Superconformal Quantum Mechanics from M2-branes

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Abstract

In this thesis we discuss the superconformal quantum mechanics arising from the M2-branes. We begin with a comprehensive review of the superconformal quantum mechanics and emphasize that conformal symmetry and supersymmetry in quantum mechanics contain a number of exotic and enlightening properties which do not occur in higher dimensional field theories. We see that superfield and superspace formalism is available for $\mathcal{N} \leq 8$ superconformal mechanical models. We then discuss the M2-branes with a focus on the world-volume descriptions of the multiple M2-branes which are superconformal three-dimensional Chern-Simons matter theories. Finally we argue that the two topics are connected in M-theoretical construction by considering the multiple M2-branes wrapped around a compact Riemann surface and study the emerging IR quantum mechanics. We establish that the resulting quantum mechanics realizes a set of novel $\mathcal{N} \geq 8$ superconformal quantum mechanical models which have not been reached so far. Also we discuss possible applications of the superconformal quantum mechanics to mathematical physics.

Contents

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Chapter 1

Introduction and Overview

1.1 Historical background

The pursue of Theory of Everything that describes our nature and achieves unification of all fundamental interactions, i.e. electroweak, strong and gravitational interactions has been a mission agitating theoretical physicists. Historically much of significant developments in theoretical physics were achieved by overcoming the inconsistency between existing concepts. The special relativity was established from the crisis between classical mechanics and electrodynamics, the general relativity was proposed by reconciling the special relativity and Newtonian gravity and quantum field theory was acquired by combining quantum mechanics and the special relativity. However, we are now confronting another contradiction between quantum field theory and the general relativity. Although a quantum field theory as the standard model successfully describes and predicts almost all phenomena controlled by electroweak and strong forces, the general relativity describing the gravity that is the remaining fundamental interaction seriously disagrees with the quantum field theory. This indicates that quantum field theory cannot lead to the correct quantization of gravity. Therefore it is expected that the standard model is regarded as the low-energy effective theory of a more fundamental theory.

String theory has been proposed as the promising candidate for such a fundamental theory since it can naturally describe all fundamental interactions. In string theory all particles are recognized as various vibrational modes of only two different types of fundamental strings; the open strings which have two endpoints and the the closed strings which have no endpoint. One of the most beautiful structures in string theory is that Yang-Mills gauge theories which govern the electroweak and strong interactions as the standard model arise from the vibrations of the open strings while the general relativity that describes the gravitational interaction appears from the vibrations of the closed string. Among the massless states of the open string there are spin-one particles which behave as gauge bosons while among the massless states of the closed string there is a spin-two particle which can be identified with graviton [1, 2].

However, the bosonic string theory is not realistic since the vibrations of the bosonic strings yield only bosonic particles. The lack of fermionic particles can be resolved by introducing supersymmetry in string theory, i.e. the superstring theory. The spectrums of superstrings contain both bosonic and fermionic particles. Hence string theory supports the existence of the supersymmetry.

One of the most fascinating predictions in superstring theory is the existence of the extra dimension in space-time. It turns out that the unitarity and the Lorentz invariance of space-time in which the superstrings live are guaranteed only for ten-dimensional space-time. In other words, flat space superstrings can only exist in ten dimensions. In order to reconcile the difference between the ten-dimensional space-time in string theory and the four-dimensional space-time in our instinctive knowing physics, the notion of compactification has been proposed. The idea is that since the six extra dimensional compact spatial directions are much smaller than the four-dimensional space-time, the original $(1+9)$-dimensional space-time effectively looks like $(1+3)$-dimensional space-time. For the six-dimensional spaces Calabi-Yau manifolds are known to possess phenomenologically promising properties.

Ten-dimensional superstring theory is not a single theory but rather a set of possible five theories; (i) type IIA, (ii) type IIB, (iii) type I (iv) $SO(32)$ heterotic (v) $E_8 \times E_8$ heterotic. When the both left-moving and right-moving modes are taken as superstrings, there are two possibilities; opposite handedness or the same handedness. The former theory is called type IIA while the latter is called type IIB. Type I superstring theory is obtained by the orientifold projection that mods out left-right symmetry of type IIB superstring theory. When the left-moving mode is chosen as the bosonic string and the right-moving mode is taken as the superstring, consistency allows only two different theories; $SO(32)$ heterotic and $E_8 \times E_8$ heterotic superstring theories. The type II superstring theories have $d = 10$, $\mathcal{N} = 2$ supersymmetry and the type I and heterotic superstring theories possess $d = 10$, $\mathcal{N} = 1$ supersymmetry [1].

It has been argued that the five superstring theories are connected with each other. T-duality relates a pair of the two type II superstring theories and also a pair

[1]For $d = 10$, $\mathcal{N} = 1$ supersymmetry a consistent local gauge symmetry group is characterized by the Lie algebras $\mathfrak{so}(32)$ and $E_8 \times E_8$.

of the two heterotic superstring theories. S-duality relates the type I superstring theory to the $SO(32)$ heterotic superstring theory and the type IIB superstring theory to itself. T-dualities and S-dualities generates a discrete non-abelian group, the so-called U-duality group [3, 4]. Remarkably the U-duality groups are recognized as discretization of global symmetry groups of supergravity. In fact it is known that type IIA and IIB superstring theories are the ultraviolet (UV) completions of $d = 10$ type IIA and IIB supergravities [2] whereas type I and heterotic superstring theories are the UV completions of $d = 10$ type I supergravities.

From the supergravity point of view, it is interesting to note that $d = 10$ type IIA supergravity arises by dimensional reduction of $d = 11$ supergravity [8, 9, 10]. $d = 11$ supergravity [11] is furnished with a particular interest since $d = 11$ is the highest space-time dimension which realizes a consistent supersymmetric theory containing particles with spins ≥ 2 [12]. $d = 11$ supergravity possesses a single 32-component spinor supercharge and its Lagrangian is unique if we require that the theory contains at most two-derivative interactions. Therefore the relation between superstring theory and $d = 10$ supergravity indicates the existence of the UV completion of $d = 11$ supergravity. It has been argued that in the strong string coupling limit an eleventh direction arises in type IIA superstring theory and the resulting eleven-dimensional theory is referred to as M-theory [13, 4, 14] [3]. Conversely M-theory reduces to type IIA superstring theory upon the compactification on a spatial circle. Up until now M-theory is the most prospective candidate for the fundamental theory in that it may explain the origin of strings and unify the five superstring theory. However, a familiar perturbative method in string theory is not applicable because M-theory describes the strong coupling region of string theory.

As the fundamental string is a fundamental object in ten-dimensional superstring theory, the membrane appears to play a fundamental role in M-theory. This membrane is called M2-brane. Indeed $d = 11$ supergravity contains a three-form gauge field, which leads to two stable extended objects as solitonic solutions; electric membrane and magnetic five-brane. Moreover it has been pointed out [15] that the M2-brane is identified with the fundamental string when M-theory is compactified on a circle and reduces to type IIA superstring theory. In spite of the prospective importance for the membranes in M-theory a number of attempts for the quantization of the membranes does not work well hitherto.

[2] Originally type IIB supergravity was discovered [5] and constructed [6, 7] as the low-energy limit of type IIB superstring theory.

[3] The letter "M" proposed by E. Witten embodies several possible meanings; membrane, matrix, magic and mother.

Although M-theory is much less understood than string theory due to the difficulty of the quantization of the membranes, we can still obtain several insights and clues from string theory and supergravity. In addition to the fundamental string, string theory also contains extended objects, the so-called D-branes on which open strings can end [16]. In fact ten-dimensional supergravities possess the solutions describing the geometries around such branes. There is a remarkable conjecture, the so-called AdS/CFT correspondence [17, 18, 19] which states that there is the correspondence between string/M-theory on certain supergravity geometries with anti-de-Sitter (AdS) factors and certain conformally invariant quantum field theories. The most basic evidence for the AdS/CFT correspondence is the equivalence between type IIB superstring theory on the $\mathrm{AdS}_5 \times S^5$ supergravity geometry constructed as a set of N coincident D3-branes and the $d = 4$, $\mathcal{N} = 4$ superconformal $U(N)$ Yang-Mills gauge theory. Namely the low-energy dynamics for the world-volume of the planar N D3-branes in flat space-time can be effectively described by $(1+3)$-dimensional maximally supersymmetric $U(N)$ Yang-Mills gauge theory [20].

Similarly, in the near-horizon limit $d = 11$ supergravity solutions describing planar M2-branes in flat space-time contain the AdS_4 factors and the low-energy dynamics of the M2-branes are expected to be described by the $(1+2)$-dimensional conformal field theories. As the eleven-dimensional flat background geometry can possess 32 space-time supercharges, the world-volume effective field theory of planar M2-branes should preserve half of the supersymmetry. Also the gauge degrees of freedom are needed to describe the internal degrees of freedom of multiple M2-branes. Thus the low-energy effective field theories of planar M2-branes are $d = 3$, $\mathcal{N} = 8$ superconformal gauge theories. The candidates for such effective field theories of world-volume dynamics of multiple M2-branes have been proposed as three-dimensional superconformal Chern-Simons matter theories, the so-called BLG-model [21, 22, 23, 24, 25] and the ABJM model [26].

In order to obtain new AdS/CFT examples it is desirable to consider more general supergravity solutions describing the wrapped branes around certain cycles which may be curved. However, in the generic setup where the branes are wrapping an arbitrary manifold, all of the supersymmetries are destined to break down. In other words, specific background geometries of branes and specific cycles wrapped by branes must be chosen to preserve supersymmetry. Mathematically the supersymmetric cycles are characterized by the calibration [27]. There is a remarkable observation [28] that topologically twisted field theories may give rise to the world-volume theories of wrapped branes. For the Euclidean D3-branes wrapping four-manifold there are three calibrated cycles embedded in special holonomy manifolds; (i) special Lagrangian submanifold in Calabi-Yau four-fold, (ii) coassociative submanifold in G_2 manifold and (iii) Cayley submanifold in $Spin(7)$ manifold. Each of them corresponds to three distinct topological twisting procedures [4]; (i) geometric Langlands (GL) twist [33, 34, 35, 36], (ii) Vafa-Witten twist [37] and (iii) Donaldson-Witten twist [38]. The world-volume of D3-branes can be put on the product of two Riemann surfaces $\mathcal{C} \times \Sigma_g$. For the compact Riemann surface Σ_g of genus $g > 1$ the field theories on the D3-branes are partially twisted on the curved Riemann surface to preserve supersymmetry. Since the compact manifold wrapped by branes introduces into the theory the typical energy scale as its volume, one can consider an additional limit where the energy is much smaller than the inverse size of the cycles. The resulting effective field theories then reduce to the two-dimensional topological sigma-models whose target space is specified by the BPS equations [28, 39, 40, 36, 41, 42, 43, 44].

When the Euclidean M2-branes wrap a compact curved three-manifold, the three-dimensional effective theories on the branes are fully twisted [45]. The $SO(3)$ Euclidean symmetry on the world-volume is topologically twisted in terms of the $SO(3)$ subgroup of the R-symmetry. For the M2-branes wrapping compact Riemann surface Σ_g of genus g supersymmetry is unbroken if the Riemann surface is chosen as holomorphic curve in Calabi-Yau manifold, which are the only known supersymmetric two-cycles. From the supergravity point of view, the solutions which describe the M2-branes wrapping compact Riemann surfaces have been studied [46, 47, 48] by using the gauged supergravity method [40]. The basic observation [49, 50] is that the dimensional reduction of $d = 11$ supergravity on a seven-sphere can be truncated to give rise to the four-dimensional $SO(8)$ gauged supergravity where $SO(8)$ gauge symmetry corresponds to the isometry of the seven-sphere. Since the planar M2-branes take the form of $\mathrm{AdS}_4 \times S^7$, the non-trivial coupling of the external $SO(8)$ gauge field which is nothing but an R-symmetry of the world-volume theory of the planar M2-branes may realize the curved geometries of the form $\mathrm{AdS}_2 \times \Sigma_g$ instead of AdS_4. Thus the uplift of the four-dimensional $SO(8)$ supergravity solutions can be used to construct the $d = 11$ supergravity solutions describing the M2-branes wrapped on holomorphic curves. Correspondingly the three-dimensional effective superconformal field theories are partially twisted for $g \neq 0$ [51]. The $SO(2)$ Euclidean symmetry on the curved Riemann surface is topologically twisted in terms of the $SO(2)$ subgroup of the

[4] Also see [29, 30, 31, 32] for interesting applications of the topological twisted $\mathcal{N} = 4$ super Yang-Mills theories.

R-symmetry. Furthermore it is shown [51] that in the limit where the size of the Riemann surface goes to zero, superconformal quantum mechanical models arise as the low-energy effective theories. This thesis explores the new connection between the M2-branes and the superconformal quantum mechanics.

We should note that conformal symmetry and supersymmetry in one-dimensional field theory, i.e. quantum mechanics, contain a bunch of intriguing properties which do not appear in higher dimensional field theories, as we will discuss in this thesis.

Although supersymmetric quantum mechanics was originally studied as the simple testing model for non-perturbative breaking of supersymmetry [52, 53], supersymmetric quantum mechanics is much more interesting itself. Supersymmetry is closely related to the translational symmetry as the square of the supercharges generates the momentum. However, in one dimension there are no spatial directions and the translational symmety generator is just the Hamiltonian, which reflects the reduced Poincaré symmetry in one-dimension. The reduced Poincaré symmetry looses the constraints for supersymmetry in one dimension and leads to richer structures than higher dimensional field theories. Indeed there exist supersymmetric quantum mechanical models which cannot reached via naive dimensional reductions from higher dimensional field theories. In parallel with that, there may be a large number of supermultiplets in one-dimension (see Table 3.4) and there is no relationship between the physical bosonic degrees of freedom and fermionic degrees of freedom in supersymmetric quantum mechanics. These properties are special in one dimension.

Conformal symmetry in one-dimension also exhibits unique features. The reduced Poincaré symmetry identifies the generator of a translation with the Hamiltonian H and does not allow for the generator of a rotational symmetry. Therefore the one-dimensional conformal symmetry is generated by the Hamiltonian H, the dilatation generator D and the conformal boost generator K, all of which together form the $\mathfrak{sl}(2,\mathbb{R})$ algebra. Therefore the one-dimensional conformal group is $SL(2,\mathbb{R}) \cong SO(1,2)$. The first detailed analysis of conformal mechanics appeared in [54]. The conformal mechanical models are typically characterized by the inverse-square potential [5]. Inverse-square potential in quantum mechanics is a jewellery box in theoretical physics and mathematics containing black hole physics [65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77], $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence [78, 79, 80, 81, 82, 83, 84, 85, 86, 87], QCD [88, 89], quantum Hall effect [90, 91],

[5] The treatment of the inverse-square potential in quantum mechanics was discussed in [55, 56, 57, 58, 59, 60, 61, 62, 63, 64].

Tomonaga-Luttinger liquid [92], string theory [93, 94, 95, 96, 97, 98], spin chains [99, 100, 101, 102, 103, 104, 105, 106], Efimov effect [107, 108, 109, 110], mesoscopic physics [111, 112], quantum dot [113], quantum chaos [114], fractional exclusion statistics [115, 116], random matrix model [117, 118, 119, 120], Seiberg-Witten theory [121, 122], Jack polynomial [123, 124, 125, 126, 127, 128, 129], and relevant algebraic and integrable structures [130, 131, 132, 133]. One of the well-known such quantum mechanical models is the Calogero model [134, 135] which is the multi-particle system with the pairwise inverse-square interaction [6]. It was firstly proven in [148, 149] that the Calogero model has the $SL(2,\mathbb{R})$ conformal symmetry. The Calogero model and its generalizations can be viewed as a system of free indistinguishable particles [145]. The indistinguishableness implies that the permutation group acting on the configuration of the particles is treated as a discrete gauge symmetry in the system and therefore the Calogero model and its generalized models can be obtained from gauged matrix models [150, 151]. This observation is used to find new conformal mechanical systems by starting gauged matrix models or gauged quantum mechanical models and reducing the systems via Hamiltonian reduction [152, 137].

Since the appearance of the seminal works [153, 154] on superconformal quantum mechanics (SCQM), there has been a great deal of efforts to construct superconformal quantum mechanics. The superconformal quantum mechanical models are characterized by the superconformal group, i.e. the Lie supergroup which contains one-dimensional conformal group $SL(2,\mathbb{R})$ and R-symmetry group as factored bosonic subgroups. One of the most powerful methods to build up superconformal mechanical systems is the superspace and superfield formalism. In fact for $\mathcal{N} \leq 8$ supersymmetric cases it does work and several superconformal quantum mechanical systems are constructed. For $\mathcal{N} = 1$ supersymmetric case, the superconformal group is $OSp(1|2)$ and there is no non-trivial one particle superconformal quantum mechanics. For $\mathcal{N} = 2$ supersymmetric case, the superconformal group is $OSp(2|2) \cong SU(1,1|1)$ and the simplest one particle model is the pioneering work of [153, 154]. For $\mathcal{N} = 4$ supersymmetric case, the generic superconformal group is $D(2,1;\alpha)$ which is a one-parameter family of supergroup. The superspace and superfield formalism keeping track of the exceptional supergroup can be derived by the non-linearlizations technique [155, 156, 157] and several models have been constructed. In the case of $\mathcal{N} = 8$ there exist four different superconformal groups; $SU(1,1|4)$, $OSp(8|2)$, $OSp(4^*|4)$ and $F(4)$. Such several

[6] See [136, 137, 138] for the enlightening reviews on (super)conformal mechanics and also see [139, 140, 141, 142, 143, 144, 145, 146, 147] for excellent reviews on the Calogero model.

choices of superconformal group cannot occur in higher dimensional field theories and thus present a various families of $\mathcal{N} = 8$ superconformal quantum mechanics. However, for $\mathcal{N} > 8$ the superspace and superfield formalism is not unrealistic and unsuccessful. One of the signals for such difficulty is that the number of bosonic and fermionic component fields in the supermultiplets typically becomes greater than the number of supersymmetry when $\mathcal{N}$ is larger than eight (see Table 3.4). In spite of the depression of the superspace and superfield formalism, several $\mathcal{N} > 8$ superconformal quantum mechanical models have been constructed via reduction of the three-dimensional quiver type superconformal Chern-Simons matter theories [51]. As mentioned before, these superconformal quantum mechanical models may capture the low-energy dynamics of the multiple M2-braens wrapped on a compact Riemann surface. We will spell out the details of these superconformal quantum mechanical models in this thesis.

1.2 What I did

The organization of this thesis consists of three parts. In part I and II we will review two main subjects; the superconformal quantum mechanics and the M2-branes. The original part of the author's work based on [51] is part III, in which we will discuss the new connection between the two subjects, that is the superconformal quantum mechanics emerging from M2-branes.

Part I contains two chapters; chapter 2 and 3. In chapter 2 we will discuss various aspects of conformal quantum mechanics. We will start with section 2.1 by studying the DFF-model [54] and its $SL(2,\mathbb{R})$ conformal symmetry and then in section 2.2 we will explore the quantum properties of the system. In section 2.5 we will see that the conformal mechanical models can be derived from the gauged mechanical system via Hamiltonian reduction or Routh reduction. In section 2.6 we will review the observation [65] that in the near horizon of the extreme Reissner-Nordström black hole the motion of the charged particle can be described by the conformal mechanics (2.6.10). In section 2.7 we will present the non-linear realization technique which is useful to construct (super)conformal quantum mechanical models and then review the statement in [74] that DFF-model (2.1.2) is equivalent to the black hole conformal mechanics (2.6.10) in [65]. We will extend the analysis to the multi-particle models, i.e. the sigma-models in section 2.8. We will review the discussion in [69] that the target space of the conformal sigma-model possesses a homothety vector field whose associated one-form is closed. We will argue that the gauging procedure for the multi-particle model, the matrix model yield the Calogero model in section 2.9. In chapter 3 we will turn to the discussion on the superconformal quantum mechanics. We will recall the Lie superalgebra and Lie supergroup and then discuss the one-dimensional superconformal group (see in Table 3.2) in section 3.1. In section 3.2 we will explain the exisotic structures of supersymmetry in one-dimension, which allows us to construct various supermultiplets.

Part II, which is comprised of two chapters; chapter4 and 5, is devoted to the low-energy effective field theories of the M2-branes. We will review the BLG-model in chapter 4 and the ABJM-model in chapter 5. We will present our notations and conventions and also the several conjectural statements for the BLG-model and the ABJM-model in these chapters.

Part III is the most important part of this thesis. It is based on the author's work of [51], in which we will engage in the superconformal quantum mechanical models arising from the M2-branes. We consider the multiple membranes on a compact Riemann surface and study the IR quantum mechanics by taking the limit where the energy scale is much lower than the inverse size of the Riemann surface. In chapter 6 we will demonstrate that the resulting quantum mechanics from the BLG-model compactified on a torus is the $\mathcal{N} = 16$ superconformal gauged quantum mechanics. Furthermore we will find the $OSp(16|2)$ superconformal quantum mechanics from the reduced system. Similarly in chapter 7 we will investigate the IR quantum mechanics from the ABJM-model compactified on a torus, which turns out to be the $\mathcal{N} = 12$ superconformal gauged quantum mechanics. By the Hamiltonian reduction, or the Routh reduction we will also find the $SU(1,1|6)$ superconformal quantum mechanics from the gauged quantum mechanics. In chapter 8 we will present various examples of the topological twisting, which is the important concept to describe curved branes in string theory and M-theory. In chapter 9 we will survey the M2-branes wrapped on a curved Riemann surface which is taken as a holomorphic curve in a Calabi-Yau manifold to preserve supersymmetry. We will present a prescription of the topological twisting for the case where the Calabi-Yau space is constructed as the direct sum of the line bundles over the Riemann surface. In chapter 10 we will complete the analysis of the M2-branes wrapped around the holomorphic Riemann surface in a K3 surface. We will find the $\mathcal{N} = 8$ superconformal gauged quantum mechanics which may describe the motion of the two M2-branes wrapping holomorphic curve in a K3 surface. Finally in chapter 11 we will present conclusion and discuss the future directions.

Part I

Superconformal Mechanics

Chapter 2

Conformal Mechanics

In this chapter we will review the conformal quantum mechanics. The simplest model is the so-called DFF model [54]. In section 2.1, 2.2, 2.3 and 2.4 we will learn from the DFF model several remarkable features of the conformal symmetry in one-dimension, which cannot occur in higher dimensional field theories. Then in section 2.5 we will argue the alternative formulation of the conformal mechanical models as the gauged mechanical models. As an interesting application of the conformal quantum mechanics we will discuss the relationship between the conformal mechanics and black hole in section 2.6 and introduce the non-linear realization method to construct (super)conformal quantum mechanics in section 2.7. Finally we will extend the analysis to the multi-particle conformal mechanical models in section 2.8 and 2.9.

2.1 $SL(2,\mathbb{R})$ conformal symmetry

In d-dimensions a scale invariant Lagrangian for a scalar field ϕ has the form

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \gamma\phi^{\frac{2d}{d-2}} \tag{2.1.1}$$

where γ is a dimensionless coupling constant. In one-dimensional case we get the Lagrangian

$$L = \frac{1}{2}\left(\dot{x}^2 - \frac{\gamma}{x^2}\right). \tag{2.1.2}$$

This simple quantum mechanical model is the so-called DFF-model [54]. To keep particles from falling into the origin, the coupling constant γ should be positive classically. As we will see in the following discussion, quantum mechanically the uncertain principle gives rise to the minimum value $\gamma = -\frac{1}{4}$, however, the normalizability of the wavefunction of the ground state requires that γ is positive. So we will denote $\gamma = g^2$ for convenience. The Lagrangian (2.1.2) leads to the equation of motion

$$\ddot{x} = \frac{g^2}{x^3}. \tag{2.1.3}$$

The action

$$S = \int L \, dt = \frac{1}{2} \int dt \left(\dot{x}^2 - \frac{g^2}{x^2} \right) \tag{2.1.4}$$

is invariant under the transformations

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \tag{2.1.5}$$

$$x'(t') = \frac{1}{\gamma t + \delta} x(t) \tag{2.1.6}$$

where the real numbers α, β, γ and δ form a real 2×2 matrix with determinant one

$$A = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}, \qquad \det A = 1. \tag{2.1.7}$$

1. **translation**

   The subgroup of the matrix (2.1.7)

$$\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \tag{2.1.8}$$

   with $\alpha = 1$, $\beta = a$, $\gamma = 0$, $\delta = 1$ yields

$$\begin{aligned} t' &= t + a, \\ x'(t') &= x(t). \end{aligned} \tag{2.1.9}$$

   This corresponds to the translation.

2. **dilatation**

   The subgroup of the matrix (2.1.7)

$$\begin{pmatrix} e^{\frac{b}{2}} & 0 \\ 0 & e^{-\frac{b}{2}} \end{pmatrix} \tag{2.1.10}$$

   with $\alpha = e^{\frac{b}{2}}$, $\beta = 0$, $\gamma = 0$, $\delta = e^{-\frac{b}{2}}$ generates the transformations

$$\begin{aligned} t' &= e^{b} t, \\ x'(t') &= e^{\frac{b}{2}} x(t). \end{aligned} \tag{2.1.11}$$

   This is the dilatation.

3. **conformal boost**

   The subgroup of the matrix (2.1.7)

$$\begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix} \tag{2.1.12}$$

   with $\alpha = 1$, $\beta = 0$, $\gamma = -c$, $\delta = 1$ corresponds to the transformations

$$\begin{aligned} t' &= \frac{t}{-ct+1}, \\ x'(t') &= \frac{x(t)}{-ct+1}. \end{aligned} \tag{2.1.13}$$

   This is the conformal boost transformation.

From a set of finite transformations (2.1.9), (2.1.11) and (2.1.13) we see that the action (2.1.4) is invariant under the infinitesimal one-dimensional conformal transformations

$$\delta t = f(t) = a + bt + ct^2, \tag{2.1.14}$$

$$\delta x = \frac{1}{2}\dot{f}x = \frac{1}{2}(b+2ct)x. \tag{2.1.15}$$

The passive transformations (2.1.14) and (2.1.15) lead to the active transformations

$$\delta t = 0, \tag{2.1.16}$$

$$\delta x = \frac{1}{2}\dot{f}x - f\dot{x}. \tag{2.1.17}$$

Noting that a, b and c are the infinitesimal parameters of the translation, the dilatation and the conformal boost, we can compute the Noether charges, i.e. the Hamiltonian H, the dilatation operator D and the conformal boost operator K

$$H = \frac{p^2}{2} + \frac{g^2}{2x^2}, \tag{2.1.18}$$

$$D = tH - \frac{1}{4}\left(xp + px\right), \tag{2.1.19}$$

$$K = t^2 H - \frac{1}{2}t(xp+px) + \frac{1}{2}x^2 \tag{2.1.20}$$

where $p = \dot{x}$ is the canonical momentum. The operators D and K are the constants of motion in the sense that

$$\frac{\partial D}{\partial t} + [H, D] = 0, \qquad \frac{\partial K}{\partial t} + [H, K] = 0. \tag{2.1.21}$$

One can carry out the canonical quantization by establishing the equal time commutation relation

$$[x, p] = i. \tag{2.1.22}$$

Using the commutation relation (2.1.22), we can show that

$$[H, D] = iH, \tag{2.1.23}$$

$$[K, D] = -iK, \tag{2.1.24}$$

$$[H, K] = 2iD \tag{2.1.25}$$

and

$$i[H, x(t)] = \dot{x}(t), \tag{2.1.26}$$

$$i[D, x(t)] = t\dot{x}(t) - \frac{1}{2}x(t), \tag{2.1.27}$$

$$i[K, x(t)] = t^2\dot{x}(t) - tx(t). \tag{2.1.28}$$

If we express the time independent part of D and K as

$$D_0 := -\frac{1}{4}\left(xp + px\right), \tag{2.1.29}$$

$$K_0 := \frac{1}{2}x^2, \tag{2.1.30}$$

then the equations (2.1.26), (2.1.27) and (2.1.28) are rewritten as

$$i[H, x(t)] = \dot{x}(t), \tag{2.1.31}$$

$$i[D_0, x(t)] = -\frac{1}{2}x(t), \tag{2.1.32}$$

$$i[K_0, x(t)] = 0. \tag{2.1.33}$$

These equations are regarded as the Heisenberg equations. The equation (2.1.31) is familiar for general quantum mechanical systems and yields the variation of the operator with respect to time while the equation (2.1.32) gives rise to the scale dimension of the operator.

Note that the explicit time dependence of D and K can be absorbed into the similarity transformations

$$D = e^{itH}D_0e^{-itH}, \qquad K = e^{itH}K_0e^{-itH} \tag{2.1.34}$$

So we will use the time independent parts as the explicit expressions for D and K and drop off the subscripts.

Defining

$$T_0 = \frac{1}{2}\left(\frac{K}{a} + aH\right), \tag{2.1.35}$$

$$T_1 = D, \tag{2.1.36}$$

$$T_2 = \frac{1}{2}\left(\frac{K}{a} - aH\right) \tag{2.1.37}$$

where a is a constant with dimension of length, we find from (2.1.23)-(2.1.25) the explicit representation of the $\mathfrak{so}(1,2)$ algebra

$$[T_i, T_j] = i\epsilon_{ijk}T^k \tag{2.1.38}$$

where ϵ_{ijk} is a three-index anti-symmetric tensor with $\epsilon_{012} = 1$ and $g_{ij} = \mathrm{diag}(1,-1,-1)$. If we introduce

$$L_0 = \frac{1}{2}\left(\frac{K}{a} + aH\right) = T_0, \tag{2.1.39}$$

$$L_\pm = \frac{1}{2}\left(\frac{K}{a} - aH \pm 2iD\right) = T_2 \pm iT_1, \tag{2.1.40}$$

then we get the explicit representation of the $\mathfrak{sl}(2,\mathbb{R})$ algebra in the Virasoro form

$$[L_n, L_m] = (m-n)L_{m+n} \tag{2.1.41}$$

with $m, n = 0, \pm$. Note that

$$H = \frac{1}{a}\left[L_0 - \frac{1}{2}(L_+ + L_-)\right], \tag{2.1.42}$$

$$D = \frac{1}{2i}(L_+ - L_-), \tag{2.1.43}$$

$$K = a\left[L_0 + \frac{1}{2}(L_+ + L_-)\right]. \tag{2.1.44}$$

Recall that in the representation theory the Casimir invariants play an important role since their eigenvalues may characterize the representations. The one-dimensional conformal group $SL(2,\mathbb{R})$ is of rank one and therefore possesses one independent second-order Casimir invariant. The second-order Casimir operator $\mathcal{C}_2$ of the $\mathfrak{sl}(2,\mathbb{R})$ algebra is given by

$$\begin{aligned}\mathcal{C}_2 &= T_0^2 - T_1^2 - T_2^2 \\ &= L_0(L_0 - 1) - L_+L_- \\ &= \frac{1}{2}(HK + KH) - D^2 \\ &= \frac{g^2}{4} - \frac{3}{16}.\end{aligned} \tag{2.1.45}$$

2.2 Spectrum

It is known that the quantum formalism based on the Hamiltonian H is awkward to describe the conformal quantum mechanics. The spectrum of H is continuous due to the existence of D [1]. This is because if $|E\rangle$ is an eigenstate of energy E, then $e^{i\alpha D}|E\rangle$ is that of energy $e^{2\alpha}E$ with α being an arbitrary real parameter. Thus the spectrum contains all $E > 0$ eigenvalues of H.

The corresponding wave functions are given by

$$\psi_E(x) = C\sqrt{x}J_{\sqrt{g^2+\frac{1}{4}}}\left(\sqrt{2E}x\right) \tag{2.2.1}$$

where C is a normalization factor and J_α is the Bessel function of the first kind. For each of the eigenstates with the eigenvalues $E > 0$ there exists a normalizable plane wave.

On the other hand, the wavefunction of the zero energy state is given by [2]

$$\psi_0(x) = Cx^{-\frac{1}{2}+\frac{\sqrt{1+4g^2}}{2}} \tag{2.2.2}$$

where C is a constant value. To make matters worse, this eigenfunction is not even plane wave normalizable and this makes it difficult for us to regard the state with $E = 0$ as the ground state.

However. it is important to note that [54] any combination

$$G = uH + vD + wK \tag{2.2.3}$$

of the three conformal generators is a constant of motion in the sense that

$$\frac{\partial G}{\partial t} + i[H, G] = 0. \tag{2.2.4}$$

This implies that the transformations generated by G leave the action invariant. Hence we may use the operator G as the new Hamiltonian to study the evolution of the system.

The switching from H to the new evolution operator G can be interpreted as a redefinition of the time and the coordinate. Let us introduce a new time parameter

$$d\tau = \frac{dt}{u + vt + wt^2} \tag{2.2.5}$$

[1] For the DFF-model (2.1.2) this can be readily seen from the behavior of the inverse-square potential as $x \to \infty$ (Figure 2.2).

[2] Due to the infinite repulsive potential at the origin, we here consider the solution $\psi_0(x)$ satisfying the boundary condition $\psi_0(x)|_{x=0} = 0$.

and a new variable

$$q(\tau) = \frac{x(t)}{\sqrt{u + vt + wt^2}}. \tag{2.2.6}$$

Then we find the action of G on the operator and on the state given by

$$\frac{dq(\tau)}{d\tau} = i[G, q(\tau)], \tag{2.2.7}$$

$$G|\Psi(\tau)\rangle = i\frac{d}{d\tau}|\Psi(\tau)\rangle \tag{2.2.8}$$

as required. Although the operator G may describe the evolution in the new time τ, it is not yet complete to justify the passing to the new description. We need to examine whether the new Hamiltonian and the new coordinates cover the whole evolution in time from $t = -\infty$ to $+\infty$. From (2.2.5) we can express the new time parameter as

$$\tau = \int_{t_0}^{t} \frac{dt'}{u + vt' + wt'^2} + \tau_0. \tag{2.2.9}$$

This integral depends on the zeros of the denominater and the result is classified by the discriminant

$$\Delta = v^2 - 4uw. \tag{2.2.10}$$

We find

$$\tau = \begin{cases} \frac{1}{\sqrt{\Delta}} \left(\ln \left| \frac{2wt+v-\sqrt{\Delta}}{2wt+v+\sqrt{\Delta}} \right| - \ln \left| \frac{v-\sqrt{\Delta}}{v+\sqrt{\Delta}} \right| \right) & \text{for } \Delta > 0 \\ \\ \frac{2}{\sqrt{|\Delta|}} \left(\tan^{-1} \frac{2wt+v}{\sqrt{|\Delta|}} - \tan^{-1} \frac{v}{\sqrt{|\Delta|}} \right) & \text{for } \Delta < 0 \\ \\ -\left(\frac{2}{2wt+v} - \frac{2}{v} \right) & \text{for } \Delta = 0 \end{cases} \tag{2.2.11}$$

where we normalize as $\tau_0 = 0$. For $\Delta > 0$, the parameter τ cannot sweep the whole time region $-\infty \leq t \leq \infty$. This unpleasant feature is associated with the fact that the corresponding operators are non-compact and their spectrums are physically unacceptable. The dilatation operator D belongs to this class. When $\Delta < 0$, τ can be defined over the whole time interval $-\infty \leq t \leq \infty$. The corresponding operators in this case generate a compact rotation and their spectrums have physically satisfactory characteristics. In the case of $\Delta = 0$, the whole time interval $-\infty \leq t \leq \infty$ can be described over $-\infty \leq \tau \leq \infty$, however, there exists

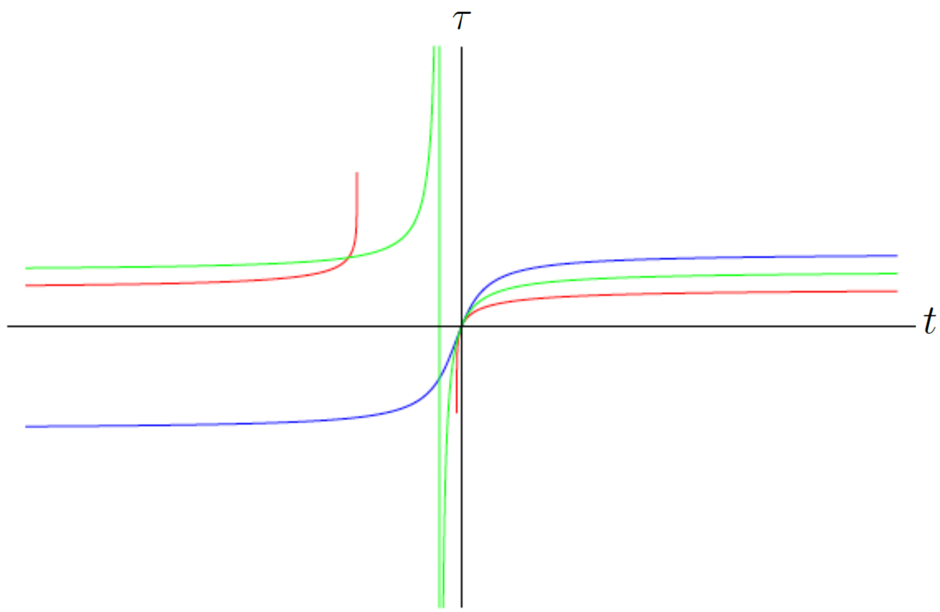


Figure 2.1: The new time parameter τ as a function of the original time t. The red curve represents the non-compact case $\Delta > 0$, which cannot sweep over the whole time t. The blue curve corresponds to the compact case $\Delta < 0$ covering all the time region. Green curve denotes the case $\Delta = 0$, which contains one singular point in τ.

one singular point in τ at $t = -\frac{v}{2w}$. This is the case for the Hamiltonian H and the conformal boost operator K. These three cases are illustrated in Figure 2.1.

In terms of the new set of coordinates (2.2.5) and (2.2.6), we can rewrite the action (2.1.4) as

$$\begin{aligned} S &= \int d\tau \left[\frac{1}{2}\dot{q}^2 - \frac{g^2}{2q^2} + \frac{1}{2}\left(\frac{v}{2} + wt\right)^2 q^2 + \frac{1}{2}(v + 2wt)q\dot{q} \right] \\ &= \int d\tau \left[\frac{1}{2}\dot{q}^2 - \frac{g^2}{2q^2} + \frac{\Delta}{8}q^2 + \frac{1}{2}\frac{d}{d\tau}\left\{ \left(\frac{v}{2} + wt\right) q^2 \right\} \right] \\ &= \int d\tau \left[\frac{1}{2}\dot{q}^2 - \frac{g^2}{2q^2} + \frac{\Delta}{8}q^2 \right] \\ &= \int d\tau L_\tau \end{aligned} \tag{2.2.12}$$

up to the total τ derivative. Note that the dot denotes τ derivative in (2.2.12). The new Lagrangian L_τ leads to the new Hamiltonian

$$\begin{aligned} H_\tau &= \dot{q}\frac{\partial L_\tau}{\partial \dot{q}} - L_\tau \\ &= \frac{1}{2}\left(\dot{q}^2 + \frac{g^2}{q^2} - \frac{\Delta}{4}q^2 \right) \end{aligned} \tag{2.2.13}$$

with

$$G(x(t), \dot{x}(t)) = H_\tau(q(\tau,)\dot{q}(\tau)). \tag{2.2.14}$$

Note that $L_0 = T_0$ is the compact generator satisfying $\Delta = -1 < 0$. Qualitatively one can see that the potential energy of this new Hamiltonian L_0 acquires the minimum and assymptotes to infinity (Figure 2.2). The new time coordinate τ and variable $q(\tau)$ associated with the generator L_0 are given by

$$\tau = 2\tan^{-1}\left(\frac{t}{a}\right), \tag{2.2.15}$$

$$q(\tau) = \sqrt{\frac{2}{a}} \frac{x(t)}{\sqrt{1 + \left(\frac{t}{a}\right)^2}}. \tag{2.2.16}$$

As we will discuss in section 2.6, in the black hole interpretation τ can be identified with the proper time of the test particle near the horizon of the extremal black hole [65].

The fact that the operator L_0 is regarded as the new Hamiltonian of the system can be paraphrased as the group theoretical statement that infinite dimensional unitary representations in terms of Hermitian operators of the non-compact group

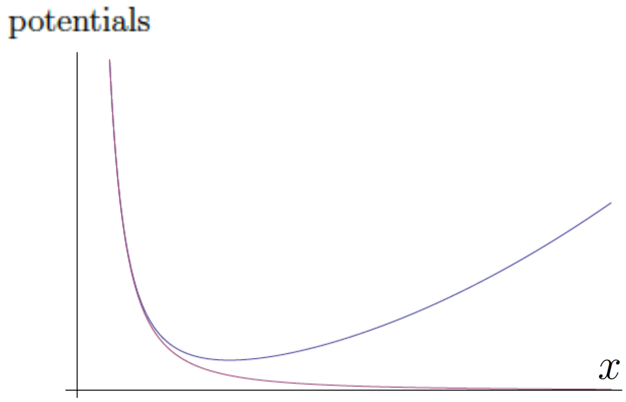


Figure 2.2: The potentials for the original Hamiltonian H and the new Hamiltonian L_0. The red line is the potential for H and the blue one is for L_0.

$SL(2,\mathbb{R})$ are characterized by the discrete eigenvalues of the Casimir operator $\mathcal{C}_2$ and of the compact generator L_0 [3].

We now look for the eigenvalues and eigenstates of L_0. Let us denote the eigenvalues and eigenstates of L_0 by r_n and $|n\rangle$

$$L_0|n\rangle = r_n|n\rangle. \tag{2.2.17}$$

From the $\mathfrak{sl}(2,\mathbb{R})$ algebra (2.1.41) one can show that

$$\begin{aligned} L_0 L_-|r_n\rangle &= (r_n-1)L_-|r_n\rangle, &(2.2.18)\\ L_0 L_+|r_n\rangle &= (r_n+1)L_+|r_n\rangle, &(2.2.19)\\ L_-L_+ &= -\mathcal{C}_2 + L_0(L_0+1), &(2.2.20)\\ L_+L_- &= -\mathcal{C}_2 + L_0(L_0-1). &(2.2.21)\end{aligned}$$

The relations (2.2.18) and (2.2.19) imply that the operators L_- and L_+ play the role of the annihilation and creation operators respectively. Since the norm of the states $L_\pm|r_n\rangle$ must be positive or zero, we require that

$$\begin{aligned} 0 &\le |L_\pm|r_n\rangle|^2 \\ &= -\mathcal{C}_2 + r_n(r_n \pm 1). \end{aligned} \tag{2.2.22}$$

Assuming that there exists one positive eigenvalue r_n among the allowed eigenvalues, the creation operator L_+ yields the infinite chain of states

$$|r_n\rangle, \quad |r_n+1\rangle, \quad |r_n+2\rangle, \quad \cdots . \tag{2.2.23}$$

[3]The diagonalization of the non-compact operator requires the continuous basis [158, 159, 160, 161].

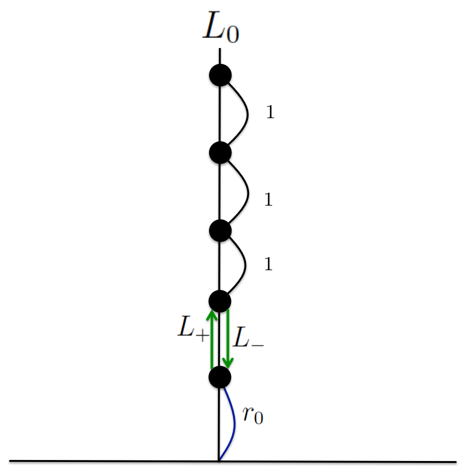


Figure 2.3: The L_0 spectrum. The ground state has eigenvalue r_0 and the excited states generated by L_+ are equally spaced with unit one.

If we require the existence of the ground state, the spectrum need to be bounded below and the chain must terminate. This means that

$$L_-|r_0\rangle = 0 \tag{2.2.24}$$

and

$$L_+L_-|r_0\rangle = \left[-\mathcal{C}_2 + r_0(r_0-1)\right]|r_0\rangle = 0. \tag{2.2.25}$$

Therefore the eigenvalues of L_0 are given by a discrete series (see Figure 2.3)

$$r_n = r_0 + n, \quad n = 0, 1, 2, \cdots \tag{2.2.26}$$

$$\mathcal{C}_2 = r_0(r_0-1). \tag{2.2.27}$$

In the following discussion we will thus simplify the expression as $|n\rangle = |r_n\rangle$.

Combining the relation (2.2.27) and the expression (2.1.45), we find two possible values for r_0 as the choice of the positive or negative signs for the square root. However, it turns out that only the positive sign for the square root should be selected.

$$r_0 = \frac{1}{2}\left(1 + \sqrt{g^2 + \frac{1}{4}}\right). \tag{2.2.28}$$

To see this let us determine the lowest eigenfunction $\psi_0(x)$. From the equation (2.2.24) and the explicit expressions for $L_\pm$

$$L_- = L_0 - \frac{x^2}{2a} - \frac{1}{2}x\frac{d}{dx} - \frac{1}{4}, \tag{2.2.29}$$

$$L_+ = L_0 - \frac{x^2}{2a} + \frac{1}{2}x\frac{d}{dx} + \frac{1}{4}, \tag{2.2.30}$$

we see that $\psi_0(x)$ satisfies the equation

$$\left[x\frac{d}{dx}+\frac{x^2}{a}-\left(2r_0-\frac{1}{2}\right)\right]\psi_0(x)=0. \tag{2.2.31}$$

Let us choose our units so that $a=1$. The generic solution is given by

$$\psi_0(x)=Ce^{-\frac{x^2}{2}}x^{2r_0-\frac{1}{2}} \tag{2.2.32}$$

where C is the constant of integration. The presence of the infinitely repulsive potential barrier at the origin and the confinement property of the wavefunction requires that

$$\lim_{x\to 0}\psi_0(x)=0, \tag{2.2.33}$$

$$\lim_{x\to 0}\psi_0'(x)=0. \tag{2.2.34}$$

These conditions lead to

$$r_0>\frac{3}{4} \tag{2.2.35}$$

which is only satisfied by the positive root solution. Note that (2.2.35) is equivalent to the condition $\gamma>0$ for the coupling constant as we mentioned. Also one can determine C by the normalization condition $\int_0^\infty|\psi_0(x)|^2\,dx=1$ as

$$C=\sqrt{\frac{2}{\Gamma(2r_0)}}. \tag{2.2.36}$$

Therefore the eigenfunction of the ground state is given by

$$\psi_0(x)=\sqrt{\frac{2}{\Gamma(2r_0)}}e^{-\frac{x^2}{2}}x^{\frac{1}{2}+\sqrt{g^2+\frac{1}{4}}}. \tag{2.2.37}$$

This is illustrated in Figure 2.4. Curiously a particle in the L_0 ground state has zero probability of existing at $x=0$.

From (2.2.22) and (2.2.27) one can see that the raising and lowering operators $L_\pm$ act as

$$L_\pm|n\rangle=\sqrt{r_n(r_n\pm 1)-r_0(r_0-1)}|n\pm 1\rangle, \tag{2.2.38}$$

which leads to the relation

$$|n\rangle=\sqrt{\frac{\Gamma(2r_0)}{n!\Gamma(2r_0+n)}}(L_+)^n|0\rangle. \tag{2.2.39}$$

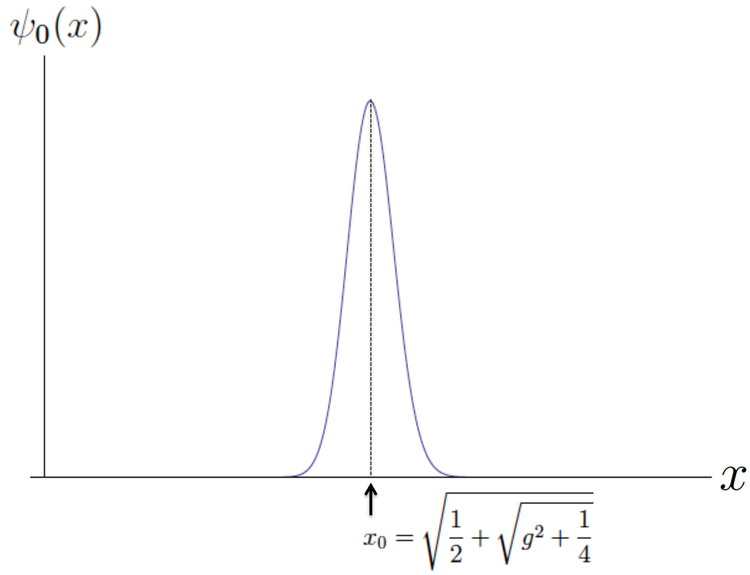


Figure 2.4: Wavefunction $\psi_0(x)$ of the L_0 ground state.

Upon the repeated application of the creation operator L_+ on the ground state $|0\rangle$, the eigenfunctions of the excited states found to be [54]

$$\psi_n(x) = \sqrt{\frac{\Gamma(n+1)}{2\Gamma(n+2r_0)}} x^{-\frac{1}{2}} \left(\frac{x^2}{a}\right)^{r_0} e^{-\frac{x^2}{2a}} L_n^{2r_0-1}\left(\frac{x^2}{a}\right) \tag{2.2.40}$$

where $L_n^{2r_0-1}$ is the associated Laguerre polynomial.

Now consider the thermodynamical aspect of the DFF-model. As we have been discussing, it has been proposed that $L_0 = \frac{1}{2}(aH + \frac{K}{a})$ is treated as the new Hamiltonian instead of the original Hamiltonian H in the DFF-model. The surface of the constant value of L_0 in the classical phase space is given by

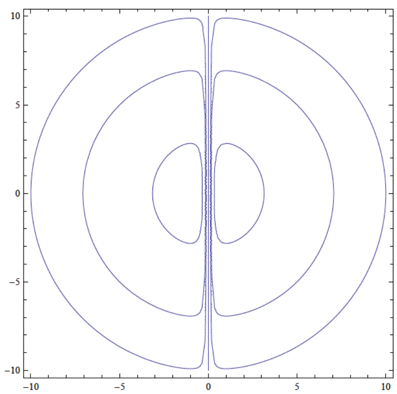
$$p = \pm\sqrt{2L_0 - \frac{g}{x^2} - a^2x^2} \tag{2.2.41}$$

and illustrated in Figure 2.5 [4].

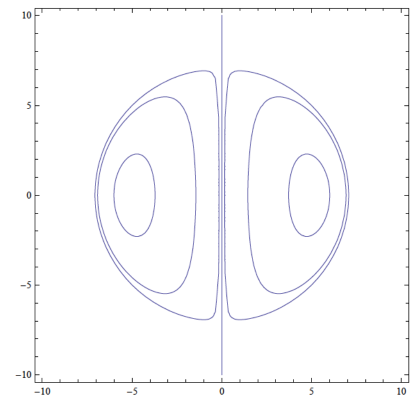
Thus the volume of the phase space with the "energy" below L_0 can be evaluated to be

$$\begin{aligned}\Gamma(L_0) &= 2\int_0^\infty dx\sqrt{2L_0 - \frac{g}{x^2} - a^2x^2} \\ &= \pi\left(\frac{L_0}{a} - g\right). \end{aligned} \tag{2.2.42}$$

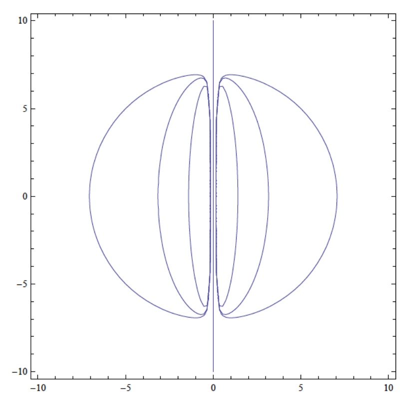
[4] Note that the phase space is restricted to either $x > 0$ or $x < 0$ region due to the infinite potential at the origin.



(a) $L_0 = 10, 50, 100; g = 1, a = 1.$



(b) $g^2 = 1, 100, 500; L_0 = 50, a = 1.$



(c) $a^2 = 1, 5, 25; L_0 = 50, g = 1.$

Figure 2.5: The surface of the constant value of L_0 in the classical phase space. The horizontal axis denotes the canonical variable x while the vertical axis represents the canonical momentum p. The volume of the phase space with "the energy" below L_0 decrease with increase in the coupling constant g and the deformation parameter a. Qualitatively g keeps a particle away from the origin whereas a sucks it into the origin.

According to the additional term $-\pi g$, the result is slightly modified from a simple harmonic oscillator. This corresponds to the fact that the L_0-ground state of the DFF-model is raised by the increase of the coupling constant g as in (2.2.28). As seen form Figure 2.5, the volume of the phase space with "the energy" below L_0 decrease with increase in the coupling constant g and the deformation parameter a. Therefore qualitatively g keeps a particle away from the origin whereas a sucks it into the origin. These features are in accord with the behavior of the wavefunction $\psi_0(x)$ of the ground state given in (2.2.37) (see also Figure 2.4).

In quantum mechanics the L_0-spectrum is the discrete value given in (2.2.26). By summing over the spectrum one obtains the partition function

$$Z = \sum_{n=0}^{\infty} e^{-\beta L_0} = \frac{e^{-\beta r_0}}{1 - e^{-\beta}}. \tag{2.2.43}$$

2.3 Time evolution

So far the DFF-model (2.1.2) has been studied in the x space, i.e. the stationary problem at $t = 0$. Now let us consider the state $|t\rangle$ which is characterized by the time t. Let us define the time-dependent function

$$\beta_n := \langle t|n\rangle, \tag{2.3.1}$$

on which the action of the Hamiltonian is realized with the time derivative

$$H = i\frac{d}{dt}. \tag{2.3.2}$$

Combining the expression (2.3.2) with the $\mathfrak{sl}(2,\mathbb{R})$ algebra (2.1.23) and the form of the Casimir operator (2.1.45), one finds the action of the dilatation operator D and the conformal boost operator K on β_n as

$$D = \left(it\frac{d}{dt} + ir_0 \right), \tag{2.3.3}$$

$$K = \left(it^2\frac{d}{dt} + 2ir_0 t \right). \tag{2.3.4}$$

Thus the compact operator L_0 acts on $\beta_n(t)$ as

$$L_0 = \frac{i}{2}\left[\left(a + \frac{t^2}{a} \right) \frac{d}{dt} + 2r_0\frac{t}{a} \right]. \tag{2.3.5}$$

From the expressions (2.3.2)-(2.3.5) we can write the actions of the corresponding operators on the state $|t\rangle$ as

$$H|t\rangle = -i\frac{d}{dt}|t\rangle, \tag{2.3.6}$$

$$D|t\rangle = -i\left(t\frac{d}{dt} + r_0\right)|t\rangle, \tag{2.3.7}$$

$$K|t\rangle = -i\left(t^2\frac{d}{dt} + 2r_0 t\right)|t\rangle, \tag{2.3.8}$$

$$L_0|t\rangle = -\frac{i}{2}\left[\left(a + \frac{t^2}{a}\right)\frac{d}{dt} + 2r_0\frac{t}{a}\right]|t\rangle. \tag{2.3.9}$$

Then the explicit expression (2.3.5) for the operator L_0 leads to the differential equation

$$\frac{i}{2}\left[\left(a + \frac{t^2}{a}\right)\frac{d}{dt} + 2r_0\frac{t}{a}\right]\beta_n = r_n\beta_n. \tag{2.3.10}$$

and its solution is given by

$$\beta_n(t) = (-1)^n\left[\frac{\Gamma(2r_0+n)}{n!}\right]^{\frac{1}{2}}\left(\frac{a-it}{a+it}\right)^{r_n}\frac{1}{\left(1+\frac{t^2}{a^2}\right)^{r_0}}. \tag{2.3.11}$$

Using the above solution (2.3.11) one finds 2-point function [54]

$$\begin{aligned} F_2(t_1,t_2) &= \langle t_1|t_2\rangle \\ &= \sum_n \beta_n(t_1)\beta_n^*(t_2) \\ &= \frac{\Gamma(2r_0)a^{2r_0}}{[2i(t_1-t_2)]^{2r_0}} \propto \frac{1}{(t_1-t_2)^{2r_0}}. \end{aligned} \tag{2.3.12}$$

The expression (2.3.12) indicates that the 2-point function is the value of two operators whose effective dimensions are r_0. Note that the 2-point function satisfies the set of conditions

$$\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}\right)F_2(t_1,t_2) = 0, \tag{2.3.13}$$

$$\left(t_1\frac{\partial}{\partial t_1} + t_2\frac{\partial}{\partial t_2} + 2r_0\right)F_2(t_1,t_2) = 0, \tag{2.3.14}$$

$$\left(t_1^2\frac{\partial}{\partial t_1} + t_2^2\frac{\partial}{\partial t_2} + 2r_0(t_1+t_2)\right)F_2(t_1,t_2) = 0. \tag{2.3.15}$$

Now we want to consider the E space. The eigenstate $|E\rangle$ is defined by

$$H|E\rangle = E|E\rangle \tag{2.3.16}$$

and we can expand the eigenstate $|n\rangle$ of the compact operator L_0 as

$$|n\rangle = \int dE C_n(E)|E\rangle \tag{2.3.17}$$

where we have defined

$$C_n(E) := \langle E|n\rangle. \tag{2.3.18}$$

Note that the eigenvalue E of the original Hamiltonian H is continuous as we have already mentioned. Requiring the $\mathfrak{sl}(2,\mathbb{R})$ algebra (2.1.23) and the realization of the Casimir operator (2.1.45), we get

$$D|E\rangle = -i\left(E\frac{d}{dE} + \frac{1}{2}\right)|E\rangle, \tag{2.3.19}$$

$$K|E\rangle = \left[-E\frac{d^2}{dE^2} - \frac{d}{dE} + \left(r_0 - \frac{1}{2}\right)^2\frac{1}{E}\right]|E\rangle. \tag{2.3.20}$$

Then we can write the compact operator L_0 as

$$L_0 = \frac{1}{2}\left[-E\frac{d^2}{dE^2} - \frac{d}{dE} + E + \left(r_0 - \frac{1}{2}\right)^2\frac{1}{E}\right] \tag{2.3.21}$$

and the explicit expression for $C_n(E)$ can be found by solving the equation

$$\frac{1}{2}\left[-E\frac{d^2}{dE^2} - \frac{d}{dE} + E + \left(r_0 - \frac{1}{2}\right)^2\frac{1}{E}\right]C_n(E) = r_n C_n(E). \tag{2.3.22}$$

If we set

$$C_n = 2^{r_0}E^{r_0-\frac{1}{2}}e^{-E}\varphi_n(E), \tag{2.3.23}$$

then we see that the function φ_n satisfies the differential equation [5]

$$\eta\varphi_n'' + (2r_0 - \eta)\varphi_n' + n\varphi_n = 0 \tag{2.3.24}$$

of the associated Laguerre polynomial $L_n^{2r_0-1}$. Putting all together we obtain

$$C_n(E) = 2^{r_0}\sqrt{\frac{\Gamma(n+1)}{\Gamma(n+2r_0)}}\frac{(aE)^{r_0}}{\sqrt{E}}e^{-aE}L_n^{2r_0-1}(2aE). \tag{2.3.25}$$

[5] The associated Laguerre polynomials $L_n^k(x)$ are defined by the solution of the differential equation

$$\left[x\frac{d^{k+2}}{dx^{k+2}} + (k+1-x)\frac{d^{k+1}}{dx^{k+1}} + n\frac{d^k}{dx^k}\right]L_n^k(x) = 0$$

with $0 \leq k \leq n$.

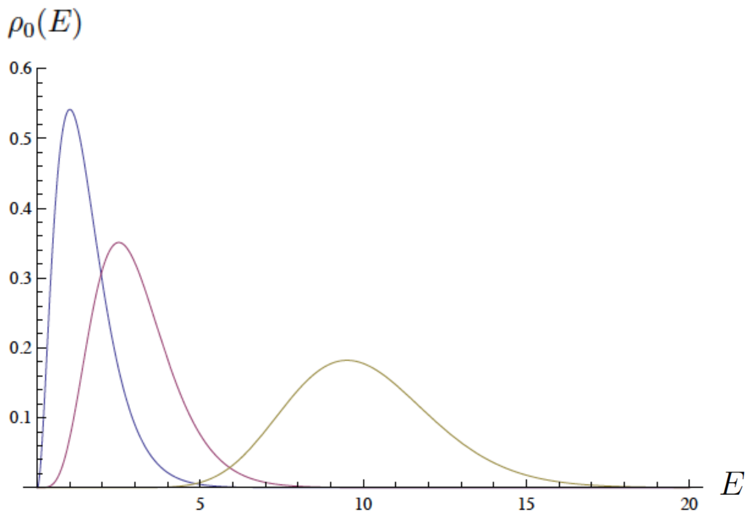


Figure 2.6: The probability density $\rho_0(E)$ of the ground state in E space. The blue, red and brown lines denote the cases with $r_0 = \frac{3}{2}$, 3 and 10. They have the maximum values at $E = r_0 - \frac{1}{2}$.

Note that the function $C_n(E)$ has the following properties:

$$\sum_n C_n(E)C_n^*(E') = \delta(E - E'), \tag{2.3.26}$$

$$\int_0^\infty dE C_n(E)C_{n'}^*(E) = \delta_{nn'}, \tag{2.3.27}$$

$$C_n(E) = 2^{r_0} E^{\frac{1}{2}-r_0} \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{iEt} \beta_n(t). \tag{2.3.28}$$

Let us discuss the probability density $\rho_n(E)$ in E space defined by

$$\rho_n(E) := |C_n(E)|^2 . \tag{2.3.29}$$

For $n = 0$, i.e. for the ground state of L_0, the probability density $\rho_0(E)$ is given by

$$\rho_0(E) = \frac{4r^{r_0}}{\Gamma(2r_0)} E^{2r_0-1} e^{-2E} \left[L^{2r_0-1}(2E) \right]^2 \tag{2.3.30}$$

with $a = 1$. This is shown in Figure 2.6.

The distribution of E of the ground state is peaked at

$$E_0 = r_0 - \frac{1}{2} \tag{2.3.31}$$

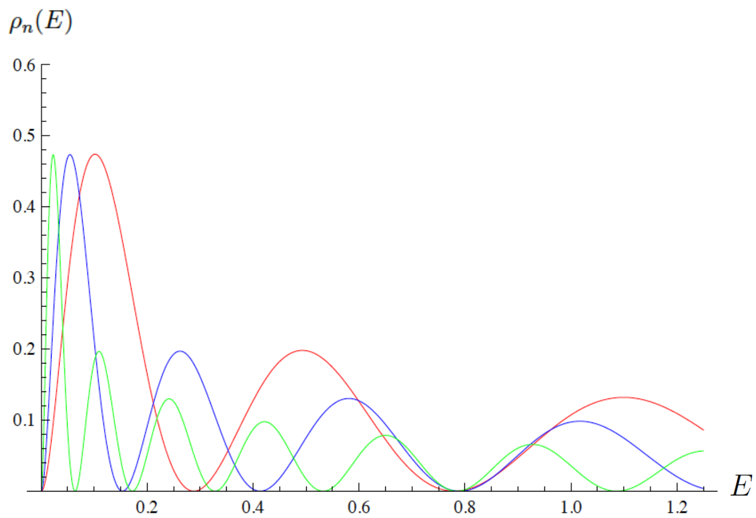


Figure 2.7: The probability densities $\rho_n(E)$ of the excited states for $r_0 = \frac{3}{2}$ in E space. The red, blue and green lines correspond to the densities with $n = 10, 20$ and 30. The number of peaks increases with the increase of the excited levels.

and its effective width is [6]

$$\Gamma = 2\sqrt{E_0} = 2\sqrt{r_0 - \frac{1}{2}}. \tag{2.3.32}$$

(2.3.31) shows that the peak of the distribution of E increases with the increase of r_0 or g. (2.3.32) implies that the width grow and the probability dense spread in E space with r_0.

The expectation values for the ground state $|0\rangle$ can be evaluated and we find

$$\langle H \rangle_0 = \langle 0|H|0\rangle = r_0 \tag{2.3.33}$$

$$(\Delta E)^2 := \langle H^2 \rangle_0 - \langle H \rangle_0^2 = \frac{r_0^2}{2}. \tag{2.3.34}$$

For $n > 0$ the probability density $\rho_n(E)$ with $r_0 = \frac{3}{2}$ is illustrated in Figure 2.7. The red, blue and green lines represent the case with $n = 10, 20$ and 30 respectively. In this case several peaks appear due to the presence of the n-th order polynomial. The expectation value of H in the excited state $|n\rangle$ is calculated to be [54]

$$\langle H \rangle_n = \langle n|H|n\rangle = r_n. \tag{2.3.35}$$

The state $|E\rangle$ provides us with the further properties of the state $|t\rangle$. From the expression (2.3.12) ant the relation (2.3.28) with the use of the Hankel integral

[6] The effective width Γ is defined by $\frac{8}{\Gamma} := -\frac{\rho_0(E_0)''}{\rho_0(E_0)}$.

representation of the gamma function

$$\frac{1}{\Gamma(z)} = \frac{i}{2\pi} \int_C dt(-t)^{-z} e^{-t}, \tag{2.3.36}$$

we can show that

$$\langle t|E\rangle = 2^{-r_0} \frac{(aE)^{r_0}}{\sqrt{E}} e^{-iEt}. \tag{2.3.37}$$

Using the two relations (2.3.28) and (2.3.37) we see that

$$\int_{-\infty}^{\infty} \frac{dt}{2\pi} |t\rangle\langle t| = \frac{1}{H} \left(\frac{aH}{2}\right)^{2r_0}. \tag{2.3.38}$$

This indicates the incompleteness of the state $|t\rangle$.

2.4 Operator

Now consider the tensor operator $B(t)$ with the mass dimension Δ. As seen from the set of the Heisenberg equations (2.1.31)-(2.1.33), the $\mathfrak{sl}(2, \mathbb{R})$ invariance of the operator implies that

$$i[H, B(t)] = \frac{d}{dt} B(t), \tag{2.4.1}$$

$$i[D, B(t)] = t\frac{d}{dt} B(t) + \Delta B(t), \tag{2.4.2}$$

$$i[K, B(t)] = t^2 \frac{d}{dt} B(t) + 2t\Delta B(t) \tag{2.4.3}$$

where $D = tH + D_0$ and $K = t^2 H - \frac{1}{2}\{x, p\} + K_0$. This is the $SO(1,2)$ Wigner-Eckart theorem. We now want to compute the 3-point function

$$\begin{aligned} F_3(t; t_2, t_1) &= \langle t_2|B(t)|t_1\rangle \\ &= \sum_{n_1, n_2} \beta_{n_2}(t_2)\beta^*_{n_1}(t_1)\langle n_2|B(t)|n_1\rangle. \end{aligned} \tag{2.4.4}$$

In analogy with (2.2.15) and (2.2.16) it is convenient to introduce the new variables

$$\tau = 2\tan^{-1} t, \tag{2.4.5}$$

$$b(\tau) = B(t)(1 + t^2)^{\Delta} \tag{2.4.6}$$

with the relations

$$\frac{db(\tau)}{d\tau} = i[L_0, b(\tau)], \tag{2.4.7}$$

$$b(\tau) = e^{iL_0\tau} b(0) e^{-iL_0\tau}. \tag{2.4.8}$$

By using the above expressions we can show that

$$\langle n_2|B(t)|n_1\rangle = \frac{1}{(1+t^2)^\Delta}\left(\frac{1-it}{1+it}\right)^{n_1-n_2}\langle n_2|B(0)|n_1\rangle. \tag{2.4.9}$$

The equation (2.4.9) enables us to rewrite the 3-point function (2.4.4) as

$$\begin{aligned} F_3(t;t_2,t_1) = \sum_{n_1,n_2}(-1)^{n_1+n_2}&\sqrt{\frac{\Gamma(n_1+2r_0)\Gamma(n_2+2r_0)}{n_1!n_2!}} \\ &\times z_1^{-n_1}z_2^{n_2}\frac{(1+z_1)^{2r_0}(1+z_2)^{2r_0}}{2^{4r_0}z_1^{2r_0}} \\ &\times 2^{-2\Delta}\left|\frac{1+z}{z}\right|^{2\Delta} z^{n_1-n_2}\langle n_2|B(0)|n_1\rangle \end{aligned} \tag{2.4.10}$$

where we have defined

$$z := \frac{1-it}{1+it} = e^{-i\tau}, \qquad z_i := \frac{1-it_i}{1+it_i} = e^{-i\tau_i}. \tag{2.4.11}$$

On the other hand, the one-dimensional conformal $\mathfrak{sl}(2,\mathbb{R})$ covariance of the 3-point function implies that

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}\right)F_3(t;t_2,t_1)=0, \tag{2.4.12}$$

$$\left(t\frac{\partial}{\partial t}+t_1\frac{\partial}{\partial t_1}+t_2\frac{\partial}{\partial t_2}+2\Delta t+2r_0(t_1+t_2)\right)F_3(t;t_2,t_1)=0, \tag{2.4.13}$$

$$\left(t^2\frac{\partial}{\partial t}+t_1^2\frac{\partial}{\partial t_1}+t_2^2\frac{\partial}{\partial t_2}+2\Delta t+2r_0(t_1+t_2)\right)F_3(t;t_2,t_1)=0. \tag{2.4.14}$$

As the first condition (2.4.12) says that $F_3(t;t_2,t_1) = F_3(t-t_1, t-t_2, t_1-t_2)$, we impose the ansatz $F_3 = f(t-t_1)^{\alpha_1}(t-t_2)^{\alpha_2}(t_1-t_2)^{\alpha_3}$ with f being an arbitrary constant value. Then the remaining two conditions (2.4.13) and (2.4.14) restrict to the form of the 3-point function as

$$\begin{aligned} F_3(t;t_2,t_1) &= f i^{2r_0+\Delta}\frac{1}{(t-t_1)^\Delta(t_2-t)^\Delta(t_1-t_2)^{-\Delta+2r_0}} \\ &= 2^{-\Delta-2r_0}f\frac{(1+z)^{2\Delta}(1+z_1)^{2r_0}(1+z_2)^{2r_0}}{(z-z_1)^\Delta(z_2-z)^\Delta(z_1-z_2)^{-\Delta+2r_0}}. \end{aligned} \tag{2.4.15}$$

Combining the two expressions (2.4.10) and (2.4.15) for the 3-point function, we obtain the relation

$$\begin{aligned} \sum_{n_1,n_2}(-1)^{n_1+n_2}\sqrt{\frac{\Gamma(n_1+2r_0)\Gamma(n_2+2r_0)}{n_1!n_2!}}z_1^{-n_1}z_2^{n_2}2^{-\Delta}\left|\frac{1+z}{z}\right|^{2\Delta}z^{n_1-n_2}\langle n_2|B(0)|n_1\rangle \\ = 2^{r_0}fz_1^{2r_0}\frac{(1+z)^{2\Delta}}{(z-z_1)^\Delta(z_2-z)^\Delta(z_1-z_2)^{-\Delta+2r_0}}. \end{aligned} \tag{2.4.16}$$

By redefining the variables $z_i \to zz_i$, we can factor out the time t dependence and thus we shall take $t = 0$ or equivalently $z = 1$ in the following discussion. Then the left hand side can be regarded as the double Taylor series expansions and one can write the quantity $\langle n_2|B(0)|n_1\rangle$ as the coefficients of the expansions as

$$\begin{aligned}\langle n_2|B(0)|n_1\rangle &= \frac{f}{(2\pi i)^2}\sqrt{\frac{n_1!n_2!}{\Gamma(n_1+2r_0)\Gamma(n_2+2r_0)}}(-1)^{n_1+n_2}2^{2r_0+\Delta} \\ &\times \oint_{C_1} dz_1 \oint_{C_2} dz_2 \frac{z_1^{2r_0+n_1-1} z_2^{-(n_2+1)}}{(1-z_1)^{\Delta}(z_2-1)^{\Delta}(z_1-z_2)^{-\Delta+2r_0}}\end{aligned} \tag{2.4.17}$$

where C_1 is the suitable anti-clockwise contour for the coordinates $z_1 = \infty$ and C_0 is for z_2. Changing the pair of the variables $z_1 = \frac{1}{w_1}$ and $z_2 = w_2$, we find the expression

$$\begin{aligned}\langle n_2|B(0)|n_1\rangle &= \frac{f}{(2\pi i)^2}\sqrt{\frac{n_1!n_2!}{\Gamma(n_1+2r_0)\Gamma(n_2+2r_0)}}(-1)^{n_1+n_2}2^{2r_0+\Delta} \\ &\times \oint_{C_2} \frac{dw_2}{w_2^{n_2+1}} \oint_{C_1} \frac{dw_1}{w_1^{n_1+1}} \frac{(1-w_1w_2)^{\Delta-2r_0}}{(1-w_1)^{\Delta}(w_2-1)^{\Delta}}\end{aligned} \tag{2.4.18}$$

where the integral are carried out around the contours C_1 for w_1 and C_2 for w_2. Applying the Cauchy theorem, the integration (2.4.18) can be calculated to be [54]

$$\begin{aligned}\langle n_2|B(0)|n_1\rangle &= 2^{2r_0+\Delta} f \sqrt{\frac{n_1!n_2!}{\Gamma(2r_0+n_1)\Gamma(2r_0+n_2)}} \\ &\times \sum_{k=0}^{\min[n_1,n_2]} (-1)^k \begin{pmatrix} \Delta-2r_0 \\ k \end{pmatrix} \begin{pmatrix} -\Delta \\ n_1-k \end{pmatrix} \begin{pmatrix} -\Delta \\ n_2-k \end{pmatrix}.\end{aligned} \tag{2.4.19}$$

For $n_1 = n_2 = 0$ we can read off the explicit formula for the constant f as

$$f = 2^{-\Delta-2r_0}\Gamma(2r_0)\langle 0|B(0)|0\rangle. \tag{2.4.20}$$

Inserting (2.4.20) into (2.4.15) and reviving the constant factor a, we finally get the formula for the 3-point function

$$\begin{aligned}F_3(t,;t_2,t_1) &= \langle 0|B(0)|0\rangle \left(\frac{i}{2}\right)^{2r_0+\Delta} \frac{\Gamma(2r_0)a^{2r_0}}{(t-t_1)^{\Delta}(t_2-t)^{\Delta}(t_1-t_2)^{-\Delta+2r_0}} \\ &\propto \frac{1}{(t-t_1)^{\Delta}(t_2-t)^{\Delta}(t_1-t_2)^{-\Delta+2r_0}}.\end{aligned} \tag{2.4.21}$$

From the above form (2.4.21) we see that the 3-point function F_3 consists of the two operators with the same mass dimension r_0 and the third operator B with the mass dimension Δ.

As seen from (2.3.12) and (2.4.21), the structures of the 2- and 3-point functions suggest that there exists the averaging state and the corresponding operator with the mass dimension r_0. Let us define the operator $\mathcal{O}(t)$ which acts on the L_0-vacuum to create the state $|t\rangle$:

$$|t\rangle = \mathcal{O}(t)|0\rangle. \tag{2.4.22}$$

Making use of the formulae (2.3.11) and (2.2.39), we can write [7]

$$\mathcal{O}(t) = N(t) \exp\left[-\overline{z}(t)L_+\right] \tag{2.4.23}$$

where

$$N(t) = \sqrt{\Gamma(2r_0)} \left(\frac{\overline{z}(t)+1}{2}\right)^{2r_0}. \tag{2.4.24}$$

Then the 2- and 3-point functions are given by

$$F_2(t_1,t_2) = \langle t_1|t_2\rangle = \langle 0|\mathcal{O}^\dagger(t_1)\mathcal{O}(t_2)|0\rangle, \tag{2.4.25}$$

$$F_3(t;t_2,t_1) = \langle t_2|B(t)|t_1\rangle = \langle 0|\mathcal{O}^\dagger(t_1)B(t)\mathcal{O}(t_2)|0\rangle. \tag{2.4.26}$$

Therefore the averaging state is the L_0 ground state $|0\rangle$ and the corresponding operators are $\mathcal{O}^\dagger(t)$ and $\mathcal{O}(t)$. We should note that the conformal invariant correlation functions can be built up although the averaging state $|0\rangle$ is not conformally invariant and the operators $\mathcal{O}(t)$ and $\mathcal{O}^\dagger(t)$ are not primary operators. This is the significant difference from other higher dimensional conformal field theories where one can assume the existence of the normalizable and invariant vacuum states. For quantum field theories we generally treat with the Fock spaces which are underlying on the empty no-particle vacuum states. However, in quantum mechanics we deal with the Hilbert space which is the subspace of the Fock space with the fixed number of the particle. This fact prevents us from constructing the normalizable and invariant empty vacuum state in conformal quantum mechanics.

Noting that

$$[L_-, e^{-\overline{z}(t)L_+}] = e^{-\overline{z}(t)L_+}\left(-2\overline{z}(t)L_0 + \overline{z}^2(t)L_+\right), \tag{2.4.27}$$

$$[L_0, e^{-\overline{z}(t)L_+}] = -e^{-\overline{z}L_+}\overline{z}(t)L_+, \tag{2.4.28}$$

we see that the state $|t\rangle$ is the eigenstate of the operator $L_- + \overline{z}(t)L_0$

$$\left(L_- + \overline{z}(t)L_0\right)|t\rangle = -r_0\overline{z}(t)|t\rangle \tag{2.4.29}$$

[7] Such construction of the state $|t\rangle$ has also been considered in [162, 163, 164]

with the eigenvalue $-r_0\bar{z}(t)$. We see that the state $|t\rangle$ is similar to the coherent state $|a\rangle$ which satisfies $L_-|a\rangle = a|a\rangle$, however, the additional term $\bar{z}(t)L_0$ deviates from it. In fact the coherent state $|a\rangle$ can be constructed as

$$\begin{aligned}|a\rangle &= \sqrt{\Gamma(2r_0)}\sum_n \frac{a^n}{\sqrt{n!\Gamma(2r_0+n)}}|n\rangle \\ &= \Gamma(2r_0)\sum_n \frac{a^n}{n!\Gamma(2r_0+n)}(L_+)^n|0\rangle.\end{aligned} \tag{2.4.30}$$

Let us define the state

$$|\Psi\rangle := e^{-Ha}|t=0\rangle = e^{-Ha}e^{-L_+}|0\rangle. \tag{2.4.31}$$

By using the relations

$$L_0e^{-Ha} = e^{-Ha}\left(\frac{K}{2a}+iD\right), \tag{2.4.32}$$

$$\left(\frac{K}{2a}+iD\right)e^{-L_+} = e^{-L_+}\left(L_0-\frac{1}{4}L_-\right), \tag{2.4.33}$$

one finds that

$$L_0|\Psi\rangle = r_0|\Psi\rangle. \tag{2.4.34}$$

Thus the state $|\Psi\rangle$ defined by (2.4.31) is proportional to the L_0 vacuum state $|0\rangle$

$$|\Psi\rangle = C|0\rangle \tag{2.4.35}$$

where the proportional constant C can be determined by noting the relation (2.3.37) as

$$C = \frac{\sqrt{\Gamma(2r_0)}}{2^{2r_0}} \tag{2.4.36}$$

up to a phase factor. Then we obtain the alternative description for the state $|t\rangle$

$$\begin{aligned}|t\rangle &= e^{iHt}|t=0\rangle \\ &= e^{iHt}e^{Ha}\left(C|0\rangle\right) \\ &= \sqrt{\Gamma(2r_0)}2^{-2r_0}e^{(a+it)H}|0\rangle.\end{aligned} \tag{2.4.37}$$

Let us consider the 4-point function

$$\begin{aligned}F_4(t_1,t_2,t_3,t_4) &= \langle t_1|B(t_2)\tilde{B}(t_3)|t_4\rangle \\ &= \langle 0|\mathcal{O}^\dagger(t_1)B(t_2)B(t_3)\mathcal{O}(t_4)|0\rangle\end{aligned} \tag{2.4.38}$$

where the two different fields $B(t)$ and $\tilde{B}(t)$ carry the mass dimension Δ and $\tilde{\Delta}$ respectively. It is calculated to be [87]

$$
\begin{aligned}
F_4(t_1,t_2,t_3,t_4) &= \langle 0|B(0)|0\rangle\langle 0|\tilde{B}(0)|0\rangle \frac{\Gamma(2r_0)}{2^{\Delta+\tilde{\Delta}+2r_0}} \\
&\times \frac{1}{(t_{13})^{\Delta-r_0}(t_{24})^{\tilde{\Delta}-r_0}(t_{12})^{\tilde{\Delta}+r_0}(t_{34})^{\Delta+r_0}(t_{14})^{2r_0-\Delta-\tilde{\Delta}}} x^{r_0}\ {}_2F_1(\Delta,\tilde{\Delta};2r_0;x) \\
&= p(t_1,t_2,t_3,t_4)x^{r_0}\ {}_2F_1(\Delta,\tilde{\Delta};2r_0;x)
\end{aligned} \tag{2.4.39}
$$

where the parameter a set to be one and we have introduced the expressions $t_{ij} := t_i - t_j$ and $x := \frac{t_{12}t_{34}}{t_{13}t_{24}}$. ${}_2F_1(\Delta,\tilde{\Delta};2r_0;x)$ is the hypergeometric function that possesses the Mellin-Barnes representation

$$
{}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} ds \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-x)^s. \tag{2.4.40}
$$

Note that the single Mellin integral appears in the formula of the 4-point function (2.4.39). This reflects the fact that four points lead to a single invariant in one-dimension in contrast to the two invariants in higher dimensions.

It is known that 4-point functions F_4 can be expressed by the superposition of the conformal blocks, or the conformal partial waves G in higher dimensional conformal field theories [165]. The conformal block G can be determined by requiring that it is the eigenfunction of the quadratic Casimir of the conformal group. As seen from the formula (2.4.39), we can easily read off a single conformal block as [87] [8]

$$
G = x^{r_0}_2 F_1(\Delta,\tilde{\Delta};2r_0;x), \tag{2.4.41}
$$

which satisfies the differential equation

$$
\mathcal{C}_2\left[p(t_1,t_2,t_3,t_4)G\right] = r_0(r_0-1)p(t_1,t_2,t_3,t_4)G. \tag{2.4.42}
$$

2.5 Gauged conformal mechanics

It has been pointed out [137] that the DFF-model (2.1.2) can be obtained by the gauged quantum mechanics. Let us consider a simple complex free particle Lagrangian

$$
L = \frac{1}{2}\dot{z}\dot{\bar{z}} \tag{2.5.1}
$$

[8] For higher dimensional field theories, the conformal block can be obtained through the operator product expansion.

where z is a complex one-dimensional field. The system (2.5.1) is invariant under the following $U(1)$ transformations

$$z' = e^{-i\lambda} z, \qquad \overline{z}' = e^{i\lambda}\overline{z} \tag{2.5.2}$$

where λ is a real parameter. Let us gauge this symmetry by promoting $\lambda \to \lambda(t)$. Then the gauge invariant Lagrangian is given by

$$\begin{aligned} L &= \frac{1}{2} D_0 z D_0 \overline{z} + cA_0 \\ &= \frac{1}{2}\left(\dot{z} + iA_0 z\right)\left(\dot{\overline{z}} - iA_0\overline{z}\right) + cA_0 \end{aligned} \tag{2.5.3}$$

where $A_0(t)$ is the one-dimensional $U(1)$ gauge field. The term cA_0 is a Fayet-Iliopoulos (FI) term with c being a constant. This term is gauge invariant itself up to total derivative.

The action (2.5.3) is invariant under the one-dimensional conformal transformations

$$\delta t = f(t) = a + bt + ct^2, \tag{2.5.4}$$

$$\delta x = \frac{1}{2}\dot{f}x, \tag{2.5.5}$$

$$\delta A_0 = -\dot{f}A_0. \tag{2.5.6}$$

Here the transformation of the gauge field $A_0(t)$ is the same as that of the time derivative ∂_0.

Note that the Lagrangian (2.5.3) is quadratic in the $U(1)$ gauge field A_0 and contains no time derivative of A_0. This immediately implies that the gauge field A_0 is identified with the auxiliary gauge field. Hence we attempt to integrate out the auxiliary gauge field. However, we should be careful of the exclusion of the auxiliary field because it is a gauge field. We need to integrate out the auxiliary gauge field in two steps; firstly we fix a gauge to eliminate residual degrees of freedom and then solve the equation of motion of the auxiliary gauge field or impose the resulting Gauss law constraint to ensure the consistency of the gauge fixing. Let us choose the gauge such that

$$z(t) = \overline{z}(t) = x(t). \tag{2.5.7}$$

Then the Lagrangian (2.5.3) becomes

$$L = \frac{1}{2}\dot{x}^2 + \frac{1}{2}A_0^2 x^2 + cA_0. \tag{2.5.8}$$

Using the algebraic equation of motion for the auxiliary gauge field A_0

$$A_0 = -\frac{c}{x^2}, \tag{2.5.9}$$

we can integrate out gauge field and obtain the reduce Lagrangian

$$L = \frac{1}{2}\left(\dot{x}^2 - \frac{c^2}{x^2}\right). \tag{2.5.10}$$

This is nothing but (2.5.2), the DFF-model Lagrangian. Thus the conformal invariance is preserved under the gauging procedure. This procedure, i.e. the integration of the auxiliary gauge field can be interpreted as the reduction process for the mechanical systems with symmetry. Let us summarize the basic concepts of the classical theory of Hamiltonian dynamical systems.

A manifold $\mathcal{M}$ is said to be endowed with a Poisson structure if there is an operation assigning to every pair of functions $F, G \in \mathcal{F}(\mathcal{M})$ a new function $\{F, G\} \in \mathcal{F}(\mathcal{M})$ which is linear in F and G and has the following properties

1. skew symmetry

$$\{F, G\} = -\{G, F\} \tag{2.5.11}$$

2. Jacobi identity

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0 \tag{2.5.12}$$

3. Leibniz rule

$$\{F, GH\} = \{F, G\} H + \{F, H\} G. \tag{2.5.13}$$

As the above three identities (2.5.11)-(2.5.13) are the axioms of the Lie algebra, the space $\mathcal{F}(\mathcal{M})$ is nothing but an infinite dimensional Lie algebra. Then a dynamical system on $\mathcal{M}$, the so-called Hamiltonian dynamical system can be introduced as

$$\dot{x}^i = \left\{H(x), x^i\right\} = X_H^i \tag{2.5.14}$$

where x^i are local coordinates on $\mathcal{M}$, $H(x)$ is the Hamiltonian of the dynamical system and the vector field X_H^i is referred to as a Hamiltonian vector field. For such system we have

$$\dot{F} = \{H, x\} \tag{2.5.15}$$

and the functions which satisfy $\{H, F\} = 0$ are conserved quantities, i.e. the integrals of motion.

