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MEASURES WITH MAXIMUM TOTAL EXPONENT OF C^1 DIFFEOMORPHISMS WITH BASIC SETS

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Abstract. We show that any C^1-diffeomorphism with a basic set has a C^1-neighborhood satisfying the following properties. A generic element in the neighborhood has a unique measure with maximum total exponent which is of zero entropy and fully supported on the continuation of the basic set. To the contrary, we show that for $r \geq 2$ any C^r-diffeomorphism with a basic set does not have a C^r-neighborhood satisfying the above properties.

1 Introduction

This paper is a continuation of the research in [15]. We begin with stating the background of our study. In 2001, G. Contreras, A. O. Lopes and Ph. Thieullen [5] introduced Lyapunov minimizing (resp. maximizing) measures

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of an expanding map on the circle as invariant Borel probability measures minimizing (resp. maximizing) the integral of the Lyapunov exponent. A Lyapunov maximizing measure is a kind of maximizing measures for a given function, which are invariant Borel probability measures (with respect to a fixed dynamical system) maximizing the integral of the function. Maximizing measures are mainly considered in ergodic optimization formulated by O. Jenkinson [9], and have been studied in various references (*e.g.*, see [1], [4], [8], [10], [11], [12], [13], [17], [18], [22], [23]). In particular, O. Jenkinson and I. D. Morris [13] proved that a generic C^1-expanding map on the circle has a unique Lyapunov maximizing measure with zero entropy and full support. However such a measure can be defined only for maps on the circle, and their argument works only on the circle. So it is natural to ask whether the result is extended to expanding maps on a general compact manifold or not. To consider the question, in [15], we introduced the notion of measures with maximum total exponent including naturally Lyapunov maximizing measures as a special case, and gave an affirmative answer. That is, we proved that a generic C^1-expanding map on a compact manifold has a unique measure with maximum total exponent, which is of zero entropy and fully supported.

Our work in this paper is also related to measures with maximum total exponent, which are defined not only for expanding maps but also for C^1-maps with nonvanishing Jacobian. If a class of those maps is given, the following is a natural and interesting question: Which properties about measures with maximum total exponent are generic in the class? Moreover, if we consider diffeomorphisms with hyperbolic structure, then an answer to the following problem raised by O. Jenkinson (see [9] Problem 4.4) may be obtained:

Problem Let $T : X \to X$ be any transitive hyperbolic map on a compact metrizable space X with local product structure. Find an explicit example of a continuous function with a unique maximizing measure of full support.

Therefore we shall investigate diffeomorphisms with hyperbolic structure.

Now let us recall the definition of measures with maximum total exponent. In order to study diffeomorphisms, we introduce a slightly generalized definition. Let M be a compact smooth Riemannian manifold without boundary. It is also assumed to be connected throughout the paper. Let $d(\cdot, \cdot)$ denote the distance function on M induced by the Riemannian metric on M. Consider a C^1-diffeomorphism $T : M \to M$ and a compact T-invariant set Λ. We denote by $\mathcal{M}(T, \Lambda)$ the space of all T-invariant Borel probability measures supported on Λ equipped with the weak-$*$ topology. Let $DT(x)$ be the derivative of T at $x \in M$. We denote by $J(T)(x)$ the Jacobian of T at $x \in M$, *i.e.*, $J(T)(x)$ is the absolute value of the determinant of $DT(x)$. For $\mu \in \mathcal{M}(T,\Lambda)$, we define

$$\lambda(T,\mu) = \int \log J(T) d\mu,$$

and put $\lambda(T,\Lambda) = \sup_{\nu \in \mathcal{M}(T,\Lambda)} \lambda(T,\nu)$. We call μ a measure with maximum total exponent on Λ for T if $\lambda(T,\mu) = \lambda(T,\Lambda)$ holds. By virtue of the Oseledec theorem, $\lambda(T,\mu)$ is equal to the integral of the total Lyapunov exponents of T with respect to μ. Therefore we see that a measure with maximum total exponent for T is a T-invariant Borel probability measure maximizing the integral of the total Lyapunov exponents of T. Let $\mathcal{L}(T,\Lambda)$ be the set of all measures with maximum total exponent on Λ for T. Since T is a C^1-diffeomorphism and $\mathcal{M}(T,\Lambda)$ is compact, we see that $\mathcal{L}(T,\Lambda)$ is not empty. In our previous work [15], it was enough to consider only measures with maximum total exponent on M since any expanding map can not be decomposed into smaller parts because of its topological transitivity on M. To study maps without the property, we need the notion of measures with maximum total exponent not only on M but also on a compact invariant set (especially, on a basic set mentioned below).

Next, we summarize terminology and notation about hyperbolic sets. Let

Λ be an isolated compact T-invariant set with an isolating neighborhood U, *i.e.*, $\Lambda = \bigcap_{i\in\mathbb{Z}} T^i(U)$ holds. We assume further that Λ is hyperbolic for T, *i.e.*, there exists a DT-invariant splitting $T_\Lambda M = E^s \oplus E^u$ of the tangent bundle over Λ, satisfying $\|DT^n(x)|_{E^s_x}\| \leq c\lambda^n$ and $\|DT^{-n}(x)|_{E^u_x}\| \leq c\lambda^n$, for any $x \in \Lambda, n \geq 0$, with constants $c \geq 1$ and $\lambda \in (0,1)$. Such a constant λ is called a skewness of T on Λ. We call Λ a basic set for T if $T|_\Lambda : \Lambda \to \Lambda$ is topologically transitive. From Theorem 9.7.4 in [20], if S is close enough to T in the C^1-topology then $\bigcap_{i\in\mathbb{Z}} S^i(U)$ is isolated and hyperbolic for S. Moreover there exists a conjugacy map from $\bigcap_{i\in\mathbb{Z}} S^i(U)$ to Λ. Therefore, in particular, we see that if Λ is a basic set for T then $\bigcap_{i\in\mathbb{Z}} S^i(U)$ is a basic set for S. We use Λ_S to denote the set $\bigcap_{i\in\mathbb{Z}} S^i(U)$. Λ_S is called the continuation of Λ for S.

As usual, for $r \geq 1$, let $C^r(M,M)$ denote the space of all C^r-maps from M to M equipped with the C^r-topology and let $\mathit{Diff}^{\,r}(M)$ denote the set of all C^r-diffeomorphisms on M.

The following two theorems are our main results.

Theorem 1.1 *Let $T : M \to M$ be a C^1-diffeomorphism with a basic set Λ. Then there exists an open neighborhood $\mathcal{U} \subset \mathit{Diff}^{\,1}(M)$ of T such that each of the following properties is generic in $\mathcal{U}$.*

(1) *S has a unique measure with maximum total exponent on the continuation*

Λ_S of Λ for S.

(2) *Any measure with maximum total exponent on Λ_S for S has zero entropy.*

(3) *Any measure with maximum total exponent on Λ_S for S is fully supported on Λ_S.*

In particular, for a generic element S in $\mathcal{U}$, the measure with maximum total exponent on Λ_S is unique, ergodic, of zero entropy and fully supported on Λ_S.

For diffeomorphisms with higher regularity, we have the following result.

Theorem 1.2 *Let $r \geq 2$. Consider a C^r-diffeomorphism $T : M \to M$ with a basic set Λ. Then for any sufficiently small neighborhood $\mathcal{U} \subset \mathit{Diff}^{\,r}(M)$ of T, any measure with maximum total exponent on the continuation Λ_S of Λ for a generic element S in $\mathcal{U}$ is not fully supported on Λ_S unless Λ itself is a periodic orbit of T.*

From Theorem 1.2, we see that any C^r-diffeomorphism with a basic set never has a C^r-neighborhood in which the properties in Theorem 1.1 are generic. In order to prove Theorems 1.1 and 1.2, we modify the arguments used in [13] and [15]. An advantage of our proofs is that we can deal with a class of maps in which the continuations of a basic set are not constant. In the proofs, conjugacy maps mentioned above play a more important role.

So we give a precise construction of conjugacy maps. To do this, unlike that in [15], we use not Contraction Principle but the shadowing lemma.

