\}^{1/2-s} = \epsilon(1/2,\pi\otimes\eta)\{\mathrm{N}(\mathfrak{f}_\pi)\mathrm{N}(\mathfrak{f}_\eta)^2 D_F^2\}^{1/2-s}$. Hence we obtain the assertion immediately. □

The following is the main consequence of this section.

Theorem 21.9. *For any ideal $\mathfrak{n} \in \mathcal{J}_{S,\eta}$ and for any $\alpha \in \mathcal{A}_S$,*

$$\begin{aligned} &2^{-1}(-1)^{\#S+\epsilon(\eta)}\mathcal{G}(\eta)\,D_F^{-1}\,\{\mathrm{ADL}^{w}_{-}(\mathfrak{n};\alpha) + \mathrm{ADL}^{w}_{+}(\mathfrak{n};\alpha) + (\log D_F)\mathrm{AL}^{w}(\mathfrak{n};\alpha) + \mathrm{AL}^{\partial w}(\mathfrak{n};\alpha)\} \\ &= \tilde{\mathbb{W}}^{\eta}_{\mathrm{u}}(l,\mathfrak{n}|\alpha) + \mathbb{W}^{\eta}_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha). \end{aligned} \tag{21.6}$$

Here

$$\begin{aligned} \tilde{\mathbb{W}}^{\eta}_{\mathrm{u}}(l,\mathfrak{n}|\alpha) = {} & (1-(-1)^{\epsilon(\eta)}\tilde{\eta}(\mathfrak{n}))(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}\{1+(-1)^{\epsilon(\eta)}\tilde{\eta}(\mathfrak{n})i^{\tilde{l}}\delta(\mathfrak{n}=\mathfrak{o})\} \\ & \times \left(\frac{1}{2\pi i}\right)^{\#S} \int_{\mathbb{L}_S(\mathbf{c})} \tilde{\mathfrak{W}}^{\eta}_{S}(l,\mathfrak{n}|\mathbf{s})\alpha(\mathbf{s})d\mu_S(\mathbf{s}) \end{aligned} \tag{21.7}$$

with $d\mu_S(\mathbf{s}) = \prod_{v\in S} 2^{-1}\log q_v(q_v^{(1+s_v)/2} - q_v^{(1-s_v)/2})\,ds_v$ *and* $\mathbb{L}_S(\mathbf{c})$ *being the multidimensional contour* $\prod_{v\in S}\{s_v \in \mathbb{C} \mid \operatorname{Re}(s_v) = c_v\}$ *directed usually,*

(21.8)

$$\tilde{\mathfrak{W}}_S^{\eta}(l, \mathfrak{n}|\mathbf{s}) = \pi^{\epsilon(\eta)}\Upsilon_S^{\eta}(\mathbf{s})L(1,\eta)\Big\{\log(\sqrt{\mathrm{N}(\mathfrak{n})}D_F\mathrm{N}(\mathfrak{f}_\eta)) + \frac{L'(1,\eta)}{L(1,\eta)} + \mathfrak{C}(l) + \sum_{v\in S}\frac{\log q_v}{1-\eta_v(\varpi_v)q_v^{(s_v+1)/2}}\Big\},$$

$$\Upsilon_S^{\eta}(\mathbf{s}) = \prod_{v\in S}(1-\eta_v(\varpi_v)q_v^{-(1+s_v)/2})^{-1}(1-q_v^{(1+s_v)/2})^{-1},$$

$$\mathfrak{C}(l) = \sum_{v\in\Sigma_\infty}\left(\sum_{k=1}^{l_v/2-1}\frac{1}{k} - \frac{1}{2}\log\pi - \frac{1}{2}C_{\mathrm{Euler}} - \delta_{\epsilon_v,1}\log 2\right).$$

Proof. From Proposition 20.4 together with Lemma 13.4,

$$\begin{aligned}&\mathrm{CT}_{\lambda=0}\partial P_{\beta,\lambda}^{\eta}(\hat{\mathbf{\Psi}}_{\mathrm{reg}}^{l}(\mathfrak{n}|\alpha))\\ &= 2^{-1}(-1)^{\#S+\epsilon(\eta)}\mathcal{G}(\eta)\,D_F^{-1}\,\{\mathrm{ADL}_{-}^{w}(\mathfrak{n};\alpha) + \mathrm{ADL}_{+}^{w}(\mathfrak{n};\alpha) + (\log D_F)\mathrm{AL}^{w}(\mathfrak{n};\alpha) + \mathrm{AL}^{\partial w}(\mathfrak{n};\alpha)\}.\end{aligned}$$

On the other hand, from the formula (21.2), the same $\mathrm{CT}_{\lambda=0}\partial P_{\beta,\lambda}^{\eta}(\hat{\mathbf{\Psi}}_{\mathrm{reg}}^{l}(\mathfrak{n}|\alpha))$ is computed by Lemmas 21.4, 21.6 and 21.7. □

22. Extraction of the new part : the totally inert case

Let $\mathcal{I}_{S,\eta}$ be the monoid of ideals generated by prime ideals $\mathfrak{p}_v$ such that $v \in \Sigma_{\mathrm{fin}} - S \cup S(\mathfrak{f}_\eta)$ and $\tilde{\eta}(\mathfrak{p}_v) = -1$. Note that $\mathcal{I}_{S,\eta}$ is a submonoid of $\mathcal{J}_{S,\eta}$ defined in §21.1.

In this section, we use the $\mathcal{N}$-transform defined in §10.3.1 and separate the new part, i.e., the contribution of those π with $\mathfrak{f}_\pi = \mathfrak{n}$, from the total average $\mathrm{ADL}_{-}^{w}(\mathfrak{n};\alpha)$ under the condition $\mathfrak{n}\in\mathcal{I}_{S,\eta}$.

Given an ideal $\mathfrak{n}$, let $\mathfrak{n}_0$ denote the largest square-free integral ideal dividing $\mathfrak{n}$; thus, there exists the unique integral ideal $\mathfrak{n}_1$ such that

$$\mathfrak{n} = \mathfrak{n}_0\mathfrak{n}_1^2.$$

Let $\mathcal{I}$ be a set of integral ideals such that if $\mathfrak{n}\in\mathcal{I}$, then all integral ideals $\mathfrak{m}$ dividing $\mathfrak{n}$ are elements of $\mathcal{I}$. The following is a corollary of Lemma 10.12; we take the derivative at $t = 0$ of the formula in Lemma 10.12.

Corollary 22.1. *The $\mathcal{N}$-transform of the arithmetic function* $\log\mathrm{N}(\mathfrak{n})$ *on* $\mathcal{I}$ *is given by*

$$\begin{aligned}\mathcal{N}[\log \mathrm{N}](\mathfrak{n}) = &\prod_{v\in S(\mathfrak{n}_1)-S_2(\mathfrak{n})}(1-q_v^{-2})\prod_{v\in S_2(\mathfrak{n})}(1-(q_v^2-q_v)^{-1})\\ &\times\left(\log\mathrm{N}(\mathfrak{n}) + \sum_{v\in S(\mathfrak{n}_1)-S_2(\mathfrak{n})}\frac{2\log q_v}{q_v^2-1} + \sum_{v\in S_2(\mathfrak{n})}\frac{2\log q_v}{q_v^2-q_v-1}\right).\end{aligned}$$

22.1. The totally inert case over $\mathfrak{n}$: holomorphic case. Fixing a test function $\alpha\in\mathcal{A}_S$ for a while, we study the arithmetic functions $\mathrm{AL}^*(-;\alpha):\mathcal{I}_{S,\eta}\to\mathbb{C}$ and $\mathrm{ADL}_{-}^{*}(-;\alpha):\mathcal{I}_{S,\eta}\to\mathbb{C}$ defined by the formulas (0.2) and (0.3), respectively. We relate these functions to the $\mathcal{N}$-transforms of arithmetic functions $\mathrm{AL}^{w}(-;\alpha)$, $\mathrm{ADL}_{\pm}^{w}(-;\alpha)$ on $\mathcal{I}_{S,\eta}$, where $\mathcal{I}_{S,\eta}$ is the set of ideals defined in §10.3.2.

As is seen in §10.3.2, we remark that an ideal $\mathfrak{n}\in\mathcal{I}_{S,\eta}$ satisfies the condition

$$\eta_v(\varpi_v) = -1, \qquad v\in S(\mathfrak{n}).$$

We recall $\omega(\mathfrak{m},\mathfrak{b})$ (cf. §10.3.1). For any ideal $\mathfrak{c}$ and a place $v\in\Sigma_{\mathrm{fin}}$, set

$$\omega_v(\mathfrak{c}) = \begin{cases}1 & (v\in S(\mathfrak{c})),\\ \dfrac{q_v+1}{q_v-1} & (v\notin S(\mathfrak{c})).\end{cases}$$

For any pair of integral ideals $\mathfrak{m}$ and $\mathfrak{b}$, define

$$\omega(\mathfrak{m},\mathfrak{b})=\delta(\mathfrak{m}\subset\mathfrak{b})\prod_{v\in S(\mathfrak{b})}\omega_v(\mathfrak{m}\mathfrak{b}^{-1}).$$

Lemma 22.2. *Let $\mathfrak{n}\in\mathcal{I}_{S,\eta}$. For any $\pi\in\Pi_{\rm cus}(l,\mathfrak{n})$, we have the following.*

(i) *If $\mathfrak{n}\mathfrak{f}_\pi^{-1}=\mathfrak{b}^2$ with an integral ideal $\mathfrak{b}$, then*

$$\partial w_{\mathfrak{n}}^{\eta}(\pi)=\omega(\mathfrak{n},\mathfrak{n}\mathfrak{f}_\pi^{-1})\sum_{v\in S(\mathfrak{b})}(-\log q_v)\,{\rm ord}_v(\mathfrak{b}).$$

(ii) *If $\mathfrak{n}\mathfrak{f}_\pi^{-1}=\mathfrak{b}^2\mathfrak{p}_u$ with an integral ideal $\mathfrak{b}$ and a place $u\in S(\mathfrak{n})$, then*

$$\partial w_{\mathfrak{n}}^{\eta}(\pi)=\omega(\mathfrak{n},\mathfrak{n}\mathfrak{f}_\pi^{-1})\,(\log q_u)\begin{cases}{\rm ord}_u(\mathfrak{b})+\dfrac{q_u-1}{(1+a_uq_u^{1/2})(1+a_u^{-1}q_u^{1/2})} & (c(\pi_u)=0),\\ {\rm ord}_u(\mathfrak{b})+\dfrac{1}{1+q_u^{-1}\chi_u(\varpi_u)} & (c(\pi_u)=1),\\ {\rm ord}_u(\mathfrak{b})+1 & (c(\pi_u)\geqslant 2).\end{cases}$$

Except the above two cases (i) and (ii), we have $\partial w_{\mathfrak{n}}^{\eta}(\pi)=0$.

Lemma 22.3. *For any $\mathfrak{n}\in\mathcal{I}_{S,\eta}$,*

$$\begin{aligned}{\rm AL}^{w}(\mathfrak{n};\alpha)&=\sum_{\mathfrak{b}}\omega(\mathfrak{n},\mathfrak{b}^2)\,\frac{\iota(\mathfrak{n}\mathfrak{b}^{-2})}{\iota(\mathfrak{n})}\,{\rm AL}^{*}(\mathfrak{n}\mathfrak{b}^{-2};\alpha),\\ {\rm ADL}^{w}_{-}(\mathfrak{n};\alpha)&=\sum_{\mathfrak{b}}\omega(\mathfrak{n},\mathfrak{b}^2)\,\frac{\iota(\mathfrak{n}\mathfrak{b}^{-2})}{\iota(\mathfrak{n})}\,{\rm ADL}^{*}_{-}(\mathfrak{n}\mathfrak{b}^{-2};\alpha),\\ {\rm ADL}^{w}_{+}(\mathfrak{n};\alpha)&=\sum_{\mathfrak{b}}\omega(\mathfrak{n},\mathfrak{b}^2)\,\frac{\iota(\mathfrak{n}\mathfrak{b}^{-2})}{\iota(\mathfrak{n})}\,\log({\rm N}(\mathfrak{n}\mathfrak{b}^{-2})^{-1/2}\,{\rm N}(\mathfrak{f}_\eta)^{-1}D_F^{-1})\,{\rm AL}^{*}(\mathfrak{n}\mathfrak{b}^{-2};\alpha),\end{aligned}$$

where $\mathfrak{b}$ runs through all the integral ideals such that $\mathfrak{n}\subset\mathfrak{b}^2$.

Proof. This follows immediately from Lemma 10.14, which is valid for $\pi\in\Pi_{\rm cus}(l,\mathfrak{n})$. To have the last formula, we also need Proposition 21.8. □

Lemma 22.4. *For any $\mathfrak{n}\in\mathcal{I}_{S,\eta}$,*

$$\begin{aligned}{\rm AL}^{*}(\mathfrak{n};\alpha)&=\mathcal{N}[{\rm AL}^{w}(-;\alpha)](\mathfrak{n}),\\ {\rm ADL}^{*}_{-}(\mathfrak{n};\alpha)&=\mathcal{N}[{\rm ADL}^{w}_{-}(-;\alpha)](\mathfrak{n}),\\ -\log(\sqrt{{\rm N}(\mathfrak{n})}{\rm N}(\mathfrak{f}_\eta)D_F)\,{\rm AL}^{*}(\mathfrak{n};\alpha)&=\mathcal{N}[{\rm ADL}^{w}_{+}(-;\alpha)](\mathfrak{n}).\end{aligned}$$

Proof. In the same way as Lemma 10.15, we obtain the first formula by applying Proposition 10.10 with $B(\mathfrak{m})=\iota(\mathfrak{m})\,{\rm AL}^{w}(\mathfrak{m};\alpha)$ and $A(\mathfrak{m})=\iota(\mathfrak{m})\,{\rm AL}^{*}(\mathfrak{m};\alpha)$ both defined for $\mathfrak{m}\in\mathcal{I}_{S,\eta}$. The remaining two formulas are proved in the same way. □

The formula (21.6) in Theorem 21.9 can be applied to an arbitrary ideal $\mathfrak{m}\in\mathcal{I}_{S,\eta}$. In the right-hand side of the formula, we have two terms $\tilde{\mathbb{W}}^{\eta}_{\rm u}(l,\mathfrak{m}|\alpha)$ and $\mathbb{W}^{\eta}_{\rm hyp}(l,\mathfrak{m}|\alpha)$, which we regard as arithmetic functions in $\mathfrak{m}$ for a while and consider their $\mathcal{N}$-transforms $\mathcal{N}[\tilde{\mathbb{W}}^{\eta}_{\rm u}(l,-|\alpha)]$ and $\mathcal{N}[\mathbb{W}^{\eta}_{\rm hyp}(l,-|\alpha)]$.

Proposition 22.5. *For any $\mathfrak{n}\in\mathcal{I}_{S,\eta}$, we have the identity among linear functionals in $\alpha\in\mathcal{A}_S$:*

$$\begin{aligned}{\rm ADL}^{*}_{-}(\mathfrak{n};\alpha)=&\,2(-1)^{\#S+\epsilon(\eta)}\,\mathcal{G}(\eta)^{-1}\,D_F\{\mathcal{N}[\tilde{\mathbb{W}}^{\eta}_{\rm u}(l,-|\alpha)](\mathfrak{n})+\mathcal{N}[\mathbb{W}^{\eta}_{\rm hyp}(l,-|\alpha)](\mathfrak{n})\}\\ &+\log({\rm N}(\mathfrak{n})^{1/2}{\rm N}(\mathfrak{f}_\eta))\,{\rm AL}^{*}(\mathfrak{n};\alpha)-\mathcal{N}[{\rm AL}^{\partial w}(-;\alpha)](\mathfrak{n}).\end{aligned}\tag{22.1}$$

Proof. We take the $\mathcal{N}$-transform of both sides of the formula (21.6) regarding it as an identity among arithmetic functions on $\mathcal{I}_{S,\eta}$. Then apply Lemma 22.4. □

23. An error term estimate for averaged L-values

In this section we prove the first asymptotic formula (0.7) of Theorem 0.9. Recall the sets $\mathcal{I}_{S,\eta}^{\pm}$, to which $\mathfrak{n}$ should belong. We note that, by the sign of the functional equation, $L(1/2,\pi)L(1/2,\pi\otimes\eta)=0$ if $\pi\in\Pi_{\mathrm{cus}}^{*}(l,\mathfrak{n})$ unless $\mathfrak{n}\in\mathcal{I}_{S,\eta}^{+}$. Thus we restrict ourselves to those levels $\mathfrak{n}$ belonging to $\mathcal{I}_{S,\eta}^{+}$, for otherwise $\mathrm{AL}^{*}(\mathfrak{n};\alpha)=0$.

We have the following asymptotic result, whose proof is given in the next subsection.

Proposition 23.1. *Suppose $\underline{l}=\inf_{v\in\Sigma_\infty}l_v\geqslant 6$. For any ideal $\mathfrak{m}\in\mathcal{I}_{S(\mathfrak{a}),\eta}^{+}$, we have*

$$\mathrm{AL}^{w}(\mathfrak{m};\alpha_{\mathfrak{a}})=4D_F^{3/2}\,L_{\mathrm{fin}}(1,\eta)\,\mathrm{N}(\mathfrak{a})^{-1/2}\delta_{\square}(\mathfrak{a}_\eta^{-})d_1(\mathfrak{a}_\eta^{+})+\mathcal{O}_{\epsilon,l,\eta}(\mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\mathrm{N}(\mathfrak{m})^{-c+\epsilon})$$

for any ideal $\mathfrak{a}$ prime to $\mathfrak{f}_\eta$, where $c=d_F^{-1}(\underline{l}/2-1)$.

From this, we can deduce the asymptotic formula for the primitive part $\mathrm{AL}^{*}(\mathfrak{n};\alpha_{\mathfrak{a}})$ stated in Theorem 0.9. Indeed, we apply the first formula of Lemma 22.4 substituting the expression of AL^{w} given in Proposition 23.1. The main and the error terms are computed by Lemmas 10.12 and 10.13, respectively. This completes the proof of the asymptotic formula (0.7). □

23.1. Proof of Proposition 23.1.

For any place $v\in\Sigma_{\mathrm{fin}}$, we define a function $\Lambda_v:F_v-\{0,-1\}\to\mathbb{Z}$ by setting

$$\Lambda_v(b)=\delta(b\in\mathfrak{o}_v)\{\mathrm{ord}_v(b(b+1))+1\}.$$

For an integral ideal $\mathfrak{b}$, we set

$$\tau^{S(\mathfrak{b})}(b)=\{\prod_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{b})}\Lambda_v(b)\}\prod_{v\in S(\mathfrak{b})}\delta(b\in\mathfrak{b}^{-1}\mathfrak{o}_v),\quad b\in F-\{0,-1\}.$$

For an even integer $k\,(\geqslant 4)$ and a real valued character ε of $\mathbb{R}^\times$, let $J^{\varepsilon}(k;b)$ $(b\in\mathbb{R}-\{0,-1\})$ be the integral studied in §17.3; they are evaluated explicitly in Lemma 17.15 as

$$J^{\mathbf{1}}(k;b)=\begin{cases}(1+b)^{-k/2}\,\frac{2\Gamma(k/2)^2}{\Gamma(k)}\,{}_2F_1(k/2,k/2;k;(b+1)^{-1}) & (b(b+1)>0),\\ 2\log|(b+1)/b|\,P_{k/2-1}(2b+1)-\sum_{m=1}^{[k/4]}\frac{8(k-4m+1)}{(2m-1)(k-2m)}\,P_{k/2-2m}(2b+1) & (b(b+1)<0),\end{cases}$$

$$J^{\mathrm{sgn}}(k;b)=\begin{cases}0 & (b(b+1)>0),\\ 2\pi i\,P_{k/2-1}(2b+1) & (b(b+1)<0),\end{cases}$$

where $P_n(x)$ is the Legendre polynomial of degree n.

Lemma 23.2. *Let k be an even integer greater than 2 and η_v a real valued character of $\mathbb{R}^\times$. Then, for any $\epsilon>0$, we have the estimate*

$$|b(b+1)|^{\epsilon}\,|J^{\eta_v}(k;b)|\ll_{\epsilon,k}(1+|b|)^{-k/2+2\epsilon},\quad b\in\mathbb{R}-\{0,-1\}$$

with the implied constant depending on k and ϵ.