The important class of phase spaces is known as a symplectic manifold $(\mathcal{M}, \omega)$, which possesses closed nondegenerate differential two-form ω, i.e. symplectic forms and their Poisson structures are given by

$$\begin{aligned} \{F(x), G(x)\} &= \omega^{ij}\partial_i F \partial_j G \\ &= \omega(X_F, X_G). \end{aligned} \tag{2.5.16}$$

Suppose that a Lie group G acts on $\mathcal{M}$. Then one can represent the corresponding Lie algebra $\mathfrak{g}$ of G in the Lie algebra of vector fields on $\mathcal{M}$. In other words there is a vector field X_ξ on $\mathcal{M}$ to each $\xi \in \mathfrak{g}$. If one can associate a function H_ξ on $\mathcal{M}$ to each X_ξ obeying the conditions

$$X^i_\xi = \left\{H_\xi, x^i\right\}, \tag{2.5.17}$$

$$H_{\xi+\eta} = H_\xi + H_\eta, \tag{2.5.18}$$

$$H_{[\xi,\eta]} = \left\{H_\xi, H_\eta\right\}, \tag{2.5.19}$$

the action of G is called Hamiltonian and H_ξ the Hamiltonian function. Namely an action of G on $\mathcal{M}$ is called Hamiltonian if the map $\xi \mapsto H_\xi$ is a homomorphism of the Lie algebra $\mathfrak{g}$ into the Lie algebra $\mathcal{F}(\mathcal{M})$. It is known that any symplectic action of Lie group G is Hamiltonian if $H^2(\mathfrak{g}, \mathbb{R}) = 0$.

Since the Hamiltonian function H_ξ of G depends on $\xi \in \mathfrak{g}$ linearly we may write it as

$$H_\xi(x) = \langle \mu(x), \xi \rangle \tag{2.5.20}$$

where the notation $\langle f, \xi \rangle$ denotes the value of f at $\xi \in \mathfrak{g}$ and $\mu(x)$ belongs to $\mathfrak{g}^*$, the dual of the Lie algebra $\mathfrak{g}$. Therefore there is a map

$$\mu : \mathcal{M} \mapsto \mathfrak{g}^* \tag{2.5.21}$$

for any Hamiltonian action of G on $\mathcal{M}$. This is called the moment map .

If we have the Hamiltonian action of a Lie group G on $\mathcal{M}$ which leaves the Hamiltonian $H(x)$ invariant, the quadruple

$$\{\mathcal{M}, \{\ ,\ \}, H, G\} \tag{2.5.22}$$

is called a Hamiltonian system with G-symmetry. There is an important property of Hamiltonian system with G-symmetry [166]

If the Hamiltonian $H(x)$ is invariant under a Hamiltonian action of a Lie group G on $\mathcal{M}$, then the moment map $\mu(x)$ is an integral of motion.

This is a generalization of the well-known Noether's theorem. Since the symmetries give rise to the integrals of motion, one can reduce the dynamical system to one with fewer degrees of freedom [167, 166, 168, 169]. Suppose we have a Hamiltonian action of G on a symplectic manifold $\mathcal{M}$ and the corresponding moment map $\mu : \mathcal{M} \mapsto \mathfrak{g}^*$. We consider the inverse image of a point $c \in \mathfrak{g}^*$ for μ and represent this set by $\tilde{\mathcal{M}}_c$

$$\tilde{\mathcal{M}}_c = \mu^{-1}(c). \tag{2.5.23}$$

We require that c is a regular value of μ. This implies that either the differential of μ at every point of $\tilde{M}_c$ maps the tangent space to $\mathcal{M}$ onto $\mathfrak{g}^*$ or $\tilde{M}_c$ is empty [9]. In this case $\tilde{\mathcal{M}}_c$ is a smooth submanifold of $\mathcal{M}$. The isotropy subgroup of c consists of the elements $\mathfrak{g}$ of G for which

$$\mathrm{Ad}^*_{\mathfrak{g}} c = c. \tag{2.5.24}$$

Put in another way, the isotropy subgroup is the subgroup relative to the coadjoint action which leaves $\tilde{\mathcal{M}}_c$ invariant. Let us denote this isotropy subgroup by

$$G_c = \left\{ g : \mathrm{Ad}^*_{\mathfrak{g}} c = c \right\}. \tag{2.5.25}$$

Now that the space $\tilde{\mathcal{M}}_c$ decomposes into orbits of the action of G, we can define the reduced phase space by

$$\mathcal{M}_c = \tilde{\mathcal{M}}_c / G_c. \tag{2.5.26}$$

It has been shown in [166, 168] that if the isotropy subgroup G_c is compact and acts on $\tilde{\mathcal{M}}_c$ without fixed points, the reduced phase space (2.5.26) is shown to symplectic manifold and that the reduced field, the vector field on the reduced phase space $\mathcal{M}_c$ remains Hamiltonian vector field on it and the corresponding Hamiltonian function pulled back to $\tilde{\mathcal{M}}_c$ coincides with the original Hamiltonian function restricted to $\tilde{\mathcal{M}}_c$.

An Abelian version of the Lagrangian reduction with the integrals of motion was firstly proposed by Routh [171]. Recall that there are two formulations for the classical dynamical system; the Lagrangian formalism and the Hamiltonian formalism. The Lagrangian is a functional of coordinates and their time derivatives and it leads to the equations of motion as a set of second order differential

[9] In [170] it has been discussed that almost all c are regular values.

equations while the Hamiltonian is a functional of coordinates and their canonical momenta and leads to the equations of motion as a set of first order differential equations at the cost of the twice number of the equations.

Suppose we have a system whose Lagrangian is independent of some subset of coordinates. We will refer them as cyclic coordinates and denote by y^i and the remaining non-cyclic ones by x^i. The Lagrangian can be written as

$$L(x^i, y^i, \dot{x}^i, \dot{y}^i; t) = L(x^i, \dot{x}^i, \dot{y}^i; t). \tag{2.5.27}$$

Note that the canonical momenta of the cyclic coordinates y^i

$$p_{y^i} = \frac{\partial L}{\partial \dot{y}^i} \tag{2.5.28}$$

are conserved quantities. In this case the differential equations associated with these momenta are trivial and therefore the Hamiltonian formulation is more advantageous.

The Routhian R is regarded as the new Lagrangian , which is the mixture of the Lagrangian with the Hamiltonian. More precisely it is defined by setting $p_{y^i} = h_i =$ constant and performing a partial Legendre transformation on the cyclic coordinates y_i

$$R(x^i, \dot{x}^i, h_i; t) := L - \sum_i h_i \dot{y}^i. \tag{2.5.29}$$

Let us consider the Euler-Lagrange expressions for the Routhian

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{x}^i}\right) - \frac{\partial R}{\partial x^i} \\
=&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i} + \frac{\partial L}{\partial \dot{y}^i}\frac{\partial \dot{y}^i}{\partial \dot{x}^i}\right) - \frac{d}{dt}\left(h_i \frac{\partial \dot{y}^i}{\partial \dot{x}^i}\right) \\
&-\left(\frac{\partial L}{\partial x^i} + \frac{\partial L}{\partial \dot{y}^i}\frac{\dot{y}^i}{\partial x^i}\right) - h^i \frac{\partial \dot{y}^i}{\partial x^i}.
\end{aligned} \tag{2.5.30}$$

The first and fourth terms cancel by the original Euler-Lagrange equations and the remaining terms vanish by the definition of the canonical momenta $h_i = \frac{\partial L}{\partial \dot{y}^i}$.

This shows that the Euler-Lagrange equations for $L(x, \dot{x}, \dot{y})$ together with the conserved quantities $h_i = p_{y^i}$ are equivalent to the Euler-Lagrange equations for the Routhian $R(x, \dot{x})$. The Euler-Lagrange equations for the Routhian are called the reduced Euler-Lagrange equations because the phase space $\mathcal{M}$ with variables $\{x^i, y^i\}$ is now reduced to the small phase space $\tilde{\mathcal{M}}$ with variables $\{x^i\}$ [10]. Note

[10]In other words the naive substitution of the conserved quantities into the original Lagrangian spoils the role of the Lagrangian.

that the Hamilton equations for the cyclic coordinates yield the trivial statement; the constant property of h_i (i.e. $\dot{h}_i = 0$) and the definition of h_i (i.e. $h_i = \frac{\partial L}{\partial \dot{y}^i}$).

Now let us go back to the gauged mechanical Lagrangian (2.5.3) and apply the Routh reduction [11]. We will parametrize the complex variable z as $z = qe^{i\varphi}$ where $q \geq 0$ and $0 \leq \varphi < 2\pi$ are real variables. We then can write the Lagrangian (2.5.3) as

$$L = \frac{1}{2}\dot{q}^2 + \frac{1}{2}(q\dot{\varphi})^2 + q\dot{\varphi}A_0 + \frac{1}{2}q^2A_0^2 + cA_0. \tag{2.5.31}$$

By choosing the temporal gauge $A_0 = 0$, we get

$$L = \frac{1}{2}\dot{q}^2 + \frac{1}{2}(q\dot{\varphi})^2 \tag{2.5.32}$$

and the Gauss law constraint

$$\phi = q^2\dot{\varphi} + c = 0. \tag{2.5.33}$$

Note that the conserved quantity $h := \frac{\partial L}{\partial \dot{\varphi}} = q^2\dot{\varphi}$ appears in the Gauss law. The Gauss law constraint is the result of fixing the gauge action on the phase space. Thus it is interpreted as the moment map condition. Since the variable φ is cyclic coordinate, we can now apply the Routh reduction (2.5.29) and derive the reduced action. We find the new Lagrangian as the Routhian

$$R = \frac{1}{2}\left(\dot{q}^2 - \frac{c^2}{q^2}\right). \tag{2.5.34}$$

Again this is exactly the DFF-model Lagrangian (2.5.2) (or (2.5.10)) as expected. Therefore upon the reduction procedure of the gauged mechanical model we get the conformal mechanics (DFF-model).

2.6 Black hole

An interesting connections between black holes and conformal mechanical modelds have been proposed in [65] [12]. Let us consider the $d = 4$ Einstein-Maxwell theory which has the action

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R - F^2\right). \tag{2.6.1}$$

[11] The application of the Routh reduction in the gauged mechanical systems was discussed in [51].

[12] Also see [66] for the conjectural relation between black holes and the Calogero model.

The theory admits a single extreme Reissner-Nordström black hole solution with the metric in isotropic coordinate

$$ds^2 = -\left(1 + \frac{|Q|l_p}{\rho}\right)^{-2} dt^2 + \left(1 + \frac{|Q|l_p}{\rho}\right)^2 \left(d\rho^2 + \rho^2 d\Omega^2\right) \tag{2.6.2}$$

and the gauge field

$$A = \left(1 + \frac{|Q|l_p}{\rho}\right)^{-1} dt \tag{2.6.3}$$

where Q is the black hole charge, l_p is the Planck length with the black hole mass $M = \frac{|Q|}{l_p}$, and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the $SO(3)$ invariant metric on S^2. In the near-horizon limit the metric (2.6.2) becomes the Bertotti-Robinson (BR) metric

$$ds^2 = -\left(\frac{\rho}{|Q|l_p}\right)^2 dt^2 + \left(\frac{|Q|l_p}{\rho}\right)^2 d\rho^2 + \left(|Q|l_p\right)^2 d\Omega^2, \tag{2.6.4}$$

which is $SO(1,2) \times SO(3)$ invariant conformally flat metric on $\mathrm{AdS}_2 \times S^2$. Defining the horospherical coordinate as $(t, \phi = \frac{\rho}{Ql_p})$ for AdS_2 part, we can express the BR metric (2.6.4) as

$$ds^2 = -\phi^2 dt^2 + \frac{\left(|Q|l_p\right)^2}{\phi^2} d\phi^2 + \left(|Q|l_p\right)^2 d\Omega^2 \tag{2.6.5}$$

where the quantity $|Q|l_p$ is interpreted as the S^2 radius and also as the radius of the curvature of the AdS_2 space. To go further, let us introduce a new radial coordinate r by

$$\phi = \left(\frac{2M}{r}\right)^2. \tag{2.6.6}$$

Putting together the black hole solutions (2.6.2) and (2.4.3) now become

$$ds^2 = -\left(\frac{2M}{r}\right)^4 dt^2 + \left(\frac{2M}{r}\right)^2 dr^2 + M^2 d\Omega^2, \tag{2.6.7}$$

$$A = \left(\frac{2M}{r}\right)^2 dt \tag{2.6.8}$$

where we have chosen the unit so that $l_p = 1$ and $M = |Q_p|$.

Now we consider the test particle with mass m and charge q. The world-line action of the particle is

$$S = -m \int ds + q \int A. \tag{2.6.9}$$

Putting the black hole solutions (2.6.7) and (2.6.8) into (2.6.9), we find the action

$$S = \int dt \left(\frac{2M}{r}\right)^2 \left[q - m\sqrt{1 - \left(\frac{2M}{r}\right)^{-2} \dot{r}^2 - M^2 \left(\frac{2M}{r}\right)^{-4} (\dot{\theta}^2 + \sin^2\theta \dot{\varphi}^2)} \right]. \tag{2.6.10}$$

The action is invariant under the conformal transformations [74]

$$\delta t = f(t) + c\left(\frac{1}{M^2}\right) r^4 = a + bt + ct^2 + c\left(\frac{1}{M^2}\right) r^4, \tag{2.6.11}$$

$$\delta r = \frac{1}{2}\dot{f} r = \frac{1}{2}(b + 2ct) r, \tag{2.6.12}$$

$$\delta\theta = \delta\varphi = 0. \tag{2.6.13}$$

The corresponding conformal generators, the Hamiltonian H, the dilatation operator D and the conformal boost operator K are given by

$$\begin{aligned} H &= \left(\frac{2M}{r}\right)^2 \left[\sqrt{m^2 + \frac{r^2 p_r^2 + 4L^2}{4M^2}} - q\right] \\ &= \frac{p_r^2}{2f} + \frac{m\gamma}{2r^2 f}, \end{aligned} \tag{2.6.14}$$

$$D = -\frac{1}{4}\left(r p_r + p_r r\right), \tag{2.6.15}$$

$$K = \frac{1}{2} f r^2 \tag{2.6.16}$$

where we have introduced

$$L^2 = p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}, \tag{2.6.17}$$

$$f = \frac{1}{2}\left[\sqrt{m^2 + \frac{1}{4M^2}\left(r^2 p_r^2 + 4L^2\right)} + q\right], \tag{2.6.18}$$

$$\gamma = 4M^2 \frac{m^2 - q^2}{m} + \frac{4L^2}{m}. \tag{2.6.19}$$

It can be shown that three generators H, D and K form the one-dimensional conformal $\mathfrak{sl}(2,\mathbb{R})$ algebra under the Poisson brackets.

It has been pointed out [65] that this conformal mechanical model (2.6.10) give rise to the DFF-model (2.1.2) in the specific limit [13]. Considering the limit

$$M \to \infty, \qquad (m - q) \to 0, \qquad M^2(m-q) = \text{fixed} \tag{2.6.20}$$

[13]However, the physical meaning of this particular limit is not clear and we will see that the mechanical model (2.6.10) and the DFF-model (2.1.2) can be realized as two different non-linear realizations of the one-dimensional conformal group $SL(2,\mathbb{R})$.

and noting that $f \to m$ in this limit, we obtain the DFF Hamiltonian

$$H = \frac{p_r^2}{2m} + \frac{\gamma}{2r^2} \tag{2.6.21}$$

with the coupling constant

$$\gamma = 8M^2(m-q) + \frac{4l(l+1)}{m}. \tag{2.6.22}$$

Here $l(l+1), l \in \mathbb{Z}$ is the quantum number of the operator L^2. Note that this quantization corresponds to the freezing of the S^2 angles, θ, φ, i.e. $\theta =$ const., $\varphi =$ const. Therefore the DFF-model (2.6.21) describes the radial motion of the $\mathrm{AdS}_2 \times S^2$ particle, i.e. the particle near the horizon of the extreme Reissner-Nordström black hole in the limit (2.6.20).

Let us discuss the procedure proposed by DFF to cure the problem of the absence of the ground state for the Hamiltonian H from the perspective of the particle motion near the black hole horizon. Firstly we see that the metric (2.6.5) is singular at $\phi = 0$, however, this is just a coordinate singularity and $\phi = 0$ is a non-singular degenerate Killing horizon of the time-like Killing vector field $\frac{\partial}{\partial t}$. To see this we recall the definition of the AdS_2 space as a Lobachevski-like embedded surface in a three dimensional Minkowski space (see Figure 2.8)

$$-(x^0)^2 + (x^1)^2 - (x^2)^2 = -R^2. \tag{2.6.23}$$

Using the hypersurface coordinate (ϕ, t) defined by

$$x^0 = t\phi, \tag{2.6.24}$$

$$x^+ = x^2 + x^1 = \frac{R^2 - t^2\phi^2}{R\phi}, \tag{2.6.25}$$

$$x^- = x^2 - x^1 = R\phi, \tag{2.6.26}$$

we obtain the AdS_2 factor of the BR metric (2.6.5) with $|Q| l_p = R$. The horospherical coordinates (t, ϕ) can only cover the half of the AdS_2 region. At $\phi = 0$ the metric (2.6.5) is singular and $\phi > 0$ or $\phi < 0$ should be chosen. Correspondingly the coordinate x^- is restricted to $x^- > 0$ or $x^- < 0$. Since the coordinate x^0, x^+ and x^- are smooth on the hypersurface, at the horizon $\phi = 0$ the time coordinate t is ill-defined. To avoid such situation, let us define new coordinates

$$t_1 = \frac{1}{2}(x^+ + x^-), \qquad t_2 = x^0, \qquad \mathfrak{t} = t_1 + it_2, \tag{2.6.27}$$

$$r = \frac{1}{2}(x^+ - x^-). \tag{2.6.28}$$

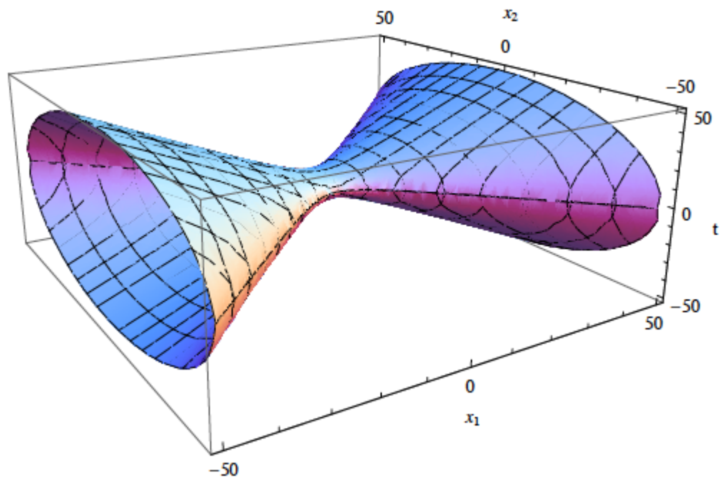


Figure 2.8: AdS_2 space viewed as a hyperboloid of one sheet in a three dimensional Minkowski space.

Then the equation (2.6.23) becomes

$$-|\mathfrak{t}|^2 + r^2 = -R^2 \tag{2.6.29}$$

and thus we can write

$$\mathfrak{t} = e^{\frac{i\tau}{R}}\sqrt{R^2 + r^2} \tag{2.6.30}$$

with $\tau \in \mathbb{R}$ being a new coordinate. In terms of these coordinates the BR metric (2.6.5) can be written as

$$ds^2 = -\left(\frac{R^2 + r^2}{R^2}\right) d\tau^2 + \left(\frac{R^2}{R^2 + r^2}\right) dr^2 + R^2 d\Omega^2. \tag{2.6.31}$$

In fact this shows that the horizon is not a true singularity as we mentioned.

Now we want to get further insights of conformal mechanics from black hole viewpoint. As seen from the expression (2.6.24), the classical analog of an eigenstate vector of the Hamiltonian H in conformal mechanics is an orbit of a time-like Killing vector $k = \frac{\partial}{\partial t}$ in the AdS_2 region outside the horizon ($\phi \neq 0$) and the energy eigenvalue E is the value of k^2. This implies that the ground state $|E = 0\rangle$ of H with $E = 0$ in conformal mechanics corresponds to the orbit of k with $k^2 = 0$ which is a null geodesic generator of the event horizon. Therefore the absence of the ground state $|E = 0\rangle$ can be interpreted as the fact that the orbit of $k^2 = 0$ cannot be covered by the static coordinate t as we discussed.

In classical general relativity it is a general procedure to demonstrate that the horizon is not a true singularity by changing the coordinate system. Note that the AdS_2 isometry $SO(1,2)$ is linearly realized on the coordinates (x^0, x^1, x^2) as rotations $\delta x^\mu = \Lambda^\mu{}_\nu x^\nu$ whose generators $J^{\mu\nu} = i x^{[\mu}\partial^{\nu]}$ form the $\mathfrak{so}(1,2)$ algebra

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(\eta^{\mu[\rho}J^{\sigma]\nu} - \eta^{\nu[\rho}J^{\sigma]\mu}\right) \tag{2.6.32}$$

with $\eta^{\mu\nu} = \mathrm{diag}(-1,+1,-1)$. Then we can find new generators in our new coordinates (t_1, t_2, r) and the corresponding operators in the DFF-model as follows:

1. rotation: $t_1 \leftrightarrow t_2$

   This rotation is expressed as the $U(1)$ rotation of the complex coordinate $\mathfrak{t}$

   $$\mathfrak{t} = e^{\frac{i\tau'}{R}}\sqrt{R^2 + r^2} = e^{i\alpha}\mathfrak{t} \tag{2.6.33}$$

   with $\alpha \in \mathbb{R}$ being the infinitesimal parameter. and thus yields the time τ translation

   $$\tau' = \tau + R\alpha. \tag{2.6.34}$$

   Since it generates compact rotation in the non-compact $SO(1,2)$ symmetry group, the corresponding generator $J^{t_1 t_2}$ is identified with the L_0 in the conformal mechanics.

2. rotation: $(t_1, t_2) \leftrightarrow r$

   In this case the rotations are expressed as two boost operations

   $$\delta\mathfrak{t} = \beta, \qquad \delta r = \frac{1}{2}\left(\beta\mathfrak{t}^* + \beta^*\mathfrak{t}\right) \tag{2.6.35}$$

   where $\beta \in \mathbb{C}$ is the infinitesimal parameter. The complexified generator $J^{t_1 r} \pm i J^{t_2 r}$ can be regarded as $L_\pm$ in the DFF conformal mechanics.

Therefore from the black hole perspective the DFF prescription can be thought of as the different choice of time coordinates in which the world-lines of static particles can pass through the black hole horizon.

2.7 Non-linear realization

The non-linear realization [155, 156, 157, 172] is a useful method to construct the non-linear invariant Lagrangian. The basic idea is the following:

1. Start from the Lie (super)group G that reflects the symmetry in the theory.

2. Find the invariants under G from the Cartan forms ω belonging to the (super)coset G/H where H is the stability subgroup of G.

3. Construct the invariant Lagrangian under G in terms of the Goldstone fields associated with the (super)coset Cartan forms ω.

By making use of the non-linear realization, it has been showed [74] that the DFF-model (2.1.2) and the black hole conformal mechanics (2.6.10) are essentially equivalent modulo redefinition of the time coordinates and the variables at classical level. Moreover the non-linear realization approach provides us with a powerful method to construct the irreducible supermultiplets for superconformal mechanical models. Much of the irreducible constraints and transformation laws can be automatically obtained from the non-linear realization technique.

Let G be a Lie (super)group and H be its subgroup. We call Y_i the generator of H. and X_i the generator of the remaining generators. We assume that the commutator $[X_i, Y_j]$ is a linear combination of X_i alone

$$[X_i, Y_j] = f_{ij}^k X_k \tag{2.7.1}$$

where f_{ij}^k are the structure constants. (2.7.1) implies that the remaining generators X_i form the representation of the subgroup H, which we will call the stability subgroup. Then a group element g of G can be represented uniquely by [155, 156, 157, 172]

$$\begin{aligned} g &= e^{x \cdot X} h \\ &= \tilde{g} h \end{aligned} \tag{2.7.2}$$

where h is an element of H, $x \cdot X := \sum_i x^i X_i$ and x^i are the coordinates parametrizing the coset space G/H. The actions of the group G can be realized by left multiplications on the coset G/H. This fact is the key statement of the non-linear realization method.

Now we want to apply the basic statement (2.7.2) to find the non-linear realization of G symmetry group and to construct the G-invariant Lagrangian. In order to achieve this, we classify the parameters x^i in two classes

$$x^i = \begin{cases} \text{(super)space coordinates} & \text{if } X_i \text{ is (super)translation} \\ \text{Goldstone (super)fields} & \text{otherwise.} \end{cases} \tag{2.7.3}$$

In other words, the (super)space and time coordinates are the parameters of the (super)translation generators while the remaining coset parameters are treated as the (super)fields. We should note that the number of the Goldstone (super)fields is not always same as the number of the coset generators. In fact some of the Goldstone (super)fields may be expressed by other Goldstone (super)fields. This phenomenon is known as the inverse Higgs effect [173].

For one-dimensional conformal algebra $\mathfrak{sl}(2,\mathbb{R})$ given by (2.1.23)-(2.1.25), the stability subgroup H is trivial and thus the coset is parametrized by the coordinates for all generators

$$\tilde{g} = e^{itH}e^{iz(t)K}e^{iu(t)D}. \tag{2.7.4}$$

Since we are now considering one-dimensional field theory, i.e. quantum mechanics, we introduce time coordinate t for the Hamiltonian H. The remaining two coordinates $z(t)$ for K and $u(t)$ for D are Goldstone fields.

Then we can find the realization of the conformal group in our coset (2.7.4). The translation H is realized by acting on g by $g_0 = e^{iaH}$ from the left

$$g_0 \cdot \tilde{g} = e^{iaH} \cdot e^{itH}e^{iz(t)K}e^{iu(t)D}. \tag{2.7.5}$$

We thus obtain the translations

$$\delta t = a, \tag{2.7.6}$$

$$\delta u = 0, \qquad \delta z = 0. \tag{2.7.7}$$

The dilatation D is realized by acting on g by $g_0 = e^{ibD}$ from the left

$$\begin{aligned} g_0 \cdot \tilde{g} &= e^{ibD} \cdot e^{itH}e^{iz(t)K}e^{iu(t)D} \\ &= (e^{ibD}e^{itH}e^{-ibD})(e^{ibD}e^{izK}e^{-ibD})e^{ibD}e^{iuD} \\ &= e^{i(t+bt)H}e^{i(z+bz)K}e^{i(u+b)D}. \end{aligned} \tag{2.7.8}$$

One finds the dilatations

$$\delta t = bt, \tag{2.7.9}$$

$$\delta u = b, \qquad \delta z = -bz. \tag{2.7.10}$$

The conformal boost K is realized by acting on g by $g_0 = e^{icK}$ from the left

$$\begin{aligned} g_0 \cdot \tilde{g} &= e^{icK} \cdot e^{itH}e^{izK}e^{iuD} \\ &= e^{itH}(e^{-itH}e^{icK}e^{itH})e^{izK}e^{iuD} \\ &= e^{i(t+ct^2)H}e^{i(z+c-2ctz)K}e^{i(u+2ct)D}. \end{aligned} \tag{2.7.11}$$

We get the conformal boost transformations

$$\delta t = ct^2, \tag{2.7.12}$$

$$\delta u = 2ct, \qquad \delta z = c - 2ctz. \tag{2.7.13}$$

Let us discuss the construction of the G-invariant expressions. To this end we introduce the Maurer-Cartan form Ω for the coset G/H defined by

$$\begin{aligned}\Omega &= \tilde{g}^{-1} d\tilde{g} \\ &= e^{-x\cdot X} d(e^{x\cdot X}) \\ &= i\omega^i X_i + i\tilde{\omega}^i Y_i.\end{aligned} \tag{2.7.14}$$

Then one can show [155, 156, 157, 172] that the forms ω^i on the coset transform homogeneously and therefore any expression constructed with ω^i is invariant under G. On the other hand it turns out [155, 156, 157, 172] that the forms $\tilde{\omega}^i$ on the stability subgroup H transform like connections and can be used to construct covariant derivatives.

The Maurer-Cartan forms for the coset (2.7.4) is

$$\Omega = \tilde{g}^{-1} d\tilde{g} = i\left(\omega_H H + \omega_K K + \omega_D D\right) \tag{2.7.15}$$

where

$$\omega_H = e^{-u}, \tag{2.7.16}$$

$$\omega_D = du - 2zdt, \tag{2.7.17}$$

$$\omega_K = e^u \left(dz + z^2 dt\right). \tag{2.7.18}$$

Alternatively the Maurer-Cartan forms associated with the generators T_i, $i = 0, 1, 2$ defined (2.1.35) are given by

$$\omega_0 = \frac{1}{m}\omega_K + m\omega_K, \tag{2.7.19}$$

$$\omega_1 = \frac{1}{m}\omega_K - m\omega_H, \tag{2.7.20}$$

$$\omega_2 = \omega_D \tag{2.7.21}$$

where m is a constant parameter. Since the form ω_1, ω_2 are the coset forms, they transform homogeneously. we can impose the following $SL(2,\mathbb{R})$ invariant conditions [14]

$$\omega_1 = 0, \tag{2.7.22}$$

$$\omega_2 = 0. \tag{2.7.23}$$

[14] Although the choice of $\omega_0 = 0$ also yields the $SL(2,\mathbb{R})$ invariant constraint, it does not lead to the good dynamical systems.

The first condition (2.7.22) turns out to be the equation of motion for the system and the second condition (2.7.23) leads to the relation

$$z = \frac{1}{2}\dot{u}, \tag{2.7.24}$$

which implies that the Goldstone field $z(t)$ can be represented by the other Goldstone field $u(t)$. This is the inverse Higgs effect [173]. In terms of the remaining Maurer-Cartan forms ω_0, one can construct the $SL(2,\mathbb{R})$ action [174]

$$\begin{aligned} S &= -c\int \omega_0 \\ &= -c\int dt\left[\frac{c}{m}e^{u}\left(\dot{z}+z^2\right)+mce^{-u}\right] \\ &= \int dt\left[x^2-\frac{c^2}{x^2}\right] \end{aligned} \tag{2.7.25}$$

where we have used the relation (2.7.24) and introduced

$$x := \mu^{-\frac{1}{2}}e^{\frac{u}{2}}, \tag{2.7.26}$$

$$\mu = \frac{m}{c} \tag{2.7.27}$$

The action (2.7.25) is just the DFF-model (2.1.4).

Let us introduce

$$\hat{K} = mK - \frac{1}{m}H, \qquad \hat{D} = mD \tag{2.7.28}$$

where m is a constant parameter. Then the one-dimensional conformal $\mathfrak{sl}(2,\mathbb{R})$ algebra can be written as

$$[H,\hat{D}] = imH, \tag{2.7.29}$$

$$[\hat{K},\hat{D}] = -2iH - im\hat{K}, \tag{2.7.30}$$

$$[H,\hat{K}] = 2i\hat{K}. \tag{2.7.31}$$

Defining the corresponding coset by

$$\tilde{g} = e^{i\tau H}e^{i\phi(\tau)\hat{D}}e^{i\Omega(\tau)\hat{K}} \tag{2.7.32}$$

and acting the corresponding elements on the coset (2.7.32) from the left, one can find the $SL(2,\mathbb{R})$ transformations for the new coordianates

$$\delta\tau = a + b + c\tau + \frac{1}{m^2}ce^{2m\phi}, \tag{2.7.33}$$

$$\delta\phi = \frac{1}{m}\left(b+2c\tau\right), \tag{2.7.34}$$

$$\delta\Omega = \frac{1}{m}ce^{m\phi} \tag{2.7.35}$$

where a, b, c are infinitesimal constant parameters. Note that the transformation of the new time coordinate τ contains the additional term $\frac{1}{m}ce^{2m\phi}$, which is similar to (2.6.11). We can read the Maurer-Cartan forms for the coset (2.7.32) [74]

$$\hat{\omega}_H = \frac{1+\Lambda^2}{1-\Lambda^2}e^{-m\phi}d\tau - 2\frac{\Lambda}{1-\Lambda^2}d\phi, \tag{2.7.36}$$

$$\hat{\omega}_D = \frac{1+\Lambda^2}{1-\Lambda^2}d\phi - 2\frac{\Lambda}{1-\Lambda^2}e^{-m\phi}d\tau, \tag{2.7.37}$$

$$\hat{\omega}_K = m\frac{\Lambda}{1-\Lambda^2}\left(\Lambda e^{-m\phi}d\tau - d\phi\right) + \frac{d\Lambda}{1-\Lambda^2} \tag{2.7.38}$$

where

$$\Lambda = \tanh\Omega. \tag{2.7.39}$$

Let us impose the $SL(2,\mathbb{R})$ invariant conditions as

$$\hat{\omega}_D = 0, \tag{2.7.40}$$

which results in the inverse Higgs effect [173]

$$\partial_\tau\phi = 2e^{-m\phi}\frac{\Lambda}{1+\Lambda^2}. \tag{2.7.41}$$

So the Goldstone field Λ or Ω can be expressed by ϕ

$$\Lambda = \partial_\tau\phi e^{m\phi}\frac{1}{1+\sqrt{1-e^{2m\phi}(\partial_\tau\phi)^2}}. \tag{2.7.42}$$

Using the non-vanishing Maurer-Cartan forms, one can construct the $SL(2,\mathbb{R})$ invariant action [74]

$$\begin{aligned} S &= \int\left[(q-\tilde{\mu})\hat{\omega}_H - \frac{2}{m}q\hat{\omega}_K\right] \\ &= -\int d\tau e^{-m\phi}\left[\tilde{\mu}\sqrt{1-e^{2m\phi}(\partial_\tau\phi)^2} - q\right]. \end{aligned} \tag{2.7.43}$$

We see that the action (2.7.43) is the conformal mechanical model (2.6.10) which describes the radial motion of the $AdS_2 \times S^2$ particle [65].

Therefore we see that the two mechanical model (2.6.10) and the DFF-model (2.1.2) can be realized as two different non-linear realizations of the one-dimensional conformal group $SL(2,\mathbb{R})$. From this point of view, we can conclude that the two conformal mechanical models are equivalent up to the redefinition of the time coordinate and the physical variable.

2.8 Multi-particle conformal mechanics

Let us study the conformal mechanical models with many degrees of freedom for different particles. Generically n-particle quantum mechanics can be viewed as a sigma-model with an n-dimensional target space $\mathcal{M}$. So we will see the conditions [69] for the target space $\mathcal{M}$ for the existence of conformal operators D and K.

Consider the Hamiltonian

$$H = \frac{1}{2} p_a^\dagger g^{ab} p_b + V(x). \tag{2.8.1}$$

Here $g_{ab}(x)$ is the metric of the target space $\mathcal{M}$ where the indices $a, b = 1, \cdots, n$ label the particles.

$$p_a = g_{ab} \dot{x}^b \tag{2.8.2}$$

are the canonical momenta obeying

$$[x^a, p_b] = i\delta_{ab}, \qquad [x^a, x^b] = 0, \qquad [p_a, p_b] = 0. \tag{2.8.3}$$

The Hermitian conjugate of p_a are

$$\begin{aligned} p_a^\dagger &= \frac{1}{\sqrt{g}} p_a \sqrt{g} \\ &= p_a - i\Gamma^b_{ba} \end{aligned} \tag{2.8.4}$$

where Γ^c_{ab} is the Christoffel symbol constructed from g_{ab}.

Let us assume that the theory has a dilatational invariance of the form

$$\delta t = bt, \qquad \delta x^a = \frac{1}{2} D^a(x) b, \tag{2.8.5}$$

which is a generalization of (2.1.14) and (2.1.15) with b being an infinitesimal parameter for the dilatation. Then the dilatation generator D is given by

$$D = \frac{1}{4}\left(D^a p_a + p_a^\dagger D^{a\dagger} \right). \tag{2.8.6}$$

Under the canonical relations (2.8.3) we find the commutation relation of the Hamiltonian (2.8.1) and the dilatation generator (2.8.6) as [69]

$$[H, D] = \frac{i}{4} p_a^\dagger \left(\mathcal{L}_D g^{ab} \right) p_b + \frac{i}{2} \mathcal{L}_D V + \frac{i}{8} \nabla^2 \nabla_a D^a \tag{2.8.7}$$

where $\mathcal{L}_d$ is the Lie derivative

$$\mathcal{L}_D g_{ab} = D^c g_{ab,c} + D^c{}_{,a} g_{cb} + D^c{}_{,b} g_{ac}. \tag{2.8.8}$$

From the $\mathfrak{sl}(2,\mathbb{R})$ algebra (2.1.23) and the expressions (2.8.1), (2.8.7), the existence of the dilatation generator D requires that

$$\mathcal{L}_D g_{ab} = 2g_{ab}, \tag{2.8.9}$$

$$\mathcal{L}_D V(x) = -2V(x), \tag{2.8.10}$$

$$\nabla^2 \nabla_a D^a = 0. \tag{2.8.11}$$

A vector field D is called homothetic vector field or similarity vector field on $\mathcal{M}$ [15]. A homothetic vector field generates a similarity transformation group. It is shown that along any integral curve of a homothetic vector field the space-like, time-like or null character of the tangent vector does not change and that there is necessarily a singularity in each orbit of the similarity transformation group [175, 176, 177].

Furthermore the remaining commutation relations (2.1.24) and (2.1.25) lead to [69]

$$\mathcal{L}_D K = 2K, \tag{2.8.13}$$

$$D_a dx^a = dK \tag{2.8.14}$$

respectively. As the solutions to the equations (2.8.13) and (2.8.14), one can express the conformal boost generator K as the norm of D^a

$$K = \frac{1}{2} g_{ab} D^a D^b. \tag{2.8.15}$$

The equation (2.8.14) means that the one-form $D = D_a dx^a$ is exact, however, it is shown [69] that closed homothety vector field D is always exact. So it is enough to impose the closeness condition for the homothetic vector field D

$$d\left(D_a dx^a\right) = 0. \tag{2.8.16}$$

Therefore we can conclude that in order to obtain conformal quantum mechanical sigma-models,

- the target space $\mathcal{M}$ must admit a homothety vector field D whose associated one-form $D_a dx^a$ is closed [16]; (2.8.9) and (2.8.16)

[15] Note that

$$X = \begin{cases} \text{conformal Killing field} & \text{if } \mathcal{L}_X g_{ab} = \rho(x) g_{ab} \\ \text{homothetic vector field} & \text{if } \mathcal{L}_X g_{ab} = c g_{ab} \\ \text{Killing vector field} & \text{if } \mathcal{L}_X g_{ab} = 0 \end{cases} \tag{2.8.12}$$

where $\rho(x)$ is a function on $\mathcal{M}$ and c is a constant on $\mathcal{M}$.

[16] The vector field D with the required properties for conformal mechanical sigma-model is referred to as a closed homothety vector field in [69]

- the potential $V(x)$ must satisfy (2.8.10)
- D^a must obey the vanishing condition (2.8.11).

2.9 Calogero model

One of the most celebrated multi-particle conformal mechanical models is the Calogero model, which is the system of multi-particles scattering on the line with inverse-square potentials [134, 135]. The Hamiltonian is given by

$$H = \sum_{i=1}^{n} \frac{1}{2} p_i^2 + \sum_{i<j} \frac{\gamma}{(x_i - x_j)^2} \tag{2.9.1}$$

where the indices $i = 1, \cdots, n$ label the particles and γ is a coupling constant.

In what follows we will discuss that the Calogero model and its generalization can be obtained from gauged matrix models. This formulation not only indicates the intimite relationship between the conformal quantum mechanical models and the gauged quantum mechanical models but also provides us with non-trivial (super)conformal mechanical models.

Let us start with the gauged matrix model action

$$S = \int dt \left[\mathrm{Tr}(DXDX) + \frac{i}{2} \left(\overline{Z}DZ - D\overline{Z}Z \right) + c\mathrm{Tr}A \right]. \tag{2.9.2}$$

Here $X_a^b(t)$, $\overline{X}_a^b = X_b^a$, $a = 1, \cdots, n$ are the bosonic Hermitian $(n \times n)$ matrices, $Z_a(t)$, $\overline{Z}^a = (\overline{Z}_a)$ are bosonic complex matrices and $A_a^b(t)$, $(\overline{A}_a^b) = A_b^a$ are the $U(n)$ gauge fields with n^2 component fields. c is a real constant parameter. In the action (2.9.2) the covariant derivatives are defined as

$$DX := \dot{X} + i[A, X], \qquad DZ := \dot{Z} + iAZ, \qquad D\overline{Z} := \dot{\overline{Z}} - i\overline{Z}A. \tag{2.9.3}$$

Note that in the third term, the Fayet-Iliopoulos term the non-abelian traceless part of the gauge field A drops out and only the $U(1)$ part has contributions.

The action (2.9.2) is invariant under the one-dimensional $SL(2,\mathbb{R})$ conformal transformations

$$\delta t = f(t), \qquad \delta\partial_0 = -\dot{f}\partial_0, \tag{2.9.4}$$

$$\delta X = \frac{1}{2}\dot{f}X, \qquad \delta Z = 0, \qquad \delta A = -\dot{f}A \tag{2.9.5}$$

where $f(t) = a + bt + ct^2$ with a, b, c being infinitesimal real parameters. The action (2.9.3) is invariant under the $U(n)$ gauge transformations

$$X \to gXg^{-1}, \tag{2.9.6}$$

$$Z \to gZ, \qquad \overline{Z} \to \overline{Z}g^{-1}, \tag{2.9.7}$$

$$A \to gAg^{-1} + i\dot{g}g^{-1} \tag{2.9.8}$$

where $g \in U(n)$. Let us impose a partial gauge fixing condition

$$X_a^b = x_a \delta_a^b \tag{2.9.9}$$

where x_a are real component fields since X are Hermitian matrices. Then the action (2.9.3) becomes

$$S = \int dt \sum_{a,b} \left[\dot{x}_a \dot{x}_a + \frac{i}{2} \left(\overline{Z}^a \dot{Z}_a - \dot{\overline{Z}}^a Z_a \right) \right.$$
$$\left. + (x_a - x_b)^2 A_a^b A_b^a - \overline{Z}^a A_a^b Z_b + c A_a^a \right]. \tag{2.9.10}$$

By noting that the action (2.9.10) is invariant under the $U(1)^n$ gauge transformations

$$x_a \to x_a, \tag{2.9.11}$$

$$Z_a \to e^{i\lambda_a} Z_a, \qquad \overline{Z}^a \to e^{-i\lambda_a} \overline{Z}^a, \tag{2.9.12}$$

$$A_a^a \to A_a^a - \dot{\lambda}_a \tag{2.9.13}$$

where $\lambda_a(t)$ are local parameters, we further impose the gauge fixing condition as

$$Z_a = \overline{Z}^a. \tag{2.9.14}$$

Then the action (2.9.10) reduces to

$$S = \int dt \sum_{a,b} \left[\dot{x}_a^2 + (x_a - x_b)^2 A_a^b A_b^a - Z_a Z_b A_a^b + c A_a^a \right]. \tag{2.9.15}$$

At this stage we attempt to integrate out the gauge field A. From the action (2.9.15) we obtain the equations of motion for A_a^a and for $A_a^b, a \neq b$ as

$$(Z_a)^2 = c \tag{2.9.16}$$

$$A_b^a = \frac{Z_a Z_b}{2(x_a - x_b)^2}. \tag{2.9.17}$$

Substituting the equations (2.9.16) and (2.9.17) into the action (2.9.15) and rescaling x_a appropriately, we obtain the Calogero model action

$$S = \frac{1}{2} \int dt \left[\sum_a \dot{x}_a^2 - \sum_{a \neq b} \frac{c^2}{(x_a - x_b)^2} \right]. \tag{2.9.18}$$

Chapter 3

Superconformal Mechanics

In this chapter we will proceed to the superextension of the conformal quantum mechanics; the superconformal quantum mechanics. Firstly in section 3.1 we will recall the basic facts about Lie superalgebra and Lie supergroup and will clarify the one-dimensional superconformal group. Then in section 3.2 we will stress that supersymmety in one-dimension possesses many peculiar properties. In section 3.3, 3.4, 3.5 and 3.6 we will review the persistent efforts to construct $\mathcal{N} = 1$, 2, 4 and 8 superconformal quantum mechanics by using the superspace and superfield formalism and also review the interesting topics which are relevant to those superconformal mechanical models.

3.1 Superalgebra and supergroup

In d-dimensional superconformal field theories the ordinary supersymmetry and the conformal symmetry lead to a second supersymmetry. The corresponding generator S_α with $\alpha, \beta, \cdots$ being spinor indices can be found by taking the commutator of the conformal boost operator K_μ with space-time indices $\mu, \nu, \cdots = 0, 1, \cdots, d-1$ and the original supersymmetry Q_α

$$[K_\mu, Q_\alpha] = (\Gamma_\mu)^\beta{}_\alpha S_\beta \tag{3.1.1}$$

where Γ_μ is a d-dimensional gamma matrix. Additionally the ani-commutator of supersymmetries Q_α and S_α generates the bosonic symmetry, the so-called R-symmetry. In general these generators form the superconformal algebras which are isomorphic to the simple Lie superalgebras. Hence it is expected that one can specify the corresponding Lie superalgebras, i.e. the superconformal algebras which characterize the superconformal field theories.

3.1.1 Lie superalgebra

A superalgebra is a $\mathbb{Z}_2$-graded algebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$. This means that if $a \in \mathfrak{g}_\alpha$, $b \in \mathfrak{g}_\beta$, $\alpha, \beta \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$, then $ab \in \mathfrak{g}_{\alpha+\beta}$. We say that a is of degree α and write $\deg a = \alpha$. $\mathfrak{g}_{\bar{0}}$ is a Lie algebra, which is called the even or bosonic part of $\mathfrak{g}$ while $\mathfrak{g}_{\bar{1}}$ is called the odd or fermionic part of $\mathfrak{g}$, which is not an algebra.

A Lie superalgebra is the superalgebra endowed with the product operation $[\ ,\ ]$ possessing the following axioms:

1. graded anticommutativity

$$[a, b] = -(-1)^{\alpha\beta}[b, a] \tag{3.1.2}$$

2. generalized Jacobi identity

$$[a, [b, c]] = [[a, b], c] + (-1)^{\alpha\beta}[b, [a, c]] \tag{3.1.3}$$

where $a \in \mathfrak{g}_\alpha$, $b \in \mathfrak{g}_\beta$. The product $[a, b]$ is referred to as the Lie superbracket or supercommutator for two elements $a, b \in \mathfrak{g}$.

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be $\mathbb{Z}_2$-graded vector space where $\dim V_{\bar{0}} = m$ and $\dim V_{\bar{1}} = n$. Then the algebra $\mathrm{End}V$ is endowed with a $\mathbb{Z}_2$-graded superalgebra structure. Hence the Lie superbracket $[\ ,\ ]$ satisfying (3.1.2) and (3.1.3) turns $\mathrm{End}V$ into a Lie superalgebra $\mathfrak{l}(m, n)$. The Lie superalgebra $l(V)$ plays the same role as the general linear Lie algebra in the theory of Lie algebra. Let $e_1, \cdots, e_m; e_{m+1}, \cdots, e_{m+n}$ be a basis of V, formed by the bases of $V_{\bar{0}}$ and $V_{\bar{1}}$. In this basis the matrices of an element a from the Lie superalgebra $\mathfrak{l}(m, n)$ can be written in the form

$$a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{3.1.4}$$

where α and δ are $\mathfrak{gl}(m)$ and $\mathfrak{gl}(n)$ matrices and β and γ are $m \times n$ and $n \times m$ rectangular matrices. On the Lie superalgebra $\mathfrak{gl}(m, n)$ the supertrace is defined by

$$\mathrm{str}(a) = \mathrm{tr}\alpha - \mathrm{tr}\delta. \tag{3.1.5}$$

In terms of the supertrace (3.1.5), we can define the bilinear form B_R associated with the representation R of $\mathfrak{g}$ by

$$B_R(a, b) = \mathrm{str}(R(a), R(b)), \quad \forall a, b \in \mathfrak{g} \tag{3.1.6}$$

where $R(a)$ is the matrix of the elements $a \in \mathfrak{g}$ in the representation R. As a special case the Killing form K can be defined as the bilinear form on $\mathfrak{g}$ associated with the adjoint representation

$$K(a, b) = \mathrm{str}(\mathrm{ad}(a), \mathrm{ad}(b)), \quad \forall a, b \in \mathfrak{g}. \tag{3.1.7}$$

The Lie superalgebra $\mathfrak{g}$ is called simple if it contains no non-trivial ideal. The Lie superalgebra $\mathfrak{g}$ is called semi-simple if it contains no non-trivial solvable ideal. If a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is simple, the representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is faithful and $\{\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}\} = \mathfrak{g}_{\bar{0}}$. If the representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is irreducible, then $\mathfrak{g}$ is simple. Unlike the Lie algebras, semi-simple Lie superalgebra cannot be written as the direct sum of simple Lie superalgebras. However, there is a construction which allows us to build finite-dimensional semi-simple Lie superalgebras in terms of simple ones [178].

It is known that simple Lie superalgebras are classified into two families; the classical Lie superalgebras and (non-classical) Cartan type superalgebras. The simple Lie superalgebra is said to be classical the representation of the Lie algebra $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is completely reducible.

For the classical Lie superalgebras there are further classifications. Firstly the representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ can be either (i) irreducible or (ii) the direct sum of two irreducible representations of $\mathfrak{g}_{\bar{0}}$. The superalgebra of the case (i) is called the type I and that of the case (ii) is called type II. In addition, the Lie superalgebra $\mathfrak{g}$ is called basic if there is a non-degenerate invariant bilinear form, the Killing form K on $\mathfrak{g}$ while strange if it is not basic. The basic Lie superalgebras is divided into (a) four infinite series: $A(m,n)$, $B(m,n)$, $C(n)$ and $D(m,n)$, that is $\mathfrak{sl}(m+1|n+1)$, $\mathfrak{osp}(2m+1|2n)$, $\mathfrak{osp}(2|2n)$ and $\mathfrak{osp}(2m|2n)$; (b) three exceptional series: 40-dimensional $F(4)$, 31-dimensional $G(3)$ and 17-dimensional $D(2,1;\alpha)$ which is a one-parameter family of superalgebras. The strange algebras split into two infinite families $P(n)$ and $Q(n)$.

For the Cartan type superalgebras there are four infinite families $W(n)$, $S(n)$, $H(n)$ $\tilde{S}(n)$, where the first three series are analogous to the corresponding series of simple infinite-dimensional Lie algebra of Cartan type and $\tilde{S}(n)$ is a deformation of $S(n)$.

Summarizing the above, the classification of simple Lie superalgebra is illustrated in Figure 3.1.

3.1.2 Lie supergroup

To begin with, let us introduce a supermatrix. A supermatrix M is defined as the matrix whose entries valued in a Grassmann algebra $\Gamma = \Gamma_{\bar{0}} \oplus \Gamma_{\bar{1}}$ of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.1.8}$$

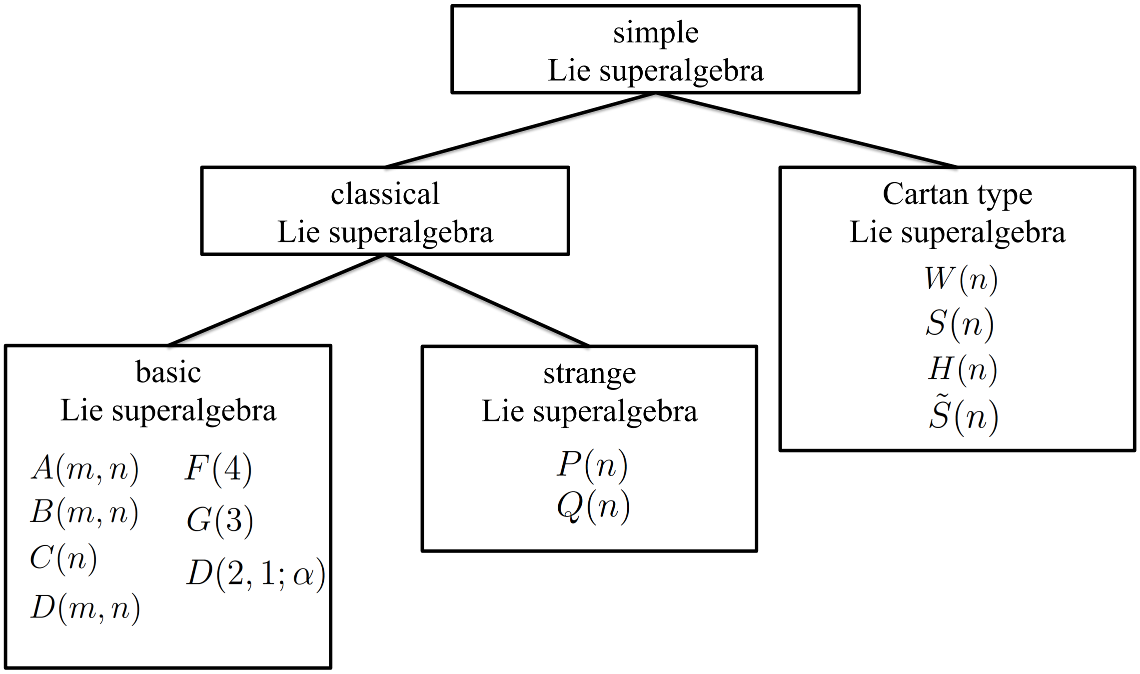


Figure 3.1: The classification of simple Lie superalgebra.

where A,B,C and D are $m \times p$, $m \times q$, $n \times p$ and $n \times q$ matrices respectively. The supermatrix M is said to be even and of degree 0 if $A, D \in \Gamma_{\bar{0}}$ and $B, C \in \Gamma_{\bar{1}}$ whereas it is called odd and of degree 1 if $A, D \in \Gamma_{\bar{1}}$ and $B, C \in \Gamma_{\bar{0}}$.

The general linear supergroup $GL(m|n)$ consists of even invertible supermatrices M and its product is defined by the multiplication rule of the supermatrices:

$$(MN)_{ij} = \sum_{k=1}^{p+q} M_{ik} N_{kj} \tag{3.1.9}$$

where M and N are two $(m+n) \times (p+q)$ and $(p+q) \times (r+s)$ supermatrices and $(MN)_{ij}$ denotes the (i, j) entry of the $(m+n) \times (r+s)$ supermatrix MN.

The operations for the supermatrices are defined as follows:

1. transpose M^t and supertranspose M^{st}

$$M^t = \begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix}, \tag{3.1.10}$$

$$M^{st} = \begin{pmatrix} A^t & (-1)^{\deg M} C^t \\ -(-1)^{\deg M} B^t & D^t \end{pmatrix} = \begin{cases} \begin{pmatrix} A^t & C^t \\ -B^t & D^t \end{pmatrix} & \text{if } M \text{ is even} \\ \begin{pmatrix} A^t & -C^t \\ B^t & D^t \end{pmatrix} & \text{if } M \text{ is odd} \end{cases} \tag{3.1.11}$$

2. supertrace $\mathrm{str}(M)$

$$\mathrm{str}(M) = \mathrm{tr}(A) - (-1)^{\deg M}\mathrm{tr}(D) = \begin{cases} \mathrm{tr}(A) - \mathrm{D} & \text{if } M \text{ is even} \\ \mathrm{tr}(A) + \mathrm{D} & \text{if } M \text{ is odd} \end{cases} \tag{3.1.12}$$

3. superdeterminant $\mathrm{sdet}(M)$

$$\mathrm{sdet}(M) = \frac{\det(A - BD^{-1}C)}{\det(D)} = \frac{\det(A)}{\det(D - CA^{-1}B)} \tag{3.1.13}$$

4. adjoint $M^\dagger$ and superadjoint $M^\ddagger$

$$M^\dagger = (M^t)^*, \tag{3.1.14}$$
$$M^\ddagger = (M^{st})^*. \tag{3.1.15}$$

The relation between the Lie superalgebra $\mathfrak{g}$ and the corresponding Lie supergroup G is analogous to the theory of the Lie algebra. Consider the complex Grassmann algebra $\Gamma(n)$ of order n with n generators $1, \theta_1, \cdots, \theta_n$ obeying the anti-commutation relations $\{\theta_i, \theta_j\} = 0$. If in the element $\eta = \sum_{0 \leq m} \sum_{i_1 < \cdots i_m} \eta_{i_1 \cdots i_m} \theta_{i_1} \cdots \theta_{i_m}$ each complex coefficient $\eta_{i_1 \cdots i_m}$ is an even (odd) value of m, the corresponding element is called even (odd). In general $\Gamma(n)$ can be decomposed into even and odd parts as a vector space; $\Gamma(n) = \Gamma(n)_{\bar{0}} \oplus \Gamma(n)_{\bar{1}}$. The Grassmann envelope $G(\Gamma)$ of the Lie superalgebra $\mathfrak{g}$ is constructed as a formal linear combinations $\sum_i \eta_i a_i$ where a_i is a basis of $\mathfrak{g}$ and $\eta_i \in \Gamma$ such that the elements a_i and η_i are both even or odd. The Lie supergroup G associated with the superalgebra $\mathfrak{g}$ is realized as the exponential mapping of the Grassmann envelope $G(\Gamma)$ of $\mathfrak{g}$; the even generators of the superalgebra $\mathfrak{g}$ corresponds to commuting parameters, i.e. even elements of the Grassmann algebra and the odd generators of the superalgebra $\mathfrak{g}$ to anti-commuting parameters, i.e. odd elements of the Grassmann algebra [179].

3.1.3 Superconformal algebra

The requirements for the corresponding superconformal algebra have been proposed in [12]:

1. The d-dimensional conformal algebra $\mathfrak{so}(d,2)$ should appear as a bosonic factored subgroup.

2. The fermionic generators should be spinor representations of the conformal algebra $\mathfrak{so}(d,2)$.

First of all we can see that these conditions can be satisfied for the simple classical Lie superalgebras. The detail list of the classical Lie superalgebras is given in Table 3.1 [180, 181, 182, 183, 184].

The unitary superalgebra $A(m-1,n-1)$ or $\mathfrak{sl}(m,n)$ with $m > n > 0$ possesses an even part $\mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus \mathfrak{u}(1)$ and an odd part $(\boldsymbol{m},\overline{\boldsymbol{n}}) \oplus (\overline{\boldsymbol{m}},\boldsymbol{n})$ as a representation of the even part. The unitary superalgbra $A(n-1,n-1)$ with $n > 1$ has an even part $\mathfrak{sl}(n) \oplus \mathfrak{sl}(n)$ and an odd part $(\boldsymbol{n},\overline{\boldsymbol{n}}) \oplus (\overline{\boldsymbol{n}},\boldsymbol{n})$.

The orthosymplectic superalgebras consist of three infinite series $B(m,n)$, $C(n+1)$ and $D(m,n)$. The superalgebra $B(m,n)$ or $\mathfrak{osp}(2m+1|2n)$ with $m \geq 0$, $n \geq 1$ possesses an even part $\mathfrak{so}(2m+1) \oplus \mathfrak{sp}(2n)$ and an odd part $(\boldsymbol{2m+1},\boldsymbol{2n})$. The superalgebra $C(n+1)$ or $\mathfrak{osp}(2|2n)$ with $n \geq 1$ contains an even part $\mathfrak{so}(2) \oplus \mathfrak{sp}(2n)$ and an odd part $\boldsymbol{2n} \oplus \boldsymbol{2n}$ as twice the fundamental representation $\boldsymbol{2n}$ of $\mathfrak{sp}(2n)$. The superalgebra $D(m,n)$ or $\mathfrak{osp}(2m|2n)$ with $m \geq 2, n \geq 1$ has an even part $\mathfrak{so}(2m) \oplus \mathfrak{sp}(2n)$ and an odd part $(\boldsymbol{2m},\boldsymbol{2n})$.

The superalgebras $D(2,1;\alpha)$ with $\alpha \neq 0,-1,\infty$ is a one-parameter family of superalgebras of rank 3 and dimension 17. It is a deformation of the superalgebra $D(2,1)$ that corresponds to the case of $\alpha = 1$. It has an even part $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ and an odd part $(\boldsymbol{2},\boldsymbol{2},\boldsymbol{2})$ as the spinor representations of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. The three $\mathfrak{sl}(2)$ factors appear as the anticommutator of the fermionic generators with the relative weights $1,\alpha$ and $1-\alpha$.

The superalgebra $F(4)$ is 40-dimensional algebra of rank 4 and possesses an even part $\mathfrak{sl}(2) \oplus \mathfrak{o}(7)$ and an odd part $(\boldsymbol{2},\boldsymbol{8})$ as the spinor representations of $\mathfrak{sl}(2) \oplus \mathfrak{o}(7)$.

The superalgebra $G(3)$ is 31-dimensional algebra of rank 3 and has an even part $\mathfrak{sl}(2) \oplus G_2$ and an odd part $(\boldsymbol{2},\boldsymbol{7})$ as the representations of $\mathfrak{sl}(2) \oplus G_2$.

By scanning through the list in Table 3.1, we can find the superconformal algebras which satisfy the required conditions. For $d = 1$ superconformal field theory, that is superconformal quantum mechanics, the bosonic conformal algebra is $\mathfrak{so}(1,2) = \mathfrak{sl}(2,\mathbb{R}) = \mathfrak{su}(1,1) = \mathfrak{sp}(2)$ and richer superconformal structures are

$\mathfrak{g}$	$\mathfrak{g}_{\bar{0}}$	$\mathfrak{g}_{\bar{1}}$	K	type
$A(m-1,n-1)$	$A_{m-1}\oplus A_{n-1}\oplus\mathfrak{u}(1)\delta_{m,n}$	$(\boldsymbol{m},\overline{\boldsymbol{n}})$ $\oplus(\overline{\boldsymbol{m}},\boldsymbol{n})$	basic	I
$\mathfrak{su}(m-p,p\vert n-q,q)$	$\mathfrak{su}(m-p,p)\oplus\mathfrak{su}(n-q,q)\oplus\mathfrak{u}(1)\delta_{m,n}$		basic	I
$\mathfrak{su}^*(2m\vert 2n)$	$\mathfrak{su}^*(2m)\oplus\mathfrak{su}^*(2n)\oplus\mathfrak{so}(1,1)\delta_{m,n}$		basic	I
$\mathfrak{sl}'(n\vert n)$	$\mathfrak{sl}(n,\mathbb{C})$		basic	I
$B(m,n)$	$B_m\oplus C_n$ $m\geq 0,n\geq 1$	$(\boldsymbol{2m+1},\boldsymbol{2n})$	basic	II
$C(n+1)$	$C_m\oplus\mathfrak{u}(1)$ $n\geq 1$	$\boldsymbol{2n}\oplus\boldsymbol{2n}$	basic	I
$D(m,n)$	$D_m\oplus C_n$ $m\geq 2,n\geq 1,m\neq n+1$	$(\boldsymbol{2m},\boldsymbol{2n})$	basic	II
$\mathfrak{osp}(m-p,p\vert n)$	$\mathfrak{so}(m-p,p)\oplus\mathfrak{sp}(n)$		basic	II
$\mathfrak{osp}(m^*\vert n-q,q)$	$\mathfrak{so}^*(m)\oplus\mathfrak{usp}(n-q,q)$		basic	II
$D(2,1;\alpha)$	$A_1\oplus A_1\oplus A_1$ $0<\alpha\leq 1$	$(\boldsymbol{2},\boldsymbol{2},\boldsymbol{2})$	basic	II
$D^p(2,1;\alpha)$	$\mathfrak{so}(4-p,p)\oplus\mathfrak{sl}(2)$		basic	II
$F(4)$	$A_1\oplus B_3$	$(\boldsymbol{2},\boldsymbol{8})$	basic	I
$F^p(4)$	$\mathfrak{so}(7-p,p)\oplus\mathfrak{sl}(2)$ $p=0,3$		basic	I
$F^p(4)$	$\mathfrak{so}(7-p,p)\oplus\mathfrak{su}(2)$ $p=1,2$		basic	I
$G(3)$	$A_1\oplus G_2$	$(\boldsymbol{2},\boldsymbol{7})$	basic	I
$G_p(3)$	$\mathfrak{g}_{2,p}\oplus\mathfrak{sl}(2)$ $p=-14,2$		basic	I
$P(m-1)$	$\mathfrak{sl}(m)$ $m\geq 3$	$(\boldsymbol{m}\otimes\boldsymbol{m})$	strange	I
$Q(m-1)$	$\mathfrak{su}(m)$ $m\geq 3$	adjoint	strange	II
$Q(m-1)$	$\mathfrak{sl}(m)$		strange	II
$Q((m-1)^*)$	$\mathfrak{su}^*(m)$		strange	II
$UQ(p,m-1-p)$	$\mathfrak{su}(p,m-p)$		strange	II

Table 3.1: The list of the classical Lie superalgebras $\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}$ with Killing forms K.

allowed due to the small conformal group. Note that $\mathfrak{so}(1,2)$ may be contained as an even part $\mathfrak{g}_{\bar{0}}$ for the series of the Lie superalgebra $\mathfrak{g} = \mathfrak{osp}(m-p,p|n)$ and thus the corresponding R-symmetry algebras are the series of the non-compact $\mathfrak{sp}(n)$. Therefore if we consider the classical Lie superalgebras with compact R-symmetry algebras, the corresponding supergroups can be represented in terms of supermatrices as

$$\begin{pmatrix} SL(2,\mathbb{R}) & B \\ C & \text{R-symmetry} \end{pmatrix}, \tag{3.1.16}$$

$$\begin{pmatrix} SU(1,1) & B \\ C & \text{R-symmetry} \end{pmatrix}, \tag{3.1.17}$$

$$\begin{pmatrix} Sp(2) & B \\ C & \text{R-symmetry} \end{pmatrix} \tag{3.1.18}$$

where B and C are fermionic matrices. Two supermatrices (3.1.17) and (3.1.18) correspond to the infinite series of the Lie superalgebra and provide us chains of the one-dimensional superconformal groups. The remaining supermatrices (3.1.16) may cover the exceptional Lie superalgebras and other special cases. The one-dimensional superconformal groups are tabulated in Table 3.2 [183, 184, 136].

In the cases of $\mathcal{N} < 4$ supersymmetry the superconformal groups are essentially unique series of $OSp(2|\mathcal{N})$ as the isomorphism $SU(1,1|1) \cong OSp(2|2)$ is taken into account.

For $\mathcal{N} = 4$ supersymmetry the structure of the superconformal group becomes large as the exceptional Lie superalgebra $D(2,1;\alpha)$ is a one-parameter family. Note that $SU(1,1|2)$ for $\mathcal{N} = 4$ case is not simple as $SU(m,n|m+n)$ is not even semi-simple. The quotient $PSU(1,1|2) \cong SU(1,1|2)/U(1)$ is simple and we denote it just by $SU(1,1|2)$. As $D(2,1;-1)$ and $D(2,1;0)$ are semi-direct product $SU(1,1|2) \rtimes SU(2)$ and they are not simple, they are excluded in the Table 3.2.

With $\mathcal{N} = 8$ supersymmetry one-dimensional superconformal groups can be realizes as four different supergroups; $OSp(8|2)$, $SU(1,1|4)$, $OSp(4^*|4)$ and $F(4)$.

When the highly extended supersymmetry with $\mathcal{N} > 8$ exists in the quantum mechanics, one can have three distinct series of one-dimensional superconformal groups for even $\mathcal{N}$; $OSp(\mathcal{N}|2)$, $SU(1,1|\frac{\mathcal{N}}{2})$ and $OSp(4^*|\frac{\mathcal{N}}{2})$. The supergroup $OSp(4^*|\frac{\mathcal{N}}{2})$ is the exceptional series which does not appear in the theories with fewer supersymmetries. It has an even part $SO^*(4) \times USp(\mathcal{N})$ where the non-compact bosonic subgroup $SO^*(4) \cong SL(2,\mathbb{R}) \times SU(2)$ contain the one-dimensional conformal group $SL(2,\mathbb{R})$.

supersymmetry	supergroup	R-symmetry
$\mathcal{N}=1$	$OSp(1\|2)$	1
$\mathcal{N}=2$	$SU(1,1\|1)$	$U(1)$
$\mathcal{N}=3$	$OSp(3\|2)$	$SU(2)$
$\mathcal{N}=4$	$SU(1,1\|2)$	$SU(2)$
	$D(2,1;\alpha), \alpha \neq -1, 0,$	$SU(2)\times SU(2)$
$\mathcal{N}=5$	$OSp(5\|2)$	$SO(5)$
$\mathcal{N}=6$	$SU(1,1\|3)$	$SU(3)\times U(1)$
	$OSp(6\|2)$	$SO(6)$
$\mathcal{N}=7$	$OSp(7\|2)$	$SO(7)$
	$G(3)$	G_2
$\mathcal{N}=8$	$OSp(8\|2)$	$SO(8)$
	$SU(1,1\|4)$	$SU(4)\times U(1)$
	$OSp(4^*\|4)$	$SU(2)\times SO(5)$
	$F(4)$	$SO(7)$
$\mathcal{N}>8$	$OSp(\mathcal{N}\|2)$	$SO(\mathcal{N})$
	$SU(1,1\|\frac{\mathcal{N}}{2})$	$SU(\frac{\mathcal{N}}{2})\times U(1)$
	$OSp(4^*\|\frac{\mathcal{N}}{2})$	$SU(2)\times Sp(\frac{\mathcal{N}}{2})$

Table 3.2: The simple classical Lie supergroups that contain the one-dimensional conformal group $SL(2,\mathbb{R})$ as a factored bosonic subgroup. For $\mathcal{N}>8$ superconformal quantum mechanics there are three different superconformal groups.

3.2 One-dimensional supersymmetry

Now we want to discuss the concrete construction of superconformal quantum mechanical models. To this end we should note that in one dimension the supersymmetry is realized containing various peculiarities which do not appear in higher dimensional cases regardless of whether a conformal symmetry exists or not. It is known that the supersymmetry of a sigma-model imposes strong restrictions on its target space. However, the restrictions of one-dimensional supersymmetric sigma-models are generically weaker than higher dimensional sigma-models. In other words, more couplings among the fields are allowed in one dimension. This is because in higher dimensional cases the Lorentz symmetry rules out particular couplings, however, in one dimension there is no Lorentz symmetry group and much more couplings are possible. Moreover we cannot expect the relation between the number of bosonic and fermionic fields as in higher dimensional supersymmetric field theories.

3.2.1 Supermultiplet

One of the most powerful methods to construct supersymmetric quantum mechanics is to appeal the superspace and superfield formalism. In what follows we will consider a particularly reasonable class of supermultiplets [185, 186, 187] and discuss how many components we need to realize the $\mathcal{N}$-extended superalgebra [1] which satisfy

$$[\delta_{\epsilon^A}, \delta_{\epsilon^B}] = -2i\epsilon^A\epsilon^B\partial_t \tag{3.2.1}$$

where $A, B, \cdots = 1, \cdots, \mathcal{N}$ denote the R-symmetry indices and ϵ^A are a set of real anti-commuting supersymmetry parameters.

Now consider the scalar multiplets Φ which consist of a set of d physical bosons $x_i(t)$ and a set of d fermions $\psi_{\hat{i}}(t)$ where $i = 1, \cdots, d$ and $\hat{i} = 1, \cdots, d$ denote the multiplicities, i.e. the numbers of the bosons and the fermions respectively and suppose that their supersymmetric transformations are given by

$$\delta_{\epsilon^A} x_i = -i\epsilon^A (L_A)_i^{\ \hat{j}} \psi_{\hat{j}}, \tag{3.2.2}$$

$$\delta_{\epsilon^A} \psi_{\hat{i}} = \epsilon^A (R_A)_{\hat{i}}^{\ j} \dot{x}_j \tag{3.2.3}$$

where $(L_A)_i^{\ \hat{j}}$ and $(R_A)_{\hat{i}}^{\ j}$ are real $d \times d$ matrices. Then the algebra (3.2.1) imposes

[1] Also see [188, 189, 190] for the classification of the supermultiplets.

constraints on the matrices L_A and R_A as

$$(L_A R_B + L_B R_A)_i^{\ j} = -2\delta_{AB}\delta_j^j, \tag{3.2.4}$$

$$(R_A L_B + R_B L_A)_{\hat{i}}^{\ \hat{j}} = -2\delta_{AB}\delta_{\hat{i}}^{\hat{j}}. \tag{3.2.5}$$

From the algebraic point of view there is no relationship between two matrices L_A and R_A, however, if we require that the kinetic action for the scalar multiplet Φ with the form

$$S = \int dt \left[\frac{1}{2}\dot{x}_i^2 - \frac{i}{2}\psi_{\hat{i}}\dot{\psi}_{\hat{i}}\right] \tag{3.2.6}$$

is invarinat under the supersymmetric transformations (3.2.1), we obtain the relation

$$(L_A^T)^{\hat{i}j} = -(R_A)^{\hat{i}j}. \tag{3.2.7}$$

Likewise let us consider the spinor multiplets Ψ which are composed of a set of d real fermions $\lambda_{\hat{i}}$ and a set of d real bosons y_i possess the supersymmetry transformations

$$\delta_{\epsilon^A}\lambda_{\hat{i}} = \epsilon^A (R_A)_{\hat{i}}^{\ j} y_j, \tag{3.2.8}$$

$$\delta_{\epsilon^A} y_i = -i\epsilon^A (L_A)_i^{\ \hat{j}}\dot{\lambda}_{\hat{j}}. \tag{3.2.9}$$

Then one finds the same constraints for the two matrices L_A and R_A as (3.2.4) and (3.2.5). In addition if we require that the quadratic part of the action for the spinor multiplet Ψ

$$S = \int dt \left[-\frac{i}{2}\lambda_{\hat{i}}\dot{\lambda}_{\hat{i}} + \frac{1}{2}y_i y_i\right] \tag{3.2.10}$$

is invariant under the supersymmetry transformations (3.2.8) and (3.2.9), then we find the precisely same relation as (3.2.7).

Hence the existence of the scalar supermultiplets Φ and Ψ is rooted in the algebra [2] defined by three conditions (3.2.4), (3.2.5) and (3.2.7). It is known that there is a minimal value of d, called $d_{\mathcal{N}}$ for which $\mathcal{N}$ linearly independent real $d \times d$ matrices L_A and R_A satisfying the relations (3.2.4), (3.2.5) and (3.2.7) exist. We see that $d_{\mathcal{N}}$ translates into the minimal number of the bosonic or fermionic

[2] In [186, 187, 191] this algebra of dimension d and rank $\mathcal{N}$ is called $\mathcal{GR}(d,\mathcal{N})$ algebra since the one of the two matrices, say L_A satisfies a general real ($\mathcal{GR}$) Pauli algebra (3.2.4), (3.2.5) with the other matrix R_A determined by the relation (3.2.7).

n	1	2	3	4	5	6	7	8
$\log_2 n$	0	1	$\log_2 3$	2	$\log_2 5$	$\log_2 6$	$\log_2 7$	3
r	0	1	2	2	3	3	3	3
$\rho(2^r)$	1	2	4	4	8	8	8	8

Table 3.3: Hurwitz-Radon function $\rho(2^r)$ where r is the nearest integer greater than or equal to $\log_2 n$.

$\mathcal{N}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$d_{\mathcal{N}}$	1	2	4	4	8	8	8	8	16	32	64	64	128	128	128	128

Table 3.4: Then minimal numbers $d_{\mathcal{N}}$ of the component fields in the $\mathcal{N}$-extended supermultiplets.

component fields in the supermultiplets for a given the number of supersymmetry $\mathcal{N}$. The value of $d_{\mathcal{N}}$ is given by [189, 187]

$$d_{\mathcal{N}} = 16^m \rho(2^r) \tag{3.2.11}$$

where the number of supersymmetry is written as a mod8 decomposition

$$\mathcal{N} = 8m + n. \tag{3.2.12}$$

Here $\rho(2^r)$ is the so-called Hurwitz-Radon function [192, 193] define by [3]

$$\rho(2^r) = \begin{cases} 2r+1 & n \equiv 0 \mod 4 \\ 2r & n \equiv 1,2 \mod 4 \\ 2r+2 & n \equiv 3 \mod 4. \end{cases} \tag{3.2.14}$$

with r being taken as the nearest integer greater than or equal to $\log_2 n$ (see Table 3.3). The results are summarized in Table 3.4 From Table 3.4 one can see that when $\mathcal{N} = 1, 2, 4, 8$ the minimal numbers $d_{\mathcal{N}}$ of the component fields coincide with

[3]The Hurwitz-Radon function $\rho(2^r)$ yields the largest integer ρ for which the square identities can

$$\left(a_1^2 + \cdots + a_\rho^2\right)\left(b_1^2 + \cdots + b_{2^r}^2\right) = c_1^2 + \cdots c_{2^r}^2 \tag{3.2.13}$$

hold where $a_1, \cdots, a_\rho$ and $b_1, \cdots, b_{2^r}$ are the independent indeterminates and c_i is a bilinear form in $a_1, \cdots, a_\rho$ and $b_1, \cdots, b_{2^r}$. The Hurwitz-Radon function also appears in topology [194] and linear algebra [195]. See also [196, 197, 198].

the numbers $\mathcal{N}$ of supersymmetries. As we will see in the following, the superspace and superfield formalism works well for these four cases. Note that when $\mathcal{N} > 8$ the minimal numbers $d_{\mathcal{N}}$ of the supermultiplets are greater than the numbers $\mathcal{N}$ of supersymmetries and the corresponding supermultiplets become much more complicated and the superspace and superfield formalism is unsuccessful at present.

3.2.2 Automorphic duality

One of the most significant features in one-dimensional supersymmetric field theories, i.e. quantum mechanical models is the fact that the the equal number of bosonic and fermionic physical degrees of freedom, which is valid in higher dimensional field theories, does not take place. This is because in one dimension there is the duality which allows us to convert any physical field to auxiliary field and vice versa [186, 187, 191]. Consequently even if we consider the $\mathcal{N} = 1, 2, 4, 8$ supersymmetric cases, where $d_{\mathcal{N}} = \mathcal{N}$ is realized, a number of supermultiplets can be constructed in one-dimension.

To see this let us take the most basic $d = 1$ $\mathcal{N} = 1$ superalgebra

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = -2i\epsilon_1\epsilon_2\partial_t. \tag{3.2.15}$$

We introduce $\mathcal{N} = 1$ superspace $\mathbb{R}^{(1|1)}$ parametrized by

$$\mathbb{R}^{(1|1)} = (t, \theta) \tag{3.2.16}$$

where t is time and θ is a real Grassmann coordinate. The covariant superderivative D is defined by [4]

$$D = i\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial t}, \qquad \{D, D\} = -2i\partial_t \tag{3.2.17}$$

and the supercharge Q is realized as

$$Q = i\frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial t}, \qquad \{Q, Q\} = 2i\partial_t \tag{3.2.18}$$

in the superspace.

In this case there are two irreducible representations of (3.2.15); the scalar multiplet Φ and the spinor multiplet Ψ. The scalar multiplet contains a real bosonic field x as the lowest component and a real fermion ψ as the highest component

[4] This convention yields $\{Q, Q\} = 2H$ and leads to simple forms of the supersymmetric Lagrangian and its supersymmetric transformation.

while the spinor multiplet Ψ includes a real fermion λ as the lowest component and a real boson y as the highest component. Namely the multiplets can be described by

$$\Phi = x + i\theta\psi, \tag{3.2.19}$$

$$\Psi = \lambda + \theta y. \tag{3.2.20}$$

The supersymmetry transformation laws for the scalar multiplet Φ are $\delta\Phi = -i[\epsilon Q, \Phi]$, which yield

$$\delta_\epsilon x = i\epsilon\psi, \tag{3.2.21}$$

$$\delta_\epsilon \psi = \epsilon \dot{x} \tag{3.2.22}$$

and those for the spinor multiplet Ψ are $\delta\Psi = -i[\epsilon Q, \Psi]$, which give rise to

$$\delta_\epsilon \lambda = \epsilon y, \tag{3.2.23}$$

$$\delta_\epsilon y = i\epsilon\dot{\lambda}. \tag{3.2.24}$$

One can write the supersymmetric action for the scalar multiplet Φ as

$$S = -\frac{1}{2}\int dt d\theta \; D\Phi\dot{\Phi} \tag{3.2.25}$$

and also write that for the spinor multiplet Ψ as

$$S = -\frac{i}{2}\int dt d\theta \Psi D\Psi. \tag{3.2.26}$$

In component fields the above supersymmetric action (3.2.25) and (3.2.26) can be expressed by

$$S = \frac{1}{2}\int dt \;\left[\dot{x}^2 + i\dot{\psi}\psi\right] \tag{3.2.27}$$

and

$$S = \frac{1}{2}\int dt \;\left[i\dot{\lambda}\lambda + y^2\right] \tag{3.2.28}$$

respectively.

As described in [187], there is a useful operation which maps between the two irreducible $\mathcal{N} = 1$ multiplets

$$-D\Phi \leftrightarrow \Psi. \tag{3.2.29}$$

In component fields this map is realized by performing the following replacements

$$(\dot{x}, \psi) \leftrightarrow (y, \lambda). \tag{3.2.30}$$

We see that the supersymmetry transformations (3.2.21) for the scalar multiplet and the transformations (3.2.23) for the spinor multiplet are exchanged under the replacement (3.2.29) and that the action (3.2.27) for the scalar multiplet and the action (3.2.28) for the spinor multiplet transform into the other under the operation (3.2.29). Therefore a map (3.2.29) or (3.2.30) is the operation which replace a scalar multiplet Φ with a spinor multiplet Ψ and vice-verse. This is called the automorphic duality (AD) map because the operation corresponds to the automorphism on the space of the representations of the superalgebra. Intriguingly the AD map (3.2.30) make it possible to convert the physical field x into the auxiliary field y and vice versa. It has been pointed out [191] that this remarkable property in quantum mechanics can be interpreted as the Hodge duality in one-dimension. In general the Hodge duality maps a differential p-form Ω_p in d-dimension into a differential $(d-p-2)$-form Ω_{d-p-2} in d-dimension by the Hodge star operation as

$$* : d\Omega_p \to d\Omega_{d-p-2}. \tag{3.2.31}$$

If we consider a scalar field, a zero-form in one dimension, then the Hodge duality (3.2.31) gives rise to a dual (-1)-form. Formally the exterior derivative of a 0-form or a scalar x is a (-1)-form. Therefore if we denote the component field of the (-1)-form by y, we then get the relation

$$\dot{x} = y. \tag{3.2.32}$$

This is just the AD map given in (3.2.30).

According to the existence of the AD map in quantum mechanics, we will use the notation $(\boldsymbol{n}, \boldsymbol{\mathcal{N}}, \boldsymbol{\mathcal{N}} - \boldsymbol{n})$ for $\mathcal{N} = 1, 2, 4, 8$ supermultiplets. Here the first entry denoted by $\boldsymbol{n}$ is the number of physical bosons in the supermultiplet, the second number $\boldsymbol{\mathcal{N}}$ represents the number of fermions which is equal to the number of supersymmetry and the last one $\boldsymbol{\mathcal{N}} - \boldsymbol{n}$ is the number of bosonic auxiliary fields. Using this notation, the $\mathcal{N} = 1$ scalar multiplet Φ is $(\mathbf{1}, \mathbf{1}, \mathbf{0})$ and the spinor multiplet Ψ is $(\mathbf{0}, \mathbf{1}, \mathbf{1})$.

3.3 $\mathcal{N} = 1$ Superconformal mechanics

3.3.1 One particle free action

Consider the $\mathcal{N} = 1$ n particle quantum mechanical system which is described by the n-dimensional scalar superfield $(\mathbf{1}, \mathbf{1}, \mathbf{0})$. In general the $\mathcal{N} = 1$ superfield can be thought of as a map from the superspace $\mathbb{R}^{(1|1)}$ to the target space $\mathcal{M}$. In terms of component fields we can write the multiplet as

$$\Phi^i(t,\theta) = x^i(t) + i\theta\psi^i(t) \tag{3.3.1}$$

where $i, j, \cdots = 1, \cdots, n$. Also consider the $(\mathbf{0}, \mathbf{1}, \mathbf{1})$ spinor superfield Ψ^a which is a section of the bundle on $\mathcal{M}$ with rank k given by

$$\Psi^a(t,\theta) = \lambda^a(t) + \theta y^a(t) \tag{3.3.2}$$

where $a, b, \cdots = 1, \cdots, k$. We attach the mass dimension as the following:

$$\begin{aligned}
[t] &= -1, & [\theta] &= -\frac{1}{2}, \\
[\Phi] &= 0, & [\Psi] &= \frac{1}{2}, \\
[D] &= \frac{1}{2}, & [\partial_t] &= 1.
\end{aligned} \tag{3.3.3}$$

Then the most general $\mathcal{N} = 1$ action with dimensionless couplings up to cubic terms is given by [5]

$$\begin{aligned}
S = \int dt d\theta \Bigg[&-\frac{1}{2} g_{ij} D\Phi^i \dot{\Phi}^j + \frac{i}{3!} c_{ijk} D\Phi^i D\Phi^j D\Phi^k \\
&- \frac{i}{2} h_{ab} \Psi^a \nabla \Psi^b + \frac{1}{3!} l_{abc} \Psi^a \Psi^b \Psi^c + f_{ia} \dot{\Phi}^i \Psi^a \\
&+ \frac{i}{2} m_{iab} \Psi^a \Psi^b D\Phi^i + \frac{i}{2} n_{ija} D\Phi^i D\Phi^j \Psi^a \Bigg]
\end{aligned} \tag{3.3.4}$$

where g_{ij} is a metric on $\mathcal{M}$ and h_{ab} is a fibre metric on the bundle. The covariant derivative for the fermions are defined by

$$\nabla \Psi^a = D\Psi^a + D\Phi^i (A_i)^a{}_b \Psi^b \tag{3.3.5}$$

with $(A_i)^a{}_b$ being the connection on the bundle.

Note that for the one particle case where the corresponding target space $\mathcal{M} = \mathbb{R}$ has no non-trivial bundle over it, the $\mathcal{N} = 1$ superspace action is described by just a free action (3.2.27). This corresponds to the statement that it is not possible to construct one-particle $OSp(1|2)$ superconformal quantum mechanics with inverse-square type potential [65, 67, 137].

[5] See also [199, 70] for the $\mathcal{N} = 1$ superfield action.

