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GAP THEOREMS FOR COMPACT GRADIENT SASAKI-RICCI SOLITONS

HOMARE TADANO

Dedicated to the memory of Professor Shoshichi Kobayashi

Abstract. In the present paper, by using estimates for the transverse Ricci curvature in terms of the Sasaki-Futaki invariant, we shall give some gap theorems for compact gradient Sasaki-Ricci solitons by showing some necessary and sufficient conditions for the solitons to be Sasaki-Einstein. Our results may be regarded as a Sasaki geometry version of recent works by H. Li, and M. Fernández-López and E. García-Río.

1. Introduction

A *Sasaki manifold* is an odd dimensional Riemannian manifold (S, g) such that the associated cone manifold

$$(C(S), \bar{g}) := (\mathbb{R}_+ \times S, dr^2 + r^2 g)$$

is a Kähler manifold, where r is the standard coordinate on the set $\mathbb{R}_+ = \{r > 0\}$ of positive real numbers. This concept was introduced by Sasaki and Hatakeyama [26] as a special kind of contact manifolds and studied in 1960s-70s as an odd dimensional counterpart of the Kähler manifold. Recently, Sasaki-Einstein manifolds have been an attractive object, not only in mathematics but also in theoretical physics, since they play an important role in AdS/CFT correspondence stemming from superstring theory [21, 22]. To construct interesting Sasaki-Einstein metrics is one of the most important problems in Sasaki geometry. Boyer, Galicki and their collaborators [2, 3] produced many quasi-regular Sasaki-Einstein metrics. An irregular Sasaki-Einstein metric was first discovered by Gauntlett, Martelli, Sparks and Waldram [14, 15]. We refer the reader to the book [1] and the survey article [28] for recent developments of Sasaki-Einstein geometry.

In the present paper, we focus on the case where the basic first Chern class is positive definite. As with the Fano case, there are some obstructions to the existence of Sasaki-Einstein metrics. For instance, Futaki, Ono and Wang [12] extended the Futaki invariant [11] to Sasaki geometry, where a similar obstruction was studied by Boyer, Galicki and Simanca [4] in the case of canonical Sasaki metrics. As to the existence, Cho, Futaki and Ono [7] gave examples of toric Sasaki-Einstein metrics with suitable uniqueness structures. Nitta and Sekiya [25] proved a uniqueness theorem for Sasaki-Einstein metrics without toric assumptions. Recently, Mabuchi and Nakagawa [20] gave some examples of Sasaki-Einstein manifolds, including non-toric ones. Smoczyk, Wang and Zhang [27] introduced the Sasaki-Ricci flow, where they extended the results of the Kähler-Ricci flow by Cao [5] to compact Sasaki manifolds. Collins [9] and He [17] generalized independently the Perelman's functional to Sasaki geometry and proved uniformal bounds for the transverse

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scalar curvature and the transverse diameter along the Sasaki-Ricci flow. Futaki, Ono and Wang [12] defined a Sasaki-Ricci soliton as a counterpart of the Kähler-Ricci soliton and showed that there exists such a soliton on a suitable compact toric Sasaki manifold with positive basic first Chern class. As in the Kähler case [29], if the basic first Chern class is positive definite, one may expect that the Sasaki-Ricci flow will converge to a soliton under some suitable assumptions. Recently, He [17] proved such a result, if an initial metric has non-negative transverse holomorphic bisectional curvature.

In the present paper, we study a gradient Sasaki-Ricci soliton. A $(2n+1)$-dimensional Sasaki manifold (S,g) is a *gradient Sasaki-Ricci soliton* if there exists some basic function $f \in \mathcal{C}^\infty_B(S)$ on S, called a *potential function*, satisfying

$$\mathrm{Ric}^T + \mathrm{Hess}^T f = (2n+2)g^T, \tag{1.1}$$

where Ric^T and $\mathrm{Hess}^T f$ denote a *transverse Ricci curvature* and a *transverse Hessian* of f, respectively (see Section 2 below). If the potential function is constant, then the soliton appears as a Sasaki-Einstein manifold. In such a case, we say that the soliton is *trivial.*

The aim of the present paper is to give some gap theorems for compact gradient Sasaki-Ricci solitons by showing necessary and sufficient conditions for the solitons to be trivial. The same observations were made for the Ricci soliton on Kähler manifolds [18] and Riemannian manifolds [10]. We remark that all of our results hold both for quasi-regular and irregular cases. Our main theorem is the following:

Main Theorem (Theorem 3.4). *Let (S,g) be a $(2n+1)$-dimensional compact gradient Sasaki-Ricci soliton satisfying* (1.1). *Then (S,g) is Sasaki-Einstein if and only if*

$$\| \mathrm{Ric}^T - (2n+2)g^T \| \leqslant \frac{-n\mathcal{F} + \sqrt{n^2\mathcal{F}^2 + 4n(2n-1)(2n+2)\mathcal{F}}}{2(2n-1)},$$

where $\mathcal{F} = \frac{1}{\mathrm{vol}(S,g)} \int_S \|\nabla^T f\|^2$ is the Sasaki-Futaki invariant defined by (1.1).

Roughly speaking, the above result shows that if the transverse Ricci curvature of a compact gradient Sasaki-Ricci soliton is sufficiently close to that of a Sasaki-Einstein manifold, then the soliton must be trivial. Hence, this result gives us a gap phenomenon between Sasaki-Einstein manifolds and non-trivial gradient Sasaki-Ricci solitons.

This paper is organized as follows: In Section 2, by introducing notations, we summarize basic facts about transverse geometry on Sasaki manifolds. Ending with Section 3, a proof for the main theorem will be given.

