Doctor Thesis

Superconformal index on $\mathbb{RP}^2 \times S^1$ and 3d mirror symmetry

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Abstract

In three dimensional quantum field theory, it is known that there is a non-trivial duality called 3d mirror symmetry. To check the duality, the so-called superconformal index is known to be a powerful tool. It was originally defined by the field theory on $\mathbb{S}^2 \times S^1$, and computed by using supersymmetric localization technique. In this thesis, we derive new formulas for the superconformal index on $\mathbb{RP}^2 \times S^1$ by introducing supersymmetric $\mathbb{Z}_2$ parity conditions on $\mathbb{S}^2 \times S^1$. The parity transformation causes non-trivial effects and the final formula becomes different from the known superconformal index. We also apply our result to the check of the 3d mirror symmetry, and give a new evidence for the duality by using quantum binomial theorem.

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1 Introduction and Summary

Quantum Field Theory (QFT) has been a useful and fundamental tool for studying physics of large degrees of freedom like particle physics or condensed matter physics. In particle physics, the theory is described by a Lagrangian with the *Poincaré symmetry* generated by translations and rotations in order to make it compatible with special relativity. (See [1] for a good explanation.) One of the generalizations of the Poincaré symmetry, supersymmetry (SUSY), was discovered in 1971 in the context of string theory [2, 3, 4]. After that, it was applied to the usual QFT in [5, 6]. The study of SUSY gauge theories has been providing many interesting results including various non-perturbative effects and un-expected relationships with mathematics since 1990's [7, 8, 9]. SUSY has a generator $\hat{Q}$ with the fermionic statistics. One can show that the SUSY algebra is a unique extension of the Poincaré algebra under the existence of a non-trivial S-matrix [10]. If we loosen this condition, there is another extension of the Poincaré symmetry. This is called *Conformal symmetry* generated by translations, rotations, dilatation and conformal boosts. The Conformal symmetry naturally emarges in the study of IR fixed points for *renormalization group* [11]. Around each IR fixed point, there is no scale, and this scale invariance enhances to the Conformal symmetry in many cases. See for example [12]. Once we start with supersymmetric UV Lagrangian and flow the renormalization group with preserving supersymmetry, the symmetry of the IR theory is expected to enhance to *Superconformal symmetry*. The possible superconformal algebras are classified in [13], and according to it, we can define superconformal theories only within $(2,)\ 3, 4, 5, 6$ dimensions. 2d is in a special case because the algebra becomes infinite dimensional one. 3d is the lowest dimension with the *finite dimensional* superconformal algebra, and we focus on the 3d SUSY QFTs from now on.

SUSY QFTs in 3d have many interesting features. Our main interest is a non-trivial dynamics of $U(1)$ gauge theory in 3d, called *three-dimensional mirror symmetry*. It is originally proposed in [14] with $\mathcal{N} = 4$ SUSY QFT, and after that in [15] with $\mathcal{N} = 2$ SUSY case. The simplest case for the duality is an equivalence between the moduli space[1] for *Supersymmetric Quantum ElectroDynamics* (SQED), and the moduli space for a SUSY matter theory called *XYZ-model*. Branches for the moduli space of SQED, so-called Coulomb branches are deformed by the quantum effect [16] but the conjectured dual moduli space branches, called Higgs branches are not because of the

[1] It corresponds to the space of possible vacuum expectation values.

non-renormalization theorem [17, 18]. In other words, the quantum effect in one side is realized by the classical effect in the other side, so it means the full quantum effects is inevitable for the duality. This proposal is reformulated in the context of the string theory [19, 20], and 3d mirror symmetry was explained as one of the consequences of the $SL(2,\mathbb{Z})$ duality in type IIB superstring theory. In addition to it, this proposal has been checked by utilizing the parity anomaly [15]. It is an analog of 't Hooft anomaly matching condition in 4d duality [7].

Of course, these results are quite non-trivial and guarantee the validity of the proposal of 3d mirror symmetry. However, it is disireble to establish more straightforward checks including full quantum calculation. For example, the following equality is expected naively.

$$Z_{\text{XYZ}} = Z_{\text{SQED}}, \tag{1.1}$$

where Z represents the partition function for each theory. At a first glance, the exact check for (1.1) looks very hard because of the existence of the interaction. Recently, however, so-called *supersymmetric localization techniques* have been developed within 2,3,4,5 dimensional SUSY QFTs[2]. It provides us a way to perform exact path integral calculations even there are interactions. One of the interesting features for these developments is that the techniques can be applied to the theories on a *curved space*. The curved space, called *manifold* in mathematics, is not arbitrary because we have to guarantee the existence of SUSY and it exists if and only if the manifold has a simple structure. In 3d, the structure has been identified to so-called *almost integrable contact structure* [40], and the exact calculations were performed on manifolds with such a structure, product space $\mathbb{S}^2 \times S^1$ [28, 41, 42, 43], $\mathbb{D}^2 \times S^1$ [30, 33], three sphere $\mathbb{S}^3$ [27, 44, 45, 46, 47, 48, 49, 50, 51] and its orbifold $\mathbb{S}^3/\mathbb{Z}_p$ [32]. In each case, the equality (1.1) has been verified by using mathematically rigorous formulas[3]. In particular, the supersymmetric partition functions on $\mathbb{M}^2 \times S^1$ where $\mathbb{M}^2$ is a 2d manifold is known to be equivalent to the following object

$$\mathcal{I}^{\mathbb{M}^2}_{\text{Theory}}(x, \alpha_a) = \mathrm{Tr}_{\mathcal{H}(\mathbb{M}^2)} \Big((-1)^{\hat{F}} x'^{\{Q, Q^\dagger\}} x^{\hat{H} + \hat{j}_3} \prod_a \alpha_a^{\hat{f}_a} \Big), \tag{1.2}$$

called *SuperConformal Index* (SCI). $\hat{j}_3$ and $\hat{f}_a$ are an orbital angular momentum and flavor charges respectively. As reviewed in Section 3, this quantity is an analog of usual

[2] The reader can find resultsof the localization tequniques for 2d QFTs in [21, 22, 23, 24, 25, 26], for 3d QFTs in [27, 28, 29, 30, 31, 32, 33] for 4d QFTs in [34, 35, 36] for 5d QFTs in [37, 38, 39].

[3] The check or proof of the equality (1.1) by utilizing supersymmetric partition function on $\mathbb{S}^2 \times S^1$, $\mathbb{D}^2 \times S^1$, $\mathbb{S}^3$ and $\mathbb{S}^3/\mathbb{Z}_p$ can be found in [42, 53, 43, 30, 44, 32] respectively.

thermal partition function. It is expected to satisfy the following equality similar to the one in (1.1) :

$$\mathcal{I}^{\mathbb{M}^2}_{\mathrm{XYZ}}(x,\alpha) = \mathcal{I}^{\mathbb{M}^2}_{\mathrm{SQED}}(x,\alpha^{-1}). \tag{1.3}$$

As explained in Section 4, thanks to the localization techniques, the structure of exact SCI on $\mathbb{S}^2 \times S^1$ for SQED is known to be constructed by a summation over the Dirac monopoles labelled by $B \in \mathbb{Z}$. As reviewed in the first subsection of Appendix C, we have to combine these contributions and utilize fancy mathematical formulas, *Ramanujan's summation formula* and *quantum binomial formula*, in order to deform its infinite summation to the XYZ side contribution:

$$\mathcal{I}^{\mathbb{S}^2}_{\mathrm{XYZ}}(x,\alpha) \xleftarrow[\text{Ramanujan's summation formula}]{\text{quantum binomial formula +}} \mathcal{I}^{\mathbb{S}^2}_{\mathrm{SQED}}(x,\alpha^{-1}). \tag{1.4}$$

This proof was originally perfomed in [53], and it provides an explicit evidence for the 3d mirror symmetry. We can observe a mechanism for the agreement through this proof. The infinitely many terms coming from Dirac monopoles combines into one contribution in XYZ-model via the mathematical formulas.

Our main results We get the following new results.

- We define a new SCI by using $\mathbb{M}^2 = \mathbb{RP}^2$ in (1.2).
- We derive formulas for the SCI based on localization for $U(1)$ gauge theories.
- We observe the equality (1.3) and prove it in our context.

$\mathbb{RP}^2$ is called *real projective plene*, topologically, one can construct this curved surface by combining the *Möbius strip* and the hemisphere $\mathbb{D}^2$ along the boundary. $\mathbb{RP}^2$ is not isomorphic to neither $\mathbb{S}^2$ nor $\mathbb{D}^2$. $\mathbb{RP}^2$ is an example for unorientable manifold, and the field theory on it sounds somewhat exotic in usual sense. We define SUSY gauge theories on $\mathbb{RP}^2 \times S^1$ by introducing sets of supersymmetric parity condition on $\mathbb{S}^2 \times S^1$. The SCI for gauge theory on $\mathbb{RP}^2 \times S^1$ consists of a summation over contributions of $+$holonomy sector and $-$holonomy sector, and there is no infinitely many terms but just 2 terms, and differ from the SCI on $\mathbb{S}^2 \times S^1$. The equality (1.3) is checked numerically in Section 6, and we exhibit its exact proof by using *quantum binomial formula* and unnamed formula (7.1) in Appendix C.

$$\mathcal{I}^{\mathbb{RP}^2}_{\mathrm{XYZ}}(x,\alpha) \xleftarrow[\text{un-named formula (7.1)}]{\text{quantum binomial formula +}} \mathcal{I}^{\mathbb{RP}^2}_{\mathrm{SQED}}(x,\alpha^{-1}). \tag{1.5}$$

Compared with the proof of (1.4), we can observe that the agreement in (1.5) is guaranteed not by the Ramanujan's formula but another, un-named formula (7.1). We can easily understand its difference because there is no Dirac monopole on $\mathbb{RP}^2$ but $\pm$ holonomies as noted above. The use of the un-named formula (7.1) is an algebraic representation of the $\pm$ holonomies. As one can see, the use of quantum binomial formula is in common. This is also easy to understand because as a common factor, we have Wilson line phase along the thermal S^1. The use of quantum binomial formula is, therefore, an algebraic representation of the Wilson line phase along the thermal S^1.

The organization of this paper is as follows. In Section 2, we review some basics of the Quantum Mechanics (QM). This section is important because we calculate SCI (1.2) by utilizing this section's method. In Section 3, we summarize some basic facts on the 3d $\mathcal{N} = 2$ supersymmetry and review the supersymmetric localization techniques. In Section 4, we review the exact calculation for the SCI with $\mathbb{M}^2 = \mathbb{S}^2$ by localization method from the many-body QM point of view. And in Section 5, we turn to the calculation with $\mathbb{M}^2 = \mathbb{RP}^2$ and get new results. Finally, in Section 6, we check the simplest 3d mirror symmetry, equivalence between XYZ-model and SQED numerically. If one wants to know how to prove it analytically, see Appendix C. In Section 7, we summarize this thesis and comment on some ongoing projects.

2 Preliminary - Quantum Mechanics (QM)

We begin our consideration from Quantum Mechanics (QM). First, we review some representation theory for boson and fermion. Second, we turn to consider the partition function

$$Z = \mathrm{Tr}(e^{-\beta\hat{H}}). \tag{2.1}$$

Third, we generalize it by turning on a insertion of $(-1)^{\hat{F}}$ into the trace :

$$I = \mathrm{Tr}\Big((-1)^{\hat{F}} e^{-\beta\hat{H}}\Big). \tag{2.2}$$

This is called *Witten index*, a prototype of the superconformal index in later discussion. $\hat{F}$ is called fermion number operator which counts the number of fermions. And in the final subsection, we generalize it and the generalized index gives the basis for the next section.

2.1 Representation theory

We briefly review the basics of boson and fermion in QM. We emphasis the relationship between operator formalism and path integral formalism for later use.

2.1.1 Boson

Classical prescription Bosonic Lagrangian typically takes the following form:

$$L_b = \frac{1}{2}\dot{x}^2 - V(x). \tag{2.3}$$

The conjugate momentum of x is defined by

$$p = \frac{\partial L_{bos}}{\partial \dot{x}}. \tag{2.4}$$

The Hamiltonian is defined by the Legendre transformation of L_{bos}:

$$\begin{aligned} H_b &= p\dot{x} - L_{bos} \\ &= \frac{1}{2}p^2 + V(x). \end{aligned} \tag{2.5}$$

Canonical quantization We start with the representation of the bosonic algebra, Heisenberg algebra:

$$[\hat{p}, \hat{x}] = -i, \tag{2.6}$$

where $\hat{p}$ and $\hat{x}$ are momentum and position operators correspondingly. In principle, we do not need to stick on the definition of $\pm$ sign in (2.6) if we treat it in self consistent way [54]. As a basis of the Hilbert space, we can take

$$|x\rangle \quad \text{or} \quad |p\rangle \tag{2.7}$$

These states are defined by

$$\hat{x}|x\rangle = x|x\rangle, \quad \int_{-\infty}^{+\infty} dx\ |x\rangle\langle x| = 1, \tag{2.8}$$

$$\hat{p}|p\rangle = p|p\rangle, \quad \int_{-\infty}^{+\infty} dp\ |p\rangle\langle p| = 1. \tag{2.9}$$

There are two important facts. First fact is that $e^{-i\hat{p}a}$ generates translation[4]of $|x\rangle$:

$$e^{-i\hat{p}a}|x\rangle = |x+a\rangle. \tag{2.10}$$

Second fact is the explicit form of the inner product[5]

$$\langle p|x\rangle = \frac{1}{\sqrt{2\pi}} e^{-ipx}. \tag{2.11}$$

The integration constant is determined by requireing the orthonormality condition $\langle x'|x\rangle = \delta(x-x')$.

[4]It can be proved by utilizing Baker-Campbell-Hausdorff (BCH) formula:

$$\begin{aligned}\hat{x}e^{-i\hat{p}a}|x\rangle &= e^{-i\hat{p}a}\underbrace{e^{+i\hat{p}a}\hat{x}e^{-i\hat{p}a}}_{BCH}|x\rangle = e^{-i\hat{p}a}\Big(\hat{x} + \underbrace{[i\hat{p}a, x]}_{a} + \underbrace{\dots}_{0}\Big)|x\rangle \\ &= e^{-i\hat{p}a}\Big(\hat{x}+a\Big)|x\rangle = (x+a)e^{-i\hat{p}a}|x\rangle.\end{aligned}$$

[5]The simplest way to prove this is to use the differential equation. For example,

$$\begin{aligned}\frac{\partial}{\partial x}\langle p|x\rangle &= \lim_{a\to 0}\frac{\langle p|x+a\rangle - \langle p|x\rangle}{a} \\ &= \lim_{a\to 0}\frac{\langle p|e^{-i\hat{p}a}|x\rangle - \langle p|x\rangle}{a} \\ &= \lim_{a\to 0}\frac{e^{-ipa}\langle p|x\rangle - \langle p|x\rangle}{a} \\ &= -ip\langle p|x\rangle.\end{aligned}$$

2.1.2 Fermion

Classical prescription Fermionic Lagrangian typically takes the following form :

$$L_f = i\psi_+\dot{\psi}_- - V(\psi_\pm). \tag{2.12}$$

Here we treat ψ_+, ψ_- as independent Grassmann numbers :

$$\psi_+^2 = 0, \quad \psi_-^2 = 0, \quad \psi_+\psi_- = -\psi_-\psi_+. \tag{2.13}$$

The (left[6]) conjugate momentum of ψ_- is defined by

$$\Pi_- = \frac{\partial}{\partial\dot{\psi}_-}L_f. \tag{2.14}$$

The Hamiltonian is defined by the Legendre transformation of L_f :

$$\begin{aligned} H_f &= \Pi_-\dot{\psi}_- - L_f \\ &= V(\psi_\pm). \end{aligned} \tag{2.15}$$

Canonical quantization We start with the representation of the fermionic algebra, Clifford algebra[7] :

$$\{\hat{\psi}_+, \hat{\psi}_-\} = +1. \tag{2.16}$$

In contrast to the bosonic case, the sign of (rhs) in (2.16) is important to get the unitary representation[54]. As an orthonormal basis of the Hilbert space, we can take

$$\Big\{|0\rangle, |1\rangle\Big\}. \tag{2.17}$$

These states are defined by

$$\begin{aligned} &\hat{\psi}_-|0\rangle = 0, \quad \hat{\psi}_+|0\rangle = |1\rangle, \\ &\hat{\psi}_-|1\rangle = |0\rangle, \quad \hat{\psi}_+|1\rangle = 0, \\ &|0\rangle\langle 0| + |1\rangle\langle 1| = 1. \end{aligned} \tag{2.18}$$

One can regard $|0\rangle$ as a hole-state, and $|1\rangle$ as an occupied state. We cannot make $|2\rangle := \psi_+|1\rangle$ because it is automatically zero. This is the famous Pauli exclusion principle.

[6]Because of the fermionic character (2.13), we have to be careful about the order of the ψ_+ and ψ_-.

[7] In order to derive this relation from the usual canonical quantization method, we need to consider not Poisson bracket but Dirac bracket.

Coherent state basis In later discussion, we convert our formalism to the path integral formalism. In order to do so, there is a more useful basis than the basis in (2.17), the coherent state basis:

$$|\Psi\rangle = e^{-\Psi\hat{\psi}_+}|0\rangle, \quad \langle\Psi| = \langle 0|e^{\Psi\hat{\psi}_-}. \tag{2.19}$$

We should take Ψ as a Grassmann valuable, therefore $\Psi^2 = 0$ and

$$|\Psi\rangle = (1 - \Psi\hat{\psi}_+)|0\rangle. \tag{2.20}$$

These states satisfy the following relations.

$$\hat{\psi}_-|\Psi\rangle = \Psi|\Psi\rangle, \quad \langle\Psi|\hat{\psi}_+ = \langle\Psi|\Psi. \tag{2.21}$$

After a direct calculation, one can get the inner product formula

$$\langle\Psi_+|\Psi_-\rangle = e^{\Psi_+\Psi_-}, \tag{2.22}$$

and the complete relation

$$\int d\Psi_+ d\Psi_- |\Psi_-\rangle e^{-\Psi_+\Psi_-}\langle\Psi_+| = 1. \tag{2.23}$$

2.2 Partition function

One of the most interesting object to study in quantum mechanics is the partition function :

$$Z = \mathrm{Tr}(e^{-\beta\hat{H}}). \tag{2.24}$$

It contains all informations of the energy spectrum because we can extract each energy by taking following process[8]:

1. Taking $\beta \to \infty$ of Z, then $Z \sim e^{-\beta E_0}$ where E_0 is the ground state energy.
2. Subtracting $e^{-\beta E_0}$ from Z, and rename it Z_1, and
   taking $\beta \to \infty$ of Z_1, then $Z_1 \sim e^{-\beta E_1}$ where E_1 is the 1st exited state energy.
3. Repeating this procedure.

[8] This is valid if there is no degeneracy.

2.2.1 Boson sector

Partition function of the bosonic degrees of freedom is described by the Hamiltonian operator defined in (2.5) classically :

$$\hat{H} = \hat{H}_b, \quad \hat{H}_b = \frac{1}{2}\hat{p}^2 + V(\hat{x}). \tag{2.25}$$

Operator formalism description of harmonic oscillator's Z

The simplest example is

$$V(\hat{x}) = \frac{1}{2}\omega^2\hat{x}^2. \tag{2.26}$$

In this case, as well known, once we define $\hat{a}$ and $\hat{a}^\dagger$ so that

$$\hat{H}_b = \omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}), \tag{2.27}$$

and by constructing a basis

$$\Big\{|0\rangle_b, |1\rangle_b, |2\rangle_b, \dots\Big\}, \qquad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \tag{2.28}$$

then, we can diagonalize the Hamiltonian : $\hat{H}_b|n\rangle = \omega(n+\frac{1}{2})|n\rangle$. By using this basis, the partition function can be computed by utilizing the formula of power series

$$\begin{aligned}\mathrm{Tr}(e^{-\beta\hat{H}_b}) &= \sum_{n=0}^{\infty} e^{-\beta\omega(n+\frac{1}{2})} \\ &= \frac{e^{-\frac{\beta\omega}{2}}}{1-e^{-\beta\omega}} \\ &= \frac{1}{2\sinh\frac{\beta\omega}{2}}.\end{aligned} \tag{2.29}$$

The zero energy which corresponds to $n=0$ is often called Casimir energy.

Path integral formalism By Inserting the complete set (2.8) and (2.9) into the trace in (2.24), we can re-express it as

$$\begin{aligned}Z_b &= \int_{x(0)=x(\beta)} \Big(\prod_{t\in[0,\beta]} dx(t)\frac{dp(t)}{2\pi}\Big) e^{-\int_0^\beta dt\left(ip\dot{x}+\frac{1}{2}p^2+V(x)\right)} \\ &= \int_{x(0)=x(\beta)} \Big(\prod_{t\in[0,\beta]} \frac{dx(t)}{\sqrt{2\pi}}\Big) e^{-\int_0^\beta dt\left(\frac{-1}{2}x\partial_t^2x+V(x)\right)}.\end{aligned} \tag{2.30}$$

Path integral description of harmonic oscillator's Z

We have the following action

$$-\int_0^\beta dt\Big(\frac{-1}{2}x\partial_t^2 x + V(x)\Big) = -\frac{1}{2}\int_0^\beta dt\ x(-\partial_t^2+\omega^2)x. \tag{2.31}$$

Thanks to the Gaussian integral formula (A.13), we get formally,

$$Z_b = \frac{1}{\sqrt{\det_{x(0)=x(\beta)}(-\partial_t^2+\omega^2)}}. \tag{2.32}$$

The "matrix" ∂_t's eigenvectors are $x_n(t) = e^{\frac{2\pi i}{\beta}nt},\ n\in\mathbb{Z}$ because

$$\partial_t x_n = \frac{2\pi i}{\beta}n x_n. \tag{2.33}$$

Therefore, we get the following representation of the determinant.

$$\begin{aligned}\det_{x(0)=x(\beta)}(-\partial_t^2+\omega^2) &= \prod_{n=-\infty}^{\infty}\Big(\frac{(2\pi)^2}{\beta^2}n^2+\omega^2\Big)\\ &= \omega^2\Big[\prod_{n=1}^{\infty}\Big(\frac{(2\pi)^2}{\beta^2}n^2+\omega^2\Big)\Big]^2\\ &= \Big[\prod_{n=1}^{\infty}\frac{2\pi}{\beta}n\Big]^4\times\omega^2\prod_{n=1}^{\infty}\Big(1+\frac{(\beta\omega)^2}{(2\pi n)^2}\Big)^2. \end{aligned}\tag{2.34}$$

Obviously, the first factor diverges. We regularize it by using zeta-function regularization. (See Appendix A for $\zeta(0),\zeta'(0)$ values' derivation.):

$$\begin{aligned}\Big[\prod_{n=1}^{\infty}\frac{2\pi}{\beta}n\Big]^4 = \exp\Big(4\sum_{n=1}^{\infty}\log\frac{2\pi}{\beta}n\Big) &\to \exp\Big(4\big[-\zeta'(0)-\zeta(0)\log\frac{\beta}{2\pi}\big]\Big)\\ &= \exp\Big(4\big[-(-\frac{1}{2}\log 2\pi)-(-\frac{1}{2})\log\frac{\beta}{2\pi}\big]\Big)\\ &= \beta^2. \end{aligned}\tag{2.35}$$

Then, by using the infinite product formula (A.1),

$$(2.34) = \Big[(\beta\omega)\prod_{n=1}^{\infty}\Big(1+\frac{(\beta\omega)^2}{(2\pi n)^2}\Big)\Big]^2 = \Big[2\sinh\frac{\beta\omega}{2}\Big]^2. \tag{2.36}$$

It reproduces the result (2.29) :

$$Z_b = \frac{1}{\sqrt{\det_{x(0)=x(\beta)}(-\partial_t^2+\omega^2)}} = \frac{1}{2\sinh\frac{\beta\omega}{2}}. \tag{2.37}$$

2.2.2 Fermion sector

Partition function of the fermionic degrees of freedom is described by the Hamiltonian operator defined in (2.15) classically :

$$\hat{H} = \hat{H}_f, \quad \hat{H}_f = V(\hat{\psi}_\pm). \tag{2.38}$$

Operator formalism description of harmonic oscillator's Z

The simplest example is

$$V(\hat{\psi}_\pm) = \omega(\hat{\psi}_+\hat{\psi}_- - \frac{1}{2}). \tag{2.39}$$

Then, the basis (2.18) diagonalizes this Hamiltonian:

$$\hat{H}_f|n\rangle = \omega(n - \frac{1}{2})|n\rangle, \quad n = 0, 1. \tag{2.40}$$

The partition function is, therefore,

$$\begin{aligned} \mathrm{Tr}(e^{-\beta \hat{H}_f}) &= \sum_{n=0}^{1} e^{-\beta\omega(n-\frac{1}{2})} \\ &= e^{\frac{\beta\omega}{2}} + e^{-\frac{\beta\omega}{2}} \\ &= 2\cosh\frac{\beta\omega}{2}. \end{aligned} \tag{2.41}$$

There are two important discrepancies compared with the bosonic harmonic oscillator.

- The absolute value of Casimir energy is same but the sign is different.
- cosh function appears, unlike the sinh in bosonic case.

As we will see later, if we insert $(-1)^{\hat{\psi}_+\hat{\psi}_-}$ into the trace, we get sinh not cosh.