Next, we state two results obtained as applications of Theorems 1.1 and 1.2. Consider a diffeomorphism T on M. T is said to be C^1-Ω-stable if for any element S in some C^1-neighborhood of T, there exists a conjugacy map from $\Omega(S)$ to $\Omega(T)$, where $\Omega(T)$ is the nonwandering set of T. We say that T satisfies Axiom A if $\Omega(T)$ is hyperbolic and the totality of periodic points of T is dense in $\Omega(T)$. It is known that every C^1-Ω-stable diffeomorphism satisfies Axiom A (see [14] and [19]). So, by virtue of Smale's spectral decomposition theorem (see Theorem 3.5 in [3]), the nonwandering set for an C^1-Ω-stable diffeomorphism is written as the union of finitely many disjoint basic sets. In addition, from the definition, the totality of C^1-Ω-stable C^r-diffeomorphisms is open in $\mathit{Diff}^r(M)$. Therefore we easily see that our proofs of Theorems 1.1 and 1.2, following O. Jenkinson and I. D. Morris' idea, provide local properties about measures with maximum total exponent in C^1-Ω-stable C^r-diffeomorphisms. Furthermore, by using another method, we can prove the following two stronger theorems.

Theorem 1.3 *Each of the following properties is generic in C^1-Ω-stable C^1-diffeomorphisms:*

(1) *There exists a unique measure with maximum total exponent on M.*

(2) *Any measure with maximum total exponent on M has zero entropy.*

(3) *Any measure with maximum total exponent on M is fully supported on one of the basic sets in the spectral decomposition.*

In particular, a generic C^1-Ω-stable C^1-diffeomorphism has a unique measure with maximum total exponent on M, which is ergodic, of zero entropy and fully supported on one of its basic sets in the spectral decomposition.

Theorem 1.4 *Let $r \geq 2$. Then any measure with maximum total exponent on each basic set in the spectral decomposition for a generic C^1-Ω-stable C^r-diffeomorphism is not fully supported on the basic set unless the basic set itself is a periodic orbit. In particular, any measure with maximum total exponent on M for a generic C^1-Ω-stable C^r-diffeomorphism is not fully supported on any basic set unless the basic set itself is a periodic orbit.*

We see that Theorem 1.3 gives a partial answer to Problem. Indeed, measures with maximum total exponent for T are maximizing measures for the function $\log J(T)$. Moreover, the totality of Anosov diffeomorphisms with topological transitivity is open in the space of C^1-Ω-stable C^1-diffeomorphisms. Therefore, if X is a compact manifold, we obtain the following result (see also Remark after the proof of Theorem 1.4).

Corollary 1.1 *For a generic Anosov diffeomorphism $T : X \to X$ with topological transitivity,* $\log J(T)$ *is a function required in Problem.*

This paper is organized as follows. In Section 2, we summarize some fundamental results on uniform hyperbolic dynamical systems. Section 3 is devoted to the proofs of Theorems 1.1 and 1.2. In Section 4, we prove Theorems 1.3 and 1.4.

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2 Preliminaries

We summarize some properties of hyperbolic dynamical systems needed in this paper. Let $T : M \to M$ be a C^1-diffeomorphism. Let β be a positive number. Consider a sequence of points $\{x_i\}_{i=m}^n$ contained in M (we admit the case of $m = -\infty$ or $n = \infty$). If

$$d(Tx_i, x_{i+1}) < \beta \quad for\ \ any\ \ i \in \{m, m+1, \ldots, n-1\},$$

then $\{x_i\}_{i=m}^n$ is called a β-pseudo-orbit of T. For a pseudo-orbit $\{x_i\}_{i=-\infty}^{\infty}$, if there exists $n \in \mathbb{N}$ such that $x_{i+n} = x_i$ for any $i \in \mathbb{Z}$, then we say that $\{x_i\}_{i=-\infty}^{\infty}$ is periodic. Moreover, let α be a positive number. For $y \in M$, if

$$d(T^i y, x_i) < \alpha \quad for\ \ any\ \ i \in \{m, m+1, \ldots, n\},$$

then we say that y α-shadows $\{x_i\}_{i=m}^n$ by T. We treat only the case of $m = -\infty$ and $n = \infty$ in this paper. So we write as $\{x_i\}$ instead of $\{x_i\}_{i=-\infty}^{\infty}$ for the sake of simplicity. We need the following lemma (see Theorem 9.3.1 in [20] for the proof).

Lemma 2.1 *Let $T : M \to M$ be a C^1-diffeomorphism with a basic set Λ. For any $\alpha > 0$ there exist $\beta > 0$ and $\eta > 0$ such that if $\{x_i\}$ is a β-pseudo-orbit of T satisfying $d(x_i, \Lambda) < \eta$ for any $i \in \mathbb{Z}$ then there exists a unique point y in Λ which α-shadows $\{x_i\}$ by T. Moreover, if a β-pseudo-orbit $\{x_i\}$ of T is periodic, the α-shadowing point of $\{x_i\}$ is a periodic point of T.*

Consider a C^1-diffeomorphism $T : M \to M$ with a basic set Λ. Then it is well-known that there exists an open neighborhood $\mathcal{U}$ of T in the C^1-topology satisfying the following properties (see [7] and [20]):

(1) For any $S \in \mathcal{U}$, the continuation Λ_S of Λ for S is a basic set for S.

(2) There exists $\lambda' \in (0, 1)$ which is a skewness of S on Λ_S for any $S \in \mathcal{U}$.

A number $\beta > 0$ as in Lemma 2.1 is obtained depending on $\alpha > 0$ and a skewness of T on Λ. Therefore, by simple modification, we can generalize Lemma 2.1 as follows.

Lemma 2.2 *Let $T : M \to M$ be a C^1-diffeomorphism with a basic set Λ. Then there exists an open neighborhood $\mathcal{U}$ of T in the C^1-topology satisfying the following property. For any $\alpha > 0$ there exist $\beta > 0$ and $\eta > 0$ such that for any $S \in \mathcal{U}$ if $\{x_i\}$ is a β-pseudo-orbit of S satisfying $d(x_i, \Lambda_S) < \eta$ for any $i \in \mathbb{Z}$ then there exists a unique point y in Λ_S which α-shadows $\{x_i\}$ by S. Moreover, if a β-pseudo-orbit $\{x_i\}$ of S is periodic, the α-shadowing point of $\{x_i\}$ is a periodic point of S.*

Let $T : M \to M$ be a C^1-diffeomorphism with a basic set Λ. By the shadowing lemma and topological transitivity of T on Λ, we see that the set of all periodic points of T is dense in Λ. It is well-known that a hyperbolic set Ω for T is isolated if and only if Ω has local product structure, *i.e.*, there exists $r > 0$ such that $W_r^u(p) \cap W_r^s(q) \subset \Omega$ for any $p, q \in \Omega$, where

$$W_r^u(p) = \{x \in M | d(T^{-n}p, T^{-n}x) \leq r \ \ for \ \ any \ \ n \geq 0\}$$

is the local unstable manifold at p and

$$W_r^s(q) = \{x \in M | d(T^{n}q, T^{n}x) \leq r \ \ for \ \ any \ \ n \geq 0\}$$

is the local stable manifold at q. This fact enable us to obtain the spectral decomposition of Λ in the sense of Smale in the same way as in the proof of Theorem 3.5 in [3]. More precisely, Λ can be written as $X_1 \cup \cdots \cup X_n$, where the X_i are pairwise disjoint closed sets such that $T(X_i) = X_{i+1}$ for any $i \in \{1, \ldots, n-1\}$ and $T(X_n) = X_1$. Moreover, each $T^n|_{X_i}$ is topologically mixing and C-dense, *i.e.*, $W^u(p) \cap X_i$ is dense in X_i for any periodic point $p \in X_i$ of T^n, where

$$W^u(p) = \{x \in M | d((T^n)^{-m}(p), (T^n)^{-m}(x)) \to 0 \ \ as \ \ m \to \infty\}$$

is the global unstable manifold at p. Therefore we can show the following lemma in the same way as the proof of Lemma 1 and Lemma 2 in [21].

Lemma 2.3 *Let $T : M \to M$ be a C^1-diffeomorphism with a basic set Λ. Then we have the following.*

(1) *Let $\mathcal{M}_p(T, \Lambda)$ be the set of all T-invariant measures supported on a periodic orbit in Λ. Then $\mathcal{M}_p(T, \Lambda)$ is dense in $\mathcal{M}(T, \Lambda)$ in the weak-$*$ topology.*

(2) *Let Y be a proper closed subset of Λ. Then for any $\mu \in \mathcal{M}(T, \Lambda)$ with* $\mathrm{supp}(\mu) \subset Y$, *there exists a sequence $\mu_n \in \mathcal{M}_p(T, \Lambda)$ such that μ_n converges to μ in the weak-$*$ topology and* $\mathrm{supp}(\mu_n) \cap \left(\bigcap_{i=0}^{\infty} T^{-i} Y \right) = \emptyset$ *for any n.*

Next, we show that basic sets are stable (see [7] and Theorem 9.7.4 in [20]).