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2. Sasaki Geometry

In this section, we give a brief review of transverse geometry on Sasaki manifolds. Throughout this paper, we assume that all manifolds are orientable, connected and have no boundary. We refer the reader to [1, 28] for an introduction to Sasaki geometry and [1, 30] for a brief review of theory of transverse geometry. We denote by ∇ the Levi-Civita connection for a Riemannian manifold (S,g) and $R(\cdot,\cdot)$ the curvature tensor for (S,g).

2.1. **Sasaki manifolds.**

Definition 2.1. A *Sasaki manifold* is an odd dimensional Riemannian manifold (S, g) such that the associated cone manifold

$$(C(S), \bar{g}) := (\mathbb{R}_+ \times S, dr^2 + r^2 g)$$

is a Kähler manifold, where r is the standard coordinate on the set $\mathbb{R}_+ = \{r > 0\}$ of positive real numbers.

Note that any Sasaki manifold (S, g) is naturally isometrically embedded in the Kähler cone $(C(S), \bar{g})$ via the inclusion $S \simeq \{r = 1\} \subset C(S)$. Throughout this paper, we identify (S, g) with the submanifold $\{r = 1\}$ of $(C(S), \bar{g})$ and set $\dim S = 2n + 1$. For a Sasaki manifold (S, g), we can define a *Reeb vector field* ξ on S and a *contact form* η on S by

$$\xi := \left(J \frac{\partial}{\partial r} \right)\bigg|_{r=1} \quad \text{and} \quad \eta := g(\xi, \cdot),$$

respectively. Here, J denotes the complex structure of the Kähler cone $(C(S), \bar{g})$. Then, we can see that

- ξ is a Killing vector field on S and satisfies $\mathcal{L}_\xi J = 0$,
- $\nabla_\xi \xi = 0$, i.e., the integral curve of ξ is a geodesic,
- $\eta(\xi) = 1$ and $i_\xi d\eta = 0$,
- $\eta \wedge (\frac{1}{2} d\eta)^n \neq 0$, in particular, $\eta \wedge (\frac{1}{2} d\eta)^n$ is a volume form on S.

The 1-dimensional foliation $\mathcal{F}_\xi$ generated by ξ is called a *Reeb foliation.* A Sasaki manifold is said to be *quasi-regular* if all leaves of $\mathcal{F}_\xi$ are compact and *irregular* otherwise. The η above induces a $2n$-dimensional subbundle D of the tangent bundle TS, called a *contact bundle*, where at each point $p \in S$ the fiber D_p of D is defined by

$$D_p := \operatorname{Ker} \eta_p.$$

Then, the tangent bundle TS admits the following orthogonal decomposition:

$$TS = D \oplus \mathbb{R}\xi, \tag{2.2}$$

where $\mathbb{R}\xi$ is a line bundle spanned by the Reeb vector field ξ. Next, we define an endomorphism Φ of the tangent bundle TS by setting $\Phi|_D = J|_D$ and $\Phi|_{\mathbb{R}\xi} = 0$. Φ satisfies

$$\Phi^2 = -\operatorname{id} + \eta \otimes \xi \quad \text{and} \quad g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for any $X, Y \in \Gamma(TS)$. We can see that Φ may also be defined by $\Phi(X) = \nabla_X \xi$. In view of (2.3), the pair $(g|_{D \times D}, \Phi|_D)$ defines a Hermite structure on D. Then, the Sasaki metric g is written as

$$g(X, Y) = \frac{1}{2} d\eta(X, \Phi(Y)) + \eta(X)\eta(Y) \tag{2.4}$$

for any $X, Y \in \Gamma(TS)$. We see that $\frac{1}{2}(d\eta)|_{D \times D}$ is a symplectic form on D associated to the Hermitian metric $g|_{D \times D}$. Moreover, D admits a transverse Kähler structure, as we will describe in the next subsection. We call the quadruple $\{g, \xi, \eta, \Phi\}$ a *Sasaki structure* of S. We conclude this subsection by summarizing well-known facts on Sasaki geometry that may be used as a definition of Sasaki manifolds.

- The associated cone manifold

$$(C(S), \bar{g}) := (\mathbb{R}_+ \times S, dr^2 + r^2 g)$$

of (S, g) is a Kähler manifold.

- There exists a unit Killing vector field ξ on S such that the $(1,1)$-type tensor field Φ as defined by $\Phi(X) = \nabla_X \xi$ satisfies

$$(\nabla_X \Phi)(Y) = \eta(Y)X - g(X,Y)\xi \tag{2.5}$$

for any $X, Y \in \Gamma(TS)$.
- There exists a unit Killing vector field ξ on S such that the curvature tensor satisfies

$$R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi$$

for any $X, Y \in \Gamma(TS)$.
- There exists a unit Killing vector field ξ on S such that any sectional curvature containing ξ equals one.

2.2. **Transverse geometry.** Let (S, g) be a $(2n+1)$-dimensional Sasaki manifold with a Sasaki structure $\{g, \xi, \eta, \Phi\}$. First, note that the Sasaki metric g is bundle-like with respect to the Reeb foliation $\mathcal{F}_\xi$, since ξ is a Killing vector field. We identify the normal bundle $\nu(\mathcal{F}_\xi)$ of $\mathcal{F}_\xi$ with D. Recall that the contact bundle D has the metric $g^T := g|_{D\times D}$ with the associated symplectic form $\frac{1}{2}(d\eta)|_{D\times D}$. We call g^T a *transverse metric*. To section $X \in \Gamma(TS), Y \in \Gamma(D)$, we associate a *transverse Levi-Civita connection* ∇^T by