Proof. For $J^{\mathrm{sgn}}(k;b)$, the estimate is obvious. As for $J^{\mathbf{1}}(k;b)$, we only have to note that the estimate ${}_2F_1(k/2,k/2;k;(b+1)^{-1})=\mathcal{O}(|\log b|)$ for small $b>0$ ([24, p.49]) and the functional equation $J^{\mathbf{1}}(k;b)=(-1)^{k/2}J^{\mathbf{1}}(k;-b-1)$ for $b<-1$. The functional equation is proved by the transformation formula of the hypergeometric function ${}_2F_1(a,b;c;z)=(1-z)^{-a}{}_2F_1(a,c-b;c;\frac{z}{z-1})$ ([24, p.47]). Indeed, the formula gives the identity $J^{\mathbf{1}}(k;b)=2\Gamma(k/2)^2\Gamma(k)^{-1}b^{-k/2}{}_2F_1(k/2,k/2;k;-b^{-1})=(-1)^{k/2}J^{\mathbf{1}}(k;-b-1)$. □

Given relatively prime integral ideals $\mathfrak{n}$ and $\mathfrak{b}$ and for $\epsilon \geqslant 0$, we set

$$\mathfrak{I}_{\epsilon}^{\eta}(l,\mathfrak{n},\mathfrak{b}) = \sum_{b\in\mathfrak{n}\mathfrak{b}^{-1}-\{0,-1\}} \tau^{S(\mathfrak{b})}(b)^2\, |\mathrm{N}(b(b+1))|^{\epsilon} \prod_{v\in\Sigma_\infty} |J^{\eta_v}(l_v;b)|.$$

Proposition 23.3. *Suppose $\underline{l} \geqslant 6$. Let $\mathfrak{b}$ and $\mathfrak{n}$ be relatively prime ideals. For any sufficiently small $\epsilon \geqslant 0$ and any $\epsilon' > 0$, we have*

$$\mathfrak{I}_{\epsilon}^{\eta}(l,\mathfrak{n},\mathfrak{b}) \ll_{\epsilon,\epsilon',l} \mathrm{N}(\mathfrak{b})^{1+c+\epsilon'}\,\mathrm{N}(\mathfrak{n})^{-c+2\epsilon+\epsilon'}$$

with the implied constant independent of $\mathfrak{b}$ and $\mathfrak{n}$.

Proof. By Lemma 19.4 and Lemma 23.2, we have

$$\mathfrak{I}_{\epsilon}^{\eta}(l,\mathfrak{n},\mathfrak{b}) \ll_{\epsilon,\epsilon',l} \mathrm{N}(\mathfrak{b})^{4\epsilon'} \sum_{b\in\mathfrak{n}\mathfrak{b}^{-1}-\{0\}} \prod_{v\in\Sigma_\infty} (1+|b_v|)^{-l_v/2+2\epsilon+4\epsilon'} = \mathrm{N}(\mathfrak{b})^{4\epsilon'}\Theta(\mathfrak{n}\mathfrak{b}^{-1})$$

for any $\epsilon \geqslant 0$ and any $\epsilon' > 0$, where we regard the fractional ideal $\mathfrak{n}\mathfrak{b}^{-1}$ as a $\mathbb{Z}$-lattice in the Euclidean space $F_\infty = F\otimes_{\mathbb{Q}}\mathbb{R}$ and $\Theta(\Lambda)$ is constructed for $\{l_v-4\epsilon-8\epsilon'\}_{v\in\Sigma_\infty}$ in place of l (see §27). If $\epsilon \geqslant 0$ and $\epsilon' > 0$ are small enough, then we can apply the theory in §27 to this $\Theta(\Lambda)$. The desired estimate follows if we apply Theorem 27.1 with $\Lambda = \mathfrak{n}\mathfrak{b}^{-1}$ and $\Lambda_0 = \mathfrak{b}^{-1}$ noting $D(\mathfrak{n}\mathfrak{b}^{-1}) = \mathrm{N}(\mathfrak{n})\mathrm{N}(\mathfrak{b})^{-1}$, $D(\mathfrak{b}^{-1}) = \mathrm{N}(\mathfrak{b})^{-1}$ and $r(\mathfrak{b}^{-1}) \leqslant r(\mathfrak{o})$. □

Proposition 23.4. *Suppose $\underline{l} \geqslant 6$. Given integral ideals $\mathfrak{n}$ and $\mathfrak{a} = \prod_{v\in S(\mathfrak{a})} \mathfrak{p}_v^{n_v}$ relatively prime to each other, for any $\epsilon > 0$, we have*

$$|\mathbb{J}_{\mathrm{hyp}}^{\eta}(l,\mathfrak{n}|\alpha_{\mathfrak{a}})| \ll_{\epsilon,l,\eta} \mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\mathrm{N}(\mathfrak{n})^{-c+\epsilon}$$

with the implied constant independent of $\mathfrak{a}$ and $\mathfrak{n}$.

Proof. Let $v \in S(\mathfrak{a})$ and $n \in \mathbb{N}_0$. By (0.5),

$$\alpha_{\mathfrak{p}_v^n}(\nu) = \frac{z^{n+1}-z^{-(n+1)}}{z-z^{-1}} = \sum_{m=0}^{[n/2]} \alpha_v^{(n-2m)}(\nu) - \delta(n\in 2\mathbb{N}_0) \tag{23.1}$$

with $\alpha_v^{(m)}(\nu) = z^m + z^{-m}$, $z = q_v^{\nu/2}$. By Lemma 17.3, we have

$$|J_v^{\eta_v}(b,\alpha_v^{(m)})| \ll (1+m)^2\delta(|b|_v \leqslant q_v^m) q_v^{\delta(m>0)-m/2}\{1+\Lambda_v(b)\}, \quad b\in F^\times - \{-1\}$$

with the implied constant independent of $m \in \mathbb{N}_0$ and v. Hence if $n > 0$,

$$\begin{aligned}|J_v^{\eta_v}(b,\alpha_{\mathfrak{p}_v^n})| &\ll \delta(|b|_v \leqslant q_v^n)\{\sum_{m=0}^{n}(1+m)^2 q_v^{1-m/2}\}\{1+\Lambda_v(b)\}\\ &\leqslant \delta(|b|_v \leqslant q_v^n) q_v \left(\sum_{m=0}^{\infty}(1+m)^2 2^{-m/2}\right)\{1+\Lambda_v(b)\}.\end{aligned}$$

Thus we have a constant C independent of $v\in S(\mathfrak{a})$ and $n\in\mathbb{N}_0$ such that

$$|J_v^{\eta_v}(b,\alpha_{\mathfrak{p}_v^n})| \leqslant C\, q_v^{\delta(n>0)}\,\delta(|b|_v \leqslant q_v^n)\,\{1+\Lambda_v(b)\}, \quad b\in F^\times - \{0,-1\}. \tag{23.2}$$

Combining (23.2) with Proposition 23.3 and Lemmas 17.4, 17.5 and Corollary 17.11, we obtain

$$\begin{aligned}
|\mathbb{J}^{\eta}_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha_{\mathfrak{a}})| &\leqslant C^{\#S(\mathfrak{a})}\{\prod_{v\in S(\mathfrak{a})} q_v^{\delta(n_v>0)}\}\sum_{I\subset S(\mathfrak{a})}\sum_{b\in\mathfrak{n}(\prod_{v\in I}\mathfrak{p}_v^{n_v})^{-1}\mathfrak{f}_\eta^{-1}} \tau^{S(\prod_{v\in I}\mathfrak{p}_v^{n_v})}(b)\prod_{v\in\Sigma_\infty}|J_v^{\eta_v}(l_v;b)| \\
&\leqslant C^{\#S(\mathfrak{a})}\,\mathrm{N}(\mathfrak{a})\sum_{I\subset S(\mathfrak{a})}\mathfrak{I}_0(l,\mathfrak{n},\mathfrak{f}_\eta\prod_{v\in I}\mathfrak{p}_v^{n_v}) \\
&\ll_{\epsilon,l} C^{\#S(\mathfrak{a})}\,\mathrm{N}(\mathfrak{a})\sum_{I\subset S(\mathfrak{a})}\mathrm{N}(\mathfrak{f}_\eta\prod_{v\in I}\mathfrak{p}_v^{n_v})^{1+c+\epsilon}\mathrm{N}(\mathfrak{n})^{-c+\epsilon} \\
&\ll_{\epsilon,l,\eta} C^{\#S(\mathfrak{a})}\mathrm{N}(\mathfrak{a})\times 2^{\#S(\mathfrak{a})}\mathrm{N}(\mathfrak{a})^{1+c+\epsilon}\mathrm{N}(\mathfrak{n})^{-c+\epsilon}.
\end{aligned}$$

By the estimate $(2\,C)^{\#S(\mathfrak{a})}\ll_{\epsilon,\eta}\mathrm{N}(\mathfrak{a})^{\epsilon}$, we are done. □

Lemma 23.5. *Set* $\Upsilon_v^{\eta_v}(s)=(1-\eta_v(\varpi_v)q_v^{-(1+s)/2})^{-1}(1-q_v^{(1+s)/2})^{-1}$. *For* $n\in\mathbb{N}_0$,

$$\frac{1}{2\pi i}\int_{\sigma-2\pi i(\log q_v)^{-1}}^{\sigma+2\pi i(\log q_v)^{-1}}\Upsilon_v^{\eta_v}(s)\,\alpha_{\mathfrak{p}_v^n}(s)\,d\mu_v(s)=-q_v^{-n/2}\begin{cases}\delta(n\in 2\mathbb{N}_0) & (\eta_v(\varpi_v)=-1),\\ n+1 & (\eta_v(\varpi_v)=+1),\end{cases}$$

$$\frac{\log q_v}{2\pi i}\int_{\sigma-2\pi i(\log q_v)^{-1}}^{\sigma+2\pi i(\log q_v)^{-1}}\frac{\Upsilon_v^{\eta_v}(s)\,\alpha_{\mathfrak{p}_v^n}(s)}{1-\eta_v(\varpi_v)q_v^{(s+1)/2}}\,d\mu_v(s)=q_v^{-n/2}(\log q_v)\begin{cases}(-1)^n\left[\frac{n+1}{2}\right] & (\eta_v(\varpi_v)=-1),\\ \frac{n(n+1)}{2} & (\eta_v(\varpi_v)=+1).\end{cases}$$

Proof. The second integral is $\tilde{U}_v^{\eta_v}(\alpha_{\mathfrak{p}_v^n})$ defined by (26.11). Then we have the second formula using (23.1) and Lemma 26.13 by a direct computation. The first formula is confirmed in the same way by using Proposition 18.1. □

To show Proposition 23.1, we apply Theorem 16.1 taking $S=S(\mathfrak{a})$. From the first formula of Lemma 23.5,

$$\begin{aligned}
\tilde{\mathbb{J}}^{\eta}_{\mathrm{u}}(l,\mathfrak{n}|\alpha_{\mathfrak{a}}) =&2(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}(1+\delta(\mathfrak{n}=\mathfrak{o}))L_{\mathrm{fin}}(1,\eta) \\
&\times(-1)^{\#S(\mathfrak{a})}\prod_{v\in S(\mathfrak{a}_\eta^-)}q_v^{-n_v/2}\delta(n_v\in 2\mathbb{N}_0)\prod_{v\in S(\mathfrak{a}_\eta^+)}q_v^{-n_v/2}(n_v+1) \\
=&2(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}(1+\delta(\mathfrak{n}=\mathfrak{o}))L_{\mathrm{fin}}(1,\eta)\times(-1)^{\#S(\mathfrak{a})}\mathrm{N}(\mathfrak{a})^{-1/2}\delta_{\square}(\mathfrak{a}_\eta^-)d_1(\mathfrak{a}_\eta^+).
\end{aligned}$$

We use Proposition 23.4 to estimate $\mathbb{J}^{\eta}_{\mathrm{hyp}}(l,\mathfrak{n}|\alpha_{\mathfrak{a}})$, which yields the error term. This completes the proof of Proposition 23.1. □

24. An error term estimate for averaged derivative of L-values

Let $\mathcal{I}^+_{S,\eta}$ and $\mathcal{I}^-_{S,\eta}$ be the same as in §0.1 and let $\mathfrak{a}=\prod_{v\in S(\mathfrak{a})}\mathfrak{p}_v^{n_v}$ be an integral ideal. In this section we prove the asymptotic formula of $\mathrm{ADL}^*_-(\mathfrak{n};\alpha_{\mathfrak{a}})$ for $\mathfrak{n}\in\mathcal{I}^-_{S(\mathfrak{a}),\eta}$ stated in Theorem 0.9. We remark that $\mathrm{ADL}^*_-(\mathfrak{n};\alpha_{\mathfrak{a}})=0$ if $\mathfrak{n}\in\mathcal{I}^+_{S(\mathfrak{a}),\eta}$. Indeed, for such $\mathfrak{n}$, $\epsilon(1/2,\pi)\epsilon(1/2,\pi\otimes\eta)=+1$ for all $\pi\in\Pi^*_{\mathrm{cus}}(l,\mathfrak{n})$, which means $\epsilon(1/2,\pi)=-1$ and hence $L(1/2,\pi)=0$ for all π occurring in the sum $\mathrm{ADL}^*_-(\mathfrak{n};\alpha_{\mathfrak{a}})$.

Starting from the formula (22.1) with α specialized to $\alpha_{\mathfrak{a}}$, we examine the four terms in the right-hand side separately as follows.

(i) We compute the term $\mathcal{N}[\tilde{\mathbb{W}}^{\eta}_{\mathrm{u}}(l,-|\alpha_{\mathfrak{a}})](\mathfrak{n})$ explicitly by using Lemma 26.13, Lemma 10.12 and Corollary 22.1, which yields the main term of the formula (modulo a part of the error term); see §24.1 for detail.

(ii) We prove

$$\mathcal{N}[\mathbb{W}^{\eta}_{\mathrm{hyp}}(l,-|\alpha_{\mathfrak{a}})](\mathfrak{n})=\mathcal{O}_{\epsilon,l,\eta}(\mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\mathrm{N}(\mathfrak{n})^{-\inf(1,c)+\epsilon})$$

by using the explicit formula of local terms given in §26; see §24.2 for detail.

(iii) Since $\mathfrak{n} \in \mathcal{I}^{-}_{S(\mathfrak{a}),\eta}$, the term $\mathrm{AL}^{*}(\mathfrak{n};\alpha_{\mathfrak{a}})$ vanishes by the reason of the sign of the functional equations.

(iv) We prove

$$\mathcal{N}[\mathrm{AL}^{\partial w}(-;\alpha_{\mathfrak{a}})](\mathfrak{n}) = \mathcal{O}_{\epsilon,l,\eta}\left(\mathrm{N}(\mathfrak{a})^{-1/2+\epsilon}\,X(\mathfrak{n}) + \mathrm{N}(\mathfrak{a})^{c+2}\mathrm{N}(\mathfrak{n})^{-\inf(1,c)+\epsilon}\right).$$

This part is most subtle and the term $X(\mathfrak{n})$ arises from this stage; see §24.3 for detail.

Combining these considerations, we obtain the second formula in Theorem 0.9 immediately.

24.1. Computation of $\mathcal{N}[\tilde{\mathbb{W}}^{\eta}_{\mathrm{u}}(l,-|\alpha_{\mathfrak{a}})](\mathfrak{n})$. Let us describe the procedure (i). We take α to be the function $\alpha_{\mathfrak{a}}$. Set $S = S(\mathfrak{a})$. From (21.7), we have that $\mathcal{N}[\tilde{\mathbb{W}}^{\eta}_{\mathrm{u}}(l,-|\alpha_{\mathfrak{a}})](\mathfrak{n})$ is the sum of the following two integrals:

$$2(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\mathcal{N}[\tilde{\mathfrak{W}}^{\eta}_{S}(l,-|\mathbf{s})](\mathfrak{n})\alpha_{\mathfrak{a}}(\mathbf{s})\,d\mu_S(\mathbf{s}), \tag{24.1}$$

$$2(-1)^{\epsilon(\eta)}\mathcal{G}(\eta)D_F^{1/2}\left(\frac{1}{2\pi i}\right)^{\#S}\int_{\mathbb{L}_S(\mathbf{c})}\mathcal{N}[D\,\tilde{\mathfrak{W}}^{\eta}_{S}(l,-|\mathbf{s})](\mathfrak{n})\,\alpha_{\mathfrak{a}}(\mathbf{s})d\mu_S(\mathbf{s}), \tag{24.2}$$

where $\tilde{\mathfrak{W}}^{\eta}_{S}(l,-|\mathbf{s})$ is the quantity (21.8) viewed as an arithmetic function in $\mathfrak{n}$ and D is an arithmetic function given by $D(\mathfrak{n}) = (-1)^{\epsilon(\eta)}\tilde{\eta}(\mathfrak{n})\delta(\mathfrak{n}=\mathfrak{o})i^{\tilde{l}}$. By the formula (21.8),

$$\begin{aligned}\mathcal{N}[\tilde{\mathfrak{W}}^{\eta}_{S}(l,-|\mathbf{s})](\mathfrak{n}) = \pi^{\epsilon(\eta)}\Upsilon^{\eta}_{S}(\mathbf{s})\,L(1,\eta)\Biggl\{&2^{-1}\mathcal{N}[\log\mathrm{N}](\mathfrak{n})\\ &+\left(\log(D_F\mathrm{N}(\mathfrak{f}_\eta)) + \frac{L'(1,\eta)}{L(1,\eta)} + \mathfrak{C}(l) + \sum_{v\in S}\frac{\log q_v}{1-\eta_v(\varpi_v)q_v^{(s_v+1)/2}}\right)\mathcal{N}[1](\mathfrak{n})\Biggr\}.\end{aligned}$$

By Lemma 10.12 and Corollary 22.1, we have formulas of $\mathcal{N}[\log\mathrm{N}](\mathfrak{n})$ and of $\mathcal{N}[1](\mathfrak{n})$; substituting these, and by using Lemma 23.5, we complete the evaluation of the integral (24.1).

The evaluation of the integral (24.2) is similar; instead of $\mathcal{N}[\log\mathrm{N}]$ and $\mathcal{N}[1]$, we need $\mathcal{N}[D\log\mathrm{N}]$ and $\mathcal{N}[D]$, which are much easier. Indeed, in the expression

$$\begin{aligned}\mathcal{N}[D\log\mathrm{N}](\mathfrak{n}) = (-1)^{\epsilon(\eta)}i^{\tilde{l}}\sum_{I\subset S(\mathfrak{n}_1)}(-1)^{\#I}\{\prod_{v\in I\cap S_1(\mathfrak{n}_1)}\omega_v(\mathfrak{n}_0)\}\,\frac{\iota(\mathfrak{n}\prod_{v\in I}\mathfrak{p}_v^{-2})}{\iota(\mathfrak{n})}\\ \times\tilde{\eta}(\mathfrak{n}\prod_{v\in I}\mathfrak{p}_v^{-2})\,\delta(\mathfrak{n}\prod_{v\in I}\mathfrak{p}_v^{-2}=\mathfrak{o})\log\mathrm{N}(\mathfrak{n}\prod_{v\in I}\mathfrak{p}_v^{-2})\end{aligned}$$

for $\mathfrak{n} = \mathfrak{n}_0\mathfrak{n}_1^2$ with square-free ideal $\mathfrak{n}_0$, the sum survives only if $\mathfrak{n} = \prod_{v\in S(\mathfrak{n})}\mathfrak{p}_v^2$ and $I = S(\mathfrak{n})$. A similar remark is applied to $\mathcal{N}[D](\mathfrak{n})$. Hence,

$$\mathcal{N}[D\log\mathrm{N}](\mathfrak{n}) = \delta(S(\mathfrak{n}) = S_2(\mathfrak{n}))\{\prod_{v\in S(\mathfrak{n})}\frac{q_v+1}{q_v-1}\}\,(-1)^{\epsilon(\eta)}i^{\tilde{l}}\,\frac{(-1)^{\#S(\mathfrak{n})}}{\iota(\mathfrak{n})}\log\mathrm{N}(\mathfrak{o}) = 0,$$

$$\mathcal{N}[D](\mathfrak{n}) = \delta(S(\mathfrak{n}) = S_2(\mathfrak{n}))\{\prod_{v\in S(\mathfrak{n})}\frac{q_v+1}{q_v-1}\}\,(-1)^{\epsilon(\eta)}i^{\tilde{l}}\,\frac{(-1)^{\#S(\mathfrak{n})}}{\iota(\mathfrak{n})}.$$

Since $\iota(\mathfrak{n})^{-1} = \mathcal{O}(\mathrm{N}(\mathfrak{n})^{-1})$, the integral (24.2) amounts at most to $\mathrm{N}(\mathfrak{n})^{-1+\epsilon}\mathrm{N}(\mathfrak{a})^{-1/2+\epsilon}$.