Lemma 2.4 *Let $T : M \to M$ be a C^1-diffeomorphism with a basic set Λ. If S is a C^1-diffeomorphism close enough to T in the C^1-topology, then S is hyperbolic on Λ_S and there exists a homeomorphism $h_S : \Lambda_S \to \Lambda$ such that $T \circ h_S = h_S \circ S$ on Λ_S. Furthermore both $\sup_{x \in \Lambda_S} d(h_S(x), x)$ and $\sup_{x \in \Lambda} d(h_S^{-1}(x), x)$ go to 0 if S is in some neighborhood of T in the C^1-topology and S goes to T in the C^0-topology.*

Proof. As mentioned above, if S is close enough to T in the C^1-topology, then Λ_S is a basic set for S. We just construct a homeomorphism $h_S : \Lambda_S \to \Lambda$ as desired.

Since Λ is a basic set for T, as shown in [16], there exist a neighborhood $\mathcal{U}_1$ of T in the C^1-topology and a positive number α such that each element $S \in \mathcal{U}_1$ is expansive on Λ_S with expansive constant α, *i.e.*, $x, y \in \Lambda_S$ and $x \neq y$ yields that $d(S^i x, S^i y) > \alpha$ for some $i \in \mathbb{Z}$. Moreover, by Lemma 2.2, we see that there exist a neighborhood $\mathcal{U}_2 \subset \mathcal{U}_1$ of T independent of α, $0 < \beta < \alpha/2$ and $\eta > 0$ such that if $S \in \mathcal{U}_2$ then any β-pseudo-orbit of S contained in an η-neighborhood of Λ_S is $\alpha/2$-shadowed by S by a unique point in Λ_S. Now we take an isolating neighborhood U of Λ such that

$$\bigcap_{i=-N_1}^{N_1} T^i(U) \subset B(\Lambda, \eta) := \{x \in M | d(x, \Lambda) < \eta\} \ \ for \ \ some \ \ N_1 \in \mathbb{N}.$$

Then we can take a neighborhood $\mathcal{U} \subset \mathcal{U}_2$ of T in the C^1-topology such that if $S \in \mathcal{U}$ then

$$\bigcap_{i=-N_1}^{N_1} S^i(U) \subset B(\Lambda, \eta)$$

and $\sup_{x \in M} d(Sx, Tx) < \beta$. For any $x \in \Lambda_S$, $\{S^i x\}$ is a β-pseudo-orbit of T contained in an η-neighborhood of Λ. Therefore there exists a unique point $h_S(x)$ in Λ such that $d(T^i(h_S(x)), S^i x) < \alpha/2$ for any $i \in \mathbb{Z}$. This means that

$$\{h_S(x)\} = \bigcap_{i \in \mathbb{Z}} T^{-i}\left(B\left(S^i x, \frac{\alpha}{2}\right)\right).$$

We define the map $h_S : \Lambda_S \to \Lambda$ in this way.

Next, we verify that $h_S : \Lambda_S \to \Lambda$ is continuous. For any $\gamma > 0$, there exists $N_2 \in \mathbb{N}$ with

$$\bigcap_{i=-N_2}^{N_2} T^{-i}\left(B\left(S^i x, \frac{\alpha}{2}\right)\right) \subset B\left(h_S(x), \frac{\gamma}{2}\right).$$

Therefore, if $y \in \Lambda_S$ is close enough to x then we have

$$\{h_S(y)\} \subset \bigcap_{i=-N_2}^{N_2} T^{-i}\left(B\left(S^i y, \frac{\alpha}{2}\right)\right) \subset B(h_S(x), \gamma).$$

Thus, we see that $h_S : \Lambda_S \to \Lambda$ is continuous. Moreover, for any $x \in \Lambda_S$, we obtain that $d(T^i(h_S(Sx)), S^i(Sx)) < \alpha/2$ for any $i \in \mathbb{Z}$. This means that $d(T^{i+1}(T^{-1}(h_S(Sx))), S^{i+1}x) < \alpha/2$ for any $i \in \mathbb{Z}$. From the uniqueness of the shadowing point $h_S(x)$, we have $T \circ h_S = h_S \circ S$ on Λ_S.

Next, we verify that $h_S : \Lambda_S \to \Lambda$ is a homeomorphism. Note that we may assume that for any $S \in \mathcal{U}$, Λ is contained in $B(\Lambda_S, \eta)$ (see Theorem 7.3 in [7]). For any $x \in \Lambda$, $\{T^i x\}$ is a β-pseudo-orbit of S contained in an η-neighborhood of Λ_S. Therefore, there exists a unique point $g_S(x)$ in Λ_S such that $d(S^i(g_S(x)), T^i x) < \alpha/2$ for any $i \in \mathbb{Z}$. We define the map $g_S : \Lambda \to \Lambda_S$ in this way. Since $g_S(x) \in \Lambda_S$ for any $x \in \Lambda$, we have $d(T^i(h_S(g_S(x))), S^i(g_S(x))) < \alpha/2$ for any $i \in \mathbb{Z}$. Therefore we obtain that $d(T^i(h_S(g_S(x))), T^i x) < \alpha$ for any $i \in \mathbb{Z}$. Thus, from the property of α, we have $h_S(g_S(x)) = x$ for any $x \in \Lambda$. Similarly we have that $g_S(h_S(x)) = x$ for any $x \in \Lambda_S$. Hence we see that $h_S : \Lambda_S \to \Lambda$ is a homeomorphism and $h_S^{-1} = g_S$. Moreover, by Lemma 2.2, we see that if $S \in \mathcal{U}$ and $\sup_{x \in M} d(Sx, Tx)$ goes to 0 then both $\sup_{x \in \Lambda_S} d(h_S(x), x)$ and $\sup_{x \in \Lambda} d(h_S^{-1}(x), x)$ go to 0. Therefore we see that the last assertion is valid. □

Next, we summarize some facts on invariant measures and entropy. As usual, $h(T,\mu)$ denotes the metric entropy of T with respect to the T-invariant measure μ.

Lemma 2.5 *Let $T : M \to M$ be a C^1-diffeomorphism with a basic set Λ. Let $\{T_n\}$ be a sequence of C^1-diffeomorphisms close enough to T in the C^1-topology and converging to T in the C^0-topology. Then we have the following.*

(1) *Any $\mu \in \mathcal{M}(T,\Lambda)$ is the weak-$*$ limit of a sequence of measures $\{\mu_n\}$ satisfying $\mu_n \in \mathcal{M}(T_n,\Lambda_{T_n})$ for any n.*

(2) *If $\{\mu_n\}$ is a sequence of measures satisfying $\mu_n \in \mathcal{M}(T_n,\Lambda_{T_n})$ for any n, then any weak-$*$ accumulation point of $\{\mu_n\}$ belongs to $\mathcal{M}(T,\Lambda)$.*

(3) *If $\{\mu_n\}$ is a sequence of measures satisfying $\mu_n \in \mathcal{M}(T_n,\Lambda_{T_n})$ for any n converging to μ in the weak-$*$ topology, then we have*

$$\limsup_{n\to\infty} h(T_n,\mu_n) \leq h(T,\mu).$$

Proof. For each $n \in \mathbb{N}$, let $h_n : \Lambda_{T_n} \to \Lambda$ be the homeomorphism as in Lemma 2.4.

(1) For each $n \in \mathbb{N}$, put $\mu_n = \mu \circ h_n$. Then we see that μ_n is in $\mathcal{M}(T_n,\Lambda_{T_n})$. Moreover we can show that $\{\mu_n\}$ converges to μ in the weak-$*$ topology in the same way as the proof of the assertion (a) of Lemma 3 in [13].

(2) Without loss of generality, we may assume that $\{\mu_n\}$ converges in the weak-$*$ topology. For each $n \in \mathbb{N}$, put $\nu_n = \mu_n \circ h_n^{-1}$. Then we see that ν_n is in $\mathcal{M}(T, \Lambda)$. Furthermore we can show that $\{\mu_n\}$ and $\{\nu_n\}$ converge to the same measure in the weak-$*$ topology in the same way as the proof of the assertion (b) of Lemma 3 in [13]. Since $\mathcal{M}(T, \Lambda)$ is compact, we see that the limit of $\{\mu_n\}$ belongs to $\mathcal{M}(T, \Lambda)$.