Path integral formalism When we derive fermion's path integral representation of the partition function, we have to be careful about the periodicity.

$$\begin{aligned} Z_f &= \mathrm{Tr}(e^{-\beta \hat{H}_f}) \\ &= \int d\Psi_+ d\Psi_- \; e^{\Psi_+\Psi_-} \langle \Psi_+ | e^{-\beta V(\hat{\psi}_+, \hat{\psi}_-)} | \Psi_- \rangle. \end{aligned} \tag{2.42}$$

Now, we divide β into N pieces : $\epsilon = \frac{\beta}{\mathrm{N}}$, then, we can write, say N=2

$$
\begin{aligned}
(2.42) &= \int d\Psi_+ d\Psi_- \int d\Lambda_+ d\Lambda_- \; e^{\Psi_+\Psi_-} \langle \Psi_+ | e^{-\epsilon V(\hat{\psi}_+,\hat{\psi}_-)} | \Lambda_- \rangle e^{-\Lambda_+\Lambda_-} \langle \Lambda_+ | e^{-\epsilon V(\hat{\psi}_+,\hat{\psi}_-)} | \Psi_- \rangle \\
&= \int d\Psi_+ d\Psi_- \int d\Lambda_+ d\Lambda_- \; e^{\Psi_+\Psi_-} e^{-\epsilon V^{(W)}(\Psi_+,\Lambda_-)} \langle \Psi_+ | \Lambda_- \rangle e^{-\Lambda_+\Lambda_-} \langle \Lambda_+ | \Psi_- \rangle e^{-\epsilon V(\Lambda_+,\Psi_-)} \\
&= \int d\Psi_+ d\Psi_- \int d\Lambda_+ d\Lambda_- \; e^{\Psi_+\Psi_- + \Psi_+\Lambda_- - \Lambda_+\Lambda_- + \Lambda_+\Psi_-} e^{-\epsilon V(\Psi_+,\Lambda_-) - \epsilon V(\Lambda_+,\Psi_-)}.
\end{aligned}
\tag{2.43}
$$

We rename fermionic valuables:

$$
\Psi_+ = \Psi_+^2, \quad \Lambda_- = \Psi_-^2, \quad \Lambda_+ = \Psi_+^1, \quad \Psi_- = \Psi_-^1, \tag{2.44}
$$

then we get

$$
(2.43) = \int d\Psi_+^2 d\Psi_-^2 d\Psi_+^1 d\Psi_-^1 \; e^{\Psi_+^2\Psi_-^1 + \Psi_+^2\Psi_-^2 - \Psi_+^1\Psi_-^2 + \Psi_+^1\Psi_-^1 - \epsilon V^{(W)}(\Psi_+^2,\Psi_-^2) - \epsilon V^{(W)}(\Psi_+^1,\Psi_-^1)}. \tag{2.45}
$$

Now, we regard each $\Psi_\pm^n$ as $\Psi_\pm(t_n) = \Psi_\pm^n$, where $t_n = \epsilon n$. In this N=2 case,

$$
\begin{aligned}
&\Psi_+^2\Psi_-^1 + \Psi_+^2\Psi_-^2 - \Psi_+^1\Psi_-^2 + \Psi_+^1\Psi_-^1 \\
&= \Psi_+(t_2)\Big(\underbrace{\Psi_-(t_1)}_{\Psi_-(0)+\epsilon\dot{\Psi}_-(0)} + \Psi_-(t_2) \Big) - \Psi_+(t_1)\Big(\underbrace{\Psi_-(t_2)}_{\Psi_-(t_1)+\epsilon\dot{\Psi}_-(t_1)} - \Psi_-(t_1) \Big) \\
&= \Psi_+(t_2)\Big(\epsilon\dot{\Psi}_-(0) + \underbrace{\big[\Psi_-(0) + \Psi_-(t_2)\big]}_{\text{we have to make it zero.}} \Big) - \Psi_+(t_1)\Big(\epsilon\dot{\Psi}_-(t_1) + \big[\underbrace{\Psi_-(t_1) - \Psi_-(t_1)}_{0}\big] \Big)
\end{aligned}
$$

As we can see above, in order to drop the $\mathcal{O}(\epsilon^0)$ term, we have to take

$$
\Psi_-(t_2) = \Psi_-(\beta) = -\Psi_-(0). \tag{2.46}
$$

Therefore, corresponding fermionic field $\Psi_\pm(t)$ are anti-periodic[9] under the translation $t \to t + \beta$. By using

$$
\dot{\Psi}(0) = \frac{d}{dt}\Big|_{t=0} \Psi(t) = -\frac{d}{dt}\Big|_{t=0} \Psi(t+\beta) = -\dot{\Psi}(t_2), \tag{2.47}
$$

and taking N$\to\infty$ limit, we arrive at

$$
(2.45) = \int_{\Psi_\pm(0) = -\Psi_\pm(\beta)} \Big(\prod_{t\in[0,\beta]} d\Psi_+(t) d\Psi_-(t) \Big) e^{-\int_0^\beta dt \left(\Psi_+\dot{\Psi}_- + V^{(W)}(\Psi_+,\Psi_-) \right)}. \tag{2.48}
$$

[9]We have checked it only with Ψ_-, but we can understand the case for Ψ_+ in similar way.

Path integral description of harmonic oscillator's Z

$$\begin{aligned}\mathrm{Tr}(e^{-\beta\hat{H}_f}) &= \int_{\Psi_\pm(0)=-\Psi_\pm(\beta)} \Big(\prod_{t\in[0,\beta]} d\Psi_+(t)d\Psi_-(t)\Big) e^{-\int_0^\beta dt\ \Psi_+(\partial_t+\omega)\Psi_-} \\ &= \det_{\Psi_\pm(0)=-\Psi_\pm(\beta)}(\partial_t+\omega). \end{aligned}\tag{2.49}$$

We used the Gaussian integral formula for fermionic variables (A.16). In this anti-periodic sector, the eigenvectors of ∂_t are $\psi_n(t) = e^{\frac{2\pi i}{\beta}(n-\frac{1}{2})t}$ with $n \in \mathbb{Z}$. Therefore,

$$\begin{aligned}\det_{\Psi_\pm(0)=-\Psi_\pm(\beta)}(\partial_t+\omega) &= \prod_{n=-\infty}^{\infty}\Big(\frac{2\pi i}{\beta}(n-\frac{1}{2})+\omega\Big) \\ &= \prod_{n=1}^{\infty}\Big(\frac{(2\pi)^2}{\beta^2}(n-\frac{1}{2})^2+\omega^2\Big) \\ &= \Big[\prod_{n=1}^{\infty}\frac{2\pi}{\beta}(n-\frac{1}{2})\Big]^2 \times \prod_{n=1}^{\infty}\Big(1+\frac{(\beta\omega)^2}{(2\pi[n-\frac{1}{2}])^2}\Big). \end{aligned}\tag{2.50}$$

The first factor diverges, so we have to regularize it.One might think that the zeta-function regularization works, however in this case, we should calculate carefully:

$$\begin{aligned}\Big[\prod_{n=1}^{\infty}\frac{2\pi}{\beta}(n-\frac{1}{2})\Big]^2 &= \Big[\prod_{n=1}^{\infty}\frac{2\pi}{\beta}n\Big]^2 \times \Big[\prod_{n=1}^{\infty}\frac{\frac{2\pi}{\beta}(n-\frac{1}{2})}{\frac{2\pi}{\beta}n}\Big]^2 \\ &= \Big[\prod_{n=1}^{\infty}\frac{2\pi}{\beta}n\Big]^2 \times \Big[\prod_{n=1}^{\infty}\frac{\frac{\pi}{\beta}(2n-1)}{\frac{\pi}{\beta}(2n)}\Big]^2 \\ &= \underbrace{\Big[\prod_{n=1}^{\infty}\frac{2\pi}{\beta}n\Big]^2}_{\to\beta} \times\frac{\pi}{\beta}\times \underbrace{\Big[\prod_{n=1}^{\infty}\frac{\frac{\pi}{\beta}(2n-1)\times\frac{\pi}{\beta}(2n+1)}{\frac{\pi^2}{\beta^2}(2n)^2}\Big]}_{\frac{2}{\pi}} \\ &\to 2, \end{aligned}\tag{2.51}$$

where we used Wallis' formula. And, of course, another part of (2.50) can be calculated by using infinite product formula for cosh (A.2) :

$$\prod_{n=1}^{\infty}\Big(1+\frac{(\beta\omega)^2}{(2\pi[n-\frac{1}{2}])^2}\Big) = \cosh\frac{\beta\omega}{2}.\tag{2.52}$$

Gathering all, we recover the result (2.41)

$$\mathrm{Tr}(e^{-\beta\hat{H}_f}) = 2\cosh\frac{\beta\omega}{2}.\tag{2.53}$$

2.3 Witten index

As we have surveyed briefly, the partition function of the harmonic oscillator can be calculated easily. However, once we turn on the cubic or more higher interaction in V, we cannot hope for the possibility of the exact calculation. In addition to it, naive zeta-function regularization does not work in the fermionic sector as we have observed in previous page. However, we can overcome such a situation by considering

$$I = \mathrm{Tr}\Big((-1)^{\hat{F}} e^{-\beta \hat{H}}\Big), \quad \text{where } \hat{F} \text{ is a fermion number operator,} \tag{2.54}$$

instead of Z. This is called Witten index [55].

Fermion number operator $\hat{F}$ is an operator which counts the number of fermion excitation, 0 or 1. Explicitly, we can write it as

$$\hat{F} = \hat{\psi}_+ \hat{\psi}_- . \tag{2.55}$$

As one can check easily,

$$(-1)^{\hat{F}} = \begin{cases} +1 & \text{bosonic state} \\ -1 & \text{fermionic state} \end{cases} . \tag{2.56}$$

Therefore, within only bosonic sector, I and Z are identical :

$$I_b = \mathrm{Tr}_b\Big((-1)^{\hat{F}} e^{-\beta \hat{H}_b}\Big) = \mathrm{Tr}_b(e^{-\beta \hat{H}_b}) = Z_b, \tag{2.57}$$

and nothing different happens compared with the partition function. However, the fermion sector's behavior changes drastically.

2.3.1 Fermion sector

Let us see what happens in the operator formalism first by using the harmonic oscillator example.

Operator formalism description of harmonic oscillator's I

Let us remind the calculation in (2.41). We can get I as

$$\begin{aligned} \mathrm{Tr}\Big((-1)^{\hat{F}} e^{-\beta \hat{H}_f}\Big) &= \sum_{n=0}^{1} (-1)^n e^{-\beta\omega(n-\frac{1}{2})} \\ &= e^{\frac{\beta\omega}{2}} - e^{-\frac{\beta\omega}{2}} \\ &= 2\sinh\frac{\beta\omega}{2}. \end{aligned} \tag{2.58}$$

Path integral formalism After a simple calculation, one can verify that

$$\begin{aligned} I &= \mathrm{Tr}\Big((-1)^{\hat{F}} e^{-\beta \hat{H}_f}\Big) \\ &= \int d\Psi_+ d\Psi_- \; e^{-\Psi_+\Psi_-} \langle \Psi_+ | e^{-\beta \hat{H}_f} \Psi_- \rangle. \end{aligned} \tag{2.59}$$

Compering with the partition function (2.42), one can see that the sign of the exponential factor is different. This insertion causes periodic boundary conditions of the fermionic field $\Psi_\pm(t)$ under $t \to t+\beta$ because the sign in the first term in (2.46) changes. In summary,

$$I_f = \int_{\Psi_\pm(0)=\Psi_\pm(\beta)} \Big(d\Psi_+(t) d\Psi_-(t)\Big) e^{-\int_0^\beta \Big(\Psi_+\partial_t\Psi_- + V^{(W)}(\Psi_+,\Psi_-)\Big)}. \tag{2.60}$$

In this case, we can recover the result (2.58) as follows.

Path integral description of harmonic oscillator's I

$$\begin{aligned} I_f &= \int_{\Psi_\pm(0)=\Psi_\pm(\beta)} \Big(d\Psi_+(t) d\Psi_-(t)\Big) e^{-\int_0^\beta \Psi_+(\partial_t+\omega)\Psi_-} \\ &= \det_{\Psi_\pm(0)=\Psi_\pm(\beta)} (\partial_t + \omega) \\ &= \prod_{n=-\infty}^{\infty} \Big(\frac{2\pi i}{\beta} n + \omega\Big) \\ &= \omega \prod_{n=1}^{\infty} \Big(\frac{(2\pi n)^2}{\beta^2} + \omega^2\Big). \end{aligned} \tag{2.61}$$

The same infinite product in the bosonic partition function (2.34) emerges. Therefore, by repeating zeta-function regularization procedure, we arrive at

$$I_f = 2\sinh\frac{\beta\omega}{2}. \tag{2.62}$$

2.3.2 Supersymmetric quantum mechanics

What happens when we consider the Witten index

$$I = \mathrm{Tr}\Big((-1)^{\hat{F}} e^{-\beta \hat{H}}\Big), \tag{2.63}$$

with harmonic oscillator Hamiltonians $\hat{H} = \hat{H}_b + \hat{H}_f$? The answer is extremely simple;

$$\begin{aligned} I &= I_b \times I_f \\ &= Z_b \times I_f \\ &= \frac{1}{2\sinh\frac{\beta\omega}{2}} \times 2\sinh\frac{\beta\omega}{2} \\ &= 1. \end{aligned} \tag{2.64}$$

Note that if we turn on different frequencies ω_b, ω_f for boson and fermion respectively, we get

$$I = \frac{\sinh\frac{\beta\omega_f}{2}}{\sinh\frac{\beta\omega_b}{2}}, \tag{2.65}$$

and it does depend on β. Therefore, the β independence is equivalent to the condition $\omega_b = \omega_f$. It is strongly related to the concept of *supersymmetry*. In other words, the Hamiltonian

$$\hat{H} = \omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \omega(\hat{\psi}_+\hat{\psi}_- - \frac{1}{2}) = \omega(\hat{a}^\dagger\hat{a} + \hat{\psi}_+\hat{\psi}_-) \tag{2.66}$$

defines supersymmetric quantum mechanics. The physical meaning is also extremely simple : the state $|0\rangle$ only contributes. The numerics $I = 1$ means that there is one unique vacuum in the system. We can learn other facts of supersymmetry from this extremely simple example by defining

$$\hat{Q} := \sqrt{\omega}\ \hat{a}^\dagger\hat{\psi}_-, \quad \hat{Q}^\dagger := \sqrt{\omega}\ \hat{a}\hat{\psi}_+. \tag{2.67}$$

These operators are called *supercharges* which satisfy the following equation.

$$\hat{H} = \{\hat{Q}, \hat{Q}^\dagger\}. \tag{2.68}$$

By using this expression, the reason for β independence of the supersymmetric Witten index becomes clear because the differential of the index with respect to β becomes zero:

$$\begin{aligned} \frac{d}{d\beta}\mathrm{Tr}(-1)^{\hat{F}} e^{-\beta\hat{H}} &= \frac{d}{d\beta}\mathrm{Tr}(-1)^{\hat{F}} e^{-\beta\{\hat{Q},\hat{Q}^\dagger\}} \\ &= -\mathrm{Tr}(-1)^{\hat{F}}(\hat{Q}\hat{Q}^\dagger + \hat{Q}^\dagger\hat{Q})e^{-\beta\{\hat{Q},\hat{Q}^\dagger\}} \\ &= -\mathrm{Tr}(-1)^{\hat{F}}(\hat{Q}\hat{Q}^\dagger - \hat{Q}\hat{Q}^\dagger)e^{-\beta\{\hat{Q},\hat{Q}^\dagger\}} = 0. \end{aligned} \tag{2.69}$$

We can construct a somewhat more non-trivial Hamiltonian (e.g. [55, 56, 57]) which contains interaction terms. In such case, supersymmetric Witten index counts the number of degeneracy of ground states, or more technically speaking, it counts the number of *BPS states*.

2.3.3 Generalized index

In (2.69), we use the following facts:

$$[\hat{H}, \hat{Q}] = [\hat{H}, \hat{Q}^\dagger] = 0. \tag{2.70}$$

It means $\hat{Q}$ and $\hat{Q}^\dagger$ generate symmetry of the system. Suppose there is another generator $\hat{J}$ which commutes with the supercharges:

$$[\hat{Q}, \hat{J}] = 0, \quad [\hat{Q}^\dagger, \hat{J}] = 0, \tag{2.71}$$

then following trace

$$\mathrm{Tr}\Big((-1)^{\hat{F}} e^{-\beta\{\hat{Q},\hat{Q}^\dagger\}} e^{-i\mu\hat{J}}\Big) \tag{2.72}$$

also does not depend on β. In later section, we introduce the concept of *Super Conformal Index* (SCI). SCI can be regarded such a generalized index. $e^{-i\mu\hat{J}}$ insertion makes $x(t)$ and $\Psi_\pm(t)$ not periodic but as follows.

Twisted boundary conditions

$$x(t+\beta) = e^{i\mu J_x} x(t), \quad \Psi_\pm(t+\beta) = e^{i\mu J_\psi} \Psi_\pm(t), \tag{2.73}$$

where J_x, J_ψ are eigenvalues of $\hat{J}$ operator. The reason is as follows. For bosonic degrees of freedom, (2.72) can be expressed

$$\begin{aligned}
\mathrm{Tr}\Big((-1)^{\hat{F}} e^{-\beta\{\hat{Q},\hat{Q}^\dagger\}} e^{-i\mu\hat{J}}\Big) &= \int dx \langle x|(-1)^{\hat{F}} e^{-\beta\{\hat{Q},\hat{Q}^\dagger\}} e^{-i\mu\hat{J}}|x\rangle \\
&= \int dx \langle x|e^{-\beta\hat{H}}|e^{-i\mu J_x} x\rangle \\
&= \int dx dp dx_1 \langle x|e^{-(\beta-\epsilon)\hat{H}}|x_1\rangle \underbrace{\langle x_1|e^{-\epsilon\hat{H}}|p\rangle}_{e^{-\epsilon H(x_1,p)+ipx_1}} \underbrace{\langle p|e^{-i\mu J_x} x\rangle}_{e^{-ipe^{-i\mu J_x} x}},
\end{aligned} \tag{2.74}$$

and at the edge, we have

$$e^{-\epsilon H(x_1,p)+ipx_1-ipe^{-i\mu J_x} x}. \tag{2.75}$$

In order to get rid of $\mathcal{O}(\epsilon^0)$ term,

$$\begin{aligned}
+ipx_1 - ipe^{-i\mu J_x} x &= ip(x_1 - e^{-i\mu J_x} x) \\
&= ip\Big(x(t=\epsilon) - e^{-i\mu J_x} x(t=\beta)\Big) \\
&= ip\Big(\epsilon\dot{x}(0) \underbrace{+x(t=0) - e^{-i\mu J_x} x(t=\beta)}_{\text{we have to make it zero.}}\Big).
\end{aligned} \tag{2.76}$$

This is the origin of the twisted boundary condition in (2.73).

3 3d Superconformal indeces on $\mathbb{M}^2 \times S^1_\beta$

In this section, we review basics for the recent calculations of the 3d superconformal index

$$\mathcal{I}^{\mathbb{M}^2}_{\text{Theory}}(x, \alpha_a) = \text{Tr}_{\mathcal{H}(\mathbb{M}^2)}\Big((-1)^{\hat{F}} x'^{\{Q,Q^\dagger\}} x^{\hat{H}+\hat{j}_3} \prod_a \alpha_a^{\hat{f}_a}\Big), \tag{3.1}$$

based on supersymmetric localization principle. In 3.1, we give the physical meaning for the SCI (3.1), and represent it in the path integral formalism. In 3.2, we turn to define the SUSY QFT on $\mathbb{M}^2 \times S^1_\beta$ where β corresponds to the inverse temperature. It gives the precise definition for the SCI in the path integral formalism. In 3.3, we explain the supersymmetric localization principle. We will perform the exact calculations in later sections based on this principle.

3.1 Superconformal index

First, we consider the physical meaning of the SCI (3.1) in operator formalism. After that, we turn to the path integral representation of SCI by quoting the results in Section 2.

3.1.1 Operator formalism description

As one can find in [58, 59, 28], the following operators

$$\hat{H} + \hat{j}_3, \quad \hat{f}_a, \; a = 1, ..., N_f \tag{3.2}$$

commute[10] with both of $\hat{Q}$ and $\hat{Q}^\dagger$, therefore, each operator can play a role of $\hat{J}$ in (2.71) and SCI turns to one of the generalized indices and does not depends on x'. It means that states which satisfy

$$\{\hat{Q}, \hat{Q}^\dagger\}|\text{phys}\rangle = 0 \tag{3.4}$$

[10] One may wonder why $\hat{H}$ alone does not commute with $\hat{Q}$ and $\hat{Q}^\dagger$. For example, we can find the same SUSY algebra in [60]:

$$[P_a, Q_\alpha] = -\frac{1}{2r}(\gamma_a)_\alpha^{\ \beta} Q_\beta, \quad [M, Q_\alpha] = -\frac{1}{2}(\gamma_3)_\alpha^{\ \beta} Q_\beta, \tag{3.3}$$

where r represents $\mathbb{S}^2$ radius which we take $r = 1$. Our operators $\hat{H}, \hat{j}_3$ correspond to $P_3, -M$ respectively. Therefore, the combination $\hat{H} + \hat{j}_3$ is a consequence of the curvature, and if we recover the r, we should write it as $\hat{H} + \frac{1}{r}\hat{j}_3$. The character $[\hat{Q}, \hat{f}_a] = 0$ is easily understood because the supercharges act only operators with Lorentz indices, spacetime vector, spinor, R-symmetry etc, and the flavor index a is not in the class.

called BPS states [61, 62] only contribute to the SCI. Now, we define subspace of the Hilbert space $\mathcal{H}$:

$$\mathcal{H}^{BPS} := \left\{ |\text{phys}\rangle \in \mathcal{H} \middle| \{\hat{Q}, \hat{Q}^\dagger\} |\text{phys}\rangle = 0 \right\}. \tag{3.5}$$

Then, we can rewrite SCI as follows

$$\mathcal{I}(x, \alpha_a) = \mathrm{Tr}_{\mathcal{H}^{BPS}} \Big((-1)^{\hat{F}} x^{\hat{H}+\hat{j}_3} \prod_a \alpha_a^{\hat{f}_a} \Big). \tag{3.6}$$

For simplicity, we suppose here the index a runs for $a = 1$ only, and omit this index, then SCI reduces to

$$\mathcal{I}(x, \alpha) = \mathrm{Tr}_{\mathcal{H}^{BPS}} \Big((-1)^{\hat{F}} x^{\hat{H}+\hat{j}_3} \alpha^{\hat{f}} \Big). \tag{3.7}$$

$\hat{H} + \hat{j}_3$ and $\hat{f}$ are conserved charges so we can divide $\mathcal{H}^{BPS}$ into more basic ingredients

$$\mathcal{H}^{BPS}_{J,f} := \left\{ |BPS\rangle \in \mathcal{H}^{BPS} \middle| \begin{array}{l} (\hat{H} + \hat{j}_3)|BPS\rangle = J|BPS\rangle \\ \hat{f}|BPS\rangle = f|BPS\rangle \end{array} \right\}. \tag{3.8}$$

Then, SCI can be represented by each Witten index of (J, f) sector $I_{(J,f)}$:

$$\mathcal{I}(x, \alpha) = \sum_{J,f} x^J \alpha^f \times \underbrace{\mathrm{Tr}_{\mathcal{H}^{BPS}_{J,f}} (-1)^{\hat{F}}}_{I_{(J,f)}}. \tag{3.9}$$

Therefore, once we know the exact form of the $\mathcal{I}(x, \alpha)$, we can extract the number $I_{(J,f)}$ by expanding it around $x = \alpha = 0$. Compared with the usual Witten index, SCI gives us finer informations of the theory because this is not just a number but a polynomial (or function) with respect to fugacities x, α.

3.1.2 Path integral description

In order to convert the path integral description, it is useful to introduce $\beta_1, \beta_2, \beta, \mu_a$ as follows.

$$x' = e^{-\beta_1}, \quad x = e^{-\beta_2}, \quad \alpha_a = e^{-i\mu_a}, \quad \beta = \beta_1 + \beta_2. \tag{3.10}$$

By utilizing the $\mathcal{N} = 2$ SUSY algebra [58, 59, 28, 60], we get the relation

$$\{\hat{Q}, \hat{Q}^\dagger\} = \hat{H} + \hat{R} - \hat{j}_3, \tag{3.11}$$

where $\hat{R}$ is called *R-charge*. We will assign R-charge to each field. (See Table 1 in later discussion.) Then we can rewrite the SCI as follows:

$$\mathcal{I}(x,\alpha_a)=\mathrm{Tr}_{\mathcal{H}}\Big((-1)^{\hat{F}}e^{-\beta\hat{H}}\cdot e^{-\beta_1(\hat{R}-\hat{j}_3)}e^{-\beta_2\hat{j}_3}e^{-\sum_a i\mu_a\hat{f}_a}\Big). \tag{3.12}$$

As we have already mentioned in Section 2, the $e^{-\beta\hat{H}}$ generates translation along the β circle, $(-1)^{\hat{F}}$ makes the all sets of degrees of freedom periodic, and other insertions $e^{-\beta_1(\hat{R}-\hat{j}_3)}e^{-\beta_2\hat{j}_3}e^{\sum_a i\mu_a\hat{f}_a}$ define twisted boundary condition for each field. (See also (2.73).):

$$x(t+\beta)=e^{\beta_1(\hat{R}-\hat{j}_3)}e^{\beta_2\hat{j}_3}e^{\sum_a i\mu_a\hat{f}_a}x(t),\quad \text{for boson}, \tag{3.13}$$

$$\Psi_\pm(t+\beta)=e^{\beta_1(\hat{R}-\hat{j}_3)}e^{\beta_2\hat{j}_3}e^{\sum_a i\mu_a\hat{f}_a}\Psi_\pm(t),\quad \text{for fermion}. \tag{3.14}$$

Therefore, by repeating the derivation of the path integral descriptions of the Witten index, or generalized index, we arrive at the path integral definition of SCI:

$$\mathcal{I}(x,\alpha_a)=\int\Big(\prod_{t\in[0,\beta]}dx(t)d\Psi_+(t)d\Psi_-(t)\Big)e^{-S_b-S_f}, \tag{3.15}$$

with conditions (3.13), (3.14).