$$\nabla^T_X Y := \begin{cases} \pi(\nabla_X Y) & \text{if } X \text{ is a section of } D, \\ \pi([X,Y]) & \text{if } X \text{ is a section of } \mathbb{R}\xi, \end{cases}$$

where $\pi : TS \to D$ is the natural projection to the first factor in (2.2). Note that ∇^T is a unique connection on D satisfying

$$\nabla^T_X Y - \nabla^T_Y X = \pi([X,Y]) \quad \text{and} \quad X g^T(Y,Z) = g^T(\nabla^T_X Y, Z) + g^T(Y, \nabla^T_X Z) \tag{2.6}$$

for any $X, Y, Z \in \Gamma(D)$. A *transverse curvature*, a *transverse Riemannian curvature*, a *transverse Ricci curvature* and a *transverse scalar curvature* are defined by

$$R^T(X,Y)Z := \nabla^T_X \nabla^T_Y Z - \nabla^T_Y \nabla^T_X Z - \nabla^T_{[X,Y]} Z, \quad \mathrm{Rm}^T(X,Y,Z,W) := g^T(R^T(X,Y)Z, W),$$

$$\mathrm{Ric}^T(X,Y) := \sum_{i=1}^{2n} \mathrm{Rm}^T(e_i, X, Y, e_i) \quad \text{and} \qquad R^T := \sum_{i=1}^{2n} \mathrm{Ric}^T(e_i, e_i),$$

respectively. Here, $\{e_i\}_{i=1}^{2n}$ is an orthonormal basis of D. By a direct calculation, we easily see that the first two curvatures satisfy the following identities:

Proposition 2.7 ([1])**.** *For any $X, Y, Z, W \in \Gamma(D)$,*
(1) $R^T(X,Y)Z + R^T(Y,Z)X + R^T(Z,X)Y = 0$,
(2) $\mathrm{Rm}^T(Y,X,Z,W) = -\mathrm{Rm}^T(X,Y,Z,W)$, $\mathrm{Rm}^T(X,Y,W,Z) = -\mathrm{Rm}^T(X,Y,Z,W)$,
(3) $\mathrm{Rm}^T(X,Y,Z,W) = \mathrm{Rm}^T(Z,W,X,Y)$,
(4) $(\nabla^T_X R^T)(Y,Z)W + (\nabla^T_Y R^T)(Z,X)W + (\nabla^T_Z R^T)(X,Y)W = 0$.

We call (1) and (4) above *the first* and *the second transverse Bianchi identity*, respectively. Moreover, Sasaki manifolds may admit the following Myers type theorem:

Theorem 2.8 (Hasegawa-Seino [16], Nitta [24])**.** *Let (S, g) be a $(2n+1)$-dimensional complete Sasaki manifold. If $\mathrm{Ric}^T \geqslant K g^T$ for some positive constant $K > 0$, then (S, g) is compact and the fundamental group of S is finite. Moreover,*

$$\mathrm{diam}(S, g) \leqslant 2\pi \sqrt{\frac{2n-1}{K}}. \tag{2.9}$$

Furthermore, from (2.5) and (2.6), we obtain

$$\nabla^T \Phi = 0.$$

Hence, the triple $(D, g^T, \Phi|_D)$ gives a *transverse Kähler structure* [23] for the Reeb foliation $\mathcal{F}_\xi$ with the transverse Käher form $\frac{1}{2}(d\eta)|_{D\times D}$.

2.3. **Transverse Hodge theory.** Let (S, g) be a $(2n+1)$-dimensional Sasaki manifold with a Sasaki structure $\{g, \xi, \eta, \Phi\}$. In this subsection, we suppose that (S, g) is compact unless otherwise specified.

Definition 2.10. A p-form $\omega \in \Omega^p(S)$ on S is called *basic* if

$$i_\xi \omega = 0 \quad \text{and} \quad \mathcal{L}_\xi \omega = 0,$$

where i_ξ and $\mathcal{L}_\xi$ denote the inner product and the Lie derivative given by ξ, respectively. Let Λ^p_B be the sheaf of germs of basic p-forms on S and $\Omega^p_B(S) = \mathit{\Gamma}(S, \Lambda^p_B)$ the set of all global sections of Λ^p_B. A smooth function $f \in \mathcal{C}^\infty(S)$ on S is called *basic* if and only if $\xi f = 0$. We denote by $\mathcal{C}^\infty_B(S)$ the set of all basic functions on S.

It is clear that the exterior derivative $d : \Lambda^p \to \Lambda^{p+1}$ preserves basic forms and induces a well-defined operator $d_B : \Lambda^p_B \to \Lambda^{p+1}_B$. As in Riemannian geometry, we obtain the following *basic de Rham complex*:

$$0 \longrightarrow \mathcal{C}^\infty_B(S) \xrightarrow{d_B} \Omega^1_B(S) \xrightarrow{d_B} \cdots \xrightarrow{d_B} \Omega^{2n}_B(S) \xrightarrow{d_B} 0.$$

Let $\delta_B : \Lambda^{p+1}_B \to \Lambda^p_B$ be the adjoint operator of $d_B : \Lambda^p_B \to \Lambda^{p+1}_B$. A *basic Laplacian* Δ_B on forms is defined by

$$-\Delta_B := d_B\delta_B + \delta_B d_B.$$

We can see that the basic Laplacian on functions coincides with the restriction of the ordinary Laplacian on the space of basic functions. We can check that

$$\Delta|_{\mathcal{C}^\infty_B(S)} = \Delta_B = (g^T)^{ij}\nabla_i\nabla_j. \tag{2.11}$$

Next, we consider the complexified bundle $D \otimes \mathbb{C}$. The complex structure $\Phi|_D$ on D induces the decomposition of the bundle $D \otimes \mathbb{C}$ into two subbundles:

$$D \otimes \mathbb{C} = D^{1,0} \oplus D^{0,1},$$

where

$D^{1,0} := \{X \in D \otimes \mathbb{C} : \Phi(X) = \sqrt{-1}X\}$ and $D^{0,1} := \{X \in D \otimes \mathbb{C} : \Phi(X) = -\sqrt{-1}X\}$.