24.2. Estimation of the term $\mathcal{N}[\mathbb{W}^{\eta}_{\rm hyp}(l,-|\alpha_{\mathfrak{a}})](\mathfrak{n})$. Let us describe the procedure (ii). We need the following estimate, which we prove in §26.4.

Proposition 24.1. *For any small $\epsilon>0$,*

$$|\mathbb{W}^{\eta}_{\rm hyp}(l,\mathfrak{n}|\alpha_{\mathfrak{a}})|\ll_{\epsilon,l,\eta}\mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\mathrm{N}(\mathfrak{n})^{-c+\epsilon},\quad \mathfrak{n}\in\mathcal{I}^{-}_{S(\mathfrak{a}),\eta},$$

where the implied constant is independent of the ideal $\mathfrak{a}$.

From this proposition and Lemma 10.13,

$$|\mathcal{N}[\mathbb{W}^{\eta}_{\rm hyp}(l,-|\alpha_{\mathfrak{a}})](\mathfrak{n})|\leqslant\mathcal{N}^{+}[|\mathbb{W}^{\eta}_{\rm hyp}(l,-|\alpha_{\mathfrak{a}})|](\mathfrak{n})\ll_{\epsilon,l,\eta}\mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\mathcal{N}^{+}[\mathrm{N}^{-c+\epsilon}](\mathfrak{n})\ll_{\epsilon}\mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+3\epsilon}.$$

24.3. Estimation of the term $\mathcal{N}[\mathrm{AL}^{\partial w}(-;\alpha_{\mathfrak{a}})](\mathfrak{n})$. Let us describe the procedure (iv).

Lemma 24.2. *Let $\alpha\in\mathcal{A}_S$. Then for any $\mathfrak{n}\in\mathcal{I}^{-}_{S(\mathfrak{a}),\eta}$, we have the inequality*

$$|\mathrm{AL}^{\partial w}(\mathfrak{n};\alpha)|\leqslant\sum_{(\mathfrak{b},u)}D(\mathfrak{n};\mathfrak{b},u)\,\frac{\iota(\mathfrak{n}\mathfrak{b}^{-2}\mathfrak{p}_u^{-1})}{\iota(\mathfrak{n})}\,|\mathrm{AL}^{*}(\mathfrak{n}\mathfrak{b}^{-2}\mathfrak{p}_u^{-1};\alpha)|,$$

where $(\mathfrak{b},u)$ runs through all the pairs of an integral ideal $\mathfrak{b}$ and a place u such that $\mathfrak{n}\subset\mathfrak{b}^2\mathfrak{p}_u$. For such $(\mathfrak{b},u)$, we set

$$D(\mathfrak{n};\mathfrak{b},u)=\omega(\mathfrak{n},\mathfrak{b}^2\mathfrak{p}_u)\,(\log q_u)\,\left(\mathrm{ord}_u(\mathfrak{b})+\frac{q_u^{1/2}+1}{q_u^{1/2}-1}\right).$$

Proof. By Lemma 22.2, the π-summand of $\mathrm{AL}^{\partial w}(\mathfrak{n};\alpha)$ vanishes unless the conductor $\mathfrak{f}_\pi$ satisfies either (i) $\mathfrak{n}\mathfrak{f}_\pi^{-1}=\mathfrak{b}^2$ with some $\mathfrak{n}\subset\mathfrak{b}$, or (ii) $\mathfrak{n}\mathfrak{f}_\pi^{-1}=\mathfrak{b}^2\mathfrak{p}_u$ with some $\mathfrak{n}\subset\mathfrak{b}$ and $u\in S(\mathfrak{n})$. In the case (i), the π-summand vanishes. Indeed, $\mathfrak{f}_\pi$ belongs to $\mathcal{I}^{-}_{S(\mathfrak{a}),\eta}$ and thus $L(1/2,\pi)L(1/2,\pi\otimes\eta)=0$ by the functional equation. In the second case (ii), by the Ramanujan bound $|a_v|=1$ by [1] and the obvious relation $|\chi_v(\varpi_v)|=1$, we have

$$\begin{aligned}|\partial w^{\eta}_{\mathfrak{n}}(\pi)|&\leqslant\omega(\mathfrak{n},\mathfrak{b}^2\mathfrak{p}_u)\,(\log q_u)\begin{cases}\mathrm{ord}_u(\mathfrak{b})+\frac{q_u-1}{(1-q_u^{1/2})^2} & (c(\pi_u)=0),\\ \mathrm{ord}_u(\mathfrak{b})+\frac{1}{1-q_u^{-1}} & (c(\pi_u)=1),\\ \mathrm{ord}_u(\mathfrak{b})+1 & (c(\pi_u)\geqslant 2)\end{cases}\\ &\leqslant\omega(\mathfrak{n},\mathfrak{b}^2\mathfrak{p}_u)\,(\log q_u)\,\left(\frac{q_u^{1/2}+1}{q_u^{1/2}-1}+\mathrm{ord}_v(\mathfrak{b})\right)=D(\mathfrak{n};\mathfrak{b},u).\end{aligned}$$

Here, we use $\frac{1}{1-q_u^{-1}}<\frac{q_u-1}{(1-q_u^{1/2})^2}=\frac{q_u^{1/2}+1}{q_u^{1/2}-1}$ to have the second inequality. □

Lemma 24.3. *For any small $\epsilon\in(0,1)$, we have*

$$\sum_{(\mathfrak{b},u)}\mathrm{N}(\mathfrak{b}^2\mathfrak{p}_u)^{\epsilon}\frac{\iota(\mathfrak{n}\mathfrak{b}^{-2}\mathfrak{p}_u^{-1})}{\iota(\mathfrak{n})}\,\mathrm{N}(\mathfrak{n}\mathfrak{b}^{-2}\mathfrak{p}_u^{-1})^{-\inf(c,1)+\epsilon}\ll_{\epsilon}\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+2\epsilon},\tag{24.3}$$

$$\sum_{(\mathfrak{b},u)}\mathrm{N}(\mathfrak{b})^{\epsilon}\left(\frac{q_u+1}{q_u-1}\right)^2(\log q_u)\frac{\iota(\mathfrak{n}\mathfrak{b}^{-2}\mathfrak{p}_u^{-1})}{\iota(\mathfrak{n})}\ll_{\epsilon}X(\mathfrak{n}),\tag{24.4}$$

where $(\mathfrak{b},u)$ runs through the same range as in Lemma 24.2.

Proof. A direct computation gives us the first estimate. Let us show the second estimate. By the inequality $\iota(\mathfrak{n}\mathfrak{b}^{-2}\mathfrak{p}_u^{-1})/\iota(\mathfrak{n}) \leqslant \mathrm{N}(\mathfrak{b}^{-2}\mathfrak{p}_u^{-1})$,

$$\begin{aligned}\sum_{(\mathfrak{b},u)} \mathrm{N}(\mathfrak{b})^{\epsilon}\left(\frac{q_u+1}{q_u-1}\right)^2 (\log q_u)\frac{\iota(\mathfrak{n}\mathfrak{b}^{-2}\mathfrak{p}_u^{-1})}{\iota(\mathfrak{n})} &\leqslant \sum_{(\mathfrak{b},u)} \mathrm{N}(\mathfrak{b})^{-2+\epsilon}\left(\frac{q_u+1}{q_u-1}\right)^2 \frac{\log q_u}{q_u} \\ &\leqslant \{\sum_{\mathfrak{b}\subset\mathfrak{o}} \mathrm{N}(\mathfrak{b})^{-2+\epsilon}\}\{\sum_{u\in S(\mathfrak{n})}\left(\frac{q_u+1}{q_u-1}\right)^2 \frac{\log q_u}{q_u}\} \\ &= \zeta_{F,\mathrm{fin}}(2-\epsilon)\{\sum_{u\in S(\mathfrak{n})}\frac{\log q_u}{q_u}+\sum_{u\in S(\mathfrak{n})}\frac{4\log q_u}{(q_u-1)^2}\}.\end{aligned}$$

Since $\zeta_{F,\mathrm{fin}}(2-\epsilon)$ is convergent, we are done. □

Proposition 24.4. *For any sufficiently small $\epsilon>0$,*

$$|\mathrm{AL}^{\partial w}(\mathfrak{n};\alpha_{\mathfrak{a}})| \ll_{\epsilon,l,\eta} \mathrm{N}(\mathfrak{a})^{-1/2}d_1(\mathfrak{a}_\eta^+)\delta_\square(\mathfrak{a}_\eta^-)\,X(\mathfrak{n})+\mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}, \quad \mathfrak{n}\in\mathcal{I}_{S(\mathfrak{a}),\eta}^-.$$

Proof. Let $\epsilon>0$. From $\frac{x+1}{x-1}\ll_\epsilon x^\epsilon$ for $x\geqslant 2$, we have

$$\omega(\mathfrak{n},\mathfrak{b}^2\mathfrak{p}_u)\leqslant\left(\prod_{v\in S(\mathfrak{b})}\frac{q_v+1}{q_v-1}\right)\frac{q_u+1}{q_u-1}\ll_\epsilon \mathrm{N}(\mathfrak{b})^\epsilon\,\frac{q_u+1}{q_u-1}$$

with the implied constant independent of $\mathfrak{n}$ and $(\mathfrak{b},u)$. By this,

$$D(\mathfrak{n};\mathfrak{b},u)\ll_\epsilon \mathrm{N}(\mathfrak{b})^\epsilon(\log q_u)\left(\frac{q_u+1}{q_u-1}\right)^2\ll_\epsilon \mathrm{N}(\mathfrak{b}^2\mathfrak{p}_u)^\epsilon$$

with the implied constant independent of $\mathfrak{n}$ and $(\mathfrak{b},u)$. Using these estimates, we have the desired bound by (0.7) and Lemmas 24.2 and 24.3. □

Proposition 24.5. *For any sufficiently small $\epsilon>0$,*

(24.5)
$$\left|\mathcal{N}[\mathrm{AL}^{\partial w}(-;\alpha_{\mathfrak{a}})](\mathfrak{n})\right| \ll_{\epsilon,l,\eta} \mathrm{N}(\mathfrak{a})^{-1/2}d_1(\mathfrak{a}_\eta^+)\delta_\square(\mathfrak{a}_\eta^-)\,X(\mathfrak{n})+\mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-\inf(1,c)+\epsilon}, \quad \mathfrak{n}\in\mathcal{I}_{S(\mathfrak{a}),\eta}^-.$$

Proof. From Proposition 24.4, we have

$$|\mathcal{N}[\mathrm{AL}^{\partial w}(-;\alpha_{\mathfrak{a}})](\mathfrak{n})| \ll_\epsilon \mathrm{N}(\mathfrak{a})^{-1/2}d_1(\mathfrak{a}_\eta^+)\delta_\square(\mathfrak{a}_\eta^-)\mathcal{N}^+[X](\mathfrak{n})+\mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\,\mathcal{N}^+[\mathrm{N}^{-\inf(1,c)+\epsilon}](\mathfrak{n})$$

for all $\mathfrak{n}\in\mathcal{I}_{S(\mathfrak{a}),\eta}^-$. Since $X(\mathfrak{m})\leqslant X(\mathfrak{n})$ if $\mathfrak{n}\subset\mathfrak{m}\subset\mathfrak{o}$, we have

$$\begin{aligned}\mathcal{N}^+[X](\mathfrak{n}) &\leqslant X(\mathfrak{n})\,\mathcal{N}^+[1](\mathfrak{n}) \\ &= X(\mathfrak{n})\,\{\prod_{v\in S(\mathfrak{n}_1)-S_2(\mathfrak{n})}(1+q_v^{-2})\}\{\prod_{v\in S_2(\mathfrak{n})}(1+(1-q_v^{-1})^{-1}q_v^{-2})\} \\ &\leqslant X(\mathfrak{n})\{\prod_{v\in\Sigma_{\mathrm{fin}}}(1+q_v^{-2})\}\{\prod_{v\in\Sigma_{\mathrm{fin}}}(1+(1-q_v^{-1})^{-1}q_v^{-2})\}\ll X(\mathfrak{n})\end{aligned}$$

since the Euler products occurring are convergent.

From the proof of Lemma 10.13, we have $\mathcal{N}^+[\mathrm{N}^{-\inf(c,1)+\epsilon}](\mathfrak{n})\ll_\epsilon \mathrm{N}(\mathfrak{n})^{-\inf(c,1)+3\epsilon}$. Consequently, for any sufficiently small $\epsilon\in(0,1)$, we obtain the estimate

$$|\mathcal{N}[\mathrm{AL}^{\partial w}(-;\alpha_{\mathfrak{a}})](\mathfrak{n})| \ll_\epsilon \mathrm{N}(\mathfrak{a})^{-1/2}d_1(\mathfrak{a}_\eta^+)\delta_\square(\mathfrak{a}_\eta^-)X(\mathfrak{n})+\mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\mathrm{N}(\mathfrak{n})^{-\inf(1,c)+3\epsilon}$$

with the implied constant independent of $\mathfrak{n}$ and $\mathfrak{a}$. Since ϵ is arbitrary, we are done. □

25. An estimation of number of cusp forms

Recall that we set $c = d_F^{-1}(\underline{l}/2-1)$. For each ideal $\mathfrak{a} \subset \mathfrak{o}$, we fix a set $\mathcal{J}_\mathfrak{a}$ consisting of ideals prime to $\mathfrak{f}_\eta \mathfrak{a}$. We suppose that a family $\{\mathcal{J}_\mathfrak{a}\}_\mathfrak{a}$ satisfies $\mathcal{J}_\mathfrak{a} \subset \mathcal{J}_{\mathfrak{a}'}$ for any $\mathfrak{a} \subset \mathfrak{a}'$. Moreover, we suppose that there exists a family of real numbers $\{\omega_\mathfrak{n}(\pi) |\, \pi \in \Pi_{\rm cus}^*(l,\mathfrak{n})\}$ for each $\mathfrak{n} \in \mathcal{J}_\mathfrak{a}$ which satisfies the following estimate for any $\epsilon > 0$:

(25.1)

$$\left|\sum_{\pi \in \Pi_{\rm cus}^*(l,\mathfrak{n})} \omega_\mathfrak{n}(\pi) \prod_{v \in S(\mathfrak{a})} X_{n_v}(\lambda_v(\pi)) - \prod_{v \in S(\mathfrak{a})} \mu_{v,\eta_v}(X_{n_v})\right| \ll_{\epsilon,l,\eta} \frac{{\rm N}(\mathfrak{a})^{-1/2+\epsilon}}{\log {\rm N}(\mathfrak{n})} + {\rm N}(\mathfrak{a})^{c+2+\epsilon}{\rm N}(\mathfrak{n})^{-\inf(c,1)+\epsilon},$$

with the implied constant independent of $\mathfrak{a}$ and $\mathfrak{n} \in \mathcal{J}_\mathfrak{a}$. Moreover we impose the non-negativity condition:

$$\omega_\mathfrak{n}(\pi) \geqslant 0 \qquad \text{for all } \pi \in \Pi_{\rm cus}^*(l,\mathfrak{n}) \text{ and } \mathfrak{n} \in \mathcal{J}_\mathfrak{a}. \tag{25.2}$$

Let $\mathfrak{q}$ be a prime ideal relatively prime to $\mathfrak{f}_\eta$. In what follows, we abuse the symbol $\mathfrak{q}$ to denote the corresponding place $v_\mathfrak{q}$ of F; for example, we write $\nu_\mathfrak{q}(\pi)$, $\lambda_\mathfrak{q}(\pi)$ in place of $\nu_{v_\mathfrak{q}}(\pi)$, $\lambda_{v_\mathfrak{q}}(\pi)$, etc. Let $S = \{v_1, \dots, v_r\}$ be a finite subset of $\Sigma_{\rm fin} - S(\mathfrak{f}_\eta\mathfrak{q})$ and set $\mathfrak{a}_S = \prod_{v\in S} \mathfrak{p}_v$. Let $\mathbf{J} = \{J_j\}_{j=1}^r$ be a family of closed subintervals of $(-2,2)$. For each J_j, we choose an open interval J_j' such that $\overline{J_j'} \subset J_j^\circ$ and C^∞-function $\chi_j : \mathbb{R} \to [0,\infty)$ with the following properties:

- $\chi_j(x) \neq 0$ for all $x \in J_j'$.
- $\operatorname{supp}(\chi_j) \subset J_j$.
- $\int_{-2}^2 \chi_j(x) d\mu_{v,\eta_v}(x) = 1$, where

$$d\mu_{v,\eta_v}(x) = \begin{cases} \dfrac{q_v - 1}{(q_v^{1/2}+q_v^{-1/2}-x)^2}\, d\mu_{\rm ST}(x) & (\eta_v(\varpi_v) = +1), \\ \dfrac{q_v + 1}{(q_v^{1/2}+q_v^{-1/2})^2-x^2}\, d\mu_{\rm ST}(x) & (\eta_v(\varpi_v) = -1). \end{cases}$$

Here $d\mu_{\rm ST}(x) = (2\pi)^{-1}\sqrt{4-x^2}dx$. Fixing such a family of functions $\{\chi_j\}$, we set

$$\Omega_\mathfrak{n}(\pi) = \omega_\mathfrak{n}(\pi) \prod_{j=1}^r \chi_j(\lambda_{v_j}(\pi)), \quad \pi \in \Pi_{\rm cus}^*(l,\mathfrak{n}),\, \mathfrak{n} \in \mathcal{J}_{\mathfrak{q}\mathfrak{a}_S}.$$

Lemma 25.1. *For any sufficiently small $\epsilon > 0$, there exists $N_{\epsilon,S,l} > 0$ such that*

(25.3)

$$\left|\sum_{\pi \in \Pi_{\rm cus}^*(l,\mathfrak{n})} \Omega_\mathfrak{n}(\pi) X_n(\lambda_\mathfrak{q}(\pi)) - \mu_{\mathfrak{q},\eta_\mathfrak{q}}(X_n)\right| \ll_{\epsilon,l,\eta,S,\mathbf{J}} \frac{n+1}{(\log {\rm N}(\mathfrak{n}))^3} + \frac{{\rm N}(\mathfrak{q}^n)^{-1/2+\epsilon}}{\log {\rm N}(\mathfrak{n})} + {\rm N}(\mathfrak{q}^n)^{2+c+\epsilon}{\rm N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}$$

for $n \in \mathbb{N}_0$ and $\mathfrak{n} \in \mathcal{J}_{\mathfrak{q}\mathfrak{a}_S}$ with ${\rm N}(\mathfrak{n}) > N_{\epsilon,S,l}$. Here the implied constant is independent of n and $\mathfrak{n}$. Moreover,

$$\Omega_\mathfrak{n}(\pi) \geqslant 0 \qquad \textit{for all } \pi \in \Pi_{\rm cus}^*(l,\mathfrak{n}) \textit{ and } \mathfrak{n} \in \mathcal{J}_{\mathfrak{q}\mathfrak{a}_S}. \tag{25.4}$$