To quantum field theory The above explanation is almost correct, but more precisely speaking, we should add two spacial dimensions represented by x^i $(i=1,2)$ which is a set of coordinates for two-dimensional manifold $\mathbb{M}^2$, and consider not quantum mechanical degrees of freedom but quantum field theoretical degrees of freedom:

$$x(t)\to\phi(x^i,t),\qquad \Psi_+(t)\to\overline{\psi}(x^i,t),\quad \Psi_-(t)\to\psi(x^i,t). \tag{3.16}$$

And of course the twisted boundary conditions (3.13) and (3.14) are lifted to

$$\phi(x^i,t+\beta)=e^{\beta_1(\hat{R}-\hat{j}_3)}e^{\beta_2\hat{j}_3}e^{\sum_a i\mu_a\hat{f}_a}\phi(x^i,t),\quad \text{for bosons}, \tag{3.17}$$

$$\psi(x^i,t+\beta)=e^{\beta_1(\hat{R}-\hat{j}_3)}e^{\beta_2\hat{j}_3}e^{\sum_a i\mu_a\hat{f}_a}\psi(x^i,t),\quad \text{for fermions}. \tag{3.18}$$

Therefore, we get the path integral representation as

$$\mathcal{I}(x,\alpha_a)=\int\mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\overline{\psi}\ e^{-S_b-S_f}. \tag{3.19}$$

3.2 Supersymmetric field theories on curved manifold $\mathbb{M}^2 \times S^1_\beta$

Finally, we can begin to discuss main part of this thesis. Our main interest is to calculate SCI (3.1) by using the path integral formalism (3.19) with the twisted boundary conditions (3.17) and (3.18). In order to do so, it is useful to make our discussion of supersymmetry to the off-shell formalism. We use so-called three-dimensional $\mathcal{N} = 2$ supersymmetries. There are two irreducible representations, called vector multiplet and matter multiplet. From now on, we take two-dimensional manifold $\mathbb{M}^2$ as round sphere $\mathbb{S}^2$ or real projective space $\mathbb{RP}^2$:

$$\mathbb{S}^2 : ds^2_{\mathbb{M}^2} = d\vartheta^2 + \sin^2\vartheta d\varphi^2, \quad \begin{cases} 0 \leq \vartheta \leq \pi \\ 0 \leq \varphi < 2\pi \end{cases}, \tag{3.20}$$

$$\mathbb{RP}^2 : ds^2_{\mathbb{M}^2} = d\vartheta^2 + \sin^2\vartheta d\varphi^2, \quad \begin{cases} 0 \leq \vartheta \leq \pi \\ 0 \leq \varphi < 2\pi \\ (\vartheta, \varphi) \sim (\pi - \vartheta, \pi + \varphi) \end{cases}. \tag{3.21}$$

As one can see, the difference between $\mathbb{S}^2$ and $\mathbb{RP}^2$ is the global information of antipodal identification $(\vartheta, \varphi) \sim (\pi - \vartheta, \pi + \varphi)$. Therefore, once we can construct a supersymmetry on $\mathbb{S}^2$, if and only if its representation is based on local Lagrangian description, we can project it into the theory on $\mathbb{RP}^2$. Its projection might looks trivial, however it is not true. For example, in mathematical point of view, we have the following 2nd homology groups

$$H_2(\mathbb{S}^2) = \mathbb{Z}, \quad H_2(\mathbb{RP}^2) = 0. \tag{3.22}$$

This means that the classical gauge field on $\mathbb{S}^2$ is labeled by the 1st Chern number, or equivalently monopole number. In addition to it, the fundamental groups are as follows.

$$\pi_1(\mathbb{S}^2) = 0, \quad \pi_1(\mathbb{RP}^2) = \mathbb{Z}_2. \tag{3.23}$$

This fact means that the classical gauge field on $\mathbb{RP}^2$ is labeled by the $\mathbb{Z}_2$-holonomy, or equivalently (discretized) Wilson line phases.

3.2.1 Our convention for spinors

We consider the following dreibein :

$$e^1 = d\vartheta, \quad e^2 = \sin\vartheta d\varphi, \quad e^3 = dt. \tag{3.24}$$

We use alphabets $a, b, c,$ for the local Lorentz indices.

Covariant derivative The 3d covariant derivative is defined by

$$\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\hat{\mathcal{J}}_{ab} \tag{3.25}$$

where ω_μ^{ab} is the spin connection computed from the dreibein (3.24),

$$de^a + \omega^{ab} \wedge e^b = 0, \quad \omega^{ba} = -\omega^{ab}, \quad \omega^{ab} = \omega_\mu^{ab} dx^\mu. \tag{3.26}$$

$\hat{\mathcal{J}}_{ab}$ are Lorentz generators of the fields characterized by its spin:

$$\begin{aligned} &\text{spin } 0 && \Rightarrow \hat{\mathcal{J}}_{ab} = 0, \\ &\text{spin } 1/2 && \Rightarrow \hat{\mathcal{J}}_{ab} = \gamma_{ab}, \\ &\text{spin } 1 && \Rightarrow (\hat{\mathcal{J}}_{ab})^c{}_d = 2(\delta^{ac}\delta^{bd} - \delta^{bc}\delta^{ad}), \end{aligned} \tag{3.27}$$

where γ_{ab} are antisymmetrized gamma matrices defined in (3.28).

Gamma matrices The gamma matrices γ_a are defined by the Pauli matrices

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_{ab} = \frac{1}{2}(\gamma_a\gamma_b - \gamma_b\gamma_a). \tag{3.28}$$

Spinor bilinear Our convention is as follows. Let us denote generic spinors by $\epsilon, \bar{\epsilon}$, and λ. We take spinor bilinears as

$$\epsilon\lambda = \begin{pmatrix} \epsilon_1 & \epsilon_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \quad \epsilon\gamma_a\lambda = \begin{pmatrix} \epsilon_1 & \epsilon_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \gamma_a \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}.$$

Using this convention, one can prove the following formulas:

$$\begin{aligned} &\epsilon\lambda = (-1)^{1+|\epsilon|\cdot|\lambda|}\lambda\epsilon, \quad \epsilon\gamma_a\lambda = (-1)^{|\epsilon|\cdot|\lambda|}\lambda\gamma_a\epsilon, \quad (\gamma_a\epsilon)\lambda = -\epsilon\gamma_a\lambda, \\ &\bar{\epsilon}(\epsilon\lambda) + (-1)^{1+|\epsilon|\cdot|\bar{\epsilon}|}\epsilon(\bar{\epsilon}\lambda) + (\bar{\epsilon}\epsilon)\lambda = 0, \quad (-1)^{1+|\epsilon|\cdot|\bar{\epsilon}|}\epsilon(\bar{\epsilon}\lambda) + 2(\bar{\epsilon}\epsilon)\lambda + (-1)^{1+|\lambda|\cdot|\epsilon|}(\bar{\epsilon}\gamma_a\lambda)\gamma^a\epsilon = 0, \end{aligned}$$

where $|\epsilon|$ means the spinor ϵ's statistics such that $|\epsilon| = 0$ for a bosonic ϵ and $|\epsilon| = 1$ for a fermonic ϵ.

3.2.2 Killing spinors

Now what we want to do is to construct SUSY QFTs on $\mathbb{M}^2 \times S^1_\beta$ with the metric

$$ds^2 = ds^2_{\mathbb{M}^2} + dt^2. \tag{3.29}$$

As well known, so-called superspace formalism is very useful to construct SUSY theories on flat space [63]. However, the curved superspace formalism is still under construction. (See [60, 64] for theories on 2,3 spheres.) So we take an ad-hoc way here.

One of the sufficient conditions to construct supersymmetry is the existence of Killing spinors [40]. With our metric (3.29) and dreibein (3.24), the following two spinors

$$\epsilon(\vartheta,\varphi,t) = e^{\frac{1}{2}(t+i\varphi)}\begin{pmatrix}\cos\frac{\vartheta}{2}\\ \sin\frac{\vartheta}{2}\end{pmatrix}, \quad \bar{\epsilon}(\vartheta,\varphi,t) = e^{\frac{-1}{2}(t+i\varphi)}\begin{pmatrix}\sin\frac{\vartheta}{2}\\ \cos\frac{\vartheta}{2}\end{pmatrix} \tag{3.30}$$

satisfy the following equations

$$\nabla_\mu\epsilon = \frac{1}{2}\gamma_\mu\gamma_3\epsilon, \quad \nabla_\mu\bar{\epsilon} = \frac{-1}{2}\gamma_\mu\gamma_3\bar{\epsilon}. \tag{3.31}$$

These spinors are Killing spinors in our case. In later discussion, we use these spinors $\bar{\epsilon}, \epsilon$.

3.2.3 $\mathcal{N}=2$ vector multiplet

Vector multiplet is constructed from a gauge field A_μ, an adjoint scalar field σ, an auxiliary field D, and adjoint 2-component spinors $\overline{\lambda}, \lambda$:

$$V := (A_\mu, \sigma, D \mid \overline{\lambda}, \lambda). \tag{3.32}$$

$\mathcal{N}=2$ supersymmetry is defined as follows [44]:

$$\delta_\epsilon A_\mu = -\frac{i}{2}\overline{\lambda}\gamma_\mu\epsilon, \quad \delta_{\bar{\epsilon}} A_\mu = -\frac{i}{2}\bar{\epsilon}\gamma_\mu\lambda, \tag{3.33}$$

$$\delta_\epsilon\sigma = +\frac{1}{2}\overline{\lambda}\epsilon, \quad \delta_{\bar{\epsilon}}\sigma = +\frac{1}{2}\bar{\epsilon}\lambda, \tag{3.34}$$

$$\delta_\epsilon\lambda = \frac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} - D\epsilon + i\gamma^\mu\epsilon\mathcal{D}_\mu\sigma + \frac{2i}{3}\sigma\gamma^\mu\nabla_\mu\epsilon, \quad \delta_{\bar{\epsilon}}\lambda = 0, \tag{3.35}$$

$$\delta_\epsilon\overline{\lambda} = 0, \quad \delta_{\bar{\epsilon}}\overline{\lambda} = \frac{1}{2}\gamma^{\mu\nu}\bar{\epsilon}F_{\mu\nu} + D\bar{\epsilon} - i\gamma^\mu\bar{\epsilon}\mathcal{D}_\mu\sigma - \frac{2i}{3}\sigma\gamma^\mu\nabla_\mu\bar{\epsilon}, \tag{3.36}$$

$$\delta_\epsilon D = +\frac{i}{2}\mathcal{D}_\mu\overline{\lambda}\gamma^\mu\epsilon - \frac{i}{2}[\overline{\lambda}\epsilon, \sigma] + \frac{i}{6}\overline{\lambda}\gamma^\mu\nabla_\mu\epsilon, \delta_{\bar{\epsilon}} D = -\frac{i}{2}\bar{\epsilon}\gamma^\mu\mathcal{D}_\mu\lambda + \frac{i}{2}[\bar{\epsilon}\lambda, \sigma] - \frac{i}{6}\nabla_\mu\bar{\epsilon}\gamma^\mu\lambda. \tag{3.37}$$

The covariant derivative is defined as

$$\mathcal{D}_\mu = \nabla_\mu - i[A_\mu, \circ]. \tag{3.38}$$

One can verify the following algebraic structure:

$$\{\delta_\epsilon, \delta_\epsilon\} = 0, \quad \{\delta_{\bar\epsilon}, \delta_{\bar\epsilon}\} = 0, \tag{3.39}$$

$$\{\delta_\epsilon, \delta_{\bar\epsilon}\} A_\mu = \xi^\nu \partial_\nu A_\mu + \partial_\mu \xi^\nu A_\nu + \mathcal{D}_\mu \Lambda, \tag{3.40}$$

$$\{\delta_\epsilon, \delta_{\bar\epsilon}\} \sigma = \xi^\mu \partial_\mu \sigma + i[\Lambda, \sigma], \tag{3.41}$$

$$\{\delta_\epsilon, \delta_{\bar\epsilon}\} \lambda = \xi^\mu \partial_\mu \lambda + \frac{1}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\lambda + i[\Lambda, \lambda] + \alpha\lambda, \tag{3.42}$$

$$\{\delta_\epsilon, \delta_{\bar\epsilon}\} \overline{\lambda} = \xi^\mu \partial_\mu \overline{\lambda} + \frac{1}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\overline{\lambda} + i[\Lambda, \overline{\lambda}] - \alpha\overline{\lambda}, \tag{3.43}$$

$$\{\delta_\epsilon, \delta_{\bar\epsilon}\} D = \xi^\mu \partial_\mu D + i[\Lambda, D]. \tag{3.44}$$

(3.40) - (3.44) relations mean

$$\{\delta_\epsilon, \delta_{\bar\epsilon}\} = \delta^{\xi}_{\text{Translation}} + \delta^{\Theta}_{\text{Rotation}} + \delta^{\Lambda}_{\text{Gauge transformation}} + \delta^{\alpha}_{\text{R-symmetry}}, \tag{3.45}$$

where each parameter is defined as follows.

$$\xi^\mu = i\bar\epsilon\gamma^\mu\epsilon, \tag{3.46}$$

$$\Theta^{\mu\nu} = \nabla^{[\mu}\xi^{\nu]} + \xi^\lambda \omega^{\mu\nu}_\lambda, \tag{3.47}$$

$$\Lambda = -A_\mu \xi^\mu + \sigma\bar\epsilon\epsilon, \tag{3.48}$$

$$\alpha = \frac{i}{3}(\nabla_\mu \bar\epsilon\gamma^\mu\epsilon - \bar\epsilon\gamma^\mu\nabla_\mu\epsilon). \tag{3.49}$$

3.2.4 $\mathcal{N} = 2$ matter multiplet

Matter multiplet is constructed from scalar fields $\phi, \overline{\phi}$, spinor fields $\psi, \overline{\psi}$, and auxiliary fields $F, \overline{F}$:

$$\Phi := (\phi, F \mid \psi), \quad \overline{\Phi} := (\overline{\phi}, \overline{F} \mid \overline{\psi}). \tag{3.50}$$

We can couple these fields to the vector multiplet (3.32) in supersymmetric way. In addition to it, we can assign arbitrary conformal dimension Δ to the matter multiplet (3.50). $\mathcal{N} = 2$ supersymmetry is defined as follows [44]:

$$\delta_\epsilon \phi = 0, \quad \delta_{\bar\epsilon}\phi = \bar\epsilon\psi, \tag{3.51}$$

$$\delta_\epsilon \overline{\phi} = \epsilon\overline{\psi}, \quad \delta_{\bar\epsilon}\overline{\phi} = 0, \tag{3.52}$$

$$\delta_\epsilon \psi = i\gamma^\mu\epsilon\mathcal{D}^A_\mu\phi + i\epsilon\sigma\phi + \frac{2\Delta i}{3}\gamma^\mu\nabla_\mu\epsilon\ \phi, \quad \delta_{\bar\epsilon}\psi = \bar\epsilon F, \tag{3.53}$$

$$\delta_\epsilon \overline{\psi} = \overline{F}\epsilon, \quad \delta_{\bar\epsilon}\overline{\psi} = i\gamma^\mu\bar\epsilon\mathcal{D}^A_\mu\overline{\phi} + i\overline{\phi}\sigma\bar\epsilon + \frac{2\Delta i}{3}\overline{\phi}\gamma^\mu\nabla_\mu\bar\epsilon, \tag{3.54}$$

$$\delta_\epsilon F = \epsilon(i\gamma^\mu\mathcal{D}^A_\mu\psi - i\sigma\psi - i\lambda\phi) + \frac{i}{3}(2\Delta - 1)\nabla_\mu\epsilon\gamma^\mu\psi, \quad \delta_{\bar\epsilon}F = 0, \tag{3.55}$$

$$\delta_\epsilon \overline{F} = 0, \quad \delta_{\bar\epsilon}\overline{F} = \bar\epsilon(i\gamma^\mu\mathcal{D}^A_\mu\overline{\psi} - i\overline{\psi}\sigma + i\overline{\phi\lambda}) + \frac{i}{3}(2\Delta - 1)\nabla_\mu\bar\epsilon\gamma^\mu\overline{\psi}. \tag{3.56}$$

We define the covariant derivative $\mathcal{D}_\mu^A$ as

$$\mathcal{D}_\mu^A \Phi = \mathcal{D}_\mu \Phi - iA_\mu \Phi, \quad \mathcal{D}_\mu^A \overline{\Phi} = \mathcal{D}_\mu \overline{\Phi} + i\overline{\Phi} A_\mu. \tag{3.57}$$

One can verify the following relations:

$$\{\delta_\epsilon, \delta_\epsilon\} = 0, \quad \{\delta_{\overline{\epsilon}}, \delta_{\overline{\epsilon}}\} = 0, \tag{3.58}$$

$$\{\delta_{\overline{\epsilon}}, \delta_\epsilon\}\phi = \xi^\mu \partial_\mu \phi + i\Lambda\phi - \Delta\alpha\phi, \tag{3.59}$$

$$\{\delta_{\overline{\epsilon}}, \delta_\epsilon\}\overline{\phi} = \xi^\mu \partial_\mu \overline{\phi} - i\overline{\phi}\Lambda + \Delta\alpha\overline{\phi}, \tag{3.60}$$

$$\{\delta_{\overline{\epsilon}}, \delta_\epsilon\}\psi = \xi^\mu \partial_\mu \psi + \frac{1}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\psi + i\Lambda\psi + (1-\Delta)\alpha\psi, \tag{3.61}$$

$$\{\delta_{\overline{\epsilon}}, \delta_\epsilon\}\overline{\psi} = \xi^\mu \partial_\mu \overline{\psi} + \frac{1}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\overline{\psi} - i\overline{\psi}\Lambda + (\Delta-1)\alpha\overline{\psi}, \tag{3.62}$$

$$\{\delta_{\overline{\epsilon}}, \delta_\epsilon\}F = \xi^\mu \partial_\mu F + i\Lambda F + (2-\Delta)\alpha F, \tag{3.63}$$

$$\{\delta_{\overline{\epsilon}}, \delta_\epsilon\}\overline{F} = \xi^\mu \partial_\mu \overline{F} - i\overline{F}\Lambda + (\Delta-2)\alpha\overline{F}. \tag{3.64}$$

Of course, we can interpret these relations in (3.45) way.

3.2.5 SUSY invariant Lagrangians

We summarize here the SUSY invariant Lagrangians which will be relevant in later discussion of this thesis.

Supersymmetric Yang-Mills term This action is automatically SUSY invariant because of the fact that it can be rewrite as $S_{YM} = \delta_\epsilon \mathcal{V}_V = \delta_{\overline{\epsilon}} \tilde{\mathcal{V}}_V$ for certain $\mathcal{V}_V, \tilde{\mathcal{V}}_V$, and thanks to the nilpotent character of $\delta_\epsilon, \delta_{\overline{\epsilon}}$ (3.39).

$$\begin{aligned} S_{YM} = \int d^3x\sqrt{g}\ \mathrm{Tr}\Big(+ \frac{1}{2}F_{\mu\nu}F^{\mu\nu} + D^2 + \mathcal{D}_\mu\sigma\cdot\mathcal{D}^\mu\sigma + \epsilon^{3\rho\sigma}\sigma F_{\rho\sigma} + \sigma^2 \\ + i\overline{\lambda}\gamma^\mu\mathcal{D}_\mu\lambda - i\overline{\lambda}[\lambda,\sigma] - \frac{i}{2}\overline{\lambda}\gamma_3\lambda\Big) \end{aligned} \tag{3.65}$$

Supersymmatric matter kinetic term This action is automatically SUSY invariant because of the fact that it can be rewrite as $S_{mat} = \delta_\epsilon \mathcal{V}_M = \delta_{\overline{\epsilon}} \tilde{\mathcal{V}}_M$ for certain $\mathcal{V}_M, \tilde{\mathcal{V}}_M$, and thanks to the nilpotent character of $\delta_\epsilon, \delta_{\overline{\epsilon}}$ (3.58).

$$\begin{aligned} S_{mat} = \int d^3x\sqrt{g}\ \Big(- i(\overline{\psi}\gamma^\mu\mathcal{D}^A_\mu\psi) + i(\overline{\psi}\sigma\psi) - i\overline{\phi}(\overline{\lambda}\psi) - \frac{i(2\Delta-1)}{2}(\overline{\psi}\gamma_3\psi) + \overline{F}F + i(\overline{\psi}\lambda)\phi \\ + \mathcal{D}^A_\mu\overline{\phi}\mathcal{D}^\mu_A\phi + \overline{\phi}\sigma^2\phi + i\overline{\phi}D\phi\ - (2\Delta-1)\overline{\phi}\mathcal{D}^A_3\phi - \frac{\Delta(2\Delta-1)}{2}\overline{\phi}\phi + \frac{\Delta}{4}R\overline{\phi}\phi\Big) \end{aligned} \tag{3.66}$$

Superpotential term We do not know how to construct superpotential terms on the curved space systematically. However, it must be possible in a certain way. For example, such a construction can be found in [60, 64]. In later discussion, we will use this term, however the result does not depends on this term thanks to the powerful calculation method, localization.

Killing spinor	ϵ	$\overline{\epsilon}$
spin	1/2	1/2
$\hat{R}$	$+1$	-1

Field	A_μ	σ	λ	$\overline{\lambda}$	D	ϕ	$\overline{\phi}$	ψ	$\overline{\psi}$	F	$\overline{F}$
spin	1	0	1/2	1/2	0	0	0	1/2	1/2	0	0
$\hat{R}$	0	0	$+1$	-1	0	$-\Delta$	Δ	$-(\Delta-1)$	$\Delta-1$	$-(\Delta-2)$	$\Delta-2$

Table 1: Charge assignments for each field. $\hat{R}$ is the R-charge appeared in (3.11).

3.3 Supersymmetric localization techniques

The mirror symmetry conjecture predicts an equivalence between two theories with *non-trivial interactions*. Therefore, the *exact check* sounds impossible in usual sense. However, a very interesting method had been introduced in [65] which provides an exact calculation method for path integrals of interacting SUSY theories on flat 4d space. This method is called *supersymmetric localization techniques*. After the discovery of it, this technique had been extended to the SUSY theories on four-sphere [34], three-sphere [27, 45, 44], and deformed spheres [47, 46, 50], and other various dimensional manifolds. We utilize this method on $\mathbb{M}^2 \times S^1_\beta$ [28, 42, 29] which give SCI. $\mathbb{M}^2$ represents two-sphere $\mathbb{S}^2$ or real projective plane $\mathbb{RP}^2$. The lower index β corresponds to the inverse temperature. The localization technique is applicable if there are

$$\begin{aligned} &\text{A SUSY} : \delta, \quad \text{A functional} : \mathcal{V}, \\ &\text{A SUSY exact action} : S = \delta\mathcal{V}, \quad \text{such that} \begin{cases} \delta S = 0 \\ S_{boson} \geq 0 \end{cases} . \end{aligned}$$

Note that the actions defined in (3.65) and (3.66) satisfy this condition. Then, the path integral

$$\int \mathcal{D}\phi\mathcal{D}\psi \; e^{-S[\phi,\psi]} \tag{3.67}$$

can be computed from

$$I(t) = \int \mathcal{D}\phi\mathcal{D}\psi \; e^{-tS[\phi,\psi]} \tag{3.68}$$

because $I(t)$ does not depend on t. One can derive this fact as follows.

$$\begin{aligned} \frac{dI(t)}{dt} &= \int \mathcal{D}\phi\mathcal{D}\psi(-S) \; e^{-tS} \\ &= \int \mathcal{D}\phi\mathcal{D}\psi(-\delta\mathcal{V}) \; e^{-tS} \\ &= -\int \mathcal{D}\phi\mathcal{D}\psi \; \delta(\mathcal{V}e^{-tS}) = 0. \end{aligned} \tag{3.69}$$

In order to perform the path integral (3.68), we can take the ultimate limit $t \to \infty$ because $I(t)$ does not depend on t! Then, the field configurations ϕ_0, ψ_0 which give

$$S[\phi_0, \psi_0] = \frac{\partial S}{\partial \phi}[\phi_0, \psi_0] = \frac{\partial S}{\partial \psi}[\phi_0, \psi_0] = 0, \tag{3.70}$$

dominate. We call them *locus* in the context of localization. Therefore, we can expand each field around the locus :

$$\phi = \phi_0 + \frac{1}{\sqrt{t}}\tilde{\phi},, \quad \psi = \psi_0 + \frac{1}{\sqrt{t}}\tilde{\psi}, \tag{3.71}$$

then the action becomes

$$tS[\phi,\psi] = \underbrace{\frac{1}{2}\tilde{\phi}\frac{\partial S}{\partial\phi\partial\phi}[\phi_0,\psi_0]\tilde{\phi} + \frac{1}{2}\tilde{\psi}\frac{\partial S}{\partial\psi\partial\psi}[\phi_0,\psi_0]\tilde{\psi}}_{:=\tilde{S}[\phi_0,\psi_0;\tilde{\phi},\tilde{\psi}]} + \mathcal{O}(t^{1/2}). \tag{3.72}$$

By taking $t \to \infty$, only the first two parts contribute. We define it as $\tilde{S}[\phi_0,\psi_0;\tilde{\phi},\tilde{\psi}]$. After taking into account the cancellation of t in the measure $\mathcal{D}\phi\mathcal{D}\psi$, the original path integral can be calculated by summing up all Gaussian contributions around the locus.

$$\int \mathcal{D}\phi\mathcal{D}\psi\ e^{-S[\phi,\psi]} = \sum_{\phi_0,\psi_0}\int \mathcal{D}\tilde{\phi}\mathcal{D}\tilde{\psi}\ e^{-\tilde{S}[\phi_0,\psi_0;\tilde{\phi},\tilde{\psi}]}. \tag{3.73}$$

Roughly speaking, this is the analog of the steepest decent method in usual integral on complex plane. We will utilize this method, and perform the exact check of (1.3).