According to this decomposition, the set of all basic forms on S splits as

$$\Lambda^r_B \otimes \mathbb{C} = \bigoplus_{p+q=r} \Lambda^{p,q}_B,$$

where $\Lambda^{p,q}_B$ denotes the sheaf of germs of basic forms of type (p, q). Then we can define operators

$$\partial_B : \Lambda^{p,q}_B \to \Lambda^{p+1,q}_B \quad \text{and} \quad \bar\partial_B : \Lambda^{p,q}_B \to \Lambda^{p,q+1}_B$$

satisfying $d_B = \partial_B + \bar\partial_B$. We further consider the form defined by $\rho^T(X, Y) = \mathrm{Ric}^T(\Phi(X), Y)$ for any $X, Y \in \mathit{\Gamma}(D)$. As in the Kähler case, we can see that

$$\rho^T = -\sqrt{-1}\partial_B\bar\partial_B \log\det(g^T),$$

and hence, ρ^T defines a basic cohomology class $[\frac{1}{2\pi}\rho^T]_B$. This class is called a *basic first Chern class* and denoted by $c^B_1(S)$. We say that the basic first Chern class $c^B_1(S)$ is *positive*, *null* and *negative* if it contains a positive, a null and a negative representation, respectively. We denote these conditions by $c_1^B(S) > 0, c_1^B(S) = 0$ and $c_1^B(S) < 0$, respectively. A transverse metric g^T is called a *transverse Kähler-Einstein metric* if

$$\mathrm{Ric}^T = \tau g^T$$

for some constant τ.

Definition 2.12. A $(2n+1)$-dimensional Sasaki manifold (S, g) is called a *Sasaki-Einstein manifold* if the Ricci curvature satisfies $\mathrm{Ric} = 2ng$.

The Einstein condition of Sasaki manifolds can be translated into those of the Käher cone and the transverse Käher structure:

Proposition 2.13 ([1]). *Let (S, g) be a $(2n+1)$-dimensional Sasaki manifold. Then the following three conditions are equivalent:*
- *The Sasaki manifold (S, g) is a Sasaki-Einstein manifold,*
- *The cone manifold $(C(S), \bar{g})$ of (S, g) is a Calabi-Yau manifold, i.e.,* $\mathrm{Ric}_{\bar{g}} = 0$,
- *$(D, g^T, \Phi|_D)$ satisfies the transverse Kähler-Einstein equation* $\mathrm{Ric}^T = (2n+2)g^T$.

Note that by definition, any Sasaki-Einstein manifold is necessarily Ricci positive and has positive basic first Chern class. It is known that there is a further necessary condition for the existence of transverse Kähler-Einstein metrics:

Proposition 2.14 (Futaki-Ono-Wang [12]). *The basic first Chern class is represented by $\tau d\eta$ for some constant τ if and only if $c_1(D) = 0$.*

A *transverse gradient vector field* $\nabla^T f$ of a basic function $f \in \mathcal{C}_B^\infty(S)$ is defined by

$$g^T(\nabla^T f, X) := d_B f(X), \quad X \in \Gamma(D). \tag{2.15}$$

A *transverse Hessian* $\mathrm{Hess}^T f(\cdot,\cdot)$ of a basic function $f \in \mathcal{C}_B^\infty(S)$ is defined by

$$\mathrm{Hess}^T(X, Y) := g^T(\nabla_X^T \nabla^T f, Y), \quad X, Y \in \Gamma(D). \tag{2.16}$$

Then, as in the Riemannian case, we have the following *transverse Bochner formula*:

$$\frac{1}{2}\Delta_B \|\nabla^T f\|^2 = \|\mathrm{Hess}^T f\|^2 + g^T(\nabla^T f, \nabla^T \Delta_B f) + \mathrm{Ric}^T(\nabla^T f, \nabla^T f). \tag{2.17}$$

2.4. **Eigenvalue problems.** Let (S, g) be a compact $(2n+1)$-dimensional Sasaki manifold. We here consider the eigenvalue problem for the basic Laplacian on compact Sasaki manifolds. We refer the reader to [6, 8] for basic facts about the ordinary eigenvalue problem. The following description can be found in the book [30]:

Definition 2.18. A non-zero basic function $0 \neq f \in \mathcal{C}_B^\infty(S)$ is called an *eigenfunction* for the basic Laplacian Δ_B associated with an *eigenvalue* $\lambda \in \mathbb{R}$ if

$$\Delta_B f + \lambda f = 0. \tag{2.19}$$

Theorem 2.20. *The eigenvalue is non-negative and discrete. Moreover, by counting multiplicities, the positive eigenvalue can be arranged such that*

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots .$$

Furthermore, there exists a complete orthonormal basis for $L_B^2(\mathcal{C}_B^\infty(S))$ consisting of smooth eigenfunctions of the basic Laplacian.