3.3.2 Multi-particle model

Let us focus on the sigma-model action constructed only from the $(\mathbf{1},\mathbf{1},\mathbf{0})$ scalar supermultiplet Φ^i [199, 200, 69, 136] [6]

$$S = \int dtd\theta \left[-\frac{1}{2} g_{ij} D\Phi^i \dot{\Phi}^j + \frac{i}{3!} c_{ijk} D\Phi^i D\Phi^j D\Phi^k \right]$$
$$= \int dt \left[\frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + \frac{i}{2} \psi^i \left(g_{ij} \frac{D\psi^j}{dt} - \dot{x}^k c_{ijk} \psi^j \right) - \frac{1}{6} \partial_l c_{ijk} \psi^l \psi^i \psi^j \psi^k \right] \tag{3.3.6}$$

where the covariant derivative is defined as

$$\frac{D\psi^i}{dt} := \dot{\psi}^i + \dot{x}^j \Gamma^i_{jk} \psi^k \tag{3.3.7}$$

with Γ^i_{jk} being the Christoffel symbol on $\mathcal{M}$.

Instead of the space-time indices i for the fermions ψ^i we shall introduce the tangent space indices $\alpha = 1, \cdots, n$ by redefining the fermions ψ^α as

$$\psi^i = e^i{}_\alpha \psi^\alpha. \tag{3.3.8}$$

Note that ψ^α commute with x^i and p^i while ψ^i does not commute with x^i and p^i

$$[p_i, \lambda^j] = -i \left(\omega_{i\,k}^{\,j} - \Gamma^j_{ik} \right) \psi^k. \tag{3.3.9}$$

Then the action (3.3.6) can be written as

$$S = \int dt \Bigg[\frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + \frac{i}{2} \left(\delta_{\alpha\beta} \psi^\alpha \dot{\psi}^\beta + \dot{x}^i \omega_{i\alpha\beta} \psi^\alpha \psi^\beta \right)$$
$$- \frac{i}{2} \dot{x}^i c_{i\beta\gamma} \psi^\alpha \psi^\beta - \frac{1}{6} e^l{}_\delta \partial_l c_{ijk} e^i{}_\alpha e^j{}_\beta e^k{}_\gamma \psi^\delta \psi^\alpha \psi^\beta \psi^\gamma \Bigg] \tag{3.3.10}$$

where ω is the spin connection and $c_{i\alpha\beta} := c_{ijk} e^j{}_\alpha e^k{}_\beta$. From the fermionic kinetic terms in the action (3.3.10) we see that the covariant derivatives of the fermions contains the connection with torsion c. Although this is similar to the two-dimensional $(1,0)$ supersymmetric sigma models [201], the torsion c here is not necessarily closed as opposed to two-dimensional case. This indicates that there exist new supermultiplets in one dimension which have no higher-dimensional ancestors. The canonical momenta p_i is expressed as

$$p_i = g_{ij} \dot{x}^j + \frac{i}{2} \left(\omega_{ijk} - c_{ijk} \right) \psi^j \psi^k \tag{3.3.11}$$

[6] The $(\mathbf{1},\mathbf{1},\mathbf{0})$ supermultiplet is also called $\mathcal{N} = 1B$ superfield.

where $\omega_{ijk} := \omega_{i\gamma}^{\beta} e_{j\beta} e_k{}^{\gamma}$. The action (3.3.6) is invariant under the supersymmetry transformations

$$\delta x^i = -i\epsilon\psi^i, \tag{3.3.12}$$

$$\delta\psi^i = \epsilon\dot{x}^i. \tag{3.3.13}$$

By means of the Noether's method we find the supercharge

$$Q = \psi^i \Pi_i - \frac{i}{3} c_{ijk}\psi^i\psi^j\psi^k \tag{3.3.14}$$

where we have defined

$$\Pi_i = g_{ij}\dot{x}^j. \tag{3.3.15}$$

Note that the supercharge Q is Hermitian though Π_i is not Hermitian. Using the canonical relation for the fermions

$$\left\{\psi^\alpha, \psi^\beta\right\} = \delta^{\alpha\beta} \tag{3.3.16}$$

and the relations (2.8.3) for bosons, one finds

$$\{Q, Q\} = 2H. \tag{3.3.17}$$

where the Hamiltonian is

$$H = \frac{1}{2} p_a^\dagger g^{ab} p_b, \tag{3.3.18}$$

which agrees with the bosonic sigma-model Hamiltonian (2.8.1) with the bosonic potential $V(x)$ vanishing.

At this stage we consider the condition so that the theory (3.3.10) is the $OSp(1|2)$ superconformal quantum mechanics. The corresponding $\mathfrak{osp}(1|2)$ superalgebra is characterized by the following (anti)commutation relations:

$$[H, D] = iH, \qquad [K, D] = -iD, \qquad [H, K] = 2iD, \tag{3.3.19}$$

$$[Q, H] = 0, \qquad [Q, D] = -\frac{i}{2}Q, \qquad [Q, K] = -iS, \tag{3.3.20}$$

$$[S, H] = iQ, \qquad [S, D] = \frac{i}{2}S, \qquad [S, K] = 0, \tag{3.3.21}$$

$$\{Q, Q\} = 2H, \qquad \{Q, S\} = -2D, \qquad \{S, S\} = 2K. \tag{3.3.22}$$

From the commutation relation (3.3.20) and the expressions (3.3.14) and (2.8.15) we can read the superconformal charge S

$$S = \psi^i D_i \tag{3.3.23}$$

where D_i has been introduced in (2.8.5) as the generalized dilatation. From the anti-commutator (3.3.22) we obtain the modified dilatation generator

$$D = \frac{1}{4}\left(D^i \Pi_i + \Pi^\dagger D_i^\dagger\right) \tag{3.3.24}$$

with p_i being replaced with Π_i. Using the new dilatation generator (3.3.24), $[S, D] = \frac{i}{2}S$ is satisfied, however, $[Q, D]$ yields

$$[Q, D] = -\frac{i}{2}Q - \frac{i}{2}c_{ijk}D^i\psi^j p^k + \mathcal{O}(\psi^3). \tag{3.3.25}$$

Thus the $OSp(1|2)$ superconformal symmetry imposes the condition so that the second quadratic term in ψ must vanish

$$D^i c_{ijk} = 0, \tag{3.3.26}$$

which means that c is orthogonal to D. With the constraint (3.3.26), the commutator (3.3.25) becomes

$$[Q, D] = -\frac{i}{2}Q - \frac{1}{12}\psi^i\psi^j\psi^k\left(\mathcal{L}_D - 2\right)c_{ijk}, \tag{3.3.27}$$

which implies that

$$\mathcal{L}_D c_{ijk} = 2c_{ijk}. \tag{3.3.28}$$

Then one can check that the remaining (anti)commutation relations (3.3.19)-(3.3.22) are satisfied and there are no further constraints for the $OSp(1|2)$ symmetry imposed on the target space $\mathcal{M}$.

Therefore the conditions so that the $\mathcal{N} = 1$ sigma-model action (3.3.10) realizes the $OSp(1|2)$ superconformal quantum mechanics are the conformal condition (2.8.9), (2.8.16) and the additional two constraints on the torsion c

$$D^i c_{ijk} = 0, \tag{3.3.29}$$

$$\mathcal{L}_D c_{ijk} = 2c_{ijk}. \tag{3.3.30}$$

3.3.3 Gauged superconformal mechanics

As a generalization of the gauged mechanics (2.5.3) for the DFF-model and the gauged matrix model (2.9.2) for the Calogero model, we will discuss the superextension of the $\mathcal{N} = 1$ supersymmetric gauged mechanical model. As we will see this gauging procedure allows for the explicit construction of the non-trivial

$\mathcal{N} = 1$ superconformal quantum mechanics [152, 137]. Consider the matrix superfield gauged mechanics action

$$S = -i \int dt d\theta \left[\mathrm{Tr} \left(\nabla_t \mathcal{X} \mathcal{D} \mathcal{X} \right) + \frac{i}{2} \left(\overline{\mathcal{Z}} \mathcal{D} \mathcal{Z} - \mathcal{D} \overline{\mathcal{Z}} \mathcal{Z} \right) + c \mathrm{Tr} \mathcal{A} \right]. \tag{3.3.31}$$

Here we have introduced

- the $\mathcal{N} = 1$ Grassmann-even Hermitian $n \times n$ matrix superfield $\mathcal{X}_a^b(t, \theta)$ which satisfies $(\mathcal{X})^\dagger = \mathcal{X}$ and transforms as the adjoint representation of $U(n)$
- the $\mathcal{N} = 1$ Grassmann-even complex superfield $\mathcal{Z}_a(t, \theta)$ which satisfies $\overline{\mathcal{Z}} = \mathcal{Z}^\dagger$ and transform as the fundamental representation of $U(n)$
- the $\mathcal{N} = 1$ Grassmann-odd anti-Hermitian $n \times n$ matrix superfield $\mathcal{A}_a^b(t, \theta)$ which satisfies $(\mathcal{A})^\dagger = -\mathcal{A}$ and transforms as the adjoint representation of $U(n)$.

The covariant derivatives are defined by

$$\nabla_t \mathcal{X} = D\mathcal{X} + i[\mathcal{A}_t, \mathcal{X}], \tag{3.3.32}$$

$$\mathcal{D}\mathcal{X} = D\mathcal{X} + i[\mathcal{A}, \mathcal{X}], \tag{3.3.33}$$

$$\mathcal{D}\mathcal{Z} = D\mathcal{Z} + i\mathcal{A}\mathcal{Z} \tag{3.3.34}$$

where [7]

$$D = \frac{\partial}{\partial \theta} + i\theta \frac{\partial}{\partial t}, \qquad \{D, D\} = 2i\partial_t, \tag{3.3.35}$$

$$\mathcal{A}_t = -iD\mathcal{A} - \mathcal{A}\mathcal{A}. \tag{3.3.36}$$

The superconformal boost transformations are found to be

$$\delta t = -i\eta\theta t, \qquad \delta\theta = \eta t, \tag{3.3.37}$$

$$\delta(dt d\theta) = -i\eta\theta(dt d\theta), \qquad \delta D = i\eta\theta D, \tag{3.3.38}$$

$$\delta\mathcal{X} = -i\eta\theta\mathcal{X}, \qquad \delta\mathcal{A} = i\eta\theta\mathcal{A}, \tag{3.3.39}$$

$$\delta\mathcal{Z} = 0. \tag{3.3.40}$$

The action (3.3.31) is invariant under the $U(n)$ gauge transformations

$$\mathcal{X} \to e^{i\Lambda} \mathcal{X} e^{-i\Lambda}, \tag{3.3.41}$$

$$\mathcal{Z} \to e^{i\Lambda} \mathcal{Z}, \tag{3.3.42}$$

$$\mathcal{A} \to e^{i\Lambda} \mathcal{A} e^{-i\Lambda} - i e^{i\Lambda} \left(D e^{-i\Lambda} \right), \tag{3.3.43}$$

$$\mathcal{A}_t \to e^{i\Lambda} \mathcal{A}_t e^{-i\Lambda} - i e^{i\Lambda} \left(\partial_t e^{-i\Lambda} \right) \tag{3.3.44}$$

[7] Note that the notation here is different from (3.2.17).

where $\Lambda_a^b(t,\theta)$ is the Hermitian $n \times n$ matrix gauge parameter. The $\mathcal{N}=1$ superfields $\mathcal{X}$, $\mathcal{Z}$ and $\mathcal{A}$ can be expanded in the component fields as

$$\mathcal{X}_a^b = x_a^b + i\theta\psi_a^b, \tag{3.3.45}$$

$$\mathcal{Z}_a = z_a + \theta\xi_a, \tag{3.3.46}$$

$$\mathcal{A}_a^b = i(\zeta_a^b + \theta A_a^b). \tag{3.3.47}$$

From the gauge transformation (3.3.43) we can fix the gauge so that

$$\mathcal{A}_a^b = i\theta(A_0)_a^b(t). \tag{3.3.48}$$

Inserting (3.3.48) into the action (3.3.31), performing the integration over θ and integrating out the auxiliary fields $\xi, \overline{\xi}$, we find the $\mathcal{N}=1$ gauged superconformal matrix model action

$$S = \int dt \left[\mathrm{Tr}\left(DxDx\right) - i\mathrm{Tr}\left(\psi D\psi\right) + \frac{i}{2}\left(\overline{z}Dz - D\overline{z}z\right) + c\mathrm{Tr}A_0 \right] \tag{3.3.49}$$

where the covariant derivative is defined by

$$Dx = \dot{x} + i[A_0, x], \qquad D\psi = \dot{\psi} + [A_0, \psi]. \tag{3.3.50}$$

Note that the action (3.3.49) is the supersymmetric generalization of (2.9.2) that describes the Calogero model.

Instead of the gauge choice (3.3.48), we can fix the gauge as

$$\mathcal{X}_a^b = \mathcal{X}_a\delta_a^b, \tag{3.3.51}$$

$$\mathcal{Z}_a = \overline{\mathcal{Z}}^a \tag{3.3.52}$$

as we have discussed in (2.9.9) and (2.9.14) for the bosonic gauged matrix model. In this gauge the theory contains n^2 real $\mathcal{N}=1$ superfields $\mathcal{A}_a^b$, $a \neq b$ and $\mathcal{X}_a$ while the superfields $\mathcal{Z}_a$ and $\mathcal{A}_a^a$ are auxiliary. The superfield action (3.3.31) reads [137]

$$\begin{aligned} S = -i\int dtd\theta \Bigg[&\sum_a \dot{\mathcal{X}}_a D\mathcal{X}_a + \frac{i}{2}\sum_a \left(\overline{\mathcal{Z}}^a D\mathcal{Z}_a - D\overline{\mathcal{Z}}^a\mathcal{Z}_a\right) - i\sum_{a,b}\left(\mathcal{X}_a - \mathcal{X}_b\right)^2 D\mathcal{A}_a^b\mathcal{A}_b^a \\ &- \sum_{a,b}\left(\mathcal{X}_a - \mathcal{X}_b\right)^2 \left(\mathcal{A}\mathcal{A}\right)_a^b \mathcal{A}_b^a + \sum_{a,b}\overline{\mathcal{Z}}^a\mathcal{A}_a^b\mathcal{Z}_b + c\sum_a \mathcal{A}_[a]^a \Bigg]. \end{aligned} \tag{3.3.53}$$

For $n=1$, one particle case, the action (3.3.53) becomes free action

$$S = -i\int dtd\theta\dot{\mathcal{X}}D\mathcal{X} \tag{3.3.54}$$

and the theory has no bosonic potential in the component action.

In the case of $n = 2$, that is two particles case, the action (3.3.53) is written as

$$S = -i \int dt d\theta \left[\frac{1}{2} \dot{\mathcal{X}}_+ D \mathcal{X}_+ - \frac{1}{2} \mathcal{A}_- D \mathcal{A}_- \right.$$
$$\left. + \frac{1}{2} \dot{\mathcal{X}}_- D \mathcal{X}_- - \frac{1}{2} \mathcal{A}_+ D \mathcal{A}_+ - c \epsilon_1 \epsilon_2 \frac{\mathcal{A}_+}{\mathcal{X}_-} \right] \tag{3.3.55}$$

where

$$\mathcal{X}_- := \mathcal{X}_1 - \mathcal{X}_2, \qquad \mathcal{X}_+ := \mathcal{X}_1 + \mathcal{X}_2, \tag{3.3.56}$$
$$\mathcal{A}_+ := \mathcal{X}(\mathcal{A}_1^2 + \mathcal{A}_2^1), \qquad \mathcal{A}_- := i\mathcal{X}(\mathcal{A}_1^2 - \mathcal{A}_2^1) \tag{3.3.57}$$

and $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$ are the constans appearing in the constraint $\mathcal{Z}_1 \mathcal{Z}_2 = -\frac{c\epsilon_1\epsilon_2}{2}$. Note that the superfield action (3.3.55) is a sum of two free $\mathcal{N} = 1$ supermultiplets $(\mathcal{X}_+, \mathcal{A}_-)$ and two interacting $\mathcal{N} = 1$ supermultiplets $(\mathcal{X}_-, \mathcal{A}_+)$. It has been argued that the superfield action (3.3.55) is the $\mathcal{N} = 1$ superfield form of the off-shell $\mathcal{N} = 2$ superconformal mechanics based on the supermultiplet $(\mathbf{1}, \mathbf{2}, \mathbf{1})$ [152, 137].

For $n = 3$ it has been shown [152, 137] that the $\mathcal{N} = 1$ superfield action (3.3.55) cannot be connected to the known $\mathcal{N} = 2$ or $\mathcal{N} = 3$ superconformal mechanical modelds and that in the bosonic limit it yields the three particle Calogero model for the component fields $x_a = \mathcal{X}_a|$.

3.4 $\mathcal{N} = 2$ Superconformal mechanics

3.4.1 One particle model

The $\mathcal{N} = 2$ superspace $\mathbb{R}^{(1|2)}$ contains time coordinate t and two Grassmann coordinate $\theta, \bar{\theta}$

$$\mathbb{R}^{(1|2)} = (t, \theta, \bar{\theta}). \tag{3.4.1}$$

The covariant superderivatives D and $\overline{D}$ are

$$D = i\frac{\partial}{\partial\theta} - \bar{\theta}\frac{\partial}{\partial t}, \qquad \overline{D} = i\frac{\partial}{\partial\bar{\theta}} - \theta\frac{\partial}{\partial t}, \qquad \{D, \overline{D}\} = -2i\partial_t \tag{3.4.2}$$

while the two supercharges Q and $\overline{Q}$ are given by

$$Q = i\frac{\partial}{\partial\theta} + \bar{\theta}\frac{\partial}{\partial t}, \qquad \overline{Q} = i\frac{\partial}{\partial\bar{\theta}} + \theta\frac{\partial}{\partial t}, \qquad \{Q, \overline{Q}\} = 2i\partial_t \tag{3.4.3}$$

in the superspace.

Now consider the $\mathcal{N}=2$ superfield $(\mathbf{1},\mathbf{2},\mathbf{1})$ in the superspace (3.4.1)

$$\Phi(t,\theta,\overline{\theta}) = x(t) + i\theta\psi(t) + i\overline{\theta}\overline{\psi}(t) + \theta\overline{\theta}y(t). \tag{3.4.4}$$

The $(\mathbf{1},\mathbf{2},\mathbf{1})$ supermultiplet is also called $\mathcal{N}=2A$ multiplet. This supermultiplet is related to the two-dimensional $(1,1)$ supersymmetry. Making use of the $(\mathbf{1},\mathbf{2},\mathbf{1})$ supermultiplet (3.4.4), we can write $\mathcal{N}=2$ supersymmetric action in the form

$$S = \frac{1}{2}\int dt d\theta d\overline{\theta}\;\left[\overline{D}\Phi D\Phi - W(\Phi)\right] \tag{3.4.5}$$

where $W(\Phi)$ is a superpotential that is some function of the superfield Φ. In component the superfield action (3.4.5) can be written as

$$S = \frac{1}{2}\int dt\;\left[\dot{x}^2 + i\dot{\overline{\psi}}\psi - i\overline{\psi}\dot{\psi} + y^2 - W'(x)y - W''(x)\psi\overline{\psi}\right]. \tag{3.4.6}$$

To obtain the conformal invariant action, let us consider the superpotential in the form

$$W(\Phi) = f\ln\Phi^2. \tag{3.4.7}$$

Then the action (3.4.6) becomes

$$S = \frac{1}{2}\int dt\;\left[\dot{x}^2 + i\dot{\overline{\psi}}\psi - i\overline{\psi}\dot{\psi} + y^2 - \frac{2fy}{x} - \frac{2f\psi\overline{\psi}}{x^2}\right]. \tag{3.4.8}$$

By solving the algebraic equation of motion of y, one can integrate out the auxiliary field y. Then we find the one-particle $\mathcal{N}=2$ $OSp(2|2)$ superconformal mechanical model [153, 154]

$$S = \frac{1}{2}\int dt\;\left[\dot{x}^2 + i\dot{\overline{\psi}}\psi - i\overline{\psi}\dot{\psi} - \frac{f(f-2\psi\overline{\psi})}{x^2}\right]. \tag{3.4.9}$$

In the superspace the generators of the superconformal group can be realized by the following expressions [8]

$$H = i\frac{\partial}{\partial t}, \tag{3.4.10}$$

$$D = i\left(t\frac{\partial}{\partial t} + \frac{1}{2}\theta\frac{\partial}{\partial\theta} + \frac{1}{2}\overline{\theta}\frac{\partial}{\partial\overline{\theta}} + \Delta\right), \tag{3.4.11}$$

$$K = i\left(t^2\frac{\partial}{\partial t} + t\theta\frac{\partial}{\partial\theta} + t\overline{\theta}\frac{\partial}{\partial\overline{\theta}} + 2t\Delta\right), \tag{3.4.12}$$

$$Q = i\frac{\partial}{\partial\theta} + \overline{\theta}\frac{\partial}{\partial t}, \qquad \overline{Q} = i\frac{\partial}{\partial\overline{\theta}} + \theta\frac{\partial}{\partial t}, \tag{3.4.13}$$

$$S = tQ - \theta\overline{\theta}\frac{\partial}{\partial\theta} + 2\Delta\overline{\theta}, \qquad \overline{S} = t\overline{Q} - \overline{\theta}\theta\frac{\partial}{\partial\overline{\theta}} + 2\Delta\theta, \tag{3.4.14}$$

$$B = -i\theta\frac{\partial}{\partial\theta} + i\overline{\theta}\frac{\partial}{\partial\overline{\theta}}. \tag{3.4.15}$$

One can show that these generators form the $\mathfrak{su}(1,1|1)$ superalgebra

$$[H,D] = iH, \quad [K,D] = -iK, \quad [H,K] = 2iD, \tag{3.4.16}$$

$$[B,H] = 0, \quad [B,D] = 0, \quad [B,K] = 0, \tag{3.4.17}$$

$$\begin{array}{lll} [H,Q] = 0, & [D,Q] = -\frac{i}{2}Q, & [K,Q] = -iS, \\ [H,\overline{Q}] = 0, & [D,\overline{Q}] = -\frac{i}{2}\overline{Q}, & [K,\overline{Q}] = i\overline{S}, \end{array} \tag{3.4.18}$$

$$\begin{array}{lll} [H,S] = iQ, & [D,S] = \frac{i}{2}S, & [K,S] = 0, \\ [H,\overline{S}] = i\overline{Q} & [D,\overline{S}] = \frac{i}{2}\overline{S}, & [K,\overline{S}] = 0, \end{array} \tag{3.4.19}$$

$$\{Q,\overline{Q}\} = 2H, \quad \{S,\overline{S}\} = 2K, \quad \{Q,\overline{S}\} = 2D - B, \tag{3.4.20}$$

$$\begin{array}{ll} [B,Q] = iQ, & [B,S] = iS, \\ [B,\overline{Q}] = -i\overline{Q}, & [B,\overline{S}] = -i\overline{S}. \end{array} \tag{3.4.21}$$

The supersymmetry transformations for the $(\mathbf{1},\mathbf{2},\mathbf{1})$ multiplet which follow from $\delta\Phi = -i[\epsilon\overline{Q} + \overline{\epsilon}Q, \Phi]$ are expressed in the component fields as

$$\delta x = i\epsilon\overline{\psi} + i\overline{\epsilon}\psi, \tag{3.4.22}$$

$$\delta\psi = \epsilon\dot{x} - i\epsilon\frac{f}{x}, \tag{3.4.23}$$

$$\delta\overline{\psi} = \overline{\epsilon}\dot{x} + i\overline{\epsilon}\frac{f}{x}. \tag{3.4.24}$$

[8] Note that in [153] the Hamiltonian is expresses by $\{Q,\overline{Q}\} = -2H$ while in our notation the additional sign does not appear.

Applying the Noether's method, we find the explicit expressions for the supercharges $Q, \overline{Q}$, the three $SL(2,\mathbb{R})$ conformal generators H, D, K and we also introduce the superconformal charges $S, \overline{S}$ and the $SO(2)$ R-symmetry generator B as follows:

$$Q = \psi\left(-ip + \frac{f}{x}\right), \qquad \overline{Q} = \overline{\psi}\left(ip + \frac{f}{x}\right), \tag{3.4.25}$$

$$S = x\psi, \qquad \overline{S} = x\overline{\psi}, \tag{3.4.26}$$

$$H = \frac{1}{2}\left[p^2 + \frac{f(f+2B)}{x^2}\right] \tag{3.4.27}$$

$$D = -\frac{1}{4}(xp + px), \tag{3.4.28}$$

$$K = \frac{1}{2}x^2, \tag{3.4.29}$$

$$B = \frac{1}{2}[\psi, \overline{\psi}]. \tag{3.4.30}$$

Note that the potential in the Hamiltonian H is shifted as a quantum effect. Under the canonical relations

$$[x, p] = i, \qquad \{\psi, \overline{\psi}\} = 1, \tag{3.4.31}$$

the set of operators (3.4.25)-(3.4.30) form the $\mathfrak{osp}(2|2)$ superalgebra (3.4.16)-(3.4.20).

Let us study the spectrum of the one-particle $OSp(2|2)$ superconformal quantum mechanics (3.4.9). In general supersymmetric quantum mechanics has the Hamiltonian H which can be written as the sum of squares of the Hermitian supercharges Q_A,$A = 1, \cdots, \mathcal{N}$. This implies that the energy of any state is positive or zero [202, 52]. If $H|\Omega\rangle = 0$, then we have $0 = \langle\Omega|H|\Omega\rangle = \sum_A \langle\Omega|Q_A^2|\Omega\rangle = \sum_A |Q_A|\Omega\rangle|^2$, which is only possible if $Q_A|\Omega\rangle$ for any A. Conversely if a state $|\Omega\rangle$ is annihilated by Q_A, then $H|\Omega\rangle = Q_A^2|\Omega\rangle = 0$, i.e. its energy is zero. Therefore the supersymmetry generated by Q_A is broken if the system has no normalizable ground state of H. Now consider the equations defining the ground state of H

$$Q|\Omega\rangle = \overline{Q}|\Omega\rangle = 0. \tag{3.4.32}$$

Using the explicit expressions (3.4.25) and (3.4.26), the equation (3.4.32) is written as

$$\left(2iBp - \frac{f}{x^2}\right)|\Omega\rangle = 0 \tag{3.4.33}$$

which can be interpreted as the first order differential equation of x. Then the generic solution of (3.4.33) leads to the x-depgroundence of the ground state of H

as

$$|\Omega\rangle = x^{-2fB}|\text{phys}\rangle \tag{3.4.34}$$

where $|\text{phys}\rangle$ is any x independent state. Noting that the $SO(2)$ R-symmetry operator B has eigenvalue $+\frac{1}{2}$ and $-\frac{1}{2}$, we see that the ground state of H may have the two different x dependence

$$|\Omega\rangle = \begin{cases} x^{-f}|\text{phys}\rangle & \text{for} \quad B = \frac{1}{2} \\ x^{f}|\text{phys}\rangle & \text{for} \quad B = -\frac{1}{2}. \end{cases} \tag{3.4.35}$$

As the wavefunction will blow up for either large or small x region, there is no normalizable state of H and therefore the supersymmetry generated by Q, $\overline{Q}$ is spontaneously broken. Note that the wavefunction with $E > 0$ energy can be exactly solved by using the result of DFF-model. Comparing the quantum Hamiltonian (3.4.27) with the DFF-model Hamiltonian (2.1.18), we find the relation

$$g^2 = f(f+2B) = \begin{cases} f(f+1) & \text{for} \quad B = \frac{1}{2} \\ f(f-1) & \text{for} \quad B = -\frac{1}{2}. \end{cases} \tag{3.4.36}$$

The appearance of two sectors, i.e. the doublet structure of the eigenstates of H corresponds to the fact that H commutes with two operators Q and $\overline{Q}$. From the expression (2.2.1) we find the eigenfunctions

$$\psi_{E,B}(x) = \begin{cases} C\sqrt{x}J_{\sqrt{f+\frac{1}{2}}}\left(\sqrt{2E}x\right) & \text{for} \quad B = \frac{1}{2} \\ C\sqrt{x}J_{\sqrt{f-\frac{1}{2}}}\left(\sqrt{2E}x\right) & \text{for} \quad B = -\frac{1}{2}. \end{cases} \tag{3.4.37}$$

These wavefunctions are shown in Figure 3.2. From Figure 3.2 we see that there are several peaks of the wavefunctions with the nearest one from the origin being the maximum value. For large coupling constant f the relative positions of the particle gradually become far from the origin. At high energy E the number of peaks increases and the probability of the position of the particle is averaged.

Then we can follow the previous discussion for the DFF-model to solve the problem of the absence of the ground state. Instead of the original Hamiltonian we now regard the compact operator $L_0 = \frac{1}{2}(H+K)$ as the new Hamiltonian. Looking at the formulae (2.2.26), (2.2.28) and the relation (3.4.36), one finds

$$r_n = \begin{cases} \frac{1}{2}\left(\frac{3}{2}+f\right) & \text{for} \quad B = \frac{1}{2} \\ \frac{1}{2}\left(\frac{1}{2}+f\right) & \text{for} \quad B = -\frac{1}{2}. \end{cases} \tag{3.4.38}$$

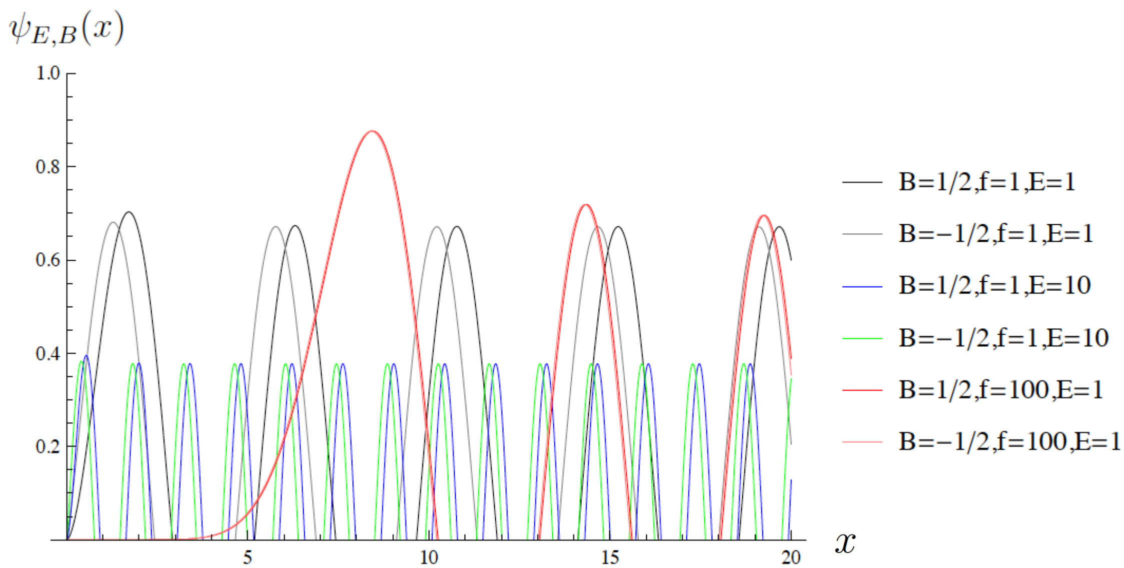


Figure 3.2: The eigenfunctions $\psi_{E,B}(x)$ of the original Hamiltonian H with $E \neq 0$. There are two sectors labeled by $B = \frac{1}{2}$ and $B = -\frac{1}{2}$.

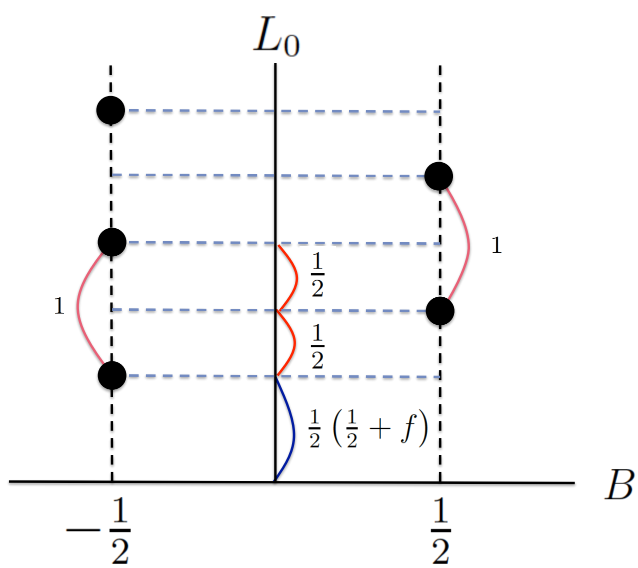


Figure 3.3: The level structure of the spectrum of the new Hamiltonian L_0. The spectrum is equally spaced. For a fixed B the equal space is 1 while the space with $\Delta B \neq 1$ is $\frac{1}{2}$.

The level structure of the spectrum of L_0 has two series corresponding to the two different eigenvalues $B = -\frac{1}{2}, \frac{1}{2}$. So it can be represented on the plane of the eigenvalue of B and L_0 (see Figure 3.3). In order to understand the appearance of the half integer shift in an algebraic way, let us define the fermionic operators [154]

$$M = Q - S = \psi \left(-ip + \frac{f}{x} - x \right), \tag{3.4.39}$$

$$\overline{M} = \overline{Q} - \overline{S} = \overline{\psi} \left(ip + \frac{f}{x} - x \right), \tag{3.4.40}$$

$$N = \overline{Q} + \overline{S} = \overline{\psi} \left(ip + \frac{f}{x} + x \right), \tag{3.4.41}$$

$$\overline{N} = Q + S = \psi \left(-ip + \frac{f}{x} + x \right). \tag{3.4.42}$$

Then we find the following anti-commutation relations

$$\{M, \overline{M}\} =: 4T_1 = 4L_0 + 2B - 2f, \tag{3.4.43}$$

$$\{N, \overline{N}\} =: 4T_2 = 4L_0 - 2B + 2f, \tag{3.4.44}$$

$$\{M, N\} = 4L_+ = 2\left(H - K + 2iD\right), \tag{3.4.45}$$

$$\{\overline{M}, \overline{N}\} = 4L_- = 2\left(H - K - 2iD\right), \tag{3.4.46}$$

$$\{M, \overline{N}\} = \{\overline{M}, N\} = 0. \tag{3.4.47}$$

and the commutation relations

$$[L_0, M] = -\frac{1}{2}M, \qquad [L_0, \overline{M}] = \frac{1}{2}\overline{M}, \tag{3.4.48}$$

$$[L_0, N] = -\frac{1}{2}N, \qquad [L_0, \overline{N}] = \frac{1}{2}\overline{N}, \tag{3.4.49}$$

$$[T_1, N] = -N, \qquad [T_1, \overline{N}] = \overline{N}, \tag{3.4.50}$$

$$[T_2, N] = -N, \qquad [T_2, \overline{N}] = \overline{N}, \tag{3.4.51}$$

$$[T_1, M] = [T_1, \overline{M}] = 0, \tag{3.4.52}$$

$$[T_2, N] = [T_2, \overline{N}] = 0, \tag{3.4.53}$$

$$[T_1, L_-] = -L_-, \qquad [T_1, L_+] = L_+, \tag{3.4.54}$$

$$[T_2, L_-] = -L_-, \qquad [T_2, L_+] = L_+. \tag{3.4.55}$$

Let us consider the ground states eliminated by the supercharges. Since there are now three sets of the supercharges; $(Q, \overline{Q})$, $(M, \overline{M})$ and $(N, \overline{N})$, we find six possible candidates for the x dependence of the ground states $|\Omega\rangle$:

$$|\Omega\rangle = \begin{cases} x^{-f}|\text{phys}\rangle & \text{for } (H, Q, \overline{Q}, B = \frac{1}{2}) \\ x^{f}|\text{phys}\rangle & \text{for } (H, Q, \overline{Q}, B = -\frac{1}{2}) \\ x^{-f}e^{\frac{x^2}{2}}|\text{phys}\rangle & \text{for } (T_1, M, \overline{M}, B = \frac{1}{2}) \\ x^{f}e^{-\frac{x^2}{2}}|\text{phys}\rangle & \text{for } (T_1, M, \overline{M}, B = -\frac{1}{2}) \\ x^{-f}e^{-\frac{x^2}{2}}|\text{phys}\rangle & \text{for } (T_2, N, \overline{N}, B = \frac{1}{2}) \\ x^{f}e^{\frac{x^2}{2}}|\text{phys}\rangle & \text{for } (T_2, N, \overline{N}, B = -\frac{1}{2}) \end{cases} \tag{3.4.56}$$

where $|\text{phys}\rangle$ is a x independent state. We see that only the set of generators $(T_1, M, \overline{M}, B = -\frac{1}{2})$ can yield the normalizable eigenfunction of the ground state. In order to obtain the normalizable ground state, $|\text{phys}\rangle$ need to be the eigenstate with $B = -\frac{1}{2}$. Let us define a state $|0\rangle$ annihilated by the operator $\overline{\psi}$

$$\overline{\psi}|0\rangle = 0. \tag{3.4.57}$$

Then $B|0\rangle = -\frac{1}{2}|0\rangle$ and we thus can choose the state $|0\rangle$ as $|\text{phys}\rangle$. Given the state $|0\rangle$, one can build up a tower of states by multiplying the operator ψ. Since the square of the Grassmann variable is zero $\psi^2 = 0$, the fermionic generators form the two-dimensional space spanned by

$$|0\rangle, \quad \psi|0\rangle \tag{3.4.58}$$

and $\overline{\psi}$ and ψ are identified with the lowering operator and raising operator for fermionic excitation respectively. Therefore we obtain the normalizable ground state

$$|\Omega\rangle = x^{f}e^{-\frac{x^2}{2}}|0\rangle \tag{3.4.59}$$

which satisfies

$$M|\Omega\rangle = \overline{M}|\Omega\rangle = 0, \tag{3.4.60}$$

$$N|\Omega\rangle = 0, \tag{3.4.61}$$

$$\overline{\psi}|\Omega\rangle = 0. \tag{3.4.62}$$

Having found the eigenfunction of L_0, we see from (2.2.32) and (3.4.59) that the ground state $|\Omega\rangle$ is the eigenstate of L_0 with the eigenvalue

$$r_0 = \frac{1}{2}\left(f + \frac{1}{2}\right) \tag{3.4.63}$$

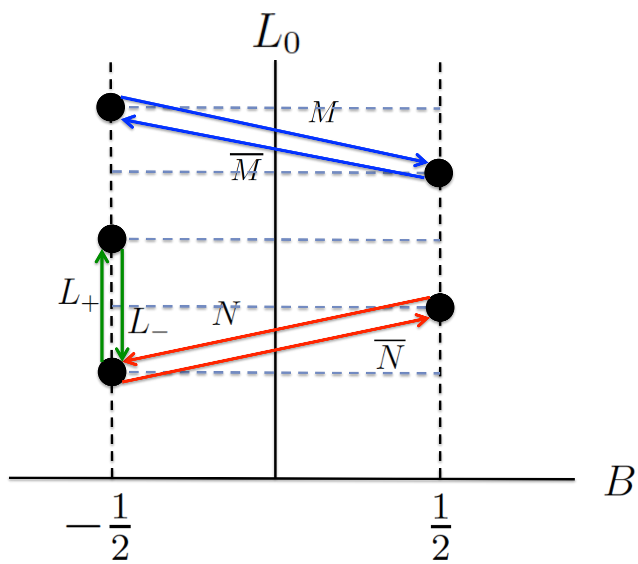


Figure 3.4: The bosonic excitations and the fermionic excitations in the L_0 spectrum. For a fixed B, i.e. for the bosonic excitation generated by L_+ and L_-, the space is one unit. For a fermionic excitation generated by $\overline{N}$, N, $\overline{M}$ and M the space is half a unit.

and obtain the two series (3.4.38) labeled by B. We observe from the commutation relations (3.4.49) that the fermionic generator M, N decreases L_0 by $\frac{1}{2}$ while $\overline{M}, \overline{N}$ increase L_0 by $\frac{1}{2}$. As seen from the relations (3.4.60), the fermionic excitation for the ground state $|\Omega\rangle$ can be generated by only $\overline{N}$. In addition, there are bosonic excitations. As in the DFF-model, L_+ increases L_0 by one and L_- decreases L_0 by one. While the fermionic excitations shift the eigenvalue of B, the bosonic excitations does not. The excitations in the L_0 spectrum are drawn in Figure 3.4.

From the relations (3.4.43), (3.4.44), (3.4.52) and (3.4.53) one can see that the two sets of new supercharges $(M, \overline{M})$ and $(N, \overline{N})$ yield the bosonic operators T_1 and T_2 respectively. Since the bosonic operators T_1 and T_2 are compact, one may also use T_1 or T_2 as the new Hamiltonian. However, unlike the compact operator L_0, T_1 and T_2 enjoy the double structures of their spectrums according to the commutation relations (3.4.52) and (3.4.53).

Now consider the spectrum of T_1. By noting the relations (3.4.43) and (3.4.60), we see that the ground state $|\Omega\rangle$ has zero eigenvalue of T_1. According to the commutation relations (3.4.50) and (3.4.54), one finds that for the T_1 spectrum the bosonic and fermionic excitations have the same spacing equal to one, which are generated by L_+, L_- and $\overline{N}, N$ respectively. Note that $\overline{M}, M$ commute with T_1 and do not play the role of the raising and lowering operators. The T_1 spectrum is

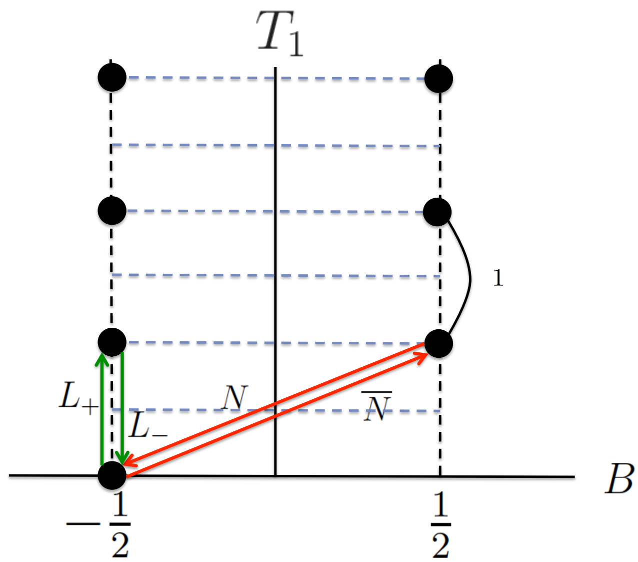


Figure 3.5: The level structure of T_1 spectrum and its bosonic and fermionic excitations. Each of the bosonic and fermionic excitations has the equal space of one unit. The ground state $|\Omega\rangle$ has zero eigenvalue.

given by the two series

$$T_1 = \begin{cases} 0, 1, 2, \cdots & \text{for } B = -\frac{1}{2} \\ 1, 2, \cdots & \text{for } B = \frac{1}{2}, \end{cases} \tag{3.4.64}$$

which is illustrated in Figure 3.5. For all non-zero T_1 states, there are degenerate structures. In other words the bosonic and fermionic states are always paired at the excited level of T_1. This is due to the relations (3.4.52), which ensure the preservation of the supersymmetry generated by M and $\overline{M}$. Therefore one can interpret the pairing structure of T_1 spectrum at excited states as the consequence of the preserved supersymmetry generated by M and $\overline{M}$.

Similarly the spectrum T_2 holds the doublet structure because T_2 commute with N and $\overline{N}$ and the corresponding supersymmetry is preserved as seen from (3.4.53). In this case the bosonic excitation is generated by L_+, L_- whereas the fermionic one is generated by $\overline{M}, M$. Also one can see from (3.4.51) and (3.4.55) that both bosonic and fermionic excitations are produced with equal spacing of one unit. In this case, however, there is no normalizable zero T_2 state. The normalizable ground state $|\Omega\rangle$ has the eigenstate of T_2 with the eigenvalue $(4f+2)$. The T_2 spectrum is

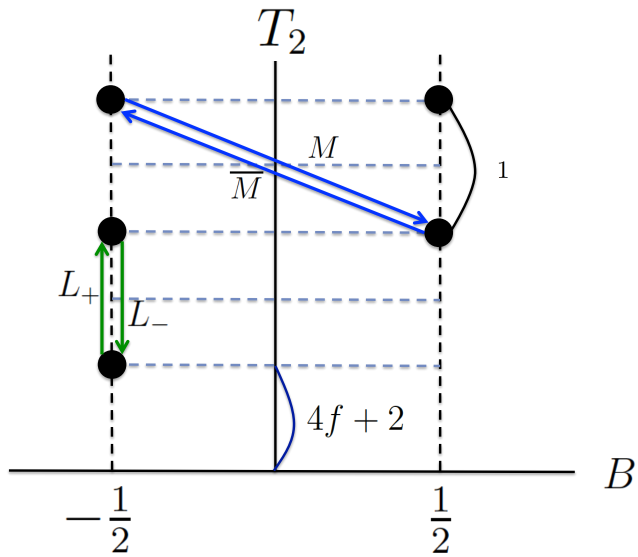


Figure 3.6: The level structure of T_2 spectrum and its bosonic and fermionic excitations. Each of the bosonic and fermionic excitations has the equal space of one unit. The ground state $|\Omega\rangle$ has eigenvalue $(4f+2)$.

given by

$$T_2 = \begin{cases} 4f+2+n & \text{for } B = -\frac{1}{2} \\ 4f+3+n & \text{for } B = \frac{1}{2} \end{cases} \tag{3.4.65}$$

where $n = 0, 1, 2, \cdots$. The T_2 spectrum and its excitation are shown in Figure 3.6.

3.4.2 Multi-particle model

Now we want to discuss the $\mathcal{N} = 2$ superconformal sigma-model. Let us start with n $(\mathbf{1}, \mathbf{2}, \mathbf{1})$ supermultiplets $\Phi_a, a = 1, \cdots, n$ [9]. The generic action without superpotential terms takes the form [200]

$$S = \frac{1}{2} \int dt d^2\theta \left[(g+b)_{ij} \overline{D}\Phi^i D\Phi^j + l_{ij} D\Phi^i D\Phi^j + m_{ij} \overline{D}\Phi^i \overline{D}\Phi^j \right] \tag{3.4.66}$$

where g_{ij} is the metric and b_{ij}, l_{ij} and m_{ij} are the two-forms on the target space $\mathcal{M}$. Note that the terms of l_{ij} and m_{ij} correspond to the non-Lorentz invariant terms in two-dimensions. Notice that the target space $\mathcal{M}$ of the $(\mathbf{1}, \mathbf{2}, \mathbf{1})$ supermultiplet, or $\mathcal{N} = 2A$ multiplet is a real manifold.

[9] The $(\mathbf{1}, \mathbf{2}, \mathbf{1})$ supermultiplet is also called $\mathcal{N} = 2A$ multiplet while $(\mathbf{2}, \mathbf{2}, \mathbf{0})$ chiral supermultiplet is also called $\mathcal{N} = 2B$ multiplet [200].

We have already defined the two covariant derivatives D and $\overline{D}$ for $\mathcal{N} = 2$ supersymmetry in (3.4.2), however, more generally in terms of the two $\mathcal{N} = 1$ covariant derivatives D_1, D_2 the $\mathcal{N} = 2$ two covariant derivatives can be chosen as

$$D_2\Phi^i = I^i{}_j D_1 \Phi^j \tag{3.4.67}$$

where I is an endomorphism of the tangent bundle of $\mathcal{M}$. Then the anti-commutation relations $\{D_i, D_j\} = 2i\partial_t$ impose the conditions

$$I^2 = -1, \tag{3.4.68}$$

$$N(I) = 0 \tag{3.4.69}$$

where $N(I)$ is the Nijenhuis tensor of the endomorphism I. The condition (3.4.68) implies that I is the almost complex structure and the condition (3.4.69) further shows that the I is an (integrable) complex structure. Thus $\mathcal{N} = 2$ supersymmetry requires a complex structure I on the target space [199].

To go further let us follow the strategy in [201] and express the second supersymmetry transformation in terms of the $\mathcal{N} = 1$ superspace formalism as [69]

$$\delta x^i = -i\epsilon I^i{}_j \psi^j, \tag{3.4.70}$$

$$\delta\psi^i = -\epsilon \left[I^i{}_j \dot{x}^j - i\psi^k \left(\partial_k I^i{}_j \right) \psi^j \right]. \tag{3.4.71}$$

Following the Noether's procedure, we obtain the second supercharge

$$Q_2 = \psi^i I_i{}^j \Pi_j - \frac{i}{2}\psi^i I^j{}_i c_{jkl}\psi^j\psi^k - \frac{i}{6}\psi^i\psi^j\psi^k I^l{}_i I^m{}_j I^n{}_k c_{lmn} - \frac{i}{2}\psi^i c_{ijk} I^{jk} \tag{3.4.72}$$

where $\Pi_i := g_{ij}\dot{x}^j$. Then it turns out that the $\mathcal{N} = 1$ action (3.3.6) is invariant under the $\mathcal{N} = 2$ supersymmetry transformations if we have [200, 203]

$$g_{ij} = I^k{}_i I^l{}_j g_{kl}, \tag{3.4.73}$$

$$\nabla^{(+)}_{(i} I^k{}_{j)} = 0, \tag{3.4.74}$$

$$\partial_{[i} \left(I^m{}_j c_{|m|kl]} \right) - 2I^m{}_{[i}\partial_{[m} c_{jkl]]} = 0 \tag{3.4.75}$$

where $\nabla^{(+)}_i$ is the connection with torsion c on $\mathcal{M}$; $\Gamma^i_{jk} + c^i{}_{jk}$. The first constraint (3.4.73) requires that the metric g on $\mathcal{M}$ is Hermitian with respect to the complex structure I. The second condition (3.4.74) is a generalized Yano tensor condition with torsion [10]. This corresponds to the vanishing of $\{Q_1, Q_2\}$ where Q_1 is the $\mathcal{N} = 1$ supercharge given by (3.3.14). The third condition (3.4.75) yields the restriction

[10]For vanishing torsion the equation (3.4.74) coincides with the Yano tensor condition as in [204]

on torsion and complex structure, however it has no geometrical interpretation so far.

It is known that the $\mathcal{N} = 2$ supermultiplets in one dimension are related to the $\mathcal{N} = 1$ supersymmetry in two dimensions

$$\begin{aligned} &1\text{d } \mathcal{N} = 2A \Leftrightarrow 2\text{d } \mathcal{N} = (1,1), \\ &1\text{d } \mathcal{N} = 2B \Leftrightarrow 2\text{d } \mathcal{N} = (2,0) \end{aligned} \tag{3.4.76}$$

by the dimensional reduction. Note that two-dimensional $(2,0)$ supersymmetry sigma-models requires the first condition (3.4.73), however, the last two conditions (3.4.74) and (3.4.75) do not appear in two-dimensional $(2,0)$ sigma-models. Instead of (3.4.74), there appears the covariant constant condition of I with respect to the connection $\nabla^{(+)}$

$$\nabla_i^{(+)} I^j{}_k = 0. \tag{3.4.77}$$

Now we consider the $\mathcal{N} = 2$ superconformal condition. Promoted from the $\mathfrak{osp}(1|2)$ algebra (3.3.19)-(3.3.22), the $\mathfrak{su}(1,1|1)$ algebra (3.4.18)-(3.4.21) contain the $U(1)$ R-symmetry generator B. From the commutation of the supercharges Q_1 in (3.3.14) and Q_2 in (3.4.72) with the conformal boost generator K we can read the superconformal charges

$$S_1 = \psi^i D_i, \qquad\qquad S_2 = \psi^i I^j{}_i D_j. \tag{3.4.78}$$

Then the R-symmetry generator B can be found from the commutator of Q and S_2 as

$$B = D_2^i \Pi_i - i I_{ij} \psi^i \psi^j - i D_2^i c_{ijk} \psi^j \psi^k. \tag{3.4.79}$$

The constraint can be found from the commutation relation $[D, Q_2] = \frac{i}{2} Q_2$, which leads to

$$\mathcal{L}_D I^j{}_i = 0. \tag{3.4.80}$$

This implies that D preserves the complex structure I, that is D acts holomorphically. Combining the constraint (3.4.80) with the other required conditions (3.3.29) and (3.4.74), we also find

$$\mathcal{L}_{\tilde{D}} I^j{}_i = 0, \qquad\qquad \mathcal{L}_{\tilde{D}} g_{ij} = 0, \tag{3.4.81}$$

which means that $\tilde{D}^i := D^j I^i{}_j$ generates a holomorphic isometry.

Therefore the $SU(1,1|1)$ superconformal quantum sigma-model with vanishing bosonic potential can be realized if the conformal invariant conditions (2.8.9), (2.8.16), the $\mathcal{N} = 2$ supersymmetry invariant conditions (3.4.73)-(3.4.75) and the $SU(1,1|1)$ superconformal invariant conditions (3.4.80), (3.4.81) are satisfied. The last additional constraints on the target space $\mathcal{M}$ require that D acts holomorphically and $\tilde{D}^i : D^j I^i{}_j$ generates a holomorphic isometry.

3.4.3 Freedman-Mende model

Let us consider n $(\mathbf{1},\mathbf{2},\mathbf{1})$ supermultiplets Φ_a,$a = 1, \cdots, n$ and a simple superfield action given by

$$S = \frac{1}{2}\int dtd^2\theta \left[\sum_{a=1}^{n} \overline{D}\Phi_a D\Phi_a - W(\Phi)\right] \tag{3.4.82}$$

where $W(\Phi)$ is the superpotential. In terms of the component fields the action (3.4.82) is expressed as

$$S = \frac{1}{2}\int dt \left[\sum_{a=1}^{n}\left(\dot{x}_a^2 + i\dot{\overline{\psi}}_a\psi_a - i\overline{\psi}_a\dot{\psi}_a\right) - \frac{1}{4}\sum_{a=1}^{N}\partial_a W\partial_a W - \sum_{a,b}\left(\partial_a\partial_b W\right)\psi_a\overline{\psi}_b\right] \tag{3.4.83}$$

where $\partial_a := \frac{\partial}{\partial x_a}$. Taking into account the superconformal boost transformation on the $(\mathbf{1},\mathbf{2},\mathbf{1})$ multiplet

$$\delta\Phi_a = -i\left(\eta\overline{\theta} + \overline{\eta}\theta\right)\Phi_a \tag{3.4.84}$$

and the invariance of the measure $\delta\left(dtd\theta\right) = 0$ we find that the action (3.4.83) is invariant under the superconformal boost transformation only if we have

$$\Phi_a\partial_a W(\Phi) = c \tag{3.4.85}$$

with c being a constant. It has been shown [205] that c characterizes the central charge in $\mathfrak{su}(1,1|1)$ superconformal algebra and that the superpotential $W(\Phi)$ is a harmonic function of Φ_a if quantum Hamiltonian contain boson-fermion interaction but no boson-boson interaction.

It is interesting to note that the superpotential [206]

$$W(\Phi) = f\sum_{a\neq b}\ln\left(\Phi_a - \Phi_b\right) \tag{3.4.86}$$

where f is a constant gives rise to the Freedman-Mende model [207]

$$S = \frac{1}{2}\int dt \left[\sum_{a=1}^{n}\left(\dot{x}_a^2 + i\dot{\overline{\psi}}_a\psi_a - i\overline{\psi}_a\dot{\psi}_a\right) - \sum_{a\neq b}\frac{f^2 + 4f\psi_a\overline{\psi}_b}{4(x_a - x_b)^2}\right]. \tag{3.4.87}$$

This is the $\mathcal{N} = 2$ superconformal generalization of the Calogero model. For the Freedman-Mende model the central charge Z in the $\mathfrak{su}(1,1|1)$ superconformal algebra can be identified with

$$Z = n(n-1)f. \tag{3.4.88}$$

The Freedman-Mende model is the supersymmetric rational A_{n+1} Calogero model in the sense that the original Calogero model is obtained by projecting the supersymmetric Hamiltonian onto the zero fermion sector.

If we have the superpotential

$$W(\Phi) = f\ln\left(\sum_a \Phi_a\Phi_a\right) \tag{3.4.89}$$

with f being a constant, then we find [208]

$$S = \frac{1}{2}\int dt\sum_{a=1}^{N}\left[\dot{x}_a^2 + i\dot{\overline{\psi}}_a\psi_a - i\overline{\psi}_a\dot{\psi}_a - \frac{f(f-2\psi_a\overline{\psi}_a)}{x_a^2}\right]. \tag{3.4.90}$$

Unlike the Freedman-Mende model (3.4.88), the interaction terms are not pairwise but still possess the inverse square behavior. This is the $\mathcal{N} = 2$ superconformal mechanics describing the motion of the n-particle center of mass and the corresponding central charge Z in the superconformal algebra $\mathfrak{su}(1,1|1)$ is [208]

$$Z = 2f. \tag{3.4.91}$$

3.4.4 Gauged superconformal mechanics

We start with the $\mathcal{N} = 2$ matrix superfield gauged mechanical action [152, 209, 137]

$$S = \int dtd^2\theta\left[\mathrm{Tr}\left(\overline{\mathcal{D}}\mathcal{X}\mathcal{D}\mathcal{X}\right) + \frac{1}{2}\overline{\mathcal{Z}}e^{2V}\mathcal{Z} - c\mathrm{Tr}V\right]. \tag{3.4.92}$$

Here we have

- the $\mathcal{N} = 2$ Grassmann-even Hermitian $n\times n$ matrix superfield $\mathcal{X}_a^b(t,\theta,\overline{\theta})$ which satisfies $(\mathcal{X})^\dagger = \mathcal{X}$ and transforms as the adjoint representation of $U(n)$; the $(\mathbf{1},\mathbf{2},\mathbf{1})$ supermultiplet

- the $\mathcal{N}=2$ Grassmann-even chiral superfield $\mathcal{Z}_a(t_L,\theta)$, $\overline{\mathcal{Z}}^a(t_R,\overline{\theta})$, $t_{L,R}=t\pm i\theta\overline{\theta}$ which transform as the fundamental representations of $U(n)$; the $(\mathbf{2},\mathbf{2},\mathbf{0})$ supermultiplets

- the $\mathcal{N}=2$ Grassmann-even complex $n\times n$ matrix bridge superfield $b_a^b(t,\theta,\overline{\theta})$ which satisfies $\overline{b}:=b^\dagger$.

Note that gauge superfields are described by the complex $n\times n$ matrix bridge superfields b or by the prepotential V defined by

$$e^{2V}=e^{-i\overline{b}}e^{ib}. \tag{3.4.93}$$

The covariant derivatives are defined by

$$\mathcal{D}\mathcal{X}=D\mathcal{X}+i[\mathcal{A},\mathcal{X}], \qquad \overline{\mathcal{D}}\mathcal{X}=\overline{D}\mathcal{X}+i[\overline{\mathcal{A}},\mathcal{X}] \tag{3.4.94}$$

where [11]

$$D=\frac{\partial}{\partial\theta}+i\overline{\theta}\frac{\partial}{\partial t}, \qquad \overline{D}=-\frac{\partial}{\partial\overline{\theta}}+i\theta\frac{\partial}{\partial t}, \qquad \{D,\overline{D}\}=-2i\partial_t \tag{3.4.95}$$

where the connections $\mathcal{A}$ are deduced from the bridge superfields

$$\mathcal{A}=-ie^{i\overline{b}}\left(De^{-i\overline{b}}\right), \qquad \overline{\mathcal{A}}=-ie^{ib}\left(\overline{D}e^{-ib}\right). \tag{3.4.96}$$

The superconformal boost transformations are [210]

$$\delta t=-i\left(\eta\overline{\theta}+\overline{\eta}\theta\right)t, \qquad \delta\theta=-\eta(t+i\theta\overline{\theta}), \tag{3.4.97}$$

$$\delta\overline{\theta}=-\eta(t-i\theta\overline{\theta}), \qquad \delta(dtd^2\theta)=0, \tag{3.4.98}$$

$$\delta\mathcal{X}=-i\left(\eta\overline{\theta}+\overline{\eta}\theta\right)\mathcal{X}, \qquad \delta\mathcal{Z}=0, \tag{3.4.99}$$

$$\delta b=0, \qquad \delta V=0. \tag{3.4.100}$$

The action (3.4.92) is invariant under the $U(n)$ transformations [152, 209, 137]

$$e^{ib}\to e^{i\Lambda}e^{ib}e^{-i\lambda}, \qquad e^{i\overline{b}}\to e^{i\Lambda}e^{i\overline{b}}e^{-i\overline{\lambda}}, \qquad e^{2V}\to e^{i\overline{\lambda}}e^{2V}e^{-i\lambda}, \tag{3.4.101}$$

$$\mathcal{X}\to e^{i\Lambda}\mathcal{X}e^{-i\Lambda}, \qquad \mathcal{Z}\to e^{i\lambda}\mathcal{Z}, \qquad \overline{\mathcal{Z}}\to\overline{\mathcal{Z}}e^{-i\overline{\lambda}}. \tag{3.4.102}$$

Here Λ is the Hermitian $n\times n$ matrix gauge parameter and λ are the complex $n\times n$ gauge parameters.

Alternatively if we use the Λ gauge invariant superfields V, $\mathcal{Z}$ and the new Hermitian $n\times n$ matrix superfield

$$\mathcal{Y}=e^{-ib}\mathcal{X}e^{i\overline{b}}, \tag{3.4.103}$$

[11] The notation here is different from (3.4.2).

the action (3.4.92) can be written as

$$S = \int dt d^2\theta \left[\mathrm{Tr} \left(\overline{\mathcal{D}} \mathcal{Y} e^{2V} \mathcal{D} \mathcal{Y} e^{2V} \right) + \frac{1}{2} \overline{\mathcal{Z}} e^{2V} \mathcal{Z} - c \mathrm{Tr} V \right] \tag{3.4.104}$$

where the covariant derivatives are

$$\mathcal{D}\mathcal{Y} = D\mathcal{Y} + e^{-2V}(De^{2V})\mathcal{Y}, \qquad \overline{\mathcal{D}}\mathcal{Y} = \overline{D}\mathcal{Y} - \mathcal{Y}e^{2V}\left(\overline{D}e^{-2V}\right). \tag{3.4.105}$$

The $\mathcal{N} = 2$ superfields V, $\mathcal{Y}$, $\mathcal{Z}$ and $\overline{\mathcal{Z}}$ can be expressed in terms of the component fields as

$$V = v + \theta\xi - \overline{\theta}\overline{\xi} + \theta\overline{\theta}A, \tag{3.4.106}$$
$$\mathcal{Y} = x + \theta\psi - \overline{\theta}\overline{\psi} + \theta\overline{\theta}y, \tag{3.4.107}$$
$$\mathcal{Z} = z + 2i\theta\zeta + i\theta\overline{\theta}\dot{z}, \tag{3.4.108}$$
$$\overline{\mathcal{Z}} = \overline{z} + 2i\overline{\theta}\overline{\zeta} - i\theta\overline{\theta}\dot{\overline{z}}. \tag{3.4.109}$$

According to the gauge transformation (3.4.101), let us choose the gauge so that

$$V(t, \theta, \overline{\theta}) = \theta\overline{\theta}A_0(t). \tag{3.4.110}$$

After integrating out the auxiliary fields $\zeta, \overline{\zeta}$ and performing the Grassmann integrations, we get the $\mathcal{N} = 2$ superconformal gauged mechanical action

$$\begin{aligned} S = \int dt \Bigg[& \mathrm{Tr}\left(DxDx\right) + \frac{i}{2}\left(\overline{z}Dz - D\overline{z}z\right) \\ & + i\mathrm{Tr}\left(\overline{\psi}D\psi - D\overline{\psi}\psi\right) - c\mathrm{Tr}A_0 \Bigg] \end{aligned} \tag{3.4.111}$$

where the covariant derivatives are

$$Dx = \dot{x} + i[A_0, x], \qquad Dz = \dot{z} + iA_0 z, \tag{3.4.112}$$
$$D\psi = \dot{\psi} + i[\psi, A_0], \qquad D\overline{\psi} = \dot{\overline{\psi}} + i[\overline{\psi}, A_0]. \tag{3.4.113}$$

The action (3.4.111) is the supersymmetric generalization of the Calogero model whose bosonic part agrees with the Calogero model (2.9.2). The action is invariant with respect to the $U(n)$ gauge transformations

$$x \to gXg^{-1}, \qquad z \to gz, \tag{3.4.114}$$
$$\psi \to g\psi g^{-1}, \qquad A_0 \to gA_0 g^{-1} + i\dot{g}g^{-1} \tag{3.4.115}$$

where $g \in U(n)$. By fixing the gauge as in (2.9.9) and (2.9.14), $z, \overline{z}$ and the non-diagonal part of x are eliminated and thus we have

$$\begin{aligned} &n \text{ physical bosons } x_a^a, \\ &2n^2 \text{ physical fermions } \psi_a^b, \overline{\psi}_a^b. \end{aligned} \tag{3.4.116}$$

This is different from the Freedman-Mende model (3.4.88) which possesses n physical bosons and $2n$ physical fermions. which can be realized as

$$(\boldsymbol{n}, \boldsymbol{2n^2}, \boldsymbol{2n^2 - n}) = n(\mathbf{1}, \mathbf{2}, \mathbf{1}) \oplus (n^2 - n)(\mathbf{0}, \mathbf{2}, \mathbf{2}). \tag{3.4.117}$$

It has been pointed out [211] that the supermultiplet (3.4.117) can be obtained from n $(\mathbf{1}, \mathbf{2}, \mathbf{1})$ supermultiplets by gauging procedure.

3.5 $\mathcal{N} = 4$ Superconformal mechanics

As we have discussed in subsection 3.1.3, the most general superconformal algebra of $\mathcal{N} = 4$ superconformal quantum mechanics is $D(2,1;\alpha)$. As opposed to the $\mathcal{N} = 1$ superconformal algebra $\mathfrak{osp}(1|2)$ and $\mathcal{N} = 2$ superconformal algebra $\mathfrak{su}(1,1|1) \cong \mathfrak{osp}(2|2)$, the Lie superalgebra $D(2,1;\alpha)$ is a one-parameter family of superalgebra characterized by a real parameter α. In order to construct the corresponding family of $\mathcal{N} = 4$ superconformal quantum mechanical modelds parametrized by α, it is desirable to find the inequivalent irreducible off-shell supermultiplets in a systematic way.

To this end there is the methodical way proposed in [212] by means of the non-linear realizations technique [155, 156, 157]. We shall start from the superconformal algebra $D(2,1;\alpha)$, wihch contains three conformal charges H, D, K which form $\mathfrak{sl}(2,\mathbb{R})$, four supercharges $Q^i, \overline{Q}_i$, $i = 1,2$, four superconformal charges $S^i, \overline{S}_i$ and two commuting sets of $\mathfrak{su}(2)$ R-symmetry generators $J, \overline{J}, J_3$ and $I, \overline{I}, I_3$ [12]

$$[H, D] = iH, \quad [K, D] = -iK, \quad [H, K] = 2iD, \tag{3.5.1}$$

$$[H, Q^i] = 0, \qquad [D, Q^i] = -\frac{i}{2} Q^i, \qquad [K, Q^i] = -iS^i, \tag{3.5.2}$$

$$[H, S^i] = iQ^i, \qquad [D, S^i] = \frac{i}{2} Q^i, \qquad [K, S^i] = 0, \tag{3.5.3}$$

[12] Here we use the notation in [183, 213, 212], which is slightly different from our previous $\mathcal{N} = 1$ and $\mathcal{N} = 2$ cases in that the signs of the anti-commutators (3.5.4) and the covariant derivatives (3.5.28) and the supercharges (3.5.29).

$$\begin{aligned}
&\{Q^i,\overline{Q}_j\} = -2\delta^i_j H, \quad \{S^i,\overline{S}_j\} = -2\delta^i_j K, && \{Q^i,S^j\} = -2(1+\alpha)\epsilon^{ij}\overline{I},\\
&\{Q^1,\overline{S}_2\} = 2\alpha\overline{J} \qquad \{Q^1,\overline{S}_1\} = -2D - 2\alpha J_3 + 2(1+\alpha)I_3, &&\\
&\{Q^2,\overline{S}_1\} = -2\alpha J, \quad \{Q^2,\overline{S}_2\} = -2D + 2\alpha J_3 + 2(1+\alpha)I_3, &&
\end{aligned} \tag{3.5.4}$$

$$\begin{aligned}
&[J_3,Q^1] = -\frac{i}{2}Q^1, \quad [J_3,Q^2] = \frac{i}{2}Q^2, \quad [J,Q^1] = -iQ^2, \quad [J,\overline{Q}_2] = i\overline{Q}_1,\\
&[J_3,S^1] = -\frac{i}{2}S^1, \quad [J_3,S^2] = \frac{i}{2}S^2, \quad [J,S^1] = -iS^2, \quad [J,\overline{S}_2] = i\overline{S}_1,\\
&[I_3,Q^i] = -\frac{i}{2}Q^i, \quad [I,Q^i] = -i\overline{Q}^i, \quad [I_3,S^i] = -\frac{i}{2}S^i, \quad [I,S^i] = -i\overline{S}^i,
\end{aligned} \tag{3.5.5}$$

$$\begin{aligned}
&[J_3,J] = iJ, \qquad [J_3,\overline{J}] = -i\overline{J}, \qquad [J,\overline{J}] = -2iJ_3,\\
&[I_3,I] = iI, \qquad [I_3,\overline{I}] = -i\overline{J}, \qquad [I,\overline{I}] = -2iJ_3.
\end{aligned} \tag{3.5.6}$$

The R-symmetry group contains two $SU(2)$ factors generated by $J, \overline{J}, J_3$ and $I, \overline{I}, I_3$. Looking at the commutation relations (3.5.5), J corresponds to the rotations indices i of θ_i while I mixes θ_i with their complex conjugates.

Here we take bosonic conformal generators H, D, K as Hermitian operators

$$(H)^\dagger = H, \qquad (D)^\dagger = D, \qquad (K)^\dagger = K \tag{3.5.7}$$

while the other operators are chosen so that

$$(J)^\dagger = \overline{J}, \qquad (J_3)^\dagger = -J_3, \tag{3.5.8}$$

$$(I)^\dagger = \overline{I}, \qquad (I_3)^\dagger = -I_3, \tag{3.5.9}$$

$$\overline{(Q^i)} = \overline{Q}_i, \qquad \overline{(S^i)} = \overline{S}_i. \tag{3.5.10}$$

The parameter α only appears in the anti-commutation relations (3.5.4), from which we see that two $\mathfrak{su}(2)$ R-symmetry algebras appear with relative weights α and $-(1+\alpha)$. Note that the conformal algebra $\mathfrak{sl}(2,\mathbb{R})$ has relative weight 1. Thus the transformation

$$\alpha \leftrightarrow -(1+\alpha) \tag{3.5.11}$$

exchanges the role of two R-symmetry algebras; $J \leftrightarrow I$. On the other hand, the transformation

$$\alpha \leftrightarrow \frac{1}{\alpha} \tag{3.5.12}$$

is not well-defined for our real $D(2,1;\alpha)$ superalgebra because it exchanges the role of the non-compact conformal algebra $\mathfrak{sl}(2,\mathbb{R})$ and the compact R-symmetry algebra $\mathfrak{su}(2)$.

In particular we have the isomorphism

$$D(2,1;\alpha) \cong \begin{cases} \mathfrak{su}(1,1|2) + \mathfrak{su}(2) & \text{for } \alpha = -1, 0 \\ \mathfrak{osp}(4^*|2) & \text{for } \alpha = 1, -2 \\ \mathfrak{osp}(4|2) & \text{for } \alpha = -\frac{1}{2} \end{cases} \tag{3.5.13}$$

At $\alpha = -1$ and $\alpha = 0$ one of the R-symmetry $\mathfrak{su}(2)$ algebra is decoupled and the superalgebra $D(2,1;-1)$ is isomorphic to the semi-direct sum $\mathfrak{su}(1,1|2) + \mathfrak{su}(2)$ [13]. In this case one can extend the $\mathfrak{su}(1,1|2)$ superalgebra by adding the central charges. To see this let us put the $\mathfrak{su}(2)$ generators J, $\overline{J}$ and J_3 as

$$Z \equiv \alpha J, \qquad \overline{Z} \equiv \alpha \overline{J}, \qquad Z_3 \equiv \alpha J_3 \tag{3.5.14}$$

where Z, $\overline{Z}$ and Z_3 commute with everything. Then the new generators Z, $\overline{Z}$ and Z_3 only appear in the anti-commutations (3.5.4) and they now become

$$\begin{aligned} &\{Q^i, \overline{Q}_j\} = -2\delta^i_j H, && \{S^i, \overline{S}_j\} = -2\delta^i_j K, && \{Q^i, S^j\} = -2\epsilon^{ij}\overline{I}, \\ &\{Q^1, \overline{S}_2\} = 2\overline{Z}, && \{Q^1, \overline{S}_1\} = -2D - 2Z_3 + 2I_3, && \\ &\{Q^2, \overline{S}_1\} = -2Z, && \{Q^2, \overline{S}_2\} = -2D + 2Z_3 + 2I_3. && \end{aligned} \tag{3.5.15}$$

Hence the three generators Z, $\overline{Z}$ and Z_3 are identified with the central charges. Note that we can only have single nonvanishing central charge by taking into account the $SU(2)$ transformation on the three central charges.

As its name suggests, $D(2,1;\alpha)$ is regarded as a deformation of the superalgebra $D(2,1) = \mathfrak{osp}(4|2)$ that corresponds to the case $\alpha = 1$, however, we are now considering the even part of $D(2,1;\alpha)$ as $\mathfrak{sl}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ not as $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. The first $\mathfrak{sl}(2)$ factor is the conformal algebra and the remaining two factors are replaced with the compact algebras $\mathfrak{su}(2)$. Consequently $\mathfrak{so}^*(4)$, the non-compact version of the original factor $\mathfrak{so}(4)$ shows up for $\alpha = 1$. We see that the case of $\alpha = -\frac{1}{2}$ is self-dual under the transformation (3.5.11). In our case this degenerate case realizes the $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ factor and all the other cases can be thought of as the deformations of $D(2,1;\frac{1}{2}) = \mathfrak{osp}(4|2)$.

Using the generators of $D(2,1;\alpha)$, let us consider the supercoset of $D(2,1;\alpha)$

$$g = e^{itH} e^{iuD} e^{izK} e^{\theta_i Q^i + \overline{\theta}^i \overline{Q}_i} e^{\psi_i S^i + \overline{\psi}^i \overline{S}_i} e^{i\varphi J + i\overline{\varphi}\overline{J}} e^{\phi J_3} \tag{3.5.16}$$

where the parameters $t, \theta_i, \overline{\theta}^i$ are the coordinates of the $\mathcal{N} = 4$ superspace $\mathbb{R}^{(1|4)}$ and the other parameters $u, z, \psi_i, \overline{\psi}^i, \varphi, \overline{\varphi}$ and ϕ are the $\mathcal{N} = 4$ Goldstone superfields. Note that the R-symmetry group $SU(2)$ generated by $(I, \overline{I}, I_3)$, which mixes

[13] We use $\oplus$ for the direct sum and $+$ for the semi-direct sum.

the fermionic charges with their conjugates, is taken into the supercoset but considered as the stability subgroup. Note that our chice of the supercoset (3.5.16) is allowed for the case of $\alpha \neq 0$ where the generators $(J, \overline{J}, J_3)$ exist.

From the supercoset one can extract left-covariant Cartan one-form Ω

$$\Omega = g^{-1}dg. \tag{3.5.17}$$

Expanding Ω over the generators, we find the the corresponding one-forms [212]

$$\omega_D = idu - 2\left(\overline{\psi}^i d\theta_i + \psi_i d\overline{\theta}^i\right) - 2izd\tilde{t}, \tag{3.5.18}$$

$$\omega_V = \frac{e^{-i\phi}}{1+\Lambda\overline{\Lambda}}\left(id\Lambda + \hat{\omega}_J + \Lambda^2\hat{\overline{\omega}}_J - \Lambda\hat{\omega}_{J_3}\right), \tag{3.5.19}$$

$$\overline{\omega}_V = \frac{e^{i\phi}}{1+\Lambda\overline{\Lambda}}\left(id\overline{\Lambda} + \hat{\overline{\omega}}_J + \overline{\Lambda}^2\hat{\omega}_J + \overline{\Lambda}\hat{\omega}_{J_3}\right), \tag{3.5.20}$$

$$\omega_{J_3} = d\phi + \frac{1}{1+\Lambda\overline{\Lambda}}\left[i\left(d\Lambda\overline{\Lambda} - \Lambda d\overline{\Lambda}\right) + \left(1-\Lambda\overline{\Lambda}\right)\hat{\omega}_{J_3} - 2\left(\Lambda\hat{\overline{\omega}}_J - \overline{\Lambda}\hat{\omega}_J\right)\right] \tag{3.5.21}$$

where

$$\hat{\omega}_J = 2\alpha\left[\psi_2 d\overline{\theta}^1 - \overline{\psi}^1\left(d\theta_2 - \psi_2 d\tilde{t}\right)\right], \tag{3.5.22}$$

$$\hat{\overline{\omega}}_J = 2\alpha\left[\overline{\psi}^2 d\theta_1 - \psi_1\left(d\overline{\theta}^2 - \overline{\psi}^2 d\tilde{t}\right)\right], \tag{3.5.23}$$

$$\hat{\omega}_{J_3} = 2\alpha\left[\psi_1 d\overline{\theta}^1 - \overline{\psi}^2 d\theta_1 - \psi_2 d\overline{\theta}^2 + \overline{\psi}^2 d\theta_2 + \left(\overline{\psi}^1\psi_1 - \overline{\psi}^2\psi_2\right)d\tilde{t}\right], \tag{3.5.24}$$

$$d\tilde{t} = dt + i\left(\theta_i d\overline{\theta}^i + \overline{\theta}^i d\theta_i\right), \tag{3.5.25}$$

$$\Lambda = \frac{\tan\sqrt{\varphi\overline{\varphi}}}{\sqrt{\varphi\overline{\varphi}}}. \tag{3.5.26}$$

For the $\mathcal{N}=4$ superspace $\mathbb{R}^{(1|4)}$ parametrized by [210]

$$\mathbb{R}^{(1|4)} = (t, \theta_i, \overline{\theta}^j), \qquad (\theta_i)^\dagger = \overline{\theta}^i, \qquad i,j = 1,2 \tag{3.5.27}$$

we will introduce the covariant derivatives

$$D^i = \frac{\partial}{\partial\theta_i} + i\overline{\theta}^i\frac{\partial}{\partial t}, \qquad \overline{D}_j = \frac{\partial}{\partial\overline{\theta}^j} + i\theta_j\frac{\partial}{\partial t}, \qquad \left\{D^i, \overline{D}_j\right\} = 2i\delta^i_j\partial_t. \tag{3.5.28}$$

The supercharges Q and $\overline{Q}$ can be expressed by

$$Q^i = \frac{\partial}{\partial\theta_i} - i\overline{\theta}^i\frac{\partial}{\partial t}, \qquad \overline{Q}_j = \frac{\partial}{\partial\overline{\theta}^j} - i\theta_j\frac{\partial}{\partial t}, \qquad \left\{Q^i, \overline{Q}_j\right\} = -2i\delta^i_j\partial_t \tag{3.5.29}$$

in the superspace.

By acting a particular element on the supercoset element (3.5.16) from the left, we can find the corresponding transformations.

1. supersymmetry transformations

   Acting the element

$$g_\epsilon = e^{\epsilon_i Q^i + \overline{\epsilon}^i \overline{Q}_i} \in D(2,1;\alpha), \tag{3.5.30}$$

   we obtain the supersymmetry transformations

$$\delta t = i\left(\theta\overline{\epsilon} - \epsilon\overline{\theta}\right), \tag{3.5.31}$$

$$\delta\theta_i = \epsilon_i, \tag{3.5.32}$$

$$\delta\overline{\theta}^i = \overline{\epsilon}^i. \tag{3.5.33}$$

2. superconformal boost transformations

   Acting the element

$$g_\eta = e^{\eta_i S^i + \overline{\eta}_i \overline{S}_i}, \tag{3.5.34}$$

   one finds the superconformal boost transformations [213, 212, 214]

$$\delta t = -it\left(\eta\overline{\theta} + \overline{\eta}\theta\right) + (1+2\alpha)\left(\theta\overline{\theta}\right)\left(\eta\overline{\theta} - \overline{\eta}\theta\right), \tag{3.5.35}$$

$$\delta\theta_i = \eta_i t - 2i\alpha\theta_i(\theta\overline{\eta}) + 2i\left(1+\alpha\right)\theta_i\left(\overline{\theta}\eta\right) - i(1+2\alpha)\eta_i(\theta\overline{\theta}), \tag{3.5.36}$$

$$\delta u = -2i\left(\eta\overline{\theta} + \overline{\eta}\theta\right), \tag{3.5.37}$$

$$\begin{aligned}\delta\phi = 2\alpha\Bigg[&\overline{\eta}^1\theta_1 - \overline{\eta}^2\theta_2 - \eta_1\overline{\theta}^1 + \eta_2\overline{\theta}^2 \\ &+ \left(\overline{\eta}^2\theta_1 - \eta_1\overline{\theta}^2\right)\Lambda + \left(\overline{\eta}^1\theta_2 - \eta_2\overline{\theta}^1\right)\overline{\Lambda}\Bigg],\end{aligned} \tag{3.5.38}$$

$$\begin{aligned}\delta\Lambda = 2i\alpha\Bigg[&\theta_2\overline{\eta}^1 - \overline{\theta}^1\eta_2 + \left(\overline{\theta}^2\eta_1 - \theta_1\overline{\eta}^2\right)\Lambda^2 \\ &+ \left(\overline{\theta}^1\eta_1 - \theta_1\overline{\eta}^1 + \theta_2\overline{\eta}^2 - \overline{\theta}^2\eta_2\right)\Lambda\Bigg]\end{aligned} \tag{3.5.39}$$

   and

$$\delta(dtd^4\theta) = 2i\left(\eta\overline{\theta} + \overline{\eta}\theta\right)dtd^4\theta, \tag{3.5.40}$$

$$\begin{aligned}\delta D^i = {}& i\left[(2+\alpha)(\eta\overline{\theta}) + \alpha(\theta\overline{\eta})\right]D^i \\ &- 2i(1+\alpha)(\overline{\eta}\overline{\theta})\overline{D}^i - 2i\alpha\left[\eta^{(i}\overline{\theta}_{k)} + \theta^{(i}\overline{\epsilon}_{k)}\right]D^k,\end{aligned} \tag{3.5.41}$$

$$\begin{aligned}\delta\overline{D}_i = {}& i\left[(2+\alpha)(\overline{\eta}\theta) + \alpha(\overline{\theta}\eta)\overline{D}_i\right] \\ &- 2i(1+\alpha)(\theta\eta)D_i - 2i\alpha\left[\eta_{(i}\overline{\theta}_{k)} + \theta_{(i}\overline{\epsilon}_{k)}\right]\overline{D}^k.\end{aligned} \tag{3.5.42}$$

At this stage we are ready to derive the irreducible off-shell supermultiplets which allow us to construct the $D(2,1;\alpha)$ superconformal mechanics. The strategy is to extract the irreducible superfields from the Goldstone superfields $u, z, \psi_i, \overline{\psi}^i, \varphi, \overline{\varphi}$ by imposing the appropriate constraints. Since the number of the fermionic Goldstone superfields is four which coincides with the minimal number of the fermionic fields in $\mathcal{N}=4$ supermultiplets, we attempt to reduce the number of the bosonic Goldstone superfields. It has been discussed [213, 212] that such irreducibility condition can be achieved by requiring that all spinor derivatives of all bosonic superfields are expressed in terms of the fermionic fields ψ_i and $\overline{\psi}^i$. From the equations (3.5.18)-(3.5.22), we see that this requirement corresponds to the constraints on the corresponding Cartan forms ω_D, ω_J, $\overline{\omega}_V$ ω_{J_3}.

3.5.1 $(4,4,0)$ supermultiplet

Let us begin with the most general case where the supercoset (3.5.17) holds all four bosonic Goldstone superfields u, φ ,$\overline{\varphi}$ and ϕ. If we require that the all spinor covariant derivatives of these bosonic superfields can be expressed by ψ_i,$\overline{\psi}^i$, then (3.5.18)-(3.5.22) lead to

$$\omega_D = \omega_J| = \overline{\omega}_J| = \omega_{J_3} = 0 \tag{3.5.43}$$

where $|$ represents the restriction to spinor projection. The set of constraints (3.5.43) can be rewritten as

$$D^{(i}q^{j)} = 0, \qquad \overline{D}^{(i}q^{j)} = 0, \qquad D^{(i}\overline{q}^{j)} = 0, \qquad \overline{D}^{(i}\overline{q}^{j)} = 0 \tag{3.5.44}$$

where

$$\begin{aligned} q^1 &= \frac{e^{\frac{1}{2}(\alpha u - i\phi)}}{\sqrt{1+\Lambda\overline{\Lambda}}}\Lambda, & q^2 &= \frac{e^{\frac{1}{2}(\alpha u - i\phi)}}{\sqrt{1+\Lambda\overline{\Lambda}}}, \\ \overline{q}_1 &= \frac{e^{\frac{1}{2}(\alpha u + i\phi)}}{\sqrt{1+\Lambda\overline{\Lambda}}}, & \overline{q}_2 &= \frac{e^{\frac{1}{2}(\alpha u + i\phi)}}{\sqrt{1+\Lambda\overline{\Lambda}}} \end{aligned} \tag{3.5.45}$$

are identified with four $\mathcal{N}=4$ superfields. This multiplet was discussed in [199, 200, 215, 216, 212, 217, 211, 218, 219] and was considered in $\mathcal{N}=4$ harmonic superspace [214]. The constraints (3.5.44) lead to the following independent fields:

$$\begin{cases} q^i & \text{4 physical bosons} \\ D_i q^i, \overline{D}_i q^i, D_i \overline{q}^i, \overline{D}_i \overline{q}^i & \text{4 fermions.} \end{cases} \tag{3.5.46}$$

The superfield q_i contains 4 bosonic, 4 fermionic fields and no auxiliary fields and is diagnosed as the $(\mathbf{4},\mathbf{4},\mathbf{0})$ supermultiplet. Since q^i and their set of constraints (3.5.44) are similar to the $d = 4$ $\mathcal{N} = 2$ hypermultiplet, it is also called hypermultiplet. However, the conditions (3.5.44) for the $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet defines the off-shell multiplet as opposed to the $d = 4$ $\mathcal{N} = 2$ hypermultiplet.

Remarkably it has been discussed that all other $\mathcal{N} = 4$ supermultiplets can be obtained from $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet via reduction process either on the component action [217] or on the superfield action [211, 218, 219]. Accordingly the $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet can be viewed as a fundamental multiplet.