4 Localization calculous of SCI with $\mathbb{M}^2 = \mathbb{S}^2$

In this section, we mainly review the calculations performed in [28, 41, 42, 43]. If we consider the $U(1)$ gauge theory, the action (3.65) defines free theory. It may sound not so interesting, however we can turn on the matter gauge coupling in (3.66) like usual QED, this is very nontrivial theory. Once we turn on the non-commutativity, there exist some different points in the argument, however the essence is same. Therefore, we focus on the gauge theory with abelian gauge field for simplicity.

4.1 Vector multiplet

Locus Now, let us remind that the Lagrangian (3.65), SUSY exact Lagrangian for vector multiplet. One can easily check that the Lagrangian defined in (3.65) can be deformed to

$$\begin{aligned}
&\mathcal{L}_{\text{YM}} = \mathcal{F}_\mu \mathcal{F}^\mu + D^2 + i\overline{\lambda}\gamma^\mu \mathcal{D}_\mu \lambda - \frac{i}{2}\overline{\lambda}\gamma_3\lambda, \\
&\mathcal{F}^\mu = \frac{1}{2}\epsilon^{\mu\rho\sigma}F_{\rho\sigma} + \partial^\mu\sigma + \delta^\mu_3\sigma.
\end{aligned} \tag{4.1}$$

The bosonic terms are obviously positive definite. Therefore, we can use this action as the $S = \delta\mathcal{V}$ term in (3.73), and the localization locus, which corresponds to the pair of configurations ϕ_0, ψ_0 in (3.70), is determined by the following equations:

$$0 = \mathcal{F}^\mu = D. \tag{4.2}$$

We can solve this BPS equation by taking

$$A = A_{\text{mon}} + \frac{\theta}{\beta}dt, \quad \sigma = -\frac{B}{2}, \tag{4.3}$$

where A_{mon} is defined as

$$A_{\text{mon}} = \frac{B}{2}(\kappa - \cos\vartheta)d\varphi, \quad \kappa = \begin{cases} +1 & \text{for } 0 \le \vartheta < \pi \\ -1 & \text{for } 0 < \vartheta \le \pi \end{cases}. \tag{4.4}$$

Thanks to the gauge symmetry, the parameters B, θ are constrained as[11]

$$B \in \mathbb{Z}, \quad \theta \in [0, 2\pi]. \tag{4.5}$$

[11] The reason for $B \in \mathbb{Z}$ is explained in the Appendix B. The condition for the θ can be also derived by the gauge symmetry.

As explained in the introduction, in the context of the supersymmetric localization, we expand field V around the locus V_0 which is parametrized by B, θ :

$$V = V_0[B,\theta] + \tilde{V}, \tag{4.6}$$

where $\tilde{V}$ represents fluctuation. It means that the path integral is composed from the summation over $B \in \mathbb{Z}$, integral over $\theta \in [0, 2\pi]$, and path integral over the fluctuation $\tilde{V}$:

$$\int \mathcal{D}V e^{-S_{YM}[V]} = \sum_{B\in\mathbb{Z}} \int_0^{2\pi} \frac{d\theta}{2\pi} \int \mathcal{D}\tilde{V} e^{-\tilde{S}_{YM}[\tilde{V}]}. \tag{4.7}$$

Action for the fluctuation $\tilde{V}$ We show here the action $\tilde{S}_{YM}[\tilde{V}]$ explicitly.

$$\tilde{\mathcal{S}}_{\text{boson}} = \int dt \int \sin\vartheta d\vartheta d\varphi \Big(\frac{1}{2}[\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu]^2 + (\partial_\mu \tilde{\sigma})^2 + \epsilon^{3\mu\nu} \tilde{\sigma}[\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu] + \tilde{\sigma}^2 \Big), \tag{4.8}$$

$$\tilde{\mathcal{S}}_{\text{fermion}} = \int dt \int \sin\vartheta d\vartheta d\varphi \Big(i \tilde{\bar{\lambda}} \gamma^\mu \nabla_\mu \tilde{\lambda} - \frac{i}{2} \tilde{\bar{\lambda}} \gamma_3 \tilde{\lambda} \Big). \tag{4.9}$$

For later simplicity, we will omit ˜ from now on, and divide the 3d gauge field A_μ to the S^1_β component A_t and 1-form on $\mathbb{S}^2$ $A_2 = A_\vartheta d\vartheta + A_\varphi d\varphi$ then the bosonic Lagrangian reduces to

$$\mathcal{S}_{\text{boson}} = \int dt \int \begin{pmatrix} A_2 \\ A_t \\ \sigma \end{pmatrix}^T \wedge *_2 \begin{pmatrix} - *_2 d_2 *_2 d_2 - \partial_t^2 & \partial_t d_2 & - *_2 d_2 \\ \partial_t *_2 d_2 *_2 & - *_2 d_2 *_2 d_2 & 0 \\ *_2 d_2 & 0 & - *_2 d_2 *_2 d_2 - \partial_t^2 + 1 \end{pmatrix} \begin{pmatrix} A_2 \\ A_t \\ \sigma \end{pmatrix}, \tag{4.10}$$

where $*_2$ is the Hodge star operator [66, 67, 68] on $\mathbb{S}^2$ defined by

$$*_2 1 = \sin\vartheta d\vartheta \wedge d\varphi, \quad *_2 d\vartheta = \sin\vartheta d\varphi, \quad *_2 d\varphi = -d\vartheta, \quad *_2 \sin\vartheta d\vartheta \wedge d\varphi = 1, \tag{4.11}$$

and d_2 is the exterior derivative along $\mathbb{S}^2$:

$$d_2 = \frac{\partial}{\partial\vartheta} d\vartheta + \frac{\partial}{\partial\varphi} d\varphi. \tag{4.12}$$

Gauge fixing procedure In order to calculate the path integral, even it is Gaussian, gauge fixing procedure is necessary. In usual procedure, one introduces Fadeev-Popov ghost fields, and construct BRST symmetry, etc. Here, we take more simpler root performed in [46, 69, 23]. The gauge orbit can be represented as follows.

$$\text{Gauge mode : } \begin{pmatrix} A_2^{(\eta)} \\ A_t^{(\eta)} \end{pmatrix} := \begin{pmatrix} i d_2 \eta \\ i \partial_t \eta \end{pmatrix}. \tag{4.13}$$

It gives *zero modes* for the fluctuation integral. We have to get rid of the mode from the path integral. It can be achieved by inserting

$$\delta(A^{(\eta)}). \tag{4.14}$$

into the path integral. However, the precise insertion is

$$\delta(\eta) \tag{4.15}$$

where η is the generator of the gauge transformation mode in (4.13). The Fadeev-Popov determinant is the factor recovering its discrepancy:

$$\delta(\eta) = \Delta_{FP}\ \delta(A^{(\eta)}). \tag{4.16}$$

The easiest way to calculate Δ_{FP} is as follows.

$$\begin{aligned} 1 = \int \mathcal{D}A^{(\eta)}\ e^{-\frac{1}{2}\langle A^{(\eta)}, A^{(\eta)}\rangle} &= \Delta_{FP} \int \mathcal{D}\eta\ e^{-\frac{1}{2}\langle A^{(\eta)}, A^{(\eta)}\rangle} \\ &= \Delta_{FP} \int \mathcal{D}\eta\ e^{-\frac{1}{2}\langle d\eta, d\eta\rangle} \\ &= \Delta_{FP} \int \mathcal{D}\eta\ e^{-\frac{1}{2}\langle \eta, d^\dagger d\eta\rangle} \end{aligned} \tag{4.17}$$

where the inner product for the gauge fields are defined by

$$\langle A, B\rangle = \int dt \int \sin\vartheta d\vartheta d\varphi\ A^\mu B_\mu. \tag{4.18}$$

Now, the precise measure for the gauge theory is

$$\Delta_{FP}\ \delta(A^{(\eta)})\mathcal{D}A^{(\eta)}\mathcal{D}A_\perp = \Delta_{FP}\ \mathcal{D}A_\perp, \tag{4.19}$$

where $A_\perp$ represents the modes perpendicular to the gauge mode $A^{(\eta)}$:

$$\langle A_\perp, A^{(\eta)}\rangle = 0. \tag{4.20}$$

As such mode, we can construct

$$\begin{pmatrix} A_2^{(\omega)} \\ A_t^{(\omega)} \end{pmatrix} := \begin{pmatrix} \partial_t d_2\omega \\ \Delta_0\omega \end{pmatrix}, \qquad \text{where } \Delta_0 = -*_2\, d_2 *_2 d_2. \tag{4.21}$$

This mode gives $\mathcal{S}_{\text{boson}} = \frac{1}{2}\langle A^{(\omega)}, d^\dagger dA^{(\omega)}\rangle$, and it gives

$$\int \mathcal{D}A^{(\omega)}\ e^{-\mathcal{S}_{\text{boson}}} = \frac{1}{\triangle_{FP}}. \tag{4.22}$$

Therefore, if we can identify the remaining modes which are perpendicular to both of (4.13) and (4.21), we can forget complicated effects of the gauge fixing procedure. And the modes are represented as follows.

$$A_t = 0, \quad *_2 d_2 *_2 A_2 = 0. \tag{4.23}$$

The second condition is equivalent to the Coulomb gauge condition

$$\nabla_i A^i = 0, \tag{4.24}$$

where i runs for ϑ, φ. In summary, what we have to consider is the path integral over $(A_i, \sigma \mid \overline{\lambda}, \lambda)$ weighted by the following actions.

Actions for the fluctuation fields

$$\mathcal{S}^{gf}_{\text{boson}} = \int dt \int \begin{pmatrix} A_2 \\ \sigma \end{pmatrix}^T \wedge *_2 \begin{pmatrix} - *_2 d_2 *_2 d_2 - \partial_t^2 & - *_2 d_2 \\ *_2 d_2 & - *_2 d_2 *_2 d_2 - \partial_t^2 + 1 \end{pmatrix} \begin{pmatrix} A_2 \\ \sigma \end{pmatrix}, \tag{4.25}$$

$$\mathcal{S}_{\text{fermion}} = \int dt \int \sin\vartheta d\vartheta d\varphi \, \overline{\lambda}\Big(i\gamma^i \nabla_i + i\gamma_3 (\partial_t - \frac{1}{2})\Big)\lambda, \tag{4.26}$$

constrained by (4.24).

4.1.1 QFT on $\mathbb{S}^2 \times S^1_\beta \to$ QM on S^1_β

Now, we take the following eigenfunction expansion:

$$A^i(\vartheta, \varphi, t) = \sum_{j=1}^{\infty} \sum_{m=-j}^{j} V^i_{jm}(\vartheta, \varphi) A_{jm}(t), \tag{4.27}$$

$$\sigma(\vartheta, \varphi, t) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} Y_{jm}(\vartheta, \varphi) \sigma_{jm}(t), \tag{4.28}$$

$$\lambda(\vartheta, \varphi, t) = \sum_{j=1/2}^{\infty} \sum_{m=-j}^{j} \sum_{\epsilon} \Upsilon^\epsilon_{jm}(\vartheta, \varphi) \lambda^\epsilon_{jm}(t), \tag{4.29}$$

$$\overline{\lambda}(\vartheta, \varphi, t) = \sum_{j=1/2}^{\infty} \sum_{m=-j}^{j} \sum_{\epsilon} {\Upsilon^\epsilon_{jm}}^\dagger(\vartheta, \varphi) \overline{\lambda}^\epsilon_{jm}(t), \tag{4.30}$$

where $V^i_{jm}, Y_{jm}, \Upsilon^\epsilon_{jm}$ are spherical harmonics with zero monopole $B = 0$ explained in the Appendix B. Then, the actions (4.25) and (4.26) gives many-body quantum mechanics :

$$\mathcal{S}^{gf}_{\text{boson}} = \sum_{j=1}^{\infty}\sum_{m=-j}^{j}\int dt\ \begin{pmatrix} A_{jm} & \sigma_{jm} \end{pmatrix} \begin{pmatrix} -\partial_t^2 + j(j+1) & \sqrt{j(j+1)} \\ \sqrt{j(j+1)} & -\partial_t^2 + j(j+1) + 1 \end{pmatrix} \begin{pmatrix} A_{jm} \\ \sigma_{jm} \end{pmatrix} + \int dt\ \sigma_0(-\partial_t^2 + 1)\sigma_0, \tag{4.31}$$

$$\mathcal{S}_{\text{fermion}} = \sum_{j=1/2}^{\infty}\sum_{m=-j}^{j}\int dt\ \begin{pmatrix} \overline{\lambda}^{-}_{jm} & \overline{\lambda}^{+}_{jm} \end{pmatrix} \begin{pmatrix} j+\frac{1}{2} & i(\partial_t - \frac{1}{2}) \\ i(\partial_t - \frac{1}{2}) & -(j+\frac{1}{2}) \end{pmatrix} \begin{pmatrix} \lambda^{-}_{jm} \\ \lambda^{+}_{jm} \end{pmatrix} \tag{4.32}$$

The periodicities for each set factor can be read from the definition of SCI (3.1) and Table 1, then,

$$A_{jm}(t+\beta) = e^{-(\beta_1-\beta_2)m}A_{jm}(t), \quad \sigma_{jm}(t+\beta) = e^{-(\beta_1-\beta_2)m}\sigma_{jm}(t), \tag{4.33}$$

$$\overline{\lambda}^{\epsilon}_{jm}(t+\beta) = e^{(-1-m)\beta_1+m\beta_2}\overline{\lambda}^{\epsilon}_{jm}(t), \quad \lambda^{\epsilon}_{jm}(t+\beta) = e^{(+1-m)\beta_1+m\beta_2}\lambda^{\epsilon}_{jm}(t). \tag{4.34}$$

Then, we can calculate the contributions explicitly as follows.

Bosonic part

$$\begin{aligned}\int \mathcal{D}A_2\mathcal{D}\sigma\ e^{-\mathcal{S}^{gf}_{\text{boson}}} &= \int \prod_{t\in[0,\beta]}\Big(d\sigma_0(t)\prod_{j=1}^{\infty}\prod_{m=-j}^{j} dA_{jm}(t)d\sigma_{jm}(t)\Big)e^{-\mathcal{S}^{gf}_{\text{boson}}} \\ &= \prod_{\tilde{j}=1}^{\infty}\prod_{\tilde{m}=-\tilde{j}}^{\tilde{j}-1}\frac{1}{\Big(2\sinh\frac{\beta\omega_{\tilde{j},-\tilde{m}}}{2}\Big)\Big(2\sinh\frac{\beta\omega_{\tilde{j}\tilde{m}}}{2}\Big)},\end{aligned} \tag{4.35}$$

where

$$\omega_{jm} = \frac{\beta_1-\beta_2}{\beta}m + j. \tag{4.36}$$

Note that the $\tilde{m}$ in resulting product runs for $(-\tilde{j}) \sim (\tilde{j}-1)$ not $(-\tilde{j}) \sim (\tilde{j})$. One can derive this results as follows. For simplicity let us denote $(\tilde{m},\tilde{j}) := \prod_{n\in\mathbb{Z}}\Big(\Big[\frac{2\pi}{\beta}n + i\frac{\beta_1-\beta_2}{\beta}\tilde{m}\Big]^2 + \tilde{j}^2\Big)$, then the denominator of (4.35) is a square root of products of the following towers:

$$
\begin{aligned}
&(0,1)\\
(\mp 1,1),&(0,1)\\
&(0,2),(\pm 1,2)\\
(\mp 2,2),(\mp 1,2),&(0,2)\\
&(0,3),(\pm 1,3),(\pm,2,3)\\
(\mp 3,3),(\mp 2,3),(\mp 1,3)&(0,3)\\
&\cdots
\end{aligned}
\tag{4.37}
$$

Easily noticed, $(-\tilde{m},\tilde{j})=(\tilde{m},\tilde{j})$, so we get the result after the zeta-function regularization.

Fermionic part

$$
\begin{aligned}
\int \mathcal{D}\overline{\lambda}\mathcal{D}\lambda\ e^{-\mathcal{S}_{\text{fermion}}} &= \int \prod_{t\in[0,\beta]} \Big(\prod_{j=1/2}^{\infty} \prod_{m=-j}^{j} d\overline{\lambda}^{+}_{jm}(t) d\overline{\lambda}^{-}_{jm}(t) d\lambda^{+}_{jm}(t) d\lambda^{-}_{jm}(t) \Big) e^{-\mathcal{S}_{\text{fermion}}}\\
&= \prod_{\tilde{j}=1}^{\infty} \prod_{\tilde{m}=-\tilde{j}}^{\tilde{j}-1} \Big(2\sinh\frac{\beta\omega_{\tilde{j},-\tilde{m}}}{2}\Big)\Big(2\sinh\frac{\beta\omega_{\tilde{j}\tilde{m}}}{2}\Big),
\end{aligned}
\tag{4.38}
$$

where ωs are same ones in (4.36).

Therefore, the numerator and the denominator in (4.35) and (4.38) cancel out, and we get somewhat trivial 1-loop determinant.

Total

$$
\begin{aligned}
&\int \mathcal{D}A_2\mathcal{D}\sigma\mathcal{D}\overline{\lambda}\mathcal{D}\lambda\ e^{-\mathcal{S}^{gf}_{\text{boson}}-\mathcal{S}_{\text{fermion}}}\\
&=1.
\end{aligned}
\tag{4.39}
$$

In later section, we will see non-trivial contribution emerges when we consider the theory not on $\mathbb{S}^2$ but $\mathbb{RP}^2$.

4.2 Matter multiplet

First of all, the matter field in gauge theory is defined by assigning a certain representation of the gauge group. With $U(1)$ gauge group, the matter representation becomes the $U(1)$ charge $\boldsymbol{q}\in\mathbb{R}$.

Locus The matter Lagrangian (3.66) defines the trivial locus.

$$0 = \phi = \psi = F, \quad 0 = \overline{\phi} = \overline{\psi} = \overline{F}. \tag{4.40}$$

So, there is no need for summation for matter sector. And we get the following actions for the fluctuation fields. We omit ˜ and integrate out the auxiliary fields for simplicity.

Actions for the fluctuation fields

$$\mathcal{S}_{\text{boson}} = \int dt \int \sin\vartheta d\vartheta d\varphi \Big(\mathfrak{D}_\mu \overline{\phi} \mathfrak{D}^\mu \phi + \frac{(\boldsymbol{q}B)^2}{2^2} \overline{\phi}\phi - (2\Delta - 1)\overline{\phi}\mathfrak{D}_t \phi - \Delta(\Delta-1)\overline{\phi}\phi \Big), \tag{4.41}$$

$$\mathcal{S}_{\text{fermion}} \int dt \int \sin\vartheta d\vartheta d\varphi \Big(- i(\overline{\psi}\gamma^\mu \mathfrak{D}_\mu \psi) - i\frac{\boldsymbol{q}B}{2}(\overline{\psi}\psi) - \frac{i(2\Delta - 1)}{2}(\overline{\psi}\gamma_3\psi) \Big), \tag{4.42}$$

where $\mathfrak{D}_\mu$ represent the covariant derivative with respect to the locus gauge field (4.3):

$$\mathfrak{D}_i = \nabla_i - i\boldsymbol{q}A_i^{\text{mon}}, \quad (i = \vartheta, \varphi) \tag{4.43}$$

$$\mathfrak{D}_t = \partial_t - i\boldsymbol{q}\frac{\theta}{\beta}. \tag{4.44}$$

The charge $\boldsymbol{q}$ must be in integers in order to make the gauge transformation of the matter fields as single valued function.

4.2.1 QFT on $\mathbb{S}^2 \times S^1_\beta \to$ QM on S^1_β

As performed in the previous subsection, we expand the component fields as follows:

$$\phi(\vartheta, \varphi, t) = \sum_{j=\frac{|\boldsymbol{q}B|}{2}}^{\infty} \sum_{m=-j}^{j} Y_{\frac{|\boldsymbol{q}B|}{2}, jm}(\vartheta, \varphi)\phi_{jm}(t) \tag{4.45}$$

$$\psi(\vartheta, \varphi, t) = \sum_{j=\frac{|\boldsymbol{q}B|}{2}+1/2}^{\infty} \sum_{m=-j}^{j} \sum_{\epsilon} \Upsilon^{\epsilon}_{\frac{|\boldsymbol{q}B|}{2}, jm}(\vartheta, \varphi)\psi^{\epsilon}_{jm}(t) + \sum_{m=1/2-\frac{|\boldsymbol{q}B|}{2}}^{\frac{|\boldsymbol{q}B|}{2}-1/2} \Upsilon^{0}_{\frac{|\boldsymbol{q}B|}{2}, m}(\vartheta, \varphi)\psi^{0}_{m}(t) \tag{4.46}$$

$$\overline{\phi}(\vartheta, \varphi, t) = \sum_{j=\frac{|\boldsymbol{q}B|}{2}}^{\infty} \sum_{m=-j}^{j} Y^*_{\frac{|\boldsymbol{q}B|}{2}, jm}(\vartheta, \varphi)\overline{\phi}_{jm}(t) \tag{4.47}$$

$$\overline{\psi}(\vartheta, \varphi, t) = \sum_{j=\frac{|\boldsymbol{q}B|}{2}+1/2}^{\infty} \sum_{m=-j}^{j} \sum_{\epsilon} {\Upsilon^{\epsilon}_{\frac{|\boldsymbol{q}B|}{2}, jm}}^{\dagger}(\vartheta, \varphi)\overline{\psi}^{\epsilon}_{jm}(t) + \sum_{m=1/2-\frac{|\boldsymbol{q}B|}{2}}^{\frac{|\boldsymbol{q}B|}{2}-1/2} {\Upsilon^{0}_{\frac{|\boldsymbol{q}B|}{2}, m}}^{\dagger}(\vartheta, \varphi)\overline{\psi}^{0}_{m}(t), \tag{4.48}$$

where $Y_{q,jm}, \Upsilon^{\epsilon}_{q,jm}$ are monopole harmonics explained in the Appendix B. Then, the action (5.33) and (5.34) gives many-body quantum mechanics :

$$\mathcal{S}_{\text{boson}} = \sum_{j=\frac{|\boldsymbol{q}B|}{2}}^{\infty} \sum_{m=-j}^{j} \int dt\ \overline{\phi}_{jm}(j+\Delta+\mathfrak{D}_t)(j+1-\Delta-\mathfrak{D}_t)\phi_{jm} \tag{4.49}$$

$$\begin{aligned}\mathcal{S}_{\text{boson}} = &\sum_{j=\frac{|\boldsymbol{q}B|}{2}+1/2}^{\infty} \sum_{m=-j}^{j} \int dt\ \begin{pmatrix}\overline{\psi}^{+}_{jm} & \overline{\psi}^{-}_{jm}\end{pmatrix} \begin{pmatrix} -\frac{\sqrt{(2j+1)^2-(\boldsymbol{q}B)^2}}{2} - i\frac{\boldsymbol{q}B}{2} & -i\mathfrak{D}_t - i\frac{2\Delta-1}{2} \\ -i\mathfrak{D}_t - i\frac{2\Delta-1}{2} & +\frac{\sqrt{(2j+1)^2-(\boldsymbol{q}B)^2}}{2} - i\frac{\boldsymbol{q}B}{2}\end{pmatrix} \begin{pmatrix}\psi^{+}_{jm} \\ \psi^{-}_{jm}\end{pmatrix} \\ &- i\frac{B}{|B|} \sum_{m=1/2-\frac{|\boldsymbol{q}B|}{2}}^{\frac{|\boldsymbol{q}B|}{2}-1/2} \int dt\ \overline{\psi}^{0}_{m}\Big(j+\Delta+\mathfrak{D}_t\Big)\psi^{0}_{m}\end{aligned} \tag{4.50}$$

The periodicities for each det factor can be read from the definition of SCI (3.1) and Table 1 :

$$\phi_{jm}(t+\beta) = e^{(-\Delta-m)\beta_1+m\beta_2+i\mu}\phi_{jm}(t) \tag{4.51}$$

$$\psi_{jm}(t+\beta) = e^{(-\Delta+1-m)\beta_1+m\beta_2+i\mu}\psi_{jm}(t) \tag{4.52}$$

Then each factor becomes as follows.