Proposition 2.21. *The first non-zero eigenvalue* λ_1 *satisfies*

$$\lambda_1 \int_S \|\nabla^T f\|^2 \leqslant \int_S (\Delta_B f)^2 \tag{2.22}$$

for any basic function $f \in \mathcal{C}^\infty_B(S)$ *on* S*. Moreover, the equality is attained if and only if* $f \in \mathcal{C}^\infty_B(S)$ *is an eigenfunction of the basic Laplacian associated with* λ_1*.*

Since Sasaki manifolds are transversally Riemannian, by using the same way as in [19], we can obtain the following transverse version of the Lichnerowicz estimate:

Lemma 2.23. *Let* λ *be an eigenvalue of the basic Laplacian satisfying* (2.19)*. If the transverse Ricci curvature has the lower bound* $\mathrm{Ric}^T \geqslant K g^T$ *for some constant* $K \geqslant 0$*, then* λ *has the lower bound*

$$\lambda \geqslant \frac{2n}{2n-1} K.$$

2.5. **Gradient Sasaki-Ricci solitons.** Let (S, g) be a $(2n+1)$-dimensional Sasaki manifold with a Sasaki structure $\{g, \xi, \eta, \Phi\}$. We here introduce a gradient Sasaki-Ricci soliton and provide some formulas that are useful to prove the main theorem. We refer the reader to [8, 13] for basic calculations on the Ricci soliton of Riemannian manifolds.

Definition 2.24. A $(2n+1)$-dimensional Sasaki manifold (S, g) is a *gradient Sasaki-Ricci soliton* if there exists a basic function $f \in \mathcal{C}^\infty_B(S)$ satisfying

$$\mathrm{Ric}^T + \mathrm{Hess}^T f = (2n+2) g^T. \tag{2.25}$$

The function $f \in \mathcal{C}^\infty_B(S)$ is referred to as a *potential function.* If the potential function is constant, then the gradient Sasaki-Ricci soliton is a Sasaki-Einstein manifold. In such a case, we say that the soliton is *trivial.*

Remark 2.26. Any compact gradient Sasaki-Ricci soliton (S, g) has the positive basic first Chern class and satisfies $c_1(D) = 0$. Then, the Sasaki-Futaki invariant $\mathcal{F}$ [12] is given by

$$\mathcal{F} := \frac{1}{\mathrm{vol}(S, g)} \int_S \|\nabla^T f\|^2.$$

Of course, if the Sasaki-Futaki invariant vanishes, then the soliton must be trivial.

By taking the trace of both sides of the equation (2.25), we have

$$R^T + \Delta_B f = 2n(2n+2). \tag{2.27}$$

Recall from Proposition 2.7 that any Sasaki manifold admits the transverse Bianchi identity. Hence, as in the Riemannian Ricci soliton [13, (8)], from (2.25) and the contracted transverse second Bianchi identity, we obtain

$$g^T(\nabla^T R^T, \nabla^T f) = 2\,\mathrm{Ric}^T(\nabla^T f, \nabla^T f). \tag{2.28}$$

Since the transverse scalar curvature R^T and the norm of the gradient vector field $\|\nabla^T f\|$ are basic functions, in view of [13, (9)], we have

$$R^T - 2(2n+2) f + \|\nabla^T f\|^2 = C \tag{2.29}$$

for some real constant C. Then, by taking the difference of (2.27) and (2.29), we obtain

$$\Delta_B f + 2(2n+2) f - \|\nabla^T f\|^2 = C', \tag{2.30}$$

where $C' := 2n(2n+2) - C$. The following lemma plays a crucial role in this paper:

Lemma 2.31. *Let (S, g) be a $(2n+1)$-dimensional compact gradient Sasaki-Ricci soliton satisfying* (2.25). *Then the following holds:*

$$\int_S (\Delta_B f)^2 = 2\int_S \|\mathrm{Hess}^T f\|^2 = 2\int_S \mathrm{Ric}^T(\nabla^T f, \nabla^T f), \tag{2.32}$$

$$\int_S (\Delta_B f)^2 = \int_S ((2n+2)^2 - R^T)\|\nabla^T f\|^2, \ \textit{and} \tag{2.33}$$

$$\|\nabla^T f\|^2 \leqslant R^T_{\max} - R^T, \tag{2.34}$$

where $R^T_{\max}$ denotes the maximum value of the transverse scalar curvature on the soliton.

Proof. First, we have

$$\begin{aligned}
\frac{1}{2}\Delta_B\|\nabla^T f\|^2 &= \|\mathrm{Hess}^T f\|^2 + g^T(\nabla^T f, \nabla^T \Delta_B f) + \mathrm{Ric}^T(\nabla^T f, \nabla^T f) && \text{(cf. (2.17))}\\
&= \|\mathrm{Hess}^T f\|^2 - g^T(\nabla^T f, \nabla^T R^T) + \mathrm{Ric}^T(\nabla^T f, \nabla^T f) && \text{(cf. (2.27))}\\
&= \|\mathrm{Hess}^T f\|^2 - \mathrm{Ric}^T(\nabla^T f, \nabla^T f). && \text{(cf. (2.28))}
\end{aligned}$$

By integrating both sides of the last equality and by using (2.11), we have

$$\int_S \|\mathrm{Hess}^T f\|^2 = \int_S \mathrm{Ric}^T(\nabla^T f, \nabla^T f).$$