Since we know the superconformal boost transformations (3.5.35)-(3.5.39) for the original Goldstone superfields, we can read off the superconformal boost transformations for the superfields q^i, $\overline{q}_i$

$$\delta q^i = 2i\alpha \left(\overline{\theta}^i \eta_j - \theta^i \overline{\eta}_j\right) q^j. \tag{3.5.47}$$

This leads to the transformations $\delta(q\overline{q}) = -2i\alpha(\eta\overline{\theta} + \overline{\eta}\theta)(q\overline{q})$, which cancel the transformation (3.5.40) of the integration measure. Therefore we can write superconformally invariant superfield action

$$S = \int dt d^4\theta \ \ (q\overline{q})^{\frac{1}{\alpha}}. \tag{3.5.48}$$

Note that this vanishes when $\alpha = -1$ due to the constraints (3.5.44). For $\alpha = -1$ the superconformal superfield action is given by [214, 212]

$$S = \int dt d^4\theta \ \frac{\ln(q\overline{q})}{q\overline{q}}. \tag{3.5.49}$$

It is worthwhile to remark that these two expressions (3.5.48) and (3.5.49) can be written uniformly by adding the overall factor as

$$\frac{1}{1+\alpha}\int dt d^4\theta \ \ (q\overline{q})^{\frac{1}{\alpha}}. \tag{3.5.50}$$

One can check that (3.5.50) is regular for any α and coincides with (3.5.49) at $\alpha = -1$.

Although there is a superpotential term for the $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet [214] which is a Wess-Zumino type term [14] of first order in time derivative, it does not produce any non-trivial potential for physical bosons. Therefore one cannot construct $\mathcal{N} = 4$ superconformal mechanics with the non-trivial potential for the physical bosons by using the $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet only. On the other hand, it has been discussed [211, 218] that the gauged action of the $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet generates more generic actions.

[14] The superfield Wezz-Zumino type potential term for all $\mathcal{N} = 4$ multiplets can be represented manifestly only in the harmonic superspace [137].

3.5.2 $(3,4,1)$ supermultiplet

Let us set $\phi = 0$ in the supercoset (3.5.16). This enforces us to put the corresponding subgroup $U(1) \subset SU(2)$ into the stability subgroup and thus the resulting supercoset involves $SL(2,\mathbb{R}) \times SU(2)/U(1)$. To realize the spinor covariant derivatives of the remaining bosonic superfields u, Λ, $\overline{\Lambda}$, we should impose the conditions

$$\omega_D = \omega_J| = \overline{\omega}_J| = 0. \tag{3.5.51}$$

The set of conditions (3.5.51) can be expressed as

$$D^{(i}V^{jk)} = 0, \qquad \overline{D}^{(i}V^{jk)} = 0 \tag{3.5.52}$$

where

$$V^{11} = -i\sqrt{2}e^{\alpha u}\frac{\Lambda}{1+\Lambda\overline{\Lambda}}, \quad V^{22} = i\sqrt{2}e^{\alpha u}\frac{\overline{\Lambda}}{1+\Lambda\overline{\Lambda}}, \quad V^{12} = \frac{i}{\sqrt{2}}e^{\alpha u}\frac{1-\Lambda\overline{\Lambda}}{1+\Lambda\overline{\Lambda}}. \tag{3.5.53}$$

Note that the $\mathcal{N}=4$ superfields V^{ij} is real and satisfy the relations

$$V^{ij} = V^{ji}, \qquad \overline{V^{ij}} = \epsilon_{ik}\epsilon_{jl}V^{kl}, \qquad V^2 := V^{ij}V_{ij} = e^{2\alpha u}. \tag{3.5.54}$$

The superfield V^{ij} obeying the constraints (3.5.52) was firstly introduced in [220] and later discussed in [221, 222, 70, 213, 212, 214, 211, 218]. The constraints (3.5.52) give rise to the independent components

$$\begin{cases} V^{11}, V^{12}, V^{12} & \text{3 physical bosons} \\ D^1V^{12}, D^2V^{12}, \overline{D}^1V^{12}, \overline{D}^2V^{12} & \text{4 fermions} \\ D^i\overline{D}^jV_{ij} & \text{1 auxiliary boson} \end{cases} \tag{3.5.55}$$

Thus we can identify the superfield V^{ij} with the $(\mathbf{3},\mathbf{4},\mathbf{1})$ supermultiplet. Since the constraints (3.5.52) are obtained by the dimensional reduction from the constraints of the $d=4$ $\mathcal{N}=2$ tensor multiplet [223], the $(\mathbf{3},\mathbf{4},\mathbf{1})$ multiplet is also called tensor multiplet [15].

From (3.5.37)-(3.5.39), one can read the $D(2,1;\alpha)$ superconformal boost transformations of V^{ij}

$$\delta V^{ij} = -2i\alpha\left[\left(\eta\overline{\theta}+\overline{\eta}\theta\right)V^{ij} + \left(\eta^{(i}\overline{\theta}_k - \overline{\eta}_k\theta^{(i}\right)V^{j)k} + \left(\eta_k\overline{\theta}^{(i} - \overline{\eta}^{(i}\theta_k\right)V^{j)k}\right] \tag{3.5.56}$$

[15] The superfield V^{ij} can also be obtained by the dimensional reduction from $d=4$ $\mathcal{N}=1$ vector multiplet [221] as the spatial component of $d=4$ Abelian gauge vector connection superfield.

The superfield action for the kinetic term is given by [213, 214, 212]

$$S_{\text{kin}} = \int dt d\theta \ (V^2)^{\frac{1}{2\alpha}} \tag{3.5.57}$$

where V^2 is defined in (3.5.54). The action (3.5.57) vanishes for $\alpha = -1$. The superfield action for the kinetic term in the case of $\alpha = -1$ is

$$S_{\text{kin}} = -\frac{1}{2}\int dt d^4\theta \ (V^2)^{-\frac{1}{2}} \ln V^2. \tag{3.5.58}$$

It has been pointed out [213] that both of the action (3.5.57) and (3.5.58) can be described in a unified form as

$$S_{\text{kin}} = \frac{1}{1+\alpha}\int dt d\theta \ (V^2)^{\frac{1}{2\alpha}}. \tag{3.5.59}$$

The superconformally invariant potential term for the $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ multiplet can be written as [213]

$$S_{\text{pot}} = -i\sqrt{2}\int dt d^4\theta \ \left[\int_0^1 dy \partial_y \mathcal{W} \frac{1}{\sqrt{V^2}}\right] \tag{3.5.60}$$

where $\mathcal{W}$ is the prepotential satisfying

$$V^{ij} = D^{(i}\overline{D}^{j)}\mathcal{W}, \qquad \overline{W} = -\mathcal{W}. \tag{3.5.61}$$

Note that the constraints (3.5.52) are solved by an unconstraint prepotential $\mathcal{W}$. Alternative way to obtain the potential term for the $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ multiplet has been proposed as an integral over the analytic harmonic superspace [214].

Combining the kinetic terms (3.5.59) and the potential terms (3.5.60), we find the bosonic superconformal actions in component fields as [213]

$$\begin{aligned}
S_{\text{bosonic}} &= \mu^{-1}\frac{\alpha^2}{1+\alpha}(S_{\text{kin}})_{\text{bosonic}} + \nu(S_{\text{pot}})_{\text{bosonic}} \\
&= \int dt \left[\mu^{-1}\alpha^2 e^u \dot{u}^2 + 4\mu^{-1}e^u \frac{\dot{\Lambda}\dot{\overline{\Lambda}}}{(1+\Lambda\overline{\Lambda})^2} - \frac{1}{4}\mu\nu^2 e^{-u} + i\nu \frac{\overline{\Lambda}\dot{\Lambda} - \Lambda\dot{\overline{\Lambda}}}{1+\Lambda\overline{\Lambda}}\right] \\
&= \frac{1}{2}\int dt \left[4\alpha^2\mu\dot{r}^2 + \mu r^2\left(\dot{\vartheta}^2 + \sin^2\vartheta\dot{\varphi}^2\right) - \frac{\nu^2}{\mu r^2} + 2\nu\cos\vartheta\dot{\varphi}\right] \\
&= \int dt \left[\mu g_{ij}(X)\dot{X}^i\dot{X}^j - \frac{1}{4\mu}\frac{\nu^2}{|\mathbf{X}|^2} + 2i\nu\frac{\epsilon^{3ij}X^i\dot{X}^j}{(X^2+|\mathbf{X}|)\,|\mathbf{X}|}\right]
\end{aligned} \tag{3.5.62}$$

where

$$\Lambda = \tan\frac{\vartheta}{2}e^{i\varphi}, \qquad e^{\frac{u}{2}} = \frac{1}{\sqrt{2}}\mu r, \tag{3.5.63}$$

$$g_{ij}(X) = \delta_{ij} + (4\alpha^2 - 1)\frac{X_i X_j}{|\mathbf{X}|^2}. \tag{3.5.64}$$

Observing the two explicit expressions (3.5.45) and (3.5.54) for the two superfields q^i and V^{ij} in terms of the initial Goldstone superfields, we can express the superfields V^{ij} as

$$V^{11} = -i\sqrt{2}q^1\overline{q}^1, \quad V^{22} = -i\sqrt{2}q^2\overline{q}^2, \quad V^{12} = -\frac{i}{\sqrt{2}}\left(q^1\overline{q}^2 + q^2\overline{q}^1\right). \tag{3.5.65}$$

Also one can check that if the the irreducible constraints (3.5.44) for the $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplet are satisfied by $q^i, \overline{q}^i$, then the constraints (3.5.52) for the $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ multiplet are also solved by (3.5.65) [214]. However, it is important to note that (3.5.65) are not general but rather special solutions to the$(\mathbf{3}, \mathbf{4}, \mathbf{1})$ multiplet. So the generic $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ multiplet cannot be covered by (3.5.65).

3.5.3 $(2, 4, 2)$ supermultiplet

Now we will put $u = 0, z = 0, \phi = 0$ in the supercoset (3.5.16). Then the supercoset contain only two bosonic fields φ, $\overline{\varphi}$ or equivalently Λ, $\overline{\Lambda}$, which parametrize the two-sphere $S^2 \sim SU(2)/U(1)$. The condition that the spinor covariant derivatives of φ, $\overline{\varphi}$ can be expressed in terms of ψ, $\overline{\psi}$ is

$$\omega_J| = \omega_{\overline{J}}| = 0. \tag{3.5.66}$$

For $\alpha \neq -1$ these the conditions (3.5.66) are written as

$$D^1\Lambda = -\Lambda D^2\Lambda, \qquad \overline{D}_2\Lambda = \Lambda\overline{D}_1\Lambda. \tag{3.5.67}$$

Under the constraints (3.5.67) the superfield Λ, $\overline{\Lambda}$ yields the independent component fields

$$\begin{cases} \Lambda, \overline{\Lambda} & \text{2 physical bosons} \\ -D^1\overline{\Lambda}, \overline{D}_1\Lambda, D^2\Lambda, -\overline{D}_2\overline{\Lambda} & \text{4 fermions} \\ \overline{D}_1 D^2\Lambda, \overline{D}_2 D^1\overline{\Lambda} & \text{2 auxiliary bosons,} \end{cases} \tag{3.5.68}$$

which implies the $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ supermultiplet. This multiplet is called non-linear chiral multiplet because the constraints (3.5.67) can be viewed as the modified chirality conditions so that they are also covariant with respect to $D(2, 1; \alpha)$. Note that, apart from the non-linear realization of $D(2, 1; \alpha)$, the $\mathcal{N} = 4$ chiral multiplet $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ is constructed by a complex superfields ϕ, $\overline{\phi}$ obeying the constraints

$$D^i\phi = 0, \qquad \overline{D}_j\overline{\phi} = 0. \tag{3.5.69}$$

It has been discussed that one cannot construct superconformal superfield actions out of the $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ multiplet alone due to the absence of the dilaton u [212]. In order to obtain superconformal superfield actions, the coupling to some other $\mathcal{N} = 4$ supermultiplets is needed.

In terms of the hypermultiplet q, $\overline{q}$, the superfield Λ, $\overline{\Lambda}$ can be written as

$$\Lambda = -\frac{q^1}{q^2}, \qquad \overline{\Lambda} = -\frac{\overline{q}_1}{\overline{q}_2}. \tag{3.5.70}$$

These are just the special solutions to the constraint equations (3.5.67) for the non-linear chiral multiplet.

3.5.4 $(1, 4, 3)$ supermultiplet

Let us retain the dilaton u alone in the supercoset (3.5.16). This corresponds to putting two $SU(2)$ R-symmetry factors into the stability subgroup. The irreducible condition

$$\omega_D| = 0 \tag{3.5.71}$$

just implies that the four spinor derivatives of u is expressed by the four fermionic Goldstone superfield $\psi, \overline{\psi}$. Therefore the equation (3.5.71) does not impose any constraints on the superfield u. The independent component fields are [210]

$$\begin{cases} e^u & \text{1 physical bosons} \\ D^i u, \overline{D}_i u & \text{4 fermions} \\ [D^{(i}, \overline{D}^{j)}]e^u, [D^i, \overline{D}_i]e^u & \text{3 auxiliary bosons} \end{cases} \tag{3.5.72}$$

and this means the $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ supermultiplet. However, as was shown in [210], one should impose additional irreducible constraints on the dilaton u

$$D^i D_i e^{-\alpha u} = \overline{D}_i \overline{D}^i e^{-\alpha u} = [D^i, \overline{D}_i] e^{-\alpha u} = 0 \tag{3.5.73}$$

for the minimal $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ multiplet. It has been pointed out [213] that if we build up the u superfield out of the $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ superfield V^{ij} satisfying (3.5.52) as

$$e^{-\alpha u} = \frac{1}{\sqrt{V^2}}, \tag{3.5.74}$$

then u automatically obeys the minimal constraints (3.5.73). Substituting the relation (3.5.74) into (3.5.60) and (3.5.58), we obtain the superconformal superfield action [16]

$$S=\int dtd^4\theta\ e^u \tag{3.5.75}$$

for $\alpha\neq -1$ and

$$S=\int dtd^4\theta\ e^u u \tag{3.5.76}$$

for $\alpha=-1$. By putting the overall factor, we can express the superconformal superfield actions for both cases as [213]

$$S=\frac{1}{1+\alpha}\int dtd^4\theta\ e^u. \tag{3.5.77}$$

Combining (3.5.37) and (3.5.40), one can check that the superfield action (3.5.77) is invariant under the superconformal boost transformations. Note that (3.5.77) is not defined at $\alpha=0$ because of our choice of the supercoset (3.5.16) and it should be treated separately [211, 218].

Inserting the appropriate set of component fields which solve the minimal constraints (3.5.73) into the superfield action (3.5.77), integrating over the Grassmann coordinates $\theta_i,\overline{\theta}^i$ and integrating out the auxiliary fields, one finds the one particle $D(2,1;\alpha)$ superconformal mechanical model [224, 137]

$$S=\frac{1}{2}\int dt\left[\dot{x}^2+i\left(\overline{\psi}_i\dot{\psi}^i-\dot{\overline{\psi}}_i\psi^i\right)+\frac{2}{3}(1+2\alpha)\frac{\psi^i\overline{\psi}^j\psi_{(i}\overline{\psi}_{j)}}{x^2}\right]. \tag{3.5.78}$$

Although the action (3.5.78) does not possess bosonic potential at the classical level, upon the quantization the anti-commutation for the fermions may yield a purely bosonic potential term. We see that the potential terms just flip the overall sign under the transformation α (3.5.11).

As we have already seen (3.5.13), when $\alpha=-1,0$ the $\mathcal{N}=4$ superconformal algebra $D(2,1;\alpha)$ is isomorphic to the semi-direct sum of $\mathfrak{su}(1,1|2)$ and $\mathfrak{su}(2)$, which implies that one of the $SU(2)$ symmetry is broken and the superalgebra $\mathfrak{su}(1,1|2)$ allows for the central charge. So the irreducible constraints for the bosonic Goldstone superfields can be weakened by adding the central charge [224, 137]. The constraints (3.5.73) can be modified as

$$D^iD_ie^{-\alpha u}=0,\qquad \overline{D}_i\overline{D}^ie^{-\alpha u}=0,\qquad [D^i,\overline{D}_i]e^{-\alpha u}=c \tag{3.5.79}$$

or

$$D^iD_ie^{-\alpha u}=c,\qquad \overline{D}_i\overline{D}^ie^{-\alpha u}=c,\qquad [D^i,\overline{D}_i]e^{-\alpha u}=0 \tag{3.5.80}$$

[16] In the original work in [210] only the $SU(1,1|2)$ invariant action with $\alpha=-1$ was considered.

where c is the central charge of the $\mathfrak{su}(1,1|2)$ superalgebra. The two constraints correspond to the case where the broken $SU(2)$ symmetry is taken as the rotation of θ coordinates and $\overline{\theta}$ coordinates respectively [210]. The solutions to the new constraint equations acquire the additional term proportional to $\theta\overline{\theta}c$. Then one obtains the one particle $SU(1,1|2)$ superconformal mechanical action [210, 224, 137]

$$S = \frac{1}{2}\int dt \left[\dot{x}^2 + i\left(\overline{\psi}_i\dot{\psi}^i - \dot{\overline{\psi}}_i\psi^i\right) - \frac{\left(c+\overline{\psi}_i\psi^i\right)^2}{x^2} \right]. \tag{3.5.81}$$

Note that the additional contribution from the central charge c yields the inverse square type bosonic potential at the classical level.

3.5.5 $(0,4,4)$ supermultiplet

Although we have seen that the irreducibility conditions for the supermultiplet can be systematically obtained by means of the non-linear realization method, there is a further possible supermultiplet $(\mathbf{0},\mathbf{4},\mathbf{4})$ [17]. It is described by a fermionic analytic superfield in the harmonic superspace (HSS) [214].

The harmonic superspace (HSS) is the extension of the original superspace by introducing the new commuting harmonic coordinate $u_i^{\pm}$, $i=1,2$ parametrizing the internal degrees of freedom as the two-sphere $S^2 \sim SU(2)/U(1)$ with $SU(2)$ being the R-symmetry [225].

$$\begin{aligned} \mathbb{HR}^{(1+2|4)} &= (t_A, \theta^+, \overline{\theta}^+, \theta^-, \overline{\theta}^-, u_i^+, u_k^-) \\ &= (\zeta, u_i^+, u_k^-, \theta^-, \overline{\theta}^-) \end{aligned} \tag{3.5.82}$$

where

$$t_A := t - i(\theta^+\overline{\theta}^- + \theta^-\overline{\theta}^+), \qquad \theta^{\pm} = \theta^i u_i^{\pm}, \qquad \overline{\theta}^{\pm} = \overline{\theta}^i u_i^{\pm}, \tag{3.5.83}$$

$$u^{+i}u_i^- = 1, \qquad u_i^+u_j^- - u_j^+u_i^- = \epsilon_{ij}. \tag{3.5.84}$$

The significant property is the existence of an analytic subspace (ASS), which is the quotient of $\mathbb{H}^{(1+2|4)}$ by $\{\theta^-, \overline{\theta}^-\}$

$$\begin{aligned} \mathbb{AR}^{(1+2|2)} &= (\zeta, u) \\ &= (t_A, \theta^+, \overline{\theta}^+, u_i^+, u_k^-). \end{aligned} \tag{3.5.85}$$

[17] At least the author does not know the $(\mathbf{0},\mathbf{4},\mathbf{4})$ supermultiplet based on the non-linear realization of the superconformal group $D(2,1;\alpha)$.

The covariant derivatives in the analytic basis of HSS, $(\zeta, u, \theta^-, \bar{\theta}^-)$ are defined by

$$D^+ = \frac{\partial}{\partial\theta^-}, \qquad \bar{D}^+ = -\frac{\partial}{\partial\bar{\theta}^-}, \tag{3.5.86}$$

$$D^- = -\frac{\partial}{\partial\theta^+} - 2i\bar{\theta}^-\frac{\partial}{\partial t_A}, \qquad \bar{D}^- = \frac{\partial}{\partial\bar{\theta}^+} - 2i\theta^-\frac{\partial}{\partial t_A} \tag{3.5.87}$$

and the harmonic covariant derivatives in the analytic basis of HSS are

$$D^{\pm\pm} = \partial^{\pm\pm} + 2i\theta^\pm\bar{\theta}^\pm\frac{\partial}{\partial t_A} + \theta^\pm\frac{\partial}{\partial\theta^\mp} + \bar{\theta}^\pm\frac{\partial}{\partial\bar{\theta}^\mp}. \tag{3.5.88}$$

The constraints for the $(\mathbf{0},\mathbf{4},\mathbf{4})$ superfield $\Psi^{+a}(\zeta,u)$, $a = 1,2$ [18] are givne by [214]

$$D^{++}\Psi^{+a} = 0. \tag{3.5.89}$$

The solution of the constraint (3.5.89) is written as

$$\Psi^{+a}(\zeta,u) = \psi^{ia}u_i^+ + \theta^+\xi^a + \bar{\theta}^+\bar{\xi}^a + 2i\theta^+\bar{\theta}^+\dot{\psi}^{ia}u_i^- \tag{3.5.90}$$

and the independent component fields are

$$\begin{cases} \psi^{ia} & \text{4 fermions} \\ \xi^a, \bar{\xi}^a & \text{4 auxiliary bosons.} \end{cases} \tag{3.5.91}$$

The $(\mathbf{0},\mathbf{4},\mathbf{4})$ superfield Ψ^{+a} has been discussed in [211, 218, 227]. The action takes the form

$$\begin{aligned} S &= \frac{1}{2}\int dud\zeta^{--}\Psi^{+a}\Psi_a^+ \\ &= \int dt\left[i\psi^{ia}\dot{\psi}_{ia} + \xi^a\bar{\xi}_a\right]. \end{aligned} \tag{3.5.92}$$

Although the action (3.5.92) contains only the kinetic term of the free fermions and the quadratic term of the bosonic auxiliary fields, if we appropriately couple the $(\mathbf{0},\mathbf{4},\mathbf{4})$ multiplet to the other $\mathcal{N} = 4$ supermultiplets, we may produce bosonic potentials [211, 218, 227].

3.5.6 Multi-particle model

WDVV equation

We have seen that the superspace and superfield formalism based on the nonlinear realization technique is useful to build up $\mathcal{N} = 4$ superconformal mechanical models possessing $D(2,1;\alpha)$ symmetry. However, it is known that the direct

[18] The indices $a = 1,2$ denote the doublet of the extra $SU(2)$ called the Pauli-Gürsey group [226].

generalization of the one particle analysis does not work well for the construction of the $D(2,1;\alpha)$ multi-particle superconformal mechanical systems [19]. Hence it is insightful to investigate the construction for the $\mathcal{N}=4$ multi-particle superconformal mechanics in the component level.

Let us consider N particles on $\mathbb{R}$ with canonical variables x^a and their momenta p_p where $a=1,\cdots,N$ label the particles. The $\mathcal{N}=4$ supersymmetry leads to two complex fermions $\psi^a_i, \overline{\psi}^{ai}$, $i=1,2$. In addition, we also consider a one pair of bosonic isospin variables u^i, $i=1,2$ which parametrize the internal degrees of freedom [20].

Now we impose the ansatz for the supercharges Q_i and $\overline{Q}^i$ of the form [228]

$$Q_i = p_a\psi^a_i + U_a(x)K_{ij}\psi^{aj} + iF_{abc}(x)\psi^{aj}\psi^b_j\overline{\psi}^c_i, \tag{3.5.93}$$

$$\overline{Q}^i = p_a\overline{\psi}^{ai} + U_a(x)K_{ij}\overline{\psi}^{aj} - iF_{abc}(x)\overline{\psi}^{aj}\overline{\psi}^b_j\psi^c_i \tag{3.5.94}$$

where $U_a(x)$ and $F_{abc}(x)$ are homogeneous functions of degree -1 in x^a and

$$K_{ij} = \frac{i}{2}\left(u_i\overline{u}_j + u_j\overline{u}_i\right). \tag{3.5.95}$$

Let us consider the Dirac brackets

$$\{x^a, p_b\} = \delta^a_b, \qquad \left\{\psi^a_i, \overline{\psi}^{bj}\right\} = -\frac{i}{2}\delta^k_i\delta^{ab}, \qquad \left\{u^i, \overline{u}_k\right\} = -i\delta^i_k. \tag{3.5.96}$$

Then the $\mathcal{N}=4$ superalgebra

$$\left\{Q_i, \overline{Q}^j\right\} = 2i\delta^j_i H \tag{3.5.97}$$

implies that [229, 228]

$$\partial_a U_b - \partial_b U_a = 0, \tag{3.5.98}$$

$$\partial_a F_{bcd} - \partial_b F_{acd} = 0 \tag{3.5.99}$$

and

$$F_{cae}F_{ebd} - F_{cbe}F_{ead} = 0, \tag{3.5.100}$$

$$\partial_a U_b - U_a U_b - F_{abc}U_c = 0. \tag{3.5.101}$$

[19] In the case of $\alpha=-1,0$ with $SU(1,1|2)$ symmetry, the standard $\mathcal{N}=4$ superspace description can be generalized to the multi-particle case [228].

[20] It has been discussed [228] that isospin variables is needed in order to obtain the multi-particle $D(2,1;\alpha)$ superconformal mechanics for $\alpha\neq-1,0$. See [229] for $\alpha=-1,0$, i.e. $SU(1,1|2)$ superconformal mechanics.

The first set of equations (3.5.98) and (3.5.99) can be solved by

$$U_a(x) = \partial_a U(x), \qquad F_{abc}(x) = \partial_a \partial_b \partial_c F(x) \tag{3.5.102}$$

where $U(x)$ and $F(x)$ are the prepotentials, the scalar functions defined up to polynomials of degree 0 and 2 in x^a respectively. Therefore we have two non-linear differential equations (3.5.100) and (3.5.101) for the prepotential $U(x)$ and $F(x)$. Quite interestingly the equation (3.5.100) is the so-called Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation [230, 231]. It has been established that the solution of WDVV equations determines the structure of a Frobenius manifold. The other equation (3.5.101) describes the so-called twisted periods U_a of the Frobenius manifold [232, 233] [21].

Under the conditions (3.5.100) and (3.5.101) the Hamiltonian can be written as

$$\begin{aligned} H = &\frac{1}{4} p_a p_a + \frac{1}{8} J^{ij} J_{ij} U_a U_a \\ &- i U_{ab} K_{ij} \psi^{ai} \overline{\psi}^{bj} - \frac{1}{2} F_{abcd} \psi^{ai} \psi^b_i \overline{\psi}^c_j \overline{\psi}^{dj}. \end{aligned} \tag{3.5.104}$$

We should note that the $\mathcal{N} = 4$ superconformal algebra $D(2,1;\alpha)$ has not been taken into account so far. So the WDVV equation and the twisted period equation are just the requirement for the conservation of $\mathcal{N} = 4$ supersymmetry.

To realize $D(2,1;\alpha)$ superconformal algebra let us introduce the conformal generators D, K, superconformal generators S_i, $\overline{S}^i$ and the R-symmetry generators J_{ij}

$$D = -\frac{1}{4}\left\{x^p p_a\right\}, \qquad K = x^a x_a, \tag{3.5.105}$$

$$S_i = -2x^a \psi^a_i, \qquad \overline{S}^i = -2x^a \overline{\psi}^{ia}, \tag{3.5.106}$$

$$J_{ij} = K_{ij} + 2i\psi^a_{(i}\overline{\psi}^a_{j)}, \tag{3.5.107}$$

$$I_{11} = i\psi^a_i \psi^{ia}, \qquad I_{22} = -i\overline{\psi}^{ia}\overline{\psi}^a_i, \qquad I_{12} = i\psi^a_i \overline{\psi}^{ia}. \tag{3.5.108}$$

From the dilatation invariance we require the homogeneity

$$\partial_b(x^a U_a) = (x^a \partial_a + 1) U_b = 0, \tag{3.5.109}$$

$$\partial_b(x^a F_{acd}) = (x^a \partial_a + 1) F_{bcd} = 0. \tag{3.5.110}$$

[21] Any function $\tilde{p}$ satisfying

$$\frac{\partial \xi_a}{\partial p^b} = \nu G^{cd} \frac{\partial^3 F_*(p)}{\partial p^d \partial p^a \partial p^b} \xi_c, \qquad \xi_a = \frac{\partial \tilde{p}(p;\nu)}{\partial p^a} \tag{3.5.103}$$

is called twisted period of the Frobenius manifold where p^a are periods.

The remaining $D(2,1;\alpha)$ superconformal algebra (3.5.1)-(3.5.6) then leads to

$$x^a U_a = 2\alpha, \tag{3.5.111}$$

$$x^a F_{abc} = -(1+2\alpha)\delta_{bc}. \tag{3.5.112}$$

For $\alpha \neq -\frac{1}{2}$ the prepotential F is non-vanishing and any two values of α are related by a rescaling under the transformation (3.5.11). In this sense the two conditions (3.5.111) and (3.5.112) can be viewed as the normalization conditions. In the case of $\alpha = -\frac{1}{2}$ which realizes the $OSp(4|2)$ superconformal mechanics, the prepotential F cannot be normalized. This corresponds to the fact that under the reflection (3.5.11) $\alpha = -\frac{1}{2}$ is self-dual [22]. Therefore we can utilize the families of the solutions (U, F) along with the expression (3.5.104) to construct $\mathcal{N} = 4$ multi-particle superconformal mechanics. Since the number of independent equations are given by[137]

$$\begin{cases} \frac{1}{12}(N-1)(N-2)^2(N-3) & \text{for WDVV equation} \\ \frac{1}{2}(N-1)(N-2) & \text{for twisted periods,} \end{cases} \tag{3.5.113}$$

when $N \geq 4$, i.e. the system contains more than four particles, the non-trivial WDVV equation (3.5.100) appears and the twisted periods equation (3.5.101) gives rise to the non-trivial conditions.

At this stage we with to look for the solution F to the WDVV equation (3.5.100) and the twisted periods U_a defined by (3.5.101). However, up to date it is an open mathematical problem to list up all the solutions to the WDVV equation and only part of the solutions are known [234, 235, 236, 237, 238, 239, 240]. In [234] it was shown that one can construct the solutions to the WDVV equation (3.5.100) by imposing the ansatz

$$F(x) = \sum_{\boldsymbol{\alpha}} f_{\boldsymbol{\alpha}} K(\boldsymbol{\alpha} \cdot \boldsymbol{x}) \tag{3.5.114}$$

where

$$K(z) = \begin{cases} -\frac{1}{4}z^2 \ln z^2 & \text{rational case} \\ -\frac{1}{4}\mathrm{Li}_3(e^{2iz}) + \frac{1}{6}z^3 & \text{trigonometric case} \\ -\frac{1}{4}\mathcal{L}i_3(e^{2iz}|\tau) & \text{elliptic case} \end{cases} \tag{3.5.115}$$

with $f_{\boldsymbol{\alpha}} \in \mathbb{R}$ and $\boldsymbol{\alpha} \cdot x = \boldsymbol{\alpha}^a x_a$. Here $\{\boldsymbol{\alpha}\}$ are the covectors constructing a deformed Lie (super)algebra root system [23]. Li_3 is the trilogarithm and $\mathcal{L}i_3$ is an elliptic generalization [242, 243, 244, 245]. Among the above known solutions to the WDVV

[22]The induced metric defined in (3.5.117) is degenerate for $\alpha = -\frac{1}{2}$.

[23]It is known that the root systems of some Lie superalgebras give rise to the solutions to the WDVV equation [237, 241].

equation, only the rational case satisfies the normalization conditions (3.5.111) and (3.5.112). Thus the $D(2,1;\alpha)$ superconformal models may arise for

$$F(x) = -\frac{1}{4}\sum_{\boldsymbol{\alpha}} f_{\boldsymbol{\alpha}} (\boldsymbol{\alpha}\cdot x)^2 \ln|\boldsymbol{\alpha}\cdot x|^2 \tag{3.5.116}$$

The ansatz (3.5.116) defines the constant metric

$$g_{ab} = -x^c F_{cab} = \sum_{\boldsymbol{\alpha}} f_{\boldsymbol{\alpha}} \boldsymbol{\alpha}\otimes\boldsymbol{\alpha}. \tag{3.5.117}$$

Then it was established [235] that certain deformations of root systems can solve the WDVV equation (3.5.100) and the corresponding collections of covectors $\{\boldsymbol{\alpha}\}$ is called $\vee$-systems [246].

On the other hand, it was observed [233] that the ansatz for the twisted periods U_a

$$U(x) = \sum_{\beta} u_\beta \ln P_\beta(x) \tag{3.5.118}$$

can solve the equation (3.5.101) where $P_\beta(x)$ are homogeneous polynomials of degree n_β in x and u_β is chosen so that $\sum_\beta n_\beta u_\beta = 2\alpha$.

Now let us assume that $\alpha \neq -\frac{1}{2}$ and consider the special solutions to the twisted periods as the form

$$U(x) = \sum_{\boldsymbol{\alpha}} u_{\boldsymbol{\alpha}} \ln(\boldsymbol{\alpha}\cdot x) \tag{3.5.119}$$

where the same covectors α are chosen for $U(x)$ and $F(x)$. Then the normalization conditions (3.5.111) and (3.5.112) reduce to

$$\sum_{\boldsymbol{\alpha}} u_{\boldsymbol{\alpha}} = 2\alpha, \tag{3.5.120}$$

$$\sum_{\boldsymbol{\alpha}} f_{\boldsymbol{\alpha}} \boldsymbol{\alpha}_a \boldsymbol{\alpha}_b = (1+2\alpha)\delta_{ab} \tag{3.5.121}$$

and we get the potential term

$$V(x) = \frac{K^{ij}K_{ij}}{8}\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}} u_{\boldsymbol{\alpha}} u_{\boldsymbol{\beta}} \frac{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}{(\boldsymbol{\alpha}\cdot x)\,(\boldsymbol{\beta}\cdot x)}. \tag{3.5.122}$$

By requiring the invariance under permutations of the particle labels, the WDVV solutions F based on deformed root systems of the Lie algebras A_n, BCD_n and EF_n and the Lie superalgebras have been discussed [247]. It is an interesting question to reveal the geometrical understanding for the relevant WDVV solutions and the relation to the construction of the $\mathcal{N}=4$ superconformal mechanical models.

On the contrary, we cannot apply the same method to the $OSp(4|2)$ superconformal mechanical models for $\alpha = -\frac{1}{2}$ since some formulae become singular. One of the illness is the degenerate induced metric

$$\sum_{\boldsymbol{\alpha}} f_{\boldsymbol{\alpha}} \boldsymbol{\alpha} \otimes \boldsymbol{\alpha} = 0, \tag{3.5.123}$$

which can be seen from (3.5.117) and (3.5.121). Since this implies the degenerate covectors $\boldsymbol{\alpha}$, it is natural to consider the degenerate limit of the deformed root systems which solve the WDVV equation. By observing that there exists a degenerate limit in the moduli space of the deformed A_n root systems, the prepotentials for the $OSp(4|2)$ superconformal mechanics have been proposed as [247]

$$F(x) = \frac{1}{4N} \sum_{a<b} (x^a - x^b)^2 \ln(x^a - x^b)^2 - \frac{1}{4N^2} \sum_a (Nx^a - X)^2 \ln(Nx^a - X)^2, \tag{3.5.124}$$

$$U(x) = -\frac{1}{2N} \sum_a \ln (Nx^a - X) \tag{3.5.125}$$

where $X = \sum_a x^a$. Correspondingly we get the potential [247, 137]

$$\begin{aligned} V(x) &= \frac{K^{ij}K_{ij}}{8} \left[\sum_a \frac{1}{(Nx^a - X)^2} - \frac{1}{N} \left(\sum_a \frac{1}{Nx^a - X} \right)^2 \right] \\ &= \frac{K^{ij}K_{ij}}{8N} \sum_{a<b} \left[\frac{1}{Nx^a - X} - \frac{1}{Nx^b - X} \right]. \end{aligned} \tag{3.5.126}$$

We should note that the potential (3.5.126) does not take the form of the Calogero type pairwise interaction albeit it is the inverse-square type interaction.

Sigma-model

We shall study the $\mathcal{N} = 4$ superconformal sigma-model which is more general multi-particle $\mathcal{N} = 4$ superconformal quantum mechanical system [24].

In order to find the condition on the target space geometry, we assume that the second, third and fourth supersymmetry transformations are expressed as [200]

$$\delta\Phi^i = \epsilon^r (I_r)^i{}_j D\Phi^j \tag{3.5.127}$$

where Φ^i is the $(\mathbf{1}, \mathbf{2}, \mathbf{1})$ superfields and ϵ^r, $r = 1, 2, 3$ are the supersymmetry parameters and I_r are the endomorphisms of the tangent bundle of the target space.

[24]Note that in the $\mathcal{N} = 4$ superconformal multi-particle mechanical models relevant to the WDVV equation, the metric is trivial due to the ansatz (3.5.93), (3.5.94).

The corresponding $\mathcal{N}=4$ supermultiplet is referred to as $\mathcal{N}=4B$ multiplet. This is related to the two-dimensional $\mathcal{N}=(4,0)$ supersymmetry. Then the $\mathcal{N}=4$ superalgebra imposes the conditions [199, 200]

$$I_r I_s + I_s I_r = -2\delta_{rs}, \tag{3.5.128}$$

$$N(I_r, I_s) = 0 \tag{3.5.129}$$

where a $N(F,G)$ is Nijenhuis concomitant [248, 249]

$$N(F,G)(X,Y) = [FX,GY] - F[X,GY] - F[GX,Y] + FG[X,Y] + G \leftrightarrow F \tag{3.5.130}$$

where X, Y are vector fields on $\mathcal{M}$. Thus the target space $\mathcal{M}$ possesses three complex sructures I_r which have vanishing mixed Nijenhuis tensors and obey the Clifford algebra (3.5.128). Furthermore the three complex structures turn out to satisfy the algebra of imaginary unit quaternions

$$I_r I_s = -\delta_{rs} + \epsilon_{rst} I_t \tag{3.5.131}$$

or the $\mathfrak{su}(2)$ R-symmetry algebra

$$[I_r, I_s] = 2\epsilon^{rst} I^t \tag{3.5.132}$$

since one can construct a third complex structures from other two by multiplication.

Also the supersymmetry invariance of the action requires that

$$g_{ij} = (I_r)^k{}_i (I_r)^l{}_j g_{kl}, \tag{3.5.133}$$

$$\nabla^{(+)}_{(i} (I_r)^k{}_{j} = 0, \tag{3.5.134}$$

$$\partial_{[i} \left(I^m{}_j c_{|m|kl]} \right) - 2(I_r)^m{}_{[i} \partial_{[m} c_{jkl]]} = 0. \tag{3.5.135}$$

The first condition(3.5.133) implies that the metric g on $\mathcal{M}$ is Hermitian with respect to the three complex structures. The second condition (3.5.134) is a generalized Yano tensor condition with torsion and the third condition (3.5.135) is imposed on torsion and complex structures.

It has been pointed out [200] that the above constraints on the target space $\mathcal{M}$ are similar to the defining conditions for a weak hyperkähler manifold with torsion (HKT) [250]. A weak HKT manifold is a Riemannian manifold $\{\mathcal{M}, g, c\}$ with a metric g, a torsion three-form c and three complex structures I_r, $r=1,2,3$ which obey the following conditions [25]

[25]If c is closed in addition to (3.5.131), (3.5.133), (3.5.136), it is called a strong HKT[250, 200].

1. the three complex structures I_r satisfy the algebra of imaginary of unit quaternions (3.5.131)

2. the metric is Hermitian with respect to the three complex structures; (3.5.133)

3. the complex structures are covariant constant

$$\nabla_k^{(+)} (I_r)^i{}_j = 0 \tag{3.5.136}$$

with respect to the covariant derivative $\nabla^{(+)}$ with the torsion.

We see that the conditions for the weak HKT geometry are only different from the constraints on the target space $\mathcal{M}$ in that the covariant constant properties for the complex structures (3.5.136) are replaced with (3.5.134) and (3.5.135). It turns out that the equation (3.5.136) always solves the constraints (3.5.134) and (3.5.135). Therefore a weak HKT geometry satisfies the constraints (3.5.130)-(3.5.135) on the $\mathcal{N} = 4B$ supersymmetric sigma-models.

Although it is known that the $\mathcal{N} = 4$ supermultiplets in one-dimension hold the connections to the $\mathcal{N} = 2$ supersymmetry in two-dimensions as

$$\begin{aligned} &\text{1d } \mathcal{N} = 4A \Leftrightarrow \text{2d } \mathcal{N} = (2,2), \\ &\text{1d } \mathcal{N} = 4B \Leftrightarrow \text{2d } \mathcal{N} = (4,0), \end{aligned} \tag{3.5.137}$$

we have seen that the target space $\mathcal{M}$ of the $\mathcal{N} = 4B$ sigma-model is not the HKT geometry in two-dimensions, but rather a weak HKT geometry. This shows that there are one-dimensional supermultiplets which cannot be obtained from higher-dimensional supermultiplets.

Furthermore the $D(2,1;\alpha)$ superconformal algebra (3.5.1)-(3.5.6) imposes the additional conditions [69] [26]

$$\mathcal{L}_{D^r} (I_r)^i{}_j = -\frac{2}{1+\alpha} \epsilon^{rst} (I^t)_i{}^j, \qquad\qquad \mathcal{L}_{D^r} g_{ij} = 0 \tag{3.5.138}$$

where $D^r := D^i (I^r)_i{}^j \partial_j$. These conditions (3.5.138) can be viewed as the generalizations of the $\mathcal{N} = 2$ superconformal constraints (3.4.81).

3.5.7 Gauged superconformal mechanics

Consider the $\mathcal{N} = 4$ matrix superfield gauged mechanical action in the harmonic superspace [152]

$$S = S_{\mathcal{X}} + S_{\text{WZ}} + S_{\text{FI}} \tag{3.5.139}$$

[26] Here the value $\alpha = -1, 0$ are excluded.

where

$$S_{\mathcal{X}} = -\frac{1}{4(1+\alpha)} \int \mu_H \mathrm{Tr} \left(\mathcal{X}^{-\frac{1}{\alpha}} \right), \tag{3.5.140}$$

$$S_{\mathrm{WZ}} = \frac{1}{2} \int \mu_A^{(-2)} \mathcal{V}_0 \tilde{\mathcal{Z}}^+ \mathcal{Z}^+, \tag{3.5.141}$$

$$S_{\mathrm{FI}} = \frac{i}{2} c \int \mu_A^{(-2)} \mathrm{Tr} V^{++} \tag{3.5.142}$$

where the integration measures are defined

$$\mu_H = dudtd^4\theta, \qquad \mu_A^{(-2)} = dud\zeta^{(-2)} \tag{3.5.143}$$

with harmonic superspace parametrized by the coordinates (3.5.82)-(3.5.85). The superfields are

- the $\mathcal{N} = 4$ Grassmann-even Hermitian $n \times n$ matrix superfield $\mathcal{X}_a^b(t, \theta^\pm, \overline{\theta}^\pm, u^\pm)$ which obeys

$$\mathcal{D}^{++}\mathcal{X} = 0, \quad \mathcal{D}^+\mathcal{D}^-\mathcal{X} = 0, \quad \left(\mathcal{D}^+\overline{\mathcal{D}}^- + \overline{\mathcal{D}}^+\mathcal{D}^- \right) \mathcal{X} = 0, \tag{3.5.144}$$

  which is the $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ supermultiplet

- the $\mathcal{N} = 4$ Grassmann-even analytic superfield $\mathcal{Z}_a^+(\zeta, u)$ which satisfies

$$\mathcal{D}^{++}\mathcal{Z}^+ = 0, \qquad D^+\mathcal{Z}^+ = 0, \qquad \overline{D}^+\mathcal{Z}^+ = 0, \tag{3.5.145}$$

  which is the $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ supermultiplet and $\tilde{\mathcal{Z}}^+$ being its Hermitian conjugation preserving analyticity [226, 214]

- the $\mathcal{N} = 4$ Grassmann-even $n \times n$ matrix gauge superfield $V_a^{++b}(\zeta, u)$

- the unconstrained real analytic superfield $\mathcal{V}_0(\zeta, u)$ defined by

$$\int du \mathcal{V}_0(t_A, \theta^+, \overline{\theta}^+, u^\pm)|_{\theta^\pm = \theta^i u_i^\pm, \overline{\theta}^\pm = \overline{\theta}^i u_i^\pm} = \mathrm{Tr}\left(\mathcal{X}\right) \tag{3.5.146}$$

where the covariant derivative $\mathcal{D}^{++}$ is given by

$$\mathcal{D}^{++}\mathcal{X} = D^{++}\mathcal{X} + i[V^{++}, \mathcal{X}], \tag{3.5.147}$$

$$\mathcal{D}^{++}\mathcal{Z} = D^{++}\mathcal{Z}^+ + iV^{++}\mathcal{Z}^+. \tag{3.5.148}$$

The first term $S_{\mathcal{X}}$ of the action (3.5.139) is the superconformal action (3.5.77) for the $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ superfield $\mathcal{X}$. The second term S_{WZ} is the Wess-Zumino (WZ) term describing $\mathcal{Z}_a^+$ [211]. The third term S_{FI} is the Fayet-Iliopoulos (FI) term for the gauge superfield V^{++}.

The superconformal boost transformations are [214, 211]

$$\delta t_A = \alpha^{-1}\Lambda t_A, \quad \delta\theta_A = -\eta^+ t_A + 2i(1+\alpha)\eta^-\theta^+\overline{\theta}^+, \quad \delta u_i^+ = \Lambda^{++}u_i^-, \quad (3.5.149)$$

$$\delta\mu_H = \mu_H\left(2\Lambda - \frac{1+\alpha}{\alpha}\Lambda_0\right), \qquad \delta\mu_A^{(-2)} = 0, \quad (3.5.150)$$

$$\delta\mathcal{X} = -\Lambda_0\mathcal{X}, \qquad \delta\mathcal{Z}^+ = \Lambda\mathcal{Z}^+, \qquad \delta V^{++} = 0 \quad (3.5.151)$$

where

$$\lambda = 2i\alpha\left(\overline{\eta}^-\theta^+ - \eta^-\overline{\theta}^+\right), \quad (3.5.152)$$

$$\Lambda^{++} = D^{++}\Lambda = 2i\alpha\left(\overline{\eta}^+\theta^+ - \eta^+\overline{\theta}^+\right), \quad (3.5.153)$$

$$\Lambda_0 = 2\Lambda - D^{--}\Lambda^{++}. \quad (3.5.154)$$

The action (3.5.139) is invariant under the $U(n)$ transformations [152]

$$\mathcal{X} \to e^{i\Lambda}\mathcal{X}e^{-i\Lambda}, \quad (3.5.155)$$

$$\mathcal{Z}^+ \to e^{i\Lambda}\mathcal{Z}^+, \quad (3.5.156)$$

$$\tilde{\mathcal{Z}}^+ \to e^{i\Lambda}\mathcal{Z}^+, \quad (3.5.157)$$

$$V^{++} \to e^{i\Lambda}V^{++}e^{-i\Lambda} - ie^{i\Lambda}\left(D^{++}e^{-i\Lambda}\right) \quad (3.5.158)$$

where $\Lambda_a^b(\zeta, u^\pm)$ is the Hermitian analytic matrix gauge parameter. From the gauge freedom (3.5.155)-(3.5.158) let us fix the gauge as

$$V^{++} = -2i\theta^+\overline{\theta}^+A(t_A). \quad (3.5.159)$$

Integrating out the auxiliary fields by means of their algebraic equations of motion and performing the Grassmann integral, we obtain the $D(2,1;\alpha)$ superconformal mechanics [209]

$$S = \int dt\Bigg[\dot{x}^2 + \frac{i}{2}\left(\overline{z}_i z^i - c\right) - i\overline{\psi}_i\dot{\psi}^i - \dot{\overline{\psi}}_i\psi^i$$
$$- \frac{\alpha^2(\overline{z}_i z^i)^2}{4x^2} + 2\alpha\frac{\psi^i\overline{\psi}^j z_{(i}\overline{z}_{j)}}{x^2} + \frac{2}{3}(1+2\alpha)\frac{\psi^i\overline{\psi}^j\psi_{(i}\overline{\psi}_{j)}}{x^2} - A\left(\overline{z}_i z^i - c\right)\Bigg]. \quad (3.5.160)$$

Using the Noether's method the set of generators are evaluated to be [209]

$$
\begin{aligned}
H = & \frac{1}{4}p^2 + \alpha^2 \frac{(\bar{z}_i z^i)^2 + 2\bar{z}_i z^i}{4x^2} - 2\alpha \frac{z^{(i}\bar{z}^{j)}\psi_{(i}\psi_{k)}}{x^2}, \\
& - (1+2\alpha)\frac{\psi_i\psi^i\bar{\psi}^j\bar{\psi}_j}{2x^2} + \frac{(1+2\alpha)^2}{16x^2}, && (3.5.161) \\
D = & \, tH - \frac{1}{4}\{x, p\}, && (3.5.162) \\
K = & \, t^2 H - \frac{1}{2}t\{x, p\} + x^2, && (3.5.163)
\end{aligned}
$$

$$
\begin{aligned}
Q^i &= p\psi^i + 2i\alpha \frac{z^{(i}\bar{z}^{j)}\psi_j}{x} + i(1+2\alpha)\frac{\langle \psi_j \psi^j \bar{\psi}^i \rangle}{x}, && (3.5.164) \\
\bar{Q}_i &= p\bar{\psi}_i - 2i\alpha \frac{z_{(i}\bar{z}_{j)}\bar{\psi}^j}{+} i(1+2\alpha)\frac{\langle \bar{\psi}^j \bar{\psi}_j \psi_i \rangle}{x}, && (3.5.165) \\
S^i &= tQ^i - 2x\psi^i && (3.5.166) \\
\bar{S}_i &= t\bar{Q}_i - 2x\bar{\psi}_i && (3.5.167)
\end{aligned}
$$

$$
\begin{aligned}
J^{ij} &= i\left(z^{(i}\bar{z}^{k)} + 2\psi^{(i}\bar{\psi}^{k)}\right), && (3.5.168) \\
I^{11} &= -i\psi_i\psi^k, && (3.5.169) \\
I^{22} &= i\bar{\psi}^i\bar{\psi}^i, && (3.5.170) \\
I^{12} &= -\frac{i}{2}[\psi_i, \bar{\psi}^i] && (3.5.171)
\end{aligned}
$$

where $\langle \cdot \rangle$ denotes the Weyl ordering. One can show that under the canonical relations

$$
[x, p] = i, \qquad [z^i, \bar{z}_j] = \delta^i_j, \qquad \left\{\psi^i, \bar{\psi}_j\right\} = -\frac{1}{2}\delta^i_j \qquad (3.5.172)
$$

the generators form the $D(2,1;\alpha)$ superalgebra [209].

3.6 $\mathcal{N} = 8$ Superconformal mechanics

Up to now much less has been known about higher extended $\mathcal{N} > 4$ supersymmetric quantum mechanics. A study on $\mathcal{N} > 4$ supersymmetric quantum mechanics was initiated in [220] within the on-shell Hamiltonian approach. As we have discussed in subsection 3.2.1, the $\mathcal{N} = 8$ supersymmetry is the maximum case in which only the same number of supersymmetry is required for the component fields in the minimal supermultiplet [27]. In other words, the $\mathcal{N} = 8$ supersymmetry is the highest supersymmetric case in which the superspace and superfield formalism is applicable. In fact off-shell actions of the $\mathcal{N} = 8$ superconformal mechanical models are only known for a few cases. From the Table 3.2 we see that there are four different possible superconformal group for $\mathcal{N} = 8$ superconformal mechanics [28]:

1. $SU(1,1|4)$
2. $OSp(8|2)$
3. $OSp(4^*|4)$
4. $F(4)$.

As we will see, the $OSp(4^*|4)$ superconformal mechanics has been constructed from the $(\mathbf{3},\mathbf{8},\mathbf{5})$ and the $(\mathbf{5},\mathbf{8},\mathbf{3})$ supermultiplets [252, 253] and $F(4)$ superconformal mechanics has been proposed from the $(\mathbf{1},\mathbf{8},\mathbf{7})$ supermultiplet [254].

3.6.1 On-shell $SU(1,1|\frac{\mathcal{N}}{2})$ action

It has been discussed [210, 255, 224, 137] that the on-shell one particle component action of the $SU(1,1|\frac{\mathcal{N}}{2})$, $\mathcal{N} > 4$ superconformal mechanical models generically take the form

$$S = \int dt \left[\dot{x}^2 + i \left(\overline{\psi}_i \dot{\psi}^i - \dot{\overline{\psi}}_i \psi^i \right) - \frac{\left(c + \overline{\psi}_i \psi^i\right)^2}{x^2} \right] \tag{3.6.1}$$

where the fermionic fields ψ^i are the spinor representation of the R-symmetry group $SU(\frac{\mathcal{N}}{2})$. It has been pointed out [224] that the generators of the superconformal group $SU(1,1|\frac{\mathcal{N}}{2})$ can be found from those of the $SU(1,1|2)$ jus by replacing the $SU(2)$ spinor ψ^i with the $SU(\frac{\mathcal{N}}{2})$ spinors and c is a constant parameter. Correspondingly the supercharges $Q^i, \overline{Q}_i$ and the Hamiltonian H can be expressed

[27] Note that this statement has not been strictly proven without the assumptions for the particular forms of supersymmetric transformations (3.2.2), (3.2.3) and the relevant algebras.

[28] The relevant D-module representations for the $d = 1$ $\mathcal{N} = 2$, 4 and 8 superconformal algebras have been discussed in [251].

as

$$Q^i = \psi^i \left(p - 2i \frac{c + \overline{\psi}_i \psi^i}{x} \right), \tag{3.6.2}$$

$$\overline{Q}_i = \overline{\psi}_i \left(p + 2i \frac{c + \psi_i \overline{\psi}^i}{x} \right), \tag{3.6.3}$$

$$H = \frac{p^2}{4} + \frac{\left[c + \overline{\psi}_i \psi^i\right]^2}{x^2}. \tag{3.6.4}$$

However, it has not been completely understood how to realize the on-shell action (3.6.1) from the off-shell superspace and superfield formalism.

3.6.2 Superspace and supermultiplet

The $\mathcal{N} = 8$ superspace $\mathbb{R}^{(1|8)}$ is parametrized by [252, 253]

$$\mathbb{R}^{(1|8)} = (t, \theta_{ia}, \vartheta_{\alpha A}), \qquad \overline{(\theta_{ia})} = \theta^{ia}, \qquad \overline{(\vartheta_{\alpha A})} = \vartheta^{\alpha A} \tag{3.6.5}$$

with $i, a, \alpha, A = 1, 2$. In terms of (3.6.5) four commuting $SU(2)$ factors of the R-symmetry will be manifest. The covariant derivatives are defined by

$$D^{ia} = \frac{\partial}{\partial \theta_{ia}} + i\theta^{ia} \frac{\partial}{\partial t}, \qquad \nabla^{\alpha A} = \frac{\partial}{\partial \vartheta_{\alpha A}} \tag{3.6.6}$$

and they satisfy

$$\left\{ D^{ia}, D^{jb} \right\} = 2i\epsilon^{ij} \epsilon^{ab} \frac{\partial}{\partial t}, \qquad \left\{ \nabla^{\alpha A}, \nabla^{\beta B} \right\} = 2i\epsilon^{\alpha\beta} \epsilon^{AB} \frac{\partial}{\partial t}. \tag{3.6.7}$$

Although the $\mathcal{N} = 8$ superfields are useful to find the irreducibility constraints and the transformation properties, it is hard to reproduce the supersymmetric action in terms of the component fields because of the large dimension of the integration measure. The efficient strategy is to split the $\mathcal{N} = 8$ supermultiplets into the $\mathcal{N} = 4$ supermultiplets and to deal with the $\mathcal{N} = 4$ superspace and superfield formalism. Such decompositions of the $\mathcal{N} = 8$ supermultiplets in terms of the $\mathcal{N} = 4$ supermultiplets can be written as the direct sum [252, 256]

$$(\boldsymbol{n}, \mathbf{8}, \mathbf{8} - \boldsymbol{n}) = (\boldsymbol{n}_1, \mathbf{4}, \mathbf{4} - \boldsymbol{n}_1) \oplus (\boldsymbol{n}_2, \mathbf{4}, \mathbf{4} - \boldsymbol{n}_2) \tag{3.6.8}$$

with $n = n_1 + n_2$. Here n represents the number of physical bosonic fields in the $\mathcal{N} = 8$ supermultiplets while n_1 and n_2 denote the numbers of physical bosons in the two $\mathcal{N} = 4$ supermultiplets respectively.

$(\mathbf{0},\mathbf{8},\mathbf{8})$ supermultiplet

The $(\mathbf{0},\mathbf{8},\mathbf{8})$ supermultiplet is described by two real fermionic superfields Ψ^{aA}, $\Xi^{i\alpha}$ satisfying the constraints

$$\nabla^{(\alpha A}\Xi_i^{\beta)} = 0, \qquad D^{(ia}\Xi_\alpha^{j)} = 0, \tag{3.6.9}$$

$$\nabla^{\alpha(A}\Psi_a^{B)} = 0, \qquad D^{i(a}\Psi_A^{b)} = 0, \tag{3.6.10}$$

$$\nabla^{\alpha A}\Psi_A^a = D^{ia}\Xi_i^\alpha, \qquad \nabla^{\alpha A}\Xi_\alpha^i = -D^{ia}\Psi_a^A. \tag{3.6.11}$$

(3.6.11) implies that the covariant derivative with respect to $\vartheta_{\alpha A}$ can be represented by the covariant derivatives with respect to θ_{ia}.

The $(\mathbf{0},\mathbf{8},\mathbf{8})$ supermultiplet possesses a unique splitting

$$(\mathbf{0},\mathbf{8},\mathbf{8}) = (\mathbf{0},\mathbf{4},\mathbf{4}) \oplus (\mathbf{0},\mathbf{4},\mathbf{4}). \tag{3.6.12}$$

In order to describe the $(\mathbf{0},\mathbf{0},\mathbf{8})$ supermultiplet in terms of the $\mathcal{N}=4$ superfields, we pick up the appropriate $\mathcal{N}=4$ superspace as

$$\mathbb{R}^{(1|4)} = (t,\theta_{ia}) \subset \mathbb{R}^{(1|8)} = (t,\theta_{ia},\vartheta_{\alpha A}). \tag{3.6.13}$$

Expanding the superfields in ϑ^{iA}, the constraints (3.6.11) leave the independent $\mathcal{N}=4$ superfields

$$\psi^{aA} = \Psi^{aA}|_{\vartheta=0}, \qquad \xi^{i\alpha} = \Xi^{i\alpha}|_{\vartheta=0}. \tag{3.6.14}$$

Then the constraints (3.6.9) and (3.6.10) imply that

$$D^{a(i}\xi^{j)\alpha} = 0, \qquad D^{i(a}\psi^{b)A} = 0. \tag{3.6.15}$$

The conditions (3.6.15) correspond to the constraints (3.5.89) for $(\mathbf{0},\mathbf{4},\mathbf{4})$ supermultiplets on the superfields ξ^{ia}, ψ^{aA}.

The $\mathcal{N}=8$ supersymmetric action can be written as

$$S = \int dt d^4\theta \left[\theta^{ia}\theta_i^b\psi_a^A\psi_{bA} + \theta^{ia}\theta_a^j\xi_i^\alpha\xi_{j\alpha}\right]. \tag{3.6.16}$$

Although the action (3.6.16) is not manifestly invariant due to the existence of the Grassmann coordinates, one can show that it is invariant.

$(\mathbf{1},\mathbf{8},\mathbf{7})$ supermultiplet

The $(\mathbf{1},\mathbf{8},\mathbf{7})$ supermultiplet is described by a single scalar superfield $\mathcal{U}$ obeying the conditions

$$\nabla^{(\alpha i}\nabla^{\beta)j}\mathcal{U} = 0, \qquad D^{i(a}D^{jb)}\mathcal{U} = 0, \tag{3.6.17}$$

$$D^{ia}D_a^j\mathcal{U} = -\nabla^{\alpha j}\nabla_\alpha^i\mathcal{U} \tag{3.6.18}$$

The condition (3.6.18) reduce the manifest R-symmetry into three $SU(2)$ factors due to the identification of the indices A and i of the covariant derivatives $\nabla^{\alpha A}$ and D^{ia}.

The $(\mathbf{1},\mathbf{8},\mathbf{7})$ has a unique decomposition into the $\mathcal{N}=4$ multiplets as

$$(\mathbf{1},\mathbf{8},\mathbf{7})=(\mathbf{1},\mathbf{4},\mathbf{3})\oplus(\mathbf{0},\mathbf{4},\mathbf{4}). \tag{3.6.19}$$

By choosing the $\mathcal{N}=4$ superspace $\mathbb{R}^{(1|4)}$ as in (3.6.13) and expanding the superfields in ϑ^{iA}, we find the projected $\mathcal{N}=4$ superfields

$$u=\mathcal{U}|_{\vartheta=0}, \qquad \psi^{i\alpha}=\nabla^{i\alpha}\mathcal{U}|_{\vartheta=0} \tag{3.6.20}$$

obeying

$$D^{(ia}\psi^{j)\alpha}=0, \qquad D^{i(a}D^{jb)}u=0, \tag{3.6.21}$$

which are viewed as the constraint equations (3.5.89) and (3.5.72). Thus we can identify $\psi^{i\alpha}$ and u with the $(\mathbf{0},\mathbf{4},\mathbf{4})$ and $(\mathbf{1},\mathbf{4},\mathbf{3})$ superfields respectively.

The general $\mathcal{N}=8$ supersymmetric component action of the $(\mathbf{1},\mathbf{8},\mathbf{7})$ supermultiplet can be found in [190]. The harmonic superspace action can be found in [257, 258, 226].

Taking into account the decomposition (3.6.19) of the $(\mathbf{1},\mathbf{8},\mathbf{7})$ supermultiplet, $\mathcal{N}=8$ superconformal mechanical model has been constructed by combining the two supermultiplets $(\mathbf{1},\mathbf{4},\mathbf{3})$ and $(\mathbf{0},\mathbf{4},\mathbf{4})$ for $D(2,1;\alpha=-\frac{1}{3})$ [254]. Since the possible $\mathcal{N}=8$ superconformal group into which one can embed $D(2,1;\alpha=-\frac{1}{3})$ is only $F(4)$, the resulting $\mathcal{N}=8$ superconformal mechanical model is identified with $F(4)$ superconformal mechanics.

$(\mathbf{2},\mathbf{8},\mathbf{6})$ supermultiplet

The $(\mathbf{2},\mathbf{8},\mathbf{6})$ supermultiplet contains two scalar bosonic superfields $\mathcal{U},\Phi$ which satisfy

$$\nabla^{(ai}\nabla^{b)j}\mathcal{U}=0, \qquad \nabla^{a(i}\nabla^{bj)}\Phi=0, \tag{3.6.22}$$

$$\nabla^{ai}\mathcal{U}=D^{ia}\Phi, \qquad \nabla^{ai}\Phi=-D^{ia}\mathcal{U} \tag{3.6.23}$$

where the indices i,A being identified and the indices a,α being identified and thus only two $SU(2)$ factors are manifest. The $(\mathbf{2},\mathbf{8},\mathbf{6})$ multiplets can be regarded as the two $(\mathbf{1},\mathbf{8},\mathbf{7})$ multiplets with the additional conditions because the two constraints (3.6.22) and (3.6.23) lead to

$$D^{(ia}D^{j)b}\Phi=0, \qquad D^{i(a}D^{jb)}\mathcal{U}=0, \tag{3.6.24}$$

$$D^{ia}D_a^j\mathcal{U}=-\nabla^{aj}\nabla_a^i\mathcal{U}, \qquad D^{ia}D_i^b\Phi=-\nabla^{bi}\nabla_i^a\Phi. \tag{3.6.25}$$

The $(\mathbf{2},\mathbf{8},\mathbf{6})$ multiplet has two different decompositions

$$(\mathbf{2},\mathbf{8},\mathbf{6}) = \begin{cases} (\mathbf{1},\mathbf{4},\mathbf{3}) \oplus (\mathbf{1},\mathbf{4},\mathbf{3}) \\ (\mathbf{2},\mathbf{4},\mathbf{2}) \oplus (\mathbf{0},\mathbf{4},\mathbf{4}). \end{cases} \tag{3.6.26}$$

1. $(\mathbf{1},\mathbf{4},\mathbf{3}) \oplus (\mathbf{1},\mathbf{4},\mathbf{3})$

   Choosing the $\mathcal{N} = 4$ superspace (3.6.13) and expanding the superfields in ϑ^{iA}, we find from (3.6.22) and (3.6.23) the independent $\mathcal{N} = 4$ superfields

$$u = \mathcal{U}|_{\vartheta=0}, \qquad \phi = \Phi|_{\vartheta=0} \tag{3.6.27}$$

   satisfying

$$D^{i(a}D^{jb)}u = 0, \qquad D^{(ia}D^{j)b}\phi = 0, \tag{3.6.28}$$

   which are the constraints equations (3.5.73). Therefore the two superfields u, ϕ are regarded as the $(\mathbf{1},\mathbf{4},\mathbf{3})$ superfields.

   The action can be written as

$$S = \int dtd^4\theta\ F(u,\phi) \tag{3.6.29}$$

   where the function F satisfies the Laplace equation

$$\frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial \phi^2} = 0. \tag{3.6.30}$$

2. $(\mathbf{2},\mathbf{4},\mathbf{2}) \oplus (\mathbf{0},\mathbf{4},\mathbf{4})$

   To realize the decomposition $(\mathbf{2},\mathbf{8},\mathbf{6}) = (\mathbf{2},\mathbf{4},\mathbf{2}) \oplus (\mathbf{0},\mathbf{4},\mathbf{4})$ we need to modify the choice of the $\mathcal{N} = 4$ superspace and the superfields. Let us introducethe covariant derivatives

$$\mathcal{D}^{ia} = \frac{1}{\sqrt{2}}\left(D^{ia} - i\nabla^{ai}\right), \qquad \overline{\mathcal{D}}^{ia} = \frac{1}{\sqrt{2}}\left(D^{ia} + i\nabla^{ai}\right) \tag{3.6.31}$$

   and the superfields $\mathcal{V}, \overline{\mathcal{V}}$ as

$$\mathcal{V} = \mathcal{U} + i\Phi, \qquad \overline{\mathcal{V}} = \mathcal{U} - i\Phi. \tag{3.6.32}$$

   Then we find a set of constraint equations

$$D^i\mathcal{V} = 0, \qquad \nabla^i\mathcal{V} = 0, \tag{3.6.33}$$

$$\overline{D}_i\overline{\mathcal{V}} = 0, \qquad \overline{\nabla}_i\overline{\mathcal{V}} = 0, \tag{3.6.34}$$

$$D^iD_i\overline{\mathcal{V}} = \overline{\nabla}_i\overline{\nabla}^i\mathcal{V}, \qquad D^i\nabla^j\overline{\mathcal{V}} = \overline{D}^i\overline{\nabla}^j\mathcal{V} = 0 \tag{3.6.35}$$

where we have defined

$$D^i := \mathcal{D}^{i1}, \qquad \overline{D}^i := \overline{\mathcal{D}}^{i2}, \tag{3.6.36}$$

$$\nabla^i := \mathcal{D}^{i2}, \qquad \overline{\nabla}^i = -\overline{\mathcal{D}}^{i1}. \tag{3.6.37}$$

Considering a new set of coordinates for the $\mathcal{N} = 4$ superspace as

$$\mathbb{R}^{(1|4)} = (t, \theta_{i1} + i\vartheta_{i1}, \theta_{i2} - i\vartheta_{i2}) \subset \mathbb{R}^{(1|8)}, \tag{3.6.38}$$

we find from the constraints (3.6.33)-(3.6.35) the independent $\mathcal{N} = 4$ superfields

$$v = \mathcal{V}|, \qquad \overline{v} = \overline{\mathcal{V}}|, \tag{3.6.39}$$

$$\psi^i = \overline{\nabla}^i \mathcal{V}|, \qquad \overline{\psi}^i = -\nabla^i \overline{V}| \tag{3.6.40}$$

satisfying

$$D^i v = 0, \qquad \overline{D}^i \overline{v} = 0, \tag{3.6.41}$$

$$D^i \psi^j = 0, \qquad \overline{D}^i \overline{\psi}^j = 0, \qquad D^i \overline{\psi}^j = -\overline{D}^i \psi^j \tag{3.6.42}$$

Thus we can identify the two sets of the superfields, $v, \overline{v}$ and $\psi^i, \overline{\psi}^i$ with the $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ and $(\mathbf{0}, \mathbf{4}, \mathbf{4})$ superfields.

The invariant action is given by

$$S = \int dt d^4\theta v\overline{v} - \frac{1}{2} \int dt d^2\overline{\theta} \psi^i \psi_i - \frac{1}{2} \int dt d^2\theta \overline{\psi}_i \overline{\psi}^i. \tag{3.6.43}$$

We should note that the form of the action (3.6.43) depend on the choice of the $\mathcal{N} = 4$ superspace. Although the superfield action (3.6.43) looks different from the previous action (3.5.92), it turns out to be the same in the component level.

$(\mathbf{3}, \mathbf{8}, \mathbf{5})$ supermultiplet

The $(\mathbf{3}, \mathbf{8}, \mathbf{5})$ supermultiplet includes the three bosonic superfields $\mathcal{V}^{ij} = \mathcal{V}^{ji}$ obeying

$$D_a^{(i} \mathcal{V}^{jk)} = 0, \qquad \nabla_\alpha{}^{(i} \mathcal{V}^{jk)} = 0 \tag{3.6.44}$$

and three $SU(2)$ factors are manifest. (3.6.44) yield to a further condition

$$\partial_t \left(D_i^a D_{ja} V^{ij} + \nabla_i^\alpha \nabla_{j\alpha} V^{ij} \right) = 0, \tag{3.6.45}$$

which leads to [252]

$$\nabla_i^\alpha \nabla_{j\alpha} V^{ij} = 6m - D_i^a D_{ja} V^{ij} \tag{3.6.46}$$

where m is a constant parameter.

The $(\mathbf{3}, \mathbf{8}, \mathbf{5})$ multiplet has two decompositions

$$(\mathbf{3}, \mathbf{8}, \mathbf{5}) = \begin{cases} (\mathbf{3}, \mathbf{4}, \mathbf{1}) \oplus (\mathbf{0}, \mathbf{4}, \mathbf{4}) \\ (\mathbf{1}, \mathbf{4}, \mathbf{3}) \oplus (\mathbf{2}, \mathbf{4}, \mathbf{2}). \end{cases} \tag{3.6.47}$$

1. $(\mathbf{3}, \mathbf{4}, \mathbf{1}) \oplus (\mathbf{0}, \mathbf{4}, \mathbf{4})$

   Let us choose the $\mathcal{N} = 4$ superspace (3.6.13) and expand the superfields in $\vartheta_{i\alpha}$. Then the constraints (3.6.44) leave in $\mathcal{V}^{ij}$ the four bosonic and four fermionic $\mathcal{N} = 4$ superfields

$$v^{ij} = \mathcal{V}^{ij}|, \qquad \xi_\alpha^i = \nabla_{j\alpha} \mathcal{V}^{ij}|, \tag{3.6.48}$$

$$A = \nabla_i^\alpha \nabla_{j\alpha} V^{ij}| \tag{3.6.49}$$

   which obey

$$D_a^{(i} v^{jk)} = 0, \qquad D_a^{(i} \xi_\alpha^{j)} = 0, \tag{3.6.50}$$

$$A = 6m - D_i^a D_{aj} v^{ij}. \tag{3.6.51}$$

   Since (3.6.50) are identified with the constraint equations (3.5.52) and (3.5.89), we see that the superfields v^{ij} and ξ_α^i are the $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ and $(\mathbf{0}, \mathbf{4}, \mathbf{4})$ superfields respectively. The remaining equation (3.6.51) is the conservation law type condition which gives rise to a constant m. As observed in [210], this is the reminiscent of the $d = 4$ $\mathcal{N} = 1$ tensor multiplet constraints [259].

   To write down the invariant action let us project out the $\mathcal{N} = 4$ superfields $v^{(ij)}$ and ξ_α^i onto the harmonic superspace as

$$v^{++} = v^{ij} u_i^+ u_j^+, \qquad v^{+-} = \frac{1}{2} D^{--} v^{++}, \qquad v^{--} = D^{--} v^{+-}, \tag{3.6.52}$$

$$\xi^+ = \xi^i u_i^+, \qquad \overline{\xi}^+ = \overline{\xi}^i u_i^+. \tag{3.6.53}$$

   Then the $OSp(4^*|4)$ superconformal action is given by [252]

$$S = \int dt d^4\theta \sqrt{v^2} - \frac{1}{\sqrt{2}} \int du d\zeta^{--} \left[\frac{\xi^+ \overline{\xi}^+}{(1 + c^{--} \hat{v}^{++})^{\frac{3}{2}}} + 12m \frac{\hat{v}^{++}}{\sqrt{1 + c^{--} \hat{v}^{++}} (1 + \sqrt{1 + c^{--} \hat{v}^{++}})} \right] \tag{3.6.54}$$

where

$$dud\zeta^{--} = dudt_A d\theta^+ d\overline{\theta}^+, \tag{3.6.55}$$

$$c^\pm c^\pm = c^{ik} u_i^\pm u_k^\pm, \qquad c^{ik} = \text{const.}, \tag{3.6.56}$$

$$v^{++} = \hat{v}^{++} + c^{++}. \tag{3.6.57}$$

2. $(\mathbf{1},\mathbf{4},\mathbf{3}) \oplus (\mathbf{2},\mathbf{4},\mathbf{2})$

   To obtain the decomposition $(\mathbf{3},\mathbf{8},\mathbf{5}) = (\mathbf{1},\mathbf{4},\mathbf{3}) \oplus (\mathbf{2},\mathbf{4},\mathbf{2})$, we shall introduce the new covariant derivatives

$$D^a = \frac{1}{\sqrt{2}} \left(D^{1a} + i\nabla^{a1} \right), \qquad \overline{D}_a = \sqrt{1}\sqrt{2} \left(D_a^2 - i\nabla_a^2 \right), \tag{3.6.58}$$

$$\nabla^a = \frac{i}{\sqrt{2}} \left(D^{2a} + i\nabla^{a2} \right), \qquad \overline{\nabla}_a = \frac{i}{\sqrt{2}} \left(D_a^1 - i\nabla_a^1 \right) \tag{3.6.59}$$

   and the set of coordinates closed under the action of $D^a, \overline{D}_a$

$$\mathbb{R}^{(1|4)} = (t, \theta_{1a} - i\vartheta_{a1}, \theta^{1a} + i\vartheta^{a1}) \subset \mathbb{R}^{(1|8)}. \tag{3.6.60}$$

   Defining the $\mathcal{N} = 4$ superfields as

$$v = -2i\mathcal{V}^{12}, \qquad \varphi = \mathcal{V}^{11}, \qquad \overline{\varphi} = \mathcal{V}^{22}, \tag{3.6.61}$$

   we find the constraints (3.5.73) for the $(\mathbf{1},\mathbf{4},\mathbf{3})$ supermultiplet and the constraints (3.5.69) for the $(\mathbf{2},\mathbf{4},\mathbf{2})$ chiral supermultiplet

$$D^a D_a v = 0, \qquad \overline{D}_a \overline{D}^a v = 0, \tag{3.6.62}$$

$$D^a \varphi = 0, \qquad \overline{D}_a \overline{\varphi}. \tag{3.6.63}$$

   from the constraints (3.6.44). Therefore the superfields v can be viewed as the $(\mathbf{1},\mathbf{4},\mathbf{3})$ superfield and φ as the $(\mathbf{2},\mathbf{4},\mathbf{2})$ superfield. From the constraints (3.6.62) and (3.6.63) it follows that

$$\frac{\partial}{\partial t} \left[D^a, \overline{D}_a \right] v = 0. \tag{3.6.64}$$

   Combining (3.6.46) and (3.6.64), we obtain the constant m [210]

$$[D^a, \overline{D}_a] v = -2m. \tag{3.6.65}$$

   In this case the $\mathcal{N} = 8$ supersymmetric free action takes the form [252]

$$S = -\frac{1}{4} \int dt d^4\theta \left(v^2 - 2\varphi\overline{\varphi} \right). \tag{3.6.66}$$

However, the action (3.6.66) is not invariant under the superconformal transformations. Following the strategy of [260, 213], the $OSp(4^*|4)$ superconformal action is given by [252]

$$S = -\frac{1}{4}\int dt d^4\theta \left[v \ln\left(v + \sqrt{v^2 + \varphi\overline{\varphi}} \right) - \sqrt{v^2 + 4\varphi\overline{\varphi}} \right] \tag{3.6.67}$$

whose bosonic part is

$$S_{\text{bosonic}} = \int dt \frac{1}{\sqrt{v^2 + 4\varphi\overline{\varphi}}} \left[\dot{v}^2 + 4\dot{\varphi}\dot{\overline{\varphi}} - m^2 - 2im\dot{v} - \frac{4im\varphi\dot{\overline{\varphi}}}{v + \sqrt{v^2 + \varphi\overline{\varphi}}} \right]. \tag{3.6.68}$$

Therefore the $(\mathbf{3}, \mathbf{8}, \mathbf{5})$ supermultiplet can describe the $OSp(4^*|4)$ superconformal mechanics [252]. By means of the non-linear realization method parametrize a coset of the supergroup $OSp(4^*|4)$ such that $SO(5) \subset OSp(4^*|4)$ belongs to the stability subgroup while one out of three Goldstone bosons is the coset parameter associated with the dilatation, the dilaton and the remaining two Goldstone bosons parametrize the R-symmetry coset $SU(2)_R/U(1)_R$. Although the action (3.6.54) and (3.6.68) have different manifest $\mathcal{N} = 4$ superconformal symmetries $OSp(4^*|2)$ and $SU(1,1|2)$ respectively, both of them form $OSp(4^*|4)$ superconformal group together with the hidden symmetries. Hence the two superfield actions (3.6.54) and (3.6.68) exhibit different symmetry aspects of the same $\mathcal{N} = 8$ superconformal mechanics. Note that the two actions (3.6.54) and (3.6.68) produce the same actions (3.6.68) in terms of the component fields as they can be obtained from the single $\mathcal{N} = 8$ superfield formulation.

$(\mathbf{4}, \mathbf{8}, \mathbf{4})$ supermultiplet

The $(\mathbf{4}, \mathbf{8}, \mathbf{4})$ supermultiplet includes a four superfields $\mathcal{Q}^{a\alpha}$ which obeys

$$D_i^{(a}\mathcal{Q}^{b)\alpha} = 0, \qquad\qquad \nabla_i^{(\alpha}\mathcal{Q}_a^{\beta)} = 0. \tag{3.6.69}$$

The constraints (3.6.69) are manifestly covariant with respect to the three $SU(2)$ factors for the indices i, a and α.

There are three different decompositions of the $(\mathbf{4}, \mathbf{8}, \mathbf{4})$ supermultiplet

$$(\mathbf{4}, \mathbf{8}, \mathbf{4}) = \begin{cases} (\mathbf{4}, \mathbf{4}, \mathbf{0}) \oplus (\mathbf{0}, \mathbf{4}, \mathbf{4}) \\ (\mathbf{3}, \mathbf{4}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{4}, \mathbf{3}) \\ (\mathbf{2}, \mathbf{4}, \mathbf{2}) \oplus (\mathbf{2}, \mathbf{4}, \mathbf{2}). \end{cases} \tag{3.6.70}$$

1. $(\mathbf{4},\mathbf{4},\mathbf{0}) \oplus (\mathbf{0},\mathbf{4},\mathbf{4})$

   Making the choice of the $\mathcal{N}=4$ superspace (3.6.13) and expanding the superfields in $\vartheta_{i\alpha}$, the constraints (3.6.69) yield the independent $\mathcal{N}=4$ superfields

$$q^{a\alpha} = \mathcal{Q}^{a\alpha}|, \qquad \psi^{ia} = \nabla^i_\alpha \mathcal{Q}^{a\alpha}| \tag{3.6.71}$$

   satisfying the constraint conditions (3.5.44) for the $(\mathbf{4},\mathbf{4},\mathbf{0})$ supermultiplet and (3.5.90) for the $(\mathbf{0},\mathbf{4},\mathbf{4})$ supermultiplet

$$D^{i(a}q^{b)\alpha} = 0, \qquad D^{i(a}\psi^{b)i} = 0. \tag{3.6.72}$$

   Thus the superfields $q^{i\alpha}$ and ψ^{ia} are the $(\mathbf{4},\mathbf{4},\mathbf{0})$ and $(\mathbf{0},\mathbf{4},\mathbf{4})$ superfields respectively.

2. $(\mathbf{3},\mathbf{4},\mathbf{1}) \oplus (\mathbf{1},\mathbf{4},\mathbf{3})$

   Let us introduce the $\mathcal{N}=8$ superfields $\mathcal{V}^{ab}$, $\mathcal{V}$ as

$$\mathcal{Q}^{a\alpha} = \delta^\alpha_b \mathcal{V}^{ab} - \epsilon^{a\alpha}\mathcal{V}, \qquad \mathcal{V}^{ab} = \mathcal{V}^{ba} \tag{3.6.73}$$

   and pick up the $\mathcal{N}=4$ superspace

$$\mathbb{R}^{(1|4)} = (t, \theta_{1a} + i\vartheta_{1a}, \theta_{2a} - i\vartheta_{2a}) \subset \mathbb{R}^{(1|8)}. \tag{3.6.74}$$

   Correspondingly we will consider the covariant derivatives $D^a, \overline{D}^a$ and $\overline{\nabla}^a, \nabla^a$ as

$$(D^a, \overline{D}^a) = \left(\mathcal{D}^{1a}, \overline{\mathcal{D}}^{2a}\right), \qquad (\overline{\nabla}^a, \nabla^a) = \left(\mathcal{D}^{2a}, \overline{\mathcal{D}}^{1a}\right) \tag{3.6.75}$$

   where $\mathcal{D}^{ia}, \overline{\mathcal{D}}^{ia}$ are defined in (3.6.31). Then the constraints (3.6.69) lead to the independent $\mathcal{N}=4$ superfields

$$v^{ab} = \mathcal{V}^{ab}, \qquad v = \mathcal{V} \tag{3.6.76}$$

   which are subjected to

$$D^{(a}v^{bc)} = 0, \qquad \overline{D}^{(a}v^{bc)} = 0, \tag{3.6.77}$$

$$D^{(a}\overline{D}^{b)}v = 0. \tag{3.6.78}$$

   Thus the superfields v^{ab} and v are the $(\mathbf{3},\mathbf{4},\mathbf{1})$ and $(\mathbf{1},\mathbf{4},\mathbf{3})$ superfields respectively.

   The supersymmetric invariant free action is given by

$$S = \int dt d^4\theta \left[v^2 - \frac{3}{8}v^{ab}v_{ab}\right]. \tag{3.6.79}$$

3. $(\mathbf{2},\mathbf{4},\mathbf{2}) \oplus (\mathbf{2},\mathbf{4},\mathbf{2})$

We shall define the new set of $\mathcal{N}=8$ superfields $\mathcal{W}$, Φ in terms of $\mathcal{V}$, $\mathcal{V}^{ab}$ introduced in (3.6.73) as

$$\mathcal{W} = \mathcal{V}^{11}, \qquad \overline{\mathcal{W}} = \mathcal{V}^{22}, \tag{3.6.80}$$

$$\Phi = \frac{2}{3}\left(\mathcal{V} + \frac{3}{2}\mathcal{V}^{12}\right), \qquad \overline{\Phi} = \frac{2}{3}\left(\mathcal{V} - \frac{3}{2}\mathcal{V}^{12}\right) \tag{3.6.81}$$

and the new set of the $\mathcal{N}=4$ covariant derivatives D^i, ∇^i as

$$D^i = \frac{1}{\sqrt{2}}\left(\mathcal{D}^{i1} + \overline{\mathcal{D}}^{i1}\right), \qquad \overline{D}^i = \frac{1}{\sqrt{2}}\left(\mathcal{D}^{i2} + \overline{\mathcal{D}}^{i2}\right), \tag{3.6.82}$$

$$\nabla^i = \frac{1}{\sqrt{2}}\left(\mathcal{D}^{i1} - \overline{\mathcal{D}}^{i1}\right), \qquad \overline{\nabla}^i = -\frac{1}{\sqrt{2}}\left(\mathcal{D}^{i2} - \overline{\mathcal{D}}^{i2}\right) \tag{3.6.83}$$

where $\mathcal{D}^{ia}, \overline{\mathcal{D}}^{ia}$ are introduced in (3.6.31). Then the constraints (3.6.69) provides us with the two independent $(\mathbf{2},\mathbf{4},\mathbf{2})$ superfields

$$w = \mathcal{W}|, \qquad \phi = \Phi|. \tag{3.6.84}$$

The free supersymmetric action can be written as

$$S = \int dt d^4\theta \left[w\overline{w} - \phi\overline{\phi}\right]. \tag{3.6.85}$$

The $(\mathbf{4},\mathbf{8},\mathbf{4})$ supermultiplet can be constructed by reducing two-dimensional $\mathcal{N}=(4,4)$ or heterotic $\mathcal{N}=(8,0)$ sigma model [261].

$(\mathbf{5},\mathbf{8},\mathbf{3})$ supermultiplet

The $(\mathbf{5},\mathbf{8},\mathbf{3})$ supermultiplet is described by the five bosonic superfields $\mathcal{V}_{\alpha a}, \mathcal{U}$ which satisfy

$$D^{ib}\mathcal{V}_{\alpha a} = -\delta^b_a \nabla^i_\alpha \mathcal{U}, \qquad \nabla^{\beta i}\mathcal{V}_{\alpha a} = -\delta^\beta_\alpha D^i_\alpha \mathcal{U}. \tag{3.6.86}$$

The constraints (3.6.86) are covariant not only with respect to three $SU(2)$ factors for the indices i, a, α but also with respect to the $SO(5)$ R-symmetry. The $SO(5)$ R-symmetry transformations mix the spinor derivatives

The $(\mathbf{5},\mathbf{8},\mathbf{3})$ supermultiplet may have two decompositions

$$(\mathbf{5},\mathbf{8},\mathbf{3}) = \begin{cases} (\mathbf{1},\mathbf{4},\mathbf{3}) \oplus (\mathbf{4},\mathbf{4},\mathbf{0}) \\ (\mathbf{3},\mathbf{4},\mathbf{1}) \oplus (\mathbf{2},\mathbf{4},\mathbf{2}) \end{cases} \tag{3.6.87}$$

1. $(\mathbf{1},\mathbf{4},\mathbf{3})\oplus(\mathbf{4},\mathbf{4},\mathbf{0})$

   Using the $\mathcal{N}=4$ superspace (3.6.13) and carrying out the expansion of the superfields in $\vartheta_{i\alpha}$, we find the independent $\mathcal{N}=4$ superfields

$$v_{\alpha a}=\mathcal{V}_{\alpha a}|, \qquad\qquad u=\mathcal{U}| \tag{3.6.88}$$

   which satisfy

$$D^{i(a}v^{b)\alpha}=0, \qquad\qquad D^{i(a}D_i^{b)}u=0. \tag{3.6.89}$$

   Hence we obtain the $(\mathbf{4},\mathbf{4},\mathbf{0})$ superfield $v_{a\alpha}$ and the $(\mathbf{1},\mathbf{4},\mathbf{3})$ superfield u.