Bosonic pert

$$\int \mathcal{D}\overline{\phi}\mathcal{D}\phi\ e^{-\mathcal{S}_{\text{boson}}} = \prod_{j\geq\frac{|\boldsymbol{q}B|}{2}} \prod_{m=-j}^{j} \frac{1}{\Big(2\sinh\frac{\beta\omega^{1}_{jm}}{2}\Big)\Big(2\sinh\frac{\beta\omega^{2}_{jm}}{2}\Big)}, \tag{4.53}$$

where

$$\beta\omega^{1}_{jm} = -i\boldsymbol{q}\theta + (j-m)\beta_1 + (j+\Delta+m)\beta_2 + i\mu, \tag{4.54}$$

$$\beta\omega^{2}_{jm} = -i\boldsymbol{q}\theta - (j+1+m)\beta_1 - (j+1-\Delta-m)\beta_2 + i\mu. \tag{4.55}$$

Fermionic part

$$\int \mathcal{D}\overline{\psi}\mathcal{D}\psi\ e^{-\mathcal{S}_{\text{fermion}}} = \prod_{\tilde{j}\geq\frac{|\boldsymbol{q}B|}{2}} \Big(\prod_{\tilde{m}=-\tilde{j}}^{\tilde{j}-1} 2\sinh\frac{\beta\omega^{1}_{\tilde{j}\tilde{m}}}{2}\Big)\Big(\prod_{\tilde{m}=-\tilde{j}-1}^{\tilde{j}} 2\sinh\frac{\beta\omega^{2}_{\tilde{j}\tilde{m}}}{2}\Big), \tag{4.56}$$

First term in fermionic term looks similar to the first factors of bosonic term in (4.53), but lacking the contribution of $m = j$. So this fermionic contribution cancels almost half of the bosonic contributions. Second term in fermionic part looks similar to the second factors in (4.53), there is contributions of $\tilde{m} = -\tilde{j} - 1$ in surplus. So this fermionic contribution cancels almost half of the bosonic contributions. Therefore, we get the following total contribution.

Total

$$\int \mathcal{D}\overline{\phi}\mathcal{D}\phi\mathcal{D}\overline{\psi}\mathcal{D}\psi \; e^{-\mathcal{S}_{\text{boson}}-\mathcal{S}_{\text{fermion}}} = \prod_{j \geq \frac{|\boldsymbol{q}B|}{2}} \frac{2\sinh\frac{\beta\omega^2_{j,-j-1}}{2}}{2\sinh\frac{\beta\omega^1_{j,j}}{2}}. \tag{4.57}$$

Another representation In later section, we will use more useful representation of (4.57). We can shift the product with respect to j by defining

$$J = j - \frac{|\boldsymbol{q}B|}{2}, \tag{4.58}$$

then

$$(4.57) = \prod_{J=0}^{\infty} \frac{2\sinh\frac{\beta\omega_f^{(J)}}{2}}{2\sinh\frac{\beta\omega_b^{(J)}}{2}}, \tag{4.59}$$

where we define $\beta\omega_f^{(J)}, \beta\omega_b^{(J)}$ as follows

$$\beta\omega_f^{(J)} = i(\boldsymbol{q}\theta - \mu) + 2\beta_2\Big(J + 1 + \frac{|\boldsymbol{q}B|}{2} - \frac{\Delta}{2}\Big) \tag{4.60}$$

$$\beta\omega_b^{(J)} = -i(\boldsymbol{q}\theta - \mu) + 2\beta_2\Big(J + \frac{|\boldsymbol{q}B|}{2} + \frac{\Delta}{2}\Big) \tag{4.61}$$

Here we ignore the overall sign. Now, after simple deformations, we get the following representation.

Another representation of (4.57)

$$\left(x^{(1-\Delta)}e^{-i\boldsymbol{q}\theta}\alpha^{-1}\right)^{\frac{|\boldsymbol{q}B|}{2}} \frac{(e^{-i\boldsymbol{q}\theta}\alpha^{-1}x^{2-\Delta+|\boldsymbol{q}B|}; x^2)_\infty}{(e^{i\boldsymbol{q}\theta}\alpha^{+1}x^{\Delta+|\boldsymbol{q}B|}; x^2)_\infty}, \tag{4.62}$$

where $(z, q)_\infty$ is called *quantum Pochhammer symbol* or *q-shifted factorial* [70] :

$$(A; q)_\infty = \prod_{J=0}^{\infty}(1 - Aq^J). \tag{4.63}$$

We used zeta function regularization to get the prefactor here. As one can noticed by comparing it to the calculation of free harmonic oscillator in Section 2, this part corresponds to the Casimir energy.

4.3 Result

We summarize here a toolkit for making SCI of our SUSY theories on $\mathbb{S}^2 \times S^1_\beta$.

4.3.1 Non gauge theory

In this case, we assume that there are dynamical fields,

$$\Phi_a = (\phi_a, F_a|\psi_a), \quad \overline{\Phi}_a = (\overline{\phi}_a, \overline{F}_a|\overline{\psi}_a), \quad a = 1, ..., N_f. \tag{4.64}$$

We assign each multiplet with dimension Δ_a and flavor charge $\boldsymbol{f}_a$. Our method can be applied to the theories with the following type of action:

$$\mathcal{S}[\Phi, \overline{\Phi}] = \sum_{a=1}^{N_f} \mathcal{S}^{\boldsymbol{q}=0}_{mat}[\Phi_a, \overline{\Phi}_a] + W[\Phi] + \overline{W}[\overline{\Phi}], \tag{4.65}$$

where $\mathcal{S}^{\boldsymbol{q}=0}_{mat}$ is the action (3.66) with $\boldsymbol{q} = 0$. We can take arbitrary superpotential W. The only restriction is that the flavor charge assignments $\boldsymbol{f}_a$ have to preserve W.

SCI for non gauge theory on $\mathbb{S}^2 \times S^1_\beta$

In this case, the SCI is simple:

$$\mathcal{I}(x, \alpha) = \prod_{a=1}^{N_f} \frac{(\alpha^{-\boldsymbol{f}_a} x^{2-\Delta_a}; x^2)_\infty}{(\alpha^{+\boldsymbol{f}_a} x^{\Delta_a}; x^2)_\infty}. \tag{4.66}$$

4.3.2 Gauge theory

For simplicity, we consider the $U(1)$ gauge theory with single gauge field (vector multiplet):

$$V = (A_\mu, \sigma, D|\overline{\lambda}, \lambda). \tag{4.67}$$

Of course, we can add charged matter multiplets :

$$\Phi_a = (\phi_a, F_a|\psi_a), \quad \overline{\Phi}_a = (\overline{\phi}_a, \overline{F}_a|\overline{\psi}_a), \quad a = 1, ..., N_f, \tag{4.68}$$

with $\Delta_a, \boldsymbol{f}_a$ and $U(1)$ charges $\boldsymbol{q}_a$. We assume action as follows.

$$\mathcal{S}[V; \Phi, \overline{\Phi}] = \mathcal{S}_{YM}[V] + \sum_{a=1}^{N_f} \mathcal{S}^{\boldsymbol{q}_a}_{mat}[V; \Phi_a, \overline{\Phi}_a] + W[\Phi] + \overline{W}[\overline{\Phi}], \tag{4.69}$$

where $\mathcal{S}_{YM}$ is the action (3.65) with $U(1)$ gauge group. See [28, 42] for more detail.

SCI for gauge theory on $\mathbb{S}^2 \times S^1_\beta$

We should sum up $B \in \mathbb{Z}$ and integrate $\theta \in [0, 2\pi]$:

$$\mathcal{I}(x,\alpha) = \sum_{B\in\mathbb{Z}} \int_0^{2\pi} \frac{d\theta}{2\pi} \prod_{a=1}^{N_f} \left(x^{(1-\Delta_a)} e^{-i\boldsymbol{q}_a\theta} \alpha^{-\boldsymbol{f}_a} \right)^{\frac{|\boldsymbol{q}_a B|}{2}} \frac{(e^{-i\boldsymbol{q}_a\theta}\alpha^{-\boldsymbol{f}_a} x^{2-\Delta_a+|\boldsymbol{q}_a B|}; x^2)_\infty}{(e^{i\boldsymbol{q}_a\theta}\alpha^{+\boldsymbol{f}_a} x^{\Delta_a+|\boldsymbol{q}_a B|}; x^2)_\infty}. \tag{4.70}$$

5 Localization calculous of SCI with $\mathbb{M}^2 = \mathbb{RP}^2$

In this section, we explain our main results on new SCI by taking $\mathbb{M}^2 = \mathbb{RP}^2$. The curved space $\mathbb{RP}^2 \times S^1_\beta$ can be constructed from $\mathbb{S}^2 \times S^1_\beta$ by taking the identification

$$(\pi - \vartheta, \pi + \varphi, t) \sim (\vartheta, \varphi, t). \tag{5.1}$$

And the QFT on $\mathbb{RP}^2 \times S^1_\beta$ is defined by imposing a boundary condition, we will call it *parity condition*, under the antipodal identification (5.1) on $\mathbb{S}^2 \times S^1_\beta$. However, we cannot take arbitrary parity condition because most of them break the supersymmetry and it spoils the validity for using supersymmetric localization techniques. Therefore, we start our argument from the discussion of the possible *supersymmetric parity condition* which preserves supersymmetry under the antipodal identification (5.1). This a very simple operation causes very non-trivial effects, for example the localization locus for vector multiplet drastically changes, and the resulting SCIs differ from the ones in Section 4.

5.1 Supersymmetric parity conditions

We can define field theories on $\mathbb{RP}^2 \times S^1_\beta$ by imposing appropriate parity condition under (5.1) on $\mathbb{S}^2 \times S^1_\beta$. Of course, in order to use the localization method, we have to preserve supersymmetry. As studied in [23] in the context of 2d supersymmetric field theory, we can find such parity conditions compatible with the antipodal identification (5.1) for component fields. Our guiding principles are as follows.

- The squared parity transformation becomes $+1$ for bosons and -1 for fermions.
- SUSY exact Lagrangians, (3.65) and (3.66), must be invariant under the parity.
- Supersymmetries, δ_ϵ and $\delta_{\bar{\epsilon}}$, must be consistent with the parity.

Let us comment on the second assumption. This requirement is too strong because one should assume parity invariance of not (3.65) or (3.66) alone, but *full* Lagrangian, e.g. (4.69). We will comment on this generic case in Section 7.

Vector multiplet We find a set of parity conditions for the vector multiplet as follows.

$$\begin{aligned}
&A_\vartheta(\pi-\vartheta,\pi+\varphi,t)=-A_\vartheta(\vartheta,\varphi,t), \quad A_{\varphi,t}(\pi-\vartheta,\pi+\varphi,t)=+A_{\varphi,t}(\vartheta,\varphi,t),\\
&\sigma(\pi-\vartheta,\pi+\varphi,t)=-\sigma(\vartheta,\varphi,t),\\
&\lambda(\pi-\vartheta,\pi+\varphi,t)=+i\gamma_1\lambda(\vartheta,\varphi,t), \quad \overline{\lambda}(\pi-\vartheta,\pi+\varphi,t)=-i\gamma_1\overline{\lambda}(\vartheta,\varphi,t),\\
&D(\pi-\vartheta,\pi+\varphi,t)=+D(\vartheta,\varphi,t).
\end{aligned} \tag{5.2}$$

One flavor matter multiplet The one flavor matter multiplet has two choices:

$$\begin{aligned}
&\phi(\pi-\vartheta,\pi+\varphi,t)=\pm\phi(\vartheta,\varphi,t), &&\overline{\phi}(\pi-\vartheta,\pi+\varphi,t)=\pm\overline{\phi}(\vartheta,\varphi,t),\\
&\psi(\pi-\vartheta,\pi+\varphi,t)=\mp i\gamma_1\psi(\vartheta,\varphi,t), &&\overline{\psi}(\pi-\vartheta,\pi+\varphi,t)=\pm i\gamma_1\overline{\psi}(\vartheta,\varphi,t),\\
&F(\pi-\vartheta,\pi+\varphi,t)=\pm F(\vartheta,\varphi,t), &&\overline{F}(\pi-\vartheta,\pi+\varphi,t)=\pm\overline{F}(\vartheta,\varphi,t).
\end{aligned} \tag{5.3}$$

Many flavors matter multiplets We use $a, b, ...$ as flavor indices $a = 1, ..., N_f$, then

$$\begin{aligned}
&\phi_a(\pi-\vartheta,\pi+\varphi,t)=\sum_{b=1}^{N_f}M_{ab}\phi_b(\vartheta,\varphi,t), &&\overline{\phi}_a(\pi-\vartheta,\pi+\varphi,t)=\sum_{b=1}^{N_f}N_{ab}\overline{\phi}_b(\vartheta,\varphi,t),\\
&\psi_a(\pi-\vartheta,\pi+\varphi,t)=-i\gamma_1\sum_{b=1}^{N_f}M_{ab}\psi_b(\vartheta,\varphi,t), &&\overline{\psi}_a(\pi-\vartheta,\pi+\varphi,t)=i\gamma_1\sum_{b=1}^{N_f}N_{ab}\overline{\psi}_b(\vartheta,\varphi,t),\\
&F_a(\pi-\vartheta,\pi+\varphi,t)=\sum_{b=1}^{N_f}M_{ab}F_b(\vartheta,\varphi,t), &&\overline{F}_a(\pi-\vartheta,\pi+\varphi,t)=\sum_{b=1}^{N_f}N_{ab}\overline{F}_b(\vartheta,\varphi,t),
\end{aligned} \tag{5.4}$$

where $(M_{ab})_{a,b=1,...,N_f} = \mathbf{M}$ and $(N_{ab})_{a,b=1,...,N_f} = \mathbf{N}$ are $N_f \times N_f$ matrices constrained by

$$\mathbf{N}^{\mathrm{T}}\mathbf{M}=\mathbf{1}, \quad \mathbf{M}^2=\mathbf{N}^2=\mathbf{1}. \tag{5.5}$$

Comments on the parity condition Suppose we have a doublet and the parity condition described by the 2×2 matrices

$$\mathbf{M}=\mathbf{N}=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \tag{5.6}$$

then we can lift its Lagrangian on $\mathbb{RP}^2 \times S^1_\beta$ to the one on $\mathbb{S}^2 \times S^1_\beta$ by defining a new matter multiplet on $\mathbb{S}^2 \times S^1_\beta$ as

$$\Phi(\vartheta,\varphi,t)=\begin{cases}\Phi_1(\vartheta,\varphi,t), & \vartheta\in[0,\frac{\pi}{2}]\\ \Phi_2(\vartheta,\varphi,t), & \vartheta\in[\frac{\pi}{2},\pi]\end{cases}. \tag{5.7}$$

The authors of [23] also commented on this fact. This is quite similar to the *doubling trick* in string theory. In Section 6, we use such parity condition exactly in the context of 3d mirror symmetry.

5.2 Vector multiplet contribution

We focus on the gauge theory with abelian gauge field for simplicity as same as in the previous section.

Locus Now, let us remind that the Lagrangian (3.65) again,

$$\begin{aligned}\mathcal{L}_{\mathrm{YM}} &= \mathcal{F}_\mu \mathcal{F}^\mu + D^2 + i\overline{\lambda}\gamma^\mu \mathcal{D}_\mu \lambda - \frac{i}{2}\overline{\lambda}\gamma_3\lambda,\\ \mathcal{F}^\mu &= \frac{1}{2}\epsilon^{\mu\rho\sigma}F_{\rho\sigma} + \partial^\mu\sigma + \delta_3^\mu\sigma.\end{aligned} \tag{5.8}$$

The bosonic terms are obviously positive definite. Therefore, the localization locus is determined by the following equations:

$$0 = \mathcal{F}^\mu = D. \tag{5.9}$$

However, we cannot take the Dirac monopole configuration A_{mon} in (4.3) because it breaks parity invariance under (5.2). Instead of it, we can take the flat connection $A^{(\pm)}_{\mathrm{flat}}$ on $\mathbb{RP}^2$.

$$A = A^{(\pm)}_{\mathrm{flat}} + \frac{\theta}{\beta}dt, \quad \sigma = 0, \tag{5.10}$$

where $A^{(\pm)}_{\mathrm{flat}}$ represent holonomies of $\mathbb{RP}^2$ along the non-contractible cycle $[\gamma] \neq 0 \in \pi_1(\mathbb{RP}^2)$. It is also characterized by

$$e^{i\oint_\gamma A^{(\pm)}_{\mathrm{flat}}} = \pm 1. \tag{5.11}$$

The constraint on the parameter θ is invariant.

$$\theta \in [0, 2\pi]. \tag{5.12}$$

As explained in the introduction, in the context of the supersymmetric localization, we expand field V around the locus V_0 which is parametrized by $\pm 1, \theta$:

$$V = V_0[\pm 1, \theta] + \tilde{V}, \tag{5.13}$$

where $\tilde{V}$ represents fluctuation. It means that the path integral is composed from the summation over ± 1 and integration over $\theta \in [0, 2\pi]$, and path integral over the fluctuation $\tilde{V}$:

$$\int \mathcal{D}V e^{-S_{YM}[V]} = \sum_{\pm 1} \int_0^{2\pi} \frac{d\theta}{2\pi} \int \mathcal{D}\tilde{V} e^{-\tilde{S}_{YM}[\tilde{V}]}. \tag{5.14}$$

Note that there is no monopole but ± 1 holonomies, so the summation is not infinite summation over the integers but constructed of just 2 terms, $+1$ sector and -1 sector.

5.2.1 QFT on $\mathbb{RP}^2 \times S^1_\beta \to$ QM on S^1_β

The gauge fixing procedure in the previous section also works on $\mathbb{RP}^2 \times S^1_\beta$, so we can use the Lagrangians

$$\mathcal{S}^{gf}_{\text{boson}} = \int dt \int \begin{pmatrix} A_2 \\ \sigma \end{pmatrix}^T \wedge *_2 \begin{pmatrix} - *_2 d_2 *_2 d_2 - \partial_t^2 & - *_2 d_2 \\ *_2 d_2 & - *_2 d_2 *_2 d_2 - \partial_t^2 + 1 \end{pmatrix} \begin{pmatrix} A_2 \\ \sigma \end{pmatrix}, \tag{5.15}$$

$$\mathcal{S}_{\text{fermion}} = \int dt \int \sin\vartheta d\vartheta d\varphi \, \overline{\lambda} \Big(i\gamma^i \nabla_i + i\gamma_3 (\partial_t - \frac{1}{2}) \Big) \lambda, \tag{5.16}$$

constrained by (4.24). One might think that the expansion of each field with respect to the harmonics $V^i_{jm}, \Psi^\epsilon_{jm}, Y_{jm}$ works. However it is not. Precisely speaking, the range of summation for j is constrained because of the parity condition (5.2). As one can find in the Appendix of [23], each harmonics behaves as follows[12] :

$$Y_{jm}(\pi - \vartheta, \pi + \varphi) = (-1)^j Y_{jm}(\vartheta, \varphi), \tag{5.17}$$

$$\Psi^\pm_{jm}(\pi - \vartheta, \pi + \varphi) = \mp i (-1)^{j-\frac{1}{2}} \gamma_1 \Psi^\pm_{jm}(\vartheta, \varphi), \tag{5.18}$$

$$V_{jm}(\pi - \vartheta, \pi + \varphi) = (-1)^{j+1} V_{jm}(\vartheta, \varphi). \tag{5.19}$$

We have no fermion zero mode, and we take eigenspinor Ψ for a modified Dirac operator $-i\gamma_3\gamma^i \mathfrak{D}_i$ rather than Υ for the Dirac operator $-i\gamma^i\mathfrak{D}_i$. V_{jm} is the 1-form constructed by $(V_{jm})_\vartheta d\vartheta + (V_{jm})_\varphi d\varphi$. The harmonics which preserves supersymmetric parity conditions in (5.2) only contribute to the expansion, then we get the following

[12] Our V_{jm} corresponds to C^2_{jm} in their notation.

expansions.

$$A^i(\vartheta,\varphi,t)=\sum_{\substack{j=2k+1\\k\geq 0}}\sum_{m=-j}^{j}V^i_{jm}(\vartheta,\varphi)A_{jm}(t), \tag{5.20}$$

$$\sigma(\vartheta,\varphi,t)=\sum_{\substack{j=2k+1\\k\geq 0}}\sum_{m=-j}^{j}Y_{jm}(\vartheta,\varphi)\sigma_{jm}(t), \tag{5.21}$$

$$\lambda(\vartheta,\varphi,t)=\sum_{\substack{j=2k+1/2\\k\geq 0}}\sum_{m=-j}^{j}\Psi^-_{jm}(\vartheta,\varphi)\lambda^-_{jm}(t)+\sum_{\substack{j=2k+3/2\\k\geq 0}}\sum_{m=-j}^{j}\Psi^+_{jm}(\vartheta,\varphi)\lambda^+_{jm}(t), \tag{5.22}$$

$$\overline{\lambda}(\vartheta,\varphi,t)=\sum_{\substack{j=2k+1/2\\k\geq 0}}\sum_{m=-j}^{j}\overline{\Psi}^-_{jm}(\vartheta,\varphi)\overline{\lambda}^-_{jm}(t)+\sum_{\substack{j=2k+3/2\\k\geq 0}}\sum_{m=-j}^{j}\overline{\Psi}^+_{jm}(\vartheta,\varphi)\overline{\lambda}^+_{jm}(t). \tag{5.23}$$

Then, the actions (4.25) and (4.26) give many-body quantum mechanics defined by the following actions :

$$\mathcal{S}^{gf}_{\text{boson}}=\sum_{\substack{j=2k+1\\k\geq 0}}\sum_{m=-j}^{j}\int dt\ \begin{pmatrix}A_{jm} & \sigma_{jm}\end{pmatrix}\begin{pmatrix}-\partial_t^2+j(j+1) & \sqrt{j(j+1)}\\ \sqrt{j(j+1)} & -\partial_t^2+j(j+1)+1\end{pmatrix}\begin{pmatrix}A_{jm}\\ \sigma_{jm}\end{pmatrix}, \tag{5.24}$$

$$\begin{aligned}\mathcal{S}_{\text{fermion}}=&\,i\sum_{\substack{j=2k+1/2\\k\geq 0}}\sum_{m=-j}^{j}\int dt\ \overline{\lambda}^-_{jm}\Big((j+\frac{1}{2})+(\partial_t-\frac{1}{2})\Big)\lambda^-_{jm}\\ &+i\sum_{\substack{j=2k+3/2\\k\geq 0}}\sum_{m=-j}^{j}\int dt\ \overline{\lambda}^+_{jm}\Big(-(j+\frac{1}{2})+(\partial_t-\frac{1}{2})\Big)\lambda^+_{jm}.\end{aligned} \tag{5.25}$$

The periodicity for each field can be read from the definition of SCI (3.1) and Table 1, then it becomes as

$$A_{jm}(t+\beta)=e^{-(\beta_1-\beta_2)m}A_{jm}(t),\quad \sigma_{jm}(t+\beta)=e^{-(\beta_1-\beta_2)m}\sigma_{jm}(t) \tag{5.26}$$

$$\overline{\lambda}^\epsilon_{jm}(t+\beta)=e^{(-1-m)\beta_1+m\beta_2}\overline{\lambda}^\epsilon_{jm}(t),\quad \lambda^\epsilon_{jm}(t+\beta)=e^{(+1-m)\beta_1+m\beta_2}\lambda^\epsilon_{jm}(t). \tag{5.27}$$

Therefore, we get each contribution as follows.

Bosonic part

$$\int \mathcal{D}A_2\mathcal{D}\sigma\ e^{-\mathcal{S}^{gf}_{\text{boson}}} = \int \prod_{t\in[0,\beta]} \Big(\prod_{\substack{j=2k+1\\k\geq 0}} \prod_{m=-j}^{j} dA_{jm}(t)d\sigma_{jm}(t)\Big)\ e^{-\mathcal{S}^{gf}_{\text{boson}}}$$

$$= \prod_{\substack{j=2k+1\\k\geq 0}} \prod_{m=-j}^{j} \frac{1}{\Big(2\sinh\frac{\omega_{jm}}{2}\Big)\Big(2\sinh\frac{\omega_{j+1,m}}{2}\Big)}, \tag{5.28}$$

where

$$\omega_{jm} = \frac{\beta_1-\beta_2}{\beta}m + j. \tag{5.29}$$

Fermionic part

$$\int \mathcal{D}\overline{\lambda}\mathcal{D}\lambda\ e^{-\mathcal{S}_{\text{fermion}}}$$

$$= \prod_{t\in[0,\beta]} \Big(\prod_{\substack{j=2k+1/2\\k\geq 0}}^{\infty} \prod_{m=-j}^{j} d\lambda^{-}_{jm}(t)d\overline{\lambda}^{-}_{jm}(t)\Big)\Big(\prod_{\substack{j=2k+3/2\\k\geq 0}}^{\infty} \prod_{m=-j}^{j} d\lambda^{+}_{jm}(t)d\overline{\lambda}^{+}_{jm}(t)\Big)\ e^{-\mathcal{S}_{\text{fermion}}}$$

$$= \prod_{\substack{j=2k+1\\k\geq 0}} \Big(\prod_{m=-j+1}^{j} 2\sinh\frac{\beta\omega_{jm}}{2}\Big)\Big(\prod_{m=-j-1}^{j} 2\sinh\frac{\beta\omega_{j+1,m}}{2}\Big). \tag{5.30}$$

Therefore, in contrast to the case of $\mathbb{M}^2 = \mathbb{S}^2$ (4.39), we get the following non-trivial contribution even from the vector multiplet.