Then

$$\begin{aligned}
\int_S (\Delta_B f)^2 &= -\int_S R^T \Delta_B f && \text{(cf. (2.27))}\\
&= \int_S g^T(\nabla^T R^T, \nabla^T f) = 2\int_S \mathrm{Ric}^T(\nabla^T f, \nabla^T f), && \text{(cf. (2.28))}
\end{aligned}$$

which proves (2.32). Secondly, we have

$$\begin{aligned}
\int_S (\Delta_B f)^2 &= 2\int_S \mathrm{Ric}^T(\nabla^T f, \nabla^T f) && \text{(cf. (2.32))}\\
&= 2(2n+2)\int_S \|\nabla^T f\|^2 - 2\int_S \mathrm{Hess}^T f(\nabla^T f, \nabla^T f) && \text{(cf. (2.25))}\\
&= 2(2n+2)\int_S \|\nabla^T f\|^2 - \int_S g^T(\nabla^T f, \nabla^T\|\nabla^T f\|^2)\\
&= 2(2n+2)\int_S \|\nabla^T f\|^2 + \int_S \|\nabla^T f\|^2 \Delta_B f\\
&= (2n+2)^2\int_S \|\nabla^T f\|^2 - \int_S R^T\|\nabla^T f\|^2 && \text{(cf. (2.27))}
\end{aligned}$$

and (2.33) follows. Here, in the last third equality above, we have used the second property of the transverse Levi-Civita connection in (2.6). Finally, in order to prove (2.34), recall from (2.29) that $2(2n+2)f = R^T + \|\nabla^T f\|^2 - C$ for some real constant C. By compactness of the manifold S, there exists some global maximum point $p \in S$ of the potential function. Then, it follows from (2.29) that for any point $x \in S$,

$$2(2n+2)f(p) = R^T(p) - C \geqslant 2(2n+2)f(x) = R^T(x) + \|\nabla^T f\|^2(x) - C,$$

and hence, $R^T(p) \geqslant R^T(x)$. Therefore, the transverse scalar curvature also attains its maximum at p, and we obtain (2.34). $\square$

3. Gap Theorems

In this section, we extend gap theorems for compact Kähler-Ricci solitons [18] and for compact gradient Ricci solitons [10] to the case of compact gradient Sasaki-Ricci solitons. First, note that if a $(2n+1)$-dimensional gradient Sasaki-Ricci soliton (S, g) is trivial, then the transverse scalar curvature on (S, g) satisfies

$$R^T = 2n(2n+2), \quad \text{therefore,} \quad R^T_{\max} = 2n(2n+2).$$

The following result characterizes triviality for a compact gradient Sasaki-Ricci soliton by using an upper bound for $R^T_{\max} - 2n(2n+2)$ in terms of the Sasaki-Futaki invariant:

Theorem 3.1. *Let (S, g) be a $(2n+1)$-dimensional compact gradient Sasaki-Ricci soliton satisfying* (2.25)*. Then (S, g) is Sasaki-Einstein if and only if*

$$R^T_{\max} - 2n(2n+2) \leqslant \left(1 + \frac{1}{n}\right) \mathcal{F},$$

where $\mathcal{F} = \frac{1}{\mathrm{vol}(S,g)} \int_S \|\nabla^T f\|^2$ is the Sasaki-Futaki invariant defined by (2.25)*.*

Proof. The result is obvious if the soliton is trivial, since in such a case the potential function is constant. Conversely, we have

$$\begin{aligned}
\int_S (\Delta_B f)^2 &= (2n+2)^2 \int_S \|\nabla^T f\|^2 - \int_S R^T \|\nabla^T f\|^2 && \text{(cf. (2.33))} \\
&\geqslant (2n+2)^2 \int_S \|\nabla^T f\|^2 - R^T_{\max} \int_S R^T + \int_S (R^T)^2 && \text{(cf. (2.34))} \\
&= (2n+2)^2 \int_S \|\nabla^T f\|^2 - 2n(2n+2) R^T_{\max} \mathrm{vol}(S,g) && \text{(cf. (2.27))} \\
&\quad + 4n^2(2n+2)^2 \mathrm{vol}(S,g) + \int_S (\Delta_B f)^2,
\end{aligned}$$

which yields

$$R^T_{\max} - 2n(2n+2) \geqslant \left(1 + \frac{1}{n}\right) \mathcal{F}.$$

Hence, by the assumption in the theorem, the equality just above must be achieved. This shows that the equality in (2.34) must also attain. Therefore, we have

$$2(2n+2)f - R^T + C = \|\nabla^T f\|^2 = R^T_{\max} - R^T,$$

equivalently, $2(2n+2)f = R^T_{\max} - C$. Hence, the potential function is constant and the soliton is trivial. □

In view of (2.25) and (2.32), on any compact gradient Sasaki-Ricci soliton (S, g), we have

$$\int_S \| \mathrm{Ric}^T - (2n+2) g^T \|^2 = \int_S \mathrm{Ric}^T (\nabla^T f, \nabla^T f).$$

Hence, if $\int_S \mathrm{Ric}^T(\nabla^T f, \nabla^T f) \leqslant 0$, then the soliton must be trivial. Therefore, the quantity $\int_S \mathrm{Ric}^T(\nabla^T f, \nabla^T f)$ measures the difference of the soliton from being Sasaki-Einstein. The following result characterizes triviality of the soliton by giving an upper bound of this quantity:

Theorem 3.2. *Let (S, g) be a $(2n+1)$-dimensional compact gradient Sasaki-Ricci soliton satisfying (2.25). Then (S, g) is Sasaki-Einstein if and only if*

$$\int_S \mathrm{Ric}^T(\nabla^T f, \nabla^T f) \leqslant \frac{\lambda_1}{2} \int_S \|\nabla^T f\|^2,$$

where λ_1 denotes the first non-zero eigenvalue of the basic Laplacian.