Proof. Given an integer $M > 1$, define $\chi_j^M(x) = \sum_{n=0}^M \hat{\chi}_j(n)\, X_n(x)$ for $x \in [-2,2]$ with $\hat{\chi}_j(n) = \int_{-2}^2 \chi_j(x) X_n(x) d\mu_{\rm ST}(x)$ and set

$$\chi(\mathbf{x}) = \prod_{j=1}^r \chi_j(x_j), \quad \chi^M(\mathbf{x}) = \prod_{j=1}^r \chi_j^M(x_j)$$

for $\mathbf{x}=(x_j)_{1\leqslant j\leqslant r}$ in the product space $[-2,2]^r$. Let $\mathfrak{n}\in\mathcal{J}_{\mathfrak{q}\mathfrak{a}_S}$. By the triangle inequality, the left-hand side of (25.3) is no greater than the sum of the following three terms :

$$\left|\sum_{\pi\in\Pi^*_{\mathrm{cus}}(l,\mathfrak{n})}\omega_{\mathfrak{n}}(\pi)X_n(\lambda_{\mathfrak{q}}(\pi))\{\chi(\lambda_S(\pi))-\chi^M(\lambda_S(\pi))\}\right|, \tag{25.5}$$

$$\left|\sum_{\pi\in\Pi^*_{\mathrm{cus}}(l,\mathfrak{n})}\omega_{\mathfrak{n}}(\pi)X_n(\lambda_{\mathfrak{q}}(\pi))\chi^M(\lambda_S(\pi))-\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(X_n)\,\mu_{S,\eta}(\chi^M)\right|, \tag{25.6}$$

$$|\{\mu_{S,\eta}(\chi^M)-\mu_{S,\eta}(\chi)\}\,\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(X_n)|, \tag{25.7}$$

where $\lambda_S(\pi)=(\lambda_v(\pi))_{v\in S}$ and $\mu_{S,\eta}=\otimes_{v\in S}\mu_{v,\eta_v}$. Note $\mu_{S,\eta}(\chi)=1$. We shall estimate these quantities. Since $|\hat{\chi}_j(n)|\ll_{\chi_j} n^{-5}$ for any $n>0$ by integration by parts and by $\max_{[-2,2]}|X_n|\ll n+1$, we have

$$|\chi_j^M(x)|\leqslant\sum_{n\leqslant M}|\hat{\chi}_j(n)|\,|X_n(x)|\ll_{\chi_j}\sum_{n\leqslant M}n^{-4}\leqslant\zeta(4)$$

and

$$\max_{x\in[-2,2]}|\chi_j(x)-\chi_j^M(x)|\leqslant\sum_{n>M}|\hat{\chi}_j(n)|\max_{[-2,2]}|X_n|\ll_{\chi_j}\sum_{n>M}n^{-4}\ll M^{-3}.$$

By these,

$$\max_{[-2,2]^r}|\chi(\mathbf{x})-\chi^M(\mathbf{x})|\leqslant\max_{[-2,2]^r}\left(\sum_{j=1}^{r}|\prod_{h=1}^{j-1}\chi_h^M(x_h)|\,|\chi_j(x_j)-\chi_j^M(x_j)|\right)\ll_{S,\chi}M^{-3}. \tag{25.8}$$

From (25.1) for $\mathfrak{a}=\mathfrak{o}$, noting $\mathfrak{n}\in\mathcal{J}_{\mathfrak{q}\mathfrak{a}_S}\subset\mathcal{J}_{\mathfrak{o}}$, we have the estimate $|\sum_{\pi\in\Pi^*_{\mathrm{cus}}(l,\mathfrak{n})}\omega_{\mathfrak{n}}(\pi)-1|\ll_{\epsilon,l,\eta}(\log\mathrm{N}(\mathfrak{n}))^{-1}+\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}$. Hence (25.5) is majorized by

$$\{\max_{[-2,2]}|X_n|\}\{\max_{\mathbf{x}\in[-2,2]^r}|\chi(\mathbf{x})-\chi^M(\mathbf{x})|\}\sum_{\pi\in\Pi^*_{\mathrm{cus}}(l,\mathfrak{n})}\omega_{\mathfrak{n}}(\pi)\ll_{\epsilon,l,\eta,S,\chi}(n+1)M^{-3}(1+\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}).$$

By (25.8), the quantity (25.7) is majorized by $\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(X_n)M^{-3}$, which amounts at most to $(n+1)M^{-3}$. Let us estimate (25.6). By expanding the product, $\chi^M(\mathbf{x})$ is expressed as a sum of the terms $\prod_{j=1}^r\hat{\chi}_j(n_j)\times\prod_{j=1}^r X_{n_j}(x_j)$ over all $\mathbf{n}=(n_j)_{j=1}^M\in\{0,\dots,M\}^r$. Hence by using (25.1), we can majorize (25.6) from above by

$$\begin{aligned}&\sum_{\mathbf{n}\in\{0,\dots,M\}^r}\left|\sum_{\pi\in\Pi^*_{\mathrm{cus}}(l,\mathfrak{n})}\omega_{\mathfrak{n}}(\pi)X_n(\lambda_{\mathfrak{q}}(\pi))\prod_{j=1}^{r}X_{n_j}(\lambda_{v_j}(\pi))-\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(X_n)\mu_{S,\eta}(\prod_{j=1}^{r}X_{n_j})\right|\\ &\ll_{\epsilon,l,\eta,S,\chi}\frac{\mathrm{N}(\mathfrak{a}_S^M\mathfrak{q}^n)^{-1/2+\epsilon}}{\log\mathrm{N}(\mathfrak{n})}+\mathrm{N}(\mathfrak{a}_S^M\mathfrak{q}^n)^{2+c+\epsilon}\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}.\end{aligned}$$

Combining the estimations made so far, we have that the left-hand side of (25.3) is majorized by

$$(n+1)M^{-3}(1+\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon})+\frac{\mathrm{N}(\mathfrak{a}_S^M\mathfrak{q}^n)^{-1/2+\epsilon}}{\log\mathrm{N}(\mathfrak{n})}+\mathrm{N}(\mathfrak{a}_S^M\mathfrak{q}^n)^{2+c+\epsilon}\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}. \tag{25.9}$$

Now take

$$M=\left[\frac{\epsilon}{2+c+\epsilon}\frac{\log\mathrm{N}(\mathfrak{n})}{\log\mathrm{N}(\mathfrak{a}_S)}\right].$$

Then $\mathrm{N}(\mathfrak{a}_S)^{M(2+c+\epsilon)} \leqslant \mathrm{N}(\mathfrak{n})^{\epsilon}$, and also $\mathrm{N}(\mathfrak{a}_S)^{M(-1/2+\epsilon)} \leqslant 1$ evidently. By these, (25.9) is majorized by

$$(n+1)(\log \mathrm{N}(\mathfrak{n}))^{-3} \log \mathrm{N}(\mathfrak{a}_S)^3 (1+\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}) + \frac{\mathrm{N}(\mathfrak{q}^n)^{-1/2+\epsilon}}{\log \mathrm{N}(\mathfrak{n})} + \mathrm{N}(\mathfrak{q}^n)^{2+c+\epsilon} \mathrm{N}(\mathfrak{n})^{-\inf(c,1)+2\epsilon}$$

$$\ll_{\epsilon,S} (n+1)(\log \mathrm{N}(\mathfrak{n}))^{-3} + \frac{\mathrm{N}(\mathfrak{q}^n)^{-1/2+\epsilon}}{\log \mathrm{N}(\mathfrak{n})} + \mathrm{N}(\mathfrak{q}^n)^{2+c+\epsilon} \mathrm{N}(\mathfrak{n})^{-\inf(c,1)+2\epsilon}.$$

□

Lemma 25.2. *Let $I \subset [-2,2]$ be an open interval disjoint from the set $\{\lambda_{\mathfrak{q}}(\pi) \,|\, \pi \in \Pi^*_{\mathrm{cus}}(l,\mathfrak{n}),\ \Omega_{\mathfrak{n}}(\pi) \neq 0\}$. Then for any small $\epsilon > 0$, there exists a constant $N_{\epsilon,l,\eta,S,\mathfrak{q}} > 0$ such that for any ideal $\mathfrak{n} \in \mathcal{J}_{\mathfrak{q}\mathfrak{a}_S}$ with $\mathrm{N}(\mathfrak{n}) > N_{\epsilon,l,\eta,S,\mathfrak{q}}$,*

$$\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(I) \ll_{\epsilon,l,\eta,S,\mathbf{J}} \mathrm{N}(\mathfrak{q})^{\epsilon} (\log \mathrm{N}(\mathfrak{n}))^{-1+\epsilon}$$

holds with the implied constant independent of I, $\mathfrak{n}$ and $\mathfrak{q}$.

Proof. The proof of [37, Proposition 5.1 and Lemma 5.2] goes through as it is with a small modification. We reproduce the argument for convenience.

Let $\Delta > 0$ be a parameter to be specified below and K a closed subinterval of I such that

(i) $\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(I - K) \leqslant \Delta$.

Depending on Δ and K, we choose a C^∞-function f on $\mathbb{R}$ such that

(ii) $\mathrm{supp}(f) \subset \bar{I}$,
(iii) $f(x) = 1$ if $x \in K$ and $0 \leqslant f(x) \leqslant 1$ for $x \in \mathbb{R}$,
(iv) $|f^{(k)}(x)| \ll_k \Delta^{-k}$ for $k \in \mathbb{N}_0$.

Since I does not contain the relevant $\lambda_{\mathfrak{q}}(\pi)$'s, from (ii) we have $\Omega_{\mathfrak{n}}(\pi) f(\lambda_{\mathfrak{q}}(\pi)) = 0$ for all $\pi \in \Pi^*_{\mathrm{cus}}(l,\mathfrak{n})$. Using this, from (i) and (iii), we have the inequalities

$$\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(I) \leqslant \mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(K) + \Delta \leqslant \int_{-2}^{2} f\, d\mu_{\mathfrak{q},\eta_{\mathfrak{q}}} + \Delta$$

$$\leqslant \left| \sum_{\pi \in \Pi^*_{\mathrm{cus}}(l,\mathfrak{n})} \Omega_{\mathfrak{n}}(\pi)\, f(\lambda_{\mathfrak{q}}(\pi)) - \int_{-2}^{2} f\, d\mu_{\mathfrak{q},\eta_{\mathfrak{q}}} \right| + \Delta. \tag{25.10}$$

If we set $f_M(x) = \sum_{n=0}^{M} \hat{f}(n)\, X_n(x)$, then the first term of (25.10) is bounded by the sum of the following three terms

$$\left(\sum_{\pi \in \Pi^*_{\mathrm{cus}}(l,\mathfrak{n})} |\Omega_{\mathfrak{n}}(\pi)| \right) \cdot \max_{[-2,2]} |f - f_M|, \tag{25.11}$$

$$\int_{-2}^{2} \max_{[-2,2]} |f - f_M|\, d\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}, \tag{25.12}$$

$$\left| \sum_{\pi \in \Pi^*_{\mathrm{cus}}(l,\mathfrak{n})} \Omega_{\mathfrak{n}}(\pi)\, f_M(\lambda_{\mathfrak{q}}(\pi)) - \int_{-2}^{2} f_M\, d\mu_{\mathfrak{q},\eta_{\mathfrak{q}}} \right|. \tag{25.13}$$

We remark that by the non-negativity of $\Omega_{\mathfrak{n}}(\pi)$, the absolute value in (25.11) can be deleted. Then by the estimate $|\hat{f}(n)| \ll_k n^{-k}\Delta^{-k}$ which follows from (iv) by integration by parts, and by $\max_{[-2,2]} |X_n| \ll n+1$, we have

$$\max_{[-2,2]} |f - f_M| \leqslant \sum_{n>M} |\hat{f}(n)| \max_{[-2,2]} |X_n| \ll_k \sum_{n>M} n^{-k}\Delta^{-k} n \ll M^{2-k}\Delta^{-k}$$

with $k \geqslant 3$. From (25.3) applied with $n=0$, noting $\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(X_0)=1$, we have the estimate $|\sum_{\pi\in\Pi^*_{\rm cus}(l,\mathfrak{n})}\Omega_{\mathfrak{n}}(\pi)-1| \ll_{\epsilon,l,\eta,S,\mathbf{J}} (\log \mathrm{N}(\mathfrak{n}))^{-1}+\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}$. Hence the sum of (25.11) and (25.12) is majorized by

$$\Delta^{-k}M^{2-k}\,(1+(\log \mathrm{N}(\mathfrak{n}))^{-1}+\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}) \ll \Delta^{-k}M^{2-k}$$

with the implied constant independent of Δ, M, $\mathfrak{q}$ and $\mathfrak{n}$. By (25.3) and by $|\hat f(n)|\ll 1$, the term (25.13) is majorazed by

$$\begin{aligned}
&\sum_{n=0}^{M}|\hat f(n)|\left|\sum_{\pi\in\Pi^*_{\rm cus}(l,\mathfrak{n})}\Omega_{\mathfrak{n}}(\pi)\,X_n(\lambda_{\mathfrak{q}}(\pi))-\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(X_n)\right|\\
&\ll_{\epsilon,l,\eta,S,\mathbf{J}} \sum_{n=0}^{M}\left(\frac{n+1}{(\log \mathrm{N}(\mathfrak{n}))^3}+\frac{\mathrm{N}(\mathfrak{q}^n)^{-1/2+\epsilon}}{\log \mathrm{N}(\mathfrak{n})}+\mathrm{N}(\mathfrak{q}^n)^{2+c+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}\right)\\
&\ll_{\epsilon} \frac{M^2}{(\log \mathrm{N}(\mathfrak{n}))^3}+\frac{1}{\log \mathrm{N}(\mathfrak{n})}+\mathrm{N}(\mathfrak{q})^{c'M}\,\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon},
\end{aligned}$$

where $c'=2+c+\epsilon$. Putting all relevant estimations together, we obtain

$$\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(I) \ll_{k,\epsilon,l,\eta,S,\mathbf{J}} \Delta+\Delta^{-k}M^{2-k}+\frac{1}{\log \mathrm{N}(\mathfrak{n})}+\frac{M^2}{(\log \mathrm{N}(\mathfrak{n}))^3}+\mathrm{N}(\mathfrak{q})^{c'M}\mathrm{N}(\mathfrak{n})^{-\inf(c,1)+\epsilon}$$

with the implied constant independent of I, Δ, M, $\mathfrak{q}$ and $\mathfrak{n}$. By setting $M=\left[\frac{\inf(c,1)}{2c'}\frac{\log \mathrm{N}(\mathfrak{n})}{\log \mathrm{N}(\mathfrak{q})}\right]$, this yields the estimate

$$\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(I) \ll_{k,\epsilon,l,\eta,S,\mathbf{J}} \Delta+\Delta^{-k}(\log \mathrm{N}(\mathfrak{q}))^{k-2}(\log \mathrm{N}(\mathfrak{n}))^{2-k}+(\log \mathrm{N}(\mathfrak{n}))^{-1}+\mathrm{N}(\mathfrak{n})^{-\inf(c,1)/2+\epsilon}.$$

Let $\epsilon>0$ and we let Δ vary so that it satisfies $\Delta^{-k}(\log \mathrm{N}(\mathfrak{n}))^{2-k}\asymp_k (\log \mathrm{N}(\mathfrak{n}))^{-1+\epsilon}$, or equivalently

$$\Delta \asymp_k (\log \mathrm{N}(\mathfrak{n}))^{-1+(3-\epsilon)/k}.$$

By taking $k=[3/\epsilon]+1$, we have $(\log \mathrm{N}(\mathfrak{n}))^{-1+\epsilon/2}\ll_\epsilon \Delta \ll_\epsilon (\log \mathrm{N}(\mathfrak{n}))^{-1+\epsilon}$. Hence,

$$\begin{aligned}
\mu_{\mathfrak{q},\eta_{\mathfrak{q}}}(I) &\ll_{\epsilon,l,\eta,S,\mathbf{J}} (\log \mathrm{N}(\mathfrak{n}))^{-1+\epsilon}+(\log \mathrm{N}(\mathfrak{n}))^{-1+\epsilon}(\log \mathrm{N}(\mathfrak{q}))^{k-2}+(\log \mathrm{N}(\mathfrak{n}))^{-1}+\mathrm{N}(\mathfrak{n})^{-\inf(c,1)/2+\epsilon}\\
&\ll_\epsilon \mathrm{N}(\mathfrak{q})^{\epsilon}\,(\log \mathrm{N}(\mathfrak{n}))^{-1+\epsilon}.
\end{aligned}$$

This completes the proof. □

Lemma 25.3. *Given $\epsilon>0$, there exists a positive number $N_{\epsilon,l,\eta,S,\mathfrak{q},\mathbf{J}}$ such that for any ideal $\mathfrak{n}\in\mathcal{J}_{\mathfrak{q}\mathfrak{a}_S}$ with $\mathrm{N}(\mathfrak{n})>N_{\epsilon,l,\eta,S,\mathfrak{q},\mathbf{J}}$, the inequality*

$$\#\{\lambda_{\mathfrak{q}}(\pi)|\,\pi\in\Pi^*_{\rm cus}(l,\mathfrak{n}),\,\Omega_{\mathfrak{n}}(\pi)\neq 0\,\}\geqslant \mathrm{N}(\mathfrak{q})^{-\epsilon}(\log \mathrm{N}(\mathfrak{n}))^{1-\epsilon}$$

holds.

Proof. It follows immediately in the same way as [37, Lemma 5.3]. □

25.1. Let $\Gamma=\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. We let the group Γ act on the set $(2\mathbb{N})^{\Sigma_\infty}$ by the rule ${}^{\sigma}l=(l_{\sigma^{-1}\circ v})_{v\in\Sigma_\infty}$ for $l=(l_v)_{v\in\Sigma_\infty}$ and $\sigma\in\Gamma$, regarding $\Sigma_\infty=\mathrm{Hom}(F,\mathbb{C})$. Let $\mathbb{Q}(l)$ be the fixed field of $\mathrm{Stab}_\Gamma(l)$, which is a finite extension of $\mathbb{Q}$. From [40] (see [34] also), the Satake parameter $A_v(\pi)$ belongs to $\mathrm{GL}(2,\bar{\mathbb{Q}})$ for any $v\in\Sigma_{\rm fin}-S(\mathfrak{n})$ and the set $\Pi_{\rm cus}(l,\mathfrak{n})$ has a natural action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(l))$ in such a way that $({}^{\sigma}\pi)_v\cong\pi_{\sigma^{-1}\circ v}$ for all $v\in\Sigma_\infty$ and

$$q_v^{1/2}\,A_v({}^{\sigma}\pi)=\sigma(q_v^{1/2}\,A_v(\pi))\quad\text{for all } v\in\Sigma_{\rm fin}-S(\mathfrak{n}). \tag{25.14}$$

The field of rationality of $\pi\in\Pi_{\rm cus}(l,\mathfrak{n})$, to be denoted by $\mathbb{Q}(\pi)$, is defined as the fixed field of the group

$$\{\sigma\in\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(l))\,|\,{}^{\sigma}\pi=\pi\,\}.$$

From (25.14), by the strong multiplicity one theorem for GL(2), we have

$$\mathbb{Q}(\pi) = \mathbb{Q}(l)(\{q_v^{1/2}\lambda_v(\pi)|\, v \in \Sigma_{\rm fin} - S(\mathfrak{n})\}).$$

Proposition 25.4. *Suppose that l is a parallel weight, i.e., there exists $k \in 2\mathbb{N}$ such that $l_v = k$ for all $v \in \Sigma_\infty$. Let S be a finite subset of $\Sigma_{\rm fin} - S(\mathfrak{f}_\eta)$ and $\mathbf{J} = \{J_v\}_{v\in S}$ a family of closed subintervals of $(-2, 2)$. Given a sufficiently small $\epsilon > 0$ and a prime ideal $\mathfrak{q}$ prime to $S \cup S(\mathfrak{f}_\eta)$, there exists a positive integer $N_{\epsilon,l,\eta,S,\mathfrak{q},\mathbf{J}}$ such that for any $\mathfrak{n} \in \mathcal{J}_{\mathfrak{q}\mathfrak{a}_S}$ with $\mathrm{N}(\mathfrak{n}) > N_{\epsilon,l,\eta,S,\mathfrak{q},\mathbf{J}}$, there exists $\pi \in \Pi^*_{\rm cus}(l, \mathfrak{n})$ such that $\omega_\mathfrak{n}(\pi) \neq 0$, $\lambda_v(\pi) \in J_v$ for all $v \in S$, and*

$$[\mathbb{Q}(\pi) : \mathbb{Q}] \geqslant \sqrt{\max\left\{\frac{(1-\epsilon)\log\log \mathrm{N}(\mathfrak{n})}{\log(16\sqrt{\mathrm{N}(\mathfrak{q})})} - 2\epsilon, 0\right\}}.$$

Proof. By choosing C^∞-functions $\{\chi_v\}$ as above, we construct the weight function $\Omega_\mathfrak{n}(\pi)$. We follow the proof of [37, Proposition 7.3]. Let $d(\mathfrak{n}, \Omega)$ denote the maximal degree of algebraic numbers $\lambda_\mathfrak{q}(\pi)$ ($\pi \in \Pi^*_{\rm cus}(l, \mathfrak{n})$, $\Omega_\mathfrak{n}(\pi) \neq 0$). Then,

$$\begin{aligned} d(\mathfrak{n}, \Omega) &\leqslant \max\{\, [\mathbb{Q}(\pi) : \mathbb{Q}] \mid \pi \in \Pi^*_{\rm cus}(l, \mathfrak{n}),\, \Omega_\mathfrak{n}(\pi) \neq 0\,\} \\ &\leqslant \max\{\, [\mathbb{Q}(\pi) : \mathbb{Q}] \mid \pi \in \Pi^*_{\rm cus}(l, \mathfrak{n}),\, \omega_\mathfrak{n}(\pi) \neq 0,\, \lambda_v(\pi) \in J_v\ (\forall v \in S)\,\}. \end{aligned}$$

Let $\mathcal{E}(M, d)$ denote the set of algebraic integers which, together with its conjugates, have the absolute values at most M and the absolute degrees at most d. From the parallel weight assumption, the Heck eigenvalues $\mathrm{N}(\mathfrak{q})^{1/2}\lambda_\mathfrak{q}(\pi)$ are known to be algebraic integers (cf. [40, Proposition 2.2]). Since $\sigma(\mathrm{N}(\mathfrak{q})^{1/2}\lambda_\mathfrak{q}(\pi)) = \mathrm{N}(\mathfrak{q})^{1/2}\lambda_\mathfrak{q}({}^\sigma\pi)$ from (25.14), by the Ramanujan bound by [1], we have $\mathrm{N}(\mathfrak{q})^{1/2}\lambda_\mathfrak{q}(\pi) \in \mathcal{E}(2\mathrm{N}(\mathfrak{q})^{1/2}, d(\mathfrak{n}, \Omega))$. Then the cardinality of the set $\{\mathrm{N}(\mathfrak{q})^{1/2}\lambda_\mathfrak{q}(\pi)|\, \pi \in \Pi^*_{\rm cus}(l, \mathfrak{n}),\, \Omega_{\mathfrak{n},\eta}(\pi) \neq 0\,\}$ is bounded from above by $\#\mathcal{E}(2\mathrm{N}(\mathfrak{q})^{1/2}, d(\mathfrak{n}, \Omega))$, which in turn is no greater than $(16\mathrm{N}(\mathfrak{q})^{1/2})^{d(\mathfrak{n},\Omega)^2}$ by [37, Lemma 6.2]. Combining this with the lower bound provided by Lemma 25.3, we have

$$\mathrm{N}(\mathfrak{q})^{-\epsilon}(\log \mathrm{N}(\mathfrak{n}))^{1-\epsilon} \leqslant (16\mathrm{N}(\mathfrak{q})^{1/2})^{d(\mathfrak{n},\Omega)^2}.$$

By taking logarithms, we are done. □

Remark : The parallel weight assumption can be removed if the integrality of the Hecke eigenvalues $q_v^{1/2}\lambda_v(\pi)$ for all $v \in \Sigma_{\rm fin} - S(\mathfrak{f}_\pi)$ is known in a broader generality.