Total

$$\int \mathcal{D}A_2\mathcal{D}\sigma\mathcal{D}\overline{\lambda}\mathcal{D}\lambda\ e^{-\mathcal{S}^{gf}_{\text{boson}}-\mathcal{S}_{\text{fermion}}}$$

$$= \prod_{\substack{j=2k+1\\k\geq 0}} \frac{2\sinh\frac{\beta\omega_{j+1,-(j+1)}}{2}}{2\sinh\frac{\beta\omega_{j,-j}}{2}} = x^{\frac{1}{4}}\frac{(x^4;x^4)_\infty}{(x^2;x^4)_\infty}. \tag{5.31}$$

5.3 Matter multiplet

Locus The matter Lagrangian (3.66) defines the trivial field contents:

$$0 = \phi = \psi = F, \quad 0 = \overline{\phi} = \overline{\psi} = \overline{F}. \tag{5.32}$$

Actions for the fluctuation fields

$$\mathcal{S}_{\text{boson}} = \int dt \int \sin\vartheta d\vartheta d\varphi \Big(\mathbb{D}_\mu \overline{\phi} \mathbb{D}^\mu \phi - (2\Delta - 1)\overline{\phi}\mathbb{D}_t \phi - \Delta(\Delta-1)\overline{\phi}\phi \Big), \quad (5.33)$$

$$\mathcal{S}_{\text{fermion}} = \int dt \int \sin\vartheta d\vartheta d\varphi \Big(-i(\overline{\psi}\gamma^\mu \mathbb{D}_\mu \psi) - \frac{i(2\Delta-1)}{2}(\overline{\psi}\gamma_3\psi) \Big), \quad (5.34)$$

where $\mathbb{D}_\mu$ represent the covariant derivative with respect to the locus gauge field (5.10):

$$\mathbb{D}_i = \nabla_i - i\boldsymbol{q}A_i^{\text{flat}} \quad (i = \vartheta, \varphi), \quad (5.35)$$

$$\mathbb{D}_t = \mathfrak{D}_t = \partial_t - i\boldsymbol{q}\frac{\theta}{\beta}. \quad (5.36)$$

5.3.1 QFT on $\mathbb{RP}^2 \times S^1_\beta \to$ QM on S^1_β

Here, for simplicity, we focus on the following two cases.

One-flavor matter multiplet First, we treat the $e^{i\oint_\gamma \boldsymbol{q}A_{\text{flat}}} = +1$ case in (5.3). In this case, we have to restrict j as follows:

$$\phi(\vartheta,\varphi,t) = \sum_{\substack{j=2k\\k\geq 0}}^{\infty} \sum_{m=-j}^{j} e^{i\int^x \boldsymbol{q}A_{\text{flat}}} Y_{jm}(\vartheta,\varphi)\phi_{jm}(t), \quad (5.37)$$

$$\psi(\vartheta,\varphi,t) = \sum_{\substack{j=2k+1/2\\k\geq 0}} \sum_{m=-j}^{j} e^{i\int^x \boldsymbol{q}A_{\text{flat}}} \Psi^+_{jm}(\vartheta,\varphi)\psi^+_{jm}(t) + \sum_{\substack{j=2k+3/2\\k\geq 0}} \sum_{m=-j}^{j} e^{i\int^x \boldsymbol{q}A_{\text{flat}}} \Psi^-_{jm}(\vartheta,\varphi)\psi^-_{jm}(t), \quad (5.38)$$

$$\overline{\phi}(\vartheta,\varphi,t) = \sum_{\substack{j=2k\\k\geq 0}}^{\infty} \sum_{m=-j}^{j} e^{-i\int^x \boldsymbol{q}A_{\text{flat}}} Y^*_{jm}(\vartheta,\varphi)\overline{\phi}_{jm}(t), \quad (5.39)$$

$$\overline{\psi}(\vartheta,\varphi,t) = \sum_{\substack{j=2k+1/2\\k\geq 0}} \sum_{m=-j}^{j} e^{-i\int^x \boldsymbol{q}A_{\text{flat}}} \overline{\Psi}^+_{jm}(\vartheta,\varphi)\overline{\psi}^+_{jm}(t) + \sum_{\substack{j=2k+3/2\\k\geq 0}} \sum_{m=-j}^{j} e^{-i\int^x \boldsymbol{q}A_{\text{flat}}} \overline{\Psi}^-_{jm}(\vartheta,\varphi)\overline{\psi}^-_{jm}(t), \quad (5.40)$$

where $Y_{jm}, \Psi^{\pm}_{jm}$ are harmonices explained in the Appendix B. Then, each action (5.33), (5.34) gives many-body quantum mechanics :

$$\mathcal{S}_{\text{boson}} = \sum_{\substack{j=2k\\k\geq 0}} \sum_{m=-j}^{j} \int dt\ \overline{\phi}_{jm}(j+\Delta+\mathfrak{D}_t)(j+1-\Delta-\mathfrak{D}_t)\phi_{jm}, \tag{5.41}$$

$$\begin{aligned}\mathcal{S}_{\text{boson}} = i &\sum_{\substack{j=2k+1/2\\k\geq 0}} \sum_{m=-j}^{j} \int dt\ \overline{\psi}^{+}_{jm}\Big((j+\frac{1}{2})-(\mathfrak{D}_t+\frac{2\Delta-1}{2})\Big)\psi^{+}_{jm}\\ &+ i \sum_{\substack{j=2k+3/2\\k\geq 0}} \sum_{m=-j}^{j} \int dt\ \overline{\psi}^{-}_{jm}\Big(-(j+\frac{1}{2})-(\mathfrak{D}_t+\frac{2\Delta-1}{2})\Big)\psi^{-}_{jm}.\end{aligned} \tag{5.42}$$

The periodicities can be read from the definition of SCI (3.1) and Table 1 :

$$\phi_{jm}(t+\beta) = e^{(-\Delta-m)\beta_1+m\beta_2+i\mu}\phi_{jm}(t) \tag{5.43}$$

$$\psi_{jm}(t+\beta) = e^{(-\Delta+1-m)\beta_1+m\beta_2+i\mu}\psi_{jm}(t) \tag{5.44}$$

Then each contribution becomes as follows.

Bosonic part

$$\begin{aligned}\int \mathcal{D}\phi\mathcal{D}\overline{\phi}\ e^{-\mathcal{S}_{\text{boson}}} &= \int \prod_{t\in[0,\beta]} \Big(\prod_{\substack{j=2k\\k\geq 0}} \prod_{m=-j}^{j} d\overline{\phi}_{jm}(t)d\phi_{jm}(t)\Big)\ e^{-\mathcal{S}_{\text{boson}}}\\ &= \prod_{\substack{j=2k\\k\geq 0}} \prod_{m=-j}^{j} \frac{1}{\Big(2\sinh\frac{\beta\omega^1_{jm}}{2}\Big)\Big(2\sinh\frac{\beta\omega^2_{jm}}{2}\Big)},\end{aligned} \tag{5.45}$$

where

$$\beta\omega^1_{jm} = -i\boldsymbol{q}\theta + (j-m)\beta_1 + (j+\Delta+m)\beta_2 + i\mu, \tag{5.46}$$

$$\beta\omega^2_{jm} = -i\boldsymbol{q}\theta - (j+1+m)\beta_1 - (j+1-\Delta-m)\beta_2 + i\mu. \tag{5.47}$$

Fermionic part

$$
\begin{aligned}
&\int \mathcal{D}\overline{\psi}\mathcal{D}\psi \; e^{-\mathcal{S}_{\text{fermion}}} \\
&= \int \prod_{t\in[0,\beta]} \Big(\prod_{\substack{j=2k+1/2\\ k\geq 0}} \prod_{m=-j}^{j} d\overline{\psi}^{+}_{jm}(t) d\psi^{+}_{jm}(t) \Big) \Big(\prod_{\substack{j=2k+3/2\\ k\geq 0}} \prod_{m=-j}^{j} d\overline{\psi}^{-}_{jm}(t) d\psi^{-}_{jm}(t) \Big) \; e^{-\mathcal{S}_{\text{fermion}}} \\
&= \prod_{\substack{j=2k\\ k\geq 0}} \prod_{m=-j-1}^{j} \Big(2\sinh \frac{\beta\omega^{2}_{jm}}{2} \Big) \times \prod_{\substack{j=2k+2\\ k\geq 0}} \prod_{m=-j}^{j-1} \Big(2\sinh \frac{\beta\omega^{1}_{jm}}{2} \Big). \qquad (5.48)
\end{aligned}
$$

Total contribution for $e^{i\oint_\gamma \boldsymbol{q}A_{\text{flat}}} = +1$ sector

$$
\begin{aligned}
\int \mathcal{D}\overline{\phi}\mathcal{D}\phi\mathcal{D}\overline{\psi}\mathcal{D}\psi \; e^{-\mathcal{S}_{\text{boson}}-\mathcal{S}_{\text{fermion}}} &= \prod_{\substack{j=2k\\ k\geq 0}} \frac{2\sinh \frac{\beta\omega^{1}_{j,-j-1}}{2}}{2\sinh \frac{\beta\omega^{2}_{j,j}}{2}} \\
&= x^{+\frac{\Delta-1}{4}} e^{+\frac{i}{4}\boldsymbol{q}\boldsymbol{\theta}} \alpha^{+\frac{1}{4}\boldsymbol{f}} \frac{(e^{-i\boldsymbol{q}\boldsymbol{\theta}}\alpha^{-\boldsymbol{f}}x^{(2-\Delta)};x^4)_\infty}{(e^{+i\boldsymbol{q}\boldsymbol{\theta}}\alpha^{+\boldsymbol{f}}x^{\Delta};x^4)_\infty}.
\end{aligned} \qquad (5.49)
$$

Now, we turn to the contribution for $e^{i\oint_\gamma \boldsymbol{q}A_{\text{flat}}} = -1$ sector. The only difference is the range for j in bosonic sector. After repeating similar procedure, we get the following contribution.

Total contribution for $e^{i\oint_\gamma \boldsymbol{q}A_{\text{flat}}} = -1$ sector

$$
\int \mathcal{D}\overline{\phi}\mathcal{D}\phi\mathcal{D}\overline{\psi}\mathcal{D}\psi \; e^{-\mathcal{S}_{\text{boson}}-\mathcal{S}_{\text{fermion}}} = x^{-\frac{\Delta-1}{4}} e^{-\frac{i}{4}\boldsymbol{q}\boldsymbol{\theta}} \alpha^{-\frac{1}{4}\boldsymbol{f}} \frac{(e^{-i\boldsymbol{q}\boldsymbol{\theta}}\alpha^{-\boldsymbol{f}}x^{(4-\Delta)};x^4)_\infty}{(e^{+i\boldsymbol{q}\boldsymbol{\theta}}\alpha^{+\boldsymbol{f}}x^{(2+\Delta)};x^4)_\infty}. \qquad (5.50)
$$

Two-flavor matter multiplets with (5.6)-type parity matrix.

In this case, as we have noted in (5.7), we can construct one-flavor matter multiplet on $\mathbb{S}^2 \times S^1_\beta$ with zero monopole, therefore we easily get the result from (4.62).

Total contribution for a doublet with parity condition (5.6)

$$
\begin{aligned}
&\int_{\mathbb{RP}^2\times S^1_\beta}[\mathcal{D}\overline{\phi}_1\mathcal{D}\phi_1\mathcal{D}\overline{\psi}_1\mathcal{D}\psi_1]\ [\mathcal{D}\overline{\phi}_2\mathcal{D}\phi_2\mathcal{D}\overline{\psi}_2\mathcal{D}\psi_2]\ e^{-\mathcal{S}_{\text{boson1}}-\mathcal{S}_{\text{fermion1}}-\mathcal{S}_{\text{boson2}}-\mathcal{S}_{\text{fermion2}}} \\
&=\int_{\mathbb{S}^2\times S^1_\beta}\mathcal{D}\overline{\phi}\mathcal{D}\phi\mathcal{D}\overline{\psi}\mathcal{D}\psi\ e^{-\mathcal{S}_{\text{boson}}-\mathcal{S}_{\text{fermion}}} \\
&=\frac{(e^{-i\boldsymbol{q}\theta}\alpha^{-\boldsymbol{f}}x^{2-\Delta};x^2)_\infty}{(e^{i\boldsymbol{q}\theta}\alpha^{+\boldsymbol{f}}x^{\Delta};x^2)_\infty}.
\end{aligned}
\tag{5.51}
$$

5.4 Result

We summarize here the toolkit for making SCI of our SUSY theories on $\mathbb{RP}^2 \times S^1_\beta$, focusing on the multiple of two types matter multiplets discussed in previous sub subsection.

5.4.1 Non gauge theory

In this case, we assume the following dynamical fields.

$$
\Phi_a = (\phi_a, F_a|\psi_a), \quad \overline{\Phi}_a = (\overline{\phi}_a, \overline{F}_a|\overline{\psi}_a), \quad a = 1, ..., N_f^{\text{single}} \quad \text{with } +1 \text{ in (5.3)}, \tag{5.52}
$$

$$
\Phi^A_{1,2} = (\phi^A_{1,2}, F^A_{1,2}|\psi^A_{1,2}), \quad \overline{\Phi}^A_{1,2} = (\overline{\phi}^A_{1,2}, \overline{F}^A_{1,2}|\overline{\psi}^A_{1,2}), \quad A = 1, ..., N_f^{\text{double}} \quad \text{with (5.6) in (5.4)}. \tag{5.53}
$$

We assign each multiplet with dimension Δ_a, Δ_A and flavor charge $\boldsymbol{f}_a, \boldsymbol{f}_A$. Our method can be applied to the theories with the following action:

$$
\mathcal{S}[\Phi, \overline{\Phi}] = \sum_{a=1}^{N_f^{\text{single}}} \mathcal{S}^{\boldsymbol{q}=0}_{mat}[\Phi_a, \overline{\Phi}_a] + \sum_{A=1}^{N_f^{\text{double}}} \mathcal{S}^{\boldsymbol{q}=0}_{mat}[\Phi^A_{1,2}, \overline{\Phi}^A_{1,2}] + W[\Phi] + \overline{W}[\overline{\Phi}], \tag{5.54}
$$

where $\mathcal{S}^{\boldsymbol{q}=0}_{mat}$ is the action (3.66) with $\boldsymbol{q} = 0$. We can take arbitrary superpotential W off it is invariant under the parity conditions. The flavor charge assignments $\boldsymbol{f}_a, \boldsymbol{f}_A$ have to preserve W.

SCI for non gauge theory on $\mathbb{RP}^2 \times S^1_\beta$

$$
\mathcal{I}(x,\alpha) = \prod_{a=1}^{N_f^{\text{single}}} x^{+\frac{\Delta_a - 1}{4}}\alpha^{+\frac{1}{4}\boldsymbol{f}_a}\frac{(\alpha^{-\boldsymbol{f}_a}x^{(2-\Delta_a)};x^4)_\infty}{(\alpha^{+\boldsymbol{f}_a}x^{\Delta_a};x^4)_\infty}\prod_{A=1}^{N_f^{\text{double}}}\frac{(\alpha^{-\boldsymbol{f}_A}x^{(2-\Delta_A)};x^2)_\infty}{(\alpha^{+\boldsymbol{f}_A}x^{\Delta_A};x^2)_\infty} \tag{5.55}
$$

5.4.2 Gauge theory

We consider the $U(1)$ gauge theory with single gauge field (vector multiplet):

$$V = (A_\mu, \sigma, D|\overline{\lambda}, \lambda). \tag{5.56}$$

Of course, we can add charged matter multiplets. But for simplicity, we consider matter singlets only :

$$\Phi_a = (\phi_a, F_a|\psi_a), \quad \overline{\Phi}_a = (\overline{\phi}_a, \overline{F}_a|\overline{\psi}_a), \quad a = 1, ..., N_f \quad \text{with } +1 \text{ in (5.3)}, \tag{5.57}$$

with $\Delta_a, \boldsymbol{f}_a$ and $U(1)$ charges $\boldsymbol{q}_a$. Our assuming action is

$$\mathcal{S}[V; \Phi, \overline{\Phi}] = \mathcal{S}_{YM}[V] + \sum_{a=1}^{N_f} \mathcal{S}_{mat}^{\boldsymbol{q}_a}[V; \Phi_a, \overline{\Phi}_a] + W[\Phi] + \overline{W}[\overline{\Phi}], \tag{5.58}$$

where $\mathcal{S}_{YM}$ is the action (3.65) with $U(1)$ gauge group. See [29] for more detail. We have to sum up all locus contributions. It means that we should sum up $\pm$ sector's contributions and integrate $\theta \in [0, 2\pi]$:

SCI for gauge theory on $\mathbb{RP}^2 \times S^1_\beta$

$$\begin{aligned}\mathcal{I}(x, \alpha) = &\int_0^{2\pi} \frac{d\theta}{2\pi} \prod_{a=1}^{N_f} x^{+\frac{\Delta_a - 1}{4}} e^{+\frac{i}{4}\boldsymbol{q}\theta} \alpha^{+\frac{1}{4}\boldsymbol{f}_a} \frac{(e^{-i\boldsymbol{q}\theta}\alpha^{-\boldsymbol{f}_a} x^{(2-\Delta_a)}; x^4)_\infty}{(e^{+i\boldsymbol{q}\theta}\alpha^{+\boldsymbol{f}_a} x^{\Delta_a}; x^4)_\infty} \times x^{\frac{1}{4}} \frac{(x^4; x^4)_\infty}{(x^2; x^4)_\infty} \\ &+ \int_0^{2\pi} \frac{d\theta}{2\pi} \prod_{a=1}^{N_f} x^{-\frac{\Delta_a - 1}{4}} e^{-\frac{i}{4}\boldsymbol{q}\theta} \alpha^{-\frac{1}{4}\boldsymbol{f}_a} \frac{(e^{-i\boldsymbol{q}\theta}\alpha^{-\boldsymbol{f}_a} x^{(4-\Delta_a)}; x^4)_\infty}{(e^{+i\boldsymbol{q}\theta}\alpha^{+\boldsymbol{f}_a} x^{(2+\Delta_a)}; x^4)_\infty} \times x^{\frac{1}{4}} \frac{(x^4; x^4)_\infty}{(x^2; x^4)_\infty}.\end{aligned} \tag{5.59}$$

6 An application : 3d abelian mirror symmetry

In this section, we apply the exact results of SCI to the check of a conjectural duality called three-dimensional mirror symmetry [14, 15, 71] , duality between two distinct quantum field theories, SQED and XYZ-model.

6.1 Conjectual Duality between SQED and XYZ-model

First, let us survey each theory's Lagrangian, global symmetries, etc.

6.1.1 XYZ-model

Degrees of freedom This is a non gauge theory constructed of three matter multiplets

$$X = (\phi_X, F_X, |\psi_X), \quad Y = (\phi_Y, F_Y, |\psi_Y), \quad Z = (\phi_Z, F_Z, |\psi_Z), \quad \text{and their conjugates.} \tag{6.1}$$

Dimensions Each multiplet have the following dimensions:

$$\Delta_X = \Delta_Y = 1 - \Delta, \quad \Delta_Z = 2\Delta. \tag{6.2}$$

Lagrangian Lagrangian is as follows.

$$\begin{aligned} &\mathcal{S}_{XYZ}[X, Y, Z] \\ &= \mathcal{S}^{\boldsymbol{q}=0}_{mat}[X] + \mathcal{S}^{\boldsymbol{q}=0}_{mat}[Y] + \mathcal{S}^{\boldsymbol{q}=0}_{mat}[Z] + \int dx^3 (XYZ)|_{\theta\theta} + \int dx^3 (\overline{XYZ})|_{\overline{\theta\theta}} \end{aligned} \tag{6.3}$$

The assignments of the dimension comes from the superpotential XYZ term.

Global symmetries There are two global symmetries called $U(1)_V$ and $U(1)_A$. We denote here the corresponding flavor charges as $\boldsymbol{f}_V, \boldsymbol{f}_A$. See Table 2.

	X	Y	Z
$\boldsymbol{f}_V$	$+1$	-1	0
$\boldsymbol{f}_A$	$+1$	$+1$	-2

Table 2: Flavor charge assignments

Parameters of the vacua As well known, scalars can take vacuum expectation values (VEVs). In this case there are three scalars. Therefore, the parameters of the vacua are the following three VEVs:

$$\langle\phi_X\rangle, \quad \langle\phi_Y\rangle, \quad \langle\phi_Z\rangle. \tag{6.4}$$

6.1.2 SQED

Degrees of freedom This is a gauge theory constructed from one vector multiplet and two charged matter multiplets. Q has a charge $+1$, and $\tilde{Q}$ has a charge -1.

$$V = (A_\mu, \sigma, D|\overline{\lambda}, \lambda), \tag{6.5}$$

$$Q = (\phi_Q, F_Q, |\psi_Q), \quad \tilde{Q} = (\phi_{\tilde{Q}}, F_{\tilde{Q}}, |\psi_{\tilde{Q}}), \quad \text{and their conjugates.} \tag{6.6}$$

Dimensions Each multiplet has the following dimensions:

$$\Delta_Q = \Delta_{\tilde{Q}} = \Delta. \tag{6.7}$$

Dual photon In 3 dimension, d.o.f. of the massless vector is equivalent to the d.o.f. of a real scalar ρ through the following equation :

$$\frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho} = \partial_\mu\rho. \tag{6.8}$$

The real scalar field ρ is called *dual photon*.

Lagrangian Lagrangian is as follows.

$$\mathcal{S}_{SQED}[V, Q, \tilde{Q}] = \mathcal{S}_{YM}[V] + \mathcal{S}_{mat}^{\boldsymbol{q}=+1}[V; Q] + \mathcal{S}_{mat}^{\boldsymbol{q}=-1}[V; \tilde{Q}]. \tag{6.9}$$

Global symmetries There are two global symmetries called $U(1)_J$ and $U(1)_A$. We denote here the corresponding flavor charges as $\tilde{\boldsymbol{f}}_J, \tilde{\boldsymbol{f}}_A$. See Table 3.

	$e^{\sigma+i\rho}$	$e^{-(\sigma+i\rho)}$	Q	$\tilde{Q}$
$\tilde{\boldsymbol{f}}_J$	$+1$	-1	0	0
$\tilde{\boldsymbol{f}}_A$	0	0	$+1$	$+1$

Table 3: Flavor charge assignments

Parameters of the vacua The scalar VEV have to preserve the gauge symmetry, so the meson field, the latest component of $\tilde{Q}Q$ is one of the good coordinates. The other ones are $e^{\sigma \pm i\rho}$. Therefore, there are three relevant VEVs.

$$\langle e^{\sigma+i\rho}\rangle, \quad \langle e^{-(\sigma+i\rho)}\rangle, \quad \langle \phi_{\tilde{Q}}\phi_Q\rangle. \tag{6.10}$$

6.2 Check of $\mathbb{M}^2 = \mathbb{S}^2$ case

At the beginning of the discovery of this duality, there were some indirect checks, moduli space equivalence, parity anomaly matching, etc [14, 15]. After the developments of the exact calculation of BPS sectors based on localization techniques, we can see its duality in the form of mathematical formula. For example, through the sphere partition function Z, the equivalence $Z_{XYZ} = Z_{SQED}$ reduces to the identity [44, 43]

$$\frac{1}{\cosh \frac{p}{2}} = \int_{-\infty}^{\infty} dx \frac{e^{ipx}}{\cosh \pi x}. \tag{6.11}$$

This is, the Fourier transformation of the $\cosh^{-1}$ function. In this section, we review recent developments of the precision check of the duality by using superconformal index on $\mathbb{S}^2 \times S^1_\beta$. In this section, for simplicity, we turn on only the fugacity for $U(1)_A$ global symmetries.

6.2.1 SCI of XYZ-model

According to the formula in (4.66) and the charge assignments in Table 2, we get

$$\mathcal{I}^{\Delta}_{\mathrm{XYZ}}(x,\alpha) = \Big(\frac{(\alpha^{-1}x^{(1+\Delta)};x^2)_\infty}{(\alpha^{+1}x^{(1-\Delta)};x^2)_\infty}\Big)^2 \frac{(\alpha^{+2}x^{2(1-\Delta)};x^2)_\infty}{(\alpha^{-2}x^{2\Delta};x^2)_\infty}. \tag{6.12}$$

For example, we can expand it with respect to x by taking spatial values for $\Delta = 1/2, \alpha = 1$ as follows

$$\mathcal{I}^{1/2}_{\mathrm{XYZ}}(x,1) = 1 + 2x^{1/2} + 3x + 2x^{3/2} + x^2 + 2x^{5/2} + 4x^3 + 4x^{7/2} - 2x^{9/2} \dots \tag{6.13}$$

This means that there are infinitely many BPS states (3.4) as summarized in Table 4.