(3) Since T is expansive on Λ as mentioned in the proof of Lemma 2.4, the entropy map $\mathcal{M}(T, \Lambda) \to \mathbb{R}; \mu \mapsto h(T, \mu)$ is upper semi-continuous as shown in Theorem 8.2 in [24]. For each $n \in \mathbb{N}$, we have that $\mu_n \circ h_n^{-1}$ is in $\mathcal{M}(T, \Lambda)$ and $h(T_n, \mu_n) = h(T, \mu_n \circ h_n^{-1})$. Moreover, since $\{\mu_n \circ h_n^{-1}\}$ converges to μ in the weak-$*$ topology, we obtain

$$\limsup_{n \to \infty} h(T_n, \mu_n) = \limsup_{n \to \infty} h(T, \mu_n \circ h_n^{-1}) \leq h(T, \mu). \ \square$$

Finally, we state a fact on measures with maximum total exponent.

Lemma 2.6 *Let $T : M \to M$ be a C^1-diffeomorphism with a basic set Λ. If $\{T_n\}$ is a sequence of C^1-diffeomorphisms converging to T in the C^1-topology and $\{\mu_n\}$ is a sequence of measures satisfying $\mu_n \in \mathcal{L}(T_n, \Lambda_{T_n})$ for any n, then any weak-$*$ accumulation point of $\{\mu_n\}$ belongs to $\mathcal{L}(T, \Lambda)$.*

Proof. For any sufficiently large n, T_n satisfies the condition of Lemma 2.4. Without loss of generality, we may assume that $\{\mu_n\}$ converges to a measure μ in the weak-$*$ topology. By the assertion (2) of Lemma 2.5, we have that μ is in $\mathcal{M}(T,\Lambda)$. Take any $\nu \in \mathcal{M}(T,\Lambda)$. Since $\{T_n\}$ converges to T in the C^1-topology, $\{J(T_n)\}$ converges to $J(T)$ uniformly on M. Thus, by using the assertion (1) of Lemma 2.5, we can show $\lambda(T,\mu) \geq \lambda(T,\nu)$ in the same way as the proof of Lemma 5 in [13]. $\square$

3 Proofs of Theorems 1.1 and 1.2

In this section, we shall prove Theorem 1.1 and Theorem 1.2. We need a theorem on perturbation of C^1-diffeomorphisms to prove these two theorems. We prepare a special fundamental neighborhood system in the C^1-topology to state the theorem. Consider a C^1-map $T : M \to M$. Let $\{(\varphi_j, U_j)\}_{j=1}^{J}$ be a C^∞-atlas of M. Let $C_j \subset U_j$ be a compact subset with $\bigcup_{j=1}^{J} C_j = M$. For each $j \in \{1,\dots,J\}$, there exists a family $\{C_{j,l}\}_{l=1}^{L_j}$ of compact subsets such that we have that $\bigcup_{l=1}^{L_j} C_{j,l} = C_j$ and $T(C_{j,l}) \subset U_{k(j,l)}$ for some $k(j,l) \in \{1,\dots,J\}$.

For $\epsilon > 0$, we define

$$\begin{aligned}\mathcal{N}(T,\epsilon) :=& \{S \in C^1(M,M) \mid S(C_{j,l}) \subset U_{k(j,l)} \quad for\ any\ j \in \{1,\dots,J\}\ and \\ & l \in \{1,\dots,L_j\}, \sup_{x\in M} d(S(x),T(x)) < \epsilon\ and \\ & \max_{1\le j\le J} \max_{1\le l\le L_j} \sup_{x\in C_{j,l}} \|D(\varphi_{k(j,l)} \circ S \circ \varphi_j^{-1})(\varphi_j(x)) \\ & - D(\varphi_{k(j,l)} \circ T \circ \varphi_j^{-1})(\varphi_j(x))\| < \epsilon\}.\end{aligned}$$

It is well-known that $\{\mathcal{N}(T,\epsilon) | \epsilon > 0\}$ is a fundamental neighborhood system of T composed of open sets in the C^1-topology. Now we state a perturbation theorem which we need to prove Theorem 1.1 and Theorem 1.2.

Theorem 3.1 *Let $T : M \to M$ be a C^1-diffeomorphism. Let $x_0 \in M$ be a periodic point of T with least period p. Then there exists $\epsilon_0 > 0$ depending only on T such that for $0 < \epsilon < \epsilon_0$ and $\gamma > 0$, there exists $\delta_0 > 0$ such that for $0 < \delta < \delta_0$, there exist neighborhood U_δ^i of $T^i x_0$ for any $0 \le i \le p-1$ and a C^1-diffeomorphism T_δ satisfying the following properties:*

(1) $T_\delta^i x_0 = T^i x_0$ *for any* $i \in \mathbb{Z}$.

(2) $U_\delta^i \cap U_\delta^j = \emptyset$ *if* $i \ne j$, $\overline{U_{\delta'}^i} \subset U_\delta^i$ *if* $0 < \delta' < \delta$, *and* $\bigcap_{0<\delta<\delta_0} U_\delta^i = \{T^i x_0\}$.

(3) $T_\delta x = Tx$ *for any* $x \in M \setminus \bigcup_{i=0}^{p-1} U_\delta^i$.

(4) $\sup_{x\in M} d(T_\delta(x), T(x)) < K_0\delta$ *and* $T_\delta \in \mathcal{N}(T, K_1\epsilon)$, *where* K_0 *and* K_1 *are constants independent of* ϵ, γ *and* δ.

(5) *Define* $G_\delta : M \to \mathbb{R}; x \mapsto \log(J(T_\delta)(x)/J(T)(x))$. *Then* $G_\delta(T_\delta^i x_0) = \epsilon$ *for any* $i \in \mathbb{Z}$.

(6) $\sup_{x\in M} G_\delta(x) < \epsilon + \gamma$.

Such a kind of perturbation theorem for C^1-maps with nonvanishing Jacobian is proved in Section 3 in [15] with $\epsilon_0 = 1$. As a corollary to the theorem we also obtain the corresponding result for C^1-expanding maps (see Corollary 1 in Section 3 in [15]). The corollary is proved by applying the perturbation theorem with ϵ_0 so small that the perturbed map can be expanding. In the present situation, it suffices to apply the same perturbation theorem to a C^1-diffeomorphism with ϵ_0 so small that the perturbed C^1-map can be a diffeomorphism. Thus we see that Theorem 3.1 is an easy consequence of the perturbation theorem in Section 3 in [15]. So we omit the proof.

Theorem 3.1 might remind us of the Franks lemma (see Lemma 1.1 in [6]). Like Theorem 3.1, the Franks lemma is a theorem on perturbation of C^1-diffeomorphisms on a finite set (not necessary a periodic orbit different from Theorem 3.1). But the Franks lemma does not insist that we can take a sequence of C^1-diffeomorphisms converging to given C^1-diffeomorphism in the C^0-topology keeping the property (5) in Theorem 3.1. Therefore our

Theorem 3.1 is more useful for us than the Franks lemma in this sense.

Now we prove Theorem 1.1. Fix a C^1-diffeomorphism $T : M \to M$ with a basic set Λ. As in the proof of Lemma 2.4, there exist a C^1-neighborhood $\mathcal{U}_1 \subset \mathit{Diff}^{\,1}(M)$ of T and $\alpha > 0$ such that each element $S \in \mathcal{U}_1$ is expansive on Λ_S with expansive constant α. By Lemma 2.2, there exist a neighborhood $\mathcal{U}_2 \subset \mathcal{U}_1$ of T independent of α, $0 < \beta < \alpha/4$ and $\eta > 0$ such that if $S \in \mathcal{U}_2$ then any β-pseudo-orbit of S contained in η-neighborhood of Λ_S is $\alpha/4$-shadowed by S by a unique point in Λ_S. Then we construct a C^1-neighborhood $\mathcal{U} \subset \mathcal{U}_2$ of T as in the proof of Lemma 2.4.

Let ρ be a metric on the space $\mathcal{M}(M)$ of all Borel probability measures on M not necessary invariant inducing the weak-$*$ topology such that for any $\mu, \nu \in \mathcal{M}(M)$ and any $\lambda \in [0, 1]$, we have

$$\rho(\mu, (1-\lambda)\mu + \lambda\nu) \leq \lambda. \tag{3.1}$$

It is well-known that such a metric ρ exists. We prove the following proposition to show that a generic element in $\mathcal{U}$ satisfies the properties (1) and (2) of Theorem 1.1.