Proof. The result is obvious if the soliton is trivial. To prove that the soliton is trivial, note that any basic function $f \in \mathcal{C}_B^\infty(S)$ satisfies (2.22). By (2.32) and the assumption in the theorem, we see that the potential function f satisfies

$$\lambda_1 \int_S \|\nabla^T f\|^2 = \int_S (\Delta_B f)^2, \tag{3.3}$$

and hence, the function f is an eigenfunction of the basic Laplacian associated with λ_1. Then, it follows from (2.30) that $(2(2n+2) - \lambda_1)f = \|\nabla^T f\|^2 + C'$. In the case that $2(2n+2) - \lambda_1 \neq 0$, since $\nabla^T f$ vanishes at any local extrema of f, we have

$$f_{\max} = f_{\min} = \frac{C'}{2(2n+2) - \lambda_1},$$

which shows that the potential function is constant and the soliton is trivial. In the case that $2(2n+2) - \lambda_1 = 0$, since we have $0 = \|\nabla^T f\|^2 + C'$, the same argument as in the previous case allows us to obtain $C' = 0$, which shows that the potential function is also constant and the soliton is trivial. □

The following result shows that if the transverse Ricci curvature of a compact gradient Sasaki-Ricci soliton is sufficiently close to that of a Sasaki-Einstein manifold, then the soliton must be trivial. See [18, 10] for the same gap theorems on compact Kähler-Ricci solitons and on compact Riemannian Ricci solitons, respectively.

Theorem 3.4. *Let (S, g) be a $(2n+1)$-dimensional compact gradient Sasaki-Ricci soliton satisfying (2.25). Then (S, g) is Sasaki-Einstein if and only if*

$$\|\mathrm{Ric}^T - (2n+2)g^T\| \leqslant \frac{-n\mathcal{F} + \sqrt{n^2\mathcal{F}^2 + 4n(2n-1)(2n+2)\mathcal{F}}}{2(2n-1)},$$

where $\mathcal{F} = \frac{1}{\mathrm{vol}(S,g)} \int_S \|\nabla^T f\|^2$ is the Sasaki-Futaki invariant defined by (2.25).

Proof. The result is obvious if the soliton is trivial. For simplicity, put

$$c := \frac{-n\mathcal{F} + \sqrt{n^2\mathcal{F}^2 + 4n(2n-1)(2n+2)\mathcal{F}}}{2(2n-1)}.$$

We easily see that $2n+2 \geqslant c$. To prove that the soliton is trivial, note that the transverse Ricci curvature satisfies $(2n+2-c)g^T \leqslant \mathrm{Ric}^T \leqslant (2n+2+c)g^T$. Then, by Lemma 2.23, we see that the first eigenvalue of the basic Laplacian has the lower bound $\lambda_1 \geqslant \frac{2n}{2n-1}(2n+2-c)$. It follows from the assumption in the theorem that $\|\mathrm{Hess}^T f\|^2 \leqslant c^2$. Hence, we have

$$\begin{aligned} c^2 &\geqslant \frac{1}{\mathrm{vol}(S,g)} \int_S \|\mathrm{Hess}^T f\|^2 = \frac{1}{2} \cdot \frac{1}{\mathrm{vol}(S,g)} \int_S (\Delta_B f)^2 && \text{(cf. (2.32))} \\ &\geqslant \frac{1}{2} \cdot \frac{\lambda_1}{\mathrm{vol}(S,g)} \int_S \|\nabla^T f\|^2 \geqslant \frac{1}{2} \cdot \frac{2n}{2n-1}(2n+2-c) \cdot \mathcal{F}. && \text{(cf. (2.22))} \end{aligned} \tag{3.5}$$

On the other hand, by definition of c, we have

$$c^2 = \frac{1}{2} \cdot \frac{2n}{2n-1}(2n+2-c) \cdot \mathcal{F},$$

which shows the third equality in (3.5) must be achieved. Hence, as we have seen in the previous theorem, the function f is an eigenfunction of the basic Laplacian associated with λ_1, and hence, the potential function is constant and the soliton is trivial. $\square$

We say that a compact gradient Sasaki-Ricci soliton (S, g) is *normalized* if its potential function $f \in \mathcal{C}_B^\infty(S)$ satisfies

$$\int_S f = 0. \tag{3.6}$$

The following gap theorem is regarded as a Sasaki geometry version of that in [18]:

Theorem 3.7. *Let (S, g) be a $(2n+1)$-dimensional compact gradient Sasaki-Ricci soliton satisfying (2.25). Suppose that the soliton is normalized in sense of (3.6). Then, there exists a non-negative constant $\delta \ll 1$ such that if*

$$\mathrm{Ric}^T \geqslant (2n+2-\delta)g^T,$$

then (S, g) is Sasaki-Einstein.

Remark 3.8. In Theorem 3.7 above, $\delta \geqslant 0$ depends only on n and the Sasaki-Futaki invariant $\mathcal{F} = \frac{1}{\mathrm{vol}(S,g)} \int_S \|\nabla^T f\|^2$ defined by (2.25). Moreover, the proof shows that δ can be expressed explicitly in terms of n and the Sasaki-Futaki invariant $\mathcal{F}$.

Proof. We here assume that the soliton is non-trivial. By (2.30) and the normalization (3.6), we have

$$\Delta_B f + 2(2n+2)f - \|\nabla^T f\|^2 = -\mathcal{F}. \tag{3.9}$$

We assume that $\mathrm{Ric}^T \geqslant K g^T$ for some positive constant $K > 0$. Then, the transverse scalar curvature satisfies $R^T \geqslant 2nK$ and by Theorem 2.8, the diameter of (S, g) has the upper bound (2.9).