25.2. Proof of Theorem 0.11. Theorem 0.9 means the numbers

$$\omega_\mathfrak{n}(\pi) = \frac{C_l}{4D_F^{3/2}L_{\rm fin}(1, \eta)\nu(\mathfrak{n})}\frac{1}{\mathrm{N}(\mathfrak{n})}\frac{L(1/2, \pi)L(1/2, \pi \otimes \eta)}{L^{S_\pi}(1, \pi, \mathrm{Ad})}, \quad \pi \in \Pi^*_{\rm cus}(l, \mathfrak{n}),\ \mathfrak{n} \in \mathcal{I}^+_{S\cup S(\mathfrak{q}),\eta}$$

satisfy our first assumption (25.1). The second assumption (25.2) follows from [18]. Thus Theorem 0.11 is a corollary of Proposition 25.4 with this particular $\{\omega_\mathfrak{n}(\pi)\}$. □

25.3. Proof of Theorem 0.12. For any $M > 1$, let $\mathcal{I}^-_{S\cup S(\mathfrak{q}),\eta}[M]$ be the set of $\mathfrak{n} \in \mathcal{I}^-_{S\cup S(\mathfrak{q}),\eta}$ such that $\sum_{v\in S(\mathfrak{n})} \frac{\log q_v}{q_v} \leqslant M$. Theorem 0.9 means

$$\omega_\mathfrak{n}(\pi) = \frac{C_l}{4D_F^{3/2}L_{\rm fin}(1, \eta)\,\nu(\mathfrak{n})\log\sqrt{\mathrm{N}(\mathfrak{n})}}\frac{1}{\mathrm{N}(\mathfrak{n})}\frac{L(1/2, \pi)L'(1/2, \pi \otimes \eta)}{L^{S_\pi}(1, \pi, \mathrm{Ad})}, \quad \pi \in \Pi^*_{\rm cus}(l, \mathfrak{n}),\ \mathfrak{n} \in \mathcal{I}^-_{S\cup S(\mathfrak{q}),\eta}[M]$$

satisfy our first assumption (25.1). By our non-negativity assumption (0.9), the second assumption (25.2) is also available. Thus Theorem 0.12 follows from Proposition 25.4. □

Remark : In the parallel weight two case (i.e., $l_v = 2$ for all $v \in \Sigma_\infty$) with totally imaginary condition on η, the assumption (0.9) follows from [53, Theorem 6.1] due to the non-negativity of the Neron-Tate height pairing. Similar results may be expected in the parallel higher weight case (cf. [51]).

26. Computations of local terms

Let $\alpha=\otimes_{v\in S}\alpha_v$ be a decomposable element of $\mathcal{A}_S$. We examine the term $\mathbb{W}_{\rm hyp}^{\eta}(l,\mathfrak{n}|\alpha)$ appearing in the formula (21.6), which is given in Lemma 21.7. Recall that the function $\hat\Psi_l^{(0)}(\mathfrak{n}|\alpha,g)$ in adele points $g=\{g_v\}$ is a product of functions $\Psi_v(g_v)$ on local groups $\mathrm{GL}(2,F_v)$ such that $\Psi_v(g_v)=\Psi_v^{(0)}(l_v;g_v)$ for $v\in\Sigma_\infty$,

$$\Psi_v(g_v)=\frac{1}{2\pi i}\int_{L_v(c)}\Psi_v^{(0)}(s_v;g_v)\alpha(s_v)d\mu_v(s_v)$$

for $v\in S$, and $\Psi_v(g_v)=\Phi_{\mathfrak{n},v}^{(0)}(g_v)$ for $v\in\Sigma_{\rm fin}-S$ (see §12). From Lemma 21.7, by exchanging the order of integrals, we have the first equality of the formula

$$\begin{aligned}\mathbb{W}_{\rm hyp}^{\eta}(l,\mathfrak{n}|\alpha)&=\sum_{b\in F-\{0,-1\}}\int_{\mathbb{A}^\times}\hat\Psi_l^{(0)}(\mathfrak{n}|\alpha,\delta_b\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_\eta\\0&1\end{smallmatrix}\right])\eta(tx_\eta^*)\log|t|_{\mathbb{A}}d^\times t\\&=\sum_{b\in F-\{0,-1\}}\sum_{w\in\Sigma_F}\{\prod_{v\in\Sigma_F-\{w\}}J_v(b)\}\,W_w(b),\end{aligned}\tag{26.1}$$

where

$$J_v(b)=\int_{F_v^\times}\Psi_v(\delta_b\left[\begin{smallmatrix}t_v&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_{\eta,v}\\0&1\end{smallmatrix}\right])\eta_v(t_vx_{\eta,v}^*)d^\times t_v,$$

$$W_w(b)=W_w^{\eta_w}(b)=\int_{F_w^\times}\Psi_w(\delta_b\left[\begin{smallmatrix}t_w&0\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&x_{\eta,w}\\0&1\end{smallmatrix}\right])\eta_w(t_wx_{\eta,w}^*)\log|t_w|_wd^\times t_w$$

for $b\in F_v-\{0,-1\}$. The second equality of (26.1) is justified by $\sum_b\sum_w\{\prod_{v\neq w}|J_v(b)|\}|W_w(b)|<\infty$, which results from the analysis to be made in §26.4. The integrals $J_v(b)$ are studied and their explicit evaluations are obtained in §17. In what follows, we examine the integral $W_w(b)$ separating cases $w\in S$, $w\in\Sigma_{\rm fin}-S$ and $w\in\Sigma_\infty$.

26.1. Orbital integrals for hyperbolic terms : S-part. Let $v\in S$. Then the integral $W_v(b)$ depends on the test function $\alpha_v\in\mathcal{A}_v$ and the character η_v of $F_v^\times$. We write $W_v^{\eta_v}(b;\alpha_v)$ in place of $W_v(b)$ in this subsection. We have

$$W_v^{\eta_v}(b,\alpha_v)=\frac{1}{2\pi i}\int_{L_v(c)}\{\int_{F_v^\times}\Psi_v^{(0)}(s_v;\delta_b\left[\begin{smallmatrix}t&0\\0&1\end{smallmatrix}\right])\eta_v(t)\log|t|_vd^\times t\}\alpha_v(s_v)d\mu_v(s_v).$$

Lemma 26.1. *Let $v\in S$. Let $\alpha_v^{(m)}(s_v)=q_v^{ms_v/2}+q_v^{-ms_v/2}$ with $m\in\mathbb{N}_0$. Then, for any $m\in\mathbb{N}$ and any $b\in F_v-\{0,-1\}$,*

$$W_v^{\eta_v}(b;\alpha_v^{(m)})=\tilde I_v^+(m;\,b)+\eta_v(\varpi_v)\{(\log q_v)I_v^+(m;\,\varpi_v^{-1}(b+1))-\tilde I_v^+(m;\varpi_v^{-1}(b+1))\}$$

with $I_v^+(m;-)$ defined in Lemma 17.2 and

$$\begin{aligned}\tilde I_v^+(m;\,b)=\mathrm{vol}(\mathfrak{o}_v^\times)(\log q_v)\,2^{\delta(m=0)}\biggl(&-q_v^{-m/2}\,\tilde\delta_m^{\eta_v}(b)\\&+\sum_{l=\sup(0,1-\mathrm{ord}_v(b))}^{m-1}\{(m-l-1)q_v^{1-m/2}-(m-l+1)q_v^{-m/2}\}\tilde\delta_l^{\eta_v}(b)\biggr),\end{aligned}$$

where we set

$$\tilde\delta_n^{\eta_v}(b)=\delta(|b|_v<q_v^n)\,\eta_v(\varpi_v^n)\eta_v(b)(-n-\mathrm{ord}_v(b))$$

for $n\in\mathbb{N}$ and

$$\tilde\delta_0^{\eta_v}(b)=\delta(|b|_v<1)\begin{cases}-2^{-1}\,\mathrm{ord}_v(b)(\mathrm{ord}_v(b)+1)&(\eta_v(\varpi_v)=1),\\4^{-1}(\eta_v(b)-1)+2^{-1}\,\mathrm{ord}_v(b)\eta_v(b)&(\eta_v(\varpi_v)=-1).\end{cases}$$

When $m=0$,

$$W_v^{\eta_v}(b;\alpha_v^{(0)})=-2\,\mathrm{vol}(\mathfrak{o}_v^\times)(\log q_v)(\tilde\delta_0^{\eta_v}(b)+\eta_v(\varpi_v)\delta_0^{\eta_v}(\varpi_v^{-1}(b+1))-\eta_v(\varpi_v)\tilde\delta_0^{\eta_v}(\varpi_v^{-1}(b+1))$$

with $\delta_0^{\eta_v}$ *defined in Lemma 17.2.*

Proof. This is proved in a similar way to Lemma 17.2. We decompose the integral into the sum $W_v(b;\alpha_v^{(m)})=\tilde I_v^+(m;b)+\tilde I_v^-(m;b)$, where $\tilde I_v^+(m;b)=\int_{t\in F_v^\times,|t|\leqslant 1}\hat\Phi_{vm}(\delta_b\left[\begin{smallmatrix} t & 0\\ 0 & 1\end{smallmatrix}\right])\eta_v(t)\log|t|_v d^\times t$ with $\hat\Phi_{vm}(g_v)$ the integral computed in Lemma 17.1. We consider the case $m>0$. By Lemma 17.1,

$$\begin{aligned}\tilde I_v^+(m;b)=&\int_{|t|\leqslant 1,\sup(1,|t|_v^{-1}|b|_v)=q_v^m}(-q_v^{-m/2})\eta_v(t)\log|t|_v d^\times t\\ &+\sum_{l=0}^{m-1}\int_{|t|\leqslant 1,\sup(1,|t|_v^{-1}|b|_v)=q_v^l}\{(m-l-1)q_v^{1-m/2}-(m-l+1)q_v^{-m/2}\}\eta_v(t)\log|t|_v d^\times t.\end{aligned}$$

We have the following three equalities:

- If $l=0$ and $\eta_v(\varpi_v)=1$,
$$\int_{|t|\leqslant 1,\sup(1,|t|_v^{-1}|b|_v)=q_v^l}\eta_v(t)\log|t|_v d^\times t=\delta(|b|_v<1)\mathrm{vol}(\mathfrak{o}_v^\times)\log q_v\frac{-\mathrm{ord}_v(b)(\mathrm{ord}_v(b)+1)}{2}.$$
- If $l=0$ and $\eta_v(\varpi_v)=-1$,
$$\int_{|t|\leqslant 1,\sup(1,|t|_v^{-1}|b|_v)=q_v^l}\eta_v(t)\log|t|_v d^\times t=\delta(|b|_v<1)\mathrm{vol}(\mathfrak{o}_v^\times)\log q_v(\frac{\eta_v(b)-1}{4}+\frac{\mathrm{ord}_v(b)\eta_v(b)}{2}).$$
- If $l>0$,
$$\begin{aligned}\int_{|t|\leqslant 1,\sup(1,|t|_v^{-1}|b|_v)=q_v^l}\eta_v(t)\log|t|_v d^\times t&=\delta(|b|<q_v^l)\int_{|t|_v=q_v^{-l}|b|_v}\eta_v(t)\log|t|_v d^\times t\\ &=-\delta(|b|_v\leqslant q_v^l)\mathrm{vol}(\mathfrak{o}_v^\times)(\log q_v)\eta_v(\varpi_v^l b)(l+\mathrm{ord}_v(b)).\end{aligned}$$

Furthermore, $\tilde I_v^-(m;b)$ is transformed into

$$\begin{aligned}\tilde I_v^-(m;b)=&\int_{|t|_v>1}\hat\Phi_{vm}(\delta_b\left[\begin{smallmatrix} t & 0\\ 0 & 1\end{smallmatrix}\right])\eta_v(t)\log|t|_v d^\times t\\ =&\int_{|y|_v<1}\hat\Phi_{vm}(\delta_b\begin{bmatrix}\varpi_v^{-1}y^{-1} & 0\\ 0 & 1\end{bmatrix})\eta_v(\varpi_v^{-1}y^{-1})\log|\varpi_v^{-1}y^{-1}|_v d^\times t\\ =&\eta_v(\varpi_v^{-1})\int_{|y|_v\leqslant 1}\hat\Phi_{vm}(\delta_b\begin{bmatrix}\varpi_v^{-1}y^{-1} & 0\\ 0 & 1\end{bmatrix})\eta_v(y)(\log q_v-\log|y|_v)d^\times y\\ =&\eta_v(\varpi_v)\{(\log q_v)I_v^+(m;\varpi_v^{-1}(b+1))-\tilde I_v^+(m;\varpi_v^{-1}(b+1))\}.\end{aligned}$$

From the results above, we have the lemma for $m>0$. The case $m=0$ is similar. □

Lemma 26.2. *For* $m\in\mathbb{N}$,

$$|W_v^{\eta_v}(b;\alpha_v^{(m)})|\ll(\log q_v)\delta(|b|_v\leqslant q_v^{m-1})q_v^{1-m/2}m(2m+\mathrm{ord}_v(b(b+1)))^2,\qquad b\in F_v^\times-\{-1\}.$$

When $m=0$,

$$|W_v^{\eta_v}(b;\alpha_v^{(0)})|\ll(\log q_v)\delta(|b|_v\leqslant 1)(\mathrm{ord}_v(b(b+1))+1)^2,\qquad b\in F_v^\times-\{-1\}.$$

Here the implied constants independent of v, m *and* b. *Moreover, for* $n\in\mathbb{N}_0$,

$$|W_v^{\eta_v}(b;\alpha_{\mathfrak{p}_v^n})|\ll(\log q_v)q_v\delta(|b|_v\leqslant q_v^n)(\mathrm{ord}_v(b(b+1))+2n+1)^2,\qquad b\in F_v^\times-\{-1\}$$

with the implied constant independent of v, n *and* b.

Proof. Noting (23.1), by the first and second estimates in the lemma, the last estimate is given in the same way as in the proof of Proposition 23.4. We only prove the first estimate. Suppose $m \geqslant 1$. By Lemma 17.2, $I_v^+(m, \varpi_v^{-1}(b+1))$ is estimated as

$$|I_v^+(m, \varpi_v^{-1}(b+1))| \ll \delta(|b|_v \leqslant q_v^{m-1})(m+1)^2 q_v^{1-m/2}.$$

Next we examine $\tilde{I}_v^+(m; b)$. From the definition of $\tilde{\delta}_m^{\eta_v}$ (in Lemma 26.1), we have $|\tilde{\delta}_0^{\eta_v}(b)| \leqslant \delta(|b|_v < 1)2^{-1}(\mathrm{ord}_v(b)+1)^2$. By using this,

$$\begin{aligned}
&\sum_{l=\sup(0,1-\mathrm{ord}_v(b))}^{m-1} (m-l-1)q_v^{1-m/2}|\tilde{\delta}_l^{\eta_v}(b)| \\
\leqslant&\delta(m \geqslant 1, |b|_v \leqslant q_v^{m-2})q_v^{1-m/2}\{\sum_{l=1}^{m-1}(m-l-1)|\tilde{\delta}_l^{\eta_v}(b)| + (m-1)|\tilde{\delta}_0^{\eta_v}(b)|\} \\
\leqslant&\delta(m \geqslant 1, |b|_v \leqslant q_v^{m-2})q_v^{1-m/2}\{\sum_{l=1}^{m-1}(m-l-1)(l+\mathrm{ord}_v(b)) + (m-1)|\tilde{\delta}_0^{\eta_v}(b)|\} \\
=&\delta(m \geqslant 1, |b|_v \leqslant q_v^{m-2})q_v^{1-m/2}(m-1)\{6^{-1}(m-2)m + 2^{-1}(m-2)\,\mathrm{ord}_v(b) + |\tilde{\delta}_0^{\eta_v}(b)|\} \\
\ll&\delta(m \geqslant 2, |b|_v \leqslant q_v^{m-2})q_v^{1-m/2}m(m^2 + m\,\mathrm{ord}_v(b) + (\mathrm{ord}_v(b)+1)^2) \\
\ll&\delta(m \geqslant 2, |b|_v \leqslant q_v^{m-2})q_v^{1-m/2}m(m+\mathrm{ord}_v(b))^2.
\end{aligned}$$

Similarly,

$$\sum_{l=\sup(0,1-\mathrm{ord}_v(b))}^{m-1} (m-l+1)q_v^{-m/2}|\tilde{\delta}_l^{\eta_v}(b)| \ll \delta(m \geqslant 1, |b|_v \leqslant q_v^{m-2})q_v^{-m/2}m(m+\mathrm{ord}_v(b)+1)^2.$$

Hence, we obtain

$$|\tilde{I}_v^+(m; b)| \ll (\log q_v)\delta(|b|_v \leqslant q_v^{m-1})q_v^{1-m/2}m(m+\mathrm{ord}_v(b))^2, \qquad m \in \mathbb{N},\ b \in F_v^\times - \{-1\}.$$

Furthermore,

$$\begin{aligned}
&|\tilde{I}_v^+(m; \varpi_v^{-1}(b+1))| \\
\ll&(\log q_v)\delta(|b+1|_v \leqslant q_v^{m-1})q_v^{1-m/2}m(m+\mathrm{ord}_v(b+1))^2, \qquad m \in \mathbb{N},\ b \in F_v^\times - \{-1\}.
\end{aligned}$$

As a consequence, we have the lemma. □

26.2. Orbital integrals for hyperbolic terms : $(\Sigma_{\mathrm{fin}} - S)$-part. Let $v \in \Sigma_{\mathrm{fin}} - S$. There are three cases to be considered: $v \in \Sigma_{\mathrm{fin}} - S(\mathfrak{n}\mathfrak{f}_\eta)$, $v \in S(\mathfrak{n})$ and $v \in S(\mathfrak{f}_\eta)$.