Proposition 3.1 *For $\kappa > 0$, put*

$$\mathcal{R}_\kappa = \{S \in \mathcal{U} \mid \operatorname{diam}_\rho(\mathcal{L}(S, \Lambda_S)) < \kappa\},$$

$$\mathcal{S}_\kappa = \{S \in \mathcal{U} \mid \sup_{\mu \in \mathcal{L}(S,\Lambda_S)} h(T, \mu) < \kappa h_{top}(S|_{\Lambda_S})\},$$

where $h_{top}(S|_{\Lambda_S})$ is the topological entropy of $S|_{\Lambda_S}$. Then both $\mathcal{R}_\kappa$ and $\mathcal{S}_\kappa$ are open and dense in $\mathcal{U}$.

Proof. First, we show that $\mathcal{R}_\kappa$ is open in $\mathcal{U}$. Let $S_n \in \mathcal{U} \backslash \mathcal{R}_\kappa$ be a sequence converging to $S \in \mathcal{U}$ in the C^1-topology. Then by compactness of $\mathcal{L}(S_n, \Lambda_{S_n})$ there exist $\mu_n, \nu_n \in \mathcal{L}(S_n, \Lambda_{S_n})$ with $\rho(\mu_n, \nu_n) \geq \kappa$. Taking subsequences, we may assume that μ_n and ν_n converge to μ and ν, respectively. By Lemma 2.6, we obtain that μ and ν are in $\mathcal{L}(S, \Lambda_S)$. Thus we have that $\rho(\mu, \nu) \geq \kappa$, consequently $\mathcal{R}_\kappa$ is open in $\mathcal{U}$.

Next, we show that $\mathcal{S}_\kappa$ is open in $\mathcal{U}$. By Lemma 2.4, we see that for any $S \in \mathcal{U}$, $S|_{\Lambda_S}$ is topologically conjugate to $T|_\Lambda$. Therefore we have that $h_{top}(S|_{\Lambda_S}) = h_{top}(T|_\Lambda)$ for any $S \in \mathcal{U}$. Let $S_n \in \mathcal{U} \setminus \mathcal{S}_\kappa$ be a sequence converging to $S \in \mathcal{U}$ in the C^1-topology. Then, by compactness of $\mathcal{L}(S_n, \Lambda_{S_n})$ and upper semi-continuity of the entropy map $\mathcal{M}(S_n, \Lambda_{S_n}) \to \mathbb{R}; \mu \mapsto h(S_n, \mu)$, there exists $\mu_n \in \mathcal{L}(S_n, \Lambda_{S_n})$ with $h(S_n, \mu_n) = \sup_{\mu \in \mathcal{L}(S_n, \Lambda_{S_n})} h(S_n, \mu)$. Taking a subsequence, we may assume that μ_n converges to μ. By Lemma 2.6, we obtain that μ is in $\mathcal{L}(S, \Lambda_S)$. By the assertion (3) of Lemma 2.5, we have

$$\kappa h_{top}(S|_{\Lambda_S}) \leq \limsup_{n\to\infty} h(S_n, \mu_n) \leq h(S, \mu).$$

Thus we see that $\mathcal{S}_\kappa$ is open in $\mathcal{U}$.

Finally, we show that $\mathcal{R}_\kappa$ and $\mathcal{S}_\kappa$ are dense in $\mathcal{U}$. Take any $S \in \mathcal{U}$ and any $0 < \epsilon < \epsilon_0$, where $\epsilon_0 > 0$ is as in Theorem 3.1 for S. By the assertion (1) of Lemma 2.3, we have a periodic point $x_0 \in \Lambda_S$ for S with least period p such that

$$\int \log J(S) d\mu_0 > \lambda(S, \Lambda_S) - \frac{\kappa\epsilon}{8}, \tag{3.2}$$

where $\mu_0 = (1/p) \sum_{i=0}^{p-1} \delta_{S^i x_0}$ is a periodic measure of S. Applying Theorem 3.1 to S and x_0 with $\gamma = (\kappa\epsilon)/8$, we have a perturbation S_δ along the orbit of x_0 for S. By the assertion (1) of Theorem 3.1, we see that μ_0 is in $\mathcal{M}(S_\delta, \Lambda_{S_\delta})$. Moreover for any sufficiently small $0 < \epsilon < \epsilon_0$, we see that S_δ is in $\mathcal{U}$. By the assertion (5) of Theorem 3.1 and the inequality (3.2), we obtain

$$\lambda(S_\delta, \Lambda_{S_\delta}) \geq \int \log J(S_\delta) d\mu_0 = \int \log J(S) d\mu_0 + \int G_\delta d\mu_0 > \lambda(S, \Lambda_S) + \left(1 - \frac{\kappa}{8}\right)\epsilon. \tag{3.3}$$

Take any strictly decreasing sequence $0 < \delta_n < \delta_0$ converging to 0 and any $\mu_n \in \mathcal{L}(S_{\delta_n}, \Lambda_{S_{\delta_n}})$, where $\delta_0 > 0$ is as in Theorem 3.1 for S, ϵ and γ. By the assertion (4) of Theorem 3.1, we see that S_{δ_n} converges to S in the C^0-topology. Taking a subsequence, we may assume that μ_n converges to μ. By the assertion (2) of Lemma 2.5, we see that μ is in $\mathcal{M}(S, \Lambda_S)$. Therefore for any sufficiently large n, we have

$$\begin{aligned} \int G_{\delta_n} d\mu_n &= \int \log J(S_{\delta_n}) d\mu_n - \int \log J(S) d\mu_n \\ &> \lambda(S_{\delta_n}, \Lambda_{S_{\delta_n}}) - \left(\lambda(S, \Lambda_S) + \frac{\kappa\epsilon}{8}\right). \end{aligned} \tag{3.4}$$

From the inequalities (3.3) and (3.4), for any sufficiently large n, we obtain

$$\int G_{\delta_n} d\mu_n > \left(1 - \frac{\kappa}{4}\right)\epsilon. \tag{3.5}$$

Now for any $0 < \delta < \delta_0$ and any $0 \leq i \leq p-1$, let U_δ^i be the neighborhood of $S^i x_0$ as in Theorem 3.1. Put $U_\delta = \bigcup_{i=0}^{p-1} U_\delta^i$. Since $\{\delta_n\}$ is a strictly decreasing sequence, by the assertion (2) of Theorem 3.1, we have that $\overline{U_{\delta_{n+1}}} \subset U_{\delta_n}$ for any n, and $\bigcap_{n=1}^{\infty} U_{\delta_n} = O_S(x_0)$, where $O_S(x_0)$ is the orbit of x_0 for S. Then by the assertions (6) and (3) of Theorem 3.1 together with the inequality (3.5), we obtain

$$\begin{aligned} \mu_n(U_{\delta_n}) &\geq \left(1 + \frac{\kappa}{8}\right)^{-1} \epsilon^{-1} \int_{U_{\delta_n}} G_{\delta_n} d\mu_n = \left(1 + \frac{\kappa}{8}\right)^{-1} \epsilon^{-1} \int G_{\delta_n} d\mu_n \\ &> \left(1 + \frac{\kappa}{8}\right)^{-1} \left(1 - \frac{\kappa}{4}\right) > 1 - \frac{3\kappa}{8}. \end{aligned}$$

Note that $\mu_m(U_{\delta_n}) \geq \mu_m(U_{\delta_m})$ for any $n < m$. Since μ_n converges to μ in the weak-$*$ topology, for any n, we have

$$\mu(\overline{U_{\delta_n}}) \geq \limsup_{m\to\infty} \mu_m(\overline{U_{\delta_n}}) \geq 1 - \frac{3\kappa}{8}.$$

Since $\bigcap_{n=1}^{\infty} \overline{U_{\delta_n}} = O_S(x_0)$, we have $\mu(O_S(x_0)) \geq 1 - (3\kappa)/8$. Therefore we see that $\mu = (1-(3\kappa)/8)\,\mu_0 + ((3\kappa)/8)\bar{\mu}$, where $\bar{\mu} \in \mathcal{M}(S, \Lambda_S)$. Thus, from the property (3.1) of ρ, we have

$$\lim_{n\to\infty} \rho(\mu_n, \mu_0) = \rho(\mu, \mu_0) \leq \frac{3\kappa}{8}.$$

Hence we obtain that $S_\delta \in \mathcal{R}_\kappa$ for any sufficiently small $\delta > 0$. Moreover, from the assertion (3) of Lemma 2.5, we have

$$\limsup_{n\to\infty} h(S_{\delta_n}, \mu_n) \leq h(S, \mu) = \frac{3\kappa}{8} h(S, \bar{\mu}) \leq \frac{3\kappa}{8} h_{top}(S|_{\Lambda_S}).$$

Hence we obtain that $S_\delta \in \mathcal{S}_\kappa$ for any sufficiently small $\delta > 0$. By the assertion (4) of Theorem 3.1 and the arbitrariness of $\epsilon > 0$, we see that $\mathcal{R}_\kappa$ and $\mathcal{S}_\kappa$ are dense in $\mathcal{U}$. □

Next we give a proposition which implies that a generic element in $\mathcal{U}$ satisfies the property (3) of Theorem 1.1. For $S \in \mathcal{U}$, let $h_S : \Lambda_S \to \Lambda$ be the homeomorphism constructed in Lemma 2.4. Then we have the following proposition.