Lemma 3.10. *The transverse scalar curvature is uniformly bounded from above, i.e.,*

$$R^T < \Lambda(n, K, \mathcal{F}), \tag{3.11}$$

where $\Lambda = \Lambda(n, K, \mathcal{F})$ is a constant depending only on the numbers n, K and $\mathcal{F}$. Moreover,

$$\lim_{K \to 2n+2-0} \Lambda(n, K, \mathcal{F}) < +\infty.$$

Proof of Lemma 3.10. We first observe that

$$\begin{aligned}
\|\nabla^T f\|^2 &= \Delta_B f + 2(2n+2)f + \mathcal{F} && (\text{cf. } (3.9)) \\
&= 2n(2n+2) - R^T + 2(2n+2)f + \mathcal{F} && (\text{cf. } (2.27)) \\
&\leqslant 2n(2n+2) - 2nK + 2(2n+2)f + \mathcal{F}.
\end{aligned}$$

Put $B := 2n(2n+2) - 2nK + \mathcal{F} + 1$. Note that B is constant. Then, it follows from the last inequality just above that $2(2n+2)f + B \geqslant 1$, and hence,

$$\frac{\|\nabla^T f\|^2}{2(2n+2)f + B} \leqslant 1 - \frac{1}{2(2n+2)f + B} < 1. \tag{3.12}$$

By compactness of the manifold S, there exists some global minimum point $q \in S$ of the potential function. By the normalized condition (3.6), we see that $f(q) \leqslant 0$. Therefore, for any point $x \in S$,

$$
\begin{aligned}
\sqrt{2(2n+2)f+B}(x) &- \sqrt{2(2n+2)f+B}(q) \\
&\leqslant \left(\max_S \left\| \nabla^T \sqrt{2(2n+2)f+B} \right\| \right) \cdot \operatorname{diam}(S,g) \\
&= (2n+2) \cdot \left(\max_S \frac{\|\nabla^T f\|}{\sqrt{2(2n+2)f+B}} \right) \cdot \operatorname{diam}(S,g) \\
&< (2n+2) \cdot 2\pi \sqrt{\frac{2n-1}{K}}. \qquad \text{(cf. (2.9) and (3.12))}
\end{aligned}
$$

For simplicity, we put $c_n := 2\pi(2n+2)\sqrt{2n-1}$. Then, it follows from the above that

$$
2(2n+2)f(x) < \frac{(c_n)^2}{K} + 2c_n\sqrt{\frac{B}{K}}. \tag{3.13}
$$

Therefore,

$$
\begin{aligned}
R^T &= 2n(2n+2) - \Delta_B f & \text{(cf. (2.27))} \\
&= 2n(2n+2) + 2(2n+2)f - \|\nabla^T f\|^2 + \mathcal{F} & \text{(cf. (3.9))} \\
&< 2n(2n+2) + \frac{(c_n)^2}{K} + 2c_n\sqrt{\frac{B}{K}} + \mathcal{F}. & \text{(cf. (3.13))}
\end{aligned}
$$

Hence, we can take the last number just above as $\Lambda(n, K, \mathcal{F})$. $\square$

Now, we can finish the proof of Theorem 3.7. For simplicity, we put

$$
\Omega_+ := \{x \in S : R^T(x) > 2n(2n+2)\} \quad \text{and} \quad \Omega_- := \{x \in S : R^T(x) < 2n(2n+2)\},
$$

respectively. Then, we have

$$
\begin{aligned}
&\frac{1}{\operatorname{vol}(S,g)} \int_S \|\operatorname{Hess}^T f\|^2 \\
&= \frac{1}{2} \cdot \frac{1}{\operatorname{vol}(S,g)} \int_S (\Delta_B f)^2 = \frac{1}{2} \cdot \frac{1}{\operatorname{vol}(S,g)} \int_S (R^T - 2n(2n+2))^2 \quad \text{(cf. (2.32) and (2.27))} \\
&= \frac{1}{2} \cdot \frac{1}{\operatorname{vol}(S,g)} \int_{\Omega_+} (R^T - 2n(2n+2))^2 + \frac{1}{2} \cdot \frac{1}{\operatorname{vol}(S,g)} \int_{\Omega_-} (R^T - 2n(2n+2))^2 \\
&< \frac{1}{2} \cdot \frac{1}{\operatorname{vol}(S,g)} (\Lambda - 2n(2n+2)) \int_{\Omega_+} (R^T - 2n(2n+2)) + \frac{1}{2} \cdot 4n^2 \cdot (2n+2-K)^2.
\end{aligned}
$$

On the other hand, by integrating both sides of (2.27) and by using (2.11), we have

$$
0 = \int_{\Omega_+} (R^T - 2n(2n+2)) + \int_{\Omega_-} (R^T - 2n(2n+2)).
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{\operatorname{vol}(S,g)} \int_S \|\operatorname{Hess}^T f\|^2 \\
&< \frac{1}{2} \cdot \frac{1}{\operatorname{vol}(S,g)} (\Lambda - 2n(2n+2)) \int_{\Omega_-} (2n(2n+2) - R^T) + \frac{1}{2} \cdot 4n^2 \cdot (2n+2-K)^2 \\
&\leqslant \frac{1}{2} \cdot (\Lambda - 2n(2n+2)) \cdot 2n \cdot (2n+2-K) + \frac{1}{2} \cdot 4n^2 \cdot (2n+2-K)^2,
\end{aligned}
$$

and hence,

$$\frac{1}{\mathrm{vol}(S,g)}\int_S \|\operatorname{Hess}^T f\|^2 \to 0 \quad \text{as} \quad K \to 2n+2-0. \tag{3.14}$$

However, we have

$$\begin{aligned}\frac{1}{\mathrm{vol}(S,g)}\int_S \|\operatorname{Hess}^T f\|^2 &= \frac{1}{\mathrm{vol}(S,g)}\int_S \mathrm{Ric}^T(\nabla^T f, \nabla^T f) \quad (\text{cf. } (2.32))\\ &\geqslant K\mathcal{F} > 0,\end{aligned}$$

which contradicts (3.14) when K is sufficiently close to $2n+2$. This proves Theorem 3.7. □

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