2. $(\mathbf{3},\mathbf{4},\mathbf{1})\oplus(\mathbf{2},\mathbf{4},\mathbf{2})$

   In order to present the decomposition $(\mathbf{5},\mathbf{8},\mathbf{3})=(\mathbf{2},\mathbf{4},\mathbf{2})\oplus(\mathbf{2},\mathbf{4},\mathbf{2})$, we introduce the new set of superfields $\mathcal{W}$, $\overline{\mathcal{W}}$ and $\mathcal{W}^{\alpha\beta}$ as

$$\mathcal{W}^{\alpha\beta}=\frac{1}{2}\left(\mathcal{V}^{\alpha\beta}+\mathcal{V}^{\beta\alpha}\right), \qquad\qquad \mathcal{W}=-\epsilon_{\alpha a}\mathcal{V}^{\alpha a}+i\mathcal{U} \tag{3.6.90}$$

   and the new $\mathcal{N}=4$ superspace

$$\mathbb{R}^{(1|4)}=(t,\theta_{i\alpha}+\vartheta_{\alpha i},\theta^{i\alpha}-i\vartheta^{\alpha i})\subset\mathbb{R}^{(1|8)}. \tag{3.6.91}$$

   Then the constraints (3.6.86) leave us with the independent $\mathcal{N}=4$ superfields

$$\phi=\mathcal{W}, \qquad\qquad e^{\alpha\beta}=\mathcal{W}^{\alpha\beta} \tag{3.6.92}$$

   which obey

$$\mathcal{D}^{\alpha}\overline{\phi}=0, \qquad\qquad \overline{\mathcal{D}}_{\alpha}\phi=0, \tag{3.6.93}$$

$$\mathcal{D}^{(\alpha}w^{\beta\gamma)}=0, \qquad\qquad \overline{\mathcal{D}}^{(\alpha}w^{\beta\gamma)}=0. \tag{3.6.94}$$

   Here the $\mathcal{N}=4$ covariant derivatives $\mathcal{D}^{\alpha}$, $\overline{\mathcal{D}}_{\alpha}$ are defined as

$$\mathcal{D}^{\alpha}=\mathcal{D}^{1\alpha} \qquad\qquad \overline{\mathcal{D}}_{\alpha}=\overline{\mathcal{D}}_{1\alpha} \tag{3.6.95}$$

   in terms of the covariant derivatives introduced in (3.6.31). Therefore the $\mathcal{N}=4$ superfields ϕ and $w^{\alpha\beta}$ are the $(\mathbf{2},\mathbf{4},\mathbf{2})$ superfield and the $(\mathbf{3},\mathbf{4},\mathbf{1})$ superfield respectively.

   The $\mathcal{N}=4$ supersymmetric free action is given by [252]

$$S=\int dtd^4\theta\left[w^2-\frac{3}{4}\phi\overline{\phi}\right]. \tag{3.6.96}$$

The $OSp(4^*|4)$ superconformal action can be written as [252]

$$S = 2\int dt d^4\theta \frac{\ln\left(\sqrt{w^2} + \sqrt{w^2 + \frac{1}{2}\phi\bar{\phi}}\right)}{\sqrt{w^2}} \tag{3.6.97}$$

whose bosonic part has the form

$$S_{\text{bosonic}} = \int dt \frac{\dot{w}_\alpha \dot{w}_{\alpha\beta} + \frac{1}{2}\dot{\phi}\dot{\bar{\phi}}}{\left(w^2 + \frac{1}{2}\phi\bar{\phi}\right)^{\frac{3}{2}}}. \tag{3.6.98}$$

The action (3.6.98) can be regarded as a conformal invariant type of the $SO(5)$ invariant sigma-model action of [262].

The $(\mathbf{5},\mathbf{8},\mathbf{3})$ supermultiplet can be obtained by the dimensional reduction of the $d = 4$ $\mathcal{N} = 2$ Abelian multiplet [262]. The three extra physical scalar fields originate from the spatial component fields of the $d = 4$ gauge vector potential.

Using the non-linear realization technique, it has been shown [252] that the $(\mathbf{5},\mathbf{8},\mathbf{3})$ supermultiplet can parametrize a coset of $OSp(4^*|4)$ such that the four out of five Goldstone bosons parametrize the $SO(5)/SO(4)$ coset while the remaining one Goldstone boson is the dilaton.

$(\mathbf{6},\mathbf{8},\mathbf{2})$ supermultiplet

The $(\mathbf{6},\mathbf{8},\mathbf{2})$ supermultiplet has two tensor superfields $\mathcal{V}^{ij}, \mathcal{W}^{ab}$ subjected to the conditions

$$D_a^{(i}\mathcal{V}^{jk)} = 0, \qquad \nabla_a^{(i}\mathcal{V}^{jk)} = 0, \tag{3.6.99}$$

$$D_i^{(a}\mathcal{W}^{bc)} = 0, \qquad \nabla_i^{(a}\mathcal{W}^{bc)} = 0, \tag{3.6.100}$$

$$D_j^a\mathcal{V}^{ij} = \nabla^{bi}\mathcal{W}_b^a, \qquad \nabla_j^a\mathcal{V}^{ij} = -D_b^i\mathcal{W}^{ab}. \tag{3.6.101}$$

The conditions (3.6.101) identify the eight fermions in $\mathcal{V}^{ij}$ with those in $\mathcal{W}^{ab}$ and also reduce the number of the auxiliary fields to two.

The $(\mathbf{6},\mathbf{8},\mathbf{2})$ supermultiplet can be decomposed as

$$(\mathbf{6},\mathbf{8},\mathbf{2}) = \begin{cases} (\mathbf{3},\mathbf{4},\mathbf{1}) \oplus (\mathbf{3},\mathbf{4},\mathbf{1}) \\ (\mathbf{4},\mathbf{4},\mathbf{0}) \oplus (\mathbf{2},\mathbf{4},\mathbf{2}). \end{cases} \tag{3.6.102}$$

1. $(\mathbf{3},\mathbf{4},\mathbf{1}) \oplus (\mathbf{3},\mathbf{4},\mathbf{1})$

Using the $\mathcal{N}=4$ superspace (3.6.13) and expanding the superfields in ϑ, we can project out the $\mathcal{N}=4$ superfields

$$v^{ij} = \mathcal{V}^{ij}, \qquad w^{ab} = \mathcal{W}^{ab} \tag{3.6.103}$$

obeying

$$D^{a(i}v^{jk)} = 0, \qquad D^{i(a}w^{bc)} = 0. \tag{3.6.104}$$

Thus we obtain the two $(\mathbf{3},\mathbf{4},\mathbf{1})$ superfields v^{ij} and w^{ab}.

The supersymmetric free action reads

$$S = \int dt d^4\theta \left(v^2 - w^2\right). \tag{3.6.105}$$

2. $(\mathbf{4},\mathbf{4},\mathbf{0}) \oplus (\mathbf{2},\mathbf{4},\mathbf{2})$

   This decomposition can be realized by combining the $(\mathbf{2},\mathbf{4},\mathbf{2})$ chiral multiplet $\phi,\overline{\phi}$ and the $(\mathbf{4},\mathbf{4},\mathbf{0})$ hypermultiplet q^{ia}.

   The invariant free action takes the form

$$S = \int dt d^4\theta \left(q^2 - 4\phi\overline{\phi}\right). \tag{3.6.106}$$

$(\mathbf{7},\mathbf{8},\mathbf{1})$ supermultiplet

The $(\mathbf{7},\mathbf{8},\mathbf{1})$ supermultiplet contains two different types of superfields $\mathcal{V}^{ij}, \mathcal{Q}^{a\alpha}$ which obey

$$D^{(ia}\mathcal{V}^{jk)} = 0, \qquad \nabla^{\alpha(i}\mathcal{V}^{jk)} = 0, \tag{3.6.107}$$

$$D^{i(a}\mathcal{Q}^{\alpha b)} = 0, \qquad \nabla_i^{(\alpha}\mathcal{Q}_a^{\beta)} = 0, \tag{3.6.108}$$

$$D_j^a\mathcal{V}^{ij} = i\nabla_\alpha^i\mathcal{Q}^{a\alpha}, \qquad \nabla_j^\alpha\mathcal{V}^{ij} = -iD_a^i\mathcal{Q}^{a\alpha}. \tag{3.6.109}$$

The constraints (3.6.107) extract the $(\mathbf{3},\mathbf{8},\mathbf{5})$ and $(\mathbf{4},\mathbf{8},\mathbf{4})$ supermultiplets from the superfields $\mathcal{V}^{ij}$ and $\mathcal{Q}^{a\alpha}$ respectively. The constraints (3.6.108) identify the fermions in the superfields $\mathcal{V}^{ij}$ and $\mathcal{Q}^{a\alpha}$ and reduce the number of the auxiliary fields to one.

The $(\mathbf{7},\mathbf{8},\mathbf{1})$ supermultiplet has a unique splitting

$$(\mathbf{7},\mathbf{8},\mathbf{1}) = (\mathbf{3},\mathbf{4},\mathbf{1}) \oplus (\mathbf{4},\mathbf{4},\mathbf{0}). \tag{3.6.110}$$

By using the $\mathcal{N}=4$ superspace (3.6.13) and expanding the superspace in ϑ, we find the independent $\mathcal{N}=4$ superfields

$$v^{ij} = \mathcal{V}^{ij}|, \qquad q^{a\alpha} = \mathcal{Q}^{a\alpha}| \tag{3.6.111}$$

which satisfy the constraints

$$D^{a(i}v^{jk)} = 0, \qquad\qquad D^{i(a}q^{b)\alpha} = 0. \tag{3.6.112}$$

We thus obtain the $(\mathbf{3},\mathbf{4},\mathbf{1})$ superfield v^{ij} and the $(\mathbf{4},\mathbf{4},\mathbf{0})$ superfield $q^{a\alpha}$.

The invariant free action is given by [253]

$$S = \int dtd^4\theta \left[v^2 - \frac{4}{3}q^2\right]. \tag{3.6.113}$$

$(\mathbf{8},\mathbf{8},\mathbf{0})$ supermultiplet

The $(\mathbf{8},\mathbf{8},\mathbf{0})$ supermultiplet possesses two real bosonic superfields $\mathcal{Q}^{aA}, \Phi^{i\alpha}$ which obey

$$D^{(ia}\Phi^{j)\alpha} = 0, \qquad\qquad \nabla^{(\alpha A}\Phi_i^{\beta)} = 0, \tag{3.6.114}$$

$$D^{i(a}\mathcal{Q}^{b)A} = 0, \qquad\qquad \nabla^{\alpha(A}\mathcal{Q}^{aB)} = 0, \tag{3.6.115}$$

$$\nabla^{\alpha A}\Phi^i_\alpha = D^{ia}\mathcal{Q}^A_a, \qquad\qquad \nabla^{\alpha A}\mathcal{Q}^a_A = -D^{ia}\Phi^\alpha_i. \tag{3.6.116}$$

Similar to the $(\mathbf{0},\mathbf{8},\mathbf{8})$ supermultiplet, the two conditions (3.6.115) and (3.6.116) means that the covariant derivatives with respect to $\vartheta_{\alpha A}$ can be written in terms of the covariant derivatives with respect to θ_{ia}.

The $(\mathbf{0},\mathbf{8},\mathbf{8})$ supermultiplet has a unique decomposition

$$(\mathbf{8},\mathbf{8},\mathbf{0}) = (\mathbf{4},\mathbf{4},\mathbf{0}) \oplus (\mathbf{4},\mathbf{4},\mathbf{0}). \tag{3.6.117}$$

Choosing the $\mathcal{N} = 4$ superspace (3.6.13) and expanding the superfields in ϑ, one find the independent $\mathcal{N} = 4$ superfields

$$q^{aA} = \mathcal{Q}^{aA}|, \qquad\qquad \phi^{i\alpha} = \Phi^{i\alpha}| \tag{3.6.118}$$

satisfying the constraints for $(\mathbf{4},\mathbf{4},\mathbf{0})$ supermultiplet

$$D^{a(i}\phi^{j)\alpha} = 0, \qquad\qquad D^{i(a}q^{b)A} = 0. \tag{3.6.119}$$

This implies that the $(\mathbf{8},\mathbf{8},\mathbf{0})$ supermultiplet can be decomposed as the sum of the two $(\mathbf{4},\mathbf{4},\mathbf{0})$ supermultiplets as in (3.6.117).

The invariant free action can be written as [253]

$$S = \int dtd^4\theta \left[q^2 - \phi^2\right]. \tag{3.6.120}$$

3.6.3 Multi-particle model

Let us consider the $\mathcal{N}=8$ supersymmetric sigma-model [29].

Suppose we have the extended supersymmetry transformations as the form

$$\delta\Phi^i = \epsilon^A (I_A)^i{}_j D\Phi^j \tag{3.6.121}$$

where Φ^i is the $(\mathbf{1},\mathbf{2},\mathbf{1})$ superfields and ϵ^A, $A=1,\cdots,7$ are the supersymmetry parameters and I_A are the endomorphism of the tangent bundle of the target space. This $\mathcal{N}=8$ supermultiplet is called $\mathcal{N}=8B$ multiplet. This is related to the two-dimensional $\mathcal{N}=(4,0)$ supersymmetry. The closure of the $\mathcal{N}=8$ superalgebra requires that [200]

$$I_A I_B + I_B I_A = -2\delta_{AB}, \tag{3.6.122}$$

$$N(I_A, I_B) = 0 \tag{3.6.123}$$

where a $N(F,G)$ is Nijenhuis concomitant defined in (3.5.130). Thus the target space $\mathcal{M}$ has seven complex structures I_r which have vanishing mixed Nijenhuis tensors and the underlying algebraic structure is associated with that of octonions.

The invariance of the action under the $\mathcal{N}=8B$ supersymmetry leads to

$$g_{ij} = (I_A)^k{}_i (I_A)^l{}_j g_{kl}, \tag{3.6.124}$$

$$\nabla^{(+)}_{(i} (I_A)^k{}_{j)} = 0, \tag{3.6.125}$$

$$\partial_{[i}\left(I^m{}_j c_{|m|kl]}\right) - 2(I_A)^m{}_{[i}\partial_{[m} c_{jkl]]} = 0. \tag{3.6.126}$$

The first condition(3.5.133) implies that the metric g on $\mathcal{M}$ is Hermitian with respect to the seven complex structures. The second condition (3.5.134) is a generalized Yano tensor condition with torsion and the third condition (3.5.135) is imposed on torsion and complex structures.

The Riemannian manifold $\{\mathcal{M}, g, c\}$ with a metric g, a torsion three-form c and three complex structures I_A, $A=1,\cdots,7$ which obey the conditions (3.6.122)-(3.6.126) is called Octonionic Kähler with torsion manifold (OKT) [200].

[29] The $\mathcal{N}=8$ superconformal sigma-model has not been well understood. We will only discuss the $\mathcal{N}=8$ supersymmetric sigma-model in this thesis.

Part II

M2-branes

Chapter 4

BLG-model

The dominant theme of this chapter and the next chapter is the world-volume theories of the multiple planar M2-branes [1]. We will begin in this chapter with the BLG-model [21, 22, 23, 24, 25], which is one of the candidate descriptions of the low-energy dynamics of the multiple planar M2-branes. In section 4.1 we will set our notations and conventions and review the basic properties. In section 4.2 we will focus on the study of the $\mathcal{A}_4$ BLG-model that is the non-trivial finite dimensional Lie 3-algebra with positive definite metric, which may describe two membranes.

4.1 Construction

The BLG-model is a three-dimensional $\mathcal{N} = 8$ supersymmetric Chern-Simons matter theory found by Bagger, Lambert [21, 22, 23] and Gustavsson [24, 25]. It is characterized by a Lie 3-algebra $\mathcal{A}$, which is a generalization of a Lie algebra. The action has a manifest $\mathcal{N} = 8$ supersymmetry and the $SO(8)_R$ R-symmetry. It has been shown [264] that the $SO(4)$ BLG theory has an $OSp(4|8)$ superconformal symmetry at the classical level.

The field content is

- 8 real scalar fields $X^I = X^I_a T^a$
- 16 (8 on-shell) real fermionic fields $\Psi_{\dot{A}} = \Psi_{\dot{A}a} T^a$
- gauge fields $A_\mu = A_{\mu ab} \mathcal{T}^{ab}$.

[1] See [263] for the excellent review on the world-volume theories of the multiple planar M2-branes.

Here T^a, $a = 1, \cdots, \dim\mathcal{A}$ is a basis of the Lie 3-algebra $\mathcal{A}$ and $\mathcal{T}^{ab}$ is the fundamental object in $\mathcal{A}$ which will be introduced in (4.1.18). Under the $SO(8)_R$ R-symmetry the bosonic scalar fields X^I, $I = 1, \cdots, 8$ are the vector representations $\mathbf{8}_v$ while the fermionic fields $\Psi_{\dot{A}}$, $\dot{A} = 1, \cdots, 8$ are the conjugate spinor representations $\mathbf{8}_c$ respectively.

They also carry the $(\dim\mathcal{A})$-dimensional representations of the Lie 3-algebra. The gauge fields $A_{\mu ab}$ are 3-algebra $\mathcal{A}$ valued world-volume vector fields. They are antisymmetric under two indices a, b of the Lie 3-algebra; $A_{\mu ab} = -A_{\mu ba}$ [2]. The mass dimensions of the field content and the supersymmetry parameter ϵ are given by

$$[X^I_a] = \frac{1}{2}, \quad [\Psi_a] = 1, \quad [A_\mu] = 1, \quad [\epsilon] = -\frac{1}{2} \tag{4.1.1}$$

$\Psi_{\dot{A}a}$ is defined as an $SO(1,10)$ Majorana fermion and its conjugate is given by

$$\overline{\Psi} := \Psi^T \mathcal{C}, \tag{4.1.2}$$

where $\mathcal{C}$ is a $SO(1,10)$ charge conjugation matrix satisfying

$$\mathcal{C}^T = -\mathcal{C}, \qquad \mathcal{C}\Gamma^M\mathcal{C}^{-1} = -(\Gamma^M)^T. \tag{4.1.3}$$

Gamma matrix Γ^M is the representation of eleven-dimensional Clifford algebra

$$\{\Gamma^M, \Gamma^N\} = 2g^{MN} \tag{4.1.4}$$

$$\Gamma^{10} := \Gamma^{0\cdots 9}, \tag{4.1.5}$$

where $g^{MN} = \eta^{MN} = \mathrm{diag}(-1, +1, +1, \cdots, +1)$. Γ^M can be decomposed as

$$\begin{cases} \Gamma^\mu = \gamma^\mu \otimes \tilde{\Gamma}^9 & \mu = 0, 1, 2 \\ \Gamma^I = \mathbb{I}_2 \otimes \tilde{\Gamma}^{I-2} & I = 3, \cdots, 10 \end{cases} \tag{4.1.6}$$

where

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad \gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3 \tag{4.1.7}$$

and $\tilde{\Gamma}^I$ is an $SO(8)$ 16×16 gamma matrix whose chirality matrix is $\tilde{\Gamma}^9 := \tilde{\Gamma}^{1\cdots 8}$. The fermionic field Ψ is a real $\frac{1}{2} \cdot 2^{[\frac{11}{2}]} = 32$-component Majorana spinor of eleven-dimensional space-time, obeying the chirality condition[3]

$$\Gamma^{012}\Psi = -\Psi. \tag{4.1.8}$$

[2]For the $\mathcal{A}_4$ algebra we have a one-to-one correspondence between the fundamental object $\mathcal{T}$ and the element T^{ab} of the associated Lie algebra $\mathfrak{so}(4)$. Hence $A_{\mu ab}$ is Lie $\mathfrak{so}(4)$-valued. Moreover matter fields $X^I_a, \Psi_{\dot{A}a}$ are interpreted as the fundamental representations $\mathbf{4}$ of $\mathfrak{so}(4)$.

[3] 32 supercharges in M-theory is broken to 16 due to the existence of M2-branes and Ψ is identified with the Goldstino corresponding to the broken supersymmetry. Therefore the chirality condition on Ψ is opposite to that of supersymmetry parameters ϵ.

Although at this stage Ψ contains 16 independent real components, the number is reduced to 8 when we treat it on-shell. From (4.1.6) it follows that

$$\Gamma^{012} = \Gamma^{34\cdots 10} = \mathbb{I}_2 \otimes \tilde{\Gamma}^9 \tag{4.1.9}$$

and

$$\Gamma^{34\cdots 10}\Psi = -\Psi. \tag{4.1.10}$$

Thus Ψ is the conjugate spinor representation $\mathbf{8}_c$ of the $SO(8)_R$ R-symmetry group.

4.1.1 Lie 3-algebra

The construction of the BLG model is based on the Lie 3-algebra $\mathcal{A}$. The Lie 3-algebra is an N-dimensional vector space endowed with the totally antisymmetric multi-linear triple product $[A, B, C]$ satisfying the fundamental identity

$$[A, B[C, D, E]] = [[A, B, C], D, E] + [C, [A, B, D], E] + [C, D, [A, B, E]], \tag{4.1.11}$$

which is a generalization of the Jacobi identity in Lie algebra and requires that the gauge symmetry $\delta_{AB}X = [A, B, X]$ acts as the derivation[4]

$$\delta_{AB}([C, D, E]) = [\delta_{AB}C, D, E] + [C, \delta_{AB}D, E] + [C, D, \delta_{AB}E]. \tag{4.1.12}$$

The supersymmetry algebra of the BLG model is closed on-shell when the fundamental identity (4.1.11) is satisfied [22]. Let us introduce the basis $\{T^a\}_{1\le a\le N}$ of 3-algebra. Then the 3-algebra is specified by the metric h^{ab} and the structure constant $f^{abc}{}_d$

$$h^{ab} = (T^a, T^b), \tag{4.1.13}$$

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d. \tag{4.1.14}$$

In terms of the structure constant, the fundamental identity (4.1.11) can be expressed as

$$f^{abc}{}_g f^{deg}{}_f = f^{dea}{}_g f^{bcg}{}_f + f^{deb}{}_g f^{cag}{}_f + f^{dec}{}_g f^{abg}{}_f \tag{4.1.15}$$

$$= 3 f^{de[a}{}_g f^{bc]g}{}_f, \tag{4.1.16}$$

which turns out to be equivalent to the relation [265]

$$f^{[abc}{}_g f^{d]eg}{}_f = 0. \tag{4.1.17}$$

[4] Jacobi identity $[A, [B, C]] = [[A, B], C] + [B, [A, C]]$ ensures that the transformation $\delta_A X = [A, X]$ behaves as derivation $\delta_A[B, C] = [\delta_A B, C] + [B, \delta_A C]$.

Here we will define the fundamental object $\mathcal{T} = \mathcal{T}^{ab}$ as

$$\mathcal{T} \cdot X := [T^a, T^b, X], \quad \forall X \in \mathcal{A}. \tag{4.1.18}$$

The fundamental object induces derivation and gives the adjoint map

$$\mathrm{ad}_{T^a T^b} : X \mapsto [T^a, T^b, X], \quad \forall X \in \mathcal{A}. \tag{4.1.19}$$

If we require that the action of the derivation on the scalar product is invariant

$$\mathcal{T} \cdot (T^c, T^d) = (\mathcal{T} \cdot T^c, T^d) + (T^c, \mathcal{T} \cdot T^d) = 0, \tag{4.1.20}$$

then we obtain the relation

$$(T^a, [T^b, T^c, T^d]) = -([T^a, T^b, T^c], T^d). \tag{4.1.21}$$

A Lie 3-algebra is called "metric" if it satisfies the relation (4.1.21). This metric property is assumed for all of the BLG theories. In terms of the structure constant, (4.1.21) is rewritten as

$$f^{abcd} = f^{[abcd]}. \tag{4.1.22}$$

This antisymmetry of f^{abcd} indicates that the symmetry algebra is contained in $\mathfrak{so}(N)$. To be more precise, we rewrite the fundamental identity (4.1.11) as

$$\mathrm{ad}_{AB}(\mathrm{ad}_{CD} X) - \mathrm{ad}_{CD}(\mathrm{ad}_{AB} X) = \mathrm{ad}_{([A,B,C],D)+(C,[A,B,D])} X \tag{4.1.23}$$

or equivalently

$$\mathrm{ad}_{\mathcal{T}}(\mathrm{ad}_{\mathcal{S}} X) - \mathrm{ad}_{\mathcal{S}}(\mathrm{ad}_{\mathcal{T}} X) = \mathrm{ad}_{\mathcal{T}\mathcal{S}} X, \quad \forall \mathcal{T}, \mathcal{S} \in \wedge^2 \mathcal{A}, \quad X \in \mathcal{A}. \tag{4.1.24}$$

Introducing the coordinates of the $(\dim \mathcal{A} \times \dim \mathcal{A})$ matrices $[T^{a_1}, T^{a_2}, \ ] =: \mathcal{T}^{a_1 a_2} =: \mathrm{ad}_{a_1 a_2} \in \mathrm{End}\mathcal{A}$ as

$$\mathrm{ad}_{\mathcal{T}^{a_1 a_2}}{}^l{}_k = (\mathcal{T}^{a_1 a_2})^l_k := f^{a_1 a_2 l}{}_k \tag{4.1.25}$$

$$\mathcal{T}^{a_1 a_2} \cdot T^k = [T^{a_1}, T^{a_2}, T^k] = f^{a_1 a_2 k}{}_l T^l, \tag{4.1.26}$$

then the equations (4.1.23) and (4.1.24) may be written in the form

$$[(\mathcal{T}^{a_1 a_2}), (\mathcal{T}^{b_1 b_2})]^s_k = -f^{a_1 a_2 [b_1 l} f^{b_2]}{}_{lk}{}^s, \tag{4.1.27}$$

which means that

$$[(\mathcal{T}^{a_1 a_2}), (\mathcal{T}^{b_1 b_2})]^s_k = \frac{1}{2} C^{a_1 a_2 b_1 b_2}{}_{c_1 c_2} (\mathcal{T}^{c_1 c_2})^s_k \tag{4.1.28}$$

where

$$C^{a_1 a_2 b_1 b_2}{}_{c_1 c_2} = f^{a_1 a_2 [b_1}{}_{[c_1} \delta^{b_2]}_{c_2]}. \tag{4.1.29}$$

Although (4.1.28) is the same form of the commutator in the Lie algebra, this does not mean $C^{a_1 a_2 b_1 b_2}{}_{c_1 c_2}$ are the structure constants of the Lie algebra $\mathfrak{g}$ because

1. $C^{a_1a_2b_1b_2}{}_{c_1c_2}$ may not be antisymmetric under $(a_1, a_2) \leftrightarrow (b_1, b_2)$
2. $\mathcal{T}^{c_1c_2}$ may not be the basis of the Lie algebra $\mathfrak{g}$.

However, it has been shown [266] that when the Lie 3-algebra is simple, $C^{a_1a_2b_1b_2}{}_{c_1c_2}$ are antisymmetric in the upper indices

$$f^{a_1a_2[b_1}{}_{[c_1}\delta^{b_2]}_{c_2]} = -f^{b_1b_2[a_1}{}_{[c_1}\delta^{a_2]}_{c_2]} \tag{4.1.30}$$

and define the structure constants of Lie algebra $\mathfrak{g}$. Moreover one can find the cases where $\mathcal{T}^{c_1c_2}$ can be viewd as the basis of $\mathfrak{g}$.

4.1.2 Lagrangian

The BLG-model Lagrangian is

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}D^\mu X^{Ia} D_\mu X^I_a + \frac{i}{2}\overline{\Psi}^a_{\dot{A}}\Gamma^\mu_{\dot{A}\dot{B}} D_\mu \Psi_{\dot{B}a} \\ &+ \frac{i}{4}\overline{\Psi}_{\dot{A}b}\Gamma^{IJ}_{\dot{A}\dot{B}} X^I_c X^J_d \Psi_{\dot{B}a} f^{abcd} - V(X) + \mathcal{L}_{TCS}\end{aligned} \tag{4.1.31}$$

where

$$V(X) = \frac{1}{12} f^{abcd} f^{efg}{}_d X^I_a X^J_b X^K_c X^I_e X^J_f X^K_g, \tag{4.1.32}$$

$$\mathcal{L}_{TCS} = \frac{1}{2}\epsilon^{\mu\nu\lambda}\left(f^{abcd} A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{2}{3} f^{cda}{}_g f^{efgb} A_{\mu ab} A_{\nu cd} A_{\lambda ef}\right). \tag{4.1.33}$$

The covariant derivative is defined as

$$\begin{aligned}D_\mu X_a &:= \partial_\mu X_a - A_{\mu cd}[T^c, T^d, X]_a \\ &= \partial_\mu X_a - \tilde{A}^b_{\mu a} X_b\end{aligned} \tag{4.1.34}$$

where $\tilde{A}^a_{\mu b} := f^{cda}{}_b A_{\mu cd}$. Alternatively we can express the Lagrangian in terms of the trace and the triple product of Lie 3-algebra:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}(D_\mu X^I, D^\mu X^I) + \frac{i}{2}(\overline{\Psi}, \Gamma^\mu D_\mu \Psi) \\ &+ \frac{i}{4}\left(\overline{\Psi}\Gamma^{IJ}[X^I, X^J, \Psi]\right) - \frac{1}{12}\left([X^I, X^J, X^K], [X^I, X^J, X^K]\right) \\ &+ \frac{1}{2}\epsilon^{\mu\nu\lambda}\left[\mathrm{Tr}\left(A_{\mu ab}\partial_\nu \tilde{A}^{ab}_\lambda\right) + \frac{2}{3}\mathrm{Tr}\left(A_{\mu ab}\tilde{A}^a_{\nu g}\tilde{A}^b_{\lambda g}\right)\right]\end{aligned} \tag{4.1.35}$$

Although the kinetic term of the gauge fields is similar to the conventional Chern-Simons term, it is twisted by the structure constant of the 3-algebra. Notice that the gauge fields are non-propagating since it has at most first order derivative terms. This is consistent with the degrees of freedom required from supersymmetry.

From (4.1.31), we obtain the equations of motion

$$
\begin{aligned}
D^\mu D_\mu X^I_a - \frac{i}{2}\overline{\Psi}_c \Gamma^{IJ} X^J_d \Psi_b f^{cdb}{}_a + \frac{1}{2} f^{bcd}{}_a f^{efg}{}_d X^J_b X^K_c X^I_e X^J_f X^K_g =&0, &(4.1.36)\\
\Gamma^\mu D_\mu \Psi_a + \frac{1}{2}\Gamma^{IJ} X^I_c X^J_d \Psi_b f^{cdb}{}_a =&0, &(4.1.37)\\
\tilde{F}^b_{\mu\nu a} + \epsilon_{\mu\nu\lambda}(X^J_c D^\lambda X^J_d + \frac{i}{2}\overline{\Psi}_c \Gamma^\lambda \Psi_d) f^{cdb}{}_a =&0. &(4.1.38)
\end{aligned}
$$

Here the field strength of the gauge field is defined as

$$\tilde{F}^b_{\mu\nu a} X_b := [D_\mu, D_\nu] X_a. \tag{4.1.39}$$

Combining the definition (4.1.34) of the covariant derivative, we can express it as

$$\tilde{F}^b_{\mu\nu a} = \partial_\nu \tilde{A}^b_{\mu a} - \partial_\mu \tilde{A}^b_{\nu a} - \tilde{A}^b_{\mu c}\tilde{A}^c_{\nu a} + \tilde{A}^b_{\nu c}\tilde{A}^c_{\mu a}. \tag{4.1.40}$$

The field strength satisfies Bianchi identity

$$\epsilon^{\mu\nu\lambda} D_\mu \tilde{F}^a_{\nu\lambda b} = 0. \tag{4.1.41}$$

The stress-energy tensor can be computed as

$$T_{\mu\nu} = D_\mu X^I_a D_\nu X^{Ia} - \eta_{\mu\nu}\left(\frac{1}{2} D_\lambda X^{Ia} D^\lambda X^I_a + V(X)\right) \tag{4.1.42}$$

where we set fermionic fields to zero. Thus bosonic part of the Hamiltonian density is

$$\mathcal{H} = T_{00} = \frac{1}{2} D_0 X^{Ia} X_0 X^I_a + \frac{1}{2} D_\alpha X^{Ia} D^\alpha X^I_a + V(X) \tag{4.1.43}$$

and the momentum density is

$$p_\alpha = T_{0\alpha} = D_0 X^{Ia} D_\alpha X^I_a. \tag{4.1.44}$$

4.1.3 Gauge transformation

The gauge transformations of the BLG-model are given by

$$
\begin{aligned}
\delta_\Lambda X^I_a =& \Lambda_{cd}[T^c, T^d, X^I]_a \\
=& \Lambda_{cd} f^{cdb}{}_a X^I_b = \tilde{\Lambda}^b_a X^I_b, &(4.1.45)\\
\delta_\Lambda \Psi_a =& \Lambda_{cd}[T^c, T^d, \Psi]_a \\
=& \Lambda_{cd} f^{cdb}{}_a \Psi_b = \tilde{\Lambda}^b_a \Psi_b, &(4.1.46)\\
\delta_\Lambda \tilde{A}^b_{\mu a} =& \partial_\mu \tilde{\Lambda}^b_{\mu a} - \tilde{\Lambda}^b_c \tilde{A}^c_{\mu a} + \tilde{A}^b_{\mu c}\tilde{\Lambda}^c_a \\
=& D_\mu \tilde{\Lambda}^b_a, &(4.1.47)\\
\delta \tilde{F}^b_{\mu\nu a} =& -\tilde{\Lambda}^b_c \tilde{F}^c_{\mu\nu a} + \tilde{F}^b_{\mu\nu c}\tilde{\Lambda}^c_a, &(4.1.48)
\end{aligned}
$$

where $\tilde{\Lambda}_a^b := f^{cdb}{}_a\Lambda_{cd}$ is a gauge parameter. Lagrangian (4.1.31) is invariant up to a total derivative terms under the above gauge transformations.

4.1.4 Supersymmetry transformation

The $\mathcal{N} = 8$ supersymmetry transformations of the BLG-model are

$$\delta X_a^I = i\bar{\epsilon}_A \Gamma^I_{A\dot{B}} \Psi_{\dot{B}a}, \tag{4.1.49}$$

$$\delta \Psi_{\dot{A}a} = D_\mu X_a^I \Gamma^\mu \Gamma^I_{\dot{A}B} \epsilon_B - \frac{1}{6} X_b^I X_c^J X_d^K f^{bcd}{}_a \Gamma^{IJK}_{\dot{A}B} \epsilon_B, \tag{4.1.50}$$

$$\delta \tilde{A}^b_{\mu a} = i\bar{\epsilon}_A \Gamma_\mu \Gamma^I_{A\dot{B}} X_c^I \Psi_{\dot{B}d} f^{cdb}{}_a. \tag{4.1.51}$$

Here ϵ_A, $A = 1, \cdots, 8$ is the unbroken supersymmetry parameter obeying the chirality condition

$$\Gamma^{012}\epsilon = \Gamma^{34\cdots 10}\epsilon = \epsilon. \tag{4.1.52}$$

This implies that ϵ_A is a two component three-dimensional Majorana spinor and transforms as the spinor representation $\mathbf{8}_s$ of the $SO(8)_R$ R-symmetry. Lagrangian (4.1.31) is invariant under the supersymmetric transformations up to a total derivative.

Using the equations of motion (4.1.36), (4.1.37) and (4.1.38), we find the following relations from (4.1.49), (4.1.50) and (4.1.51):

$$[\delta_1, \delta_2] X_a^I = v^\lambda D_\lambda X_a^I + \tilde{\Lambda}_a^b X_b^I, \tag{4.1.53}$$

$$[\delta_1, \delta_2] \Psi_a = v^\lambda D_\lambda \Psi_a + \tilde{\Lambda}_a^b \Psi_b, \tag{4.1.54}$$

$$[\delta_1, \delta_2] \tilde{A}^b_{\mu a} = v^\lambda \tilde{F}^b_{\mu\lambda a} + D_\mu \tilde{\Lambda}_a^b \tag{4.1.55}$$

where $v^\lambda = -2i\bar{\epsilon}_2 \Gamma^\lambda \epsilon_1$ and $\tilde{\Lambda}_a^b = -2i\bar{\epsilon}_2 \Gamma^{JK} \epsilon_1 X_c^J X_d^K f^{cdb}{}_a$ are identified with a translation parameter and a gauge parameter respectively. Thus the supersymmetry transformations close into a translation (the first term) and a gauge transformation (the second term) on-shell and the theory is invariant under 16 supersymmetries and $SO(8)_R$ R-symmetry at the classical level.

Allowing the supersymmetry parameter ϵ to has x dependence and taking supersymmetry variations of the action, we obtain

$$\delta S = -i \int d^3x D_\mu \bar{\epsilon} \left(D_\nu X_a^I \Gamma^\nu \Gamma^I \Gamma^\mu \Psi^a + \frac{1}{6} X_a^I X_b^J X_c^K f^{abcd} \Gamma^{IJK} \Gamma^\mu \Psi_d \right). \tag{4.1.56}$$

This gives

$$J^\mu = -D_\nu X_a^I \Gamma^\nu \Gamma^I \Gamma^\mu \Psi^a - \frac{1}{6} X_a^I X_b^J X_c^K f^{abcd} \Gamma^{IJK} \Gamma^\mu \Psi_d. \tag{4.1.57}$$

Then the supercharge is

$$Q = \int dx^2 x J^0 = -\int d^2x \left(D_\nu X_a^I \Gamma^\nu \Gamma^I \Gamma^0 \Psi^a + \frac{1}{6} X_a^I X_b^J X_c^K f^{abcd} \Gamma^{IJK} \Gamma^0 \Psi_d \right). \quad (4.1.58)$$

From (4.1.1) one can check that Q has the correct mass dimension $[Q] = \frac{1}{2}$ and J^0 has $[J^0] = \frac{5}{2}$. The supercharge Q is the SUSY generator in the sense that[5]

$$\delta_\epsilon \Phi = i[\bar{\epsilon} Q, \Phi\} = \begin{cases} i\bar{\epsilon}_{\dot{P}}[Q^{\dot{P}}, \Phi_B] & \text{(bosonic field)} \\ i\bar{\epsilon}_{\dot{P}}\{Q^{\dot{P}}, \Phi_F^{\dot{Q}}\} & \text{(fermionic field)} \end{cases} \quad (4.1.59)$$

where $\dot{P}, \dot{Q}, \cdots$ are 11-dimensional spinor indices. As an example, we can generate the SUSY transformation for the scalar fields X^I

$$\begin{aligned} \delta X^I =& i\bar{\epsilon}[Q, X^I] \\ =& i\bar{\epsilon}\left[-\int d^2x \partial_\nu X^J(x) \Gamma^\nu \Gamma^J \Gamma^0 \Psi(x), X^I(x') \right] \\ =& -i\bar{\epsilon}\Gamma^0 \Gamma^J \Gamma^0 \int d^2x \Psi(x) \left[\partial_0 X^J(x), X^I(x') \right] \\ =& i\bar{\epsilon}\Gamma^J \int d^2x \Psi(x) \delta^{IJ} \delta(x - x') = i\bar{\epsilon}\Gamma^I \Psi. \end{aligned} \quad (4.1.60)$$

4.1.5 M2-brane algebra

Now we want to discuss the algebraic structure of the M2-brane by studying the BLG-model. Noting that [6]

$$\begin{aligned} i\bar{\epsilon}_{\dot{P}}\{Q^{\dot{P}}, Q^{\dot{Q}}\} &= \int d^2x \bar{\epsilon}_{\dot{P}} \left\{ Q^{\dot{P}}, J^{0\dot{Q}}(x) \right\} \\ &= \int d^2x (\delta_\epsilon J^{0\dot{Q}}(x)), \end{aligned} \quad (4.1.61)$$

we obtain [267, 268]

$$\begin{aligned} \left\{ Q^{\dot{P}}, Q^{\dot{Q}} \right\} =& -2P_\mu (\Gamma^\mu \Gamma^0)^{\dot{P}\dot{Q}} + Z_{IJ} \Gamma^{IJ} \Gamma^0 \\ &+ Z_{\alpha IJKL} (\Gamma^{IJKL} \Gamma^\alpha \Gamma^0)^{\dot{P}\dot{Q}} + Z_{IJKL} (\Gamma^{IJKL})^{\dot{P}\dot{Q}} \end{aligned} \quad (4.1.62)$$

where α is the two-dimensional spatial indice of the M2-brane world-volume and P^μ is the energy momnetum vector $P^\mu := \int d^2x T^{0\mu}$. The central charges are given

[5] The symbol $[A, B\}$ means $AB - (-1)^{AB} BA$ in a $\mathbb{Z}_2$-graded algebra.

[6] The central charges are proportional to the world-volume of M2-branes and can be infinite for infinitely extended M2-branes. Focusing on the charge density, we can avoid the infinities.

by

$$Z_{IJ} = -\int d^2x \mathrm{Tr}\left(D_\alpha X^I D_\beta X^J \epsilon^{\alpha\beta} - D_0 X^K [X^I, X^J, X^K]\right), \tag{4.1.63}$$

$$Z_{\alpha IJKL} = \frac{1}{3}\int d^2x \mathrm{Tr}\left(D_\beta X^{[I}[X^J, X^K, X^{L]}]\epsilon^{\alpha\beta}\right), \tag{4.1.64}$$

$$Z_{IJKL} = \frac{1}{4}\int d^2x \mathrm{Tr}\left([X^M, X^{[I}, X^J], [X^M, X^K, X^{L]}]\right). \tag{4.1.65}$$

Introducing the expression

$$\tilde{\Gamma}^I = \begin{pmatrix} 0 & \overline{\Gamma}^I_{\dot{A}A} \\ \overline{\Gamma}^I_{B\dot{B}} & 0 \end{pmatrix} \tag{4.1.66}$$

where $(\overline{\Gamma}^I_{A\dot{A}})^T = \overline{\Gamma}^I_{\dot{A}A}$ are 8×8 real gamma matrices satisfying

$$\begin{aligned} \overline{\Gamma}^I_{A\dot{A}}\overline{\Gamma}^J_{\dot{A}B} + \overline{\Gamma}^J_{A\dot{A}}\overline{\Gamma}^I_{\dot{A}B} &= 2\delta^{IJ}\delta_{AB}, \\ \overline{\Gamma}^I_{\dot{A}A}\overline{\Gamma}^J_{A\dot{B}} + \overline{\Gamma}^J_{\dot{A}A}\overline{\Gamma}^I_{A\dot{B}} &= 2\delta^{IJ}\delta_{\dot{A}\dot{B}} \end{aligned} \tag{4.1.67}$$

we can rewrite (4.1.63) and (4.1.64) as surface integrals [267]

$$Z^{[AB]} = -\int d^2x \partial_\alpha \mathrm{Tr}\left(X^I, D_\beta X^J\right)\epsilon^{\alpha\beta}(\overline{\Gamma}^{IJ})^{AB}, \tag{4.1.68}$$

$$Z^{(AB)}_\mu = -\frac{1}{12}\int d^2x \partial_\alpha \mathrm{Tr}\left(X^I, [X^J, X^K, X^L]\right)\epsilon^{0\alpha}{}_\mu(\overline{\Gamma}^{IJKL})^{AB} \tag{4.1.69}$$

where the symmetric central charge is traceless $\delta_{AB}Z^{(AB)}_\mu = 0$ and $A, B, \cdots = 1, \cdots, 8$ are the $SO(8)$ indices. (4.1.62) and (4.1.63)-(4.1.65) are the field realization of the M2-brane algebra and the central charges [269]. These are useful tools to investigate five constitutes in M-theory, that is M-wave, M2-brane, M5-brane, M-KK monopole, M9-brane.

1. $Z^{[AB]}$

   $Z^{[AB]}$ is a world-volume 0-form transforming **28** of $SO(8)$. 0-form corresponds to a 0-brane (point) on the M2-brane. **28** defines a 2-form or 6-form in the transverse space to the M2-brane. In the case of 2-form, 0-brane is the result of the intersection with two another M2-brane over a point and defines the 2-plane along which the second M2-brane is aligned [270].

$$\begin{array}{cccccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \mathrm{M2} & \circ & \circ & \circ & \times & \times & \times & \times & \times & \times & \times & \times \\ \mathrm{M2} & \circ & \times & \times & \circ & \circ & \times & \times & \times & \times & \times & \times \end{array} \tag{4.1.70}$$

When choosing 6-form, 0-brane acquires the interpretation as the intersection of M2-brane with M-KK monopole over a point [271].

$$\begin{array}{cccccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \text{M2} & \circ & \circ & \circ & \times & \times & \times & \times & \times & \times & \times & \times \\ \text{MKK} & \circ & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ & \times & \times \end{array} \tag{4.1.71}$$

2. $Z_\mu^{(AB)}$

   $Z_\mu^{(AB)}$ is a world-volume 1-form and $\mathbf{35}^+$ of $SO(8)$. 1-form corresponds to a 1-brane (string) on the M2-brane. $\mathbf{35}^+$ defines a 4-form in the 8-dimensional transverse space. 1-brane is determined by 4-plane along which four of the spatial spaces of the M5-brane are aligned. Thus 1-brane has the interpretation as the intersection of M2-brane with M5-brane.

$$\begin{array}{cccccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \text{M2} & \circ & \circ & \circ & \times & \times & \times & \times & \times & \times & \times & \times \\ \text{M5} & \circ & \circ & \times & \circ & \circ & \circ & \circ & \times & \times & \times & \times \end{array} \tag{4.1.72}$$

3. Z_{IJKL}

   Due to the total antisymmetry and the fundamental identity, $Z_{IJKL} = Z_{[IJKL]}$ vanishes when we consider trace elements.

   However, it is discussed [268] that if we take into account constant background configurations of X^I that take values in non-trace elements[7], such configurations may give rise to BPS charges although non-abelian fields are infinite dimensional and have an infinite norm[8].

4. P_μ

   P_μ is a 1-form on a world-volume and a singlet $\mathbf{1}$ of $SO(8)$. 1-form corresponds to a 1-brane (string) on the M2-brane. $\mathbf{1}$ defines a 0-form or 8-form in the transverse space. In the case of 0-form, 1-brane can be viewed as the

[7] Configurations with non-trace elements are discussed in the matrix theory conjecture for M-theory in the light-cone quantization [272].

[8] By the novel Higgs mechanism, we can reduce Z_{IJKL} to the form $\mathrm{Tr}[X^I, X^J][X^I, X^J]$ which is similar to D4-brane charge in the D0-brane action in the matrix model. It is natural to think that Z_{IJKL} is identified with D6-brane charge because the BLG theory action reduces to the D2-brane action rather than D0-brane action. Furthermore D6-brane is uplifted to M-KK monopole, so Z_{IJKL} is expected to produce the energy bound of the configuration of M2-brane and M-KK monopole.

intersection of M2-brane with an M-wave over a 1-dimensional string.

$$\begin{array}{lccccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \text{M2} & \circ & \circ & \circ & \times & \times & \times & \times & \times & \times & \times & \times \\ \text{MW} & \circ & \circ & \times & \times & \times & \times & \times & \times & \times & \times & \times \end{array} \tag{4.1.73}$$

In the case of 8-form, 1-brane is the intersection of the M2-brane with M9-brane over a string.

$$\begin{array}{lccccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \text{M2} & \circ & \circ & \circ & \times & \times & \times & \times & \times & \times & \times & \times \\ \text{M9} & \circ & \circ & \times & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \end{array} \tag{4.1.74}$$

4.2 $\mathcal{A}_4$ BLG-theory

If we assume that

1. the metric h^{ab} of the 3-algebra $\mathcal{A}$ is positive definite so that the kinetic term and the potential term are all positive,

2. the dimension N of 3-algebra $\mathcal{A}$ is finite,

then the 3-algebra $\mathcal{A}$ is uniquely determined by [273, 274]

$$f^{abcd} = \frac{2\pi}{k}\epsilon^{abcd} = f\epsilon^{abcd}, \tag{4.2.1}$$

$$h^{ab} = \delta^{ab} \tag{4.2.2}$$

with $a, b = 1, \cdots, 4$. Here ϵ^{abcd} is an antisymmetric tensor and k is the integer determined by the quantization of the Chern-Simons level for a non-simply connected gauge group $SO(4)$ [275]. The correct normalization can be checked by using the expression (4.2.14) and noting that the coefficient of the Chern-Simons term is $\frac{k}{4\pi}$.

The 3-algebra characterized by (4.2.1) and (4.2.2) is called the $\mathcal{A}_4$ algebra. For the $\mathcal{A}_4$ algebra we do not distinguish superscripts and subscripts since gauge indices $a, b, \cdots$ are raised and lowered with Kronecker delta. However, A and $\tilde{A}$ should be distinguished because of the existence of f. The corresponding BLG theory has no continuous coupling constant but admit a discrete coupling k. The uniqueness up to the Chern-Simons level k makes it difficult to describe an arbitrary number of coincident M2-branes because the rank of the gauge algebra is expected to be related to the number of M2-branes in analogy with D-branes.

In terms of the antisymmetric tensor ϵ_{abcd} let us introduce the dual generators

$$M_{a_1 a_2} := \frac{1}{2}\epsilon_{a_1 a_2 b_1 b_2}\mathcal{T}^{b_1 b_2} \tag{4.2.3}$$

for the fundamental object $\mathcal{T}$. Then from the relation

$$\epsilon^{i_1\cdots i_n}_{j_1\cdots j_n} = \sum_{k=1}^{n}(-1)^{k+1}\delta^{i_1}_{j_k}\epsilon^{i_2\cdots i_n}_{j_1\cdots\hat{j}_k\cdots j_n} = \sum_{k=1}^{n}(-1)^{k+n}\epsilon^{i_1\cdots i_{n-1}}_{j_1\cdots\hat{j}_k\cdots j_n}\delta^{i_n}_{j_k}, \tag{4.2.4}$$

we obtain the commutation relations

$$[M_{a_1 a_2}, M_{b_1 b_2}] = -\delta_{a_1 b_2}M_{a_2 b_1} - \delta_{a_2 b_1}M_{a_1 b_2} + \delta_{a_1 b_1}M_{a_2 b_2} + \delta_{a_2 b_2}M_{a_1 b_1} \tag{4.2.5}$$

The algebraic relation (4.2.5) is recognized as commutators of semisimple $\mathfrak{so}(4)$ algebra. Thus from the ordinary Lie algebra point of view, the $\mathcal{A}_4$ BLG theory is based on the $\mathfrak{so}(4)$ gauge algebra. It has been discussed [276] that for the $\mathcal{A}_4$ BLG-model there are two possible inequivalent gauge groups G;

$$G = \begin{cases} SO(4) & \cong (SU(2)\times SU(2))/\mathbb{Z}_2 \\ Spin(4) & \cong SU(2)\times SU(2). \end{cases} \tag{4.2.6}$$

4.2.1 Quiver gauge structure

Now we want to discuss the connection between the BLG-model based on the Lie 3-algebras and the ordinary gauge theories based on the Lie algebras. This has been accomplished by the remarkable observation [277] that the $\mathcal{A}_4$ BLG-model can be rewritten as an ordinary gauge theory with quiver type gauge group and matters in the bifundamental representation [9].

Since in the $\mathcal{A}_4$ theory the Higgs fields X^I and Ψ are the fundamental representation $\mathbf{4}$ of the $\mathfrak{so}(4)$ we can denote them by the four-vectors

$$X^I = \begin{pmatrix} x_1^I \\ x_2^I \\ x_3^I \\ x_4^I \end{pmatrix}, \qquad \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix}. \tag{4.2.7}$$

In terms of the Pauli matrices σ_i[10], one may express these in the bi-fundamental

[9] This alternative expression of the $\mathcal{A}_4$ BLG-model triggered the discovery of the ABJM-model.
[10] Pauli matrices σ_i are given in (4.1.7) and normalized such that $\mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}$.

representation $(\mathbf{2},\mathbf{2})$ of the $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ gauge algebra as

$$\begin{aligned} X^I =&\frac{1}{2}(x_4^I\mathbb{I}_2 + ix_i^I\sigma^i) = \frac{1}{2}\begin{pmatrix} x_4^I + ix_3^I & x_2^I + ix_1^I \\ -x_2^I + ix_1^I & x_4^I - ix_3^I \end{pmatrix}, \\ \Psi =&\frac{1}{2}(\Psi_4\mathbb{I}_2 + i\Psi_i\sigma^i) = \frac{1}{2}\begin{pmatrix} \Psi_4 + i\Psi_3 & \Psi_2 + i\Psi_1 \\ -\Psi_2 + i\Psi_1 & \Psi_4 - i\Psi_3 \end{pmatrix}. \end{aligned} \tag{4.2.8}$$

They obey the reality conditions

$$\begin{aligned} X^I_{\alpha\dot{\beta}} =&\epsilon_{\alpha\beta}\epsilon_{\dot{\beta}\dot{\alpha}}(X^{I\dagger})^{\dot{\alpha}\beta}, \\ \Psi_{\alpha\dot{\beta}} =&\epsilon_{\alpha\beta}\epsilon_{\dot{\beta}\dot{\alpha}}(\Psi^{\dagger})^{\dot{\alpha}\beta}, \end{aligned} \tag{4.2.9}$$

where $\alpha,\beta = 1,2$ and $\dot{\alpha},\dot{\beta} = 1,2$ denote bi-fundamental representation $(\mathbf{2},\mathbf{2})$ of the $\mathfrak{su}(2)\times\mathfrak{su}(2)$ gauge algebra.

In order to find the adjoint gauge field for each $\mathfrak{su}(2)$ gauge symmetry factor, we decompose gauge fields $A_{\mu ab}$ into the sum of the selfdual and anti-selfdual parts

$$A_{\mu ab} := -\frac{1}{2f}(A^+_{\mu ab} + A^-_{\mu ab}) \tag{4.2.10}$$

where

$$\begin{aligned} *A^+_{\mu ab} =&\frac{1}{2}\epsilon_{ab}{}^{cd}A^+_{\mu cd} = A^+_{\mu ab}, \\ *A^-_{\mu ab} =&\frac{1}{2}\epsilon_{ab}{}^{cd}A^-_{\mu cd} = -A^-_{\mu ab} \end{aligned} \tag{4.2.11}$$

and $*$ is the Hodge star acting on the gauge indices and satisfying $*^2 = 1$. Noting that $\tilde{A}^{ab}_\mu = f^{cdab}A_{\mu cd}$, we also have

$$\tilde{A}^{cd}_\mu = -(A^{+cd}_\mu - A^{-cd}_\mu). \tag{4.2.12}$$

Then we define

$$\begin{aligned} A_\mu :=&A^+_{\mu 4i}\sigma_i, \\ \hat{A}_\mu :=&A^-_{\mu 4i}\sigma_i. \end{aligned} \tag{4.2.13}$$

Using the expressions (4.2.8) and (4.2.13), we rewrite the original BLG-theory Lagrangian (4.1.31) as

$$\begin{aligned}\mathcal{L} = &- \operatorname{Tr}\left(D^{\mu}X^{I\dagger}D_{\mu}X^{I}\right) + i\operatorname{Tr}\left(\overline{\Psi}^{\dagger}\Gamma^{\mu}D_{\mu}\Psi\right) \\ &- \frac{2}{3}if\operatorname{Tr}\left(\overline{\Psi}^{\dagger}\Gamma^{IJ}X^{I}X^{J\dagger}\Psi + \overline{\Psi}^{\dagger}\Gamma^{IJ}X^{J}\Psi^{\dagger}X^{I} + \overline{\Psi}^{\dagger}\Gamma^{IJ}\Psi X^{I\dagger}X^{J}\right) \\ &- \frac{8}{3}f^{2}\operatorname{Tr}\left(X^{[I}X^{J\dagger}X^{k]}X^{K\dagger}X^{J}X^{I\dagger}\right) \\ &+ \frac{1}{2f}\epsilon^{\mu\nu\lambda}\operatorname{Tr}\left(A_{\mu}\partial_{\nu}A_{\lambda} + \frac{2}{3}iA_{\mu}A_{\nu}A_{\lambda}\right) - \frac{1}{2f}\epsilon^{\mu\nu\lambda}\operatorname{Tr}\left(\hat{A}_{\mu}\partial_{\nu}\tilde{A}_{\lambda} + \frac{2}{3}i\hat{A}_{\mu}\hat{A}_{\nu}\hat{A}_{\lambda}\right).\end{aligned} \tag{4.2.14}$$

Here the covariant derivative is defined by

$$D_{\mu}X^{I} = \partial_{\mu}X^{I} + iA_{\mu}X^{I} - iX^{I}\hat{A}_{\mu}. \tag{4.2.15}$$

Notice that now the twisted Chern-Simons terms in the original BLG-model is decomposed into two ordinary Chern-Simons terms for A and $\hat{A}$ with opposite signs. This observation was crucial for the discovery of the ABJM-model as it opens up the highly extended supersymmetric Chern-Simons matter theories with quiver type gauge group. We see that the theory becomes weakly coupled in the large k limit since after rescaling $A \to \sqrt{f}A$, all interaction terms are proportional to positive power of $f = \frac{2\pi}{k}$.

The Lagrangian (4.2.14) is invariant under a new set of supersymmetry transformations

$$\delta X^{I} = i\bar{\epsilon}\Gamma^{I}\Psi, \tag{4.2.16}$$

$$\delta\Psi = D_{\mu}X^{I}\Gamma^{\mu}\Gamma^{I}\epsilon + \frac{4\pi}{3}X^{I}X^{J\dagger}X^{K}\Gamma^{IJK}\epsilon, \tag{4.2.17}$$

$$\delta A_{\mu} = f\bar{\epsilon}\Gamma^{I}(X^{I}\Psi^{\dagger} - \Psi X^{I\dagger}), \tag{4.2.18}$$

$$\delta\hat{A}_{\mu} = f\bar{\epsilon}\Gamma_{\mu}\Gamma^{I}(\Psi^{\dagger}X^{I} - X^{I\dagger}\Psi). \tag{4.2.19}$$

4.2.2 Superconformal symmetry

It has been proven [264] that the $\mathcal{A}_4$ BLG theory has $OSp(8|4)$ superconformal symmetry that contains the $SO(8)_R$ R-symmetry group and the three-dimensional $Sp(4) \cong Spin(2,3)$ conformal symmetry group as bosonic factor groups at the classical level. To see the superconformal symmetry explicitly, we replace supersymmetry parameter ϵ_A by $\Gamma^{\mu}x_{\mu}\eta_A$ where η_A is a superconformal symmetry parameter and add a term $-\Gamma^{I}X^{I}_{a}\eta$ to $\delta\Psi_a$ in the supersymmetry transformations of the BLG-model. Then the superconformal symmetry is given by

$$\delta X_a^I = i\bar{\eta}\Gamma^\mu x_\mu \Gamma^I \Psi_a, \tag{4.2.20}$$

$$\delta \Psi_a = D_\mu X_a^I \Gamma^\mu \Gamma^I \Gamma^\nu x_\nu \eta - \frac{1}{6} X_b^I X_c^J X_d^K f^{bcd}{}_a \Gamma^{IJK} \Gamma^\nu x_\nu \eta - \Gamma^I X_a^I \eta, \tag{4.2.21}$$

$$\delta \tilde{A}_{\mu a}^b = i\bar{\eta} x_\nu \Gamma^\nu \Gamma_\mu \Gamma^I X_c^I \Psi_d f^{cdb}{}_a \tag{4.2.22}$$

and one can check that the action (4.1.31) is invariant under the superconformal transformations (4.2.20)-(4.2.22) up to total derivative terms.

4.2.3 Parity invariance

Although Chern-Simons theories are parity violating, we can make the $\mathcal{A}_4$ BLG Lagrangian (4.2.14) parity invariant by defining parity transformation as a spatial reflection together with interchange of two $SU(2)$ gauge groups [264, 22, 277]. This implies that we assign an odd parity to f^{abcd}. In particular, under the reflection $x^2 \to -x^2$ we require that

$$X_a^I \to X_a^I, \qquad \tilde{A}_{2b}^a \to -\tilde{A}_{2b}^a, \tag{4.2.23}$$

$$\tilde{A}_{0b}^a \to \tilde{A}_{0b}^a, \qquad f^{abcd} \to -f^{abcd}, \tag{4.2.24}$$

$$\tilde{A}_{1b}^a \to \tilde{A}_{1b}^a, \qquad \Psi_a \to \Gamma_2 \Psi_a. \tag{4.2.25}$$

Then (4.2.14) turns out ot be parity conserving.

4.2.4 Moduli space

The vacuum moduli space of the theory is the configuration space that minimise the potential modulo gauge transformations. For the $\mathcal{A}_4$ BLG-model it was initially investigated in [277, 278, 275]. Since $\mathcal{A}_4$ BLG theory has the Euclidean inner product, the potential is positive definite and the potential is minimal when

$$[X^I, X^J, X^K] = 0. \tag{4.2.26}$$

From the fact that the bosonic scalar fields X_a^I are eight vectors in an $\mathbb{R}^4$ rotated by the gauge symmetry $SO(4)$, the triple product $X_a^I X_b^J X_c^K$ produces a new vector perpendicular to the three vectors X_a^I, X_b^J and X_c^K whose length is the signed volume of the parallelepiped spanned by the three vectors in $\mathbb{R}^4$ (see Figure 4.1).

The bosonic potential is proportional to the square of this volume summed over each possible triple of vectors. Therefore the bosonic potential vanishes if and only if all the three vectors lie in the same plane. This space is labeled by ordered sets

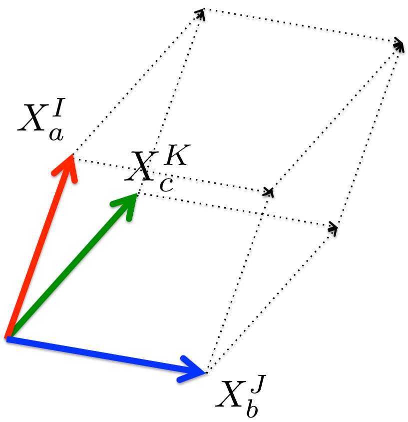


Figure 4.1: The parallelepiped spanned by the three vectors X_a^I, X_b^J and X_c^K. A new vector produced by the triple product has the length as the signed volume of the parallelepiped. The triple product is zero if and only if all the vectors lie in the same plane.

of eight vectors in the same plane. One can assume that all vectors lie in the x_1-x_2 plane without losing generality where x_a are the coordinates of T^a. Then eight x_1 coordinates r_1^I and the eight x_2 coordinates r_2^I form two octuplets which are rotated into each other by the residual $O(2)$ symmetry. Thus, up to gauge transformation, the vacuum moduli space is parametrized by

$$X_a^I = r_1^I T^1 + r_2^I T^2 = \begin{pmatrix} r_1^I \\ r_2^I \\ 0 \\ 0 \end{pmatrix}, \quad r_1^I, r_2^I \in \mathbb{R}^8. \tag{4.2.27}$$

In the bi-fundamental notation (4.2.8), (4.2.27) is expressed as

$$X^I = \frac{1}{\sqrt{2}} \begin{pmatrix} z^I & 0 \\ 0 & \bar{z}^I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} r_1 + ir_2^I & 0 \\ 0 & r_1^I - ir_2^I \end{pmatrix}. \tag{4.2.28}$$

Then one can see that the residual gauge symmetries $g \in SO(4)$ that preserve the form X^I is the block diagonal form

$$g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \tag{4.2.29}$$

where $g_1, g_2 \in O(2)$ act on (x_1, x_2) and (x_3, x_4) respectively, with $\det g_1 = \det g_2$. Since g_2 acts trivially on (4.2.27), we can ignore it and simply look at $g_1 \in O(2)$.

Let us discuss the residual gauge symmetry in the diagonal configurations (4.2.28). The residual $O(2)$ gauge symmetry in which g_1 is contained consists of two types of symmetries:

1. simultaneous rotation on z^I (continuous symmetry)

$$U(1)_{12} : z^I \to e^{i\theta} z^I, \quad \theta \in [0, 2\pi) \tag{4.2.30}$$

2. simultaneous complex conjugation (discrete symmetry)

$$z^I \to \bar{z}^I \tag{4.2.31}$$

However, the continuous symmetry U_{12} is generically broken down for the diagonal configuration (4.2.27). Therefore the remaining component of gauge field become massive by the Higgs mechanism. To see this we shall write down the effective action. Let us firstly define the gauge field B_μ associated with the broken $U(1)_{12}$ that rotate z^I and the preserved gauge field C_μ associated with the preserved $U(1)$ by

$$B_\mu := \frac{4\pi}{k} A_\mu^{34}, \tag{4.2.32}$$

$$C_\mu := \frac{4\pi}{k} A_\mu^{12}. \tag{4.2.33}$$

Note that because of ϵ^{abcd} in the covariant derivative (4.1.34) the broken $U(1)_{12}$ gauge field is associated with A_μ^{34} not A_μ^{12}. Substituting the configurations (4.2.27), (4.2.32) and (4.2.33) into the BLG Lagrangian (4.1.31), one can write the kinetic terms on the moduli space and the twisted Chern-Simons terms as [278, 275]

$$\mathcal{L}_{kin} + \mathcal{L}_{TCS} = -\frac{1}{2} |\mathcal{D}_\mu z^I|^2 + \frac{k}{2\pi} \epsilon^{\mu\nu\lambda} B_\mu \partial_\nu C_\lambda \tag{4.2.34}$$

where $\mathcal{D}_\mu z^I = \partial_\mu z^I + i B_\mu z^I$.

Moreover we can replace the unbroken gauge field C with its dual photon σ that plays a role of a Lagrange multiplier to impose the Bianchi identity $\epsilon^{\mu\nu\lambda}\partial_\mu G_{\nu\lambda}$ on the field strength $G_{\mu\nu} := \partial_\mu C_\nu - \partial_\nu C_\mu$ by introducing the additional term [11]

$$\mathcal{L}_{dual} = \frac{1}{4\pi} \sigma \epsilon^{\mu\nu\lambda} \partial_\mu G_{\nu\lambda}. \tag{4.2.35}$$

[11] In the original work of [278, 275] the normalization is chosen as $\mathcal{L}_{dual} = \frac{1}{8\pi}\sigma\epsilon^{\mu\nu\lambda}\partial_\mu G_{\nu\lambda}$ so that $\sigma \in [0, 2\pi)$. However, this does depend on the two choice of the gauge group; $SU(2) \times SU(2)$ and $(SU(2) \times SU(2))/\mathbb{Z}_2$ as pointed in [276].

Combining (4.2.34) and (4.2.35), we can write the low-energy effective action as [278, 275]

$$\mathcal{L}_{kin} + \mathcal{L}_{TCS} + \mathcal{L}_{dual} = -\frac{1}{2}|\mathcal{D}_\mu z^I|^2 + \frac{1}{4\pi}\epsilon^{\mu\nu\lambda}(kB_\mu - \partial_\mu\sigma)G_{\nu\lambda}. \tag{4.2.36}$$

The action (4.2.36) is invariant under the $U(1)_{12}$ gauge symmetry transformations

$$z^I \to e^{i\theta}z^I, \qquad \sigma \to \sigma + k\theta, \qquad B_\mu \to B_\mu + \partial_\mu\theta. \tag{4.2.37}$$

Using the equation of motion for $G_{\mu\nu}$

$$B_\mu = \frac{\partial_\mu\sigma}{k}, \tag{4.2.38}$$

the action (4.2.36) further reduces to

$$\mathcal{L} = -\frac{1}{2}|\partial_\mu z^I - \frac{i}{k}z^I\partial_\mu\sigma|^2. \tag{4.2.39}$$

By defining the fields

$$w^I := e^{-\frac{i\sigma}{k}}z^I, \tag{4.2.40}$$

we can absorb the Lagrange multiplier σ and the action (4.2.39) finally becomes

$$\mathcal{L} = -\frac{1}{2}\partial_\mu w^I\partial^\mu\overline{w}^I. \tag{4.2.41}$$

As a next step we need to determine the periodicity of σ which yields the gauge symmetry of the moduli parameter z^I as seen from the redefinition (4.2.40). The periodicity of σ occurs from the Dirac quantization of the flux of the field strength. Let us consider the case where some field ϕ couples to a $U(1)$ gauge field A_μ as $D_\mu\phi = \partial_\mu\phi + iA_\mu\phi$. If we go around a closed path γ, then ϕ is parallel transported into $\phi_\gamma = e^{i\oint_\gamma A}\phi = e^{i\int_\Sigma F}\phi$ where D is a two-dimensional surface whose boundary is γ and F is the field strength of A. Since the choice of the surface Σ is not unique, we require that $q := \int_\Sigma F = 2\pi\mathbb{Z}$. This is the Dirac quantization for the charge q. Now we are interested in the Dirac quantization of the field strength $G = dC$ of the preserved gauge field C since it yields the periodicity for σ as we see from the action (4.2.36). However, in our case the charge of the field strength $G = dC$ turns out to be different as the Dirac value. The result is given by [275, 276]

$$\int_\Sigma G \in \begin{cases} 4\pi\mathbb{Z} & \text{for } Spin(4) = SU(2)\times SU(2) \\ 2\pi\mathbb{Z} & \text{for } SO(4) = (SU(2)\times SU(2))/\mathbb{Z}_2. \end{cases} \tag{4.2.42}$$

This is because at the generic point of the moduli space the $U(1)$ gauge field C sits inside the diagonal $SO(3) \in (SU(2) \times SU(2))/\mathbb{Z}_2$ or $SU(2) \times SU(2)$ and the Higgs fields does not transform as the adjoint representations of the $U(1)$ but that of the $SO(3)$. This situation is similar to the 't Hooft-Polyakov monopoles [279] where all the fields transform in the adjoint representation of $SU(2) \cong SO(3)$. For the $SU(2) \times SU(2)$ group G is thus essentially the sum of two independent field strengths and we need the additional factor 2 as $\int_\Sigma G \in 4\pi\mathbb{Z}$. For the $(SU(2) \times SU(2))/\mathbb{Z}_2$ gauge group the phase is equal to one only up to a $\mathbb{Z}_2$ action and we require that $\int_\Sigma G \in 2\pi\mathbb{Z}$. Noting that $dG = \frac{1}{2}\epsilon^{\mu\nu\lambda}\partial_\mu G_{\nu\lambda}$ and lifting the relation (4.2.42) to the integral of dG over the three-manifold, we get

$$\frac{1}{4\pi}\int \epsilon^{\mu\nu\lambda}\partial_\mu G_{\nu\lambda} \in \begin{cases} 2\mathbb{Z} & \text{for } Spin(4) = SU(2) \times SU(2) \\ \mathbb{Z} & \text{for } SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2. \end{cases} \tag{4.2.43}$$

Since σ appears in the action (4.2.36) as the coupling to $\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}\partial_\mu G_{\nu\lambda}$, which takes the discrete value in (4.2.42), σ must be periodic as

$$\sigma \sim \begin{cases} \sigma + \pi & \text{for } Spin(4) = SU(2) \times SU(2) \\ \sigma + 2\pi & \text{for } SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2. \end{cases} \tag{4.2.44}$$

Combining the periodicity (4.2.44) and the expression (4.2.40), we can read the gauge identification of z^I from the continuous transformation (4.2.30) as

$$z^I \cong \begin{cases} e^{\frac{\pi i}{k}} z^I & \text{for } Spin(4) = SU(2) \times SU(2) \\ e^{\frac{2\pi i}{k}} z^I & \text{for } SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2. \end{cases} \tag{4.2.45}$$

At this stage we have two types of the gauge equivalences; one is (4.2.45) from the continuous symmetry (4.2.45) yielding $\mathbb{Z}_{2k}$ or $\mathbb{Z}_k$ and the other is from the discrete one (4.2.31) corresponding to $\mathbb{Z}_2$. Since both of them do not commute, we finally obtain the moduli space $\mathcal{M}_k$ of the $\mathcal{A}_4$ BLG-model with the Chern-Simons level k as [276]

$$\mathcal{M}_k = \begin{cases} \frac{\mathbb{R}^8 \times \mathbb{R}^8}{D_{4k}} & \text{for } Spin(4) = SU(2) \times SU(2) \\ \frac{\mathbb{R}^8 \times \mathbb{R}^8}{D_{2k}} & \text{for } SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2. \end{cases} \tag{4.2.46}$$

For generic k we do not know whether these moduli spaces can have a geometrical interpretation of the M2-branes. However, for $k = 1, 2, 4$ there is a conjectural space-time interpretation of the M2-branes.

Chapter 5

ABJM-model

In this chapter we will review the ABJM-model [26] which may describe an arbitrary number of M2-branes. We will introduce the notations and conventions in section 5.1. We will turn to the analysis of the moduli space in section 5.2. Then we will discuss the conjectural duality between the BLG-model and the ABJM-model in section 5.3.

5.1 Construction

The ABJM-model is a three-dimensional $\mathcal{N}=6$ superconformal $U(N)_k \times \hat{U}(N)_{-k}$ Chern-Simons-matter theory proposed as a generalization of the BLG-model in that it may describe the dynamics of an arbitrary number of coincident M2-branes [26]. The theory has manifestly only $\mathcal{N}=6$ supersymmetry and the corresponding $SU(4)_R$ R-symmetry at the classical level. It has been discussed that [26, 280, 281] at $k=1$ and $k=2$ these symmetries are enhanced to $\mathcal{N}=8$ supersymmetry and $SO(8)_R$ R-symmetry as a quantum effect. The theory contains

- 4 complex scalar fields Y^A
- 4 Weyl spinors ψ_A
- 2 types of gauge fields $A_\mu, \hat{A}_\mu$.

Here the upper and lower indices $A, B, \cdots = 1, 2, 3, 4$ denote $\mathbf{4}$ and $\overline{\mathbf{4}}$ of the $SU(4)_R$ respectively. The matter fields are $N \times N$ matrices so that Y^A and ψ_A transform as $(\boldsymbol{N}, \overline{\boldsymbol{N}})$ bi-fundamental representations of $U(N)_k \times \hat{U}(N)_{-k}$ gauge group, while $Y_A^\dagger$ and $\psi^{\dagger A}$ do as $(\overline{\boldsymbol{N}}, \boldsymbol{N})$. A_μ is a Chern-Simons $U(N)$ gauge field of level $+k$ and $\hat{A}_\mu$ is that of level $-k$. Also in the theory there is a $U(1)_B$ flavor symmetry and the corresponding baryonic charges are assigned $+1$ for bi-fundamental fields,

	$U(N)$	$\hat{U}(N)$	$SU(4)_R$	$U(1)_B$
Y^A	$\boldsymbol{N}$	$\overline{\boldsymbol{N}}$	$\mathbf{4}$	$+1$
$Y_A^\dagger$	$\overline{\boldsymbol{N}}$	$\boldsymbol{N}$	$\bar{\mathbf{4}}$	-1
ψ_A	$\boldsymbol{N}$	$\overline{\boldsymbol{N}}$	$\bar{\mathbf{4}}$	$+1$
$\psi^{\dagger A}$	$\overline{\boldsymbol{N}}$	$\boldsymbol{N}$	$\mathbf{4}$	-1
A_μ	$\boldsymbol{N}^2$	$\mathbf{1}$	$\mathbf{1}$	0
$\hat{A}_\mu$	$\mathbf{1}$	$\boldsymbol{N}^2$	$\mathbf{1}$	0

Table 5.1: The symmetries and their representations for fields in the ABJM-model. The bold letters for $U(N)$, $\hat{U}(N)$ and $SU(4)_R$ symmetries denote the representations for the symmetry groups and the quantities for $U(1)_B$ symmetry are the corresponding charges.

-1 for anti-bi-fundamental fields and 0 for gauge fields. The symmetries in the ABJM-model are summarized in Table 5.1.

5.1.1 Lagrangian

The Lagrangian of the ABJM-model is given by [282]

$$\begin{aligned}\mathcal{L}_{\mathrm{ABJM}} =& -\mathrm{Tr}(D_\mu Y_A^\dagger D^\mu Y^A) - i\mathrm{Tr}(\psi^{\dagger A}\gamma^\mu D_\mu \psi_A) - V_{\mathrm{ferm}} - V_{\mathrm{bos}} \\ &+ \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}\mathrm{Tr}\left[A_\mu\partial_\nu A_\lambda + \frac{2i}{3}A_\mu A_\nu A_\lambda - \hat{A}_\mu\partial_\nu\hat{A}_\lambda - \frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\right] \end{aligned}\tag{5.1.1}$$

where

$$\begin{aligned}V_{\mathrm{ferm}} =& -\frac{2\pi i}{k}\mathrm{Tr}\Big(Y_A^\dagger Y^A\psi^{\dagger B}\psi_B - \psi^{\dagger B}Y^A Y_A^\dagger\psi_B \\ &- 2Y_A^\dagger Y^B\psi^{\dagger A}\psi_B + 2Y^A Y_B^\dagger\psi_A\psi^{\dagger B} \\ &- \epsilon^{ABCD}Y_A^\dagger\psi_B Y_C^\dagger\psi_D + \epsilon_{ABCD}Y^A\psi^{\dagger B}Y^C\psi^{\dagger D}\Big),\end{aligned}\tag{5.1.2}$$

$$\begin{aligned}V_{\mathrm{bos}} =& -\frac{4\pi^2}{3k^2}\mathrm{Tr}\Big(Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + Y_A^\dagger Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C \\ &+ 4Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger - 6Y^A Y_B^\dagger Y^B Y_A^\dagger Y^C Y_C^\dagger\Big).\end{aligned}\tag{5.1.3}$$

Here we use the Dirac matrix $(\gamma^\mu)_\alpha{}^\beta = (i\sigma_2, \sigma_1, \sigma_3)$. The spinor indices are raised, $\theta^\alpha = \epsilon^{\alpha\beta}\theta_\beta$, and lowered, $\theta_\alpha = \epsilon_{\alpha\beta}\theta^\beta$ with $\epsilon^{12} = -\epsilon_{12} = 1$. Note that this makes the Dirac matrix $\gamma^\mu_{\alpha\beta} := (\gamma^\mu)_\alpha{}^\gamma\epsilon_{\beta\gamma} = (-\mathbb{I}_2, -\sigma_3, \sigma_1)$ symmetric and guarantees the

Hermiticity of the fermionic kinetic term. The covariant derivatives are defined by

$$\begin{aligned} D_\mu Y^A &= \partial_\mu Y^A + iA_\mu Y^A - iY^A \hat{A}_\mu, \quad D_\mu \psi_A = \partial_\mu \psi_A + iA_\mu \psi_A - i\psi_A \hat{A}_\mu, \\ D_\mu Y_A^\dagger &= \partial_\mu Y_A^\dagger - iA_\mu Y_A^\dagger + iY_A^\dagger \hat{A}_\mu, \quad D_\mu \psi^{\dagger A} = \partial_\mu \psi^{\dagger A} - iA_\mu \psi^{\dagger A} + i\psi^{\dagger A} \hat{A}_\mu. \end{aligned} \tag{5.1.4}$$

5.1.2 Supersymmetry transformation

The supersymmetry transformation laws are

$$\delta Y^A = i\omega^{AB}\psi_B, \tag{5.1.5}$$

$$\delta Y_A^\dagger = i\psi^{\dagger B}\omega_{AB}, \tag{5.1.6}$$

$$\delta\psi_A = -\gamma^\mu \omega_{AB} D_\mu Y^B + \frac{2\pi}{k}\left[-\omega_{AB}(Y^C Y_C^\dagger Y^B - Y^B Y_C^\dagger Y^C) + 2\omega_{CD} Y^C Y_A^\dagger Y^D\right], \tag{5.1.7}$$

$$\delta\psi^{\dagger A} = D_\mu Y_B^\dagger \omega^{AB}\gamma^\mu + \frac{2\pi}{k}\left[-(Y_B Y^C Y_C^\dagger - Y_C^\dagger Y^C Y_B^\dagger)\omega^{AB} + 2Y_D^\dagger Y^A Y_C^\dagger \omega^{CD}\right], \tag{5.1.8}$$

$$\delta A_\mu = \frac{\pi}{k}\left(-Y^A\psi^{\dagger B}\gamma_\mu\omega_{AB} + \omega^{AB}\gamma_\mu\psi_A Y_B^\dagger\right), \tag{5.1.9}$$

$$\delta\hat{A}_\mu = \frac{\pi}{k}\left(-\psi^{\dagger A} Y^B\gamma_\mu\omega_{AB} + \omega^{AB}\gamma_\mu Y_A^\dagger\psi_B\right). \tag{5.1.10}$$

The parameter ω_{AB} is defined by

$$\omega_{AB} := \epsilon_i(\Gamma^i)_{AB}, \quad \omega^{AB} := \epsilon_i(\Gamma^{i*})^{AB} \tag{5.1.11}$$

where the $SL(2,\mathbb{R})$ spinor ϵ^i, $i = 1, \cdots, 6$ transforms as the representation **6** under the $SU(4)_R$ and Γ^i is the six-dimensional 4×4 matrix satisfying

$$(\Gamma^i)_{AB} = -(\Gamma^i)_{BA}, \tag{5.1.12}$$

$$\frac{1}{2}\epsilon^{ABCD}(\Gamma^i)_{CD} = -(\Gamma^{i\dagger})^{AB} = (\Gamma^{i*})^{AB}, \tag{5.1.13}$$

$$\left\{\Gamma^i, \Gamma^j\right\} = 2\delta_{ij}. \tag{5.1.14}$$

Note that the supersymmetry parameter ω_{AB} obeys

$$\omega^{AB} = \omega_{AB}^* = \frac{1}{2}\epsilon^{ABCD}\omega_{CD}. \tag{5.1.15}$$

5.2 Moduli space

In order to determine the vacuum moduli space of the $U(N)_k \times \hat{U}(N)_{-k}$ ABJM-model, we need to consider the minimum of the scalar potential. Since the potential turns out to be a perfect square, the potential is minimal when the potential vanishes. The vanishing condition of the bosonic potential is given by

$$Y^C Y_C^\dagger Y^B = 0, \tag{5.2.1}$$

$$Y^C Y_A^\dagger Y^D = 0. \tag{5.2.2}$$

The generic solution is given by diagonal configurations

$$Y^A = \mathrm{diag}(y_1^A, \cdots, y_N^A) \tag{5.2.3}$$

up to gauge equivalences. The configurations (5.2.3) are the full moduli space because for generic diagonal elements one obtains positive definite mass matrix for the off-diagonal elements and all off-diagonal elements turn out to be massive. The solutions (5.2.3) break the gauge group $U(N) \times \hat{U}(N)$ to $U(1)^N \times U(1)^N \times S_N$ where S_N is the Weyl group of $U(N)$ that permutes the diagonal elements of all matrices. At a generic point of the moduli space, only a $U(1)^N$ subgroup that does not act on the eigenvalues remains unbroken and its gauge transformations keep There are gauge transformations that Y^A diagonal. Quotienting by such gauge symmetries, one finds The moduli space of the $U(N)_k \times \hat{U}(N)_{-k}$ ABJM-model is [26]

$$\mathcal{M}_{N,k} = \frac{(\mathbb{C}^4/\mathbb{Z}_k)^N}{S_N} = \mathrm{Sym}^N(\mathbb{C}^4/\mathbb{Z}_k). \tag{5.2.4}$$

This can be identified with the moduli space of N indistinguishable M2-branes moving in $\mathbb{C}^4/\mathbb{Z}_k$ transverse space. Therefore the ABJM-model is expected to describe the low-energy world-volume theory of N coincident M2-branes probing an orbifold $\mathbb{C}^4/\mathbb{Z}_k$. The four complex scalar fields Y^A represent the positions of the membranes in $\mathbb{C}^4$.

The orbifold $\mathbb{Z}_k$ acts on the four complex coordinates y^A as

$$y^A \to e^{\frac{2\pi i}{k}} y^A. \tag{5.2.5}$$

This preserves $SU(4)$ rotational symmetry, which is realized as the R-symmetry in the ABJM theory. The action of the $\mathbb{Z}_k$ on the fermionic fields is

$$\psi \to e^{\frac{2\pi(s_1+s_2+s_3+s_4)}{k}} \psi \tag{5.2.6}$$

where $s_i = \pm\frac{1}{2}$ are the spinor weights. The chirality projection implies that the sum of all s_i must be even, which produces an eight-dimensional representation. The spinors that are left invariant by the orbifold have $\sum_{i=1}^4 s_i = 0, \mod k$. This selects six out of eight spinors, so the M2-brane theory has 12 supercharges. This agrees with ABJM theory. Therefore this is consistent to the conjecture that the ABJM theory is dual to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ with N units of flux [26].

5.3 Duality between BLG and ABJ(M)

In [276] it has been discussed that if N and k are co-prime, then the vacuum moduli space of the $U(N)_k \times \hat{U}(N)_{-k}$ theory is equivalent to that of the $SU(N) \times SU(N)/\mathbb{Z}_N$ theory. Consequently there are conjectural dualities between the ABJ(M) theory and the BLG theory

$$U(2)_1 \times \hat{U}(2)_{-1} \text{ ABJM theory} \Leftrightarrow SO(4) \text{ BLG theory with } k = 1, \tag{5.3.1}$$

$$U(2)_2 \times \hat{U}(2)_{-2} \text{ ABJM theory} \Leftrightarrow Spin(4) \text{ BLG theory with } k = 2, \tag{5.3.2}$$

$$U(3)_2 \times \hat{U}(2)_{-2} \text{ ABJ theory} \Leftrightarrow SO(4) \text{ BLG theory with } k = 4. \tag{5.3.3}$$

These proposed dualities have been tested by the computations of the superconformal indices [283]. Hence we may regard the $SO(4)$ BLG-model with $k = 1$ as the world-volume theory of two planar M2-branes propagating in a flat space.