Proposition 3.2 *For a nonempty proper closed subset Y of Λ, put*

$$M^1(Y) = \{S \in \mathcal{U} \mid \operatorname{supp}(\mu) \subset h_S^{-1}(Y) \ \textit{for some} \ \mu \in \mathcal{L}(S, \Lambda_S)\}.$$

Then $M^1(Y)$ is closed and nowhere dense in $\mathcal{U}$.

Proof. First, we show that $M^1(Y)$ is closed in $\mathcal{U}$. Let $S_n \in M^1(Y)$ be a sequence converging to $S \in \mathcal{U}$ in the C^1-topology. Let $\mu_n \in \mathcal{L}(S_n, \Lambda_{S_n})$ be a sequence with $\operatorname{supp}(\mu_n) \subset h_{S_n}^{-1}(Y)$. Then, by Lemma 2.6, we see that any weak-$*$ accumulation point μ of $\{\mu_n\}$ is in $\mathcal{L}(S, \Lambda_S)$. Taking a subsequence, we may assume that μ_n converges to μ. Since $h_S^{-1} \circ h_{S_n} : \Lambda_{S_n} \to \Lambda_S$ is a unique homeomorphism such that $S \circ h_S^{-1} \circ h_{S_n} = h_S^{-1} \circ h_{S_n} \circ S_n$ on Λ_{S_n} and $d(S^i(h_S^{-1} \circ h_{S_n}(x)), S_n^i(x)) < \alpha/2$ for any $x \in \Lambda_{S_n}$ and any $i \in \mathbb{Z}$, we have that $h_{S_n}^{-1}$ converges to h_S^{-1} on Λ in the C^0-topology. Therefore we see that $\mu_n \circ h_{S_n}^{-1}$ converges to $\mu \circ h_S^{-1}$ in the weak-$*$ topology. Since Y is closed, we have

$$\mu(h_S^{-1}(Y)) \geq \limsup_{n \to \infty} \mu_n(h_{S_n}^{-1}(Y)) = 1.$$

Thus $M^1(Y)$ is closed in $\mathcal{U}$.

Next, we show that $M^1(Y)$ is nowhere dense in $\mathcal{U}$. Take any $S \in M^1(Y)$ and any $0 < \epsilon < \epsilon_0$, where $\epsilon_0 > 0$ is as in Theorem 3.1 for S. Let μ be a measure in $\mathcal{L}(S, \Lambda_S)$ with $\operatorname{supp}(\mu) \subset h_S^{-1}(Y)$. By the assertion (2) of Lemma

2.3, we have a periodic point $x_0 \in \Lambda_S$ for S with least period p such that $O_S(x_0) \cap \left(\bigcap_{i=0}^{\infty} S^{-i}(h_S^{-1}(Y)) \right) = \emptyset$ and

$$\int \log J(S) d\mu_0 > \lambda(S, \Lambda_S) - \epsilon, \tag{3.6}$$

where $\mu_0 = (1/p) \sum_{i=0}^{p-1} \delta_{S^i x_0}$. Applying Theorem 3.1 to S and x_0 with $\gamma = 1$, we have a perturbation S_δ along the orbit of x_0 for S. By the assertions (1) and (4) of Theorem 3.1, we see that μ_0 is in $\mathcal{M}(S_\delta, \Lambda_{S_\delta})$ and $\sup_{x \in M} d(S_\delta(x), S(x)) \leq K_0\delta$, where $K_0 > 0$ is as in Theorem 3.1 for S. Moreover, for any sufficiently small $0 < \epsilon < \epsilon_0$, we see that S_δ is in $\mathcal{U}$. Now take a positive integer N_0 with $O_S(x_0) \cap \left(\bigcap_{i=0}^{N_0} S^{-i}(h_S^{-1}(Y)) \right) = \emptyset$. Put $Y_{N_0} = \bigcap_{i=0}^{N_0} S^{-i}(h_S^{-1}(Y))$ and $Y_{N_0,\delta} = \bigcup_{R \in \mathcal{U}: d_0(R,S) \leq K_0\delta} \bigcap_{i=0}^{N_0} R^{-i}(h_R^{-1}(Y))$, where $d_0(R, S) = \sup_{x \in M} d(R(x), S(x))$. Note that $Y_{N_0} = \bigcap_{\delta > 0} Y_{N_0,\delta}$. Take $0 < \delta' < \delta < \delta_0$ such that

$$\inf_{y \in Y_{N_0,\delta}} d(O_S(x_0), y) > K_0\delta,$$

$$U^i_{\delta'} \subset B(S^i x_0, K_0\delta) \ \ for \ \ any \ \ 0 \leq i \leq p-1,$$

where $\delta_0 > 0$ is as in Theorem 3.1 for S, ϵ and γ, and $U^i_{\delta'}$ is the neighborhood of $S^i x_0$ as in Theorem 3.1. Then, by the assertion (3) of Theorem 3.1, we see that $S_{\delta'} = S$ on $Y_{N_0,\delta}$. Therefore, by continuity, we obtain that $S_{\delta'} = S$ on $\overline{Y_{N_0,\delta}}$. We see that if $\mu \in \mathcal{M}(S_\delta, \Lambda_{S_\delta})$ and $\mu(h_{S_\delta}^{-1}(Y)) = 1$ then $\mu(\overline{Y_{N_0,\delta}}) = 1$.

Therefore for any measure μ with $\mu(\overline{Y_{N_0,\delta}}) = 1$, we see that μ is $S_{\delta'}$-invariant if and only if μ is S-invariant. Since $S_{\delta'} = S$ on $\overline{Y_{N_0,\delta}}$, for any $\mu \in \mathcal{M}(S_{\delta'}, \Lambda_{S_{\delta'}})$ with $\mu(h_{S_{\delta'}}^{-1}(Y)) = 1$, we have

$$\int \log J(S_{\delta'}) d\mu = \int \log J(S) d\mu \leq \lambda(S, \Lambda_S). \tag{3.7}$$

From the inequality (3.6) and the assertion (5) of Theorem 3.1, we have

$$\lambda(S, \Lambda_S) < \int \log J(S) d\mu_0 + \epsilon = \int \log J(S_{\delta'}) d\mu_0. \tag{3.8}$$

Therefore, from the inequalities (3.7) and (3.8), we obtain

$$\int \log J(S_{\delta'}) d\mu < \int \log J(S_{\delta'}) d\mu_0.$$

Thus we have that μ is not in $\mathcal{L}(S_{\delta'}, \Lambda_{S_{\delta'}})$, consequently $S_{\delta'}$ is not in $M^1(Y)$. By the assertion (4) of Theorem 3.1 and the arbitrariness of $\epsilon > 0$, we see that $M^1(Y)$ is nowhere dense in $\mathcal{U}$. □

Now we are in a position to finish the proof of Theorem 1.1. By Proposition 3.1, $\bigcap_{n=1}^{\infty} \mathcal{R}_{1/n}$ and $\bigcap_{n=1}^{\infty} \mathcal{S}_{1/n}$ are residual subsets of $\mathcal{U}$ and any elements in $\bigcap_{n=1}^{\infty} \mathcal{R}_{1/n}$ and $\bigcap_{n=1}^{\infty} \mathcal{S}_{1/n}$ satisfy the property (1) and (2) of Theorem 1.1, respectively. This completes the proof that a generic element in $\mathcal{U}$ satisfies (1) and (2) of Theorem 1.1.

Next let $\{B_n\}$ be a countable open basis of Λ. We may assume that $B_n \neq \emptyset$ for any n. Put $Y_n = \Lambda \setminus B_n$. By Proposition 3.2, we see that $\bigcap_{n=1}^{\infty} (\mathcal{U} \setminus M^1(Y_n))$ is a residual subset of $\mathcal{U}$ and any element in $\bigcap_{n=1}^{\infty} (\mathcal{U} \setminus M^1(Y_n))$ satisfies the property (3) of Theorem 1.1. This completes the proof that a generic element in $\mathcal{U}$ satisfies (3) of Theorem 1.1.