$\hat{H}+\hat{j}^3$	0	1/2	1	3/2	2	5/2	3	7/2	9/2	...
$\#_b - \#_f$ in BPS states	1	2	3	2	1	2	4	4	−2	...

Table 4: (6.13) indicates these BPS spectrum

6.2.2 SCI of SQED

According to the formula (4.70) and the charge assignments in Table 3, we get[13]

$$\begin{aligned}&\mathcal{I}^{\Delta}_{\text{SQED}}(x,\alpha^{-1})\\&=\sum_{B\in\mathbb{Z}}\int_0^{2\pi}\frac{d\theta}{2\pi}\Big(x^{(1-\Delta)}\alpha\Big)^{|B|}\frac{(e^{-i\theta}\alpha x^{2-\Delta+|B|};x^2)_\infty}{(e^{i\theta}\alpha^{-1}x^{\Delta+|B|};x^2)_\infty}\times\frac{(e^{i\theta}\alpha x^{2-\Delta+|B|};x^2)_\infty}{(e^{-i\theta}\alpha^{-1}x^{\Delta+|B|};x^2)_\infty}.\end{aligned}\tag{6.14}$$

By using mathematica, we can get numerical value for $\Delta=1/2,\alpha=1$ as follows:

$$\mathcal{I}^{1/2}_{\text{SQED}}(x,1)=1+2x^{1/2}+3x+2x^{3/2}+x^2+2x^{5/2}+4x^3+4x^{7/2}-2x^{9/2}+\dots\tag{6.15}$$

As one can see, this looks in agreement with (6.13). In fact, one can find the analytic proof of

$$\mathcal{I}^{\Delta}_{\text{XYZ}}(x,\alpha)=\mathcal{I}^{\Delta}_{\text{SQED}}(x,\alpha^{-1}),\tag{6.16}$$

in Appendix C.1.

6.3 Check of $\mathbb{M}^2=\mathbb{RP}^2$ case

We can also check the duality through SCI on $\mathbb{RP}^2\times S^1_\beta$. This case, we have to identify supersymmetric parity conditions in each side. The hint for it is the correspondence of the VEVs [15].

$$\langle\phi_X\rangle=\langle e^{\sigma+i\rho}\rangle,\quad\langle\phi_Y\rangle=\langle e^{-(\sigma+i\rho)}\rangle,\quad\langle\phi_Z\rangle=\langle\phi_{\tilde{Q}}\phi_Q\rangle.\tag{6.17}$$

Now, let us remind our parity conditions for component fields in vector multiplet (5.2). As one can simply check,

$$\sigma+i\rho\ \to\ -(\sigma+i\rho)\tag{6.18}$$

[13] The reason for taking α^{-1} not α in (6.14) is that the sign of the conserved current for $U(1)_A$ is reversed under the mirror symmetry [43].

under the antipodal identification (5.1). And we choose here the parity for matter fields in SQED as

$$\phi_Q \;\to\; \phi_Q, \quad \phi_{\tilde{Q}} \;\to\; \phi_{\tilde{Q}}, \tag{6.19}$$

then, (6.17) suggests the following parity conditions for XYZ-model:

$$\phi_X \;\leftrightarrows\; \phi_Y, \quad \phi_Z \;\to\; \phi_Z. \tag{6.20}$$

The parity conditions (6.20) mean the matter multiplets X and Y form the doublet with the parity matrix (5.6). The condition (6.20) means that the matter multiplet Z is singlet under the antipodal identification.

6.3.1 SCI of XYZ-model

According to the formula in (5.55) and the charge assignments in Table 1, we get

$$\mathcal{I}^{\Delta}_{\mathrm{XYZ}}(x,\alpha) = \left(x^{+\frac{2\Delta-1}{4}}\alpha^{\frac{-1}{2}}\right)\frac{(\alpha^2 x^{2(1-\Delta)};x^4)_\infty}{(\alpha^{-2}x^{2\Delta};x^4)_\infty} \times \frac{(\alpha^{-1}x^{(1+\Delta)};x^2)_\infty}{(\alpha x^{(1-\Delta)};x^2)_\infty} \tag{6.21}$$

The spatial value for $\Delta = 1/2, \alpha = 1$ becomes

$$\mathcal{I}^{1/2}_{\mathrm{XYZ}}(x,1) = 1 + x^{1/2} + x + x^{5/2} + x^3 - x^4 + 2x^5 + x^{11/2} - x^6 - x^{13/2} + x^7 + \dots \tag{6.22}$$

This gives totally different contributions compared with (6.13).

6.3.2 SCI of SQED

According to the formula (5.59) and the charge assignments in Table 2, we get

$$\begin{aligned}
&\mathcal{I}^{\Delta}_{\mathrm{SQED}}(x,\alpha^{-1}) \\
&= \int_0^{2\pi}\frac{d\theta}{2\pi}\left(x^{+\frac{2\Delta-1}{4}}\alpha^{\frac{-1}{2}}\right)\frac{(e^{-i\theta}\alpha x^{(2-\Delta)};x^4)_\infty}{(e^{i\theta}\alpha^{-1}x^{\Delta};x^4)_\infty} \times \frac{(e^{i\theta}\alpha x^{(2-\Delta)};x^4)_\infty}{(e^{-i\theta}\alpha^{-1}x^{\Delta};x^4)_\infty} \times \frac{(x^4;x^4)_\infty}{(x^2;x^4)_\infty} \\
&+ \int_0^{2\pi}\frac{d\theta}{2\pi}\left(x^{-\frac{2\Delta-3}{4}}\alpha^{\frac{1}{2}}\right)\frac{(e^{-i\theta}\alpha x^{(4-\Delta)};x^4)_\infty}{(e^{i\theta}\alpha^{-1}x^{(2+\Delta)};x^4)_\infty} \times \frac{(e^{i\theta}\alpha x^{(4-\Delta)};x^4)_\infty}{(e^{-i\theta}\alpha^{-1}x^{(2+\Delta)};x^4)_\infty} \times \frac{(x^4;x^4)_\infty}{(x^2;x^4)_\infty}.
\end{aligned} \tag{6.23}$$

This gives

$$\mathcal{I}^{1/2}_{\mathrm{SQED}}(x,1) = 1 + x^{1/2} + x + x^{5/2} + x^3 - x^4 + 2x^5 + x^{11/2} - x^6 - x^{13/2} + x^7 + \dots \tag{6.24}$$

The reader can find the exact proof for this equality in Appendix C.2.

7 Concluding remarks

In this thesis, we performed exact calculations of the SCI based on the supersymmetric localization method. We considered supersymmetric QFT on $\mathbb{S}^2 \times S^1_\beta$ in Section 4, on $\mathbb{RP}^2 \times S^1_\beta$ in Section 5. By integrating out the degrees of freedom along the 2-dimensional surface, we got many-body quantum mechanics. The families of many particles coming from the reduction along the $\mathbb{S}^2$ are different from the ones along the $\mathbb{RP}^2$. In this sense, we may be able to regard that the difference between the SCI on $\mathbb{S}^2 \times S^1_\beta$ and the SCI on $\mathbb{RP}^2 \times S^1_\beta$ is the difference of the Hilbert space $\mathcal{H}$ in (3.1). And we also applied these two SCI's to check the conjectural duality, 3d mirror symmetry or equivalence between XYZ-model (6.3) and SQED (6.9). As one can find in Appendix C, the equivalence can be recognized by the uses of the mathematical formulas.

$$
\begin{aligned}
&\mathbb{S}^2 \times S^1_\beta \ \text{ case :} \quad &&\begin{cases} \bullet \ \text{Ramanujan's summation formula (C.7)} \\ \bullet \ \text{q-binomial formula (C.13)} \end{cases} \\
&\mathbb{RP}^2 \times S^1_\beta \ \text{ case :} \quad &&\ \ \bullet \ \text{q-binomial formula (C.13)}
\end{aligned}
$$

Naively speaking, the use of Ramanujan's summation formula is necessary for summing up the monopole numbers $B \in \mathbb{Z}$. And the use of q-binomial formula is necessary for summing up the contributions from the residue integrals, so it comes from the integral over $\theta \in [0, 2\pi]$. In later case, as one can notice, the following unnamed formulas are important.

$$
(A;q)_{2l} = (A;q^2)_l (Aq;q^2)_l, \quad (A;q)_{2l+1} = (1-A)(Aq;q^2)_l (Aq^2;q^2)_l, \quad \text{for } l \in \mathbb{N}. \tag{7.1}
$$

Instead of the existence of the Dirac monopole on $\mathbb{S}^2$, this formulas are algebraic representations of the $\pm$ holonomies along $\mathbb{RP}^2$. In summary, in the context of the mirror symmetry, there are the following correspondences between algebraic mathematical formula and geometric physical object.

$$
\text{Ramanujan's summation formula} \Leftrightarrow \text{Monopoles on } \mathbb{S}^2, \tag{7.2}
$$

$$
\text{No name formulas in (7.1)} \Leftrightarrow \text{Holonomies along } \mathbb{RP}^2, \tag{7.3}
$$

$$
\text{q-binomial formula} \Leftrightarrow \text{Holonomy along } S^1_\beta. \tag{7.4}
$$

Thanks to the duality between two QFTs is realized in such way, we can observe how the duality works in mathematically rigorous way. These kinds of understandings of

QFT are inevitable for studying the non-perturbative structures of QFT, and conversely, the duality provides unexpected relationships between different mathematical objects. I guess, no one can imagine that the above formulas, including (6.11), are related under the concept of mirror symmetry. Therefore, the study of the dualities in quantum physics is fruitful and very interesting research area definitely.

One more comment on ongoing project As noted in Section 5, there may be different supersymmetric parity conditions. This is as follows.

$$\begin{aligned}
&A_\vartheta(\pi-\vartheta,\pi+\varphi,t)=+A_\vartheta(\vartheta,\varphi,t),\quad A_{\varphi,t}(\pi-\vartheta,\pi+\varphi,t)=-A_{\varphi,t}(\vartheta,\varphi,t),\\
&\sigma(\pi-\vartheta,\pi+\varphi,t)=+\sigma(\vartheta,\varphi,t),\\
&\lambda(\pi-\vartheta,\pi+\varphi,t)=-i\gamma_1\lambda(\vartheta,\varphi,t),\quad \overline{\lambda}(\pi-\vartheta,\pi+\varphi,t)=+i\gamma_1\overline{\lambda}(\vartheta,\varphi,t),\\
&D(\pi-\vartheta,\pi+\varphi,t)=-D(\vartheta,\varphi,t).
\end{aligned}\tag{7.5}$$

This condition also preserves SUSY and $U(1)$ Yang-Mills action (3.65). However, it breaks the invariance of the following differential operator.

$$(\partial-iA)^2,\tag{7.6}$$

because under the above transformation, we get

$$(\partial-iA)^2\to(\partial+iA)^2.\tag{7.7}$$

In order to overcome such problem, we have to turn on two matters with $\pm$ charges respectively. Happily, we have such mattes in SQED, Q and $\tilde{Q}$. We are now trying to check our above consideration's validity based on the check of mirror symmetry. We seem to be close at the correct understandings, however, we still have not get answer. According to our calculation, the SCI of XYZ-model becomes

$$\begin{aligned}
&\frac{1}{\sqrt[4]{x}}+\left(\alpha+\frac{2}{\alpha}\right)\sqrt[4]{x}+\frac{(\alpha^4+2)\,x^{3/4}}{\alpha^2}+\frac{(\alpha^6-\alpha^2+2)\,x^{5/4}}{\alpha^3}\\
&+\left(\alpha^4+\frac{2}{\alpha^4}-2\right)x^{7/4}+\left(\alpha^5+\frac{2}{\alpha^5}-\alpha-\frac{2}{\alpha}\right)x^{9/4}+\left(\alpha^6+\frac{2}{\alpha^6}-\alpha^2-\frac{2}{\alpha^2}+2\right)x^{11/4}+\dots.
\end{aligned}\tag{7.8}$$

And the SCI of SQED becomes

$$\begin{aligned}
&\frac{1}{\sqrt[4]{x}}+\frac{2\sqrt[4]{x}}{\alpha}+\frac{(\alpha^4+2)\,x^{3/4}}{\alpha^2}+\frac{2x^{5/4}}{\alpha^3}\\
&+\left(\alpha^4+\frac{2}{\alpha^4}-2\right)x^{7/4}+\frac{(2-2\alpha^4)\,x^{9/4}}{\alpha^5}+\frac{(\alpha^{12}-\alpha^8-2\alpha^4+2)\,x^{11/4}}{\alpha^6}+\dots.
\end{aligned}\tag{7.9}$$

There are many terms in agreement, but still, there are many junks. There seem to be something missed. We hope this problem to be solved in near future.

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A Some arithmetics for the thesis

In this appendix, we summarize and sometimes derive some mathematical formulas which are relevant in the thesis.

A.1 Trigonometric functions

As well known, the trigonometric functions can be represented as infinite products :

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \Big(1 - \frac{z^2}{n^2}\Big), \quad \sinh \pi z = \pi z \prod_{n=1}^{\infty} \Big(1 + \frac{z^2}{n^2}\Big) \tag{A.1}$$

$$\cos \pi z = \prod_{n=1}^{\infty} \Big(1 - \frac{z^2}{(n-\frac{1}{2})^2}\Big), \quad \cosh \pi z = \prod_{n=1}^{\infty} \Big(1 + \frac{z^2}{(n-\frac{1}{2})^2}\Big). \tag{A.2}$$

One interesting application is an infinite product formula for π:

$$\begin{aligned} 1 = \sin \frac{\pi}{2} = \frac{\pi}{2} \prod_{n=1}^{\infty} \Big(1 - \frac{(\frac{1}{2})^2}{n^2}\Big) &= \frac{\pi}{2} \prod_{n=1}^{\infty} \Big(\frac{(2n)^2 - 1}{(2n)^2}\Big) \\ &= \frac{\pi}{2} \prod_{n=1}^{\infty} \Big(\frac{(2n-1)(2n+1)}{(2n)^2}\Big). \end{aligned} \tag{A.3}$$

This is called *Wallis' formula.*

A.2 Zeta function

We use the *zeta function regularization* throughout this thesis. This regularization corresponds to introducing a soft cutoff to the UV momenta [72]. The zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad \text{for } Re(s) > 1, \tag{A.4}$$

and is analytically continued to whole complex plane $s \in \mathbb{C}$. One can try to calculate particular value for fixed s by introducing soft UV cutoff for n. For example,

$$\begin{aligned}
\zeta(0) \sim \sum_{n=1}^{\infty} 1 \xleftarrow{0 \leftarrow \epsilon} \sum_{n=1}^{\infty} e^{-\epsilon n} \\
&= \frac{e^{-\epsilon}}{1-e^{-\epsilon}} = \frac{1}{e^{\epsilon}-1} = \frac{1}{\epsilon\Big(1+\frac{1}{2}\epsilon+\mathcal{O}(\epsilon^2)\Big)} \\
&= \frac{1}{\epsilon}\Big(1-\frac{1}{2}\epsilon+\mathcal{O}(\epsilon^2)\Big) = \frac{1}{\epsilon} - \frac{1}{2} + \mathcal{O}(\epsilon),
\end{aligned} \tag{A.5}$$

in this regularization, the “scale” for the cutoff corresponds to ϵ and UV limit is $\epsilon \to 0$. Obviously, the divergent first term in (A.5) represent UV divergence. Now we take the following regularization:

$$\zeta(0) := \lim_{\epsilon \to 0}\Big[\sum_{n=1}^{\infty} e^{-\epsilon n} - \frac{1}{\epsilon}\Big] = -\frac{1}{2}. \tag{A.6}$$

In fact, it is known that this procedure reproduces the precise analytic continued value for $\zeta(0)$. We would like to derive the value for $\zeta'(0)$. By differentiating (A.4) with s, we can get

$$\zeta'(s) = -\sum_{n=1}^{\infty} n^{-s} \log n. \tag{A.7}$$

So the value for $s = 0$ may be

$$\zeta'(0) \sim -\sum_{n=1}^{\infty} \log n = -\log \prod_{n=1}^{\infty} n. \tag{A.8}$$

This divergence can be regularize by using Wallis' formula and the regularized value of $\zeta(0)$ as follows. 1st, by deforming Wallis' formula,

$$
\begin{aligned}
\frac{\pi}{2} &= \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)} = \prod_{n=1}^{\infty} \frac{(2n)^4}{(2n)^2(2n-1)(2n+1)} \\
&= \Big(\prod_{n=1}^{\infty} 2^4\Big)\Big(\prod_{n=1}^{\infty} n\Big)^4 \Big(\prod_{n=1}^{\infty} \frac{1}{(2n)(2n-1)}\Big)\Big(\prod_{n=1}^{\infty} \frac{1}{(2n)(2n+1)}\Big) \\
&= \Big(2^{4\sum_{n=1}^{\infty} 1}\Big)\Big(\prod_{n=1}^{\infty} n\Big)^4 \Big(\prod_{n=1}^{\infty} \frac{1}{n}\Big)\Big(\prod_{n=1}^{\infty} \frac{1}{n}\Big) \\
&\sim \Big(2^{4\zeta(0)}\Big)\Big(\prod_{n=1}^{\infty} n\Big)^2 = \Big(2^{-2}\Big)\Big(\prod_{n=1}^{\infty} n\Big)^2,
\end{aligned}
\tag{A.9}
$$

2nd, by taking $\sqrt{\ }$, we arrive at

$$\prod_{n=1}^{\infty} n \sim \sqrt{2\pi}. \tag{A.10}$$

Then, by substituting it to (A.8), we get

$$\zeta'(0) = -\frac{1}{2}\log 2\pi. \tag{A.11}$$

A.3 Gaussian integrals

The gaussian integral

$$\int_{-\infty}^{\infty} dx\ e^{-\frac{1}{2}x^2} = \sqrt{2\pi} \tag{A.12}$$

is the most important integral in this thesis. Here, we summarize basic facts of Gaussian integrals of bosonic degrees of freedom x_i and sermonic degrees of freedom ψ_i.

Bosonic case

$$\text{Real Gaussian}: \int \prod_i \frac{dx_i}{\sqrt{2\pi}} e^{-\frac{1}{2}\sum_{ij} x_i M_{ij} x_j} = \frac{1}{\sqrt{\det M_{ij}}} \tag{A.13}$$

$$\text{Complex Gaussian}: \int \prod_i \frac{dz_i d\bar{z}_i}{2\pi} e^{-\frac{1}{2}\sum_{ij} \bar{z}_i M_{ij} z_j} = \frac{1}{\det M_{ij}} \tag{A.14}$$

Fermionic case

$$\text{Real Gaussian}: \int \prod_i d\psi_i e^{-\frac{1}{2}\sum_{ij} \psi_i M_{ij} \psi_j} = \sqrt{\det M_{ij}} \tag{A.15}$$

$$\text{Complex Gaussian}: \int \prod_i d\psi_i d\bar{\psi}_i e^{-\frac{1}{2}\sum_{ij} \bar{\psi}_i M_{ij} \psi_j} = \det M_{ij} \tag{A.16}$$

B Monopole spherical harmonics

As well known in the context of Schödinger equation with spherically symmetric system and Laplace equation etc, the spherical harmonics $Y_{jm}(\vartheta,\varphi)$ diagonalizes the Laplacian on $\mathbb{S}^2$:

$$\begin{aligned}\nabla_i\nabla^i Y_{jm}(\vartheta,\varphi) &= \Big(\frac{1}{\sin\vartheta}\partial_\vartheta \sin\vartheta\partial_\vartheta + \frac{1}{\sin^2\vartheta}\partial_\varphi^2\Big)Y_{jm}(\vartheta,\varphi) \\ &= -j(j+1)Y_{jm}(\vartheta,\varphi). \end{aligned} \tag{B.1}$$

This is a consequence of the fact that the Laplacian $\nabla_i\nabla^i$ on $\mathbb{S}^2$ can be regarded as the squared orbital angular momentum $\vec{L}^2$. Here, let us remind the definition for the orbital angular momentum operators:

$$L_1 \pm iL_2 = e^{i\varphi}\Big(\pm\partial_\vartheta + i\cot\vartheta\partial_\varphi\Big), \quad L_3 = -i\partial_\varphi. \tag{B.2}$$

Of course, L_1, L_2, L_3 satisfy the $SU(2)$ algebra:

$$[L_A, L_B] = i\epsilon_{ABC}L_C. \tag{B.3}$$

The spectrum of $-\nabla_i\nabla^i = \vec{L}^2 = L_1^2 + L_2^2 + L_3^2$ is purely determined by this $SU(2)$ algebraic structure:

$$\vec{L}^2 Y_{jm}(\vartheta,\varphi) = j(j+1)Y_{jm}(\vartheta,\varphi), \tag{B.4}$$

$$L_3 Y_{jm}(\vartheta,\varphi) = mY_{jm}(\vartheta,\varphi). \tag{B.5}$$

In this appendix, we review extensions of this construction.

Monopole background Consider a background $U(1)$ gauge field

$$A_{\rm mon} = \frac{B}{2}(\kappa - \cos\vartheta)d\varphi, \tag{B.6}$$

where κ is $+1$ when we take a coordinate patch around north pole ; $0 \le \vartheta < \pi$, and -1 when we take a coordinate patch around south pole ; $0 < \vartheta \le \pi$. The gauge field around north pole, say A^n, and the gauge field around south pole, say A^s are related by the following gauge transformation:

$$A^n_{\rm mon} = A^s_{\rm mon} + ig^{-1}dg, \quad g = e^{Bi\varphi}. \tag{B.7}$$

Now, in order to define the gauge transformation g as single valued function on $\mathbb{S}^2$, we have to take $B \in \mathbb{Z}$. This is famous Dirac's quantization condition for monopole charge.

Monopole harmonics By using the background gauge field (B.6), we can generalize the orbital angular momentum operators (B.2) :

$$J_1 \pm iJ_2 = e^{i\varphi}\Big(\pm \partial_\vartheta + i\cot\vartheta(\partial_\varphi - iA_\varphi) + \frac{B}{2}\sin\vartheta\Big), \quad J_3 = -i\partial_\varphi \mp \frac{B}{2}. \tag{B.8}$$

One may wonder the physical meaning of this definition but it becomes clear when we represent them by using $x_1 = r\sin\vartheta\cos\varphi, x_2 = r\sin\vartheta\sin\varphi, x_3 = r\cos\vartheta$:

$$\vec{J} = \vec{r}\times\Big(-i\vec{\nabla} + \vec{A}_{\rm mon}\Big) + \frac{B}{2}\frac{\vec{r}}{r}. \tag{B.9}$$

$\vec{J}$ is composed of orbital angular momentum under the background gauge field (B.6) and the angular moment of the monopole itself. Note that the value for A_φ on north pole patch and south pole patch are different, so $J_1 \pm J_2$ are not usual differential operators. Precisely speaking, the operators (B.8) act on not functions but sections of certain non-trivial vector bundle. These operators satisfy

$$[J_A, J_B] = i\epsilon_{ABC}J_C. \tag{B.10}$$

In the following sub-subsections, we briefly summarize the eigenstates for $\vec{J}^2, J_3$:

$$\vec{J}^2|j,m\rangle = j(j+1)|j,m\rangle, \tag{B.11}$$

$$J_3|j,m\rangle = m|j,m\rangle, \tag{B.12}$$

with spin 0, 1/2, 1, respectively. For later use, we define monopole covariant derivative

$$\mathfrak{D}_i := \nabla_i - iA^{\rm mon}{}_i, \tag{B.13}$$

where ∇_i is defined in (3.25), the usual covariant derivative with respect to the spin connection.

B.1 Scalar harmonics $Y_{\frac{B}{2},jm}$

With a spin 0 field, scalar field, one can verify

$$\mathfrak{D}_i\mathfrak{D}^i = -\Big(\vec{J}^2 - \frac{B^2}{2^2}\Big). \tag{B.14}$$

This fact means that we can diagonalize the monopole Laplacian $\mathfrak{D}_i\mathfrak{D}^i$ on $\mathbb{S}^2$ with the state satisfying (B.11) and (B.12). Let us define the spin zero wave function as $Y_{\frac{B}{2},jm}(\vartheta,\varphi)$, then we get

$$\mathfrak{D}_i\mathfrak{D}^iY_{\frac{B}{2},jm}(\vartheta,\varphi) = -\Big(j(j+1) - \frac{B^2}{2^2}\Big)Y_{\frac{B}{2},jm}(\vartheta,\varphi). \tag{B.15}$$

By repeating well known argument of orthogonality, we can also derive

$$\int \sin\vartheta d\vartheta d\varphi \; Y^{*}_{\frac{B}{2},jm}(\vartheta,\varphi) Y_{\frac{B}{2},j'm'}(\vartheta,\varphi) = \delta_{jj'}\delta_{mm'}. \tag{B.16}$$

If and only if $j \geq |\frac{B}{2}|$, $Y_{\frac{B}{2},jm}$ is normalizable. See [73] for more details.

B.2 Spinor harmonics $\Upsilon_{\frac{B}{2},jm}, \Psi_{\frac{B}{2},jm}$

Spin 1/2 monopole angular momentum operators satisfy the following relation.

$$\vec{J}^{\,2}_{spinor} = -(\gamma^i \mathfrak{D}_i)^2 - \frac{1}{4} + \left(\frac{B}{2}\right)^2. \tag{B.17}$$

Therefore, by taking square root of this eigenvalues, we can diagonalize the monopole Dirac operator $-i\gamma^i\mathfrak{D}_i$ on $\mathbb{S}^2$ with the spin 1/2 state satisfying (B.11) and (B.12).