Part III

SCQM from M2-branes

Chapter 6

$\mathcal{N} = 16$ Superconformal Mechanics

Let us turn to the most important part of this thesis in which we will see how the two subjects discussed so far are connected with each other. We will initiate our study in this chapter by considering the BLG-model wrapped on a torus and derive the IR quantum mechanics by shrinking the torus. We will see that the IR quantum mechanics is the $\mathcal{N} = 16$ superconformal gauged quantum mechanics and also find the $OSp(16|2)$ superconformal quantum mechanics from the reduced systems.

6.1 $\mathcal{N} = 16$ gauged quantum mechanics

We shall start our analysis of the wrapped M2-branes with the case where the two membranes wrap a torus T^2 and propagate in a transverse space with an $SO(8)$ holonomy group. For a torus there is no non-trivial spin connection and the world-volume theory of M2-branes is given by the BLG action (4.1.31) defined on $M_3 = \mathbb{R} \times T^2$.

A torus is a compact Riemann surface of genus one and it is characterized by two periods which are defined as the integration of a holomorphic differential ω along two canonical homology basis a, b of a torus (see Figure 6.1). Let us define the periods by

$$\int_a \omega = 1, \qquad \int_b \omega = \tau. \tag{6.1.1}$$

Here τ is the moduli of the torus and it should not be real.

We now want to consider the limit in which T^2 has vanishingly small size and derive the low-energy effective one-dimensional theory on $\mathbb{R}$. In order to obtain such a theory we need to determine the configurations with the lowest

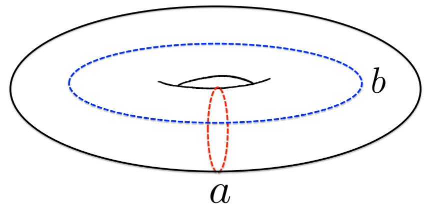


Figure 6.1: A torus with two canonical homology basis a and b.

energy. Since we are now considering supersymmetric theories, the low-energy configurations can be determined by solving the BPS equations. As we are interested in bosonic BPS configurations, we require that the background values of the fermionic fields vanish. Then the bosonic fields are automatically invariant under their supersymmetry transformations. Therefore the BPS equations correspond to the vanishing of the supersymmetry transformations (4.1.50) for fermionic fields. Also we discard the terms which include the covariant derivatives with respect to time because we are now interested in the low energy dynamics as a fluctuation around gauge invariant static configurations. Then one finds the BPS equations

$$D_z X_a^I = 0, \quad D_{\bar{z}} X_a^I = 0, \tag{6.1.2}$$

$$[X^I, X^J, X^K] = 0. \tag{6.1.3}$$

To go further we consider the $SO(4)$ BLG-model that may describe two M2-branes. In this case the Higgs fields transform as fundamental representations of the $SO(4)$ gauge group and we assume that these Higgs fields have non-zero values. Then the generic solution to (6.1.3) is given by $X_a^I = \left(X_1^I, X_2^I, 0, 0\right)^T$. For these solutions, the remaining BPS equations (6.1.2) reduce to

$$\partial_z X_1^I + \tilde{A}_{z2}^1 X_2^I = 0, \qquad \partial_z X_2^I - \tilde{A}_{z2}^1 X_1^I = 0, \tag{6.1.4}$$

$$\tilde{A}_{z3}^1 X_1^I + \tilde{A}_{z3}^2 X_2^I = 0, \qquad \tilde{A}_{z4}^1 X_1^I + \tilde{A}_{z4}^2 X_2^I = 0, \tag{6.1.5}$$

and their complex conjugates. First of all, the equations (6.1.4) tell us that the sum of the squares $(X_1^I)^2 + (X_2^I)^2$ for $I = 1, \cdots, 8$ is independent of the locus of the Riemann surface. Thus we can write

$$X_1^{I+2} + iX_2^{I+2} = r^I e^{i(\theta^I + \varphi(z,\bar{z}))} \tag{6.1.6}$$

where $r^I, \theta^I \in \mathbb{R}$ are constant on the torus and represent the configuration of the two membranes in the I-th direction while $\varphi(z, \bar{z})$ may depend on z and $\bar{z}$. Furthermore the equations (6.1.4) enable us to write $\tilde{A}_{z2}^1 = \partial_z \varphi$. The second set of equations (6.1.5) forces us to turn off four of six gauge fields; $\tilde{A}_{z3}^1 = \tilde{A}_{z3}^2 =$

$\tilde{A}^1_{z4} = \tilde{A}^2_{z4} = 0$. These components of the gauge field become massive by the Higgs mechanism. Note that the above set of solutions automatically satisfies the integrability condition for (6.1.2) because the gauge field $\tilde{A}^1_{z2}$ is flat.

One can find further restrictions by noting that the flat gauge fields $\tilde{A}^1_{z2}$ on a torus have specific expressions. Cutting a torus along the canonical basis a and b, the sections of a flat bundle are described by their transition functions, i.e. constant phases around a and b. Thus they can be completely classified by their twists $e^{2\pi i\xi}$, $e^{-2\pi i\zeta}$ on the homology along cycles a, b where ξ and ζ are real parameters. This space is the torus $\mathbb{C}/L_\tau$ where L_τ is the lattice generated by $\mathbb{Z}+\tau\mathbb{Z}$. It is referred to as the Jacobi variety of T^2 denoted by $\mathrm{Jac}(T^2)$. The twists on the homology can be described as a point on the Jacobi variety. Hence the flat gauge field can be expressed in the form [284]

$$\tilde{A}^1_{z2} = -2\pi\frac{\Theta}{\tau-\overline{\tau}}\omega, \qquad \tilde{A}^1_{\overline{z}2} = 2\pi\frac{\overline{\Theta}}{\tau-\overline{\tau}}\overline{\omega} \tag{6.1.7}$$

where $\Theta := \zeta + \overline{\tau}\xi$ is the complex parameter representing the twists on the homology along two cycles. Subsequently we can write

$$\varphi(z,\overline{z}) = 2\pi\frac{\overline{\Theta}}{\tau-\overline{\tau}}\overline{z} - 2\pi\frac{\Theta}{\tau-\overline{\tau}}z. \tag{6.1.8}$$

Since the angular variable $\varphi(z,\overline{z})$ in the $X^I_1X^I_2$-plane characterizes the ratio of two bosonic degrees of freedom for the two membranes, it must take same values modulo $2\pi\mathbb{Z}$ under the shifts $z\to z+1$ and $z\to z+\tau$ around two cycles. Therefore both the coordinates ξ and ζ are required to be integer values. From the expression (6.1.7), the discretization of these coordinates implies that $\tilde{A}^1_{z2}$ and $\tilde{A}^1_{\overline{z}2}$ are quantized. We therefore conclude that the generic BPS solutions are given by

$$\begin{aligned}
X^{I+2} &= \begin{pmatrix} X^I_A \\ X^I_B \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta^I+\varphi(z,\overline{z})) \\ \sin(\theta^I+\varphi(z,\overline{z})) \\ 0 \\ 0 \end{pmatrix} r^I, \\
\tilde{A}_z &= \begin{pmatrix} 0 & -2\pi\frac{\Theta}{\tau-\overline{\tau}}\omega_z & 0 & 0 \\ 2\pi\frac{\Theta}{\tau-\overline{\tau}}\omega_z & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{A}^3_{z4}(z,\overline{z}) \\ 0 & 0 & -\tilde{A}^3_{z4}(z,\overline{z}) & 0 \end{pmatrix}.
\end{aligned} \tag{6.1.9}$$

Note that the Abelian gauge fields $\tilde{A}^3_{z4}$ and $\tilde{A}^3_{\overline{z}4}$ which are associated with the preserved $U(1)$ symmetry do not have any constraints from the BPS conditions. Taking into account the bosonic configurations (6.1.9) and the supersymmetry transformations (4.1.49), we can write down the fermionic partners

$$\Psi_{\pm} = \begin{pmatrix} \Psi_{\pm A} \\ \Psi_{\pm B} \\ 0 \\ 0 \end{pmatrix}, \quad \overline{\Psi}^{\pm} = \begin{pmatrix} \overline{\Psi}^{\pm}_{A} \\ \overline{\Psi}^{\pm}_{B} \\ 0 \\ 0 \end{pmatrix} \tag{6.1.10}$$

where $\overline{\Psi}$ is the conjugate spinor defined by $\overline{\Psi} := \Psi^T \tilde{C}$ in terms of the $SO(8)$ charge conjugation matrix $\tilde{C}$. Ψ^a_+ and $\overline{\Psi}^{+a}$ are the $SO(2)_E$ spinors with the positive chiralities while Ψ^a_- and $\overline{\Psi}^{-a}$ carry the negative ones. Both of them transform as $\mathbf{8}_c$ of the $SO(8)_R$. The subscripts A, B are just the label of the gauge indices 1 and 2 [1].

Under retaining the above static BPS configurations (6.1.9) and (6.1.10), we now wish to consider the evolution of time and compactify the system on T^2. Substitution of the configurations (6.1.9) and (6.1.10) into the action (4.1.31) yields

$$\begin{aligned} S = \int_{\mathbb{R}} dt \int_{T^2} d^2z \Bigg[&\frac{1}{2} D_0 X^{Ia} D_0 X^I_a - \frac{i}{2} \overline{\Psi}^{\alpha a} D_0 \Psi_{\alpha a} \\ &- \frac{k}{2\pi} \tilde{A}^1_{02} \tilde{F}^3_{z\bar{z}4} - \frac{k}{4\pi} \left(\tilde{A}^1_{z2} \dot{\tilde{A}}^3_{\bar{z}4} - \tilde{A}^1_{\bar{z}2} \dot{\tilde{A}}^3_{z4} \right) \Bigg] \end{aligned} \tag{6.1.11}$$

where the Greek letters $\alpha = +, -$ denote the $SO(2)_E$ spinor indices. The terms in the first line of the action (6.1.11) come from the kinetic terms of the BLG action while those in the second correspond to the twisted topological Chern-Simons terms.

Firstly since the gauge fields $\tilde{A}^1_{z2}$ and $\tilde{A}^1_{\bar{z}2}$ are quantized and their time derivatives do not appear in the action, these fields are just auxiliary fields. Exploiting the equations of motion they can be excluded and we find the constraints $\dot{\tilde{A}}^3_{z4} = \dot{\tilde{A}}^3_{\bar{z}4} = 0$. Hence the corresponding field strength $\tilde{F}^3_{z\bar{z}4}$ has no time dependence. In order to dimensionally reduce the theory on the torus, we rescale the fields as

$$X^{I'} = R^2 X^I, \quad \Psi'_{\alpha a} = R^2 \Psi_{\alpha a}, \quad \overline{\Psi}^{\alpha a \prime} = R^2 \overline{\Psi}^{\alpha a} \tag{6.1.12}$$

where R is the circumference of the torus. Note that they get the canonical dimensions in the reduced theory; the bosonic variable $X^{I'}$ has mass dimension $-1/2$ and the fermionic variable Ψ' acquires mass dimension 0.

[1] In order to avoid the confusion coming from the various possible explicit numerical subscripts, we here relabel the gauge indices $a = 1, 2$ as $a = A, B$.

Let us carry out the integration on the torus with respect to the coordinates $z, \bar{z}$ by applying the Kaluza-Klein ansatz for the Abelian gauge field $\tilde{A}_0^{12}$ and omit the unimportant primes on the fields. Then we find the effective action

$$S = \int_{\mathbb{R}} dt \left[\frac{1}{2} D_0 X^{Ia} D_0 X_a^I - \frac{i}{2} \overline{\Psi}^{\alpha a} D_0 \Psi_{\alpha a} - k C_1(E) \tilde{A}_{02}^1 \right]. \tag{6.1.13}$$

Here

$$C_1(E) = \int_{T^2} c_1(E) := \frac{1}{2\pi} \int_{T^2} d^2 z \tilde{F}_{z\bar{z}4}^3 \tag{6.1.14}$$

is the Chern number resulting from the integration of the first Chern class $c_1(E)$ of the $U(1)$ principal bundle $E \to T^2$ over the torus, which is associated with the preserved $U(1)$ gauge field $\tilde{A}_{z4}^3$. Hence the last term in the action (6.1.13) is recognized as a Fayet-Iliopoulos (FI) term as in (2.5.3).

The action (6.1.13) is invariant under the one-dimensional conformal transformations

$$\delta t = f(t) = a + bt + ct^2, \qquad \delta \partial_0 = -\dot{f} \partial_0, \tag{6.1.15}$$

$$\delta X_a^I = \frac{1}{2} \dot{f} X_a^I, \qquad \delta \tilde{A}_{02}^1 = -\dot{f} \tilde{A}_{02}^1, \tag{6.1.16}$$

$$\delta \Psi_{\alpha a} = 0, \qquad \delta \overline{\Psi}^{\alpha a} = 0 \tag{6.1.17}$$

where $f(t)$ is a quadratic function of time with real infinitesimal parameters a, b and c.

The action (6.1.13) is also invariant under the $\mathcal{N} = 16$ supersymmetry transformations

$$\delta X_a^I = i \bar{\epsilon}^+ \tilde{\Gamma}^I \Psi_{-a} - i \bar{\epsilon}_- \tilde{\Gamma}^I \Psi_{+a}, \qquad \delta \tilde{A}_{02}^1 = 0, \tag{6.1.18}$$

$$\delta \Psi_{+a} = -D_0 X_a^I \tilde{\Gamma}^I \epsilon_-, \qquad \delta \Psi_{-a} = D_0 X_a^I \tilde{\Gamma}^I \epsilon_+. \tag{6.1.19}$$

Therefore the low-energy effective theory (6.1.13) is the $\mathcal{N} = 16$ superconformal gauged quantum mechanics with the FI term.

6.2 Reduction

As we have already argued, gauged conformal mechanics and the Calogero model reduce to conformal mechanical models with inverse-square type potentials after integrating out the auxiliary gauge fields. In fact our gauged mechanical action (6.1.13) is quadratic in the $U(1)$ gauge field $\tilde{A}_{02}^1$ and does not contain the time derivative of the Abelian gauge field. So $\tilde{A}^1_{02}$ can be identified with an auxiliary field and has no contribution to the Hamiltonian. Namely, the Hamiltonian is invariant under the action of the corresponding $U(1)$ gauge group on the phase space $\mathcal{M}$. This implies that the corresponding moment map $\mu : \mathcal{M} \to \mathfrak{u}(1)^*$ is the integral of motion [166] and we can reduce the original phase space $\mathcal{M}$ to a smaller phase space $\mathcal{M}_c = \mu^{-1}(c)$ with fewer degrees of freedom by fixing the inverse of the moment map at a point $c \in \mathfrak{u}(1)^*$ [2].

In order to obtain our reduced system, we shall eliminate the auxiliary field $\tilde{A}^1_{02}$ in two steps; first we choose a specific gauge and then impose the Gauss law constraint to ensure the consistency of the gauge fixing. Let us choose the temporal gauge $\tilde{A}_0 = 0$. Together with the solutions

$$\tilde{A}^1_{02} = \frac{kC_1(E) + \sum_I (r^I)^2 \dot{\theta}^I + i\overline{\Psi}^{\alpha}_{A}\Psi_{\alpha B}}{\sum_I (r^I)^2}, \tag{6.2.1}$$

$$\tilde{A}^1_{03} = \tilde{A}^1_{04} = \tilde{A}^2_{03} = \tilde{A}^2_{04} = 0 \tag{6.2.2}$$

to the equations of motion for $\tilde{A}_0$, we can read off the Gauss law constraint

$$\phi_0 := kC_1(E) + \sum_I (r^I)^2 \dot{\theta}^I + i\overline{\Psi}^{\alpha}_{A}\Psi_{\alpha B} = 0. \tag{6.2.3}$$

This equation is the moment map condition. To see the physical meaning of this constraint, we observe that $(r^I)^2\dot{\theta}^I$ represents the "angular momentum", the $SO(2)$-charge corresponding to the rotation in the $X^I_1 X^I_2$-plane while the fermionic bilinear term $i\overline{\Psi}^{\alpha}_{A}\Psi_{\alpha B}$ produces the charge of the $SO(2)$ rotational group of the two types of fermionic variables Ψ_A and Ψ_B. Accordingly the equation (6.2.3) says that the total $SO(2)$ charge which rotates the internal degrees of freedom for the two M2-branes is fixed by the Chern-Simons level k and the Chern number $C_1(E)$.

With the constraint function ϕ_0, one can write a new Lagrangian by adding $\lambda\phi_0$ where λ is the Lagrange multiplier. The resulting action is

$$\begin{aligned} S = \int_{\mathbb{R}} dt \Bigg[& \frac{1}{2}\sum_I (\dot{r}^I)^2 + \frac{1}{2}\sum_I (r^I\dot{\theta}^I)^2 - \frac{i}{2}\overline{\Psi}^{\alpha a}\dot{\Psi}_{\alpha a} \\ & + \lambda \left(kC_1(E) + \sum_I (r^I)^2\dot{\theta}^I + i\overline{\Psi}^{\alpha}_{A}\Psi_{\alpha B} \right) \Bigg]. \end{aligned} \tag{6.2.4}$$

The absence of the variables θ^I's in the action (6.2.4) immediately implies that they are cyclic coordinates and their canonical momenta $p_{\theta^I} = (r^I)^2\dot{\theta}^I$ are just the integrals of motion.

[2] The components of the moment map form a system being in involution since the gauge group is Abelian. So we do not need to divide by the non-trivial coadjoint isotropy subgroup to obtain the reduced phase space.

At this state we can eliminate cyclic coordinates from the Lagrangian by introducing the Routhian. As we have discussed in section 2.5, the Routhian is a hybrid between the Lagrangian and the Hamiltonian, defined by performing a Legendre transformation on the cyclic coordinates

$$R(r^I,\dot{r}^I,h^I,\Psi) := L(r^I,\dot{r}^I,\dot{\theta}^I,\Psi) - \sum_I \dot{\theta}^I p_{\theta^I}. \tag{6.2.5}$$

Due to the partial Legendre transformation, the variables r^I and Ψ still follow the Euler-Lagrange equations while the cyclic coordinates θ^I and their momenta $h^I := p_{\theta^I}$ obey the Hamilton equations. However, the latter set of equations results in trivial statements; the constant property of h^I (i.e. $\dot{h}^I = 0$) and the definition of h^I (i.e. $\dot{\theta}^I = \frac{h^I}{(r^I)^2}$). So classically the Routhian is not really $R(r^I,\dot{r}^I,h^I,\Psi)$ but rather $R(r^I,\dot{r}^I,\Psi)$ along with the integrals of motion h^I's. Hence we can rewrite (6.2.4) as

$$S = \int_{\mathbb{R}} dt \left[\frac{1}{2}\sum_I (\dot{r}^I)^2 - \frac{1}{2}\sum_I \frac{(h^I)^2}{(r^I)^2} - \frac{i}{2}\overline{\Psi}^{\alpha a}\dot{\Psi}_{\alpha a} + \lambda \left(kC_1(E) + \sum_I h^I + i\overline{\Psi}^{\alpha}_{A}\Psi_{\alpha B} \right) \right]. \tag{6.2.6}$$

Integrating out λ, we finally obtain the reduced effective action

$$S = \frac{1}{2} \int_{\mathbb{R}} dt \Bigg[\dot{q}^2 + \sum_{I\neq K} (\dot{r}^I)^2 - i\Psi^{\alpha a}\dot{\Psi}_{\alpha a} - \frac{\left[kC_1(E) + \sum_{I\neq K} h^I + i\Psi^{\alpha}_{A}\Psi_{\alpha B}\right]^2}{q^2} - \sum_{I\neq K}\frac{(h^I)^2}{(r^I)^2} \Bigg]. \tag{6.2.7}$$

Here we have taken the $SO(8)$ charge conjugation matrix $\tilde{C}$ as an identity matrix for simplicity [3]. We have defined $q := r^K$ where K denotes the specific direction in which h^K is automatically determined by other conserved quantities h^I's. Note that the terms appearing in the numerator of the potential are the integrals of motion, namely they commute with the Hamiltonian.

Let us study the classical properties of the theory (6.2.7). The action (6.2.7)

[3] For the symmetric charge conjugation matrix one can reduce it to an identity matrix by an appropriate unitary transformation.

leads to the classical equations of motion

$$\ddot{q} = \frac{[kC_1(E) + \sum_{I \neq K} h^I + i\Psi^{\alpha}_{A}\Psi_{\alpha B}]^2}{q^3}, \tag{6.2.8}$$

$$\ddot{r}^I = \frac{(h^I)^2}{(r^I)^3}, \tag{6.2.9}$$

$$\dot{\Psi}_{\alpha A} = -\frac{[kC_1(E) + \sum_{I \neq K} h^I + i\Psi^{\alpha}_{A}\Psi_{\alpha B}]}{q^2}\Psi_{\alpha B}, \tag{6.2.10}$$

$$\dot{\Psi}_{\alpha B} = \frac{[kC_1(E) + \sum_{I \neq K} h^I + i\Psi^{\alpha}_{A}\Psi_{\alpha B}]}{q^2}\Psi_{\alpha A}. \tag{6.2.11}$$

Making use of the equations of motion (6.2.10) and (6.2.11), one can check that the differentiation of the Gauss law constraint (6.2.3) with respect to time t vanishes. In other words, ϕ_0 is the constant of motion.

The canonical momenta are

$$p := \frac{\partial L}{\partial \dot{q}} = \dot{q}, \qquad\qquad p_I := \frac{\partial L}{\partial \dot{r}^I} = \dot{r}^I, \tag{6.2.12}$$

$$\pi^{\alpha a} := \frac{\vec{\partial} L}{\partial \dot{\Psi}_{\alpha a}} = \frac{i}{2}\Psi^{\alpha a}. \tag{6.2.13}$$

The fermionic momenta $\pi^{\alpha a}$ do not depend on the velocities but on the fermionic degrees of freedom themselves. Hence one can read second-class constraints

$$\phi_1^{\alpha a} := \pi^{\alpha a} - \frac{i}{2}\Psi^{\alpha a} = 0. \tag{6.2.14}$$

Under the constraints, we get the Dirac brackets

$$[q, p]_{DB} = 1, \qquad\qquad \left[r^I, p_J\right]_{DB} = \delta^I{}_J, \tag{6.2.15}$$

$$\left[\Psi_{\alpha a \dot{A}}, \pi^{\beta b \dot{B}}\right]_{DB} = \frac{1}{2}\delta_{\alpha\beta}\delta_{ab}\delta_{\dot{A}\dot{B}}, \qquad \left[\Psi_{\alpha a \dot{A}}, \Psi^{\beta b \dot{B}}\right]_{DB} = -i\delta_{\alpha\beta}\delta_{ab}\delta_{\dot{A}\dot{B}}. \tag{6.2.16}$$

The action (6.2.7) is invariant under the following one-dimensional conformal transformations

$$\delta t = f(t) = a + bt + ct^2, \qquad\qquad \delta\partial_0 = -\dot{f}\partial_0, \tag{6.2.17}$$

$$\delta q = \frac{1}{2}\dot{f}q, \qquad\qquad \delta r^I = \frac{1}{2}\dot{f}r^I, \tag{6.2.18}$$

$$\delta\Psi_{\alpha a} = 0. \tag{6.2.19}$$

Here the constant parameters a, b and c are infinitesimal parameters of translation, dilatation and conformal boost respectively. The corresponding Noether charges, the Hamiltonian H, the dilatation operator D and the conformal boost operator K are found to be

$$H = \frac{1}{2}\left[p^2 + \frac{\left(kC_1(E) + \sum_{I\neq K} h^I + i\Psi^{\alpha}_{A}\Psi_{\alpha B}\right)^2}{q^2} + \sum_{I\neq K}\left(p_I^2 + \frac{(h^I)^2}{(r^I)^2}\right)\right], \tag{6.2.20}$$

$$D = tH - \frac{1}{4}\left[(qp+pq) + \sum_{I\neq K}\left(r^I p_I + p_I r^I\right)\right], \tag{6.2.21}$$

$$K = t^2 H - \frac{1}{2}t\left[(qp+pq) + \sum_{I\neq K}\left(r^I p_I + p_I r^I\right)\right] + \frac{1}{2}\left[q^2 + \sum_{I\neq K}(r^I)^2\right]. \tag{6.2.22}$$

The action (6.2.7) is invariant under the following fermionic transformations

$$\delta q = \frac{i}{\sqrt{2}}\left(\epsilon^-\Psi_{-A} - \epsilon^+\Psi_{+A}\right) + \frac{i}{\sqrt{2}}\left(\epsilon^-\Psi_{-B} - \epsilon^+\Psi_{+B}\right), \tag{6.2.23}$$

$$\delta r^I = i\cos\theta^I\left(\epsilon^+\tilde{\Gamma}^I\Psi_{-A} - \epsilon^-\tilde{\Gamma}^I\Psi_{+A}\right) + i\sin\theta^I\left(\epsilon^+\tilde{\Gamma}^I\Psi_{-B} - \epsilon^-\tilde{\Gamma}^I\Psi_{+B}\right), \tag{6.2.24}$$

$$\delta\Psi_{+A\dot{A}} = -\frac{1}{\sqrt{2}}\left(\dot{q} - \frac{h^K}{q}\right)\epsilon_{+\dot{A}} - \frac{i}{\sqrt{2}}\frac{l}{q}\Psi_{+B\dot{A}} - \sum_{I\neq K}\left(\dot{r}^I\cos\theta^I - \sin\theta^I\frac{h^I}{r^I}\right)\tilde{\Gamma}^I\epsilon_{-\dot{A}}, \tag{6.2.25}$$

$$\delta\Psi_{-A\dot{A}} = \frac{1}{\sqrt{2}}\left(\dot{q} - \frac{h^K}{q}\right)\epsilon_{-\dot{A}} - \frac{i}{\sqrt{2}}\frac{l}{q}\Psi_{-B\dot{A}} + \sum_{I\neq K}\left(\dot{r}^I\cos\theta^I - \sin\theta^I\frac{h^I}{r^I}\right)\tilde{\Gamma}^I\epsilon_{+\dot{A}}, \tag{6.2.26}$$

$$\delta\Psi_{+B\dot{A}} = -\frac{1}{\sqrt{2}}\left(\dot{q} + \frac{h^K}{q}\right)\epsilon_{+\dot{A}} + \frac{i}{\sqrt{2}}\frac{l}{q}\Psi_{+A\dot{A}} - \sum_{I\neq K}\left(\dot{r}^I\sin\theta^I + \cos\theta^I\frac{h^I}{r^I}\right)\tilde{\Gamma}^I\epsilon_{-\dot{A}}, \tag{6.2.27}$$

$$\delta\Psi_{-B\dot{A}} = \frac{1}{\sqrt{2}}\left(\dot{q} + \frac{h^K}{q}\right)\epsilon_{-\dot{A}} + \frac{i}{\sqrt{2}}\frac{l}{q}\Psi_{-A\dot{A}} + \sum_{I\neq K}\left(\dot{r}^I\sin\theta^I + \cos\theta^I\frac{h^I}{r^I}\right)\tilde{\Gamma}^I\epsilon_{+\dot{A}} \tag{6.2.28}$$

where we have defined

$$\theta^I(t) := h^I\int^t \frac{dt'}{(r^I(t'))^2}, \tag{6.2.29}$$

$$l := \left(\Psi^{+A}\epsilon_+ - \Psi^{-A}\epsilon_-\right) - \left(\Psi^{+B}\epsilon_+ - \Psi^{-B}\epsilon_-\right). \tag{6.2.30}$$

We should note that the supersymmetry is generically non-local in the sense that the transformations contain the integrals of the function of the non-local variables r^I's with respect to time. The non-locality is the consequence of the Routh reduction. Hence the infinite number of the associated conserved charges may exist and things may become much more exotic [4]. However, as seen from the (6.2.20), the motion in the K-th direction endowed with the local supersymmetry and others with non-local ones are essentially decoupled because their Hamiltonians commute with each other. Thus we can treat them separately. This indicates that the theory possesses the local conserved supercurrents and the non-local supercurrents which are in involution.

6.3 $OSp(16|2)$ superconformal mechanics

We shall focus on the study of the motion in the K-th direction which is associated with the local charges and investigate the algebraic structure of the symmetry group in the quantum mechanics. From now on we will consider the case where the all independent conserved charges h^I's are zeros. This is realized when the internal degrees of freedom for two M2-branes are unbiased. Note that for the purpose of the exploration of the algebraic structure for the K-th motion, this specific charge assignment does not affect the following discussion since non-vanishing h^I's can only give rise to a constant shift of the coupling constant in the potential. From (6.2.7) one can read the effective action for the dynamics in the K-th direction

$$S = \frac{1}{2} \int_{\mathbb{R}} dt \left[\dot{q}^2 - i\Psi^{\alpha a}\dot{\Psi}_{\alpha a} - \frac{\left(kC_1(E) + i\Psi_A^{\alpha}\Psi_{\alpha B}\right)^2}{q^2} \right]. \tag{6.3.1}$$

We see that our reduced action (6.3.1) contains the inverse-square potential which is similar to the known $\mathcal{N} > 4$ superconformal mechanical potentials discussed in (3.6.1) (also see [210, 255, 224, 137]).

We shall study the existing symmetry in the effective action (6.3.1). The action (6.3.1) may be rewritten as $SU(1,1|16)$ superconformal quantum mechanics in the form of (3.6.1). However, we should note that the same form of the Lagrangian does not necessarily lead to the same symmetry in the theory if we have additional constraints or symmetries. In fact in our setup the bilinear terms for fermions are treated as conserved quantities due to the Gauss constraint (6.2.3). This implies that the gauge indices $a, b, \cdots = A, B$ should be distinguished from other indices $\alpha, \beta, \cdots$ and $\dot{A}, \dot{B}, \cdots$ and prevents us from forming 32 supercharges. Put it another way, our theory describes the radial motion of the wrapped membranes and

[4] The action (6.2.7) is invariant under the fermionic transformations (6.2.23)-(6.2.28), however, the Gauss constraint (6.2.14) may not be invariant under those transformations. Although in that case the original system may be modified, we here just want to study the reduced system without the auxiliary gauge field.

thus we have at most 16 supercharges on the branes by the projection. So we will only focus on the remaining $\mathcal{N} = 16$ supersymmetry due to the constraint. The simplest way to read the consistent supersymmetry for the wrapped branes is just look at the supersymmetry transformations for the original BLG-model. From (4.1.49)-(4.1.51) we see that the action (6.3.1) is invariant under the following $\mathcal{N} = 16$ supersymmetry transformation laws

$$\delta q = \frac{i}{\sqrt{2}} \left(\epsilon^- \Psi_{-A} - \epsilon^+ \Psi_{+A}\right) + \frac{i}{\sqrt{2}} \left(\epsilon^- \Psi_{-B} - \epsilon^+ \Psi_{+B}\right), \tag{6.3.2}$$

$$\delta \Psi_{+A\dot{A}} = -\frac{1}{\sqrt{2}} \left(\dot{q} + \frac{g}{q}\right) \epsilon_{+\dot{A}} - \frac{i}{\sqrt{2}} \frac{l}{q} \Psi_{+B\dot{A}}, \tag{6.3.3}$$

$$\delta \Psi_{-A\dot{A}} = \frac{1}{\sqrt{2}} \left(\dot{q} + \frac{g}{q}\right) \epsilon_{-\dot{A}} - \frac{i}{\sqrt{2}} \frac{l}{q} \Psi_{-B\dot{A}}, \tag{6.3.4}$$

$$\delta \Psi_{+B\dot{A}} = -\frac{1}{\sqrt{2}} \left(\dot{q} - \frac{g}{q}\right) \epsilon_{+\dot{A}} + \frac{i}{\sqrt{2}} \frac{l}{q} \Psi_{+A\dot{A}}, \tag{6.3.5}$$

$$\delta \Psi_{-B\dot{A}} = \frac{1}{\sqrt{2}} \left(\dot{q} - \frac{g}{q}\right) \epsilon_{-\dot{A}} + \frac{i}{\sqrt{2}} \frac{l}{g} \Psi_{-A\dot{A}} \tag{6.3.6}$$

where we have defined

$$g := kC_1(E) + i\Psi^{\alpha}_{A} \Psi_{\alpha B}. \tag{6.3.7}$$

Unlike the transformations (6.2.23)-(6.2.28), the supersymmetry transformations (6.3.2)-(6.3.6) are local and we therefore can apply the conventional Noether's procedure. By means of the Noether's method, the corresponding supercharges are calculated to be

$$Q_{+\dot{A}} = \frac{1}{\sqrt{2}} \left(p + \frac{g}{q}\right) \Psi_{+A\dot{A}} + \frac{1}{\sqrt{2}} \left(p - \frac{g}{q}\right) \Psi_{+B\dot{A}}, \tag{6.3.8}$$

$$Q_{-\dot{A}} = -\frac{1}{\sqrt{2}} \left(p + \frac{g}{q}\right) \Psi_{-A\dot{A}} - \frac{1}{\sqrt{2}} \left(p - \frac{g}{q}\right) \Psi_{-B\dot{A}}. \tag{6.3.9}$$

Since the action (6.3.1) is invariant under the conformal transformations $\delta t = f(t)$, $\delta q = \frac{1}{2} \dot{f} q$ and $\delta \Psi_{\alpha a} = 0$, three generators, the Hamiltonian H, the dilatation generator D and the conformal boost generator K are explicitly expressed as

$$H = \frac{1}{2} p^2 + \frac{\left[kC_1(E) + i\Psi^{\alpha}_{A} \Psi_{\alpha B}\right]^2}{2q^2}, \tag{6.3.10}$$

$$D = -\frac{1}{4} \{q, p\}, \tag{6.3.11}$$

$$K = \frac{1}{2} q^2 \tag{6.3.12}$$

where $\{,\}$ represents an anti-commutator.

In order to quantize the theory, we impose the (anti)commutation relations for the canonical variables obtained from the Dirac brackets (6.2.15) and (6.2.16)

$$[q, p] = i, \qquad \left\{\Psi_{\alpha a\dot{A}}, \Psi_{\beta b\dot{B}}\right\} = \delta_{\alpha\beta}\delta_{ab}\delta_{\dot{A}\dot{B}}. \tag{6.3.13}$$

The combination of the conformal symmetry and the supersymmetry leads to the superconformal symmetry. Let us define the superconformal boost generators

$$S_{+\dot{A}} = \frac{1}{\sqrt{2}}q\left(\Psi_{+A\dot{A}} + \Psi_{+B\dot{A}}\right), \tag{6.3.14}$$

$$S_{-\dot{A}} = -\frac{1}{\sqrt{2}}q\left(\Psi_{-A\dot{A}} + \Psi_{-B\dot{A}}\right). \tag{6.3.15}$$

Because of the extended supersymmetry the theory has the internal R-symmetry which rotates the fermionic charges. Let us define the R-symmetry generators by

$$(J_{\alpha\beta})_{\dot{A}\dot{B}} = i\Psi_{\alpha a\dot{A}}\Psi^{a}_{\beta\dot{B}}. \tag{6.3.16}$$

Notice that the R-symmetry generators satisfy the relations

$$(J_{++})_{\dot{A}\dot{B}} = -(J_{++})_{\dot{B}\dot{A}}, \tag{6.3.17}$$

$$(J_{--})_{\dot{A}\dot{B}} = -(J_{--})_{\dot{B}\dot{A}}, \tag{6.3.18}$$

$$(J_{+-})_{\dot{A}\dot{B}} = -(J_{-+})_{\dot{B}\dot{A}} \tag{6.3.19}$$

and therefore the matrices J_{++}, J_{--} and J_{-+} contain 28, 28 and 64 independent entries respectively while J_{-+} yields no independent ones because of the relations (6.3.19). Therefore the R-symmetry matrix totally carries $28 + 28 + 64 = 120$ elements.

Using the canonical (anti)commutation relations (6.3.13) and considering the Weyl ordering for the fermionic bilinear terms, one can find the complete set of (anti)commutators among the generators

$$[H, D] = iH, \quad [K, D] = -iK, \quad [H, K] = 2iD, \tag{6.3.20}$$

$$[(J_{\alpha\beta})_{\dot{A}\dot{B}}, H] = 0, \quad [(J_{\alpha\beta})_{\dot{A}\dot{B}}, D] = 0, \quad [(J_{\alpha\beta})_{\dot{A}\dot{B}}, K] = 0, \tag{6.3.21}$$

$$\begin{aligned}[(J_{\alpha\beta})_{\dot{A}\dot{B}}, (J_{\gamma\delta})_{\dot{C}\dot{D}}] = {} & i(J_{\gamma\beta})_{\dot{C}\dot{B}}\delta_{\alpha\delta}\delta_{\dot{A}\dot{D}} - i(J_{\alpha\delta})_{\dot{A}\dot{D}}\delta_{\beta\gamma}\delta_{\dot{B}\dot{C}}, \\ & + i(J_{\delta\beta})_{\dot{D}\dot{B}}\delta_{\alpha\gamma}\delta_{\dot{A}\dot{C}} - i(J_{\alpha\gamma})_{\dot{A}\dot{C}}\delta_{\beta\delta}\delta_{\dot{B}\dot{D}},\end{aligned} \tag{6.3.22}$$

$$[H, Q_{\alpha\dot{A}}] = 0, \quad [D, Q_{\alpha\dot{A}}] = -\tfrac{i}{2}Q_{\alpha\dot{A}}, \quad [K, Q_{\alpha\dot{A}}] = iS_{\alpha\dot{A}}, \tag{6.3.23}$$

$$[H, S_{\alpha\dot{A}}] = -iQ_{\alpha\dot{A}}, \quad [D, S_{\alpha\dot{A}}] = \frac{i}{2}S_{\alpha\dot{A}}, \quad [K, S_{\alpha\dot{A}}] = 0, \tag{6.3.24}$$

$$\begin{aligned}
\{Q_{\alpha\dot{A}}, Q_{\beta\dot{B}}\} &= 2H\delta_{\alpha\beta}\delta_{\dot{A}\dot{B}}, \\
\{S_{\alpha\dot{A}}, S_{\beta\dot{B}}\} &= 2K\delta_{\alpha\beta}\delta_{\dot{A}\dot{B}}, \\
\{Q_{\alpha\dot{A}}, S_{\beta\dot{B}}\} &= -2D\delta_{\alpha\beta}\delta_{\dot{A}\dot{B}} + (J_{\alpha\beta})_{\dot{A}\dot{B}},
\end{aligned} \tag{6.3.25}$$

$$\begin{aligned}
[(J_{\alpha\beta})_{\dot{A}\dot{B}}, Q_{\gamma\dot{C}}] &= i\left(Q_{\alpha\dot{A}}\delta_{\beta\gamma}\delta_{\dot{B}\dot{C}} - Q_{\beta\dot{B}}\delta_{\alpha\gamma}\delta_{\dot{A}\dot{C}}\right), \\
[(J_{\alpha\beta})_{\dot{A}\dot{B}}, S_{\gamma\dot{C}}] &= i\left(S_{\alpha\dot{A}}\delta_{\beta\gamma}\delta_{\dot{B}\dot{C}} - S_{\beta\dot{B}}\delta_{\alpha\gamma}\delta_{\dot{A}\dot{C}}\right).
\end{aligned} \tag{6.3.26}$$

The Hamiltonian H, the dilatation generator D and the conformal boost generator K satisfy the one-dimensional conformal algebra (6.3.20).

As the superpartners of the conformal generators there are sixteen supercharges $Q_{\alpha\dot{A}}$ and as many superconformal generators $S_{\alpha\dot{A}}$. As seen from (6.3.21) and (6.3.26), the R-symmetry generators $(J_{\alpha\beta})_{\dot{A}\dot{B}}$ commute with the bosonic generators H, D and K while they yield the rotations of the fermionic generators $Q_{\alpha\dot{A}}$ and $S_{\alpha\dot{A}}$. The commutation relation (6.3.22) implies that $(J_{\alpha\beta})_{\dot{A}\dot{B}}$ obey the $\mathfrak{so}(16)$ algebra. Therefore we can conclude that the theory (6.3.1) is the $OSp(16|2)$ invariant $\mathcal{N} = 16$ superconformal mechanics. In fact this fits in the list of the possible simple supergroup for superconformal quantum mechanics, which we have already given in Table 3.2.

It is true that the quantum mechanics (6.3.1) possesses the $\mathcal{N} = 16$ superconformal symmetry, however, it is not clear that the theory (6.3.1) actually captures the dynamics of the wrapped membranes around a torus since it is not totally same as the superconformal gauged quantum mechanics (6.1.13) due to the reduction process.

Note that the original $SO(8)$ R-symmetry is now enhanced to $SO(16)$ in our quantum mechanics. It is not so strange as a similar phenomenon has been already observed in $d = 11$ supergravity. In $d = 11$ supergravity the original tangent space symmetry $SO(1, 10)$ can break down into the subgroup $SO(1, 2) \times SO(8)$ through a partial choice of gauge for the elfbein. However, it has been pointed out in [285, 286, 287] that one can find the enhanced $SO(1, 2) \times SO(16)$ tangent space symmetry by introducing new gauge degrees of freedom. It would be an intriguing open question to investigate whether such enhanced R-symmetry of our quantum mechanics reflects that of $d = 11$ supergravity.

Chapter 7

$\mathcal{N} = 12$ Superconformal Mechanics

Similar to the previous chapter, we will consider the ABJM-model wrapped on a torus and derive the IR quantum mechanics by shrinking the torus in this chapter. We will derive the IR $\mathcal{N} = 12$ superconformal gauged quantum mechanicsãĂĂ and extract the corresponding $SU(1,1|6)$ superconformal quantum mechanics from the reduced systems.

7.1 $\mathcal{N} = 12$ gauged quantum mechanics

We now want to consider an arbitrary number of M2-branes wrapped around a torus, which may be described by the $U(N)_k \times \hat{U}(N)_{-k}$ ABJM-model on $\mathbb{R} \times T^2$. The theory may describe the dynamics of N coincident M2-branes with the world-volume $M_3 = \mathbb{R} \times T^2$ moving in a transverse space with an $SU(4)$ holonomy. The crucial point is now that the volume of the torus yields a typical energy scale in the theory and we can take a further limit where the energy is lower than the inverse of the size of the torus. Such low-effective theory describes the fluctuations around static BPS configurations obeying the BPS equations. From the supersymmetry transformations (5.1.7), (5.1.8) for fermions we find the following set of the BPS equations:

$$D_z Y^A = 0, \quad D_{\bar{z}} Y^A = 0, \tag{7.1.1}$$

$$Y^C Y^\dagger_C Y^B - Y^B Y^\dagger_C Y^C = 0, \tag{7.1.2}$$

$$Y^C Y^\dagger_A Y^D = 0. \tag{7.1.3}$$

To satisfy the algebraic equations (7.1.2) and (7.1.3), the bosonic Higgs fields Y^A and $Y^\dagger_A$ should take the diagonal form

$$Y^A = \mathrm{diag}(y^A_1, \cdots, y^A_N), \qquad Y^\dagger_A = \mathrm{diag}(\bar{y}_{A1}, \cdots, \bar{y}_{AN}) \tag{7.1.4}$$

where y^A_a is a complex scalar field. For the above diagonal configurations, all the off-diagonal elements are massive and the gauge group $U(N) \times \hat{U}(N)$ is spontaneously broken to $U(1)^N$ [26]. Let us define

$$\mathcal{A}^+_{\mu a} := A_{\mu aa} + \hat{A}_{\mu aa}, \qquad \mathcal{A}^-_{\mu a} := A_{\mu aa} - \hat{A}_{\mu aa} \tag{7.1.5}$$

where the indices $a = 1, \cdots, N$ characterize the gauge degrees of freedom, i.e. the internal degrees of freedom of the multiple M2-branes. Note that all the couplings involve the gauge fields $\mathcal{A}^-_{\mu a}$ while the other gauge fields $\mathcal{A}^+_{\mu a}$ are associated with the preserved $U(1)$ gauge group. In terms of the expressions (7.1.4) and (7.1.5), we can rewrite the equations (7.1.1) as

$$\partial_z y^A_a + i\mathcal{A}^-_{za} y^A_a = 0, \qquad \partial_z \overline{y}_{Aa} - i\mathcal{A}^-_{za} \overline{y}_{Aa} = 0, \tag{7.1.6}$$
$$\partial_{\overline{z}} y^A_a + i\mathcal{A}^-_{\overline{z}a} y^A_a = 0, \qquad \partial_{\overline{z}} \overline{y}_{Aa} - i\mathcal{A}^-_{\overline{z}a} \overline{y}_{Aa} = 0, \tag{7.1.7}$$
$$A_{zab} = \hat{A}_{zab} = A_{\overline{z}ab} = \hat{A}_{\overline{z}ab} = 0 \qquad \text{for } \ a \neq b. \tag{7.1.8}$$

The first and second lines correspond to the equations for diagonal elements and last one is for the off-diagonal elements. The general solutions to the equations (7.1.6) and (7.1.7) are given by

$$y^A_a = r^A_a e^{i\left(\varphi_a(z,\overline{z}) + \theta^A_a\right)}, \tag{7.1.9}$$
$$\mathcal{A}^-_{za} = -\partial_z \varphi_a(z, \overline{z}) \tag{7.1.10}$$

where $r^A_a, \theta^A_a \in \mathbb{R}$ have no dependence on z and $\overline{z}$ while $\varphi_a(z, \overline{z}) \in \mathbb{R}$ is a function of z and $\overline{z}$. The expression (7.1.10) ensures the flatness of the $U(1)$ gauge field $\mathcal{A}^-_z$. Hence φ_a, $\mathcal{A}^-_{za}$ and $\mathcal{A}^-_{\overline{z}a}$ take the form [284]

$$\varphi_a(z, \overline{z}) = -2\pi \frac{\Theta_a}{\tau - \overline{\tau}} z + 2\pi \frac{\overline{\Theta}_a}{\tau - \overline{\tau}} \overline{z}, \tag{7.1.11}$$

$$\mathcal{A}^-_{za} = 2\pi \frac{\Theta_a}{\tau - \overline{\tau}} \omega, \qquad \mathcal{A}^-_{\overline{z}a} = -2\pi \frac{\overline{\Theta}_a}{\tau - \overline{\tau}} \overline{\omega}. \tag{7.1.12}$$

Here τ is the moduli of the torus defined in (6.1.1) and $\Theta_a := \zeta_a + \overline{\tau}\xi_a$, $a = 1, \cdots, N$ are the coordinates of the product space of the N Jacobi varieties characterizing the N $U(1)$ flat bundles. For the bosonic Higgs fields to describe the positions of the membranes, we should impose the single-valuedness of y^A_a as

$$\begin{aligned} y^A_a(z+1, \overline{z}+1) &= y^A_a(z, \overline{z}), \\ y^A_a(z+\tau, \overline{z}+\overline{\tau}) &= y^A_a(z, \overline{z}). \end{aligned} \tag{7.1.13}$$

These conditions require that ξ_a and ζ_a are integers, which result in the quantization of the variables φ_a, $\mathcal{A}^-_{za}$ and $\mathcal{A}^-_{\overline{z}a}$. Then the resulting static BPS configurations are

$$
\begin{aligned}
Y^A &= \mathrm{diag}(y_1^A, \cdots, y_N^A) = \mathrm{diag}\left(r_1^A e^{i(\varphi_1(z,\bar{z})+\theta_1^A)}, \cdots, r_N^A e^{i(\varphi_N(z,\bar{z})+\theta_N^A)}\right), \\
Y_A^\dagger &= \mathrm{diag}(\bar{y}_{A1}, \cdots, \bar{y}_{AN}) = \mathrm{diag}\left(r_1^A e^{-i(\varphi_1(z,\bar{z})+\theta_1^A)}, \cdots, r_N^A e^{-i(\varphi_N(z,\bar{z})+\theta_N^A)}\right), \\
A_z &= \mathrm{diag}\left(A_{z11}, \cdots, A_{zNN}\right), \\
\hat{A}_z &= A_z + \partial_z \varphi = \mathrm{diag}\left(A_{z11} + \partial_z \varphi_1, \cdots, A_{zNN} + \partial_z \varphi_N\right).
\end{aligned}
\tag{7.1.14}
$$

By the supersymmetry the above bosonic configurations are paired with the fermionic fields

$$
\psi_{\pm A} = \mathrm{diag}\left(\psi_{\pm A1}, \cdots, \psi_{\pm AN}\right), \qquad \psi_{\pm}^{\dagger A} = \mathrm{diag}\left(\psi_{\pm}^{\dagger A1}, \cdots, \psi_{\pm}^{\dagger AN}\right) \tag{7.1.15}
$$

where the subscripts $\pm$ label the $SO(2)_E$ spinor representation.

Inserting the set of BPS configurations (7.1.14) and (7.1.15) into the ABJM action (5.1.1) one finds

$$
\begin{aligned}
S = \int_{\mathbb{R}} dt \int_{T^2} d^2 z \sum_A \sum_{a=1}^{N} \Bigg[& D_0 \bar{y}_A^a D_0 y_a^A - i\psi_+^{\dagger Aa} D_0 \psi_{+Aa} - i\psi_-^{\dagger Aa} D_0 \psi_{-Aa} \\
& + \frac{k}{4\pi}\left(\mathcal{A}_{0a}^- \mathcal{F}_{z\bar{z}a}^+ + \frac{1}{2}\mathcal{A}_{\bar{z}a}^- \dot{\mathcal{A}}_{za}^+ - \frac{1}{2}\mathcal{A}_{za}^- \dot{\mathcal{A}}_{\bar{z}a}^+\right)\Bigg].
\end{aligned}
\tag{7.1.16}
$$

Recall that $\mathcal{A}_z^-$ and $\mathcal{A}_{\bar{z}}^-$ are quantized and their time derivative terms do not show up in the action. Thus we can treat them as auxiliary fields and integrate out them. Consequently we get constraints $\dot{\mathcal{A}}_{za}^+ = \dot{\mathcal{A}}_{\bar{z}a}^+ = 0$, which imply that the gauge fields $\mathcal{A}_{za}^+$ and $\mathcal{A}_{\bar{z}a}^+$ on the Riemann surface have no time dependence.

Taking these constraints into account and proceeding the integration over the torus, we obtain the low-energy effective action

$$
S = \int_{\mathbb{R}} dt \left[D_0 \bar{y}_A^a D_0 y_a^A - i\psi^{\dagger\alpha Aa} D_0 \psi_{\alpha Aa} + k C_1(E_a) \mathcal{A}_{0a}^- \right]. \tag{7.1.17}
$$

Here the repeated indices are summed over and $\alpha, \beta, \cdots = +, -$ denote the $SO(2)_E$ spinor indices. The covariant derivatives are defined by

$$
\begin{aligned}
D_0 y_a^A &= \dot{y}_a^A + i\mathcal{A}_{0a}^- y_a^A, & D_0 \bar{y}_{Aa} &= \dot{\bar{y}}_{Aa} - i\mathcal{A}_{0a}^- \bar{y}_{Aa}, \\
D_0 \psi_{\alpha Aa} &= \dot{\psi}_{\alpha Aa} + i\mathcal{A}_{0a}^- \psi_{\alpha Aa}, & D_0 \psi_{\alpha a}^{\dagger A} &= \dot{\psi}_{\alpha a}^{\dagger A} - i\mathcal{A}_{0a}^- \psi_{\alpha a}^{\dagger A}.
\end{aligned}
\tag{7.1.18}
$$

and

$$
C_1(E_a) := \frac{1}{2\pi}\int_{T^2} F_{z\bar{z}aa} = \frac{1}{4\pi}\int_{T^2} \mathcal{F}_{z\bar{z}a}^+ \tag{7.1.19}
$$

is the Chern number of the a-th $U(1)$ principal bundle $E_a \to T^2$ over the torus associated with the preserved $U(1)$ gauge fields A_{zaa}.

The action (7.1.17) is invariant under the one-dimensional conformal transformations

$$\delta t = f(t) = a + bt + ct^2, \qquad \delta\partial_0 = -\dot{f}\partial_0, \tag{7.1.20}$$

$$\delta y_a^A = \frac{1}{2}\dot{f} y_a^A, \qquad \delta\overline{y}_{Aa} = \frac{1}{2}\dot{f}\overline{y}_{Aa}, \tag{7.1.21}$$

$$\delta\psi_{\alpha Aa} = 0, \qquad \delta\psi_{\alpha a}^{\dagger A} = 0, \tag{7.1.22}$$

$$\delta\mathcal{A}_{0a}^- = -\dot{f}\mathcal{A}_{0a}^- \tag{7.1.23}$$

and $\mathcal{N} = 12$ supersymmetry transformations

$$\delta y_a^A = i\omega^{\alpha AB}\psi_{\alpha Ba}, \qquad \delta\overline{y}_{Aa} = i\psi_a^{\dagger\alpha B}\omega_{\alpha AB}, \tag{7.1.24}$$

$$\delta\psi_{\alpha Aa} = \omega_{\alpha AB}D_0 y_a^B, \qquad \delta\psi_{\alpha a}^{\dagger A} = -D_0\overline{y}_{Ba}\omega_\alpha^{AB}, \tag{7.1.25}$$

$$\delta\mathcal{A}_{0a}^- = 0 \tag{7.1.26}$$

where the supersymmetry parameters $\omega_{+AB} := \epsilon_{+i}(\Gamma^i)_{AB}$ and $\omega_{-AB} := \epsilon_{-i}(\Gamma^i)_{AB}$ transform as $\mathbf{6}_+$ and $\mathbf{6}_-$ under $SU(4) \times SO(2)_E$ respectively. Therefore the low-energy effective theory is described by the $\mathcal{N} = 12$ superconformal gauged quantum mechanics (7.1.17).

7.2 Reduction

The low-energy effective action (7.1.17) is quadratic in $\mathcal{A}_{0a}^-$ and contains no time derivatives of $\mathcal{A}_{0a}^-$. So they are auxiliary fields and we want to integrate them out. Let us fix the gauge as $\mathcal{A}_{0a}^- = 0$. Then the algebraic equations of motion of $\mathcal{A}_{0a}^-$ yield the Gauss law constraints, the moment map conditions

$$\phi_{0a} := kC_1(E_a) + 2\sum_A (r_a^A)^2\dot{\theta}_a^A + \sum_A \psi^{\dagger\alpha Aa}\psi_{\alpha Aa} = 0 \tag{7.2.1}$$

for $a = 1, \cdots, N$. Note that although the set of equations (7.2.1) has the same form as that of (6.2.3), the physical meaning of these constraints are different because the angular variable θ_a^A's are defined not in the abstract space of the internal degrees of freedom as in (6.2.3), but in the actual configuration space of the a-th M2-brane in the A-th complex plane.

Defining the conserved charges $h_a^A := 2(r_a^A)^2\dot{\theta}_a^A$, using the above constraints (7.2.1) and following the reduction procedure as in the derivation of (6.2.7), we can integrate out the auxiliary gauge fields $\mathcal{A}_{0a}^{-}$ and find the reduced effective action with the inverse-square type interaction

$$
\begin{aligned}
S = \int_{\mathbb{R}} dt \sum_{a=1}^{N} \Bigg[& \dot{x}_a^2 - \frac{i}{2} \sum_{A \neq B} \left(\psi^{\dagger \alpha A a} \dot{\psi}_{\alpha A a} - \dot{\psi}^{\dagger A a} \psi_{\alpha A a} \right) \\
& + \sum_{A \neq B} (\dot{r}_a^A)^2 - \frac{i}{2} \left(\lambda^{\dagger \alpha a} \dot{\lambda}_{\alpha a} - \dot{\lambda}^{\dagger \alpha a} \lambda_{\alpha a} \right) \\
& - \frac{\left[k C_1(E_a) + \sum_{A \neq B} h_a^A + \sum_{A \neq B} \psi^{\dagger \alpha A a} \psi_{\alpha A a} + \lambda^{\dagger \alpha a} \lambda_{\alpha a} \right]^2}{4 x_a^2} - \sum_{A \neq B} \frac{(h_a^A)^2}{4 (r_a^A)^2} \Bigg].
\end{aligned}
\tag{7.2.2}
$$

Here $x_a := r_a^B$ describes the motion of the a-th M2-brane in the B-th complex plane in which the corresponding "angular momentum" h_a^B is determined by the assignment of the other preserved charges. We have also introduced the fermionic variable $\lambda_{\alpha a} := \psi_{\alpha B a}$ with $A = B$, which turns out to be the superpartner of r_a^C, $C = 1, 2, 3$, as we will see the supersymmetry transformations (7.2.27) and (7.2.28).

The action (7.2.2) leads to the following equations of motion

$$
\ddot{x}_a = \frac{\left[k C_1(E_a) + \sum_{A \neq B} h_a^A + \sum_{A \neq B} \psi^{\dagger \alpha A a} \psi_{\alpha A a} + \lambda^{\dagger \alpha a} \lambda_{\alpha a} \right]^2}{4 x_a^3}, \tag{7.2.3}
$$

$$
\ddot{r}_a^A = \frac{(h_a^A)^2}{4 (r_a^A)^3}, \tag{7.2.4}
$$

$$
\dot{\psi}_{\alpha A a} = i \frac{k C_1(E_a) + \sum_{A \neq B} h_a^A + \sum_{A \neq B} \psi^{\dagger \alpha A a} \psi_{\alpha A a} + \lambda^{\dagger \alpha a} \lambda_{\alpha a}}{2 x_a} \psi_{\alpha A a}, \tag{7.2.5}
$$

$$
\dot{\psi}^{\dagger \alpha A a} = -i \frac{k C_1(E_a) + \sum_{A \neq B} h_a^A + \sum_{A \neq B} \psi^{\dagger \alpha A a} \psi_{\alpha A a} + \lambda^{\dagger \alpha a} \lambda_{\alpha a}}{2 x_a} \psi^{\dagger \alpha A a}, \tag{7.2.6}
$$

$$
\dot{\lambda}_{\alpha a} = i \frac{k C_1(E_a) + \sum_{A \neq B} h_a^A + \sum_{A \neq B} \psi^{\dagger \alpha A a} \psi_{\alpha A a} + \lambda^{\dagger \alpha a} \lambda_{\alpha a}}{2 x_a} \lambda_{\alpha a}, \tag{7.2.7}
$$

$$
\dot{\lambda}^{\dagger \alpha a} = -i \frac{k C_1(E_a) + \sum_{A \neq B} h_a^A + \sum_{A \neq B} \psi^{\dagger \alpha A a} \psi_{\alpha A a} + \lambda^{\dagger \alpha a} \lambda_{\alpha a}}{2 x_a} \lambda^{\dagger \alpha a}. \tag{7.2.8}
$$

Using the fermionic equations of motion (7.2.5)-(7.2.8), we can check that the Gauss law constraint (7.2.1) has no time dependence, i.e. $\dot{\phi}_{0a} = 0$.

The canonical momenta are given by

$$
p^a := \frac{\partial L}{\partial \dot{x}_a} = 2 \dot{x}^a, \qquad P_A^a := \frac{\partial L}{\partial \dot{r}_a^A} = 2 \dot{r}_A^a, \tag{7.2.9}
$$

$$
\pi^{\alpha A a} := \frac{\vec{\partial} L}{\partial \dot{\psi}_{\alpha A a}} = \frac{i}{2} \psi^{\dagger \alpha A a}, \qquad \tilde{\pi}_{\alpha A a} := \frac{\vec{\partial} L}{\partial \dot{\psi}^{\dagger \alpha A a}} = \frac{i}{2} \psi_{\alpha A a}, \tag{7.2.10}
$$

$$
\Pi^{\alpha a} := \frac{\vec{\partial} L}{\partial \dot{\lambda}_{\alpha a}} = \frac{i}{2} \lambda^{\dagger \alpha a}, \qquad \tilde{\Pi}_{\alpha a} := \frac{\vec{\partial} L}{\partial \dot{\lambda}^{\dagger \alpha a}} = \frac{i}{2} \lambda_{\alpha a}. \tag{7.2.11}
$$

The fermionic canonical momenta provide the second class constraints

$$\phi_1^{\alpha Aa} := \pi^{\alpha Aa} - \frac{i}{2}\psi^{\dagger Aa} = 0, \qquad \phi_{2\alpha Aa} := \tilde{\pi}_{\alpha Aa} - \frac{i}{2}\psi_{\alpha Aa} = 0, \tag{7.2.12}$$

$$\phi_3^{\alpha a} := \Pi^{\alpha a} - \frac{i}{2}\lambda^{\dagger\alpha a} = 0, \qquad \phi_{4\alpha a} := \tilde{\Pi}_{\alpha a} - \frac{i}{2}\lambda_{\alpha a} = 0. \tag{7.2.13}$$

Taking account into the constraints (7.2.12) and (7.2.13), we find the Dirac brackets

$$[x_a, p^b]_{DB} = \delta_{ab}, \qquad [r_a^A, P_B^b]_{DB} = \delta_{AB}\delta_{ab}, \tag{7.2.14}$$

$$\left[\psi_{\alpha Aa}, \psi^{\dagger\beta Bb}\right]_{DB} = i\delta_{\alpha\beta}\delta_{AB}\delta_{ab}, \qquad \left[\lambda_{\alpha a}, \lambda^{\dagger\beta b}\right]_{DB} = i\delta_{\alpha\beta}\delta_{ab}. \tag{7.2.15}$$

The action (7.2.2) possesses the one-dimensional conformal invariance

$$\delta t = f(t) = a + bt + ct^2, \qquad \delta\partial_0 = -\dot{f}\partial_0, \tag{7.2.16}$$

$$\delta x_a = \frac{1}{2}\dot{f}x_a, \qquad \delta r_a^A = \frac{1}{2}\dot{f}r_a^A, \tag{7.2.17}$$

$$\delta\psi_{\alpha Aa} = 0, \qquad \delta\psi_a^{\dagger\alpha A} = 0, \tag{7.2.18}$$

$$\delta\lambda_{\alpha a} = 0, \qquad \delta\lambda_a^{\dagger\alpha} = 0. \tag{7.2.19}$$

Using the Noether's procedure we find the $SL(2,\mathbb{R})$ generators

$$H = \sum_{a=1}^{N}\left[\frac{p_a^2}{4} + \frac{\left(kC_1(E_a) + \sum_{A\neq B} h_a^A + \sum_A \psi^{\dagger\alpha Aa}\psi_{\alpha Aa} + \lambda^{\dagger\alpha a}\lambda_{\alpha a}\right)^2}{4x_a^2} + \sum_{A\neq B}\left(\frac{(P_a^A)^2}{4} + \frac{(h_a^A)^2}{4(r_a^A)^2}\right)\right], \tag{7.2.20}$$

$$D = -\frac{1}{4}\sum_{a=1}^{N}\left[\{x_a, p_a\} + \sum_{A\neq B}\left\{r_a^A, P_a^A\right\}\right], \tag{7.2.21}$$

$$K = \sum_{a=1}^{N}\left[x_a^2 + \sum_{A\neq B}(r_a^A)^2\right]. \tag{7.2.22}$$

Note that we have absorbed the time dependent part of D and K by using the similarity transformations (2.1.34).

Also the action (7.2.2) is invariant under the following fermionic transformations

$$\delta x_a = \frac{i}{\sqrt{2}}\left(\epsilon^{\alpha C}\psi_{\alpha C a} + \epsilon^{\dagger}_{\alpha C}\psi^{\dagger\alpha C}_a\right), \tag{7.2.23}$$

$$\delta r^C_a = \frac{i}{2}\left[\left(\omega^{\alpha CD}\psi_{\alpha D a}\right)e^{-i\theta^C_a} + \left(\psi^{\dagger\alpha D}_a\omega_{\alpha CD}\right)e^{i\theta^C_a} - \left(\epsilon^{\alpha C}\lambda_{\alpha a}\right)e^{-i\theta^C_a} - \left(\epsilon^{\dagger}_{\alpha C}\lambda^{\dagger\alpha}_a\right)e^{i\theta^C_a}\right], \tag{7.2.24}$$

$$\begin{aligned}\delta\psi_{\alpha C a} =& \left(\dot{r}^D_a + i\frac{h^D_a}{2r^D_a}\right)e^{i\theta^D_a}\omega_{\alpha CD}\\ &+\sqrt{2}\left(\dot{x}_a - i\frac{kC_1(E_a) + \sum_{D\neq B}h^D_a + \psi^{\dagger\alpha Da}\psi_{\alpha Da} + \lambda^{\dagger\alpha a}\lambda_{\alpha a}}{2x_a}\right)\epsilon^{\dagger}_{\alpha C} - \frac{i}{\sqrt{2}}\frac{l_a}{x_a}\psi_{\alpha Ca},\end{aligned} \tag{7.2.25}$$

$$\begin{aligned}\delta\psi^{\dagger\alpha C}_a =& -\left(\dot{r}^D_a - i\frac{h^D_a}{2r^D_a}\right)e^{-i\theta^D_a}\omega^{CD}_{\alpha}\\ &+\sqrt{2}\left(\dot{x}_a + i\frac{kC_1(E_a) + \sum_{D\neq B}h^D_a + \psi^{\dagger\alpha Da}\psi_{\alpha Da} + \lambda^{\dagger\alpha a}\lambda_{\alpha a}}{2x_a}\right)\epsilon^{\alpha C} + \frac{i}{\sqrt{2}}\frac{l_a}{x_a}\psi^{\dagger\alpha C}_a,\end{aligned} \tag{7.2.26}$$

$$\delta\lambda_{\alpha a} = -\epsilon^{\dagger}_{\alpha C}\left(\dot{r}^C_a + i\frac{h^C_a}{2r^C_a}\right)e^{i\theta^C_a}, \tag{7.2.27}$$

$$\delta\lambda^{\dagger\alpha}_a = -\left(\dot{r}^C_a - i\frac{h^C_a}{2r^C_a}\right)e^{-i\theta^C_a}\epsilon^{\alpha C} \tag{7.2.28}$$

with $C, D = 1, 2, 3$. Here $\epsilon^{\alpha C}$ and their Hermitian conjugate $\epsilon^{\dagger}_{\alpha C}$ are infinitesimal fermionic parameters and we have defined

$$\theta^C_a(t) = h^C_a\int^t\frac{dt'}{(r^C_a(t'))^2}, \tag{7.2.29}$$

$$l_a = \epsilon\psi_a - \epsilon^{\dagger}\psi^{\dagger}_a. \tag{7.2.30}$$

7.3 $SU(1,1|6)$ superconformal mechanics

Since the non-local quantities are included in the fermionic transformations (7.2.23)-(7.2.28), there may exist infinitely many conserved non-local charges. However, we see from (7.2.20) that the Hamiltonian describing the motion in the B-th complex plane associated with the variable x_a and the local charges commute with the others associated with the variables r^C_a's and the non-local charges. Therefore they are decoupled with one another and we thus can analyze the dynamics in the B-th direction separately. As in the subsection 6.3, it is convenient to assign the conserved charges h^A_a and $\lambda^{\dagger\alpha a}\lambda_{\alpha a}$ to be zeros. Then the low-energy dynamics in the

B-th complex plane is described by the action

$$S = \int_{\mathbb{R}} dt \sum_{a=1}^{N} \left[\dot{x}_a^2 - i\psi^{\dagger\alpha A a}\dot{\psi}_{\alpha A a} - \frac{\left(kC_1(E_a) + \psi^{\dagger\alpha A a}\psi_{\alpha A a}\right)^2}{4x_a^2} \right] \tag{7.3.1}$$

where $A = 1, 2, 3$ denote the R-symmetry indices. Note that the action (7.3.1) has the same structure as (3.6.1) [210, 224, 137] for $SU(1,1|\frac{\mathcal{N}}{2})$, $\mathcal{N} > 4$ superconformal quantum mechanics.

The action (7.3.1) has the invariance under the $\mathcal{N} = 12$ supersymmetry transformation laws

$$\delta x_a = \frac{i}{\sqrt{2}} \left(\epsilon^{\alpha A}\psi^{\alpha}_{Aa} + \epsilon^{\dagger}_{\alpha A}\psi_a^{\dagger\alpha A} \right), \tag{7.3.2}$$

$$\delta\psi_{\alpha A a} = \sqrt{2} \left(\dot{x}_a - i\frac{g_a}{2x_a} \right) \epsilon^{\dagger}_{\alpha A} - \frac{i}{\sqrt{2}} \frac{l_a}{x_a} \psi_{\alpha A a}, \tag{7.3.3}$$

$$\delta\psi_a^{\dagger\alpha A} = \sqrt{2} \left(\dot{x}_a + i\frac{g_a}{2x_a} \right) \epsilon^{\alpha A} + \frac{i}{\sqrt{2}} \frac{l_a}{x_a} \psi_a^{\dagger\alpha A} \tag{7.3.4}$$

where

$$g_a = kC_1(E_a) + \psi^{\dagger\alpha A a}\psi_{\alpha A a}. \tag{7.3.5}$$

The supersymmetry transformations (7.3.2)-(7.3.4) are generated by the supercharges

$$Q_{\alpha A} = \frac{i}{\sqrt{2}} \left(p^a - \frac{g_a}{x_a} \right) \psi_{\alpha A a}, \tag{7.3.6}$$

$$\tilde{Q}^{\alpha A} = \frac{i}{\sqrt{2}} \left(p^a + \frac{g_a}{x_a} \right) \psi^{\dagger\alpha A}. \tag{7.3.7}$$

Also the action (7.3.1) has the one-dimensional conformal invariance. The corresponding Noether charges are now expressed as

$$H = \sum_{a=1}^{N} \left[\frac{p_a^2}{4} + \frac{\left(kC_1(E_a) + \psi^{\dagger\alpha A a}\psi_{\alpha A a}\right)^2}{4x_a^2} \right], \tag{7.3.8}$$

$$D = -\frac{1}{4} \sum_{a=1}^{N} \{x_a, p^a\}, \tag{7.3.9}$$

$$K = \sum_{a=1}^{N} x_a^2. \tag{7.3.10}$$

According to the Dirac brackets (7.2.14) and (7.2.15), quantum operators of the canonical coordinates and momenta obey the quantum brackets

$$[x_a, p^b] = i\delta_{ab}, \qquad \left\{ \psi_{\alpha A a}, \psi^{\dagger\beta B b} \right\} = -\delta_{\alpha\beta}\delta_{AB}\delta_{ab}. \tag{7.3.11}$$

Combining the supercharges and the conformal generators, we find the superconformal boost generators

$$S_{\alpha A} = \sqrt{2}i \sum_a x_a \psi_{\alpha A a}, \tag{7.3.12}$$

$$\tilde{S}^{\alpha A} = \sqrt{2}i \sum_a x_a \psi_a^{\dagger \alpha A}. \tag{7.3.13}$$

The R-symmetry generator is given by

$$(J_{\alpha\beta})_{AB} = i \sum_a \psi_a^{\dagger \beta B} \psi_{\alpha A a}. \tag{7.3.14}$$

Note that (7.3.14) is a complex 6×6 matrix with $\alpha, \beta = +, -$ and $A, B = 1, 2, 3$ and it contains 36 complex valued elements.

Under the canonical relations (7.3.11) and the Weyl ordering [1], the generators form the following algebra

$$[H, D] = iH, \quad [K, D] = -iK, \quad [H, K] = 2iD, \tag{7.3.15}$$

$$[(J_{\alpha\beta})_{AB}, H] = 0, \quad [(J_{\alpha\beta})_{AB}, D] = 0, \quad [(J_{\alpha\beta})_{AB}, K] = 0, \tag{7.3.16}$$

$$[(J_{\alpha\beta})_{AB}, (J_{\gamma\delta})_{CD}] = i(J_{\alpha\delta})_{AD}\delta_{\beta\gamma}\delta_{BC} - i(J_{\gamma\beta})_{CB}\delta_{\alpha\delta}\delta_{AD}, \tag{7.3.17}$$

$$\begin{aligned} &[H, Q_{\alpha A}] = 0, \quad [D, Q_{\alpha A}] = -\tfrac{i}{2} Q_{\alpha A}, \quad [K, Q_{\alpha A}] = iS_{\alpha A}, \\ &[H, \tilde{Q}^{\alpha A}] = 0, \quad [D, \tilde{Q}^{\alpha A}] = -\tfrac{i}{2} \tilde{Q}^{\alpha A}, \quad [K, \tilde{Q}^{\alpha A}] = i\tilde{S}^{\alpha A}, \end{aligned} \tag{7.3.18}$$

$$\begin{aligned} &[H, S_{\alpha A}] = -iQ_{\alpha A}, \quad [D, S_{\alpha A}] = \tfrac{i}{2} S_{\alpha A}, \quad [K, S_{\alpha A}] = 0, \\ &[H, \tilde{S}^{\alpha A}] = -i\tilde{Q}^{\alpha A}, \quad [D, \tilde{S}^{\alpha A}] = \tfrac{i}{2} \tilde{S}^{\alpha A}, \quad [K, \tilde{S}^{\alpha A}] = 0, \end{aligned} \tag{7.3.19}$$

$$\begin{aligned} \{Q_{\alpha A}, \tilde{Q}^{\beta B}\} &= 2H\delta_{\alpha\beta}\delta_{AB}, \\ \{S_{\alpha A}, \tilde{S}^{\beta B}\} &= 2K\delta_{\alpha\beta}\delta_{AB}, \\ \{Q_{\alpha A}, \tilde{S}^{\beta B}\} &= -2D\delta_{\alpha\beta}\delta_{AB} - 2(J_{\alpha\beta})_{AB}, \\ \{\tilde{Q}^{\alpha A}, S_{\beta B}\} &= -2D\delta_{\alpha\beta}\delta_{AB} - 2(J^{\dagger}_{\alpha\beta})_{AB}, \end{aligned} \tag{7.3.20}$$

$$\begin{aligned} &[(J_{\alpha\beta})_{AB}, Q_{\gamma C}] = iQ_{\alpha A}\delta_{\beta\gamma}\delta_{BC}, \qquad [(J_{\alpha\beta})_{AB}, S_{\gamma, C}] = iS_{\alpha A}\delta_{\beta\gamma}\delta_{BC}, \\ &[(J_{\alpha\beta})_{AB}, \tilde{Q}^{\gamma C}] = -i\tilde{Q}^{\alpha A}\delta_{\beta\gamma}\delta_{BC}, \qquad [(J_{\alpha\beta})_{AB}, \tilde{S}^{\gamma, C}] = -i\tilde{S}^{\alpha A}\delta_{\beta\gamma}\delta_{BC}. \end{aligned} \tag{7.3.21}$$

[1] One needs to pick up constant shifts as a quantum effect.

The Hamiltonian H, the dilatation generator D and the conformal boost generator form the one-dimensional conformal algebra $\mathfrak{so}(1,2) = \mathfrak{sl}(2,\mathbb{R}) = \mathfrak{su}(1,1)$. As each of the supercharges $Q_{\alpha A}$ and $\tilde{Q}^{\alpha A} = -(Q_{\alpha A})^{\dagger}$ contain six real components, there exist twelve supercharges. They are the square roots of the Hamiltonian H. In addition, there are as many superconformal charges $S_{\alpha A}$ and $\tilde{S}^{\alpha A}$, which are the square roots of the conformal boost generator K. The anti-commutators of the fermionic charges generate an extra bosonic R-symmetry generators $(J_{\alpha\beta})_{AB}$. They form the $\mathfrak{u}(6)$ algebra (7.3.17). Thus the action (7.3.1) describes the $SU(1,1|6)$ invariant $\mathcal{N} = 12$ superconformal mechanics. In fact this belongs to the list of the simple supergroup for superconformal quantum mechanics, which we have shown in Table 3.2.

Following the $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence, we expect that the superconformal quantum mechanical models (6.1.13) and (7.1.17) may be related to $\mathrm{AdS}_2 \times T^2$ solutions, the so-called magnetic brane solutions [288, 289]. It may be interesting to check those correspondences.

Chapter 8

Curved Branes and Topological Twisting

In this chapter we will investigate the topological twisting and its relevant application as the world-volume description of curved branes in string theory and M-theory, which was firstly pointed out in [28]. In section 8.1 we will discuss various topological twisting procedures. In section 8.2 we will explain that the topologically twisted theories may yield the world-volume theories of the curved branes.

8.1 Topological twisting

Topological twisting is a modification of the Euclidean rotational group of a supersymmetric theory through an embedding into a global symmetry of the theory. The resulting theory will be topological if the twisted supersymmetry generators include at least one space-time scalar. Equivalently one can regard the twisting procedure as a gauging of an internal symmetry group in which a global symmetry is promoted to a space-time symmetry. In many cases, gauging can be performed by coupling of the internal symmetry current to the spin connection of the underlying manifold to the Lagrangian. We will give many examples of the topological twisting in the following.

8.1.1 $d = 4$, $\mathcal{N} = 2$ SYM theories

Let us consider topological twisting of $d = 4$, $\mathcal{N} = 2$ super Yang-Mills (SYM) theories [290]. We take $M_4 = \mathbb{R}^4$ whose rotational symmetry group is $Spin(4)_E \cong$

$SU(2)_l \times SU(2)_r$. The global symmetry of the theory is $U(2)_R \simeq SU(2) \times U(1)$ R-symmetry. The field content is

- complex scalar field ϕ
- 2 complex fermionic fields $\lambda^i_\alpha, \overline{\lambda}_{\dot{\alpha} i}$
- gauge field $A_{\alpha\dot{\alpha}}$

where α are indices of the fundamental representation of $SU(2)_l$ and $\dot{\alpha}$ are indices of the fundamental representation of $SU(2)_r$. i denotes the fundamental representation of $SU(2)_R$. These indices are raised and lowered with the antisymmetric tensor $\epsilon_{\alpha\beta}, \epsilon_{\dot{\alpha}\dot{\beta}}, \epsilon_{ij}$ such that $\epsilon_{12} = \epsilon^{12} = 1$. All fields are the adjoint representation of compact group G. The scaling dimensions are

$$[\phi] = 1, \quad [\psi] = [\lambda] = \frac{3}{2}, \quad [A] = 1, \qquad [\epsilon] = -\frac{1}{2} \tag{8.1.1}$$

where ϵ is a supersymmetry parameter.

The supersymmetry transformations are

$$\delta A_\mu = -i\overline{\lambda}^{\dot{\alpha}}_i \sigma_{\mu\alpha\dot{\alpha}} \epsilon^{\alpha i} + i\overline{\epsilon}^{\dot{\alpha}}_i \sigma_{\mu\alpha\dot{\alpha}} \lambda^{\alpha i}, \tag{8.1.2}$$

$$\delta\lambda^i_\alpha = \sigma^{\mu\nu}_{\alpha\beta} \epsilon^{\beta i} F_{\mu\nu} + i\epsilon^i_\alpha [\phi, \overline{\phi}] + i\sqrt{2}\sigma^\mu_{\alpha\dot{\alpha}} D_\mu \phi \epsilon^{ij} \overline{\epsilon}^{\dot{\alpha}}_j, \tag{8.1.3}$$

$$\delta\overline{\lambda}_{\dot{\alpha} i} = \sigma^{\mu\nu}_{\dot{\alpha}\dot{\beta}} \overline{\epsilon}^{\dot{\beta}}_i F_{\mu\nu} - i\overline{\epsilon}_{\dot{\alpha} i} [\phi, \overline{\phi}] + i\sqrt{2} D_\mu \sigma^\mu_{\dot{\alpha}\alpha} \overline{\phi} \epsilon_{ij} \epsilon^{\alpha j}, \tag{8.1.4}$$

$$\delta\phi = \sqrt{2}\epsilon^{\alpha i} \lambda_{\alpha i}, \tag{8.1.5}$$

$$\delta\overline{\phi} = \sqrt{2}\overline{\epsilon}^{\dot{\alpha}}_i \overline{\lambda}^i_{\dot{\alpha}} \tag{8.1.6}$$

where ϵ^i_α and $\overline{\epsilon}_{\dot{\alpha} i}$ are supersymmetry parameters that transform as $(\mathbf{2}, \mathbf{1}, \mathbf{2})$ and $(\mathbf{1}, \mathbf{2}, \mathbf{2})$ respectively.

The Lorentzian action is given by

$$\begin{aligned}\mathcal{L} =& \frac{1}{e^2} \int_M d^4x \mathrm{Tr} \Bigg(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i\overline{\lambda}^{\dot{\alpha}}_i \sigma^{\mu\nu}_{\alpha\dot{\alpha}} D_\mu \lambda^{\alpha i} - D_\mu \overline{\phi} D^\mu \phi \\ & - \frac{1}{2} [\overline{\phi}, \phi]^2 - \frac{1}{\sqrt{2}} \overline{\varphi} \epsilon_{ij} [\lambda^{\alpha i}, \lambda^j_\alpha] + \frac{i}{\sqrt{2}} \phi \epsilon^{ij} [\overline{\lambda}_{\dot{\alpha} i}, \overline{\lambda}^{\dot{\alpha}}_i] \Bigg). \end{aligned} \tag{8.1.7}$$

Here Tr is an invariant quadratic form on the Lie algebra.

The classical $\mathcal{N} = 2$ theory has a $U(2)$ symmetry acting on the two fermion $(\lambda, \overline{\lambda})$. The center $U(1)_R \subset U(2)$ is anomalous. On a given 4-manifold M_4 and for a given instanton number$k$, the total violation ΔU of the $U(1)_R$ charge is given by the dimension of the Yang-Mills instanton moduli space [290]. For $SU(2)$ this is

$$\Delta U = \dim \mathcal{M} = 8k - \frac{3}{2}(\chi + \sigma) \tag{8.1.8}$$

where χ and σ are the Euler characteristic and signature of M_4[1]. This was first discussed in [290].

The fields and supersymmetry parameters transform under $SO(4)_E \times U(2)_R \simeq SU(2)_l \times SU(2)_r \times SU(2)_R \times U(1)_R$ as

$$\phi :(\mathbf{1},\mathbf{1},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_{-2} \tag{8.1.9}$$
$$\psi, \lambda :(\mathbf{2},\mathbf{1},\mathbf{2})_1 \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_{-1} \tag{8.1.10}$$
$$A_\mu :(\mathbf{2},\mathbf{2},\mathbf{1})_0 \tag{8.1.11}$$
$$\epsilon :(\mathbf{2},\mathbf{1},\mathbf{2})_1 \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_{-1}. \tag{8.1.12}$$

To perform the topological twisting, we leave $SU(2)_l$ undisturbed and pick a homomorphism

$$\pi : SU(2)_r \to SU(2)_R, \tag{8.1.13}$$

and replace $SU(2)_r$ by a diagonal subgroup $SU(2)'_r = (1+\pi)(SU(2)) \subset SU(2)_r \times SU(2)_R$. Then under the new rotational symmetry $SO(4)'_E \simeq SU(2)_l \times SU(2)'_r$, the fields and supersymmetry parameters transform as

$$\phi \to (\mathbf{1},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{1})_{-2} \tag{8.1.14}$$
$$\psi, \lambda \to (\mathbf{2},\mathbf{2})_1 \oplus (\mathbf{1},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{3})_{-1} \tag{8.1.15}$$
$$A_\mu \to (\mathbf{2},\mathbf{2})_0 \tag{8.1.16}$$
$$\epsilon \to (\mathbf{2},\mathbf{2})_1 \oplus (\mathbf{1},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{3})_{-1}. \tag{8.1.17}$$

Thus the bosonic field content is

- complex scalar field ϕ: $(\mathbf{1},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{1})_{-2}$
- gauge field A_μ: $(\mathbf{2},\mathbf{2})_0$

and the fermionic field content is

- scalar field η: $(\mathbf{1},\mathbf{1})_{-1}$
- 1-form ψ_μ: $(\mathbf{2},\mathbf{2})_1$
- 2-form (self-dual antisymmetric 2-tensor) $\chi^+_{\mu\nu}$: $(\mathbf{1},\mathbf{3})_{-1}$.

[1]The quantity $\frac{\chi+\sigma}{2}$ is always integer.

From (8.1.17), one can see that there exists one BRST charge.

In $d=4$, $\mathcal{N}=2$ SYM theories, the possible anomalies are related to the global $SU(2)$ anomaly [291], which only appear when the corresponding moduli space is not orientable [290]. In Donaldson-Witten theory the moduli space is given by anti-self-dual connections, which is orientable [292]. Thus the twisted theory is anomaly free.

The twisted Lagrangian is

$$\mathcal{L} = \mathrm{Tr}\Bigg(\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2}D_\alpha\phi D^\alpha\sigma - i\eta D_\alpha\psi^\alpha + iD_\alpha\psi_\beta\cdot\chi^{\alpha\beta} - \frac{i}{8}\phi[\chi_{\alpha\beta},\chi^{\alpha\beta}] - \frac{i}{2}\sigma[\psi_\alpha,\psi^\alpha] - \frac{i}{2}\phi[\eta,\eta] - \frac{1}{8}[\phi,\sigma]^2\Bigg). \tag{8.1.18}$$

For the closure of supersymmetry algebra, it it necessary to introduce an auxiliary field $T_{ij}=T_{ji}$. It has scaling dimension $[T]=2$ and transform $(\mathbf{1},\mathbf{1},\mathbf{3})_0$ under $SU(2)_l\times SU(2)_r\times SU(2)_R\times U(1)_R$. After twisting they transform $(\mathbf{1},\mathbf{3})_0$ and identified with a 2-form.

Twisted $\mathcal{N}=2$ supersymmetric gauge theories have an off-shell formulation such that the action can be expressed as a Q-exact expression up to a θ-term[2], where Q is the BRST charge.

8.1.2 $d=4$, $\mathcal{N}=2$ SCFT on $C\times\Sigma$

We now consider a four-dimensional $\mathcal{N}=2$ superconformal field theory (SCFT) on $M_4=C\times\Sigma$ whose holonomy group is reduced to $U(1)_C\times U(1)_\Sigma$, where C and Σ are Riemann surfaces. This has been discussed in [293]. The global symmetry of the theory is $SU(2)_R\times U(1)_R$ R-symmetry and $U(1)_B$ symmetry. The field content is

- complex scalar field in the adjoint representation: $\varphi,\overline{\varphi}$
- 2 complex scalar fields in representation $R,R^\vee$ of G (squarks): $q,\tilde{q}$
- gauge field A_μ
- 2 gauginos: ψ,λ
- 2 complex left-handed quarks in representation $R^\vee,R$ of G: $\psi_q,\psi_{\tilde{q}}$

[2] Because of the chiral anomaly inherent to the R-symmetry of $\mathcal{N}=2$ SYM theories, observables are independent of θ-term up to rescaling. Thus one can ignore θ-term.

	$U(1)'_R$
φ	0
$\overline{\varphi}$	0
ψ	$+$
λ	$-$
q	$+$
$\tilde{q}$	$-$
ψ_q	0
$\psi_{\tilde{q}}$	0

Table 8.1: The $U(1)'_R \subset SU(2)_R$ charge assignments for $d = 4$, $\mathcal{N} = 2$ SCFT field content.

- 2 complex right-handed quarks in representation $R, R^\vee$ of G: $\overline{\psi_q}, \overline{\psi_{\tilde{q}}}$.