In order to verify the second assertion in Theorem 1.1, it suffices to show that the unique measure with maximum total exponent on Λ_S for S satisfying (1) of Theorem 1.1 is ergodic with respect to S. To this end, let μ be the unique measure with maximum total exponent on Λ_S for S. Then by virtue of the ergodic decomposition theorem, we see that there exists a probability measure τ on the set $\mathcal{E}(S, \Lambda_S)$ of all ergodic measures with respect to $S|_{\Lambda_S}$ such that

$$\int \log J(S) d\mu = \int_{\mathcal{E}(S,\Lambda_S)} \left(\int \log J(S) d\nu \right) d\tau(\nu).$$

Since μ is a measure with maximum total exponent on Λ_S for S, we see that τ-a.e. ν must be a measure with maximum total exponent on Λ_S for S. Therefore we can easily see that the uniqueness of μ yields its ergodicity.

Finally, we prove Theorem 1.2. We need the following lemma (see Theorem 4.7 in [9]).

Lemma 3.1 *Let $T : M \to M$ be a C^1-diffeomorphism with a basic set Λ. Then for any Hölder continuous function $f : \Lambda \to \mathbb{R}$, there exists a continuous function $\varphi : \Lambda \to \mathbb{R}$ such that*

$$f + \varphi - \varphi \circ T \leq \sup_{\mu \in \mathcal{M}(T,\Lambda)} \int f d\mu \quad on \ \Lambda.$$

We omit the proof (for the proof, see [2]).

Proof of Theorem 1.2. If Λ consists of a periodic orbit of T, then Theorem 1.2 follows by Lemma 2.4. We assume that Λ is not a periodic orbit of T. Since T is C^r $(r \geq 2)$, $J(T)$ is a Lipschitz continuous function on Λ. Therefore, by Lemma 3.1, there exists a continuous function $\varphi : \Lambda \to \mathbb{R}$ such that

$$J(T) + \varphi - \varphi \circ T \leq \lambda(T, \Lambda) \quad on \ \Lambda.$$

If there exists a $\mu \in \mathcal{L}(T, \Lambda)$ such that $\mathrm{supp}(\mu) = \Lambda$ then we must have

$$J(T) + \varphi - \varphi \circ T = \lambda(T, \Lambda) \quad on \ \Lambda.$$

Therefore we obtain $\mathcal{L}(T, \Lambda) = \mathcal{M}(T, \Lambda)$. Since Λ is not a periodic orbit of T, there exist two distinct periodic orbits of T contained in Λ. Therefore, by perturbing along a periodic orbit, we can construct a C^r-diffeomorphism

S close enough to T in the C^r-topology satisfying $\mathcal{L}(S,\Lambda_S) \neq \mathcal{M}(S,\Lambda_S)$. Moreover, for any sufficiently small neighborhood $\mathcal{U} \subset \mathit{Diff}^{\,r}(M)$ of T, the set of all C^r-diffeomorphisms $S \in \mathcal{U}$ such that $\mathrm{supp}(\mu) = \Lambda_S$ for some $\mu \in \mathcal{L}(S,\Lambda_S)$ is closed in $\mathcal{U}$. Thus, we see that this set is closed and nowhere dense in $\mathcal{U}$. $\square$

4 Proofs of Theorems 1.3 and 1.4

In this section, we show Theorems 1.3 and 1.4. Consider a C^1-diffeomorphism $T : M \to M$. For simplicity, we write $\lambda(T)$, $\mathcal{L}(T)$ instead of $\lambda(T,M)$, $\mathcal{L}(T,M)$, respectively. We say that a point $x \in M$ is nonwandering for T if for any open neighborhood U of x there exists $n \geq 1$ with $T^{-n}(U) \cap U \neq \emptyset$. Note that $\Omega(T)$ as in Section 1 is the set of all points which are nonwandering for T. It is well-known that $\Omega(T) \neq \emptyset$, $T(\Omega(T)) = \Omega(T)$ and $\Omega(T)$ is closed. Recall that a C^1-diffeomorphism $T : M \to M$ is C^1-Ω-stable if there exists a neighborhood $\mathcal{U}$ of T in the C^1-topology such that for any $S \in \mathcal{U}$ there exists a homeomorphism $h : \Omega(S) \to \Omega(T)$ such that $T \circ h = h \circ S$ on $\Omega(S)$. For $r \geq 1$, we denote by $\mathcal{T}^r$ the totality of C^1-Ω-stable C^r-diffeomorphisms. From the definition, we see that the set $\mathcal{T}^r$ is open in $\mathit{Diff}^{\,r}(M)$. It is known that every element in $\mathcal{T}^1$ satisfies Axiom A (see [14] and [19]). Moreover, by Smale's spectral decomposition theorem (see Theorem 3.5 in [3]), for any Axiom A diffeomorphism T, its nonwandering set $\Omega(T)$ can be written as $\Lambda_1 \cup \cdots \cup \Lambda_k$, where each Λ_i is a basic set for T in our sense. Note that some authors use the term 'basic set' only for the basic set appearing in the spectral decomposition. Let T be an element in $\mathcal{T}^1$. Since T satisfies Axiom A, it has the spectral decomposition $\Omega(T) = \Lambda_1 \cup \cdots \cup \Lambda_k$. If $S \in \mathcal{T}^1$ is sufficiently close to T, then we have the following:

(1) By definition, there exists a homeomorphism $h : \Omega(S) \to \Omega(T)$ such that $T \circ h = h \circ S$ on $\Omega(S)$.

(2) By Lemma 2.4, we can consider the continuation $\Lambda_{i,S}$ of Λ_i for S and a homeomorphism $h_{i,S} : \Lambda_{i,S} \to \Lambda_i$ satisfying $T \circ h_{i,S} = h_{i,S} \circ S$ on $\Lambda_{i,S}$ for each $i \in \{1, \ldots, k\}$.

In particular, $h|_{\Lambda_{i,S}} = h_{i,S}$ for each $i \in \{1, \ldots, k\}$ and $\Omega(S) = \Lambda_{1,S} \cup \cdots \cup \Lambda_{k,S}$ is the spectral decomposition for S. Therefore we can apply Theorem 1.1 to the space $\mathcal{T}^1$.

Proof of Theorem 1.3. Consider the subset $\mathcal{T}_0^1$ of $\mathcal{T}^1$ consisting of all elements T such that there exists a basic set Λ satisfying $\lambda(T) = \lambda(T, \Lambda) >$

$\lambda(T, \Lambda')$ for the other basic sets Λ' in the spectral decomposition, *i.e.*,

$$\mathcal{T}_0^1 = \{T \in \mathcal{T}^1 \mid \mathcal{L}(T) = \mathcal{L}(T, \Lambda) \text{ holds for some basic set } \Lambda \text{ in the spectral decomposition}\}.$$

Obviously such a basic set Λ is determined uniquely by $T \in \mathcal{T}_0^1$. So we denote it by $\Lambda(T)$ in the sequel.

First we show that $\mathcal{T}_0^1$ is open and dense in $\mathcal{T}^1$. Let $T_n \in \mathcal{T}^1 \setminus \mathcal{T}_0^1$ be a sequence converging to $T \in \mathcal{T}^1$ in the C^1-topology. Since T is in $\mathcal{T}^1$, we may assume that any T_n is topologically conjugate to T on the respective nonwandering sets. Let $\Omega(T) = \Lambda_1 \cup \cdots \cup \Lambda_k$ be the spectral decomposition for T. For $n \in \mathbb{N}$, we may assume that T_n has the spectral decomposition $\Omega(T_n) = \Lambda_{n,1} \cup \cdots \cup \Lambda_{n,k}$ such that for each $i \in \{1, \dots, k\}$, $\Lambda_{n,i}$ is the continuation of Λ_i for T_n. Taking a subsequence and renumbering, we may assume further that $\lambda(T_n, \Lambda_{n,1}) = \lambda(T_n, \Lambda_{n,2}) = \lambda(T_n)$ for any $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let μ_n be a measure in $\mathcal{L}(T_n, \Lambda_{n,1})$. Then, by Lemma 2.6, we see that any weak-$*$ accumulation point of $\{\mu_n\}$ is in $\mathcal{L}(T, \Lambda_1)$. Therefore we see that $\lambda(T_n, \Lambda_{n,1})$ converges to $\lambda(T, \Lambda_1)$. Similarly $\lambda(T_n, \Lambda_{n,2})$ converges to $\lambda(T, \Lambda_2)$. Since $\lambda(T_n, \Lambda_{n,1}) = \lambda(T_n, \Lambda_{n,2}) = \lambda(T_n)$ for any $n \in \mathbb{N}$, we obtain that $\lambda(T, \Lambda_1) = \lambda(T, \Lambda_2) = \lambda(T)$. Thus we have that T is not in $\mathcal{T}_0^1$, consequently $\mathcal{T}_0^1$ is open in $\mathcal{T}^1$.