Eigenspinors for $-i\gamma^i\mathfrak{D}_i$

As one can notice, there must be two modes:

$$-i\gamma^i\mathfrak{D}_i \Upsilon^{\pm}_{\frac{B}{2},jm}(\vartheta,\varphi) = \pm\mu_{j\frac{B}{2}} \Upsilon^{\pm}_{\frac{B}{2},jm}(\vartheta,\varphi), \quad \mu_{j\frac{B}{2}} = \frac{\sqrt{(2j+1)^2 - B^2}}{2} \tag{B.18}$$

where the two modes are exchanged by the multiplication of γ_3:

$$\gamma_3 \Upsilon^{\pm}_{\frac{B}{2},jm}(\vartheta,\varphi) = \Upsilon^{\mp}_{\frac{B}{2},jm}(\vartheta,\varphi). \tag{B.19}$$

And the normalizability requires $j \geq \frac{|B|}{2} - \frac{1}{2}$. When $j = \frac{|B|}{2} - \frac{1}{2}$, we have one zero mode:

$$-i\gamma^i\mathfrak{D}_i \Upsilon^{0}_{\frac{B}{2},jm}(\vartheta,\varphi) = 0, \tag{B.20}$$

$$\gamma_3 \Upsilon^{0}_{\frac{B}{2},jm}(\vartheta,\varphi) = \mathrm{sign}(B) \Upsilon^{0}_{\frac{B}{2},jm}(\vartheta,\varphi). \tag{B.21}$$

$\Psi^{\epsilon=\pm,0}_{\frac{B}{2},jm}(\vartheta,\varphi)$ are orthonormal:

$$\int \sin\vartheta d\vartheta d\varphi \; \Upsilon^{\epsilon}_{\frac{B}{2},jm}(\vartheta,\varphi)^{\dagger} \Upsilon^{\epsilon'}_{\frac{B}{2},j'm'}(\vartheta,\varphi) = \delta^{\epsilon\epsilon'}\delta_{jj'}\delta_{mm'}. \tag{B.22}$$

See the appendix of [74] for more details.

Eigenspinors for $-i\gamma_3\gamma^i\mathfrak{D}_i$

One can construct eigenspjnors for $-i\gamma_3\gamma^i\mathfrak{D}_i$ by taking

$$\Psi^{\pm}_{\frac{B}{2},jm}(\vartheta,\varphi) = (1 - i\gamma_3)\Upsilon^{\pm}_{\frac{B}{2},jm}(\vartheta,\varphi). \tag{B.23}$$

These spinors give following formula

$$-i\gamma_3\gamma^i\mathfrak{D}_i\Psi^{\pm}_{\frac{B}{2},jm}(\vartheta,\varphi) = \pm i\mu_{j\frac{B}{2}}\Psi^{\pm}_{\frac{B}{2},jm}(\vartheta,\varphi) \tag{B.24}$$

We define corresponding $\overline{\Psi}$ as

$$\int \sin\vartheta d\vartheta d\varphi\ \overline{\Psi}^{\epsilon}_{\frac{B}{2},jm}(\vartheta,\varphi)\gamma_3\Psi^{\epsilon'}_{\frac{B}{2},j'm'}(\vartheta,\varphi) = \delta^{\epsilon\epsilon'}\delta_{jj'}\delta_{mm'}. \tag{B.25}$$

B.3 Vector harmonics $V^i_{\frac{B}{2},jm}$

By repeating procedure similar to the case represented above, we can make vector harmonics [75]. However it is somewhat complicated, so we would like to concentrate on the case of

$$B = 0, \quad \nabla_i V^i_{jm}(\vartheta,\varphi) = 0. \tag{B.26}$$

This vector satisfies the following formulas [41]:

$$\nabla_1 V^2_{jm}(\vartheta,\varphi) - \nabla_2 V^1_{jm}(\vartheta,\varphi) = \sqrt{j(j+1)}Y_{jm}(\vartheta,\varphi), \quad (\text{for } j \geq 1) \tag{B.27}$$

$$\nabla_1 V^2_{jm}(\vartheta,\varphi) - \nabla_2 V^1_{jm}(\vartheta,\varphi) = 0, \quad (\text{for } j = -1). \tag{B.28}$$

When $j = \frac{|B|}{2}$, the mode with (B.26) becomes zero. Orthonormality condition is

$$\int \sin\vartheta d\vartheta d\varphi\ V^i_{jm}(\vartheta,\varphi)V^i_{j'm'}(\vartheta,\varphi) = \delta_{jj'}\delta_{mm'}. \tag{B.29}$$

C Proof of $\mathcal{I}^{\Delta}_{XYZ} = \mathcal{I}^{\Delta}_{SQED}$

C.1 $\mathbb{M}^2 = \mathbb{S}^2$ case

The following argument is originally found by [53]. In order to calculate this complex integral (6.14), it is useful to change the integration variable from θ to $z = e^{i\theta}$:

$$(6.14) = \sum_{B\in\mathbb{Z}} \oint \frac{dz}{2\pi i z} \Big(x^{(1-\Delta)}\alpha \Big)^{|B|} \frac{(z^{-1}\alpha x^{2-\Delta+|B|};x^2)_\infty}{(z\alpha^{-1}x^{\Delta+|B|};x^2)_\infty} \times \frac{(z\alpha x^{2-\Delta+|B|};x^2)_\infty}{(z^{-1}\alpha^{-1}x^{\Delta+|B|};x^2)_\infty}, \tag{C.1}$$

then, the problem is which poles are chosen. We assume here that

$$|\alpha^{-1}x^{\Delta+|B|}| < 1. \tag{C.2}$$

Then, the relevant residues are located at

$$z_l = x^{2l+\Delta+|B|}\alpha^{-1}, \quad l = 0,1,2,\ldots \tag{C.3}$$

and the integral becomes

$$(C.1) = \sum_{B\in\mathbb{Z}}\sum_{l=0}^{\infty} \Big(x^{(1-\Delta)}\alpha \Big)^{|B|} \frac{(\alpha^2 x^{-2(l-1+\Delta)};x^2)_\infty}{(\alpha^{-2}x^{2(l+\Delta+|B|)};x^2)_\infty} \times \frac{(x^{2(1+l)+2|B|};x^2)_\infty}{(x^2;x^2)_\infty} \times \frac{1}{(x^{-2l};x^2)_l}, \tag{C.4}$$

where $(A;q)_l = \prod_{n=0}^{l-1}(1-Aq^n)$. Now, we can observe the following fact: the $|B|$ in the series (C.4) can be replaced by B [53, 43] , and the following formula:

$$(Ax^{2B};x^2)_\infty = \frac{(A;x^2)_\infty}{(A;x^2)_B}, \tag{C.5}$$

where $(A;q)_{-l} = \prod_{n=1}^{l}(1-Aq^{-n})^{-1}$ for $l>0$. Then,

$$(C.4) = \sum_{l=0}^{\infty} \frac{(\alpha^2 x^{-2(l-1+\Delta)};x^2)_\infty}{(\alpha^{-2}x^{2(l+\Delta)};x^2)_\infty} \frac{(x^{2(1+l)};x^2)_\infty}{(x^2;x^2)_\infty} \frac{1}{(x^{-2l};x^2)_l} \underbrace{\sum_{B\in\mathbb{Z}} \Big(x^{(1-\Delta)}\alpha \Big)^{B} \frac{(\alpha^{-2}x^{2(l+\Delta)};x^2)_B}{(x^{2(1+l)};x^2)_B}}_{\boxed{\text{1st key terms}}}, \tag{C.6}$$

Now, we use the following formula in order to deform the $\boxed{\text{1st key terms}}$:

Ramanujan's summation formula [70]

$$\sum_{B\in\mathbb{Z}} z^B \frac{(a,q)_B}{(b,q)_B} = \frac{(q;q)_\infty(\frac{b}{a};q)_\infty(az;q)_\infty(\frac{q}{az};q)_\infty}{(b;q)_\infty(\frac{q}{a};q)_\infty(z;q)_\infty(\frac{b}{az};q)_\infty} \tag{C.7}$$

In our case (C.6),

$$q = x^2, \quad z = \left(x^{(1-\Delta)}\alpha\right), \quad a = (\alpha^{-2}x^{2(l+\Delta)}), \quad b = x^{2(1+l)}. \tag{C.8}$$

Then,

$$\boxed{\text{1st key terms}} = \frac{(x^2;x^2)_\infty(\alpha^2 x^{2(1-\Delta)};x^2)_\infty(\alpha^{-1}x^{2l+\Delta+1};x^2)_\infty(\alpha x^{1-\Delta-2l};x^2)_\infty}{(x^{2(1+l)};x^2)_\infty(\alpha^2 x^{-2(l+\Delta-1)};x^2)_\infty(\alpha x^{1-\Delta};x^2)_\infty(\alpha x^{1-\Delta};x^2)_\infty}. \tag{C.9}$$

By substituting it into (C.6), we get

$$\begin{aligned}
(\text{C.6}) =& \sum_{l=0}^{\infty} \frac{(\alpha^2 x^{-2(l-1+\Delta)};x^2)_\infty}{(\alpha^{-2}x^{2(l+\Delta)};x^2)_\infty} \frac{(x^{2(1+l)};x^2)_\infty}{(x^2;x^2)_\infty} \frac{1}{(x^{-2l};x^2)_l} \\
&\times \frac{(x^2;x^2)_\infty(\alpha^2 x^{2(1-\Delta)};x^2)_\infty(\alpha^{-1}x^{2l+\Delta+1};x^2)_\infty(\alpha x^{1-\Delta-2l};x^2)_\infty}{(x^{2(1+l)};x^2)_\infty(\alpha^2 x^{-2(l+\Delta-1)};x^2)_\infty(\alpha x^{1-\Delta};x^2)_\infty(\alpha x^{1-\Delta};x^2)_\infty} \\
=& \sum_{l=0}^{\infty} \frac{(-1)^l x^{-l(l+1)}\left(\alpha^2 x^{-2(-1+\Delta)}\right)^l (\alpha^{-2}x^{2(-1+\Delta)}x^2;x^2)_l(\alpha^2 x^{-2(-1+\Delta)};x^2)_\infty}{(\alpha^{-2}x^{2\Delta};x^2)_\infty/(\alpha^{-2}x^{2\Delta};x^2)_l} \frac{1}{(-1)^l x^{-l(l+1)}(x^2;x^2)_l} \\
&\times \frac{(\alpha^2 x^{2(1-\Delta)};x^2)_\infty \frac{(\alpha^{-1}x^{\Delta+1};x^2)_\infty}{(\alpha^{-1}x^{\Delta+1};x^2)_l}(-1)^l x^{-l(l+1)}\left(\alpha x^{1-\Delta}\right)^l(\alpha^{-1}x^{\Delta-1}x^2;x^2)_l(\alpha x^{1-\Delta};x^2)_\infty}{(-1)^l x^{-l(l+1)}\left(\alpha^2 x^{-2(\Delta-1)}\right)^l(\alpha^{-2}x^{2(\Delta-1)}x^2;x^2)_l(\alpha^2 x^{-2(\Delta-1)};x^2)_\infty(\alpha x^{1-\Delta};x^2)_\infty(\alpha x^{1-\Delta};x^2)_\infty} \\
=& \frac{(\alpha^2 x^{-2(-1+\Delta)};x^2)_\infty(\alpha^{-1}x^{\Delta+1};x^2)_\infty}{(\alpha^{-2}x^{2\Delta};x^2)_\infty(\alpha x^{1-\Delta};x^2)_\infty} \underbrace{\sum_{l=0}^{\infty} \frac{(\alpha^{-2}x^{2\Delta};x^2)_l}{(x^2;x^2)_l}\left(\alpha x^{1-\Delta}\right)^l}_{\boxed{\text{2nd key terms}}}
\end{aligned} \tag{C.10}$$

Here, we used the following formulas.

$$(x^{-2l};x^2)_l = (-1)^l x^{-2l(l+1)}(x^2;x^2)_l, \tag{C.11}$$

$$(Ax^{-2l};x^2)_\infty = (-1)^l x^{-2l(l+1)}A^l(A^{-1}x^2;x^2)_l(A;x^2)_\infty \tag{C.12}$$

The final key is the following formula:

q-binomial formula [70]

$$\sum_{l=0}^{\infty} \frac{(A;q)_l}{(q;q)_l} Z^l = \frac{(AZ;q)_\infty}{(Z;q)_\infty} \tag{C.13}$$

In our case, (C.10),

$$q = x^2, \quad A = \alpha^{-2} x^{2\Delta}, \quad Z = \alpha x^{1-\Delta}, \tag{C.14}$$

so we get

$$\boxed{\text{2nd key terms}} = \frac{(\alpha^{-1} x^{1+\Delta}; x^2)_\infty}{(\alpha x^{1-\Delta}; x^2)_\infty} \tag{C.15}$$

Substituting it into (C.10), we finally arrived at

$$\begin{aligned}
(\text{C.10}) &= \frac{(\alpha^2 x^{-2(-1+\Delta)}; x^2)_\infty (\alpha^{-1} x^{\Delta+1}; x^2)_\infty}{(\alpha^{-2} x^{2\Delta}; x^2)_\infty (\alpha x^{1-\Delta}; x^2)_\infty} \frac{(\alpha^{-1} x^{1+\Delta}; x^2)_\infty}{(\alpha x^{1-\Delta}; x^2)_\infty} \\
&= \left(\frac{(\alpha^{-1} x^{1+\Delta}; x^2)_\infty}{(\alpha x^{1-\Delta}; x^2)_\infty} \right)^2 \frac{(\alpha^2 x^{2(1-\Delta)}; x^2)_\infty}{(\alpha^{-2} x^{2\Delta}; x^2)_\infty}.
\end{aligned} \tag{C.16}$$

This is exactly identical to the SCI of XYZ-model (6.12).

C.2 $\mathbb{M}^2 = \mathbb{RP}^2$ case

The following argument is based on our original work [29]. In order to calculate this complex integral (6.23), it is useful to change the integration variable from θ to $z, w = e^{i\theta}$:

$$\begin{aligned}
(6.23) = &\oint \frac{dz}{2\pi i z} \Big(x^{+\frac{2\Delta-1}{4}} \alpha^{\frac{-1}{2}} \Big) \frac{(z^{-1} \alpha x^{(2-\Delta)}; x^4)_\infty}{(z \alpha^{-1} x^{\Delta}; x^4)_\infty} \times \frac{(z \alpha x^{(2-\Delta)}; x^4)_\infty}{(z^{-1} \alpha^{-1} x^{\Delta}; x^4)_\infty} \times \frac{(x^4; x^4)_\infty}{(x^2; x^4)_\infty} \\
&+ \oint \frac{dw}{2\pi i z} \Big(x^{-\frac{2\Delta-3}{4}} \alpha^{\frac{1}{2}} \Big) \frac{(w^{-1} \alpha x^{(4-\Delta)}; x^4)_\infty}{(w \alpha^{-1} x^{(2+\Delta)}; x^4)_\infty} \times \frac{(w \alpha x^{(4-\Delta)}; x^4)_\infty}{(w^{-1} \alpha^{-1} x^{(2+\Delta)}; x^4)_\infty} \times \frac{(x^4; x^4)_\infty}{(x^2; x^4)_\infty}.
\end{aligned} \tag{C.17}$$

We take same assumption (C.2) :

$$|\alpha^{-1} x^{\Delta + |B|}| < 1, \quad B = 0, 2. \tag{C.18}$$

Then, the relevant residues are

$$z_l = \alpha^{-1} x^{\Delta + 4l}, \quad l = 0, 1, 2, \ldots \quad \text{for upper integral in (C.17)}, \tag{C.19}$$

$$w_l = \alpha^{-1} x^{2+\Delta+4l}, \quad l = 0, 1, 2, \ldots \quad \text{for lower integral in (C.17)}. \tag{C.20}$$

The residue integral becomes

(C.17)

$$
\begin{aligned}
&= \sum_{l=0}^{\infty}\left(x^{+\frac{2\Delta-1}{4}}\alpha^{\frac{-1}{2}}\right)\frac{(\alpha^2 x^{(2-2\Delta-4l)};x^4)_\infty}{(\alpha^{-2}x^{2\Delta+4l};x^4)_\infty}\times\frac{(x^{(2+4l)};x^4)_\infty}{(x^4;x^4)_\infty(x^{-4l};x^4)_l}\times\frac{(x^4;x^4)_\infty}{(x^2;x^4)_\infty}\\
&\quad+\sum_{l=0}^{\infty}\left(x^{-\frac{2\Delta-3}{4}}\alpha^{\frac{1}{2}}\right)\frac{(\alpha^2 x^{(2-2\Delta-4l)};x^4)_\infty}{(\alpha^{-2}x^{(4+2\Delta+4l)};x^4)_\infty}\times\frac{(x^{(6+4l)};x^4)_\infty}{(x^4;x^4)_\infty(x^{-4l};x^4)_l}\times\frac{(x^4;x^4)_\infty}{(x^2;x^4)_\infty}\\
&= \sum_{l=0}^{\infty}\left(x^{+\frac{2\Delta-1}{4}}\alpha^{\frac{-1}{2}}\right)\frac{(-1)^l x^{-4l(l+1)}\left(\alpha^2 x^{(2-2\Delta)}\right)^l(\alpha^{-2}x^{-(2-2\Delta)}x^4;x^4)_l(\alpha^2x^{(2-2\Delta)};x^4)_\infty}{(\alpha^{-2}x^{2\Delta};x^4)_\infty/(\alpha^{-2}x^{2\Delta};x^4)_l}\\
&\qquad\times\frac{(x^2;x^4)_\infty/(x^2;x^4)_l}{(x^4;x^4)_\infty(-1)^l x^{-4l(l+1)}(x^4;x^4)_l}\times\frac{(x^4;x^4)_\infty}{(x^2;x^4)_\infty}\\
&\quad+\sum_{l=0}^{\infty}\left(x^{-\frac{2\Delta-3}{4}}\alpha^{\frac{1}{2}}\right)\frac{(-1)^l x^{-4l(l+1)}\left(\alpha^2 x^{(2-2\Delta)}\right)^l(\alpha^{-2}x^{-(2-2\Delta)}x^4;x^4)_l(\alpha^2x^{(2-2\Delta)};x^4)_\infty}{(\alpha^{-2}x^{(4+2\Delta)};x^4)_\infty/(\alpha^{-2}x^{(4+2\Delta)};x^4)_l}\\
&\qquad\times\frac{(x^6;x^4)_\infty/(x^6;x^4)_l}{(x^4;x^4)_\infty(-1)^l x^{-4l(l+1)}(x^4;x^4)_l}\times\frac{(x^4;x^4)_\infty}{(x^2;x^4)_\infty}\\
&= \left(x^{+\frac{2\Delta-1}{4}}\alpha^{\frac{-1}{2}}\right)\frac{(\alpha^2x^{(2-2\Delta)};x^4)_\infty}{(\alpha^{-2}x^{2\Delta};x^4)_\infty}\sum_{l=0}^{\infty}\frac{(\alpha^{-2}x^{(2\Delta+2)};x^4)_l(\alpha^{-2}x^{2\Delta};x^4)_l}{(x^2;x^4)_l(x^4;x^4)_l}\left(\alpha^2x^{(2-2\Delta)}\right)^l\\
&\quad+\left(x^{-\frac{2\Delta-3}{4}}\alpha^{\frac{1}{2}}\right)\frac{(\alpha^2x^{(2-2\Delta)};x^4)_\infty}{(\alpha^{-2}x^{(4+2\Delta)};x^4)_\infty}\frac{(x^6;x^4)_\infty}{(x^2;x^4)_\infty}\sum_{l=0}^{\infty}\frac{(\alpha^{-2}x^{(2+2\Delta)};x^4)_l(\alpha^{-2}x^{(4+2\Delta)};x^4)_l}{(x^4;x^4)_l(x^6;x^4)_l}\left(\alpha^2x^{(2-2\Delta)}\right)^l,
\end{aligned}
\tag{C.21}
$$

where we used the following formulas.

$$(x^{-4l};x^4)_l=(-1)^l x^{-4l(l+1)}(x^4;x^4)_l, \tag{C.22}$$

$$(Ax^{-4l};x^2)_\infty=(-1)^l x^{-4l(l+1)}A^l(A^{-1}x^4;x^4)_l(A;x^4)_\infty. \tag{C.23}$$

We can deform the pre factor of lower term in (C.21) as follows:

$$
\begin{aligned}
&\left(x^{-\frac{2\Delta-3}{4}}\alpha^{\frac{1}{2}}\right)\frac{(\alpha^2x^{(2-2\Delta)};x^4)_\infty}{(\alpha^{-2}x^{(4+2\Delta)};x^4)_\infty}\frac{(x^6;x^4)_\infty}{(x^2;x^4)_\infty}\\
&=\left(x^{+\frac{2\Delta-1}{4}}\alpha^{\frac{-1}{2}}\right)x^{-\frac{4\Delta-4}{4}}\alpha^1\frac{(\alpha^2x^{(2-2\Delta)};x^4)_\infty}{(\alpha^{-2}x^{2\Delta};x^4)_\infty}\frac{(\alpha^{-2}x^{2\Delta};x^4)_\infty}{(\alpha^{-2}x^{(4+2\Delta)};x^4)_\infty}\frac{(x^6;x^4)_\infty}{(x^2;x^4)_\infty}\\
&=\left(x^{+\frac{2\Delta-1}{4}}\alpha^{\frac{-1}{2}}\right)x^{1-\Delta}\alpha^1\frac{(\alpha^2x^{(2-2\Delta)};x^4)_\infty}{(\alpha^{-2}x^{2\Delta};x^4)_\infty}\frac{(1-\alpha^{-2}x^{2\Delta})}{1-x^2}.
\end{aligned}
\tag{C.24}
$$

Then

(C.21)

$$
\begin{aligned}
&= \left(x^{+\frac{2\Delta-1}{4}}\alpha^{\frac{-1}{2}}\right)\frac{(\alpha^2x^{(2-2\Delta)};x^4)_\infty}{(\alpha^{-2}x^{2\Delta};x^4)_\infty}\sum_{l=0}^{\infty}\Big[\frac{(\alpha^{-2}x^{(2\Delta+2)};x^4)_l(\alpha^{-2}x^{2\Delta};x^4)_l}{(x^2;x^4)_l(x^4;x^4)_l}\left(\alpha^2x^{(2-2\Delta)}\right)^l \\
&\quad + \frac{(1-\alpha^{-2}x^{2\Delta})}{1-x^2}\frac{(\alpha^{-2}x^{(2+2\Delta)};x^4)_l(\alpha^{-2}x^{(4+2\Delta)};x^4)_l}{(x^4;x^4)_l(x^6;x^4)_l}\left(\alpha^2x^{(2-2\Delta)}\right)^l\Big] \\
&= \left(x^{+\frac{2\Delta-1}{4}}\alpha^{\frac{-1}{2}}\right)\frac{(\alpha^2x^{(2-2\Delta)};x^4)_\infty}{(\alpha^{-2}x^{2\Delta};x^4)_\infty}\sum_{l=0}^{\infty}\Big[\frac{(\alpha^{-2}x^{2\Delta};x^2)_{2l}}{(x^2;x^2)_{2l}}\left(\alpha x^{(1-\Delta)}\right)^{2l} \\
&\quad + \frac{(\alpha^{-2}x^{2\Delta};x^2)_{2l+1}}{(x^2;x^2)_{2l+1}}\left(\alpha x^{(1-\Delta)}\right)^{2l+1}\Big] \\
&= \left(x^{+\frac{2\Delta-1}{4}}\alpha^{\frac{-1}{2}}\right)\frac{(\alpha^2x^{(2-2\Delta)};x^4)_\infty}{(\alpha^{-2}x^{2\Delta};x^4)_\infty}\sum_{k=0}^{\infty}\Big[\frac{(\alpha^{-2}x^{2\Delta};x^2)_k}{(x^2;x^2)_k}\left(\alpha x^{(1-\Delta)}\right)^k\Big]. \qquad \text{(C.25)}
\end{aligned}
$$

Here we used

$$
(A;q)_{2l} = (A;q^2)_l(Aq;q^2)_l, \quad (A;q)_{2l+1} = (1-A)(Aq;q^2)_l(Aq^2;q^2)_l. \qquad \text{(C.26)}
$$

Now, we can use the q-binomial formula (C.13) :

$$
\sum_{k=0}^{\infty}\frac{(A;q)_k}{(q;q)_k}Z^k = \frac{(AZ;q)_\infty}{(Z;q)_\infty}, \qquad \text{(C.27)}
$$

then we arrive at

$$
\text{(C.25)} = \left(x^{+\frac{2\Delta-1}{4}}\alpha^{\frac{-1}{2}}\right)\frac{(\alpha^2x^{(2-2\Delta)};x^4)_\infty}{(\alpha^{-2}x^{2\Delta};x^4)_\infty}\frac{(\alpha^{-1}x^{1+\Delta};x^2)_\infty}{(\alpha x^{1-\Delta};x^2)_\infty}. \qquad \text{(C.28)}
$$

This is exactly (6.21).

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