Before topological twisting, fields transform under $U(1)_C \times U(1)_\Sigma \times SU(2)_R \times U(1)_R \times U(1)_B$ as

$$\varphi, \overline{\varphi} : \mathbf{1}_{0020} \oplus \mathbf{1}_{00-20} \tag{8.1.19}$$

$$q, \tilde{q} : \mathbf{2}_{000-} \oplus \mathbf{2}_{000+} \tag{8.1.20}$$

$$A_\mu : \mathbf{1}_{2000} \oplus \mathbf{1}_{-2000} \oplus \mathbf{1}_{0200} \oplus \mathbf{1}_{0-200} \tag{8.1.21}$$

$$\psi, \lambda : \mathbf{2}_{+-+0} \oplus \mathbf{2}_{-++0} \tag{8.1.22}$$

$$\overline{\psi}, \overline{\lambda} : \mathbf{2}_{---0} \oplus \mathbf{2}_{++-0} \tag{8.1.23}$$

$$\psi_q, \psi_{\tilde{q}} : \mathbf{1}_{+---} \oplus \mathbf{1}_{+--+} \oplus \mathbf{1}_{-+--} \oplus \mathbf{1}_{-+-+} \tag{8.1.24}$$

$$\overline{\psi}_q, \overline{\psi}_{\tilde{q}} : \mathbf{1}_{--++} \oplus \mathbf{1}_{++++} \oplus \mathbf{1}_{--+-} \oplus \mathbf{1}_{+++-} \tag{8.1.25}$$

. To perform the topological twisting, we pick a homomorphism $\pi : U(1)_E \to SU(2)_R \times U(1)_R \times U(1)_B$ and replace $U(1)_E$ by $U(1)'_E = (1 + \pi)(U(1)_E) \subset U(1)_E \times SU(2)_R \times U(1)_R \times U(1)_B$.

To pick a homomorphism, we consider the maximal torus $U(1)'_R$ of $SU(2)_R$

$$SU(2)_R \supset U(1)'_R. \tag{8.1.26}$$

We assign $U(1)'_R$ charge for each field as in Table 8.1[3]. Then all of the $U(1)$ charges are summarized in Table 8.2. In Table 8.2, the subscripts $\pm$ indicate the upper and lower components of spinors. If Σ is flat, we should twist only $U(1)_C$ and there

[3]Our assignment is different from that in [293] where the both charges for $q, \tilde{q}$ are $-$.

	$U(1)_C$	$U(1)_\Sigma$	$U(1)'_R$	$U(1)_R$	$U(1)_B$	section
φ	0	0	0	2	0	$\mathcal{O}_C \otimes \mathcal{O}_\Sigma$
$\overline{\varphi}$	0	0	0	-2	0	$\mathcal{O}_C \otimes \mathcal{O}_\Sigma$
ψ_+	$+$	$-$	$+$	$+$	0	$K_C^{\frac{1}{2}} \otimes K_\Sigma^{-\frac{1}{2}}$
ψ_-	$-$	$+$	$+$	$+$	0	$K_C^{-\frac{1}{2}} \otimes K_\Sigma^{\frac{1}{2}}$
λ_+	$+$	$-$	$-$	$+$	0	$K_C^{\frac{1}{2}} \otimes K_\Sigma^{-\frac{1}{2}}$
λ_-	$-$	$+$	$-$	$+$	0	$K_C^{-\frac{1}{2}} \otimes K_\Sigma^{\frac{1}{2}}$
$\overline{\psi}_+$	$-$	$-$	$-$	$-$	0	$K_C^{-\frac{1}{2}} \otimes K_\Sigma^{-\frac{1}{2}}$
$\overline{\psi}_-$	$+$	$+$	$-$	$-$	0	$K^{\frac{1}{2}} \otimes K_\Sigma^{\frac{1}{2}}$
$\overline{\lambda}_+$	$-$	$-$	$+$	$-$	0	$K_C^{-\frac{1}{2}} \otimes K_\Sigma^{-\frac{1}{2}}$
$\overline{\lambda}_-$	$+$	$+$	$+$	$-$	0	$K_C^{\frac{1}{2}} \otimes K_\Sigma^{\frac{1}{2}}$
q	0	0	$+$	0	$-$	$\mathcal{O}_C \otimes \mathcal{O}_\Sigma$
$\tilde{q}$	0	0	$-$	0	$+$	$\mathcal{O}_C \otimes \mathcal{O}_\Sigma$
ψ_{q+}	$+$	$-$	0	$-$	$-$	$K_C^{\frac{1}{2}} \otimes K_\Sigma^{-\frac{1}{2}}$
ψ_{q-}	$-$	$+$	0	$-$	$-$	$K_C^{-\frac{1}{2}} \otimes K_\Sigma^{\frac{1}{2}}$
$\psi_{\tilde{q}+}$	$+$	$-$	0	$-$	$+$	$K_C^{\frac{1}{2}} \otimes K_\Sigma^{-\frac{1}{2}}$
$\psi_{\tilde{q}-}$	$-$	$+$	0	$-$	$+$	$K_C^{-\frac{1}{2}} \otimes K_\Sigma^{\frac{1}{2}}$
$\overline{\psi}_{q+}$	$-$	$-$	0	$+$	$+$	$K_C^{-\frac{1}{2}} \otimes K_\Sigma^{-\frac{1}{2}}$
$\overline{\psi}_{q-}$	$+$	$+$	0	$+$	$+$	$K_C^{\frac{1}{2}} \otimes K_\Sigma^{\frac{1}{2}}$
$\overline{\psi}_{\tilde{q}+}$	$-$	$-$	0	$+$	$-$	$K_C^{-\frac{1}{2}} \otimes K_\Sigma^{-\frac{1}{2}}$
$\overline{\psi}_{\tilde{q}-}$	$+$	$+$	0	$+$	$-$	$K_C^{\frac{1}{2}} \otimes K_\Sigma^{\frac{1}{2}}$

Table 8.2: $U(1)_R$ charge assignments for $d = 4$, $\mathcal{N} = 2$ SCFT field content. The subscripts $\pm$ indicate the upper and lower components of spinors. We denote the trivial bundle as $\mathcal{O}$ and the canonical bundle as K.

	A-twist		B-twist	
fields	$U(1)'_C$	$U(1)'_\Sigma$	$U(1)'_C$	$U(1)_\Sigma$
φ	0	0	2	0
$\overline{\varphi}$	0	0	-2	0
ψ_+	2	$-$	2	$-$
ψ_-	0	$+$	0	$+$
λ_+	0	$-$	2	$-$
λ_-	-2	$+$	0	$+$
$\overline{\psi}_+$	-2	$-$	-2	$-$
$\overline{\psi}_-$	0	$+$	0	$+$
$\overline{\lambda}_+$	0	$-$	-2	$-$
$\overline{\lambda}_-$	2	$+$	0	$+$
q	$+$	0	0	0
$\tilde{q}$	$-$	0	0	0
ψ_{q+}	$+$	$-$	0	$-$
ψ_{q-}	$-$	$+$	-2	$+$
$\psi_{\tilde{q}+}$	$+$	$-$	0	$-$
$\psi_{\tilde{q}-}$	$-$	$+$	2	$+$
$\overline{\psi}_{q+}$	$-$	$-$	0	$-$
$\overline{\psi}_{q-}$	$+$	$+$	2	$+$
$\overline{\psi}_{\tilde{q}+}$	$-$	$-$	0	$-$
$\overline{\psi}_{\tilde{q}-}$	$+$	$+$	2	$+$

Table 8.3: The spin of the fields for A-twisted and B-twisted $d = 4$, $\mathcal{N} = 2$ SCFT on $C \times \Sigma$.

are two types of twisting

$$\text{A-twist}: U(1)_C \to U(1)'_R \tag{8.1.27}$$

$$\text{B-twist}: U(2)_C \to U(1)_R. \tag{8.1.28}$$

The field content of A-twist and B-twist listed in Table 8.3. If both C and Σ are curved, we should also twist $U(1)_\Sigma$. Although there are many possibilities for

	AA-twist		BA-twist		BB-twist		BA+-twist	
fields	$U(1)'_C$	$U(1)'_\Sigma$	$U(1)'_C$	$U(1)_\Sigma$	$U(1)'_C$	$U(1)'_\Sigma$	$U(1)'_C$	$U(1)'_\Sigma$
φ	0	0	2	0	2	2	2	0
$\overline{\varphi}$	0	0	-2	0	-2	-2	-2	0
ψ_+	2	0	2	0	2	0	2	0
ψ_-	0	2	0	2	0	2	0	2
λ_+	0	-2	2	-2	2	0	2	-2
λ_-	-2	0	0	0	0	2	0	0
$\overline{\psi}_+$	-2	-2	-2	-2	-2	-2	-2	-2
$\overline{\psi}_-$	0	0	0	0	0	0	0	0
$\overline{\lambda}_+$	0	0	-2	0	-2	-2	-2	0
$\overline{\lambda}_-$	2	2	0	2	0	0	0	2
q	$-$	$-$	0	$-$	0	0	0	0
$\tilde{q}$	$-$	$-$	0	$-$	0	0	0	0
ψ_{q+}	$+$	$-$	0	$-$	0	-2	0	-2
ψ_{q-}	$-$	$+$	-2	$+$	-2	0	-2	0
$\psi_{\tilde{q}+}$	$+$	$-$	0	$-$	0	-2	0	0
$\psi_{\tilde{q}-}$	$-$	$+$	-2	$+$	-2	0	-2	2
$\overline{\psi}_{q+}$	$-$	$-$	0	$-$	0	0	0	0
$\overline{\psi}_{q-}$	$+$	$+$	2	$+$	2	2	2	2
$\overline{\psi}_{\tilde{q}+}$	$-$	$-$	0	$-$	0	0	0	-2
$\overline{\psi}_{\tilde{q}-}$	$+$	$+$	2	$+$	2	2	2	0

Table 8.4: The spin of the fields for AA, BA, BB-twisted 4d $\mathcal{N} = 2$ SCFT on $C \times \Sigma$.

twisting, we consider the following cases

$$\text{AA-twist} : U(1)_C \to U(1)'_R, U(1)_\Sigma \to U(1)'_R \tag{8.1.29}$$

$$\text{BA-twist} : U(1)_C \to U(1)_R, U(1)_\Sigma \to U(1)'_R \tag{8.1.30}$$

$$\text{BB-twist} : U(1)_C \to U(1)_R, U(1)_\Sigma \to U(1)_R \tag{8.1.31}$$

$$\text{BA+-twist} : U(1)_C \to U(1)_R, U(1)_\Sigma \to U(1)'_R \times U(1)_B. \tag{8.1.32}$$

The results of twisting are given in Table 8.4.

A-twist

After twisting, the bosonic scalar fields $\varphi, \overline{\varphi}, q, \tilde{q}$ remain scalars and the left-handed quarks $\psi_q, \psi_{\tilde{q}}$ remain sections of

$$K_C^{\frac{1}{2}} \otimes K_\Sigma^{-\frac{1}{2}} + K_C^{-\frac{1}{2}} \otimes K_\Sigma^{\frac{1}{2}} \tag{8.1.33}$$

and the right-handed quarks $\overline{\psi}_q, \overline{\psi}_{\tilde{q}}$ are sections of

$$K_C^{-\frac{1}{2}} \otimes K_\Sigma^{-\frac{1}{2}} + K_C^{\frac{1}{2}} \otimes K_\Sigma^{\frac{1}{2}}. \tag{8.1.34}$$

On the other hand squarks $q, \tilde{q}$ become sections of $K_C^{\frac{1}{2}}$ and $K_C^{-\frac{1}{2}}$. The gauginos ψ reduces to sections of

$$K_C \otimes K_\Sigma^{-\frac{1}{2}} + \mathcal{O}_C \otimes K_\Sigma^{\frac{1}{2}} \tag{8.1.35}$$

and λ become sections of

$$\mathcal{O}_C \otimes K_\Sigma^{-\frac{1}{2}} + K_C^{-1} \otimes K_\Sigma^{\frac{1}{2}}. \tag{8.1.36}$$

Their right-handed partners are sections of

$$K_C^{-1} \otimes K_\Sigma^{-\frac{1}{2}} + \mathcal{O}_C \otimes K_\Sigma^{\frac{1}{2}} \tag{8.1.37}$$

$$\mathcal{O}_C \otimes K_\Sigma^{-\frac{1}{2}} + K_C \otimes K_\Sigma^{\frac{1}{2}}. \tag{8.1.38}$$

In the original theory, we have eight supercharges. Since the transformations of supercharges under R-symmetry are identical to those of gauginos and only scalars on C survive in the twisted theory, (8.1.35)-(8.1.38) shows that there remains four supercharges

$$\begin{aligned} &\mathcal{O}_C \otimes K_\Sigma^{\frac{1}{2}}, \quad \mathcal{O}_C \otimes K_\Sigma^{-\frac{1}{2}}, \\ &\mathcal{O}_C \otimes K_\Sigma^{\frac{1}{2}}, \quad \mathcal{O}_C \otimes K_\Sigma^{-\frac{1}{2}}. \end{aligned} \tag{8.1.39}$$

Two of them transform as spinors of positive chirality on Σ and the other two transform as those of negative chirality on Σ. Therefore if one takes into account the dimensional reduction to Σ, the theory on Σ has $(2,2)$ supersymmetry.

B-twist

After the twisting the bosonic scalars φ becomes section of K_C and squarks $q, \tilde{q}$ are unchanged. The quarks ψ_q and $\psi_{\tilde{q}}$ become sections of

$$\mathcal{O}_C \otimes K_\Sigma^{-\frac{1}{2}} + K_C^{-1} \otimes K_\Sigma^{\frac{1}{2}} \tag{8.1.40}$$

$$\mathcal{O}_C \otimes K_\Sigma^{-\frac{1}{2}} + K_C \otimes K_\Sigma^{\frac{1}{2}}. \tag{8.1.41}$$

The gauginos ψ and λ are sections of

$$K_C \otimes K_\Sigma^{-\frac{1}{2}} + \mathcal{O}_C \otimes K_\Sigma^{\frac{1}{2}} \tag{8.1.42}$$

$$K_C \otimes K_\Sigma^{-\frac{1}{2}} + \mathcal{O}_C \otimes K_\Sigma^{\frac{1}{2}} \tag{8.1.43}$$

and $\tilde{\psi}$ and $\tilde{\lambda}$ are sections of

$$K_C^{-1} \otimes K_\Sigma^{-\frac{1}{2}} + \mathcal{O}_C \otimes K_\Sigma^{\frac{1}{2}} \tag{8.1.44}$$

$$K_C^{-1} \otimes K_\Sigma^{-\frac{1}{2}} + \mathcal{O}_C \otimes K_\Sigma^{\frac{1}{2}}. \tag{8.1.45}$$

From (8.1.42)-(8.1.45), we see that there are four supercharges

$$\mathcal{O}_C \otimes K_\Sigma^{\frac{1}{2}}, \quad \mathcal{O}_C \otimes K_\Sigma^{\frac{1}{2}} \tag{8.1.46}$$

$$\mathcal{O}_C \otimes K_\Sigma^{\frac{1}{2}}, \quad \mathcal{O}_C \otimes K_\Sigma^{\frac{1}{2}}, \tag{8.1.47}$$

which transform as spinors of the positive chirality on Σ. Thus the theory on Σ can have $(4,0)$ supersymmetry.

AA-twist

After twisting, we have

$$\varphi \in \Gamma(\mathcal{O}_C \otimes \mathcal{O}_\Sigma) \tag{8.1.48}$$

$$q \in \Gamma(K_C^{\frac{1}{2}} \otimes K_\Sigma^{-\frac{1}{2}}), \quad \tilde{q} \in \Gamma(K_C^{-\frac{1}{2}} \otimes K_\Sigma^{-\frac{1}{2}}) \tag{8.1.49}$$

$$\psi_q \in \Gamma(K_C^{\frac{1}{2}} \otimes K_\Sigma^{-\frac{1}{2}} + K_C^{-\frac{1}{2}} \otimes K_\Sigma^{\frac{1}{2}}), \quad \psi_{\tilde{q}} \in (K_C^{\frac{1}{2}} \otimes K_\Sigma^{-\frac{1}{2}} + K_C^{-\frac{1}{2}} \otimes K_\Sigma^{\frac{1}{2}}) \tag{8.1.50}$$

$$\overline{\psi}_q \in \Gamma(K_C^{-\frac{1}{2}} \otimes K_\Sigma^{-\frac{1}{2}} + K_C^{\frac{1}{2}} \otimes K_\Sigma^{\frac{1}{2}}), \quad \overline{\psi}_{\tilde{q}} \in \Gamma(K_C^{-\frac{1}{2}} \otimes K_\Sigma^{-\frac{1}{2}} + K_C^{\frac{1}{2}} \otimes K_\Sigma^{\frac{1}{2}}) \tag{8.1.51}$$

$$\psi \in \Gamma(K_C \otimes \mathcal{O}_\Sigma + \mathcal{O}_C \otimes K_\Sigma), \quad \lambda \in \Gamma(\mathcal{O}_C \otimes K_\Sigma^{-1} + K_C^{-1} \otimes \mathcal{O}_\Sigma) \tag{8.1.52}$$

$$\overline{\psi} \in \Gamma(K_C^{-1} \otimes K_\Sigma^{-1} + \mathcal{O}_C \otimes \mathcal{O}_\Sigma), \quad \overline{\lambda} \in \Gamma(\mathcal{O}_C \otimes \mathcal{O}_\Sigma + K_C \otimes K_\Sigma). \tag{8.1.53}$$

In other word, the fields transform under $U(1)'_C \times U(1)'_\Sigma$ as

$$\varphi, \overline{\varphi} \to \mathbf{1}_{00} \oplus \mathbf{1}_{00} \tag{8.1.54}$$

$$\psi, \lambda \to \mathbf{1}_{20} \oplus \mathbf{1}_{02} \oplus \mathbf{1}_{0-2} \oplus \mathbf{1}_{-20} \tag{8.1.55}$$

$$\overline{\psi}, \overline{\lambda} \to \mathbf{1}_{-2-2} \oplus \mathbf{1}_{00} \oplus \mathbf{1}_{00} \oplus \mathbf{1}_{22} \tag{8.1.56}$$

$$q, \tilde{q} \to \mathbf{1}_{++} \oplus \mathbf{1}_{--} \tag{8.1.57}$$

$$\psi_q, \psi_{\tilde{q}} \to \mathbf{1}_{+-} \oplus \mathbf{1}_{-+} \oplus \mathbf{1}_{+-} \oplus \mathbf{1}_{-+} \tag{8.1.58}$$

$$\overline{\psi}_q, \overline{\psi}_{\tilde{q}} \to \mathbf{1}_{--} \oplus \mathbf{1}_{++} \oplus \mathbf{1}_{--} \oplus \mathbf{1}_{++}. \tag{8.1.59}$$

Therefore the bosonic field content is

- 2 scalar fields ϕ, σ: $\mathbf{1}_{00} \oplus \mathbf{1}_{00}$
- gauge fields A_z, A_w: $\mathbf{1}_{20} \oplus \mathbf{1}_{-20} \oplus \mathbf{1}_{02} \oplus \mathbf{1}_{0-2}$
- spinor fields $\tilde{q}, \tilde{q}$: $\mathbf{1}_{++} \oplus \mathbf{1}_{--}$

and the fermionic field content is

- scalar field η: $\mathbf{1}_{00}$
- 1-forms ψ_z, ψ_w: $\mathbf{1}_{20} \oplus \mathbf{1}_{02} \oplus \mathbf{1}_{-02} \oplus \mathbf{1}_{-20}$
- 2-form χ: $\mathbf{1}_{-2-2} \oplus \mathbf{1}_{22}$
- spinor fields $\psi_q, \psi_{\tilde{q}}, \overline{\psi}_q, \overline{\psi}_{\tilde{q}}$: $2\left(\mathbf{1}_{+-} \oplus \mathbf{1}_{-+} \oplus \mathbf{1}_{--} \oplus \mathbf{1}_{++}\right)$

Focusing on the vector multiplet, one can check that the field content is same as that of $\mathcal{N} = 2$ twisted Donaldson-Witten theory as expected.

In the twisted theory we have two supercharges. Both of them are right-handed in 4-dimensions. Noting the spin of $U(1)_C \times U(1)_\Sigma$ for $\overline{\psi}_-, \overline{\lambda}_+$

$$\overline{\psi}_- : (+,+) \xrightarrow{\text{AA-twist}} (0,0) \tag{8.1.60}$$

$$\overline{\lambda}_+ : (-,-) \xrightarrow{\text{AA-twist}} (0,0), \tag{8.1.61}$$

one can see that the two supercharges have the opposite chiralities on both C and Σ.

BA-twist

After twisting we obtain

$$\varphi \in \Gamma(K_C \otimes \mathcal{O}_\Sigma) \tag{8.1.62}$$

$$q \in \Gamma(\mathcal{O}_C \otimes K_\Sigma^{-1}), \quad \tilde{q} \in \Gamma(\mathcal{O}_C \otimes K_\Sigma^{-1}) \tag{8.1.63}$$

$$\psi_q, \psi_{\tilde{q}} \in \Gamma(\mathcal{O}_C \otimes K_\Sigma^{-\frac{1}{2}} + K_C^{-1} \otimes K_\Sigma^{\frac{1}{2}}) \tag{8.1.64}$$

$$\overline{\psi}_q, \overline{\psi}_{\tilde{q}} \in \Gamma(\mathcal{O}_C \otimes K_\Sigma^{-\frac{1}{2}} + K_C \otimes K_\Sigma^{\frac{1}{2}}) \tag{8.1.65}$$

$$\psi \in \Gamma(K_C \otimes \mathcal{O}_\Sigma + \mathcal{O}_C \otimes K_\Sigma), \quad \lambda \in \Gamma(K_C \otimes K_\Sigma^{-1} + \mathcal{O}_C \otimes \mathcal{O}_\Sigma) \tag{8.1.66}$$

$$\overline{\psi} \in \Gamma(K_C^{-1} \otimes K_\Sigma^{-1} + \mathcal{O}_C \otimes \mathcal{O}_\Sigma), \quad \overline{\lambda} \in \Gamma(K_C^{-1} \otimes \mathcal{O}_\Sigma + \mathcal{O}_C \otimes K_\Sigma). \tag{8.1.67}$$

Therefore there exists two supercharges. One is left-handed and the other is right-handed in 4-dimensions. From the fact

$$\lambda_- : (-,+) \to (0,0) \tag{8.1.68}$$

$$\overline{\psi}_- : (+,+) \to (0,0), \tag{8.1.69}$$

it turns out that two supercharges have the same chirality on Σ and the opposite chirality on C.

8.1.3 $d = 4, \mathcal{N} = 4$ SYM theories

Let us start with $d = 10$, $\mathcal{N} = 1$ SYM theory. $d = 4$, $\mathcal{N} = 4$ SYM theory is most easily derived by dimensional reduction from ten dimensions [4].

The field content of $d = 10$, $\mathcal{N} = 1$ SYM theory is

- gauge field A
- 16 fermionic fields (gauginos) Ψ

The gauge field A is a connection on a G-bundle E. The fermionic field Ψ is a positive chirality spinor field with values in the adjoint representation of G, that is a section of $S^+ \otimes \mathrm{ad}(E)$. We should note that the ten-dimensional spin representations

$$\mathbf{16}_s \text{ and } \mathbf{16}_c \text{ are } \begin{cases} \text{real and dual to each other :} & \text{Lorentz signature} \\ \text{complex conjugate :} & \text{Euclidean signature} \end{cases}. \tag{8.1.70}$$

The gaugino Ψ is a ten-dimensional positive chirality spinor field

$$\Gamma^{11}\Psi = \Psi \tag{8.1.71}$$

where

$$\Gamma^{11} := i\Gamma^{12\cdots 10}. \tag{8.1.72}$$

The conjugate is given by

$$\overline{\Psi} := \Psi^T \mathcal{C} \tag{8.1.73}$$

where $\mathcal{C}$ is a ten-dimensional charge conjugation matrix satisfying[5]

$$\mathcal{C}^T = \mathcal{C}, \quad \mathcal{C}\Gamma^M \mathcal{C}^{-1} = \Gamma^M. \tag{8.1.75}$$

---

[4] From Nahm's theorem [12], ten-dimensional is the maximum possible dimension for SYM theory.

[5] Although there is another definition given by

$$\mathcal{C}^T = -\mathcal{C}, \quad \mathcal{C}\Gamma^M \mathcal{C}^{-1} = -\Gamma^M \tag{8.1.74}$$

which corresponds to $\mathcal{C}_+$, we choose $\mathcal{C}_-$ in (8.1.74).

physics convention	mathmatics convention
$D = d - iA'$ (A':Hermitian) $F = i(D)^2 = dA' - iA' \wedge A'$	$D = d + A$ (A:anti-Hermitian), $F = D^2 = dA + A \wedge A$

Table 8.5: The physical and mathematical definitions of connection and field strength. The relations are given by $-iF' = F, -iA' = A$. Although the anti-Hermitian fields A are unnatural for $G = U(1)$, they may avoid unnatural factors i.

The Lagrangian of Euclidean $d = 10$, $\mathcal{N} = 1$ SYM theory is given by

$$\mathcal{L} = \frac{1}{e^2}\mathrm{Tr}\left(\frac{1}{4}F_{MN}F^{MN} + \frac{1}{2}\overline{\Psi}\Gamma^M D_M \Psi\right) \tag{8.1.76}$$

where $M, N, \cdots = 1, 2, \cdots, 10$ are indices of ten-dimensional space-time and we define[6]

$$D_M\Psi := \partial_M \Psi - i[A_M, \Psi], \tag{8.1.77}$$

$$F_{MN} := \partial_M A_N - \partial_N A_M - i[A_M, A_N]. \tag{8.1.78}$$

The 16 supersymmetries are

$$\delta A_M = \overline{\Psi}\Gamma_M \epsilon = -\overline{\epsilon}\Gamma_M \Psi \tag{8.1.79}$$

$$\delta\Psi = \frac{1}{2}F_{MN}\Gamma^{MN}\epsilon. \tag{8.1.80}$$

To consider the dimensional reduction of $d = 10$, $\mathcal{N} = 1$ SYM theories to four dimensions, we decompose the gamma matrices under $SO(10) \supset SO(4)_E \times SO(6)$ as

$$\begin{cases} \Gamma^\mu = \gamma^\mu \otimes \hat{\Gamma}^7 \\ \Gamma^I = \mathbb{I}_4 \otimes \hat{\Gamma}^I \end{cases} \tag{8.1.81}$$

where $\mu = 1, 2, 3, 4$ and $I = 5, 6, 7, 8, 9, 10$. $\hat{\Gamma}^I$ are six-dimensional gamma matrices satisfying

$$\{\hat{\Gamma}^I, \hat{\Gamma}^J\} = 2\delta^{IJ}, \quad (\hat{\Gamma}^I)^\dagger = \Gamma^I \tag{8.1.82}$$

$$\hat{\Gamma}^7 = i\hat{\Gamma}^{12\cdots 6} = \begin{pmatrix} \mathbb{I}_4 & 0 \\ 0 & -\mathbb{I}_4 \end{pmatrix} \tag{8.1.83}$$

[6] This is preferred convention in physics (see Table 8.5).

and γ^μ are four-dimensional gamma matrices

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}, \quad (\gamma^\mu)^\dagger = \gamma^\mu, \quad \gamma^5 := \gamma^{1\cdots 4}. \tag{8.1.84}$$

The charge conjugation matrix $\mathcal{C}$ and the chiral matrix are decomposed as

$$\mathcal{C} = C \otimes \hat{C}, \tag{8.1.85}$$

$$\Gamma^{11} = \gamma^5 \otimes \hat{\Gamma}^7 \tag{8.1.86}$$

where C is the four-dimensional charge conjugation matrix satisfying [7]

$$C^T = -C, \quad C\gamma^\mu C^{-1} = (\gamma^\mu)^T, \quad C\gamma^5 C^{-1} = (\gamma^5)^T \tag{8.1.87}$$

and $\hat{C}$ is the six-dimensional charge conjugation matrix

$$\hat{C}^T = -\hat{C}, \quad \hat{C}\hat{\Gamma}^I \hat{C}^{-1} = (\hat{\Gamma}^I)^T, \qquad C\hat{\Gamma}^7 C^{-1} = -(\hat{\Gamma}^7)^T \tag{8.1.88}$$

The global symmetry of the theory is $SU(4)_R$ R-symmetry. The field content is

- 6 real scalar fields $\phi^I (I = 5, 6, \cdots, 10)$
- 16 fermionic fields (gauginos) $\psi^A (A = 1, 2, 3, 4)$
- gauge field A_μ

where indices $I, J, \cdots$ and $A, B, \cdots$ are **6** and **4** of $SU(4)_R$ R-symmetry.

Performing the dimensional reduction of (8.1.76), we obtain the Lagrangian of $d = 4$, $\mathcal{N} = 4$ SYM theories

$$\mathcal{L} = \frac{1}{e^2}\mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi_I D^\mu\phi^I - \frac{1}{4}[\phi_I, \phi_I][\phi^I, \phi^I] \quad + \frac{1}{2}\overline{\psi}\Gamma^\mu D_\mu\psi - \frac{i}{2}\overline{\psi}\Gamma^I[\phi_I, \psi]\right) \tag{8.1.89}$$

where $\phi_I := A_I (5 \leq I \leq 10)$ and $\mu, \nu = 1, 2, 3, 4$. If G is simple and if we require that Lagrangian is quadratic in derivatives, the above Lagrangian is unique except for the change of parameter e. However, we may have θ-term that measures the topology of the G-bundle E

$$\mathcal{L}_\theta = \frac{i\theta}{16\pi^2}\mathrm{Tr}(F \wedge F) = \frac{i\theta}{16\pi^2}\mathrm{Tr}(*F_{\mu\nu}F^{\mu\nu}), \tag{8.1.90}$$

[7] Here we choose C as C_-.

which is θ times the second Chern class or instanton number of the bundle[8]. The parameter e and θ combine into a complex parameter

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}. \tag{8.1.91}$$

The Lagrangian (8.1.89) is invariant under 16 supersymmetries

$$\delta\phi_I = \overline{\psi}\Gamma_I\epsilon = -\overline{\epsilon}\Gamma_I\psi, \tag{8.1.92}$$

$$\delta A_\mu = \overline{\psi}\Gamma_\mu\epsilon = -\overline{\epsilon}\Gamma_\mu\psi, \tag{8.1.93}$$

$$\delta\psi = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}\epsilon + D_\mu\Phi_I\Gamma^\mu\Gamma^I\epsilon - \frac{i}{2}[\phi_I,\phi_J]\Gamma^{IJ}\epsilon. \tag{8.1.94}$$

Before topological twisting, fields transform under $SO(4)_E \times SU(4)_R \simeq SU(2)_l \times SU(2)_r \times SU(4)_R$ as

$$\phi :(\mathbf{1},\mathbf{1},\mathbf{6}) \tag{8.1.95}$$

$$\psi :(\mathbf{2},\mathbf{1},\overline{\mathbf{4}}) \oplus (\mathbf{1},\mathbf{2},\mathbf{4}) \tag{8.1.96}$$

$$A_\mu :(\mathbf{2},\mathbf{2},\mathbf{1}). \tag{8.1.97}$$

To perform the fully topological twisting, we pick a homomorphism $\pi : SO(4)_E \to SU(4)_R$ and replace $SO(4)_E$ by $SO(4)'_E = (1+\pi)(SO(4)_E) \subset SO(4)_E \times SO(6)_R$.

The choice of π amounts to embedding $SO(4)_E \simeq SU(2)_l \times SU(2)_r$ in $SU(4)_R$ as

$$\pi : SU(2)_l \times SU(2)_r \to \begin{pmatrix} SU(2)_l & 0 \\ 0 & SU(2)_r \end{pmatrix}, \tag{8.1.98}$$

which leads us to consider the decomposition

$$SU(4) \supset SU(2) \times SU(2) \times U(1). \tag{8.1.99}$$

Under (8.1.99), we still have several possible embedding determined by telling how the $\mathbf{4}$ of $SU(4)_R$ transforms under $SU(2)_l \times SU(2)_r$. Up to an exchange of left and right, there are three inequivalent transformations of $\mathbf{4}$ of $SU(4)_R$ under (8.1.99).

$$\begin{array}{lll} \text{(i)} & \text{GL twist} & \mathbf{4} = (\mathbf{2},\mathbf{1})_1 \oplus (\mathbf{1},\mathbf{2})_{-1} \\ \text{(ii)} & \text{VW twist} & \mathbf{4} = (\mathbf{1},\mathbf{2})_1 \oplus (\mathbf{1},\mathbf{2})_{-1} \\ \text{(iii)} & \text{DW twist} & \mathbf{4} = (\mathbf{1},\mathbf{2})_0 \oplus (\mathbf{1},\mathbf{1})_1 \oplus (\mathbf{1},\mathbf{1})_{-1} \end{array} \tag{8.1.100}$$

[8] In $\mathcal{N} = 4$ SYM theories θ-terms are observable because there is no chiral anomaly and we cannot shift them. This situation is different from $\mathcal{N} = 2$ SYM.

Geometric Langlands (GL) twist

GL twist has the branching

$$\begin{aligned}\mathbf{4} &= (\mathbf{2},\mathbf{1})_1 \oplus (\mathbf{1},\mathbf{2})_{-1}\\ \bar{\mathbf{4}} &= (\mathbf{2},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{2})_1.\end{aligned} \tag{8.1.101}$$

1. **fermionic fields**

   Noting that

$$(\mathbf{2},\mathbf{1})_0 \times ((\mathbf{2},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{2})_1) = (\mathbf{1},\mathbf{1})_{-1} \oplus (\mathbf{3},\mathbf{1})_{-1} \oplus (\mathbf{2},\mathbf{2})_1 \tag{8.1.102}$$

   and

$$(\mathbf{1},\mathbf{2})_0 \times ((\mathbf{2},\mathbf{1})_1 \oplus (\mathbf{1},\mathbf{2}_{-1})) = (\mathbf{2},\mathbf{2})_1 \oplus (\mathbf{1},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{3})_{-1}, \tag{8.1.103}$$

   one can see that the fermionic fields transform under $SU(2)'_l \times SU(2)'_r \times U(1)$ as

$$(\mathbf{1},\mathbf{1})_{-1} \oplus (\mathbf{3},\mathbf{1})_{-1} \oplus (\mathbf{2},\mathbf{2})_1 \oplus (\mathbf{2},\mathbf{2})_1 \oplus (\mathbf{1},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{3})_{-1}. \tag{8.1.104}$$

   Similarly one can obtain the transformations of supersymmetries. Thus, from (8.1.104) we see that GL twist leads to two unbroken BRST charges which have the same $U(1)$ charge.

2. **bosonic fields**

   The bosonic scalar field $\mathbf{6}_v$ of $SO(6)_R$ is produced by the product of $SO(6)$ spinor $\mathbf{8} = \mathbf{4} + \bar{\mathbf{4}}$ as

$$\begin{aligned}\mathbf{8}\times\mathbf{8} =&(\mathbf{4}+\bar{\mathbf{4}}) \times (\mathbf{4}+\bar{\mathbf{4}})\\ =&\mathbf{4}\times\mathbf{4} + \mathbf{4}\times\bar{\mathbf{4}} + \bar{\mathbf{4}}\times\mathbf{4} + \bar{\mathbf{4}}\times\bar{\mathbf{4}}\\ =&([1]+[3]) + ([0]+[2]) + ([0]+[2]) + ([1]+[3])\\ =&(\mathbf{6}+\mathbf{10}) + (\mathbf{1}+\mathbf{15}) + (\mathbf{1}+\mathbf{15}) + (\mathbf{6}+\mathbf{10}).\end{aligned} \tag{8.1.105}$$

   Note that $\mathbf{6}_v$ is the antisymmetric product of $\mathbf{4}$

$$\begin{aligned}\mathbf{6}_v =&(\mathbf{4}+\mathbf{4})_a\\ =&(((\mathbf{2},\mathbf{1})_1 \oplus (\mathbf{1},\mathbf{2})_{-1}) \otimes ((\mathbf{2},\mathbf{1})_1 \oplus (\mathbf{1},\mathbf{2})_{-1}))_a\\ =&(((\mathbf{1},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{1})_{-2} \oplus (\mathbf{2},\mathbf{2})_0) \oplus ((\mathbf{3},\mathbf{1})_2 \oplus (\mathbf{2},\mathbf{2})_0 \oplus (\mathbf{1},\mathbf{3})_{-2}))_a\\ =&(\mathbf{1},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{1})_{-2} \oplus (\mathbf{2},\mathbf{2})_0.\end{aligned} \tag{8.1.106}$$

   Thus the bosonic field content consists of

- 2 scalar fields : $(\mathbf{1},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{1})_{-2}$
- 1-form : $(\mathbf{2},\mathbf{2})_0$
- gauge field : $(\mathbf{2},\mathbf{2})_0$

and the fermionic field content is

- 2 scalar fields : $(\mathbf{1},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{1})_{-1}$
- 2 1-forms : $(\mathbf{2},\mathbf{2})_1 \oplus (\mathbf{2},\mathbf{2})_1$
- 2 2-forms [9] : $(\mathbf{3},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{3})_{-1}$.

Under decomposition $SO(10) \supset SO(4)_E \times SO(4) \times SO(2)$, the decomposition of the ten-dimensional gamma matrices is given by

$$\begin{cases} \Gamma^\mu = \gamma^\mu \otimes \gamma^5 \otimes -\sigma_3 \\ \Gamma^{\mu+4} = \mathbb{I}_4 \otimes \gamma^\mu \otimes \mathbb{I}_2 \\ \Gamma^{i+8} = \mathbb{I}_4 \otimes \gamma^5 \otimes \sigma_i \end{cases} \tag{8.1.107}$$

where $\mu = 1,2,3,4$ and $i = 1,2$. σ_i are Pauli matrices and γ^μ are four-dimensional gamma matrices defined by (8.1.84). The charge conjugation matrix $\mathcal{C}$ is decomposed as

$$\mathcal{C} = C \otimes C \otimes \sigma_1 \tag{8.1.108}$$

Under the decomposition (8.1.107), ten-dimensional chirality matrix is expressed as

$$\Gamma^{11} = -\left(\gamma^5 \otimes \gamma^5 \otimes \sigma_3\right). \tag{8.1.109}$$

1. **bosonic fields**

   For the bosonic fields we redefine

$$\Phi_\mu := \phi_{4+\mu}, \quad (\mu = 1,2,3,4), \tag{8.1.110}$$

$$A^\pm_\mu := \frac{1}{\sqrt{2}}(A_\mu \pm i\Phi_\mu), \tag{8.1.111}$$

$$\varphi := \frac{1}{\sqrt{2}}(\phi_9 + i\phi_{10}), \quad \overline{\varphi} := \frac{1}{\sqrt{2}}(\phi_9 - i\phi_{10}). \tag{8.1.112}$$

[9] Note that this is not self-dual.

2. **fermionic fields**

Noting the fermionic field content of GL twist, one can decompose the 10-dimensional fermionic field Ψ under the decomposition $SO(10) \supset SO(4)_E \times SO(4) \times SO(2)$ as [10]

$$\Psi_{pq\alpha} = \frac{1}{\sqrt{2}}\left(\eta_\alpha \mathbb{I}_{4pq} + \psi_{\mu\alpha}\gamma^\mu_{pq} + \frac{1}{2}\chi_{\mu\nu\alpha}\gamma^{\mu\nu}_{pq} + \omega_{\mu\alpha}\gamma^{\mu 5}_{pq} + \zeta_\alpha \gamma^5_{pq}\right) C^{-1} \quad (8.1.113)$$

where p, q and α are indices of $SO(4)_E, SO(4)$ and $SO(2)$ respectively. $\eta_\alpha, \zeta_\alpha$ are scalars $(\mathbf{1},\mathbf{1})_- \oplus (\mathbf{1},\mathbf{1})_-$ and $\psi_{\mu\alpha}, \omega_{\mu\alpha}$ are 1-forms $(\mathbf{2},\mathbf{2})_+ \oplus (\mathbf{2},\mathbf{2})_+$ and $\chi_{\mu\nu} = -\chi_{\nu\mu}$ is a 2-form $(\mathbf{3},\mathbf{1})_- \oplus (\mathbf{1},\mathbf{3})_-$.

From the decompositions (8.1.107) and the chirality condition (8.1.71) for the femionic fields, we see that

$$\sigma_3 \eta = \eta, \quad \sigma_3 \psi_\mu = \psi_\mu, \quad (8.1.114)$$

$$\sigma_3 \chi_{\mu\nu} = -\chi_{\mu\nu}, \quad \sigma_3 \omega_\mu = \omega_\mu, \quad (8.1.115)$$

$$\sigma_3 \zeta = -\zeta \quad (8.1.116)$$

Therefore ψ_μ, ω_μ have $U(1)$ ghost charge $+1$ and $\eta, \chi_{\mu\nu}, \zeta$ have $U(1)$ charge -1. This is consistent to the results of the fermionic field content for GL twist.

3. **supersymmetries**

For the sypersymmetries we can also expand as

$$\epsilon_{pq\alpha} = \frac{1}{\sqrt{2}}\left(\epsilon_\alpha \mathbb{I}_{4pq} + \epsilon_{\mu\alpha}\gamma^\mu_{pq} + \frac{1}{2}\epsilon_{\mu\nu\alpha}\gamma^{\mu\nu}_{pq} + \tilde{\epsilon}_{\mu\alpha}\gamma^{\mu 5} + \tilde{\epsilon}_\alpha \gamma^5_{pq}\right) C^{-1}. \quad (8.1.117)$$

From the decompositions (8.1.107) and the chirality condition for the supersymmetries, we see that

$$\sigma_3 \epsilon = -\epsilon, \quad \sigma_3 \tilde{\epsilon} = -\tilde{\epsilon} \quad (8.1.118)$$

Therefore both BRST charges ϵ and $\tilde{\epsilon}$ have $U(1)$ charge -1 as expected.

[10] The inverse of charge conjugation matrices C^{-1} is included just because of the convenience of the calculation.

The BRST transformation is given by

$$\delta A_\mu = -2(\bar{\epsilon}\omega^\mu) - 2(\bar{\tilde{\epsilon}}\psi^\mu), \tag{8.1.119}$$

$$\delta\Phi_\mu = -2i(\bar{\epsilon}\psi^\mu) - 2i(\bar{\tilde{\epsilon}}\omega^\mu), \tag{8.1.120}$$

$$\delta\varphi = 2i\bar{\epsilon}\sigma_+\zeta + 2i\bar{\tilde{\epsilon}}\sigma_+\eta, \tag{8.1.121}$$

$$\delta\tilde{\varphi} = 2i\bar{\epsilon}\sigma_-\zeta + 2i\bar{\tilde{\epsilon}}\sigma_-\eta, \tag{8.1.122}$$

$$\delta\eta = iD_\mu\Phi^\mu\tilde{\epsilon} - \frac{1}{2}[\phi^i,\phi^j](\epsilon_{ij}\epsilon), \tag{8.1.123}$$

$$\delta\psi_\mu = -iD_\mu\phi_i(\sigma_i\epsilon) - i[\Phi_\mu,\phi_i](\sigma_i\tilde{\epsilon}), \tag{8.1.124}$$

$$\begin{aligned}\delta\chi_{\mu\nu} = &-F_{\mu\nu}\epsilon + \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\tilde{\epsilon} + 2i\epsilon_{\mu\nu\rho\sigma}D^\rho\Phi^\sigma\epsilon + 2iD_\mu\Phi_\nu\tilde{\epsilon}\\ &+ i[\Phi_\mu,\Phi_\nu]\epsilon - i\epsilon_{\mu\nu\rho\sigma}[\Phi^\rho,\Phi^\sigma]\tilde{\epsilon},\end{aligned} \tag{8.1.125}$$

$$\delta\omega_\mu = -iD_\mu\phi_i(\sigma_i\tilde{\epsilon}) + i[\Phi_\mu,\phi_i](\sigma_i\epsilon), \tag{8.1.126}$$

$$\delta\zeta = -iD_\mu\Phi^\mu\epsilon - \frac{1}{2}(\epsilon_{ij}\tilde{\epsilon})[\phi_i,\phi_j] \tag{8.1.127}$$

where we introduce

$$\sigma_+ := \frac{1}{\sqrt{2}}(\sigma_1 + i\sigma_2), \tag{8.1.128}$$

$$\sigma_- := \frac{1}{\sqrt{2}}(\sigma_1 - i\sigma_2). \tag{8.1.129}$$

Vafa-Witten (VW) twist

VW twist corresponds to the following branching:

$$\begin{aligned}\mathbf{4} &= (\mathbf{1},\mathbf{2})_1 \oplus (\mathbf{1},\mathbf{2})_{-1}\\ \bar{\mathbf{4}} &= (\mathbf{1},\mathbf{2})_{-1} \oplus (\mathbf{1},\mathbf{2})_1.\end{aligned} \tag{8.1.130}$$

1. **fermionic fields**

   Noting that

$$(\mathbf{2},\mathbf{1})_0 \times ((\mathbf{1},\mathbf{2})_{-1} \oplus (\mathbf{1},\mathbf{2})_1) = (\mathbf{2},\mathbf{2})_{-1} \oplus (\mathbf{2},\mathbf{2})_1 \tag{8.1.131}$$

   and

$$(\mathbf{1},\mathbf{2})_0 \times ((\mathbf{1},\mathbf{2})_1 \oplus (\mathbf{1},\mathbf{2})_{-1}) = (\mathbf{1},\mathbf{1})_1 \oplus (\mathbf{1},\mathbf{3})_1 \oplus (\mathbf{1},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{3})_{-1}, \tag{8.1.132}$$

   it turns out that the fermionic fields transform under $SU(2)'_l \times SU(2)'_r \times U(1)$ as

$$(\mathbf{2},\mathbf{2})_{-1} \oplus (\mathbf{2},\mathbf{2})_1 \oplus (\mathbf{1},\mathbf{1})_1 \oplus (\mathbf{1},\mathbf{3})_1 \oplus (\mathbf{1},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{3})_{-1}. \tag{8.1.133}$$

   Therefore VW twist gives rise to two unbroken BRST charges which have the opposite $U(1)$ charge.

2. **bosonic fields**

   The bosonic scalar field $\mathbf{6}_v$ of $SO(6)_R$ is given by the antisymmetric product of $\mathbf{4}$

$$\begin{aligned}\mathbf{6}_v =&(\mathbf{4}\times\mathbf{4})_a\\ =&(((\mathbf{1},\mathbf{2})_1\oplus(\mathbf{1},\mathbf{2})_{-1})\times((\mathbf{1},\mathbf{2})_1\oplus(\mathbf{1},\mathbf{2})_{-1}))_a\\ =&(((\mathbf{1},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{3})_0\oplus(\mathbf{1},\mathbf{1})_2\oplus(\mathbf{1},\mathbf{1})_{-2})\oplus((\mathbf{1},\mathbf{3})_2\oplus(\mathbf{1},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{3})_0\oplus(\mathbf{1},\mathbf{3})_{-2}))_a\\ =&(\mathbf{1},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{3})_0\oplus(\mathbf{1},\mathbf{1})_2\oplus(\mathbf{1},\mathbf{1})_{-2}.\end{aligned}\tag{8.1.134}$$

   Thus the bosonic field content consists of

   - 3 scalar fields : $(\mathbf{1},\mathbf{1})_{-2}\oplus(\mathbf{1},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{1})_2$
   - 2-form : $(\mathbf{1},\mathbf{3})_0$
   - gauge field : $(\mathbf{2},\mathbf{2})_0$

   and the fermionic one is

   - 2 scalar fields : $(\mathbf{1},\mathbf{1})_+\oplus(\mathbf{1},\mathbf{1})_-$
   - 1-form : $(\mathbf{2},\mathbf{2})_-\oplus(\mathbf{2},\mathbf{2})_+$
   - 2 2-from : $(\mathbf{1},\mathbf{3})_-\oplus(\mathbf{1},\mathbf{3})_+$.

Donaldson-Witten (DW) twist

DW twist has the branching

$$\begin{aligned}\mathbf{4}&=(\mathbf{1},\mathbf{2})_0\oplus(\mathbf{1},\mathbf{1})_1\oplus(\mathbf{1},\mathbf{1})_{-1}\\ \bar{\mathbf{4}}&=(\mathbf{1},\mathbf{2})_0\oplus(\mathbf{1},\mathbf{1})_{-1}\oplus(\mathbf{1},\mathbf{1})_1.\end{aligned}\tag{8.1.135}$$

1. **fermionic fields**

   Noticing that

$$(\mathbf{2},\mathbf{1})_0\times((\mathbf{1},\mathbf{2})_0\oplus(\mathbf{1},\mathbf{1})_{-1}\oplus(\mathbf{1},\mathbf{1})_1)=(\mathbf{2},\mathbf{2})_0\oplus(\mathbf{2},\mathbf{1})_{-1}\oplus(\mathbf{2},\mathbf{1})_1\tag{8.1.136}$$

   and

$$(\mathbf{1},\mathbf{2})_0\times((\mathbf{1},\mathbf{2})_0\oplus(\mathbf{1},\mathbf{1})_1\oplus(\mathbf{1},\mathbf{1})_{-1})=(\mathbf{1},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{3})_0\oplus(\mathbf{1},\mathbf{2})_1\oplus(\mathbf{1},\mathbf{2})_{-1},\tag{8.1.137}$$

   one can see that fermionic fields transform under $SU(2)'_l\times SU(2)'_r\times U(1)$ as

$$(\mathbf{2},\mathbf{2})_0\oplus(\mathbf{2},\mathbf{1})_{-1}\oplus(\mathbf{2},\mathbf{1})_1\oplus(\mathbf{1},\mathbf{1})_0\oplus(\mathbf{2},\mathbf{3})_0\oplus(\mathbf{1},\mathbf{2})_1\oplus(\mathbf{1},\mathbf{2})_{-1},\tag{8.1.138}$$

   which implies that DW twist allows one unbroken BRST charge.

2. **bosonic fields**

The bosonic scalar field $\mathbf{6}_v$ of $SO(6)_R$ is given by the antisymmetric product of $\mathbf{4}$

$$\begin{aligned}
\mathbf{6}_v =&(\mathbf{4}\times\mathbf{4})_a\\
=&\left(((\mathbf{1},\mathbf{2})_0\oplus(\mathbf{1},\mathbf{1})_1\oplus(\mathbf{1},\mathbf{1})_{-1})\times((\mathbf{1},\mathbf{2})_0\oplus(\mathbf{1},\mathbf{1})_1\oplus(\mathbf{1},\mathbf{1})_{-1})\right)_a\\
=&\left((2(\mathbf{1},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{2})_1\oplus(\mathbf{1},\mathbf{2})_{-1})\oplus((\mathbf{1},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{3})_0\oplus(\mathbf{1},\mathbf{2})_1\oplus(\mathbf{1},\mathbf{2})_{-1}\oplus(\mathbf{1},\mathbf{1})_2\oplus(\mathbf{1},\mathbf{1})_{-2})\right)_a\\
=&2(\mathbf{1},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{2})_1\oplus(\mathbf{1},\mathbf{2})_{-1}.
\end{aligned}\tag{8.1.139}$$

Thus the bosonic field content consists of

- 2 scalar fields : $(\mathbf{1},\mathbf{1})_0\oplus(\mathbf{1},\mathbf{1})_0$
- 2 spinor fields : $(\mathbf{1},\mathbf{2})_1\oplus(\mathbf{1},\mathbf{2})_{-1}$
- gauge field : $(\mathbf{2},\mathbf{2})_0$

8.1.4 $d=4$, $\mathcal{N}=4$ SYM theories on $C\times\Sigma$

Now we discuss a $d=4$, $\mathcal{N}=4$ SYM theory on $M_4=C\times\Sigma$. We consider the twisting by using the embedding

$$U(1)_C\to U(1)_R. \tag{8.1.140}$$

The assignment of $U(1)$ charges are given in Table 8.6 where $+,-$ signs denote upper and lower components of spinors and right-handed fermions indicated with bars. Under $U(1)'_C\times U(1)'_\Sigma\times SU(2)_2$ the fields transform as

$$\begin{aligned}
\phi &\to \mathbf{1}_{00}\oplus\mathbf{3}_{00}\oplus\mathbf{1}_{20}\oplus\mathbf{1}_{-20} &(8.1.141)\\
\psi &\to \mathbf{2}_{2-}\oplus\mathbf{2}_{0-}\oplus\mathbf{2}_{0+}\oplus\mathbf{2}_{-2+} &(8.1.142)\\
\overline{\psi} &\to \mathbf{2}_{0-}\oplus\mathbf{2}_{-2-}\oplus\mathbf{2}_{2+}\oplus\mathbf{2}_{0+} &(8.1.143)\\
A_\mu &\to \mathbf{1}_{20}\oplus\mathbf{1}_{02}\oplus\mathbf{1}_{0-2}. &(8.1.144)
\end{aligned}$$

The bosonic field content is

- 2 complex scalar fields $\phi,\overline{\phi}$: $\mathbf{1}_{00}\oplus\mathbf{3}_{00}$
- 1-form $\Phi_w,\Phi_{\overline{w}}$: $\mathbf{1}_{20}\oplus\mathbf{1}_{-20}$
- gauge field $A_z,A_{\overline{z}},A_w,A_{\overline{w}}$: $\mathbf{1}_{20}\oplus\mathbf{1}_{-20}\oplus\mathbf{1}_{02}\oplus\mathbf{1}_{0-2}$

and the fermionic field content is

	$U(1)_C$	$U(1)_\Sigma$	$U(1)_R$	$U(1)'_C$	$U(1)'_\Sigma$
ϕ^1	0	0	0	0	0
ϕ^2	0	0	0	0	0
ϕ^3	0	0	0	0	0
ϕ^4	0	0	0	0	0
ϕ^5	0	0	0	2	2
ϕ^6	0	0	0	-2	-2
ψ^1_+	$+$	$-$	$+$	2	$-$
ψ^2_+	$+$	$-$	$+$	2	$-$
ψ^3_+	$+$	$-$	$+$	0	$-$
ψ^4_+	$+$	$-$	$+$	0	$-$
ψ^1_-	$-$	$+$	$+$	0	$+$
ψ^2_-	$-$	$+$	$+$	0	$+$
ψ^3_-	$-$	$+$	$-$	-2	$+$
ψ^4_-	$-$	$+$	$-$	-2	$+$
$\overline{\psi}^1_+$	$-$	$-$	$-$	0	$-$
$\overline{\psi}^2_+$	$-$	$-$	$-$	0	$-$
$\overline{\psi}^3_+$	$-$	$-$	$-$	-2	$-$
$\overline{\psi}^4_+$	$-$	$-$	$-$	-2	$-$
$\overline{\psi}^1_-$	$+$	$+$	$+$	2	$+$
$\overline{\psi}^2_-$	$+$	$+$	$+$	2	$+$
$\overline{\psi}^3_-$	$+$	$-$	$+$	0	$+$
$\overline{\psi}^4_-$	$+$	$-$	$+$	0	$+$

Table 8.6: $U(1)$ charges for VW partial twisting.

- 8 scalar fields $\eta^a_-, \chi^a_+, \overline{\eta}^a_-, \overline{\chi}^a_+ : \mathbf{2}_{0-} \oplus \mathbf{2}_{0+} \oplus \mathbf{2}_{0-} \oplus \mathbf{2}_{0+}$
- 1-form $\lambda^a_{w-}, \chi^a_{\overline{w}+}, \overline{\lambda}^a_{\overline{w}}, \overline{\chi}^a_{w+} : \mathbf{2}_{2-} \oplus \mathbf{2}_{-2+} \oplus \mathbf{2}_{-2-} \oplus \mathbf{2}_{2+}$

where $a = 1, 2$ are the indices of the fundamental representations of the unbroken $SU(2)_2$ symmetry.

One can see that there are eight BRST charges. Four of them transform as spinors with positive chirality on Σ corresponding to χ^a_+ and $\overline{\chi}^a_+$ and the others, that is η^a_- and $\overline{\eta}^a_-$ transform as those with negative chirality on Σ. Therefore we can regard the theory on Σ has $(4, 4)$ supersymmetry.

8.1.5 $d = 3$, $\mathcal{N} = 4$ SYM theories

The global symmetry of the theory is $SO(4) \simeq SU(2)_1 \times SU(2)_2$ R-symmetry. In order to understand the symmetry, it is convenient to construct $d = 3$, $\mathcal{N} = 4$ theories by dimensional reduction from $d = 6$, $\mathcal{N} = 1$ supersymmetric gauge theories. $SU(2)_1$ is the double cover of rotational symmetry $SO(3)$ in the three reduced coordinates and $SU(2)_2$ is the R-symmetry in six-dimensional $\mathcal{N} = 1$ SYM theories [294, 295, 296, 297, 298].

The field content is

- 3 scalar fields ϕ^i
- fermionic field ψ
- gauge fields A_μ

Before topological twisting, fields transform under $SU(2)_E \times SU(2)_1 \times SU(2)_2$ as

$$\phi : (\mathbf{1}, \mathbf{3}, \mathbf{1}) \tag{8.1.145}$$

$$\psi : (\mathbf{2}, \mathbf{2}, \mathbf{2}) \tag{8.1.146}$$

$$A_\mu : (\mathbf{3}, \mathbf{1}, \mathbf{1}). \tag{8.1.147}$$

To perform the fully topological twisting, we pick a homomorphism $\pi : SU(2)_E \to SU(2)_1 \times SU(2)_2$ and replace $SU(2)_E$ by $SU(2)'_E = (1 + \pi)(SU(2)_E) \subset SU(2)_E \times SU(2)_1 \times SU(2)_2$.

We many have two choices of π as

$$\begin{cases} \text{(i)A-twist} & SU(2)_E \to SU(2)_2 \\ \text{(ii)B-twist} & SU(2)_E \to SU(2)_1 \end{cases}.$$

A-twist

After twisting, the fields transform under the new symmetry $SU(2)'_E \times SU(2)_1$ as

$$\phi \to (\mathbf{1}, \mathbf{3}) \tag{8.1.148}$$

$$\psi \to (\mathbf{1}, \mathbf{2}) \oplus (\mathbf{3}, \mathbf{2}) \tag{8.1.149}$$

$$A_\mu \to (\mathbf{3}, \mathbf{1}). \tag{8.1.150}$$

There are two BRST charges and this is just the dimensional reduction of Donaldson-Witten theory (twisted $d = 4$, $\mathcal{N} = 2$ theory). The field content is same as $d = 3$ super BF model [299, 300, 301, 302] associated with the Casson invariant.

B-twist

After twisting, the fields transform under the new symmetry $SU(2)'_E \times SU(2)_2$ as

$$\phi \to (\mathbf{3}, \mathbf{1}) \tag{8.1.151}$$

$$\psi \to (\mathbf{1}, \mathbf{2}) \oplus (\mathbf{3}, \mathbf{2}) \tag{8.1.152}$$

$$A_\mu \to (\mathbf{3}, \mathbf{1}). \tag{8.1.153}$$

In B-twist we also have two BRST charges. As B-twist is related A-twist under the exchange of $SU(2)_1$ and $SU(2)_2$, it is expected that B-twist may be regarded as a mirror discription of the Casson invariant because $d = 3$, $\mathcal{N} = 4$ mirror symmetry has mirror pair under this exchange.

8.1.6 $d = 3$, $\mathcal{N} = 8$ SYM theories

The global symmetry of the theory is $Spin(7)_R$ R-symmetry. If we construct the theories by the dimensional reduction of $d = 10$, $\mathcal{N} = 1$ SYM theories, this is recognized as double cover of rotational symmetry $SO(7)$ in the seven reduced coordinates. The field content is

- 7 scalar fields ϕ^i
- fermionic field ψ
- gauge fields A_μ

Before topological twisting, fields transform under $SU(2)_E \times Spin(7)_R$ as

$$\phi : (\mathbf{1}, \mathbf{7}) \tag{8.1.154}$$

$$\psi : (\mathbf{2}, \mathbf{8}) \tag{8.1.155}$$

$$A_\mu : (\mathbf{3}, \mathbf{1}). \tag{8.1.156}$$

To perform the fully topological twisting, we pick a homomorphism $\pi : SU(2)_E \to Spin(7)_R$ and replace $SU(2)_E$ by $SU(2)'_E = (1+\pi)(SU(2)_E) \subset SU(2)_E \times Spin(7)_R$.

The homomorphism π is determined by the decomposition of $Spin(7)$ under $SU(2)$ and the embedding of $SU(2)$ in $Spin(7)$. Although there are many possible decompositions, we consider the following branchings

$$Spin(7) \supset SU(2)_1 \times SU(2)_2 \times SU(2)_3. \tag{8.1.157}$$

Under (8.1.157), **7** and **8** of $Spin(7)$ decomposed as [303]

$$\mathbf{7} = (\mathbf{2},\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{3}) \tag{8.1.158}$$
$$\mathbf{8} = (\mathbf{2},\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{2},\mathbf{2}). \tag{8.1.159}$$

Then one can consider two different types of embedding with the residual global symmetry [11] $SU(2) \times SU(2)$

$$\begin{aligned} &\text{A-twist} : SU(2)_E \to SU(2)_3 \\ &\text{B-twist} : SU(2)_E \to SU(2)_1. \end{aligned} \tag{8.1.160}$$

A-twist

After twisting the fields transform under $SU(2)'_E \times SU(2)_1 \times SU(2)_2$ as

$$\phi \to (\mathbf{1},\mathbf{2},\mathbf{2}) \oplus (\mathbf{3},\mathbf{1},\mathbf{1}) \tag{8.1.161}$$
$$\psi \to (\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{3},\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{2}) \oplus (\mathbf{3},\mathbf{1},\mathbf{2}). \tag{8.1.162}$$

The bosonic field content consists of

- 4 scalar fields: $(\mathbf{1},\mathbf{2},\mathbf{2})$
- 1-form: $(\mathbf{3},\mathbf{1},\mathbf{1})$
- gauge field: $(\mathbf{3},\mathbf{1},\mathbf{1})$

and fermionic fields are

- 4 scalar fields: $(\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{2})$
- 4 vector fields: $(\mathbf{3},\mathbf{2},\mathbf{1}) \oplus (\mathbf{3},\mathbf{1},\mathbf{2})$.

Thus in A-twist, there are four BRST charges transforming as two $SU(2)$ doublets. It turns out that A-twist topological theories is the dimensional reduction of twisted $d = 4$, $\mathcal{N} = 4$ theories with GL twist and VW twist.

[11] Note that $SU(2)_E \to SU(2)_2$ is same as B twist.

B-twist

After twisting, the fields transform under $SU(2)'_E \times SU(2)_2 \times SU(2)_3$ as

$$\phi \rightarrow (\mathbf{2}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{3}) \tag{8.1.163}$$

$$\psi \rightarrow (\mathbf{1}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{3}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{2}, \mathbf{2}, \mathbf{2}). \tag{8.1.164}$$

The bosonic field content is

- 3 scalar fields: $(\mathbf{1}, \mathbf{1}, \mathbf{3})$
- 2 spinor fields: $(\mathbf{2}, \mathbf{2}, \mathbf{1})$
- gauge fields: $(\mathbf{3}, \mathbf{1}, \mathbf{1})$

and the fermionic field content is

- 2 scalar fields: $(\mathbf{1}, \mathbf{1}, \mathbf{2})$
- 2 vector fields: $(\mathbf{3}, \mathbf{1}, \mathbf{2})$
- 4 spinor fields: $(\mathbf{2}, \mathbf{2}, \mathbf{2})$.

In B-twist, we have two BRST charges transforming as a $SU(2)$ doublet. B-twist topological theories are the dimensional reduction of twisted $d = 4, \mathcal{N} = 4$ theories with DW twist.

8.1.7 $d = 3, \mathcal{N} = 8$ SYM theories on $\mathbb{R} \times \Sigma$

Now consider a three-dimensional $\mathcal{N} = 8$ SYM theories on $M_3 = \mathbb{R} \times \Sigma$.

Before twisting, fields transform under $SO(2)_E \times Spin(7)_R$ as

$$\phi : \mathbf{7}_0 \tag{8.1.165}$$

$$\psi : \mathbf{8}_+ \oplus \mathbf{8}_- \tag{8.1.166}$$

$$A_\mu : \mathbf{1}_{-2} \oplus \mathbf{1}_0 \oplus \mathbf{1}_2. \tag{8.1.167}$$

To determine the homomorphism, we consider the decomposition of $Spin(7)$ under $SO(2)$ as

$$\text{A-twist} : Spin(7) \supset SO(5) \times SO(2) \tag{8.1.168}$$

$$\text{B-twist} : Spin(7) \supset SO(3) \times SO(4) \supset SO(3) \times SO(2)_1 \times SO(2)_2 \tag{8.1.169}$$

$$\text{C-twist} : Spin(7) \supset SO(6) \supset SO(2)_1 \times SO(2)_2 \times SO(2)_3 \tag{8.1.170}$$

A-twist

Under (8.1.168), **7** and **8** of $Spin(7)_R$ decomposed as

$$\mathbf{7} = \mathbf{5}_0 \oplus \mathbf{1}_{-2} \oplus \mathbf{1}_2 \tag{8.1.171}$$

$$\mathbf{8} = \mathbf{4}_+ \oplus \overline{\mathbf{4}}_-. \tag{8.1.172}$$

Then after twisting, fields transform under $SO(2)_E \times SO(5)_R$ as

$$\mathbf{7}_0 \to \mathbf{5}_0 \oplus \mathbf{1}_{-2} \oplus \mathbf{1}_2 \tag{8.1.173}$$

$$\mathbf{8}_+ \oplus \mathbf{8}_- \to \mathbf{4}_2 \oplus \overline{\mathbf{4}}_0 \oplus \mathbf{4}_0 \oplus \overline{\mathbf{4}}_{-2} \tag{8.1.174}$$

$$\mathbf{1}_{-2} \oplus \mathbf{1}_0 \oplus \mathbf{1}_2 \to \mathbf{1}_{-2} \oplus \mathbf{1}_0 \oplus \mathbf{1}_2. \tag{8.1.175}$$

Thus there are eight BRST charges in A-twisted $d = 3$, $\mathcal{N} = 8$ SYM theory on $\mathbb{R} \times \Sigma$.

B-twist

Under (8.1.169), **7** and **8** of $Spin(7)_R$ decomposed as

$$\mathbf{7} = \mathbf{3}_{00} \oplus \mathbf{1}_{0-2} \oplus \mathbf{1}_{02} \oplus \mathbf{1}_{20} \oplus \mathbf{1}_{-20} \tag{8.1.176}$$

$$\mathbf{8} = \mathbf{2}_{++} \oplus \mathbf{2}_{+-} \oplus \mathbf{2}_{-+} \oplus \mathbf{2}_{--}. \tag{8.1.177}$$

We normalize $SO(2)_1, SO(2)_2$ charges by dividing by two and simply take a sum of all of the charges including the original rotational $SO(2)_E$ charges. Performing this twisting, fields transform under $SO(2)'_E \times SO(3)_R$ as

$$\mathbf{7}_0 \to \mathbf{3}_0 \oplus 2(\mathbf{1}_+) \oplus 2(\mathbf{1}_-) \tag{8.1.178}$$

$$\mathbf{8}_+ \oplus \mathbf{8}_- \to \mathbf{2}_2 \oplus 2(\mathbf{2}_+) \oplus 2(\mathbf{2}_0) \oplus 2(\mathbf{2}_-) \oplus \mathbf{2}_{-2} \tag{8.1.179}$$

$$\mathbf{1}_{-2} \oplus \mathbf{1}_0 \oplus \mathbf{1}_2 \to \mathbf{1}_{-2} \oplus \mathbf{1}_0 \oplus \mathbf{1}_2. \tag{8.1.180}$$

Thus there are four BRST charges in B-twisted $d = 3$, $\mathcal{N} = 8$ SYM theory on $\mathbb{R} \times \Sigma$.

C-twist

Under (8.1.170), **7** and **8** of $Spin(7)_R$ decomposed as

$$\mathbf{7} = \mathbf{1}_{200} \oplus \mathbf{1}_{-200} \oplus \mathbf{1}_{020} \oplus \mathbf{1}_{0-20} \oplus \mathbf{1}_{002} \oplus \mathbf{1}_{00-2} \oplus \mathbf{1}_{000} \tag{8.1.181}$$

$$\mathbf{8} = \mathbf{1}_{+++} \oplus \mathbf{1}_{++-} \oplus \mathbf{1}_{+-+} \oplus \mathbf{1}_{+--} \oplus \mathbf{1}_{-++} \oplus \mathbf{1}_{-+-} \oplus \mathbf{1}_{--+} \oplus \mathbf{1}_{---}. \tag{8.1.182}$$

We normalize $SO(2)_1, SO(2)_2, SO(2)_3$ charges by dividing by three and simply take a sum of all of the charges including the original rotational $SO(2)_E$ charges. Performing this twisting, fields transform under $SO(2)'_E$ as

$$\mathbf{7}_0 \to 3(\mathbf{1}_{\frac{2}{3}}) \oplus 3(\mathbf{1}_{-\frac{2}{3}}) \oplus \mathbf{1}_0 \tag{8.1.183}$$

$$\mathbf{8}_+ \oplus \mathbf{8}_- \to 2(\mathbf{1}_0) \oplus 2(\mathbf{1}_{\frac{2}{3}}) \oplus 2(\mathbf{1}_{-\frac{2}{3}}) \oplus \mathbf{1}_{\frac{4}{3}} \oplus \mathbf{1}_{-\frac{4}{3}} \tag{8.1.184}$$

$$\mathbf{1}_{-2} \oplus \mathbf{1}_0 \oplus \mathbf{1}_2 \to \mathbf{1}_{-2} \oplus \mathbf{1}_0 \oplus \mathbf{1}_2. \tag{8.1.185}$$

Thus there are two BRST charges in B-twisted $d = 3$, $\mathcal{N} = 8$ SYM theory on $\mathbb{R} \times \Sigma$.

8.1.8 $d = 2$, $\mathcal{N} = 8$ SYM theories

The global symmetry of the theory is $Spin(8)_R$ R-symmetry. The field content is

- 8 scalar fields ϕ^i
- fermionic fields ψ
- gauge field A_μ.

Before topological twisting, fields transform under $SO(2)_E \times Spin(8)_R$ as

$$\phi : \mathbf{8}_{v0} \tag{8.1.186}$$

$$\psi : \mathbf{8}_{c+} \oplus \mathbf{8}_{s-} \tag{8.1.187}$$

$$A_\mu : \mathbf{1}_{-2} \oplus \mathbf{1}_2. \tag{8.1.188}$$

To perform the topological twisting, we pick a homomorphism $\pi : SO(2)_E \to Spin(8)_R$ and replace $SO(2)_E$ by $SO(2)'_E = (1 + \pi)(SO(2)_E) \subset SO(2)_E \times Spin(8)_R$.

The homomorphism π is determined by the decomposition of $Spin(8)$ under $SO(2)$ and the embedding of $SO(2)$ in $Spin(8)$. Although there are many possible decompositions, we consider the following two types of branching

$$\text{A-twist}: \ Spin(8)_R \supset SO(6)_R \times SO(2)_1 \tag{8.1.189}$$

$$\begin{aligned}\text{B-twist}: \ Spin(8)_R &\supset SO(4)_1 \times SO(4)_2 \supset SO(4)_1 \times SU(2)_1 \times SU(2)_2 \\ &\supset SO(4)_1 \times SU(2)_1 \times SO(2)_2\end{aligned} \tag{8.1.190}$$

A-twist

Under (8.1.189), $\mathbf{8}_v$, $\mathbf{8}_s$ and $\mathbf{8}_c$ of $Spin(8)_R$ decomposed as

$$\mathbf{8}_v = \mathbf{6}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2} \tag{8.1.191}$$

$$\mathbf{8}_s = \mathbf{4}_+ \oplus \overline{\mathbf{4}}_- \tag{8.1.192}$$

$$\mathbf{8}_c = \mathbf{4}_- \oplus \overline{\mathbf{4}}_+. \tag{8.1.193}$$

Then the choice of π amounts to $SO(2)'_E \to SO(2)_1$ and the fields transform under $SO(2)_E \times SO(6)_R$ as

$$\mathbf{8}_v \to \mathbf{6}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2} \tag{8.1.194}$$
$$\mathbf{8}_s \to \mathbf{4}_0 \oplus \overline{\mathbf{4}}_{-2} \tag{8.1.195}$$
$$\mathbf{8}_c \to \mathbf{4}_0 \oplus \overline{\mathbf{4}}_2 \tag{8.1.196}$$

There are eight supercharges transforming $\mathbf{4}$ of $SO(6)_R$. This is just the dimensional reduction of A-twisted $d = 3$, $\mathcal{N} = 8$ SYM theory.

To see this, let us consider the further decomposition

$$SO(6)_R \simeq SU(4)_R \supset SU(2)_1 \times SU(2)_2 \times U(1). \tag{8.1.197}$$

Under (8.1.197), $\mathbf{6}$ and $\mathbf{4}$ decomposed as [12]

$$\mathbf{6} = (\mathbf{2},\mathbf{2})_0 \oplus (\mathbf{1},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{1})_{-2} \tag{8.1.198}$$
$$\mathbf{4} = (\mathbf{2},\mathbf{1})_+ \oplus (\mathbf{1},\mathbf{2})_- \tag{8.1.199}$$
$$\overline{\mathbf{4}} = (\mathbf{2},\mathbf{1})_- \oplus (\mathbf{1},\mathbf{2})_+. \tag{8.1.200}$$

Then the fields transform under $SO(2)'_E \times SU(2)_1 \times SU(2)_2$ as

$$\mathbf{8}_v \to (\mathbf{2},\mathbf{2})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{1})_{-2} \tag{8.1.201}$$
$$\mathbf{8}_s \to (\mathbf{2},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{2})_0 \oplus (\mathbf{2},\mathbf{1})_{-2} \oplus (\mathbf{1},\mathbf{2})_{-2} \tag{8.1.202}$$
$$\mathbf{8}_c \to (\mathbf{2},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{2})_0 \oplus (\mathbf{2},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{2})_2 \tag{8.1.203}$$

Thus bosonic field content is

- 6 scalar fields: $2(\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{2},\mathbf{2})_0$
- 1-form: $(\mathbf{1},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{1})_{-2}$

and the fermionic field content is

- 8 scalar fields: $2(\mathbf{2},\mathbf{1})_0 \oplus 2(\mathbf{1},\mathbf{2})_0$
- 4 1-forms: $(\mathbf{2},\mathbf{1})_2 \oplus (\mathbf{2},\mathbf{1})_{-2} \oplus (\mathbf{1},\mathbf{2})_2 \oplus (\mathbf{1},\mathbf{2})_{-2}$.

The two bosonic scalars $2(\mathbf{1},\mathbf{1})_0$ correspond to the third components of the gauge field and 1-form in three dimensions. Also the four fermionic scalars $(\mathbf{2},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{2})_0$ are the third components of the 1-form in three dimensions.

[12] As in (8.1.100), we have other possible decompositions. (8.1.197) is same as GL twist.

B-twist

Under (8.1.190), $\mathbf{8}_v$, $\mathbf{8}_s$ and $\mathbf{8}_c$ of $Spin(8)$ decomposed as

$$\begin{aligned}
\mathbf{8}_v =&(\mathbf{4},\mathbf{1}) \oplus (\mathbf{1},\mathbf{4}) \\
=&(\mathbf{4},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2},\mathbf{2}) = (\mathbf{4},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{2})_+ \oplus (\mathbf{1},\mathbf{2})_- \qquad (8.1.204)\\
\mathbf{8}_s =&(\mathbf{2},\mathbf{2}) \oplus (\mathbf{2}',\mathbf{2}') \\
=&(\mathbf{2},\mathbf{2},\mathbf{1}) \oplus (\mathbf{2}',\mathbf{1},\mathbf{2}) = (\mathbf{2},\mathbf{2})_0 \oplus (\mathbf{2}',\mathbf{1})_+ \oplus (\mathbf{2}',\mathbf{1})_- \qquad (8.1.205)\\
\mathbf{8}_c =&(\mathbf{2},\mathbf{2}') \oplus (\mathbf{2}',\mathbf{2}) \\
=&(\mathbf{2},\mathbf{1},\mathbf{2}) \oplus (\mathbf{2}',\mathbf{2},\mathbf{1}) = (\mathbf{2},\mathbf{1})_+ \oplus (\mathbf{2},\mathbf{1})_- \oplus (\mathbf{2}',\mathbf{2})_0. \qquad (8.1.206)
\end{aligned}$$

Then the choice of π amounts to $SO(2)_E \to SO(2)_2$ and the fields transform under $SO(2)'_E \times SO(4)_R \times SO(2)_1$ as

$$\mathbf{8}_v \to (\mathbf{4},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{2})_+ \oplus (\mathbf{1},\mathbf{2})_- \qquad (8.1.207)$$

$$\mathbf{8}_s \to (\mathbf{2},\mathbf{2})_- \oplus (\mathbf{2}',\mathbf{1})_0 \oplus (\mathbf{2}',\mathbf{1})_{-2} \qquad (8.1.208)$$

$$\mathbf{8}_c \to (\mathbf{2},\mathbf{1})_2 \oplus (\mathbf{2},\mathbf{1})_0 \oplus (\mathbf{2}',\mathbf{2})_+. \qquad (8.1.209)$$

The bosonic field contnt is

- 4 scalar fields: $(\mathbf{4},\mathbf{1})_0$
- 4 spinors: $(\mathbf{1},\mathbf{2})_+ \oplus (\mathbf{1},\mathbf{2})_-$

and the fermionic field content is

- 4 scalar fields: $(\mathbf{2}',\mathbf{1})_0 \oplus (\mathbf{2},\mathbf{1})_0$
- 8 spinors: $(\mathbf{2},\mathbf{2})_- \oplus (\mathbf{2}',\mathbf{2})_+$
- 2 1-forms: $(\mathbf{2}',\mathbf{1})_{-2} \oplus (\mathbf{2},\mathbf{1})_2$.

There are four BRST charges in B-twist.

8.1.9 $d = 3$, $\mathcal{N} = 4$ Chern-Simons matter theories

Gaiotto and Witten gave a general prescription for coupling Chern-Simons theory to hypermultiplets, which allows for a new large class of three-dimensional $\mathcal{N} = 4$ supersymmetric gauge theories [304]. Gaiotto-Witten theory can be regarded as a three-dimensional $\mathcal{N} = 4$ gauged sigma-model with a hyperkähler target space X.

The field content is

- gauge field A^m_μ

- hypermultiplet boson q^A_α
- hypermultiplet fermion $\psi^A_{\dot\alpha}$
- twisted hypermultiplet $\tilde{q}^A_{\dot\alpha}$
- twisted hypermultiplet $\tilde{\psi}^A_\alpha$

where m is the adjoint indices raised by invariant quadratic form k^{mn} of the gauge group. The gauge group is a subgroup of $Sp(2n)$ and we denote the anti-Hermitian generators of the gauge group by $(t^m)^A_B(A, B, \cdots = 1, \cdots, 2n)$, which satisfy

$$[t^m, t^n] = f^{mn}{}_p t^p, \quad t_{AB} = \omega_{AC} t^C_B \tag{8.1.210}$$

where ω_{AB} are the anti-symmetric invariant tensor.

The hyper-multiplet fields satisfy the reality condition

$$(q^A_\alpha)^* = \epsilon^{\alpha\beta}\omega_{AB}q^B_\beta, \tag{8.1.211}$$
$$(\psi^A_{\dot\alpha})^* = \epsilon^{\dot\alpha\dot\beta}\omega_{AB}\psi^B_{\dot\beta} \tag{8.1.212}$$

where $(\alpha, \beta; \dot\alpha, \dot\beta)$ are the indices of $SU(2) \times SU(2)$ R-symmetry.

For $\mathcal{N} = 4$ supersymmetry, t^m_{AB} satisfy the fundamental identity

$$k_{mn} t^m_{(AB} t^n_{C)D} = 0 \tag{8.1.213}$$

where $A, B, C, \cdots$ are symmetrized. (8.1.213) is nothing but the Jacobi identity for three fermionic generators of a Lie superalgebra

$$[M^m, M^n] = f^{mn}{}_p M^p, \tag{8.1.214}$$
$$[M^m, Q_A] = Q_B(t^m)^B_A, \tag{8.1.215}$$
$$\{Q_A, Q_B\} = t^m_{AB} M_m. \tag{8.1.216}$$

Lagrangian of the Gaiotto-Witten theory is given by [13]

$$\begin{aligned}\mathcal{L} =&\frac{1}{2}\epsilon^{\mu\nu\lambda}\left(k_{mn}A^m_\mu\partial_\nu A^n_\lambda + \frac{1}{3}f_{mnp}A^m_\mu A^n_\nu A^p_\lambda\right)\\ &+ \omega_{AB}\left(-\epsilon^{\alpha\beta}Dq^A_\alpha Dq^B_\beta + i\epsilon^{\dot\alpha\dot\beta}\psi^A_{\dot\alpha}D_\mu\gamma^\mu\psi^B_{\dot\beta}\right)\\ &- ik_{mn}\epsilon^{\alpha\beta}\epsilon^{\dot\gamma\dot\delta}j^m_{\alpha\dot\gamma}j^n_{\beta\dot\delta} - \frac{1}{12}f_{mnp}(\mu^m)^\alpha_\beta(\mu^n)^\beta_\gamma(\mu^p)^\gamma_\alpha\end{aligned} \tag{8.1.217}$$

[13] An overall coefficients of the Lagrangian should satisfy an integrality condition to make the quantum theory well-defined. But here we suppress them.

where

$$\mu^m_{\alpha\beta} := t^m_{AB} q^A_\alpha q^B_\beta, \tag{8.1.218}$$

$$j^m_{\alpha\dot{B}} := t^m_{AB} q^A_\alpha \psi^B_{\dot{\beta}}, \tag{8.1.219}$$

$$\rho^m_{\dot{\alpha}\dot{\beta}} := t^m_{AB} \psi^A_{\dot{\alpha}} \psi^B_{\dot{\beta}} \tag{8.1.220}$$

are the momentum map multiplet.

The Euclidean Lagrangian is

$$\begin{aligned}\mathcal{L} = &- \frac{i}{2}\epsilon^{\mu\nu\lambda}\left(k_{mn}A^m_\mu \partial_\nu A^n_\lambda + \frac{1}{3}f_{mnp}A^m_\mu A^n_\nu A^p_\lambda\right) \\ &- \omega_{AB}\left(-\epsilon^{\alpha\beta}Dq^A_\alpha Dq^B_\beta + i\epsilon^{\dot{\alpha}\dot{\beta}}\psi^A_{\dot{\alpha}} D_\mu \gamma^\mu \psi^B_{\dot{\beta}}\right) \\ &+ ik_{mn}\epsilon^{\alpha\beta}\epsilon^{\dot{\gamma}\dot{\delta}} j^m_{\alpha\dot{\gamma}} j^n_{\beta\dot{\delta}} + \frac{1}{12}f_{mnp}(\mu^m)^\alpha_\beta (\mu^n)^\beta_\gamma (\mu^p)^\gamma_\alpha.\end{aligned} \tag{8.1.221}$$

Note that it differs from the Lorentzian Lagrangian (8.1.217) by the factor $(-i)$ for Chern-Simons term and an overall sign for the matter terms. Now the fermionic fields do not obey the reality conditions.

The supersymmetry transformations are

$$\delta q^A_\alpha = i\epsilon^{\dot{\alpha}}_\alpha \psi^A_{\dot{\alpha}}, \tag{8.1.222}$$

$$\delta\psi^A_{\dot{\alpha}} = \left(D^\mu \gamma_\mu q^A_\alpha + \frac{1}{3}k_{mn}(t^m)^A_B q^B_\beta (\mu^n)^\beta_\alpha\right)\epsilon^\alpha_{\dot{\alpha}}, \tag{8.1.223}$$

$$\delta A^m_\mu = i\epsilon^{\alpha\dot{\alpha}}\gamma_\mu j^m_{\alpha\dot{\alpha}}. \tag{8.1.224}$$

The supersymmetry parameter ϵ transforms as $(\mathbf{2},\mathbf{2})$ in $SU(2)\times SU(2)$ R-symmetry and satisfies the reality condition

$$(\epsilon^{\dot{\alpha}}_\alpha)^* = -\epsilon^{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\beta}}_\beta. \tag{8.1.225}$$

The supersymmetry transformations are same as in the Lorentzian case.

Furthermore we can add twisted hyper-multiplets $(\tilde{q}^A_{\dot{\alpha}}, \tilde{\psi}^A_\alpha)$ to Gaiotto-Witten theory. This is regarded as a non-linear sigma model [304]. The Lagrangian is

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\epsilon^{\mu\nu\lambda}\left(k_{mn}A^m_\mu \partial_\nu A^n_\lambda + \frac{1}{3}f_{mnp}A^m_\mu A^n_\nu A^p_\lambda\right) \\ &+ \omega_{AB}\left(-\epsilon^{\alpha\beta}Dq^A_\alpha Dq^B_\beta + i\epsilon^{\dot{\alpha}\dot{\beta}}\psi^A_{\dot{\alpha}} D_\mu\gamma^\mu\psi^B_{\dot{\beta}}\right) + \omega_{AB}\left(-\epsilon^{\dot{\alpha}\dot{\beta}}D\tilde{q}^B_{\dot{\alpha}} + i\epsilon^{\alpha\beta}\tilde{\psi}^A_\alpha D_\mu\gamma^\mu\tilde{\psi}^B_\beta\right) \\ &- ik_{mn}\left(\epsilon^{\alpha\beta}\epsilon^{\dot{\gamma}\dot{\delta}} j^m_{\alpha\dot{\gamma}} j^n_{\beta\dot{\delta}} + \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\gamma\delta}\tilde{j}^m_{\dot{\alpha}\gamma}\tilde{j}^n_{\dot{\beta}\delta} + 4\epsilon^{\alpha\gamma}\epsilon^{\dot{\beta}\dot{\delta}} j^m_{\alpha\dot{\beta}}\tilde{j}^n_{\dot{\delta}\gamma} - \epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon^{\dot{\beta}\dot{\delta}}\tilde{\mu}^m_{\dot{\alpha}\dot{\beta}}\rho^n_{\dot{\gamma}\dot{\delta}} - \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\mu^m_{\alpha\beta}\tilde{\rho}_{\gamma\delta}\right) \\ &- \frac{1}{12}f_{mnp}(\mu^m)^\alpha_\beta(\mu^n)^\beta_\gamma(\mu^p)^\gamma_\alpha + (\tilde{\mu}^m)^{\dot{\alpha}}_{\dot{\beta}}(\tilde{\mu}^n)^{\dot{\beta}}_{\dot{\gamma}}(\tilde{\mu}^p)^{\dot{\gamma}}_{\dot{\alpha}} \\ &- \frac{1}{2}\tilde{\mu}^{mn}(\mu_m)^\alpha_\beta(\mu_n)^\beta_\alpha - \frac{1}{2}\mu^{mn}(\tilde{\mu}_m)^{\dot{\alpha}}_{\dot{\beta}}(\tilde{\mu}_n)^{\dot{\beta}}_{\dot{\alpha}}\end{aligned} \tag{8.1.226}$$

where the twisted moment map is defined by

$$\mu^{mn} = \epsilon^{\alpha\beta}(t^m t^n)_{AB} q^A_\alpha q^B_\beta, \tag{8.1.227}$$

$$\tilde{\mu}^{mn} = \epsilon^{\dot{\alpha}\dot{\beta}}(\bar{t}^m \bar{t}^n)_{AB} \tilde{q}^A_{\dot{\alpha}} \tilde{q}^B_{\dot{\beta}}. \tag{8.1.228}$$

The supersymmetry transformation is

$$\delta q^A_\alpha = i\epsilon^{\dot{\alpha}}_\alpha \psi^A_{\dot{\alpha}}, \tag{8.1.229}$$

$$\delta \psi^A_{\dot{\alpha}} = \left(D_\mu \gamma^\mu q^A_\alpha + \frac{1}{3}(t_m)^A_B q^B_\beta (\mu^m)^\beta_\alpha \right) \epsilon^\alpha_{\dot{\alpha}}, \tag{8.1.230}$$

$$\delta \tilde{\psi}^A_\alpha = \left(D_\mu \gamma^\mu q^A_{\dot{\alpha}} + \frac{1}{3}(\tilde{t}_m) \tilde{q}^B_{\dot{\beta}} (\tilde{\mu}^m)^{\dot{\beta}}_{\dot{\alpha}} \right) \epsilon^{\dot{\alpha}}_\alpha - (\tilde{t}_m)^A_B \tilde{q}^B_{\dot{\beta}} (\mu^m)^\beta_\alpha \epsilon^{\beta\dot{\beta}}, \tag{8.1.231}$$

$$\delta A^m_\mu = i\epsilon^{\alpha\dot{\alpha}} \gamma_\mu (j^m_{\alpha\dot{\alpha}} + \tilde{j}^m_{\dot{\alpha}\alpha}). \tag{8.1.232}$$

The topological twisting for Gaiotto-Witten theory was discussed in [305, 306].