Next we take any $T \in \mathcal{T}^1$. Let $\Omega(T) = \Lambda_1 \cup \cdots \cup \Lambda_k$ be the spectral decomposition for T. By renumbering if necessary, we may assume that $\lambda(T) = \lambda(T, \Lambda_1)$. By Lemma 2.3, for any $\epsilon > 0$ we can find a periodic point x_0 of T in Λ_1 such that the T-invariant measure ν supported on its orbit satisfies $\lambda(T, \nu) > \lambda(T) - \epsilon/2$. Consider the positive number ϵ_0 found in Theorem 3.1 for T. Recall that we can apply Theorem 3.1 to T and x_0 if $\epsilon > 0$ is smaller than ϵ_0. Therefore given any neighborhood $\mathcal{U} \subset \mathcal{T}^1$ of T in the C^1-topology, we can choose an ϵ with $0 < \epsilon < \epsilon_0$, a periodic point x_0 of T and a C^1-diffeomorphism S such that they satisfy the following conditions.

(a) S coincides with T on the periodic orbit $O_T(x_0)$ and an open set containing $\Omega(T) \setminus \Lambda_1$.

(b) $S \in \mathcal{U}$.

(c) $\lambda(S, \Lambda_{1,S}) \geq \lambda(S, \nu) = \lambda(T, \nu) + \epsilon > \lambda(T) + \epsilon/2$, where $\Lambda_{1,S}$ is the continuation of Λ_1 for S.

Note that (a) follows from the assertions (1) and (3) in Theorem 3.1. (b) and (c) are consequences of the assertions (4) and (5), respectively. Obviously, the condition (a) implies that ν is an S-invariant measure and $\Omega(S) = \Lambda_{1,S} \cup \Lambda_2 \cup \cdots \cup \Lambda_k$ is the spectral decomposition for S. In particular, T and S

coincide on $\Lambda_2 \cup \cdots \cup \Lambda_k$. Thus we have

$$\begin{aligned}\lambda(S, \Lambda_{1,S}) &> \lambda(T) + \frac{\epsilon}{2} \\ &= \max_{1 \le i \le k} \lambda(T, \Lambda_i) + \frac{\epsilon}{2} \\ &\ge \max_{2 \le i \le k} \lambda(S, \Lambda_i) + \frac{\epsilon}{2}.\end{aligned}$$

This yields that $\lambda(S) = \lambda(S, \Lambda_{1,S}) > \max_{2 \le i \le k} \lambda(S, \Lambda_i) + \epsilon/2$ and $S \in \mathcal{T}_0^1$. Hence $\mathcal{T}_0^1$ is dense in $\mathcal{T}^1$.

Now we show that the properties (1), (2) and (3) of Theorem 1.3 are generic in $\mathcal{T}^1$. To this end it suffices to show that the properties (1), (2) and (3) are generic in $\mathcal{T}_0^1$ since $\mathcal{T}_0^1$ is open and dense in $\mathcal{T}^1$. For each $T \in \mathcal{T}_0^1$, we can apply Theorem 1.1 with $\Lambda = \Lambda(T)$. Therefore we can find an open neighborhood $\mathcal{U}_T \subset \mathcal{T}_0^1$ of T such that the properties (1), (2) and (3) of Theorem 1.1 are generic in $\mathcal{U}_T$ and $\Lambda(S)$ is the continuation $\Lambda(T)_S$ of $\Lambda(T)$ for each $S \in \mathcal{U}_T$. Combining this with the fact that $\mathcal{L}(T) = \mathcal{L}(T, \Lambda(T))$ holds for any $T \in \mathcal{T}_0^1$, we see that the properties (1), (2) and (3) of Theorem 1.3 are also generic in $\mathcal{U}_T$. Since $\mathit{Diff}^1(M)$ is second countable, we can take a countable open base $\mathfrak{U}$. Consider its subfamily $\mathfrak{V} = \{\mathcal{U} \in \mathfrak{U} \mid \overline{\mathcal{U}} \subset \mathcal{U}_T \text{ for some } T \in \mathcal{T}_0^1\}$, where $\overline{\mathcal{U}}$ denotes the closure of $\mathcal{U}$ in $\mathit{Diff}^1(M)$. Obviously $\mathfrak{V}$ is a countable family of open sets. Since $\mathit{Diff}^1(M)$ is metrizable and $\bigcup_{T \in \mathcal{T}_0^1} \mathcal{U}_T = \mathcal{T}_0^1$, we easily see that $\bigcup_{\mathcal{V}} \overline{\mathcal{V}} = \mathcal{T}_0^1$, where the union is taken over $\mathcal{V} \in \mathfrak{V}$. Now we put

$$\mathcal{T}' = \{T \in \mathcal{T}_0^1 \mid T \text{ satisfies the properties (1), (2) and (3) of Theorem 1.3}\}.$$

Then for any $T \in \mathcal{T}_0^1$, we obtain that $\mathcal{T}' \cap \mathcal{U}_T$ is a residual subset of $\mathcal{U}_T$. Since for any $\mathcal{V} \in \mathfrak{V}$ we have that $\overline{\mathcal{V}} \subset \mathcal{U}_T$ for some $T \in \mathcal{T}_0^1$, we see that $\left(\mathcal{T}' \cap \overline{\mathcal{V}}\right) \cup \left(\mathcal{T}_0^1 \setminus \overline{\mathcal{V}}\right)$ is a residual subset of $\mathcal{T}_0^1$. Since $\bigcup_{\mathcal{V}} \overline{\mathcal{V}} = \mathcal{T}_0^1$, we obtain that

$$\mathcal{T}' = \bigcap_{\mathcal{V}} \left(\left(\mathcal{T}' \cap \overline{\mathcal{V}}\right) \cup \left(\mathcal{T}_0^1 \setminus \overline{\mathcal{V}}\right)\right),$$

where the intersection is taken over $\mathcal{V} \in \mathfrak{V}$. Thus $\mathcal{T}'$ is a residual subset of $\mathcal{T}_0^1$. Hence we conclude that the properties (1), (2) and (3) of Theorem 1.3 are generic in $\mathcal{T}_0^1$.

In order to verify the second assertion in Theorem 1.3, it suffices to show that the unique measure with maximum total exponent on M for $T \in \mathcal{T}_0^1$ satisfying (1) of Theorem 1.3 is ergodic with respect to T. But this is done in the same way as in the proof of the second assertion in Theorem 1.1 by using the fact that $\mathcal{L}(T) = \mathcal{L}(T, \Lambda(T))$ holds for any $T \in \mathcal{T}_0^1$. $\square$

As in the case of Theorem 1.1, we can apply Theorem 1.2 to the space $\mathcal{T}^r$ for $r \geq 2$.

Proof of Theorem 1.4. We proved that the set of all C^r-diffeomorphisms satisfying the property stated in Theorem 1.2 is not only residual but also open and dense in some open neighborhood of T in the C^r-topology. Therefore by applying Theorem 1.2 to each basic set of T in $\mathcal{T}^r$, we can show that the set of all C^r-diffeomorphisms satisfying the property of Theorem 1.4 is dense in $\mathcal{T}^r$. Moreover, we can show that the set is open in $\mathcal{T}^r$ in the same way. □

We close this paper with the following remark.

Remark. A C^1-diffeomorphism $T : M \to M$ is structurally stable if there exists a neighborhood $\mathcal{U}$ of T in the C^1-topology such that for any $S \in \mathcal{U}$ there exists a homeomorphism $h : M \to M$ such that $T \circ h = h \circ S$, *i.e.*, any $S \in \mathcal{U}$ is topologically conjugate to T. From the definition, we see that the totality of structurally stable C^r-diffeomorphisms is open in $\mathit{Diff}^r(M)$. Since a conjugate homeomorphism maps the nonwandering set to the other one, we have that every structurally stable diffeomorphism is C^1-Ω-stable. Therefore it is obvious that Theorems 1.3 and 1.4 hold even if we replace C^1-Ω-stability with structural stability.

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