Lemma 26.3. *Let $v \in \Sigma_{\mathrm{fin}} - (S \cup S(\mathfrak{n}\mathfrak{f}_\eta))$. For $b \in F_v^\times - \{-1\}$, we have*

$$W_v^{\eta_v}(b) = \int_{F_v^\times} \Phi_{v,0}^{(0)}(\delta_b[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}])\eta_v(t)\log|t|_v\ d^\times t = \mathrm{vol}(\mathfrak{o}_v^\times)(\log q_v)\tilde{\Lambda}_v^{\eta_v}(b),$$

where

$$\tilde{\Lambda}_v^{\eta_v}(b) = \delta(|b|_v \leqslant 1)\begin{cases} \tilde{\delta}_0^{\eta_v}(b) & (|b|_v < 1), \\ -\tilde{\delta}_0^{\eta_v}(b+1) & (|b+1|_v < 1), \\ 0 & (|b|_v = |b+1|_v = 1). \end{cases}$$

In particular, $|W_v^{\eta_v}(b)| \ll (\log q_v)\delta(|b(b+1)|_v < 1)(\mathrm{ord}_v(b(b+1))+1)^2$, $b \in F_v^\times - \{-1\}$.

Proof. It follows immediately from the following computation:

$$\int_{F_v^\times} \Phi_{v,0}^{(0)}(\delta_b[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}])\eta_v(t)\log|t|_v\ d^\times t = \int_{|b|_v\leqslant|t|_v<1} \eta_v(t)\log|t|_v d^\times t + \int_{|b+1|_v^{-1}\geqslant|t|_v>1} \eta_v(t)\log|t|_v d^\times t$$
$$= \mathrm{vol}(\mathfrak{o}_v^\times)\,(\log q_v)\,\{\tilde{\delta}_0^{\eta_v}(b) - \tilde{\delta}_0^{\eta_v}(b+1)\}.$$

$\square$

Lemma 26.4. *Let $v \in S(\mathfrak{n})$. If $\eta_v(\varpi_v) = 1$, we have*

$$W_v^{\eta_v}(b) = \int_{F_v^\times} \Phi_{v,\mathfrak{n}}^{(0)}(\delta_b[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}])\eta_v(t)\log|t|_v\ d^\times t$$
$$= \mathrm{vol}(\mathfrak{o}_v^\times)(-\log q_v)\delta(b\in\mathfrak{n}\mathfrak{o}_v)\ 2^{-1}(\mathrm{ord}_v(b)+\mathrm{ord}_v(\mathfrak{n}))(\mathrm{ord}_v(b)-\mathrm{ord}_v(\mathfrak{n})+1).$$

If $\eta_v(\varpi_v) = -1$, then

$$W_v^{\eta_v}(b)$$
$$= \mathrm{vol}(\mathfrak{o}_v^\times)(-\log q_v)\delta(b\in\mathfrak{n}\mathfrak{o}_v)[2^{-1}\{\mathrm{ord}_v(\mathfrak{n})\eta_v(\varpi_v^{\mathrm{ord}_v(\mathfrak{n})}) + \mathrm{ord}_v(b)\eta_v(b)\} + 4^{-1}\{\eta_v(b) - \eta_v(\varpi_v^{\mathrm{ord}_v(\mathfrak{n})})\}].$$

In particular,

$$|W_v^{\eta_v}(b)| \leqslant \delta(b\in\mathfrak{n}\mathfrak{o}_v)(\log q_v)(\mathrm{ord}_v(b)+\mathrm{ord}_v(\mathfrak{n})+1)^2, \qquad b\in F_v^\times - \{-1\}.$$

Proof. It follows immediately from the following computation:

$$\int_{F_v^\times} \Phi_{v,\mathfrak{n}}^{(0)}(\delta_b[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}])\eta_v(t)\log|t|_v\ d^\times t = \int_{|b|_v\leqslant|t|_v<1} \delta(t\in\mathfrak{n}\mathfrak{o}_v)\eta_v(t)\log|t|_v d^\times t$$
$$= \delta(b\in\mathfrak{n}\mathfrak{o}_v)\sum_{n=\mathrm{ord}_v(\mathfrak{n})}^{\mathrm{ord}_v(b)} \int_{\mathfrak{o}_v^\times} \eta_v(\varpi_v^n u)\log|\varpi_v^n u|_v d^\times u = \delta(b\in\mathfrak{n}\mathfrak{o}_v)\mathrm{vol}(\mathfrak{o}_v^\times)(-\log q_v)\sum_{n=\mathrm{ord}_v(\mathfrak{n})}^{\mathrm{ord}_v(b)} \eta_v(\varpi_v^n)n.$$

$\square$

Lemma 26.5. *Let $v \in S(\mathfrak{f}_\eta)$ and put $f = f(\eta_v) \in \mathbb{N}$. For $b \in F_v^\times - \{-1\}$,*

$$W_v^{\eta_v}(b) = \delta(b\in\mathfrak{p}_v^{-f})\eta_v(-1)(1-q_v^{-1})^{-1}q_v^{-f-d_v/2}(\log q_v)\times[-f+$$
$$\eta_v(b(b+1))\{\delta(b\in\mathfrak{p}_v)(-f-\mathrm{ord}_v(b)) + \delta(b\in\mathfrak{o}_v^\times)(-f+\mathrm{ord}_v(b+1)) + \delta(b\notin\mathfrak{o}_v)(-f)q_v^{\mathrm{ord}_v(b)}\}].$$

In particular,

$$|W_v^{\eta_v}(b)| \leqslant 6(\log q_v)q_v^{-f}\delta(|b|_v\leqslant q_v^f)\{f+\delta(|b|_v\leqslant 1)\,\mathrm{ord}_v(b(b+1))\}, \quad b\in F_v^\times - \{-1\}.$$

Proof. We have the expression $W_v^{\eta_v}(b) = \delta(b\in\mathfrak{p}_v^{-f})(W_{v,1}^{\eta_v}(b) + W_{v,2}^{\eta_v}(b))$ with

$$W_{v,1}^{\eta_v}(b) = \int_{\substack{-t\in\varpi_v^f U_v(f) \\ |t|_v|b+1|_v\leqslant 1}} \eta_v(t\varpi_v^{-f})\log|t|_v d^\times t = \eta_v(-1)(-f\log q_v)q_v^{-f-d_v/2}(1-q_v^{-1})^{-1}$$

and

$$W_{v,2}^{\eta_v}(b) = \int_{\substack{-t\in F_v^\times - \varpi_v^f U_v(f) \\ |1+t\varpi_v^{-f}|_v|b+t\varpi_v^{-f}(b+1)|_v\leqslant|t|_v}} \eta_v(t\varpi_v^{-f})\log|t|_v d^\times t.$$

The integration domain of $W_{v,2}^{\eta_v}(b)$ is a disjoint union of the sets $D_l(b)$ ($l \in \mathbb{Z}$) defined in §17.2. By Lemmas 17.6, 17.7 and 17.8, we obtain

$$
\begin{aligned}
&W_{v,2}^{\eta_v}(b) = \sum_{l\in\mathbb{Z}}(-l\log q_v)\int_{D_l(b)}\eta_v(t\varpi_v^{-f})d^\times t\\
=&\delta(|b|_v<1=|b+1|_v)\{(-f+\mathrm{ord}_v(b+1)-\mathrm{ord}_v(b))\log q_v\}\eta_v\left(\frac{-b}{b+1}\right)(1-q_v^{-1})^{-1}q_v^{-f-d_v/2}\\
&+\delta(|b|_v=|b+1_v|\geqslant 1)(-f\log q_v)\eta_v\left(\frac{-b}{b+1}\right)(1-q_v^{-1})^{-1}q_v^{-f+\mathrm{ord}_v(b)-d_v/2}\\
&+\delta(|b+1|_v<1=|b|_v)\{(-f+\mathrm{ord}_v(b+1)-\mathrm{ord}_v(b))\log q_v\}\eta_v\left(\frac{-b}{b+1}\right)(1-q_v^{-1})^{-1}q_v^{-f-d_v/2}\\
=&\eta_v\left(\frac{-b}{b+1}\right)(1-q_v^{-1})^{-1}q_v^{-f-d_v/2}(\log q_v)\{\delta(|b|_v<1=|b+1|_v)(-f-\mathrm{ord}_v(b))\\
&+\delta(|b|_v=|b+1|_v\geqslant 1)(-f)q_v^{\mathrm{ord}_v(b)}+\delta(|b+1|_v<1=|b|_v)(-f+\mathrm{ord}_v(b+1))\}\\
=&\eta_v\left(\frac{-b}{b+1}\right)(1-q_v^{-1})^{-1}q_v^{-f-d_v/2}(\log q_v)\{\delta(b\in\mathfrak{p}_v)(-f-\mathrm{ord}_v(b))\\
&+\delta(b\in\mathfrak{o}_v^\times)(-f+\mathrm{ord}_v(b+1))+\delta(b\notin\mathfrak{o}_v)(-f)q_v^{\mathrm{ord}_v(b)}\}.
\end{aligned}
$$

This completes the proof. $\square$

26.3. Orbital integrals for hyperbolic terms : Σ_∞-part. Let $v \in \Sigma_\infty$ and fix an identification $F_v \cong \mathbb{R}$. In this paragraph, we abbreviate l_v to l omitting the subscript v. Let $\varepsilon : \mathbb{R}^\times \to \{\pm 1\}$ be a character; thus ε is the sign character or the trivial one. From the proof of Lemma 17.12, we have

$$
\begin{aligned}
W_v^\varepsilon(b) &= \int_{\mathbb{R}^\times}\left(\frac{1+it}{\sqrt{t^2+1}}\right)^l\{1+i(bt^{-1}+t(b+1))\}^{-l/2}\varepsilon(t)\log|t|_v d^\times t\\
&= \int_{\mathbb{R}^\times}(1-it)^{-l/2}(1+b+t^{-1}bi)^{-l/2}\varepsilon(t)\log|t|_v d^\times t\\
&= W_+(b)+\varepsilon(-1)\overline{W_+(b)},
\end{aligned}
$$

where we set

$$
W_+(b) = i^{l/2}(1+b)^{-l/2}\int_0^\infty (t+i)^{-l/2}\left(t+\frac{bi}{b+1}\right)^{-l/2}t^{l/2-1}\log t\ dt.
$$

Here is an explicit formula of $W_+(b)$.

Lemma 26.6. *Suppose $l \geqslant 4$. Then, for $b \in \mathbb{R}^\times - \{-1\}$, we have*

$$
W_+(b) = -\pi i\, J_+(l;b) - A(b) - i\, B(b),
$$

where

$$A(b)=\sum_{k=0}^{l/2-1}\tbinom{l/2+k-1}{k}\tbinom{l/2-1}{k}\left\{\tfrac{b^k}{2}(\log|\tfrac{b}{b+1}|)^2-\tfrac{\theta(b)^2}{2}\,b^k-\tfrac{9\pi^2}{8}(-1)^{k+l/2}(b+1)^k\right\}$$
$$+\sum_{k=0}^{l/2-1}\tbinom{l/2+k-1}{k}\sum_{j=1}^{l/2-k-1}\tbinom{l/2-1}{k+j}\tfrac{(-1)^j}{j}\left(\sum_{m=1}^{j-1}\tfrac{1}{m}\{b^k+(-1)^{k+l/2}(b+1)^k\}-b^k\log|\tfrac{b}{b+1}|\right),$$

$$B(b)=\sum_{k=0}^{l/2-1}\tbinom{l/2+k-1}{k}\tbinom{l/2-1}{k}\,b^k\log|\tfrac{b}{b+1}|\,\theta(b)$$
$$-\sum_{k=0}^{l/2-1}\tbinom{l/2+k-1}{k}\sum_{j=1}^{l/2-k-1}\tbinom{l/2-1}{k+j}\tfrac{(-1)^j}{j}\{\tfrac{3\pi}{2}(-1)^{k+l/2}(b+1)^k+b^k\theta(b)\},$$

$\theta(b)=\pi/2$ if $b(b+1)<0$, $\theta(b)=3\pi/2$ if $b(b+1)>0$ and $J_+(l;b)$ is the function defined in Lemma 17.13.

Proof. For $b\in\mathbb{R}^\times-\{-1\}$, put $g(z)=i^{l/2}(1+b)^{-l/2}(z+i)^{-l/2}\left(z+\frac{bi}{b+1}\right)^{-l/2}z^{l/2-1}(\log z)^2$, where $\log z=\log|z|+i\arg(z)$ with $\arg(z)\in[0,2\pi)$. Then, $g(z)$ is holomorphic on $\mathbb{C}-(\mathbb{R}_{\geqslant 0}\cup\{-i,\frac{-bi}{b+1}\})$. We note $\frac{-bi}{b+1}\in i\mathbb{R}-\{0,-i\}$. By Cauchy's integral theorem, we have

$$2\pi i\{\mathrm{Res}_{z=-i}+\mathrm{Res}_{z=\frac{-bi}{b+1}}\}g(z)=\int_\epsilon^R g(t)dt+\oint_{|z|=R}g(z)dz-\int_\epsilon^R g(te^{2\pi i})-\oint_{|z|=\epsilon}g(z)dz$$

with R sufficiently large and $\epsilon>0$ sufficiently small. By $\lim_{R\to\infty}\oint_{|z|=R}g(z)dz=0$, $\lim_{\epsilon\to+0}\oint_{|z|=\epsilon}g(z)dz=0$ and $(\log t+2\pi i)^2=(\log t)^2+4\pi i\log t-4\pi^2$, we also have

$$2\pi i\{\mathrm{Res}_{z=-i}+\mathrm{Res}_{z=\frac{-bi}{b+1}}\}g(z)=-4\pi iW_+(b)+4\pi^2J_+(l;b).$$

Hence, we obtain

$$W_+(b)=-\frac{1}{2}\{\mathrm{Res}_{z=-i}+\mathrm{Res}_{z=\frac{-bi}{b+1}}\}g(z)-\pi i\,J_+(l;b).$$

Furthermore, a direct computation gives us

$$\mathrm{Res}_{z=-i}\,g(z)=\sum_{k=0}^{l/2-1}\tbinom{l/2+k-1}{k}(-1)^{k+l/2}(b+1)^k$$
$$\times\left\{\tbinom{l/2-1}{k}\tfrac{-9\pi^2}{4}+2\sum_{j=1}^{l/2-k-1}\tbinom{l/2-1}{k+j}\tfrac{(-1)^j}{j}\left(\sum_{m=1}^{j-1}\tfrac{1}{m}-\tfrac{3\pi}{2}i\right)\right\}$$

and

$$\mathrm{Res}_{z=\frac{ib}{b+1}}\,g(z)=\sum_{k=0}^{l/2-1}\tbinom{l/2+k-1}{k}b^k\times\left\{\tbinom{l/2-1}{k}\left(\log|\tfrac{b}{b+1}|+\theta(b)i\right)^2\right.$$
$$\left.+2\sum_{j=1}^{l/2-k-1}\tbinom{l/2-1}{k+j}\tfrac{(-1)^j}{j}\left(\sum_{m=1}^{j-1}\tfrac{1}{m}-\log|\tfrac{b}{b+1}|-\theta(b)i\right)\right\}.$$

This completes the proof. $\square$

Lemma 26.7. *Suppose $l > 4$. For any $\epsilon > 0$, we have*

$$|b(b+1)|^{\epsilon}\,|W_v^{\epsilon}(b)| \ll_{\epsilon,l} (1+|b|)^{-l/2+2\epsilon}, \quad b \in \mathbb{R} - \{0,-1\}.$$

Proof. From Lemmas 23.2 and 26.6, for any $\epsilon > 0$, $|b(b+1)|^{\epsilon}|W_+(b)|$ is locally bounded around the points $b = 0, -1$. For b away from the set $\{0,-1\}$, we have

$$|W_+(b)| \leqslant |2b(b+1)|^{-l/4} \int_0^{\infty} t^{l/4}(t^2+1)^{-l/4}|\log t|\,\tfrac{dt}{t}$$

by $t^2(b+1)^2 + b^2 \geqslant 2|t||b(b+1)|$. Since $l > 4$, the last integral is convergent; hence the above inequality gives us $|b(b+1)|^{\epsilon}|W_+(b)| \ll_{\epsilon,l} (1+|b|)^{-l/2+2\epsilon}$ for large $|b|$. □

26.4. Proof of Proposition 24.1. We start from the formula (26.1) taking α to be $\alpha_{\mathfrak{a}}$ defined by (0.6). If we set

$$\mathbb{W}(T) = \sum_{b \in F-\{0,-1\}} \sum_{w \in T} \{\prod_{v \in \Sigma_F - \{w\}} J_v(b)\}\, W_w(b) \tag{26.2}$$

for any subset $T \subset \Sigma_F$, then (26.1) can be written in the form

$$\mathbb{W}^{\eta}_{\rm hyp}(l,\mathfrak{n}|\alpha_{\mathfrak{a}}) = \mathbb{W}(\Sigma_\infty) + \mathbb{W}(S(\mathfrak{a})) + \mathbb{W}(S(\mathfrak{n})) + \mathbb{W}(S(\mathfrak{f}_\eta)) + \mathbb{W}(\Sigma_{\rm fin} - S(\mathfrak{n}\mathfrak{a}\mathfrak{f}_\eta)).$$

We shall estimate each term in the right-hand side of this equality explicating the dependence on $\mathfrak{n}$ and $\mathfrak{a} = \prod_{v \in S(\mathfrak{a})} \mathfrak{p}_v^{n_v}$. Set $c = (\underline{l}/2 - 1)/d_F$. For convenience, we collect here all the estimates used below (other than these, we also need Lemma 26.7). Let $w_1 \in S(\mathfrak{a})$, $w_2 \in S(\mathfrak{n})$, $w_3 \in S(\mathfrak{f}_\eta)$, $w_4 \in \Sigma_{\rm fin} - S(\mathfrak{a}\mathfrak{f}_\eta)$, and $w_5 \in \Sigma_\infty$. Let $\epsilon > 0$ be a small number. Then,

$$|J_{w_1}(b)| \ll \delta(b \in \mathfrak{a}^{-1}\mathfrak{o}_{w_1})\, q_{w_1}\{1 + \Lambda_{w_1}(b)\}, \qquad |J_{w_2}(b)| \leqslant \delta(b \in \mathfrak{n}\mathfrak{o}_{w_2})\, \Lambda_{w_2}(b), \tag{26.3}$$

$$|J_{w_3}(b)| \ll \delta(b \in \mathfrak{f}_\eta^{-1}\mathfrak{o}_{w_3}), \qquad |J_{w_4}(b)| \leqslant \delta(b \in \mathfrak{o}_{w_4})\, \Lambda_{w_4}(b), \tag{26.4}$$

$$|b(b+1)|_{w_5}^{\epsilon}|J_{w_5}(b)| \ll_{\epsilon,l_{w_5}} (1+|b|_{w_5})^{-l_{w_5}/2+2\epsilon} \tag{26.5}$$

(note the difference of $\ll$ and $\leqslant$), and

$$|W_{w_1}(b)| \ll (\log q_{w_1})\, q_{w_1}\, \delta(b \in \mathfrak{a}^{-1}\mathfrak{o}_{w_1})\{2n_{w_1} + {\rm ord}_{w_1}(b(b+1)) + 1\}^2, \tag{26.6}$$

$$|W_{w_2}(b)| \ll (\log q_{w_2})\, \delta(b \in \mathfrak{n}\mathfrak{o}_{w_2})\, \{{\rm ord}_{w_2}(b) + {\rm ord}_{w_2}(\mathfrak{n}) + 1\}^2, \tag{26.7}$$

$$|W_{w_3}(b)| \ll (\log q_{w_3})\, \delta(b \in \mathfrak{f}_\eta^{-1}\mathfrak{o}_{w_3})\, \{2f(\eta_{w_3}) + {\rm ord}_{w_3}(b(b+1)) + 1\}, \tag{26.8}$$

$$|W_{w_4}(b)| \ll (\log q_{w_4})\, \delta(|b(b+1)|_{w_4} < 1)\, \Lambda_{w_4}(b)^2 \tag{26.9}$$

for $b \in F^\times$, where all the constants implied by the Vinogradov symbol are independent of the ideals $\mathfrak{n}$, $\mathfrak{a}$ and the places w_i ($1 \leqslant i \leqslant 5$). Indeed, the second estimate in (26.3) and the both estimates of (26.4) follow from Lemmas 17.4, 17.5 and Corollary 17.11 immediately. The estimate (26.5) is from Lemma 23.2. The first estimate in (26.3) is obtained in the proof of Proposition 23.4. The estimate (26.6) follows from Lemma 26.2, (26.7) is from Lemma 26.4, (26.8) is from Lemma 26.5, and (26.9) is from Lemma 26.3.

In the remaining part of this section, all the constants implied by Vinogradov symbol are independent of $\mathfrak{n}$ and $\mathfrak{a}$ (but may depend on l, η and a given small number $\epsilon > 0$).

Lemma 26.8. *We have*

$$|\mathbb{W}(\Sigma_\infty)| \ll_{\epsilon,l,\eta} {\rm N}(\mathfrak{a})^{c+2+\epsilon}\, {\rm N}(\mathfrak{n})^{-c+\epsilon}.$$

Proof. Similarly to the proof of Proposition 23.4, by Lemma 26.7, we have

$$|\mathbb{W}(\Sigma_\infty)| \ll_{\epsilon,l,\eta} C^{\#S(\mathfrak{a})}{\rm N}(\mathfrak{a}) \sum_{I \subset S(\mathfrak{a})} \mathfrak{I}_0^{\eta}(l,\mathfrak{n},\mathfrak{f}_\eta \prod_{v \in I} \mathfrak{p}_v^{n_v}).$$

Then, the desired estimate is given by Proposition 23.3. □

Lemma 26.9. *We have*

$$|\mathbb{W}(S(\mathfrak{a}))| \ll_{\epsilon,l,\eta} \mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-c+\epsilon}.$$

Proof. By the estimates recalled above, the range of b in the summation (26.2) with $T = S(\mathfrak{a})$ can be restricted to $\mathfrak{n}\mathfrak{a}^{-1}\mathfrak{f}_\eta^{-1} - \{0, -1\}$. If $b \in \mathfrak{n}\mathfrak{a}^{-1}\mathfrak{f}_\eta^{-1}$, then $b(b+1)\mathfrak{a}^2\mathfrak{f}_\eta^2$ is an ideal of $\mathfrak{o}$. From this, noting that η is unramified over $S(\mathfrak{a})$, we have the equality $\mathrm{ord}_w(b(b+1)\mathfrak{a}^2\mathfrak{f}_\eta^2) = 2n_w + \mathrm{ord}_w(b(b+1))$ for any $w \in S(\mathfrak{a})$. By taking summation over $w \in S(\mathfrak{a})$,

$$\sum_{w\in S(\mathfrak{a})} \{2n_w + \mathrm{ord}_w(b(b+1)) + 1\}\,\log q_w \leqslant \log \mathrm{N}\big(b(b+1)\mathfrak{a}^2\mathfrak{f}_\eta^2\big) + \log \mathrm{N}(\mathfrak{a}) \ll_{\epsilon,\eta} |\mathrm{N}(b(b+1))|^{\epsilon/2}\mathrm{N}(\mathfrak{a})^{\epsilon}.$$

Using this, from (26.6), (26.3) and (26.4), we obtain

$$\begin{aligned}
|\mathbb{W}(S(\mathfrak{a}))| &\ll \sum_{b\in\mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}}\ \sum_{w_1\in S(\mathfrak{a})} \{\prod_{v\in\Sigma_F-\{w_1\}} |J_v(b)|\}\,(\log q_{w_1})\,q_{w_1}\,\{\mathrm{ord}_{w_1}(b(b+1)) + 2n_{w_1} + 1\}^2 \\
&\ll_{\epsilon,\eta} C^{\#S(\mathfrak{a})}\,\mathrm{N}(\mathfrak{a})^{1+2\epsilon} \sum_{b\in\mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}} |\mathrm{N}(b(b+1))|^{\epsilon} \prod_{v\in\Sigma_\infty}|J_v(b)| \prod_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{a}\mathfrak{f}_\eta)} \Lambda_v(b) \prod_{v\in S(\mathfrak{a})}\{1+\Lambda_v(b)\} \\
&\leqslant C^{\#S(\mathfrak{a})}\mathrm{N}(\mathfrak{a})^{2\epsilon+1} \sum_{I\subset S(\mathfrak{a})} \mathfrak{I}_\epsilon^\eta(l,\mathfrak{n},\mathfrak{f}_\eta \prod_{v\in I}\mathfrak{p}_v^{n_v}),
\end{aligned}$$

where C is the implied constant in the first estimate of (26.3) and (26.6). Noting $C^{\#S(\mathfrak{a})} \ll_\epsilon \mathrm{N}(\mathfrak{a})^\epsilon$, we obtain the assertion by Proposition 23.3. □

Lemma 26.10. *We have*

$$|\mathbb{W}(S(\mathfrak{n}))| \ll_{\epsilon,l,\eta} \mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-c+\epsilon}.$$

Proof. From the estimates recalled above,

$$\begin{aligned}
&|\mathbb{W}(S(\mathfrak{n}))| \\
&\ll_{\epsilon,\eta} C^{\#S(\mathfrak{a})}\mathrm{N}(\mathfrak{a}) \sum_{b\in\mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}} \prod_{v\in\Sigma_\infty}|J_v(b)| \prod_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{a}\mathfrak{n}\mathfrak{f}_\eta)} \Lambda_v(b) \prod_{v\in S(\mathfrak{a})}\{1+\Lambda_v(b)\} \sum_{w_2\in S(\mathfrak{n})} |W_{w_2}(b)|,
\end{aligned}$$

where C is the implied constant in the first estimate of (26.3). By (26.7),

$$\begin{aligned}
\sum_{w_2\in S(\mathfrak{n})} |W_w(b)| &\ll \sum_{w_2\in S(\mathfrak{n})} (\log q_{w_2})(\mathrm{ord}_{w_2}(\mathfrak{n}) + \mathrm{ord}_{w_2}(b) + 1)^2 \\
&\ll \sum_{w_2\in S(\mathfrak{n})} \mathrm{ord}_{w_2}(\mathfrak{n})^2(\log q_{w_2}) + \sum_{w_2\in S(\mathfrak{n})} (\log q_{w_2})\Lambda_{w_2}(b)^2 \ll_\epsilon \mathrm{N}(\mathfrak{n})^\epsilon \prod_{v\in S(\mathfrak{n})}\Lambda_v(b)^2
\end{aligned}$$

for $b \in \mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}$. From this, we obtain

$$\begin{aligned}
|\mathbb{W}(S(\mathfrak{n}))| &\ll_{\epsilon,\eta} C^{\#S(\mathfrak{a})}\mathrm{N}(\mathfrak{a})\mathrm{N}(\mathfrak{n})^\epsilon \sum_{b\in\mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}} \prod_{v\in\Sigma_\infty}|J_v(b)| \prod_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{a}\mathfrak{f}_\eta)} \Lambda_v(b)^2 \prod_{v\in S(\mathfrak{a})}\{1+\Lambda_v(b)\} \\
&= C^{\#S(\mathfrak{a})}\mathrm{N}(\mathfrak{a})\mathrm{N}(\mathfrak{n})^\epsilon \sum_{I\subset S(\mathfrak{a})} \mathfrak{I}_0^\eta(l,\mathfrak{n},\mathfrak{f}_\eta \prod_{v\in I}\mathfrak{p}_v^{n_v}).
\end{aligned}$$

Then, the desired estimate is given by Proposition 23.3. □

Lemma 26.11. *We have*

$$|\mathbb{W}(S(\mathfrak{f}_\eta))| \ll_{\epsilon,l,\eta} \mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-c+\epsilon}.$$

Proof. By the same argument as in the proof of Lemma 26.9, we have

$$\sum_{w\in S(\mathfrak{f}_\eta)}\{2f(\eta_w)+\mathrm{ord}_w(b(b+1))+1\}\ \log q_w \leqslant \log \mathrm{N}\big(b(b+1)\mathfrak{a}^2\mathfrak{f}_\eta^2\big)+\log \mathrm{N}(\mathfrak{f}_\eta)$$
$$\ll_{\epsilon,\eta} |\mathrm{N}(b(b+1))|^{\epsilon}\mathrm{N}(\mathfrak{a})^{2\epsilon}$$

for $b\in\mathfrak{n}\mathfrak{a}^{-1}\mathfrak{f}_\eta^{-1}$. From the estimates recalled as above, we obtain

$$\begin{aligned}|\mathbb{W}(S(\mathfrak{f}_\eta))| &\leqslant \sum_{b\in\mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}}\ \sum_{w_3\in S(\mathfrak{f}_\eta)}\{\prod_{v\in\Sigma_F-\{w_3\}}|J_v(b)|\}\,(\log q_{w_3})\{2f(\eta_{w_3})+\mathrm{ord}_{w_3}(b(b+1))+1\}\\
&\ll_{\epsilon,l,\eta} C^{\#S(\mathfrak{a})}\mathrm{N}(\mathfrak{a})\sum_{b\in\mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}}|\mathrm{N}(b(b+1))|^{\epsilon}\mathrm{N}(\mathfrak{a})^{2\epsilon}\prod_{v\in\Sigma_\infty}|J_v(b)|\prod_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{a}\mathfrak{f}_\eta)}\Lambda_v(b)\prod_{v\in S(\mathfrak{a})}\{1+\Lambda_v(b)\}\\
&\ll_{\epsilon,l,\eta} C^{\#S(\mathfrak{a})}\mathrm{N}(\mathfrak{a})^{1+2\epsilon}\sum_{I\subset S(\mathfrak{a})}\mathfrak{I}^{\eta}_{\epsilon}(l,\mathfrak{n},\mathfrak{f}_\eta\prod_{v\in I}\mathfrak{p}_v^{n_v}).\end{aligned}$$

Then, the desired estimate is given by Proposition 23.3. □

Lemma 26.12. *We have*

$$|\mathbb{W}(\Sigma_{\mathrm{fin}}-S(\mathfrak{a}\mathfrak{n}\mathfrak{f}_\eta))|\ll_{\epsilon,l,\eta}\mathrm{N}(\mathfrak{a})^{c+2+\epsilon}\,\mathrm{N}(\mathfrak{n})^{-c+\epsilon}.$$

Proof. In the summation on the left-hand side of (26.2) with $T=\Sigma_{\mathrm{fin}}-S(\mathfrak{a}\mathfrak{n}\mathfrak{f}_\eta)$, the range of (b,w) is restricted to $b\in\mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}$ and $w\in S(b(b+1)\mathfrak{o}\cap\mathfrak{o})\cap T$, due to the estimates recalled above. Thus,

$$\begin{aligned}&|\mathbb{W}(\Sigma_{\mathrm{fin}}-S(\mathfrak{a}\mathfrak{n}\mathfrak{f}_\eta))|\\
\leqslant&\sum_{b\in\mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}}\ \sum_{w\in S(b(b+1)\mathfrak{o}\cap\mathfrak{o})-S(\mathfrak{a}\mathfrak{n}\mathfrak{f}_\eta)}\{\prod_{v\in\Sigma_F-\{w_4\}}|J_v(b)|\}\,|W_{w_4}(b)|\\
\ll_{\epsilon,\eta}&\,C^{\#S(\mathfrak{a})}\mathrm{N}(\mathfrak{a})\sum_{b\in\mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}}\prod_{v\in\Sigma_\infty}|J_v(b)|\prod_{v\in S(\mathfrak{a})}\{1+\Lambda_v(b)\}\\
&\times\sum_{w_4\in S(b(b+1)\mathfrak{o}\cap\mathfrak{o})-S(\mathfrak{a}\mathfrak{n}\mathfrak{f}_\eta)}\{\prod_{\substack{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{a}\mathfrak{f}_\eta)\\ v\neq w_4}}\Lambda_v(b)\}\,(\log q_{w_4})\Lambda_{w_4}(b)^2\\
\ll_{\epsilon,\eta}&\,C^{\#S(\mathfrak{a})}\mathrm{N}(\mathfrak{a})\sum_{b\in\mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}}\prod_{v\in\Sigma_\infty}|J_v(b)|\prod_{v\in S(\mathfrak{a})}\{1+\Lambda_v(b)\}\\
&\times\{\sum_{w_4\in S(b(b+1)\mathfrak{o}\cap\mathfrak{o})-S(\mathfrak{a}\mathfrak{n}\mathfrak{f}_\eta)}\log q_{w_4}\}\prod_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{a}\mathfrak{f}_\eta)}\Lambda_v(b)^2\\
\ll_{\epsilon,\eta}&\,C^{\#S(\mathfrak{a})}\mathrm{N}(\mathfrak{a})\sum_{b\in\mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}}\prod_{v\in\Sigma_\infty}|J_v(b)|\prod_{v\in S(\mathfrak{a})}\{1+\Lambda_v(b)\}\\
&\times\prod_{v\in\Sigma_{\mathrm{fin}}-S(\mathfrak{a}\mathfrak{f}_\eta)}\Lambda_v(b)^2\times\mathrm{N}(\mathfrak{a})^{2\epsilon}|\mathrm{N}(b(b+1))|^{\epsilon}\\
=&C^{\#S(\mathfrak{a})}\mathrm{N}(\mathfrak{a})^{1+2\epsilon}\sum_{I\subset S(\mathfrak{a})}\mathfrak{I}^{\eta}_{\epsilon}(l,\mathfrak{n},\mathfrak{f}_\eta\prod_{v\in I}\mathfrak{p}_v^{n_v}).\end{aligned}$$

Here we note

$$\sum_{w_4\in S(b(b+1)\mathfrak{o}\cap\mathfrak{o})-S(\mathfrak{a}\mathfrak{n}\mathfrak{f}_\eta)}\log q_{w_4}\ll_{\epsilon,\eta}\mathrm{N}(\mathfrak{a})^{2\epsilon}|\mathrm{N}(b(b+1))|^{\epsilon},\quad b\in\mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}-\{0,-1\}.$$

Indeed, if $b \in \mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1}$, we have $b(b+1)\mathfrak{f}_\eta^2\mathfrak{a}^2 \subset \mathfrak{o}$ and $S(b(b+1)\mathfrak{o} \cap \mathfrak{o}) - S(\mathfrak{a}\mathfrak{n}\mathfrak{f}_\eta) \subset S(b(b+1)\mathfrak{f}_\eta^2\mathfrak{a}^2)$. Hence,

$$\sum_{w_4 \in S(b(b+1)\mathfrak{o}\cap\mathfrak{o}) - S(\mathfrak{a}\mathfrak{n}\mathfrak{f}_\eta)} \log q_{w_4} \leqslant \sum_{w_4 \in S(b(b+1)\mathfrak{f}_\eta^2\mathfrak{a}^2)} \log q_{w_4} \leqslant \log \mathrm{N}(b(b+1)\mathfrak{f}_\eta^2\mathfrak{a}^2)$$
$$\ll_\epsilon |\mathrm{N}(b(b+1))\mathrm{N}(\mathfrak{f}_\eta)^2\mathrm{N}(\mathfrak{a})^2|^\epsilon, \qquad b \in \mathfrak{n}\mathfrak{f}_\eta^{-1}\mathfrak{a}^{-1} - \{0,-1\}.$$

Therefore, the assertion follows from Proposition 23.3 and from $C^{\#S(\mathfrak{a})} \ll_\epsilon \mathrm{N}(\mathfrak{a})^\epsilon$. □

As a consequence, Proposition 24.1 follows from Lemmas 26.8, 26.9, 26.10, 26.11 and 26.12. □

26.5. Unipotent terms. We compute the local terms for $\tilde{\mathbb{W}}_{\mathrm{u}}^\eta(l, \mathfrak{n}|\alpha)$ at a place $v \in S$. For $\alpha_v \in \mathcal{A}_v$, set

$$U_v^{\eta_v}(\alpha_v) = \frac{1}{2\pi i}\int_{\sigma-2\pi i(\log q_v)^{-1}}^{\sigma+2\pi i(\log q_v)^{-1}} \frac{1}{(1-\eta_v(\varpi_v)q_v^{-(s+1)/2})(1-q_v^{(s+1)/2})}\,\alpha_v(s)\,d\mu_v(s), \tag{26.10}$$

$$\tilde{U}_v^{\eta_v}(\alpha_v) = \frac{1}{2\pi i}\int_{\sigma-2\pi i(\log q_v)^{-1}}^{\sigma+2\pi i(\log q_v)^{-1}} \frac{\eta_v(\varpi_v)\,\log q_v}{(1-\eta_v(\varpi_v)q_v^{-(s+1)/2})^2(1-q_v^{-(s+1)/2})\,q_v^{s+1}}\,\alpha_v(s)\,d\mu_v(s) \tag{26.11}$$

with $d\mu_v(s) = 2^{-1}\log q_v\,(q_v^{(1+s)/2} - q_v^{(1-s)/2})ds$ and $\sigma > 0$. The integral $U_v^{\eta_v}$ is already computed in Proposition 18.1. In the same way, we have the following lemma easily.

Lemma 26.13. *For any $m \in \mathbb{N}_0$, we have*

$$\tilde{U}_v^{\eta_v}(\alpha_v^{(m)}) = -\delta(m>0)\,q_v^{-m/2}(\log q_v)\begin{cases}\left\{\frac{q_v-1}{2}m(-1)^m - \frac{3q_v+1}{4}(-1)^m + \frac{1-q_v}{4}\right\} & (\eta_v(\varpi_v) = -1),\\ \left\{\frac{(m-1)(m-2)}{2}q_v - \frac{m(m+1)}{2}\right\} & (\eta_v(\varpi_v) = +1).\end{cases}$$

27. An estimation of a certain lattice sum

Let $d \geqslant 1$ be an integer. We fix $l = (l_j)_{1\leqslant j\leqslant d} \in \mathbb{R}^d$ such that $l_d \geqslant \ldots \geqslant l_1 \geqslant 4$, and consider a positive function $f(x)$ on $\mathbb{R}^d$ defined by

$$f(x) = \prod_{j=1}^d (1+|x_j|)^{-l_j/2}, \quad x = (x_j)_{1\leqslant j\leqslant d} \in \mathbb{R}^d.$$

Given a $\mathbb{Z}$-lattice $\Lambda \subset \mathbb{R}^d$ (of full rank), we define

$$\Theta(\Lambda) = \sum_{b \in \Lambda - \{0\}} f(b).$$

Viewing this as a function in Λ, we need to compare its asymptotic size with a certain power of $D(\Lambda)$, the Euclidean volume of a fundamental domain of $\mathbb{R}^d/\Lambda$. To state the main result of this section, we need another quantity $r(\Lambda)$ given by

$$r(\Lambda) = \frac{1}{2}\min_{b\in\Lambda-\{0\}}\|b\|.$$

Theorem 27.1. *Let F be a totally real number field of degree d. Let Λ_0 and Λ be fractional ideals such that $\Lambda \subset \Lambda_0$; we regard them as $\mathbb{Z}$-lattices in $\mathbb{R}^d$ by the embedding $F \to \mathbb{R}^{\mathrm{Hom}(F,\mathbb{R})} \cong \mathbb{R}^d$. Then,*

$$\Theta(\Lambda) \ll \{1 + r(\Lambda_0)\}^{dl_d/2}\,D(\Lambda_0)^{-1}\,D(\Lambda)^{(1-l_1/2)/d}$$

with the implied constant independent of Λ and Λ_0.

The proof is given at the last part of the next subsection after several lemmas.

27.1. Proof of Theorem 27.1. Let $d\mu(\omega)$ denote the Euclidean measure on the sphere $\mathbb{S}^{d-1} = \{x = (x_j)_{1\leqslant j\leqslant d} \in \mathbb{R}^d \mid \sum_{j=1}^d x_j^2 = 1\}$.

Lemma 27.2. *For any $\lambda = (\lambda_j) \in \mathbb{C}^d$ such that $\mathrm{Re}(\lambda_j) < 1$, we have*

$$I(\lambda) = \int_{\mathbb{S}^{d-1}} \prod_{j=1}^d |\omega_j|^{-\lambda_j}\, d\mu(\omega) = 2\Gamma\left(\sum_{j=1}^d \tfrac{1-\lambda_j}{2}\right)^{-1} \prod_{j=1}^d \Gamma\left(\tfrac{1-\lambda_j}{2}\right).$$

Proof. The formula is obtained by computing the integral

$$\int_{\mathbb{R}^d} \exp(-\epsilon\|x\|^2) \prod_{j=1}^d |x_j|^{-\lambda_j}\, dx \tag{27.1}$$

in two different ways, where $\epsilon > 0$ and $\mathrm{Re}(\lambda_j) < 1$ for the absolute convergence of the integral. By expressing (27.1) as an iterating integral, we compute it as

$$\prod_{j=1}^d \int_{\mathbb{R}} e^{-\epsilon x_j^2} |x_j|^{-\lambda_j} dx_j = \prod_{j=1}^d \epsilon^{(\lambda_j-1)/2}\Gamma\left(\tfrac{1-\lambda_j}{2}\right) = \epsilon^{(\sum_{j=1}^d \lambda_j - d)/2} \prod_{j=1}^d \Gamma\left(\tfrac{1-\lambda_j}{2}\right)$$

on one hand. On the other hand, by the polar decomposition, (27.1) becomes

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{S}^{d-1}} e^{-\epsilon\rho^2} \prod_{j=1}^d |\rho\omega_j|^{-\lambda_j}\, \rho^{d-1}\, d\rho\, d\mu(\omega) \\
&= \left(\int_{\mathbb{S}^{d-1}} \prod_{j=1}^d |\omega_j|^{-\lambda_j}\, d\mu(\omega)\right) \left(\int_0^\infty e^{-\epsilon\rho^2} \rho^{-\sum_{j=1}^d \lambda_j + d - 1}\, d\rho\right) \\
&= I(\lambda)\, 2^{-1} \epsilon^{(\sum_{j=1}^d \lambda_j - d)/2}\, \Gamma\left(\sum_{j=1}^d \tfrac{1-\lambda_j}{2}\right).
\end{aligned}$$

□

Lemma 27.3. *For $t = (t_j)_{1\leqslant j\leqslant d} \in [1,\infty)^d$, set*

$$\varphi(t_1, \dots, t_d) = \int_{\mathbb{S}^{d-1}} f(t_1\omega_1, \dots, t_d\omega_d)\, d\mu(\omega).$$

For $t > 1$, let $\underline{t}$ denote the diagonal element (t_j) defined by $t_j = t$ $(1 \leqslant j \leqslant d)$. Then,

$$\varphi(\underline{t}) = \mathcal{O}(t^{1-d-l_1/2}), \quad t \in [1,\infty).$$

Proof. For $\lambda = (\lambda_j) \in \mathbb{C}^d$ such that $0 < \mathrm{Re}(\lambda_j) < 1$, we compute the multiple Mellin transform

$$\tilde{\varphi}(\lambda) = \int_0^\infty \cdots \int_0^\infty \varphi(t_1, \dots, t_d) \prod_{j=1}^d t_j^{\lambda} \frac{dt_j}{t_j}.$$

By Lemma 27.2, we compute this in the following manner.

$$\begin{aligned}
\tilde{\varphi}(\lambda) &= \int_{\mathbb{S}^{d-1}} \{\prod_{j=1}^{d} \int_0^\infty (1+t_j|\omega_j|)^{-l_j/2} t_j^{\lambda_j-1}\, dt_j\}\, d\mu(\omega) \\
&= \int_{\mathbb{S}^{d-1}} \{\prod_{j=1}^{d} |\omega_j|^{-\lambda_j} \int_0^\infty (1+t_j)^{-l_j/2} t_j^{\lambda_j-1}\, dt_j\}\, d\mu(\omega) \\
&= \left(\int_{\mathbb{S}^{d-1}} \prod_{j=1}^{d} |\omega_j|^{-\lambda_j}\, d\mu(\omega) \right) \left(\prod_{j=1}^{d} \int_0^\infty (1+t_j)^{-l_j/2} t_j^{\lambda_j-1}\, dt_j \right) \\
&= I(\lambda)\, \{\prod_{j=1}^{d} \Gamma(l_j/2)^{-1}\, \Gamma(l_j/2-\lambda_j)\, \Gamma(\lambda_j)\} \\
&= 2\Gamma\left(\sum_{j=1}^{d} \tfrac{1-\lambda_j}{2} \right)^{-1} \{\prod_{j=1}^{d} \Gamma(l_j/2)^{-1}\, \Gamma((1-\lambda_j)/2)\, \Gamma(l_j/2-\lambda_j)\, \Gamma(\lambda_j)\}.
\end{aligned}$$

By Stirling's formula, this is bounded by a constant multiple of $P(\mathrm{Im}\lambda)\exp(-\pi\sum_{j=1}^{d}|\mathrm{Im}(\lambda_j)|)$ with some polynomial $P(x_1,\ldots,x_d)$ which can be taken uniformly with $\mathrm{Re}(\lambda)$ varied compactly. Thus, by a successive application of the Mellin inversion formula, we obtain

$$\varphi(\underline{t}) = \left(\frac{1}{2\pi i}\right)^d \int_{(\sigma_1)} \cdots \int_{(\sigma_d)} 2\,\{\prod_{j=1}^{d} \frac{\Gamma\left(\frac{1-\lambda_j}{2}\right)\Gamma\left(\frac{l_j}{2}-\lambda_j\right)\Gamma(\lambda_j)}{\Gamma(l_j/2)}\}\, \frac{t^{-\sum_{j=1}^{d}\lambda_j}}{\Gamma\left(\sum_{j=1}^{d}\frac{1-\lambda_j}{2}\right)} \prod_{j=1}^{d} d\lambda_j,$$

where the contour $(\sigma_j) = \{\mathrm{Re}(\lambda)=\sigma_j\}$ should be contained in the band $0<\mathrm{Re}(\lambda_j)<1$. We shift the contours (σ_j) in some order far to the right. The residues arise when the moving contour (σ_j) passes the points in $(1+2\mathbb{Z}_{\geqslant 0})\cup(l_j/2+\mathbb{Z}_{\geqslant 0})$. Among those residues, the one with the smallest possible power of t^{-1} comes from the pole at $\lambda_1 = l_1/2$, $\lambda_j = 1\,(2\leqslant j\leqslant d)$ if $l_2>l_1$, which we assume for simplicity in the rest of the proof of this lemma. (When $l_2=l_1$, there are several terms giving the same power in t^{-1}.) The residue term is $O(t^{-(d-1+l_2/2)})$, by which the contribution from the remaining terms are majorized. This completes the proof. $\square$

Lemma 27.4. (1)

$$f(x+y) \geqslant f(x)\,f(y), \qquad x,\, y\in\mathbb{R}^d \tag{27.2}$$

(2)

$$\mathrm{vol}(\mathbb{S}^{d-1})\,(1+\rho)^{-dl_d/2} \leqslant \int_{\mathbb{S}^{d-1}} f(\rho\,\omega)\, d\mu(\omega) \ll (1+\rho)^{1-d-l_1/2}, \quad \rho>0,$$

with the implied constant depending on l and d.

Proof. (1) is immediate from the inequality $1+|x_j+y_j|\leqslant(1+|x_j|)(1+|y_j|)$. As for (2), we first note the inequality $0\leqslant|\omega_j|\leqslant 1$ for $\omega\in\mathbb{S}^{d-1}$. Using this, we have $\prod_{j=1}^{d}(1+|\rho\omega_j|)\leqslant(1+\rho)^d$. By this,

$$f(\rho\omega) \geqslant \{\prod_{j=1}^{d}(1+|\rho\omega_j|)\}^{-l_d/2} \geqslant (1+\rho)^{-dl_d/2}.$$

Taking the integral in ω, we have the estimate from below as desired. The upper bound is provided by Lemma 27.3. $\square$

We compare $\Theta(\Lambda)$ with the integral of $f(x)$ on the ball $B_\Lambda = \{x \in \mathbb{R}^d \,|\, \|x\| < r(\Lambda)\,\}$. For convenience, we set $I(D) = \int_D f(x)\,dx$ for any Borel set D in $\mathbb{R}^d$.

Lemma 27.5. *Let Λ_0 and Λ be $\mathbb{Z}$-lattices such that $\Lambda \subset \Lambda_0$. Then, we have the inequality*

$$\Theta(\Lambda) \leqslant I(B_{\Lambda_0})^{-1}\, I(\mathbb{R}^d - B_\Lambda)$$

Proof. The inequality (27.2) gives us

$$I(B_\Lambda)\,\Theta(\Lambda) \leqslant \sum_{b \in \Lambda - \{0\}} \int_{B_\Lambda} f(b+x)\,dx.$$

Since $\Lambda \subset \Lambda_0$, we have $B_{\Lambda_0} \subset B_\Lambda$, from which $I(B_{\Lambda_0}) \leqslant I(B_\Lambda)$ is obtained by the non-negativity of $f(x)$. Since $(B_\Lambda + B_\Lambda) \cap \Lambda = \{0\}$, the translated sets $B_\Lambda + b\,(b \in \Lambda - \{0\})$ are mutually disjoint. From this remark,

$$\sum_{b \in \Lambda - \{0\}} \int_{B_\Lambda} f(b+x)\,dx \leqslant \int_{\mathbb{R}^d - B_\Lambda} f(x)\,dx = I(\mathbb{R}^d - B_\Lambda).$$

Putting altogether, we are done. $\square$

Lemma 27.6. *Let Λ be a $\mathbb{Z}$-lattice.*

$$I(B_\Lambda) \geqslant \mathrm{vol}(\mathbb{S}^{d-1})\,(1 + r(\Lambda))^{-dl_d/2}\, r(\Lambda)^d/d,$$
$$I(\mathbb{R}^d - B_\Lambda) \ll r(\Lambda)^{1-l_1/2}$$

with the implied constant independent of Λ.

Proof. By Lemma 27.4 (2),

$$\begin{aligned}
I(B_\Lambda) &= \int_0^{r(\Lambda)} \int_{\mathbb{S}^{d-1}} f(\rho\omega)\,d\omega\,\rho^{d-1}\,d\rho \\
&\geqslant \mathrm{vol}(\mathbb{S}^{d-1}) \int_0^{r(\Lambda)} (1+\rho)^{-dl_d/2}\,\rho^{d-1}\,d\rho \\
&\geqslant \mathrm{vol}(\mathbb{S}^{d-1})(1 + r(\Lambda))^{-dl_d/2} \int_0^{r(\Lambda)} \rho^{d-1}\,d\rho = \mathrm{vol}(\mathbb{S}^{d-1})(1+r(\Lambda))^{-dl_d/2}\, r(\Lambda)^d/d.
\end{aligned}$$

In a similar way,

$$\begin{aligned}
I(\mathbb{R}^d - B_\Lambda) &= \int_{r(\Lambda)}^\infty \int_{\mathbb{S}^{d-1}} f(\rho\omega)\,d\omega\,\rho^{d-1}\,d\rho \\
&\ll \int_{r(\Lambda)}^\infty (1+\rho)^{1-d-l_1/2}\,\rho^{d-1}\,d\rho \leqslant \int_{r(\Lambda)}^\infty \rho^{-l_1/2}\,d\rho = (l_1/2 - 1)^{-1} r(\Lambda)^{1-l_1/2}.
\end{aligned}$$

$\square$

Lemma 27.7. *Let F be a totally real number field of degree d. There exist positive constants C_d and C'_d such that $C_d\, r(\Lambda)^d \leqslant D(\Lambda) \leqslant C'_d\, r(\Lambda)^d$ for any fractional ideal Λ.*

Proof. The first inequality follows from Minkowski's convex body theorem. The second inequality is proved as follows. For any $b \in \Lambda - \{0\}$, there exists an ideal $\mathfrak{a} \subset \mathfrak{o}$ such that $(b) = \mathfrak{a}\Lambda$, and hence $|\mathrm{N}(b)| = \mathrm{N}(\Lambda)\mathrm{N}(\mathfrak{a}) \geqslant \mathrm{N}(\Lambda)$. Thus, by the arithmetic-geometric mean inequality,

$$D(\Lambda)^{1/d} = \mathrm{N}(\Lambda)^{1/d} \leqslant \{\prod_{j=1}^d |b_j|^2\}^{1/(2d)} \leqslant \{\sum_{j=1}^d |b_j|^2/d\}^{1/2} = d^{-1/2}\|b\|$$

Hence, $D(\Lambda)^{1/d} \leqslant 2d^{-1/2} r(\Lambda)$. This shows $D(\Lambda) \leqslant C'_d\, r(\Lambda)^d$ with $C'_d = (2d^{-1/2})^d$. $\square$

Theorem 27.1 follows from Lemmas 27.5, 27.6 and 27.7 immediately. □

References

[1] D. Blasius, *Hilbert modular forms and the Ramanujan conjecture*, Noncommutative Geometry and Number Theory, Aspects Math. E37, Vieweg, Wiesbaden, 35–56, 2006.

[2] A. Borel, *Automorphic forms on* $\mathrm{SL}_2(\mathbb{R})$, Cambridge Tracts in Mathematics, 130 Cambridge University Press, 1997.

[3] N. Bourbaki, *Intégration* chap. I–IV, 3° édition, Springer, 2007.

[4] E. Carletti, G. Monti Bragadin and A. Perelli, *On general L-functions*, Acta Arith. **LXVI**, 147–179, 1994.

[5] D.L. DeGeorge and N. Wallach, *Limit formulas for multiplicities in* $L^2(\Gamma\backslash G)$, Ann. of Math. (2) **107**, 133–150, 1978.

[6] B. Feigon and D. Whitehouse, *Averages of central L-values of Hilbert modular forms with an application to subconvexity*, Duke. Math. J., **149**, 347–410, 2009.

[7] S. Gelbert and H. Jacquet, *A relation between automorphic representations of* GL(2) *and* GL(3), Ann. Sci. École Norm. Sup. 4^e série, **11**, n° 4, 471–542, 1978.

[8] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products (the seventh edition)*, Academic Press, Inc., 2007.

[9] B.H. Gross and D.B. Zagier, *Heegner points and derivatives of L-series*, Inventiones Math. **84** 225–320, 1986.

[10] J. Guo, *On the positivity of the central critical values of automorphic L-functions*, Duke Math. J. **83** No. 1, 157–190, 1996.

[11] M. Hirano, *Shintani functions on* $\mathrm{GL}(2,\mathbb{R})$, Trans. of Amer. Math. Soc. **352** No.4, 1709–1721, 2000.

[12] A. Ivić, *On sums of Hecke series in short intervals*, J. Théorie des Nombres de Bordeaux 13, no.2, 453–468, 2001.

[13] H. Iwaniec, *The spectral growth of automorphic L-functions*, J. Reine Angew. Math. **428**, 139–159, 1992.

[14] H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications vol. 53, Amer. Math. Soc. Providence, RI, 2004.

[15] H. Iwaniec and P. Sarnak, *The non-vanishing of central values of automorphic L-functions and Landau-Siegel zeros*, Israel J. Math. **120**, 155–177, 2000.

[16] H. Jacquet, *Sur un résultat de Waldspurger*, Ann. Sci. École. Norm. Sup. 4^e série, **19**, n, 185–229, 1986.

[17] H. Jacquet, *Sur un résultat de Waldspurger II*, Compos. Math. **63**, 315–389, 1986.

[18] H. Jacquet and Nan Chen, *Positivity of quadratic base change L-functions*, Bull. Soc. Math. France 129, no. 1, 33–90, 2001. MR MR1871978 (2003b:11048)

[19] M. Jutila and Y. Motohashi, *Uniform bound for Hecke L-functions*, Acta Math., **195**, 61–115, 2005.

[20] A. Knightly and C. Li, *Kuznetsov's trace formula and the Hecke eigenvalues of Hilbert modular forms*, Modular Forms on Schiermonnikoog, edited by Edixhoven, van der Geer and Moonen, Cambridge University Press, 145–188, 2008.

[21] A. Knightly and C. Li, *Kuznetsov's trace formula and the Hecke eigenvalues of Maass forms*, Mem. Amer. Math. Soc., **224**, No. 1055, 2013.

[22] C. Li, Kuznietsov trace formula and weighted distribution of Hecke eigenvalues, J. Number Theory **104**, 177–192, 2004.

[23] X. Li, *Upper bounds on L-functions at the edge of the critical strip*, Int. Math. Res. Not., 727–755, 2010.

[24] W. Magnus, F. Oberhettinger and R. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, in Einzeldarstellungen mit besonderer Berücksichtingung der Anwendungsgebiete Band **52**, Springer-Verlag, NewYork, 1966.

[25] K. Martin and D. Whitehouse, *Central values and toric periods for* GL(2) , Int. Math. Res. Not., 141–191, 2008.

[26] P. Michel and D. Ramakrishnan, *Consequences of the Gross-Zagier formulae: Stability of average L-values, subconvexity, and non-vanishing mod p*, Number theory, analysis and geometry (In memory of Serge Lang), 437–459, Springer, New York, 2012.

[27] P. Michel and A. Venkatesh, *The subconvexity problem for* GL_2, Publ. I.H.E.S., **111**, 171–271, 2010.

[28] G. Molteni, *L-functions: Siegel-type theorems and structure theorems*, Ph.D. thesis, University of Milan, Milan, 1999.

[29] G. Molteni, *Upper and lower bounds at* $s = 1$ *for certain Dirichlet series with Euler product*, Duke Math. J. **111** No. 1, 133–158, 2002.

[30] Y. Motohashi, *Spectral mean values of Maass waveform L-functions*, J. Number Theory **42**, 258–284, 1992.

[31] P. Nelson, *Stable average of central values of Rankin-Selberg L-functions: some new variants*, J. Number Theory, **133** Issue 8, 2588–2615, 2013.

[32] P. Nelson, A. Pitale and A. Saha, *Bounds for Rankin-Selberg integrals and quantum unique ergodicity for powerful levels*, J. Amer. Math. Soc. **27**, 147–191, 2014.

[33] Z. Peng, *Zeros and central values of automorphic L-functions*, Ph.D thesis, Princeton University, Princeton, 2001.

[34] A. Raghuram and N. Tanabe, *Notes of the arithmetic of Hilbert modular forms*, J. Ramanujan Math. Soc. 26, No. 3 261–319, 2011.

[35] D. Ramakrishnan and J. Rogawski, *Average values of modular L-series via the relative trace formula*, Pure and Appl. Math. Q. **1** No.4, 701–735, 2005.

[36] M. Reed and B. Simon, *Methods of Mathematical Physics I, Functional Analysis*, Academic Press, N.Y., 1972.
[37] E. Royer, *Facteurs $\mathbb{Q}$-simples de $J_0(N)$ de grande dimension et de grand rang,* Bull. Soc. Math. France 128, no. 2, 219–248, 2000.
[38] R. Schmidt, *Some remarks on local newforms for $GL(2)$*, J. Ramanujan Math. Soc. **17**, 115–147, 2002.
[39] J. P. Serre, *Répartition asymptotique des valeurs propres de l'operateur de Hecke T_p*, J. Amer. Math. Soc. 10, no. 1, 75–102, 1997.
[40] G. Shimura, *The special values of the zeta functions associated with Hilbert modular forms*, Duke Math. J. **45**, no.3, 637–679, 1978.
[41] S. Sugiyama, *Regularized periods of automorphic forms on GL(2)*, Tohoku Math. J. 65, no.3, 373–409, 2013.
[42] S. Sugiyama, *Asymptotic behaviors of means of central values of automorphic L-functions for GL(2)*, (preprint 2012). http://arxiv.org/abs/1312.2732
[43] S. Sugiyama and M. Tsuzuki, *Relative trace formulas and subconvexity estimates of L-functions for Hilbert modular forms*, (preprint 2013). http://arxiv.org/abs/1305.2261
[44] S. Sugiyama and M. Tsuzuki, *Existence of Hilbert cusp forms with nonvanishing L-values*, (preprint 2014). http://arxiv.org/abs/1406.2902
[45] E.C. Titchmarsh, revised by H. Brown, *The theory of Riemann zeta-function (second edition)*, Oxford science publications, Clarendon Press, Oxford, 1986.
[46] D. Trotabas, *Non annulation des fonctions L des formes modulaires de Hilbert au point central*, Ann. Inst. Fourier, Grenoble, **61**, 1, 187–259, 2011.
[47] M. Tsuzuki, *Spectral means of central values of automorphic L-functions for* GL(2), Mem. Amer. Math. Soc., **235**, No. 1110, 2014.
[48] J.L. Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symmetrie*, Compos. Math. **54**, 173–242, 1985.
[49] N. R. Wallach, *On the constant term of a square integrable automorphic form*, Operator algebras and group representations, vol. II Monog. Stud. Math., 18. Pitman, Boston, MA, 227-237, 1984.
[50] X. Yuan, S.W. Zhang and W. Zhang, *The Gross-Zagier Formula on Shimura Curves*, Annals of Mathematics Studies, 184, Princeton University Press, Princeton, NJ, 2013.
[51] S.W. Zhang, *Heights of Heegner cycles and derivatives of L-series*, Invent. Math. **130**, 99–152, 1997.
[52] S.W. Zhang, *Gross-Zagier formula for* GL(2), Asian J. Math. **5**, 183–290, 2001.
[53] S.W. Zhang, *Gross-Zagier formula for* GL(2) *II*, in Heegner points and Rankin L-series, Math. Sci. Res. Inst. Publ. **49** (Cambridge University Press, Cambridge), 191–214, 2004.