1. flat target space X

   If the target X is flat, Gaiotto-Witten theory has $SU(2) \times SU(2)$ R-symmetry. The topologically twisted theory is equivalent to the pure Chern-Simons theory whose gauge group is a supergroup [305]. In other words, the topologically twisted Gaiotto-Witten theory is obtained from the supergroup Chern-Simons theory by gauge fixing the odd part of the supergroup and the even part of the supergroup gives rise to gauge group G.

2. general target space X

   For general target X, Gaiotto-Witten theory has $SU(2)$ R-symmetry. The topologically twisted Gaiotto-Witten theory can be interpreted as a gauged Rozansky-Witten theory [307], that is a hybrid of Chern-Simons and Rozansky-Witten theory [305]. It is associated to a quadruple:

   (a) G: a compact Lie group

   (b) κ: invariant metric on the Lie algebra

   (c) X: hyperkähler manifold with a tri-holomorphic action of G

   (d) I: complex structure on X such that the complex moment map with respect to the complex symplectic form Ω_I is isotropic with respect to κ

Before twisting, fields and supercharges transform under $SU(2)_E \times SU(2)_1 \times$

$SU(2)_2$ as

$$q : (\mathbf{1}, \mathbf{2}, \mathbf{1}) \tag{8.1.233}$$

$$\psi : (\mathbf{2}, \mathbf{1}, \mathbf{2}) \tag{8.1.234}$$

$$Q : (\mathbf{2}, \mathbf{2}, \mathbf{2}). \tag{8.1.235}$$

We may also have fields of twisted hypermultiplet, which transform as

$$\tilde{q} : (\mathbf{1}, \mathbf{1}, \mathbf{2}) \tag{8.1.236}$$

$$\tilde{\psi} : (\mathbf{2}, \mathbf{2}, \mathbf{1}). \tag{8.1.237}$$

Depending on which $SU(2)$ factor we use, we may think two types of twisting

$$\text{A-twist} : SU(2)_E \to SU(2)_2 \tag{8.1.238}$$

$$\text{B-twist} : SU(2)_E \to SU(2)_1. \tag{8.1.239}$$

However, if both hypermultiplet and twisted hypermultiplet are present, A-twist and B-twist are the same. We call it AB-twist.

A-twist

After A-twisting, the fields and supercharges transform under $SU(2)'_E \times SU(2)$ as

$$q \to (\mathbf{1}, \mathbf{2}) \tag{8.1.240}$$

$$\psi \to (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{1}) \tag{8.1.241}$$

$$Q \to (\mathbf{1}, \mathbf{2}) \oplus (\mathbf{3}, \mathbf{1}). \tag{8.1.242}$$

Thus in the bosonic field content we have

- 2 scalar fields: $(\mathbf{1}, \mathbf{2})$

and in the fermionic field content we include

- scalar field $(\mathbf{1}, \mathbf{1})$
- 1-form $(\mathbf{3}, \mathbf{1})$.

There are two BRST charges in A-twisted theory.

1. **bosonic fields**

   In A-twist all of the hypermultiplet scalars remain scalars.

2. **fermionic fields**

   We decompose the fermionic fields under $SU(2)_E \times SU(2)_1 \times SU(2)_2$ as

$$(\psi_{\alpha'})^A_{\dot\alpha} = \frac{1}{\sqrt{2}}\left(\psi^A \mathbb{I}_{\alpha'\dot\alpha} + \Psi^A_i (\sigma_i)_{\alpha'\dot\alpha}\right)\sigma_2^{-1} \tag{8.1.243}$$

   where $\alpha', \dot\alpha$ are the indices of $SU(2)_E, SU(2)_2$ respectively. $A = 1, 2, \cdots, 2n$ is again the index of $Sp(2n)$.

3. **supersymmetries**

   For the supersymmetries we expand as

$$(\epsilon^{\alpha'})^{\dot\alpha}_{\alpha} = \frac{1}{\sqrt{2}}\left(\epsilon_\alpha \mathbb{I}^{\alpha'\dot\alpha} + \epsilon_{i\alpha}(\sigma_i)^{\alpha'\dot\alpha}\right)(\sigma_2)^{-1}. \tag{8.1.244}$$

B-twist

After B-twisting, the fields tranform under $SU(2)'_E \times SU(2)$ as

$$q \to (\mathbf{2}, \mathbf{1}) \tag{8.1.245}$$
$$\psi \to (\mathbf{2}, \mathbf{2}) \tag{8.1.246}$$
$$Q \to (\mathbf{1}, \mathbf{2}) \oplus (\mathbf{3}, \mathbf{2}). \tag{8.1.247}$$

Thus the bosonic field content is

- spinor field: $(\mathbf{2}, \mathbf{1})$

and the fermionic field content is

- 2 spinor fields $(\mathbf{2}, \mathbf{2})$.

We have two BRST charges in B-twisted theory.

AB-twist

After twisting, the fields transform under $SU(2)'_E \times SU(2)_2$ as

$$q \to (\mathbf{1}, \mathbf{2}) \tag{8.1.248}$$
$$\psi \to (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{1}) \tag{8.1.249}$$
$$\tilde{q} \to (\mathbf{2}, \mathbf{1}) \tag{8.1.250}$$
$$\tilde{\psi} \to (\mathbf{2}, \mathbf{2}) \tag{8.1.251}$$
$$Q \to (\mathbf{1}, \mathbf{2}) \oplus (\mathbf{3}, \mathbf{2}). \tag{8.1.252}$$

Thus the bosonic field content is

- 2 scalar fields : $(\mathbf{1}, \mathbf{2})$
- spinor field: $(\mathbf{2}, \mathbf{1})$

and the fermionic field content is

- scalar fields : $(\mathbf{1}, \mathbf{1})$
- 1-form : $(\mathbf{3}, \mathbf{1})$
- 2 spinor fields: $(\mathbf{2}, \mathbf{2})$.

Again we have two BRST charges in AB-twisted theory.

8.1.10 $d = 3$, $\mathcal{N} = 5$ Chern-Simons matter theories

Three-dimensional $\mathcal{N} \geq 5$ theories can be understood in the Gaiotto-Witten framework by adding twisted hypermultiplets [308]. The target spaces of $\mathcal{N} \geq 5$ theories are only flat spaces and their orbifolds.

We may consider the decomposition of $SO(5)_R$ R-symmetry under $SO(3) \simeq SU(2)$ as

$$SO(5) \supset SO(2) \times SO(3) \tag{8.1.253}$$

$$SO(5) \supset SO(4) \supset SU(2)_1 \times SU(2)_2. \tag{8.1.254}$$

Under (8.1.253), $\mathbf{5}$ of $SO(5)_R$ decomposed as

$$\mathbf{5} = \mathbf{3}_0 \oplus \mathbf{1}_{-2} \oplus \mathbf{1}_2. \tag{8.1.255}$$

Noting that

$$\mathbf{2}_0 \times (\mathbf{3}_0 \oplus \mathbf{1}_{-2} \oplus \mathbf{1}_2) = \mathbf{2}_0 \oplus \mathbf{4}_0 \oplus \mathbf{2}_2 \oplus \mathbf{2}_{-2}, \tag{8.1.256}$$

we see that there are no BRST charges.

On the other hand, under (8.1.254), $\mathbf{5}$ and $\mathbf{4}$ of $SO(5)_R$ decomposed as

$$\mathbf{5} = \mathbf{4} + \mathbf{1} = (\mathbf{2}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1}) \tag{8.1.257}$$

$$\mathbf{4} = \mathbf{4} = (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}). \tag{8.1.258}$$

This is nothing but the AB-twist in $d = 3$, $\mathcal{N} = 4$ Chern-Simons matter theory.

8.1.11 $d = 3$, $\mathcal{N} = 6$ Chern-Simons matter theories

We may consider the decomposition of $SO(6)_R$ R-symmetry under $SO(3) \simeq SU(2)$ as

$$SO(6) \supset SO(3) \times SO(3) \tag{8.1.259}$$

$$SO(6) \supset SO(2) \times SO(4) \supset SO(2) \times SU(2)_1 \times SU(2)_2. \tag{8.1.260}$$

Under (8.1.259), **5** of $SO(5)_R$ decomposed as

$$\mathbf{6} = (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}). \tag{8.1.261}$$

Noting that

$$\mathbf{2} \times \mathbf{3} = \mathbf{2} \oplus \mathbf{4} \oplus \mathbf{2} \oplus \mathbf{2}, \tag{8.1.262}$$

we see that there are no BRST charges.

On the other hand, under (8.1.260), **6** of $SO(6)_R$ decomposed as

$$\mathbf{6} = \mathbf{4}_0 \oplus \mathbf{1}_{-2} \oplus \mathbf{1}_2 = (\mathbf{2}, \mathbf{2})_0 \oplus (\mathbf{1}, \mathbf{1})_{-2} \oplus (\mathbf{1}, \mathbf{1})_2. \tag{8.1.263}$$

As seen from the appearance of $(\mathbf{2}, \mathbf{2})$, this is nothing but the AB-twist in $d = 3$, $\mathcal{N} = 4$ Chern-Simons matter theory.

8.1.12 $d = 3$, $\mathcal{N} = 8$ Chern-Simons matter theories

To perfom the topological twisting, we put the BLG theory on a three-dimensional Euclidean space. The fermionic fields and supersymmetry parameters are defined as eleven-dimensional fermions and their conjugate are given by

$$\overline{\Psi} := \Psi^T \mathcal{C} \tag{8.1.264}$$

where $\mathcal{C}$ is a eleven-dimensional matrix satisfying

$$\mathcal{C}^T = -\mathcal{C}, \quad \mathcal{C}\Gamma^M \mathcal{C}^{-1} = -(\Gamma^M)^T. \tag{8.1.265}$$

Gamma matrix Γ^M is the representation of eleven-dimensional Clifford algebra

$$\left\{\Gamma^M, \Gamma^N\right\} = 2g^{MN}, \quad \Gamma^{11} := i\Gamma^{12\cdots 10}. \tag{8.1.266}$$

Γ^M can be decomposed under $SO(11) \supset SO(3) \times SO(8)$ as

$$\begin{cases} \Gamma^i = \sigma_i \otimes \tilde{\Gamma}^9 \\ \Gamma^{I+3} = \mathbb{I} \otimes \tilde{\Gamma}^I \end{cases} \tag{8.1.267}$$

where $\tilde{\Gamma}^9 := \tilde{\Gamma}^{1\cdots 8}$. Note that

$$\Gamma^{123} = i\Gamma^{45\cdots 11} = i(\mathbb{I} \otimes \tilde{\Gamma}^9). \tag{8.1.268}$$

The fermionic fields Ψ are $\mathbf{8}_c$ of $SO(8)_R$, so they satisfy the chirality condition

$$\Gamma^{45\cdots 11}\Psi = -\Psi, \tag{8.1.269}$$

$$\Gamma^{123}\Psi = -i\Psi. \tag{8.1.270}$$

The Euclidean BLG Lagrangian is given by

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}(D_\mu X^I, D^\mu X^I) - \frac{i}{2}(\overline{\Psi}, \Gamma^\mu D_\mu \Psi) - \frac{i}{4}\left(\overline{\Psi}\Gamma^{IJ}[X^I, X^J, \Psi]\right) \\ &+ \frac{1}{12}\left([X^I, X^J, X^K], [X^I, X^J, X^K]\right) - \frac{i}{2}\epsilon^{\mu\nu\lambda}\left[\mathrm{Tr}\left(A_{\mu ab}\partial_\nu \tilde{A}^{ab}_\lambda\right) + \frac{2}{3}\mathrm{Tr}\left(A_{\mu ab}\tilde{A}^a_{\nu g}\tilde{A}^b_{\lambda g}\right)\right],\end{aligned} \tag{8.1.271}$$

which differs from Lorentzian case by the factor $(-i)$ for the Chern-Simons terms and a overall sign factors for matter terms.

The supersymmetry transformations are

$$\delta X^I_a = i\bar{\epsilon}\Gamma^I \Psi_a, \tag{8.1.272}$$

$$\delta \Psi_a = D_\mu X^I_a \Gamma^\mu \Gamma^I \epsilon - \frac{1}{6} X^I_b X^J_c X^K_d f^{bcd}{}_a \Gamma^{IJK}\epsilon, \tag{8.1.273}$$

$$\delta \tilde{A}^a_{\mu b} = i\bar{\epsilon}\Gamma_\mu \Gamma^I X^I_c \Psi_d f^{cda}{}_b. \tag{8.1.274}$$

This is exactly same as in the Lorentzian transformations (4.1.49)-(4.1.51). ϵ is the unbroken supersymmetry parameters obeying[14]

$$\Gamma^{34\cdots 11}\epsilon = \epsilon, \tag{8.1.275}$$

$$\Gamma^{123}\epsilon = i\epsilon \tag{8.1.276}$$

Before topological twisting, fields and supersymmetry parameter transform under $SU(2)_E \times SO(8)_R$ as

$$X : (\mathbf{1}, \mathbf{8}_v) \tag{8.1.277}$$

$$\Psi : (\mathbf{2}, \mathbf{8}_c) \tag{8.1.278}$$

$$\epsilon : (\mathbf{2}, \mathbf{8}_s). \tag{8.1.279}$$

Although there are many possible ways of the twisting, we consider the following decomposition $SO(8)_R$ under $SO(3)$ [303]

$$\text{A-twist} : SO(8) \supset SO(5) \times SO(3) \tag{8.1.280}$$

$$\text{B-twist} : SO(8) \supset G_2 \supset SU(2)_1 \times SU(2)_2. \tag{8.1.281}$$

[14] This convention is different from that in [306]. There are two choices for $\tilde{\Gamma}^9 = \pm\tilde{\Gamma}^{12\cdots 8}$. We take $\tilde{\Gamma}^9 = +\tilde{\Gamma}^{12\cdots 8}$.

A-twist

Under (8.1.280), we can obtain the decomposition of $\mathbf{8}_v$, $\mathbf{8}_s$ and $\mathbf{8}_c$ as

$$\mathbf{8}_v = (\mathbf{5},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \tag{8.1.282}$$
$$\mathbf{8}_s = (\mathbf{4},\mathbf{2}) \tag{8.1.283}$$
$$\mathbf{8}_c = (\mathbf{4},\mathbf{2}). \tag{8.1.284}$$

Then the transformations under $SO(3)'_E \times SO(5)_R$ of the fields are

$$X \to (\mathbf{1},\mathbf{5}) \oplus (\mathbf{3},\mathbf{1}) \tag{8.1.285}$$
$$\Psi \to (\mathbf{1},\mathbf{4}) \oplus (\mathbf{3},\mathbf{4}) \tag{8.1.286}$$
$$\epsilon \to (\mathbf{1},\mathbf{4}) \oplus (\mathbf{3},\mathbf{4}). \tag{8.1.287}$$

The bosonic field content is

- 5 scalar fields : $(\mathbf{1},\mathbf{5})$
- 1-form : $(\mathbf{3},\mathbf{1})$

and the fermionic field content is

- 4 scalar fields : $(\mathbf{1},\mathbf{4})$
- 4 1-form : $(\mathbf{3},\mathbf{4})$.

Therefore there exists four supercharges.

We decompose the gamma matrices under $SO(11) \supset SO(3)_E \times SO(5) \times SO(3)$ as

$$\begin{cases} \Gamma^i = \sigma_i \otimes \mathbb{I}_4 \otimes \mathbb{I}_2 \otimes \sigma_1 \\ \Gamma^{\mu+3} = \mathbb{I}_2 \otimes \gamma^\mu \otimes \mathbb{I}_2 \otimes \sigma_2 \\ \Gamma^{\mu+9} = \mathbb{I}_2 \otimes \mathbb{I}_4 \otimes \sigma_i \otimes \sigma_3 \end{cases} \tag{8.1.288}$$

where σ_i are Pauli matrices and γ^μ are five-dimensional gamma matrices satisfying

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}\gamma^5 := \gamma^{1234}. \tag{8.1.289}$$

The charge conjugation matrix can be expressed as

$$\mathcal{C} = \sigma_2 \otimes \mathrm{C} \otimes \sigma_2 \otimes \mathbb{I}_2 \tag{8.1.290}$$

where σ_2 is a three-dimensional charge conjugation matrix

$$(\sigma_2)^T = -\sigma_2\sigma_2\sigma_i\sigma_2^{-1} = -(\sigma_2)^T \tag{8.1.291}$$

and C is a five-dimensional charge conjugation matrix

$$(C)^T = -CC\gamma^\mu C^{-1} = (\gamma^\mu)^T. \tag{8.1.292}$$

The chirality matrix is given by

$$\Gamma^{123} == -i\Gamma^{45\cdots 11} i(\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I}\otimes\sigma_1). \tag{8.1.293}$$

1. **bosonic fields**

   For the bosonic fields we redefine

$$\begin{aligned}\phi^I &:= X^I (I = 4,5,6,7,8),\\ \Phi^\mu &:= X^{\mu+8}(\mu = 1,2,3)\end{aligned} \tag{8.1.294}$$

2. **fermionic fields**

   We expand the elven-dimensional fermionic fields Ψ under the decomposition $SO(11) \supset SO(3)_E \times SO(5) \times SO(3)$ as

$$\Psi_{p\alpha q} = \frac{1}{\sqrt{2}}\left(\psi_\alpha \mathbb{I}_{pq} + \Psi_{i\alpha}\sigma_{ipq}\right)\sigma_2^{-1} \tag{8.1.295}$$

   where p, q, α are indices of $SO(3)_E, SO(5), SO(3)$ respectively. ψ_α and $\Psi_{i\alpha}$ are scalars $(\mathbf{1},\mathbf{4})$ and a 1-form $(\mathbf{3},\mathbf{4})$.

B-twist

Under (8.1.281), we can obtain the decomposition of $\mathbf{8}_v$, $\mathbf{8}_s$ and $\mathbf{8}_c$ as

$$\mathbf{8}_v = \mathbf{7}+\mathbf{1} = (\mathbf{1},\mathbf{3})\oplus(\mathbf{2},\mathbf{2})\oplus(\mathbf{1},\mathbf{1}) \tag{8.1.296}$$

$$\mathbf{8}_s = \mathbf{7}+\mathbf{1} = (\mathbf{1},\mathbf{3})\oplus(\mathbf{2},\mathbf{2})\oplus(\mathbf{1},\mathbf{1}) \tag{8.1.297}$$

$$\mathbf{8}_c = \mathbf{7}+\mathbf{1} = (\mathbf{1},\mathbf{3})\oplus(\mathbf{2},\mathbf{2})\oplus(\mathbf{1},\mathbf{1}). \tag{8.1.298}$$

Choosing the homomorphism as $SU(2)_E \to SU(2)_1$, the transformations under $SO(3)'_E \times SU(2)'_2$ of the fields are

$$X \to (\mathbf{1},\mathbf{3})\oplus(\mathbf{2},\mathbf{2})\oplus(\mathbf{1},\mathbf{1}) \tag{8.1.299}$$

$$\Psi \to (\mathbf{2},\mathbf{3})\oplus(\mathbf{1},\mathbf{2})\oplus(\mathbf{3},2)\oplus(\mathbf{2},\mathbf{1}) \tag{8.1.300}$$

$$\epsilon \to (\mathbf{2},\mathbf{3})\oplus(\mathbf{1},\mathbf{2})\oplus(\mathbf{3},2)\oplus(\mathbf{2},\mathbf{1}). \tag{8.1.301}$$

The bosonic field content is

- 4 scalar fields : $(\mathbf{1},\mathbf{3})\oplus(\mathbf{1},\mathbf{1})$

- 2 spinor fields : $(\mathbf{2},\mathbf{2})$

and the fermionic field content is

- 2 scalar fields : $(\mathbf{1},\mathbf{2})$
- 4 spinor fields : $(\mathbf{2},\mathbf{3})\oplus(\mathbf{2},\mathbf{1})$.
- 2 1-form : $(\mathbf{3},\mathbf{2})$.

Therefore there exists two supercharges.

8.1.13 $d=2$, $\mathcal{N}=(2,2)$ non-linear sigma-model

There are two types global symmetry in the theory, which are called $U(1)_V$ vector R-symmetry and $U(1)_A$ axial R-symmetry [15].

The field content is

- scalar fields $\phi^I(z,\overline{z})$
- fermionic fields $\psi^I_+(z,\overline{z}),\psi^I_-(z,\overline{z})$

The bosonic field $\phi^I(z,\overline{z})$ is a map from 2-dimensional genus g Riemann surface Σ to a target space X of metric g

$$\phi^I(z,\overline{z}):\Sigma\to X \tag{8.1.304}$$

where $z,\overline{z}$ are the local coordinates on Σ. The fermionic field ψ^I_+ is a section of $K^{\frac{1}{2}}\otimes\phi^*(TX)$ and ψ^I_- is a section of $K^{-\frac{1}{2}}\otimes\phi^*(TX)$

$$\psi^I_\pm(z,\overline{z})\in\Gamma(K^{\pm\frac{1}{2}}\otimes\phi^*(TX)) \tag{8.1.305}$$

where TX is the holomorphic tangent bundle to X. K and K^{-1} are the canonical and anti-canonical bundle on Σ (i.e. the bundle of $(1,0)$ and $(0,1)$ forms) and $K^{\frac{1}{2}}$ and $K^{-\frac{1}{2}}$ are square roots of these.

[15]Although $SO(2)_E$ rotational symmetry acts on the variables $z,\overline{z},\theta^\pm$ and $\overline{\theta}$ simultaneously, one can construct two $U(1)$ groups that act only on a subset of variables and leave the measure invariant and keep the chiral fields to be chiral

$$U(1)_V:\begin{cases}(\theta^+,\overline{\theta}^+) & \to(e^{-i\alpha}\theta^+,e^{i\alpha}\overline{\theta}^+)\\(\theta^-,\overline{\theta}^-) & \to(e^{-i\alpha}\theta^+,e^{i\alpha}\overline{\theta}^+)\end{cases} \tag{8.1.302}$$

$$U(1)_A:\begin{cases}(\theta^+,\overline{\theta}^+) & \to(e^{-i\alpha}\theta^+,e^{i\alpha}\overline{\theta}^+)\\(\theta^-,\overline{\theta}^-) & \to(e^{i\alpha}\theta^-,e^{-i\alpha}\overline{\theta}^-)\end{cases}. \tag{8.1.303}$$

Here we consider the case where we have $\mathcal{N} = (2,2)$ supersymmetry, which require that X is Kähler. We denote the local complex coordinates by ϕ^i and their complex conjugate by $\phi^{\bar{i}} = \overline{\phi}^{\bar{i}}$. As the complexified tangent bundle TX has a decomposition as $TX = T^{1,0}X \oplus T^{0,1}X$, (8.1.305) becomes

$$\begin{aligned}
&\psi^i_+ \in \Gamma(K^{\frac{1}{2}} \otimes \phi^*(T^{1,0}X)),\\
&\psi^{\bar{i}}_+ \in \Gamma(K^{\frac{1}{2}} \otimes \phi^*(T^{0,1}X)),\\
&\psi^i_- \in \Gamma(K^{-\frac{1}{2}} \otimes \phi^*(T^{1,0}X)),\\
&\psi^{\bar{i}}_- \in \Gamma(K^{-\frac{1}{2}} \otimes \phi^*(T^{0,1}X))
\end{aligned} \tag{8.1.306}$$

and $\psi^i_\pm$ and $\psi^{\bar{i}}_\pm$ are left- and right-moving fermionic fields respectively.

The action is

$$S = 2t\int_\Sigma d^2z\left(\frac{1}{2}g_{IJ}\partial_z\phi^I\partial_{\bar{z}}\phi^J + ig_{i\bar{j}}\psi^{\bar{j}}_- D_z\psi^i_- + ig_{i\bar{j}}\psi^{\bar{j}}_+ D_{\bar{z}}\psi^i_+ + R_{i\bar{j}k\bar{l}}\psi^i_+\psi^{\bar{j}}_+\psi^k_-\psi^{\bar{l}}_-\right) \tag{8.1.307}$$

where t is a coupling constant or a string tension depending on the overall volume of X and $R_{i\bar{j}k\bar{l}}$ is the Riemann tensor of target space X.

Originally fields transform under $SO(2)_E \times U(1)_V \times U(1)_A$ as in Table 8.7 [16]. Likewise supersymmetry generators transform as Table 8.8. Depending on which R-symmetry we use, there are two homomorphisms for the twisting

$$\text{A-twist}: U(1)_E \to U(1)_V \tag{8.1.314}$$

$$\text{B-twist}: U(1)_E \to U(1)_A. \tag{8.1.315}$$

The results of the twisting are summarized in Table 8.9 and Table 8.10.

[16] From (8.1.302) and (8.1.303), the vector R-rotations and the axial R-rotations of superfield are given by

$$e^{i\alpha F_V} : \Phi(x^\mu, \theta^\pm, \overline{\theta}^\pm) \mapsto e^{i\alpha q_V}\Phi(x^\mu, e^{-i\alpha}\theta^\pm, e^{i\alpha}\overline{\theta}^\pm) \tag{8.1.308}$$

$$e^{i\beta F_A} : \Phi(x^\mu, \theta^\pm, \overline{\theta}^\pm) \mapsto e^{i\beta q_A}\Phi(x^\mu, e^{\mp i\beta}\theta^\pm, e^{\pm i\beta}\overline{\theta}^\pm) \tag{8.1.309}$$

where F_V, F_A are the generators of the vector and the axial R-symmetry and q_V, q_A are the vector and the axial R-charges respectively. Therefore we see that

$$\psi^i_{+new} = e^{i\alpha(1-q_V)}\psi^i_{+old}, \qquad \psi^i_{+new} = e^{i\beta(1-q_A)}\psi^i_{+old} \tag{8.1.310}$$

$$\psi^i_{-new} = e^{i\alpha(1-q_V)}\psi^i_{-old}, \qquad \psi^i_{-new} = e^{i\beta(-1-q_A)}\psi^i_{-old} \tag{8.1.311}$$

$$\psi^{\bar{i}}_{+new} = e^{i\alpha(-1+q_V)}\psi^{\bar{i}}_{+old}, \qquad \psi^{\bar{i}}_{+new} = e^{i\beta(-1-q_A)}\psi^{\bar{i}}_{+old} \tag{8.1.312}$$

$$\psi^{\bar{i}}_{-new} = e^{i\alpha(-1+q_V)}\psi^{\bar{i}}_{+old}, \qquad \psi^{\bar{i}}_{-new} = e^{i\alpha(1-q_A)}\psi^{\bar{i}}_{-old}. \tag{8.1.313}$$

Setting $q_V = q_A = 0$, we obtain the $U(1)_V$ and the $U(1)_A$ charges in Table 8.7.

	$U(1)_E$	$U(1)_V$	$U(1)_A$	$\mathcal{L}$
ϕ	0	0	0	$\mathcal{O}$
ψ^i_+	$+$	$-$	$+$	$K^{\frac{1}{2}}$
ψ^i_-	$-$	$-$	$-$	$K^{-\frac{1}{2}}$
$\psi^{\bar{i}}_+$	$+$	$+$	$-$	$K^{\frac{1}{2}}$
$\psi^{\bar{i}}_-$	$-$	$+$	$+$	$K^{-\frac{1}{2}}$

Table 8.7: $U(1)$ charges of the $d = 2$, $\mathcal{N} = (2,2)$ sigma model fields. $\mathcal{L}$ is the complex line bundle on Σ in which the fields take values. $\mathcal{O}$ is the trivial bundle and K is the canonical bundle.

	$U(1)_E$	$U(1)_V$	$U(1)_A$	$\mathcal{L}$
Q_+	$+$	$-$	$+$	$K^{\frac{1}{2}}$
Q_-	$-$	$-$	$-$	$K^{-\frac{1}{2}}$
$\overline{Q}_+$	$+$	$+$	$-$	$K^{\frac{1}{2}}$
$\overline{Q}_-$	$-$	$+$	$+$	$K^{-\frac{1}{2}}$

Table 8.8: $U(1)$ charges of the $d = 2$, $\mathcal{N} = (2,2)$ sigma model supersymmetry generators.

	A-twist		B-twist	
fields	$U(1)'_E$	$\mathcal{L}$	$U(1)'_E$	$\mathcal{L}$
ϕ	0	$\mathcal{O}$	0	$\mathcal{O}$
ψ^i_+	2	K	2	K
ψ^i_-	0	$\mathcal{O}$	-2	K^{-1}
$\psi^{\bar{i}}_+$	0	$\mathcal{O}$	0	$\mathcal{O}$
$\psi^{\bar{i}}_-$	-2	K^{-1}	0	$\mathcal{O}$

Table 8.9: The spin of field for A-twisted and B-twisted $d = 2$, $\mathcal{N} = (2,2)$ sigma model.

	A-twist		B-twist	
fields	$U(1)'_E$	$\mathcal{L}$	$U(1)'_E$	$\mathcal{L}$
Q_+	2	K	2	K
Q_-	0	$\mathcal{O}$	-2	K^{-1}
$\overline{Q}_+$	0	$\mathcal{O}$	0	$\mathcal{O}$
$\overline{Q}_-$	-2	K^{-1}	0	$\mathcal{O}$

Table 8.10: The spin of the supersymmetry generators for A-twisted and B-twisted $d = 2$, $\mathcal{N} = (2,2)$ sigma model.

A-model

After performing A-twist, the bundles in which the fermionic fields take values are modified as

$$\begin{aligned} &\psi^i_z \in \Gamma(K \otimes \phi^*(T^{1,0}X)), \\ &\psi^i \in \Gamma(\phi^*(T^{0,1}X)), \\ &\psi^{\bar{i}} \in \Gamma(\phi^*(T^{1,0}X)), \\ &\psi^{\bar{i}}_{\bar{z}} \in \Gamma(K^{-1} \otimes \phi^*(T^{0,1}X)). \end{aligned} \tag{8.1.316}$$

B-model

B-twist changes the bundles in which fermionic fields take values as

$$\begin{aligned} &\psi^i_z \in \Gamma(K \otimes \phi^*(T^{1,0}X)), \\ &\psi^{\bar{i}} \in \Gamma(\phi^*(T^{0,1}X)), \\ &\psi^i_{\bar{z}} \in \Gamma(K^{-1} \otimes \phi^*(T^{1,0}X)), \\ &\psi^{\bar{i}} \in \Gamma(\phi^*(T^{0,1}X)). \end{aligned} \tag{8.1.317}$$

These topological twisted theories are known as A-model and B-model topological sigma-models, or topological string theories [309, 230, 310, 311] [17].

8.2 Curved branes and twisted theories

Let us consider the gauge theories arising from the dimensional reduction of ten-dimensional $\mathcal{N} = 1$ SYM theory to $(p+1)$ dimensions. It is known that these theories describe the low energy world-volume dynamics of flat Dp-branes [20].

[17] See [312] for the detailed review on the topological string theory.

On the other hand, when one consider curved branes wrapping around a non-trivial cycle $\mathcal{C}$ in the ambient manifold X, the cycle has to be identified with a calibrated submanifold and satisfy some stringent conditions to preserve some fraction of the supersymmetries.

As discussed in [28], curved world-brane theories are obtained by topological twisting along the directions where the world-volume is curved. To see this, we remember that the bosonic scalar fields are associated with translations of the D-brane. Thus when D-brane wrap around curved cycle $\mathcal{C}$ in X, there are only $(10 - \dim X)$ actual scalar fields and the other translational modes are identified with the section of the normal bundle $N_{\mathcal{C}}$ to $\mathcal{C}$ in X. Therefore these modes should be twisted if the normal bundle is non-trivial and so are their superpartners.

From the above observations, for given supersymmetric cycles $\mathcal{C}$ and their ambient manifolds X, one can determine

1. the bosonic field content
2. the number of scalar supercharges

of the world-volume topological gauge theories of D-branes. On the contrary, one can check whether there exists supersymmetric cycles with the required properties for given topological gauge theories.

Noting that there is a global invariance under the rotational $SO(10 - \dim X)$ symmetry [18] of the uncompactified dimensions, the original R-symmetry $SO(9 - p)$ should be decomposed as

$$SO(9-p) \subset SO(10 - \dim X) \times SO(\dim X - p - 1). \tag{8.2.1}$$

Then, under the branching (8.2.1) of R-symmetry, we try to perform topological twisting by using the second factor $SO(\dim X - p - 1)$ corresponding to the normal bundle $N_{\mathcal{C}}$ in X. A relevant information is given by Table 8.11.

The preserved fraction of supersymmetries are derived as follows. The holonomy group of K3 surface is $SU(2)$, so the spinor of $SO(4)$ is decomposed under $SO(4) \supset SU(2)_H \times SU(2) \times U(1) \supset SU(2)_H$ as

$$\mathbf{4} = (\mathbf{2}, \mathbf{1})_- \oplus (\mathbf{1}, \mathbf{2})_+ = \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1}. \tag{8.2.2}$$

Thus 2 of 4 supercharges are constant spinors and we have $\frac{1}{2}$ BPS background.

[18] When one considers Euclidean D-branes, the invariant rotational symmetry is $SO(1, 9 - \dim X)$ because curved D-branes do not wrap the time direction.

ambient manifolds (dimensions)	holonomy	submanifolds	SUSY
Calabi-Yau 2-fold (4)	$SU(2) \subset SO(4)$	holomorphic curve (2)	$\frac{1}{2}$
Calabi-Yau 3-fold (6)	$SU(3) \subset SO(6)$	Lagrangian (3)	$\frac{1}{4}$
G_2 manifold (7)	$G_2 \subset SO(7)$	coassociative (4)	$\frac{1}{8}$
		associative (3)	$\frac{1}{8}$
$Spin(7)$ manifold (8)	$Spin(7) \subset SO(8)$	Cayley (4)	$\frac{1}{16}$
Calabi-Yau 4-fold (8)	$SU(4) \subset SO(8)$	Lagrangian (4)	$\frac{1}{8}$
Hyperkähler manifold (8)	$Sp(2) \subset SO(8)$		$\frac{3}{16}$
$CY_2 \times CY_2$ (8)	$SU(2) \times SU(2) \subset SO(8)$		$\frac{1}{4}$
Calabi-Yau 5-fold (10)	$SU(5) \subset SO(10)$	Lagrangian (5)	$\frac{1}{16}$

Table 8.11: The ambient manifolds and the examples of calibrated submanifolds that preserve the fraction of supersymmetry. Note that all of the Calabi-Yau manifolds include holomorphic submanifolds as calibrated submanifolds.

The holonomy group of Calabi-Yau 3-fold is $SU(3)$, so the spinor of $SO(6)$ is decomposed under $SO(6) \supset SU(3)$ as

$$\mathbf{4} = \mathbf{3} \oplus \mathbf{1}. \tag{8.2.3}$$

Thus 1 of 4 supercharges is constant spinor and we have $\frac{1}{4}$ BPS background.

The holonomy group of Calabi-Yau 4-fold is $SU(4)$, so the spinor of $SO(8)$ is decomposed under $SO(8) \supset SU(4)$ as

$$\mathbf{8}_s \oplus \mathbf{8}_c = \mathbf{6} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{4} \oplus \overline{\mathbf{4}} \tag{8.2.4}$$

Thus 2 of 16 supercharges is constant spinor and we have $\frac{1}{8}$ BPS background.

The holonomy group of G_2 manifold is G_2, so the spinor of $SO(7)$ is decomposed under $SO(7) \supset G_2$ as

$$\mathbf{8} = \mathbf{1} \oplus \mathbf{7} \tag{8.2.5}$$

Thus 1 of 8 supercharges is constant spinor and we have $\frac{1}{8}$ BPS background.

The holonomy group of $Spin(7)$ manifold is $Spin(7)$, so the spinor of $SO(8)$ is decomposed under $SO(8) \supset Spin(7)$ as

$$\mathbf{8}_s \oplus \mathbf{8}_c = \mathbf{7} \oplus \mathbf{1} \oplus \mathbf{8} \tag{8.2.6}$$

Thus 1 of 16 supercharges is constant spinor and we have $\frac{1}{16}$ BPS background.

The holonomy group of Calabi-Yau 5-fold is $SU(5)$, so the spinor of $SO(10)$ is decomposed under $SO(10) \supset SU(5)$ as

$$\mathbf{16} \oplus \mathbf{16}' = \mathbf{1}_{-5} \oplus \overline{\mathbf{5}}_3 \oplus \mathbf{10}_- \oplus \mathbf{1}_5 \oplus \mathbf{5}_{-3} \oplus \mathbf{10}_+ \tag{8.2.7}$$

Thus 2 of 32 supercharges is constant spinor and we have $\frac{1}{16}$ BPS background.

8.2.1 D3-branes and twisted $d = 4$, $\mathcal{N} = 4$ SYM theories

The first example is the low-energy effective field theories of the D3-branes wrapped on curved four-manifold. A set of these descriptions can be obtained by the three distinct topologically twisted $d = 4$, $\mathcal{N} = 4$ SYM theories.

1. **GL twist D-brane**

   The fact that $\dim \mathcal{C} = 4$ and that there are two scalar fields means that the theory describes 4-cycle $\mathcal{C}$ in $4 + (6 - 2) = 8$-dimensional manifold X. The existence of two preserved BRST charges indicates that 8-manifold preserve $\frac{2}{16} = \frac{1}{8}$ of the supersymmetry. From the above facts and Table 8.11, X is a Calabi-Yau 4-fold and $\mathcal{C}$ is a special Lagrangian submanifold[19].

   Moreover it it known that in the case where special Lagrangian submanifold is embedded in Calabi-Yau 4-fold, the normal bundle $N_{\mathcal{C}}$ can be identified with the cotangent bundle $T^*_{\mathcal{C}}$ [313]. This is consistent to the fact that the remaining four scalar fields combine to form one 1-form on $\mathcal{C}$.

   Note that a global $U(1)$ ghost number symmetry corresponds to the rotational symmetry of the two uncompactified dimensions. The two scalars having opposite $U(1)$ charges are identified with the 2-dimensional vector and the 1-form is a $U(1)$-singlet.

2. **VW twist D-brane**

   The fact that $\dim \mathcal{C} = 4$ and that there are three scalar fields means that the theory describes 4-cycle $\mathcal{C}$ in $4 + (6 - 3) = 7$-dimensional manifold X. The existence of two preserved BRST charges indicates that 7-manifold preserve $\frac{2}{16} = \frac{1}{8}$ of the supersymmetry. From the above facts and Table 8.11, X is a G_2 manifold and $\mathcal{C}$ is a coassociative submanifold.

   It is known that for a coassociative 4-submanifold in G_2 manifold, the normal bundle is $(\mathbf{1}, \mathbf{3})_0$ [313]. This is consistent to the results obtained by the twisting.

[19] Special Lagrangian submanifold is a submanifold for which the real part of the holomorphic form restricts to the volume form on the submanifold.

twist	submanifold (dimension)	ambient manifold (dimension)	SUSY
GL twist	Lagrangian (4)	Calabi-Yau 4-fold (8)	$\frac{16}{8} = 2$
VW twist	coassociative (4)	G_2 manifold (7)	$\frac{16}{8} = 2$
DW twist	Cayley (4)	$Spin(7)$ manifold (8)	$\frac{16}{16} = 1$

Table 8.12: Three types of topological twists for $d = 4$, $\mathcal{N} = 4$ SYM theories, curved D3-branes (submanifolds) and ambient manifolds.

3. **DW twist D-brane**

   The fact that $\dim \mathcal{C} = 4$ and that there are two scalar fields means that the theory describes 4-cycle $\mathcal{C}$ in $4 + (6 - 2) = 8$-dimensional manifold X. The existence of one preserved BRST charge indicates that 8-manifold preserve $\frac{1}{16} = \frac{1}{16}$ of the supersymmetry. From the above facts and Table 8.11, X is a $Spin(7)$ manifold and $\mathcal{C}$ is a Cayley submanifold.

   It it known that for the $Spin(7)$ manifold the normal bundle is $S_+ \oplus V$ where S_+ is a spin bundle of a given chirality and V is a 2-dimensional bundle [28]. When V is trivial, this becomes $S_+ \oplus S_+$, that is $(\mathbf{1}, \mathbf{2})_+ \oplus (\mathbf{1}, \mathbf{2})_-$.

These results are summarized in Table 8.12

8.2.2 D2-branes and twisted $d = 3$, $\mathcal{N} = 8$ SYM theories

The D2-branes wrapped on three-manifold are given by the topologically twisted $d = 3$, $\mathcal{N} = 8$ SYM theories.

1. **A-twist**

   The fact that $\dim \mathcal{C} = 3$ and that there are four scalar fields means that the theory describes 3-cycle $\mathcal{C}$ in $3 + (7 - 4) = 6$-dimensional manifold X. The existence of the four preserved BRST charges indicates that 6-manifold preserve $\frac{4}{16} = \frac{1}{4}$ of the supersymmetry. From the above facts and Table 8.11, X is a Calabi-Yau 3-fold and $\mathcal{C}$ is a special Lagrangian submanifold.

   Also it is known that the normal bundle $N_{\mathcal{C}}$ can be identified with the cotangent bundle $T^*_{\mathcal{C}}$ [313]. This is consistent to the fact that the remaining three scalar fields combine to form one 1-form on $\mathcal{C}$.

   A global $SU(2)_1 \times SU(2)_2 \simeq SU(4)$ ghost number symmetry corresponds to the rotational symmetry of the four uncompactified dimensions. The four scalars transform as a $\mathbf{4}_v$ of $SO(4)$ and the 1-form is an $SO(4)$-singlet.

twist	submanifold (dimension)	ambient manifold (dimension)	SUSY
A-twist	Lagrangian (3)	Calabi-Yau 3-fold (6)	$\frac{16}{4} = 4$
B-twist	associative (3)	G_2 manifold (7)	$\frac{16}{8} = 2$

Table 8.13: Two types of topological twists for $d = 3$, $\mathcal{N} = 8$ SYM theories, curved D2-branes (submanifolds) and ambient manifolds.

2. **B-twist**

   The fact that $\dim \mathcal{C} = 3$ and that there are three scalar fields means that the theory describes 3-cycle $\mathcal{C}$ in $3 + (7 - 3) = 7$-dimensional manifold X. The existence of the two preserved BRST charges indicates that 7-manifold preserve $\frac{2}{16} = \frac{1}{8}$ of the supersymmetry. From the above facts and Table 8.11, X is a G_2 manifold and $\mathcal{C}$ is an associative submanifold.

   Also it is known that for an associative 3-submanifold in G_2 manifold, the normal bundle is $N_{\mathcal{C}} = S \otimes V$ where S is a spinor bundle of $\mathcal{C}$ and V is a rank two $SU(2)$-bundle. This is consistent to the fact that the twisted bosonic spinors $(\mathbf{2}, \mathbf{2}, \mathbf{1})$ are an $SU(2)$-doublet of spinors on $\mathcal{C}$.

   Again a global $SU(2)_3 \simeq SO(3)$ symmetry corresponds to the rotational symmetry of the four uncompactified dimensions. The three scalars transform as a $\mathbf{3}_v$ of $SO(3)$ and the twisted bosonic spinors $(\mathbf{2}, \mathbf{2}, \mathbf{1})$ are $SO(3)$-singlet.

These results are summarized in Table 8.13

8.2.3 D2-branes and twisted $d = 3$, $\mathcal{N} = 8$ SYM theories on $\mathbb{R} \times \Sigma$

The low-energy effective theories of the D2-branes wrapping on the holomorphic Riemann surface Σ are the partially twisted $d = 3$, $\mathcal{N} = 8$ SYM theories:

1. **A-twist**

   The fact that $\dim \Sigma = 2$ and that there are five scalar fields means that the theory describes 2-cycle Σ in $2 + (7 - 5) = 4$-dimensional manifold X. The existence of the eight preserved BRST charges indicates that 4-manifold preserve $\frac{8}{16} = \frac{1}{2}$ of the supersymmetry. From the above facts and Table (8.11), X is a K3 surface and Σ is a holomorphic curve.

2. **B-twist**

   The fact that $\dim \Sigma = 2$ and that there are three scalar fields means that the theory describes 2-cycle Σ in $2 + (7 - 3) = 6$-dimensional manifold X.

twist	submanifold (dimension)	ambient manifold (dimension)	SUSY
A-twist	holomorphic (2)	K3 surface (4)	$\frac{16}{2}=8$
B-twist	holomorphic (2)	Calabi-Yau 3-fold (6)	$\frac{16}{4}=4$
C-twist	holomorphic (2)	Calabi-Yau 4-fold (8)	$\frac{16}{8}=2$

Table 8.14: Three types of topological twists for $d=3$, $\mathcal{N}=8$ SYM theories on $\mathbb{R}\times\Sigma$, curved D2-branes (submanifolds) and ambient manifolds.

The existence of the four preserved BRST charges indicates that 6-manifold preserve $\frac{4}{16}=\frac{1}{4}$ of the supersymmetry. From the above facts and Table (8.11), X is a Calabi-Yau 3-fold and Σ is a holomorphic curve.

3. **C-twist**

   The fact that $\dim\Sigma=2$ and that there are one scalar field means that the theory describes 2-cycle Σ in $2+(7-1)=8$-dimensional manifold X. The existence of the two preserved BRST charges indicates that 6-manifold preserve $\frac{2}{16}=\frac{1}{8}$ of the supersymmetry. From the above facts and Table (8.11), X is a Calabi-Yau 4-fold and Σ is a holomorphic curve.

These results are summarized in Table 8.14

8.2.4 Relationship between $d=4$ and $d=3$ twists

The $d=4$ twisting and the $d=3$ twisting are connected via dimensional reduction.

1. **DW twist and B-twist**

   Let us define Cayley 4-form in local coordinates $\mathbb{R}^8$ as [314, 313]

$$\begin{aligned}\Omega_{Cayley} := dx^{0123} &+ \left(dx^{01}-dx^{23}\right)\wedge\left(dx^{45}-dx^{67}\right)\\ &+\left(dx^{02}+dx^{13}\right)\wedge\left(dx^{46}+dx^{57}\right)\\ &+\left(dx^{03}-dx^{12}\right)\wedge\left(dx^{47}-dx^{56}\right)+dx^{4567}\end{aligned} \tag{8.2.8}$$

   where $dx^{ijk}:=dx^i\wedge dx^j\wedge dx^k$, etc. Then one can define $Spin(7)$ manifold to be the subgroup of $GL(8)$ that preserve Ω_{Cayley}. Integrating over the fibre x^0, we obtain

$$\begin{aligned}\pi_*\Omega_{Cayley} = dx^{123} &+ dx^1\wedge\left(dx^{45}-dx^{67}\right)\\ &+dx^2\wedge\left(dx^{46}+dx^{57}\right)\\ &+dx^3\wedge\left(dx^{47}-dx^{56}\right)\end{aligned} \tag{8.2.9}$$

On the other hand, the associative 3-form Ω_{ass} characterizing associative 3-manifolds of G_2 manifolds is defined as [314]

$$\begin{aligned}\Omega_{ass} := dx^{456} &+ dx^4 \wedge \left(dx^{01} - dx^{23}\right) \\ &+ dx^5 \wedge \left(dx^{02} + dx^{13}\right) \\ &+ dx^6 \wedge \left(dx^{03} - dx^{12}\right)\end{aligned} \tag{8.2.10}$$

Thus

$$\pi_* \Omega_{Cayley} = \Omega_{ass}. \tag{8.2.11}$$

Therefore DW twist is related to B twist by the dimensional reduction.

2. **VW twist and A-twist**

VW twist theory corresponds to coassociative submanifolds of G_2 manifolds characterized by the Hodge dual 4-form $\Omega_{coass} = *\Omega_{ass}$, which is expressed as [313]

$$\begin{aligned}\Omega_{coass} = dx^{0123} &- dx^{56} \wedge \left(dx^{01} - dx^{23}\right) \\ &+ dx^{46} \wedge \left(dx^{02} + dx^{13}\right) \\ &- dx^{45} \wedge \left(dx^{03} - dx^{12}\right).\end{aligned} \tag{8.2.12}$$

Integrating this over x^0, one obtains

$$\pi_* \Omega_{coass} = dx^{123} - dx^{156} + dx^{246} - dx^{345}. \tag{8.2.13}$$

On the other hand, the holomorphic volume form of a Calabi-Yau 3-fold characterizing special Lagrangian submanifolds is

$$\begin{aligned}\Omega_{slag} =& dz^1 \wedge dz^2 \wedge dz^3 \\ =& \left(dx^{123} - dx^{453} - dx^{156} - dx^{426}\right) + i \left(dx^{423} + dx^{513} + dx^{612} - dx^{456}\right).\end{aligned}$$

Thus

$$\pi_* \Omega_{coass} = \mathrm{Re}\Omega_{slag}. \tag{8.2.14}$$

Therefore VW twist is associated with A twist by the dimensional reduction.

3. **GL twist and A-twist**

Suppose that the Calabi-Yau 4-fold is locally of the form

$$CY_4 = CY_3 \times T^2. \tag{8.2.15}$$

Then special Lagrangian of CY_4 wrapping around one of the circles reduce to special Lagrangian submanifolds of CY_3 by the double dimensional reduction. Therefore GL twist is related to A-twist by the double dimensional reduction.

8.2.5 D1-branes and twisted $d = 2, \mathcal{N} = 8$ SYM theories

The world-volume theories of the D1-branes wrapped on holomorphic Riemann surfaces are topologically twisted $d = 2$, $\mathcal{N} = 8$ SYM theories.

1. **A-twist**

   The fact that $\dim \mathcal{C} = 2$ and that there are six scalar fields means that the theory describes 2-cycle $\mathcal{C}$ in $2 + (8 - 6) = 4$-dimensional manifold X. The existence of the eight preserved BRST charges indicates that 4-manifold preserve $\frac{8}{16} = \frac{1}{2}$ of the supersymmetry. From the above facts and Table 8.11, X is a K3 surface and $\mathcal{C}$ is a holomorphic curve.

   Let us consider the normal bundle $N_{\mathcal{C}}$. Noting that

   $$T_X = T_{\mathcal{C}} \oplus N_{\mathcal{C}}, \tag{8.2.16}$$

   $$c_1(T_X) = 0 \tag{8.2.17}$$

   for holomorphic genus g curve $\mathcal{C}$ in Calabi-Yau n-folds X, we see that

   $$c_1(N_{\mathcal{C}}) = -c_1(T_{\mathcal{C}}) = 2g - 2. \tag{8.2.18}$$

   Alternatively as $\wedge^n T_X$ is trivial, one has

   $$\wedge^n T_X = T_{\mathcal{C}} \wedge^{n-1} N_{\mathcal{C}} = 1, \tag{8.2.19}$$

   which gives the condition of the canonical bundle $K_{\mathcal{C}}$ on $\mathcal{C}$

   $$\wedge^{n-1} N_{\mathcal{C}} = K_{\mathcal{C}} \tag{8.2.20}$$

   because $T_{\mathcal{C}} = K_{\mathcal{C}}^{-1}$.

   If X is a K3 surface and $\mathcal{C}$ is a holomorphic curve, then $n = 2$ and $N_{\mathcal{C}}$ has rank one and (8.2.20) becomes

   $$N_{\mathcal{C}} = K_{\mathcal{C}}. \tag{8.2.21}$$

   This is consistent to the fact that remaining two scalar fields combine to form a single one-form on $\mathcal{C}$.

   A global $SO(6)_R$ ghost number symmetry corresponds to the rotational symmetry of the six uncompactified dimensions. The six scalars transform as a $\mathbf{6}_v$ of $SO(6)$ and the one-form is an $SO(6)$-singlet.

twist	submanifold (dimension)	ambient manifold (dimension)	SUSY
A-twist	holomorphic curve (2)	K3 surface (4)	$\frac{16}{2} = 8$
B-twist	holomorphic curve (2)	Calabi-Yau 3-fold (6)	$\frac{16}{4} = 4$

Table 8.15: Two types of topological twists for $d = 2$, $\mathcal{N} = 8$ SYM theories, curved D1-branes (submanifolds) and ambient manifolds.

2. **B-twist**

   The fact that $\dim \mathcal{C} = 2$ and that there are four scalar fields means that the theory describes 2-cycle $\mathcal{C}$ in $2 + (8 - 4) = 6$-dimensional manifold X. The existence of the four preserved BRST charges indicates that 6-manifold preserve $\frac{4}{16} = \frac{1}{4}$ of the supersymmetry. From the above facts and Table 8.11, X is a Calabi-Yau 3-fold and $\mathcal{C}$ is an holomorphic curve.

   In this case (8.2.20) becomes

$$\wedge^2 N_{\mathcal{C}} = K_{\mathcal{C}} \tag{8.2.22}$$

   and generally this is solved by

$$N_{\mathcal{C}} = K_{\mathcal{C}}^{\frac{1}{2}} \otimes V \tag{8.2.23}$$

   where V is a rank two bundle with trivial determinant.

These results are summarized in Table 8.15

8.2.6 M2-branes and twisted BLG theory

The low-energy description of the two M2-branes wrapping curved three-fold are as follows:

1. **A-twist**

   The fact that $\dim \mathcal{C} = 3$ and that there are five scalar fields means that the theory describes 3-cycle $\mathcal{C}$ in $3 + (8 - 5) = 6$-dimensional manifold X. The existence of the four preserved BRST charges indicates that 6-manifold preserve $\frac{4}{16} = \frac{1}{4}$ of the supersymmetry. From the above facts and Table 8.11, X is a Calabi-Yau 3-fold and $\mathcal{C}$ is a special Lagrangian submanifold.

   Also it is known that the normal bundle $N_{\mathcal{C}}$ can be identified with the cotangent bundle $T^*_{\mathcal{C}}$ [313]. This is consistent to the fact that the remaining three scalar fields combine to form one 1-form on $\mathcal{C}$.

twist	submanifold (dimension)	ambient manifold (dimension)	SUSY
A-twist	Lagrangian (3)	Calabi-Yau 3-fold (6)	$\frac{16}{4} = 4$
B-twist	associative (3)	G_2 manifold (7)	$\frac{16}{8} = 2$

Table 8.16: Two types of topological twists for BLG model, curved M2-branes (submanifolds) and ambient manifolds.

A global $SO(5)$ ghost number symmetry corresponds to the rotational symmetry of the four uncompactified dimensions. The five scalars transform as a $\mathbf{5}_v$ of $SO(5)$ and the 1-form is an $SO(5)$-singlet.

2. **B-twist**

   The fact that $\dim \mathcal{C} = 3$ and that there are three scalar fields means that the theory describes 3-cycle $\mathcal{C}$ in $3 + (8 - 4) = 7$-dimensional manifold X. The existence of the two preserved BRST charges indicates that 7-manifold preserve $\frac{2}{16} = \frac{1}{8}$ of the supersymmetry. From the above facts and Table 8.11, X is a G_2 manifold and $\mathcal{C}$ is an associative submanifold.

These results are summarized in Table 8.16 and same as that of D2-brane instantons (Table 8.13).

Chapter 9

Curved M2-branes and Topological Twisting

In this chapter we will return to the study of the M2-branes and discuss that the topologically twisted $\mathcal{A}_4$ BLG-model may describe the two wrapped M2-branes around a holomorphic Riemann surface in Calabi-Yau manifold based on the work of [51]. We will study the preserved supersymmetry on the wrapped branes around a holomorphic Riemann surface inside a Calabi-Yau manifold in section 9.1, 9.2 and 9.3. In section 9.4 we will specify the appropriate twisting procedures for our wrapped M2-branes.

9.1 M2-branes wrapping a holomorphic curve

Now we are ready to discuss the M2-branes wrapping curved Riemann surface. Recall that the BLG action (4.1.31) and the ABJM action (5.1.1) may describe the dynamics of probe membranes propagating in a fixed background geometry with an $SO(8)$ and an $SU(4)$ holonomy respectively. For both cases, the world-volume M_3 is considered as a flat space-time $\mathbb{R}^{1,2}$ or $\mathbb{R} \times T^2$. Now let us consider more general situations where curved M2-branes reside in some fixed curved background geometries. If we naively put the theory on a general three dimensional manifold, all supersymmetries are broken. However, here we shall wrap the M2-branes on a Riemann surface Σ_g of genus g that can preserve supersymmetry (i.e. supersymmetric two-cycles) as the form

$$M_3 = \mathbb{R} \times (\Sigma_g \subset X) \tag{9.1.1}$$

where $\mathbb{R}$ is viewed as a time direction and X is a real $2(n+1)$-dimensional space preserving supersymmetry with vanishing three-form gauge field. Thus far the only known supersymmetric two-cycles, i.e. calibrated two-cycles, in special holonomy manifolds are holomorphic two-cycles in Calabi-Yau spaces. The corresponding two-form calibrations are Kähler calibrations. Accordingly we will take the ambient space X as an $(n+1)$-dimensional Calabi-Yau space and the other space as flat. Namely the geometry of the M-theory is taken as

$$\mathbb{R}^{1,8-2n} \times CY_{n+1}. \tag{9.1.2}$$

9.2 Supersymmetry in Calabi-Yau space

As a first step to count the number of preserved supersymmetries in our setup, one should know the dimension of the vector space formed by the corresponding Killing spinor ϵ, that is the amount of supersymmetries in the background geometry. Since we are now considering the background geometries with vanishing four-form flux, the Killing spinor equation is given by

$$\nabla_M \epsilon = \left(\partial_M + \frac{1}{4} \omega_{MPQ} \Gamma^{PQ} \right) \epsilon = 0 \tag{9.2.1}$$

where ω_{MPQ}, $M, N, P, Q = 0, 1, \cdots, 10$ is an eleven-dimensional Levi-Civita spin connection. This leads to the integrability condition

$$[\nabla_M, \nabla_N]\epsilon = \frac{1}{4} R_{MNPQ} \Gamma^{PQ} \epsilon = 0, \tag{9.2.2}$$

which implies that a Killing spinor ϵ transforms as a singlet under the restricted holonomy group $H \subset Spin(1,10)$ generated by $R_{MNPQ}\Gamma^{PQ}$. Therefore the amount of preserved supersymmetries in the special holonomy manifold is equivalent to the number of singlets in the decomposition of the spinor representation **32** of $Spin(1,10)$ into the representation of the holonomy group H. In our case the background geometries are taken as Calabi-Yau $(n+1)$-folds with the holonomy $H = SU(n+1)$, $n = 1, 2, 3, 4$ and the decompositions are as follows.

1. CY_5

   In this case the geometry is of the form $\mathbb{R} \times CY_5$. This splits the $Spin(10)$ into $SU(5)$ and the corresponding decomposition of the spinor representation is given by

$$\begin{aligned} \mathbf{16} &= \mathbf{10}_- \oplus \bar{\mathbf{5}}_3 \oplus \mathbf{1}_{-5} \\ \mathbf{16}' &= \mathbf{10}_+ \oplus \mathbf{5}_{-3} \oplus \mathbf{1}_5. \end{aligned} \tag{9.2.3}$$

Here the capital letters denote the representations of the $SU(5)$ and the subscripts stand for the $U(1)$ charges which appear after the decomposition $Spin(10) \to SU(5) \times U(1)$. The existence of two singlets implies that the space $\mathbb{R} \times CY_5$ preserves $\frac{2}{32} = \frac{1}{16}$ supersymmetries.

Let us define an explicit set of projections defining the Killing spinors. To this end we need to specify how the Calabi-Yau spaces live in the eleven-dimensional space-time. We shall consider the situations where the Calabi-Yau manifolds fill in the order (x^1, x^2), (x^9, x^{10}), (x^7, x^8), (x^5, x^6) and (x^3, x^4). Then the Killig spinors can be defined by the eigenvalues ± 1 for the following set of commuting matrices

$$\Gamma^{12910}, \qquad \Gamma^{91078}, \qquad \Gamma^{7856}, \qquad \Gamma^{5634}. \tag{9.2.4}$$

The corresponding Killing spinors for CY_5 can be defined by the projection

$$\Gamma^{12910}\epsilon = \Gamma^{91078}\epsilon = \Gamma^{7856}\epsilon = \Gamma^{5634}\epsilon = -\epsilon. \tag{9.2.5}$$

Note that this implies that $\Gamma^{012}\epsilon = \epsilon$.

2. CY_4

   For this case the geometry is the product form $\mathbb{R}^{1,2} \times CY_4$. This leads to the decomposition of the $Spin(8)$ into $SU(4)$ and that of the spinor representation

$$\begin{aligned} \mathbf{8}_s &= \mathbf{6}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2} \\ \mathbf{8}_c &= \mathbf{4}_- \oplus \mathbf{4}_+. \end{aligned} \tag{9.2.6}$$

   We see that the decomposition provides two singlets from sixteen components. Thus the geometry $\mathbb{R}^{1,2} \times CY_4$ can preserve $\frac{2}{16} = \frac{1}{8}$ supersymmetries. In this case the projection for the Killing spinor is given by

$$\Gamma^{12910}\epsilon = \Gamma^{91078}\epsilon = \Gamma^{7856}\epsilon = -\epsilon. \tag{9.2.7}$$

3. CY_3

   In this case the geometry is given by $\mathbb{R}^{1,4} \times CY_3$. This decomposes the $Spin(6)$ into $SU(3)$ and correspondingly spinor representation decomposes as

$$\begin{aligned} \mathbf{4} &= \mathbf{3}_- \oplus \mathbf{1}_3 \\ \overline{\mathbf{4}} &= \mathbf{3}_+ \oplus \mathbf{1}_{-3}. \end{aligned} \tag{9.2.8}$$

The appearance of two singlets from eight components means that there are $\frac{2}{8} = \frac{1}{4}$ supersymmetries in the product space $\mathbb{R}^{1,4} \times CY_3$. Therefore the Killing spinor can be defined by the projection

$$\Gamma^{12910}\epsilon = \Gamma^{91078}\epsilon = -\epsilon. \tag{9.2.9}$$

4. CY_2

For this case the geometry is the product space $\mathbb{R}^{1,6} \times CY_2$. The decomposition of $Spin(4)$ into $SU(2) \times SU(2)$ gives rise to that of the spinor representation

$$\begin{aligned} \mathbf{2} &= (\mathbf{2}, \mathbf{1}) \\ \mathbf{2}' &= (\mathbf{1}, \mathbf{2}). \end{aligned} \tag{9.2.10}$$

The presence of two singlets under one part of the $SU(2)$ implies that there are $\frac{2}{4} = \frac{1}{2}$ supersymmetries in the geometry $\mathbb{R}^{1,6} \times CY_2$. The corresponding Killing spinors satisfy the projection

$$\Gamma^{12910}\epsilon = -\epsilon. \tag{9.2.11}$$

9.3 Calibration and supersymmetric cycle

As a next step we shall consider the situation where the M2-branes wrapping a Riemann surface Σ_g propagate in a Calabi-Yau space without back reaction. In order to hold supersymmetry on the world-volume, Σ_g turns out to be a calibrated two-cycle, i.e. holomorphic curve of a Calabi-Yau manifold. To see this let us briefly review the background material concerning a calibration. In general a calibration on a special holonomy manifold X is a differential p-form φ obeying [27]

$$d\varphi = 0, \tag{9.3.1}$$

$$\varphi|_{\mathcal{C}_p} \leq \mathrm{Vol}|_{\mathcal{C}_p}, \quad \forall \mathcal{C}_p \tag{9.3.2}$$

where $\mathcal{C}_p$ is any p-cycle in X and Vol is the volume form on the cycle induced from the metric on X. Here the inequality is defined locally, namely $\varphi|_{\mathcal{C}_p} = a \cdot \mathrm{Vol}|_{\mathcal{C}_p}$ for some $a \in \mathbb{R}$, and $\varphi|_{\mathcal{C}_p} \leq \mathrm{Vol}|_{\mathcal{C}_p}$ if $a \leq 1$. A p-cycle Σ is said to be calibrated by φ if it satisfies

$$\varphi|_{\Sigma} = \mathrm{Vol}|_{\Sigma}. \tag{9.3.3}$$

We remark that a calibrated submanifold is a minimal surface in their homology class because

$$\mathrm{Vol}(\Sigma) = \int_{\Sigma} \varphi = \int_{M_{p+1}} d\varphi + \int_{\Sigma'} \varphi = \int_{\Sigma'} \varphi \leq \mathrm{Vol}(\Sigma') \tag{9.3.4}$$

where Σ' is another p-cycle in the same homology class such that $\partial M_{p+1} = \Sigma - \Sigma'$.

It is known that Calabi-Yau $(n+1)$-folds admit two different types of calibrations; the Kähler form J and the real part of holomorphic $(n+1,0)$-form Ω. One can construct calibrations as bilinear forms of spinors [315, 316]

$$J_{MN} = i\epsilon^{\dagger}\Gamma_{MN}\epsilon, \tag{9.3.5}$$

$$\Omega_{M_1\cdots M_{n+1}} = \epsilon^T \Gamma_{M_1\cdots M_{2(n+1)}}\epsilon. \tag{9.3.6}$$

Now we consider the condition so that a bosonic configuration of membranes is supersymmetric. Since one can always add a second probe brane without breaking supersymmetry if it is wrapped on the supersymmetric cycle which the original probe brane is wrapping, a simple way to find such condition is to analyze an effective world-volume action of a single membrane [317]. The action for a supermembrane coupled to $d = 11$ supergravity is given by [318]

$$\begin{aligned} S = \int d^3x \Bigg[&\frac{1}{2}\sqrt{-h}h^{\mu\nu}\partial_\mu X^M \partial_\nu X^N g_{MN} - \frac{1}{2}\sqrt{-h} \\ &- i\sqrt{-h}h^{\mu\nu}\overline{\Theta}\Gamma_\mu \nabla_\nu \Theta + \frac{1}{6}\epsilon^{\mu\nu\lambda} C_{MNP}\partial_\mu X^M \partial_\nu X^N \partial_\lambda X^P + \cdots \Bigg] \end{aligned} \tag{9.3.7}$$

where $h_{\mu\nu}, \mu, \nu = 0, 1, 2$ is the metric of the world-volume, $h = \det(h_{\mu\nu})$, $g_{MN}, M = 0, 1, \cdots, 10$ is the $d = 11$ space-time metric. X^M is a space-time coordinate and Θ is a fermionic space-time coordinate. C_{MNP} is a three-form gauge field, which is now taken to be zero in our background geometries. The action (9.3.7) is invariant under the rigid supersymmetry transformations

$$\delta_\epsilon X^M = i\overline{\epsilon}\Gamma^M \Theta, \tag{9.3.8}$$

$$\delta_\epsilon \Theta = \epsilon \tag{9.3.9}$$

where ϵ is a constant anti-commuting eleven-dimensional spinor. Also the action (9.3.7) has a local fermionic symmetry, called κ-symmetry. The κ-symmetry transformation is given by

$$\delta_\kappa X^M = 2i\overline{\Theta}\Gamma^M P_+ \kappa(x), \tag{9.3.10}$$

$$\delta_\kappa \Theta = 2P_+\kappa(x) \tag{9.3.11}$$

where $\kappa(x)$ is a $d = 11$ spinor and the matrices

$$P_{\pm} = \frac{1}{2}\left(1 \pm \frac{1}{6\sqrt{-h}}\epsilon^{\mu\nu\lambda}\partial_\mu X^M \partial_\nu X^N \partial_\lambda X^P \Gamma_{MNP}\right) \tag{9.3.12}$$

are projection operators satisfying

$$P_{\pm}^2 = 1, \qquad P_+ P_- = 0, \qquad P_+ + P_- = 1. \tag{9.3.13}$$

To extract the physical degrees of freedom, we must choose the suitable gauge that fixes the local world-volume reparametrization and the local κ-symmetry. Let us fix the reparametrization by choosing $x^0 = X^0$. Then the projection operator (9.3.12) can be expressed as

$$P_{\pm} = \frac{1}{2}\left(1 \pm \Gamma\right) \tag{9.3.14}$$

where

$$\Gamma := \frac{1}{2\sqrt{\det(h_{\Sigma ij})}}\Gamma^0 \epsilon^{ij}\partial_i X^M \partial_j X^N \Gamma_{MN}. \tag{9.3.15}$$

Here $h_{\Sigma ij}, i, j = 1, 2$ is the metric of the Riemann surface wrapped by the M2-brane and $\sqrt{\det(h_{\Sigma ij})}$ is the area of the surface. As a next step we want to fix the local κ-symmetry on the world-volume. In order for a bosonic world-volume configuration to be supersymmetric, the global supersymmetry transformations (9.3.9) need to be compensated for by the κ-symmetry transformations (9.3.11)

$$\left(\delta_\epsilon + \delta_\kappa\right)\Theta = \epsilon + 2P_+\kappa(x) = 0. \tag{9.3.16}$$

Acting P_- on both sides we find that

$$P_-\epsilon = \frac{1-\Gamma}{2}\epsilon = 0. \tag{9.3.17}$$

Therefore the supersymmetry preserved by the M2-branes is given by the Killing spinor ϵ which obeys the projection (9.3.16). Noting that $\Gamma^2 = 1$ and $\Gamma^\dagger = \Gamma$, we find that

$$\epsilon^\dagger \frac{1-\Gamma}{2}\epsilon = \epsilon^\dagger \frac{(1-\Gamma)(1-\Gamma)}{4}\epsilon = \left|\frac{1-\Gamma}{2}\epsilon\right|^2 \geq 0. \tag{9.3.18}$$

By normalizing the Killing spinors such that $\epsilon^\dagger\epsilon = 1$, the inequality (9.3.18) can be rewritten as

$$\mathrm{Vol}(\Sigma_g) \geq \varphi \tag{9.3.19}$$

M_3	supersymmetry
$\mathbb{R} \times (\Sigma_g \subset \mathrm{K3})$	8
$\mathbb{R} \times (\Sigma_g \subset CY_3)$	4
$\mathbb{R} \times (\Sigma_g \subset CY_4)$	2
$\mathbb{R} \times (\Sigma_g \subset CY_5)$	2

Table 9.1: The amounts of the preserved supersymmetries for the M2-branes wrapping holomorphic curves Σ_g in Calabi-Yau spaces. Note that the M2-branes can wrap a holomorphic curve in a CY_5 without loss of the supersymmetries.

where $\mathrm{Vol}(\Sigma_g) = \sqrt{\det(h_{\Sigma ij})}$ is the area of the Riemann surface and φ is the differential two-form defined by

$$\varphi = -\frac{1}{2}\left(\bar{\epsilon}\Gamma_{MN}\epsilon\right) dX^M \wedge dX^N. \tag{9.3.20}$$

Hence the two-form (9.3.20) satisfies the condition (9.3.2) for the calibration and has the bilinear expression for Kähler calibration J (see (9.3.5)). Moreover it can be shown that the two-form (9.3.20) obeys the other required condition (9.3.1) for the calibration by noting the explicit expression (9.3.20) [1]. Therefore we can conclude that the two-form (9.3.20) is a Kähler calibration and that the supersymmetric two-cycle Σ_g wrapped by the M2-branes is a calibrated two-cycle, i.e. a holomorphic curve. Notice that (9.3.16) is precisely the chirality condition $\Gamma^{012}\epsilon = \epsilon$ imposed on the supersymmetry parameters in the BLG-model (see (4.1.52)).

At this stage we are ready to count the number of preserved supersymmetries in our M2-brane configurations by combining the two different types of projections; the projections (9.2.5), (9.2.7), (9.2.9) and (9.2.11) for the background Calabi-Yau manifolds and the projection (9.3.16) (or (4.1.52)) for the membranes wrapped around a calibrated two-cycle Σ_g. In most of the cases wrapped branes break half of the supersymmetries preserved by the special holonomy manifolds according to the additional projection for the branes wrapping calibrated cycles. However, for the Calabi-Yau 5-fold the projection condition (9.3.16) for the M2-branes does not give rise to a further constraint on the surviving two Killing spinors. This implies that M2-branes can wrap a holomorphic curve in a Calabi-Yau 5-fold without breaking down the supersymmetry. The amounts of preserved supersymmetries by the M2-branes wrapping holomorphic curves Σ_g in Calabi-Yau spaces are summarized in Table 9.1. Upon the dimensional reduction to $\mathbb{R}$, the arising quantum mechanics on $\mathbb{R}$ will have the same number of supersymmetries.

[1] It can also be checked by using the supersymmetry algebra [319].

9.4 Topological twisting

In general a quantum field theory on the curved M2-branes interacts with gravity, however, it is also possible to get a supersymmetric quantum field theory on $\mathbb{R} \times \Sigma_g$ by taking the appropriate decoupling limit $l_p \to 0$ while keeping the volume of Σ_g and that of X fixed. In order to derive such low-energy effective theories on the curved world-volume, we recall how the BLG-model describes the dynamics of the flat M2-branes. In the BLG-model the fields and supercharges transform under $SO(2)_E \times SO(8)_R$ as

$$\begin{aligned} X_a^I &: \mathbf{8}_{v0} \\ \Psi_a &: \mathbf{8}_{c+} \oplus \mathbf{8}_{c-} \\ \epsilon &: \mathbf{8}_{s+} \oplus \mathbf{8}_{s-}. \end{aligned} \tag{9.4.1}$$

The eight scalar fields X^I's transform as the vector representations of the R-symmetry $SO(8)_R$ which represents the rotational group of the transverse space of the M2-branes. In other words, they are sections of the normal bundle, which is trivial in this case. However, corresponding to the geometry given in (9.1.1), now the tangent bundle T_X of the ambient Calabi-Yau manifold X is decomposed as

$$T_X = T_\Sigma \oplus N_\Sigma \tag{9.4.2}$$

where T_Σ is the tangent bundle over the Riemann surface Σ_g and N_Σ is the normal bundle over the surface. Therefore we need to take into account the existence of the non-trivial normal bundle of calibrated cycles and to introduce new dynamical variables instead of the original scalar fields. These transitions from scalars, i.e. trivial normal bundle to the non-trivial normal bundles are intimately connected with the way in which the field theory on $\mathbb{R} \times \Sigma_g$ realizes supersymmetry. Along with the coupling to the curvature on the Riemann surface, there now exists a coupling to an external $SO(2n)$ gauge group, the R-symmetry background. Thus one can preserve supersymmetry on the holomorphic Riemann surface by choosing the $SO(2)$ Abelian background from the $SO(2n)$ appropriately.

There is a beautiful observation that such an effective description for curved branes can be obtained by topological twisting [28]. Here we attempt to twist the BLG-model to obtain the low-energy descriptions for the curved M2-branes [2]. Schematically topological twisting procedure can be achieved by replacing the original Euclidean rotational group $SO(2)_E$ on the Riemann surface by a different

[2] For the ABJM-model the geometric meaning of the topological twisting is less clear because the classical $SU(4)_R$ R-symmetry reflects the orbifolds. In this paper we will focus on the BLG-model.

subgroup $SO(2)'_E$ of $SO(2)_E \times SO(8)_R$. Although there are many possible ways to pick such subgroups, here we will consider the following decomposition

$$\begin{aligned} SO(8) \supset & SO(8-2n) \times SO(2n) \\ \supset & SO(8-2n) \times SO(2)_1 \times \cdots \times SO(2)_n. \end{aligned} \tag{9.4.3}$$

The $SO(8-2n)$ is a rotational group of the Euclidean space perpendicular to the Riemann surface, while the $SO(2)_i$ are diagonal subgroups of the external $SO(2n)$ gauge group. The meaning of this decomposition is that the Calabi-Yau manifold X enjoys the decomposable line bundles as the form

$$X = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n \to \Sigma_g. \tag{9.4.4}$$

Under the decomposition (9.4.3), the R-charges for $\mathbf{8}_v$, $\mathbf{8}_s$ and $\mathbf{8}_c$ are determined as follows:

1. $SO(8) \supset SO(6) \times SO(2)_1$

$$\begin{aligned} \mathbf{8}_v =& \mathbf{6}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2} \\ \mathbf{8}_s =& \mathbf{4}_+ \oplus \overline{\mathbf{4}}_- \\ \mathbf{8}_c =& \mathbf{4}_- \oplus \overline{\mathbf{4}}_+. \end{aligned} \tag{9.4.5}$$

2. $SO(8) \supset SO(4) \times SO(2)_1 \times SO(2)_2$

$$\begin{aligned} \mathbf{8}_v =& \mathbf{4}_{00} \oplus \mathbf{1}_{02} \oplus \mathbf{1}_{0-2} \oplus \mathbf{1}_{20} \oplus \mathbf{1}_{-20} \\ \mathbf{8}_s =& \mathbf{2}_{++} \oplus \mathbf{2}'_{+-} \oplus \mathbf{2}_{--} \oplus \mathbf{2}'_{-+} \\ \mathbf{8}_c =& \mathbf{2}_{-+} \oplus \mathbf{2}'_{--} \oplus \mathbf{2}_{+-} \oplus \mathbf{2}'_{++}. \end{aligned} \tag{9.4.6}$$

3. $SO(8) \supset SO(2) \times SO(2)_1 \times SO(2)_2 \times SO(2)_3$

$$\begin{aligned} \mathbf{8}_v =& \mathbf{2}_{000} \oplus \mathbf{1}_{002} \oplus \mathbf{1}_{00-2} \oplus \mathbf{1}_{020} \oplus \mathbf{1}_{0-20} \oplus \mathbf{1}_{200} \oplus \mathbf{1}_{-200} \\ \mathbf{8}_s =& \mathbf{1}_{+++} \oplus \mathbf{1}_{++-} \oplus \mathbf{1}_{+--} \oplus \mathbf{1}_{+-+} \oplus \mathbf{1}_{--+} \oplus \mathbf{1}_{---} \oplus \mathbf{1}_{-+-} \oplus \mathbf{1}_{-++} \\ \mathbf{8}_c =& \mathbf{1}_{-++} \oplus \mathbf{1}_{-+-} \oplus \mathbf{1}_{---} \oplus \mathbf{1}_{--+} \oplus \mathbf{1}_{+-+} \oplus \mathbf{1}_{+--} \oplus \mathbf{1}_{++-} \oplus \mathbf{1}_{+++}. \end{aligned} \tag{9.4.7}$$

4. $SO(8) \supset SO(2)_1 \times SO(2)_2 \times SO(2)_3 \times SO(2)_4$

$$\begin{aligned} \mathbf{8}_v =& \mathbf{1}_{0002} \oplus \mathbf{1}_{000-2} \oplus \mathbf{1}_{0020} \oplus \mathbf{1}_{00-20} \oplus \mathbf{1}_{0200} \oplus \mathbf{1}_{0-200} \oplus \mathbf{1}_{2000} \oplus \mathbf{1}_{-2000} \\ \mathbf{8}_s =& \mathbf{1}_{++++} \oplus \mathbf{1}_{++--} \oplus \mathbf{1}_{+--+} \oplus \mathbf{1}_{+-+-} \oplus \mathbf{1}_{--++} \oplus \mathbf{1}_{----} \oplus \mathbf{1}_{-+-+} \oplus \mathbf{1}_{-++-} \\ \mathbf{8}_c =& \mathbf{1}_{-+++} \oplus \mathbf{1}_{-+--} \oplus \mathbf{1}_{---+} \oplus \mathbf{1}_{--+-} \oplus \mathbf{1}_{+-++} \oplus \mathbf{1}_{+---} \oplus \mathbf{1}_{++-+} \oplus \mathbf{1}_{+++-}. \end{aligned} \tag{9.4.8}$$

With one of the decompositions (9.4.5)-(9.4.8), we can now define a new generator s', i.e. the $SO(2)'_E$ charge by

$$s' := s - \sum_{i=1}^{n} a_i T_i. \tag{9.4.9}$$

Here s denotes a generator of the original rotational group $SO(2)_E$, T_i represents a generator of the subgroup $SO(2)_i$ diagonally embedded in the external gauge group $SO(2n)$ and a_i's are the constant parameters characterizing the twisting procedures. From now on we normalize these charges s', s and T_i such that they are twice as the usual spin on the Riemann surface. Since a_i's are related to the degrees of the line bundles $\mathcal{L}_i$'s as

$$\deg(\mathcal{L}_i) = \begin{cases} 2|g-1|a_i & \text{for } \ g \neq 1 \\ a_i & \text{for } \ g = 1 \end{cases} \tag{9.4.10}$$

and the degrees correspond to the first Chern classes, the conditions that X is Calabi-Yau are given by

$$\sum_{i=1}^{n} a_i = \begin{cases} -1 & \text{for } \ g = 0 \\ 0 & \text{for } \ g = 1 \, . \\ 1 & \text{for } \ g > 1 \end{cases} \tag{9.4.11}$$

Note that the Calabi-Yau conditions (9.4.11) simultaneously ensure the existence of the covariant constant spinors in the twisted theories. One can easily check that the topological twists underlying the decompositions (9.4.5), (9.4.6), (9.4.7) and (9.4.8) preserve 8, 4, 2 and 2 supersymmetries as we expect for K3, CY_3, CY_4 and CY_5.

Therefore given the decomposable line bundle structures of the Calabi-Yau manifolds (9.4.4), we can determine the topological twisting procedure from the two conditions (9.4.10) and (9.4.11). For a K3 surface, i.e. for $a_2 = a_3 = a_4 = 0$, the local geometry is $T^*\Sigma_g$ and a single twisting parameter a_1 is uniquely determined by the Calabi-Yau condition. For other Calabi-Yau spaces the Calabi-Yau conditions are not so powerful and there are infinitely many ways of the twisting characterized by a_i, or the degrees of the line bundles.

Chapter 10

SCQM from M2-branes in a K3 surface

In this chapter we will give further detailed investigation on the wrapped M2-branes on the holomorphic Riemann surface of genus $g > 1$ in a K3 surface. Firstly we will discuss the field content and the supersymmetry in the twisted theory and their consistency in section 10.1. Then we will derive the twisted theory in section 10.2. Finally we will compactify the twisted theory on the Riemann surface and find the IR quantum mechanics in section 10.3. The theory turns out to be the $\mathcal{N} = 8$ superconformal gauged quantum mechanics.

10.1 K3 twisting

In order to obtain the world-volume description for the membranes wrapping a curved Riemann surface of genus $g > 1$ embedded in a K3 surface, we should carry out the topological twisting utilizing the decomposition (9.4.5). Requiring the existence of covariant constant spinors, the twisting procedure can be uniquely determined since the external gauge field is nothing but an $SO(2)$ Abelian background in this case. Note that the twisting for $\Sigma_g = \mathbb{P}^1$ can be realized just by the orientation reversal.

The decomposition $SO(2)_E \times SO(8)_R \to SO(2)'_E \times SO(6)_R$ yields the new field content and the supersymmetry parameters characterized by the following representations for the twisted field theory with $g > 1$:

$$\begin{aligned} X^I &: \mathbf{8}_{v0} \to \mathbf{6}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2} \\ \epsilon &: \mathbf{8}_{s+} \oplus \mathbf{8}_{s-} \to \mathbf{4}_0 \oplus \overline{\mathbf{4}}_2 \oplus \mathbf{4}_{-2} \oplus \overline{\mathbf{4}}_0 \\ \Psi &: \mathbf{8}_{c+} \oplus \mathbf{8}_{c-} \to \mathbf{4}_2 \oplus \overline{\mathbf{4}}_0 \oplus \mathbf{4}_0 \oplus \overline{\mathbf{4}}_{-2}. \end{aligned} \tag{10.1.1}$$

The results of the topological twisting for the components of fields and supersymmetry parameters are shown in Table 10.1 and 10.2 respectively. In the twisted theory the bosonic field content is six scalar fields ϕ^I transforming as $\mathbf{6}_0$ and one-forms Φ_z, $\Phi_{\overline{z}}$ transforming as $\mathbf{1}_2 \oplus \mathbf{1}_{-2}$. The fermionic field content is eight scalar fields $\psi, \tilde{\lambda}$ as $\mathbf{4}_0 \oplus \overline{\mathbf{4}}_0$ and one-forms $\Psi_z, \tilde{\Psi}_{\overline{z}}$ as $\mathbf{4}_2 \oplus \overline{\mathbf{4}}_{-2}$. The supersymmetry parameters are eight scalars $\epsilon, \tilde{\epsilon}$ as $\mathbf{4}_0 \oplus \overline{\mathbf{4}}_0$ and one-forms $\tilde{\epsilon}_z, \epsilon_{\overline{z}}$ as $\overline{\mathbf{4}}_2 \oplus \mathbf{4}_{-2}$.

Here and hereafter we distinguish $\mathbf{4}$ and $\overline{\mathbf{4}}$ in terms of tildes over the fermionic objects.

We should note that there are six bosonic scalar fields and eight fermionic scalar charges in the twisted theory. Since a Riemann surface is a real two-dimensional manifold and there are six scalar fields, the theory should describe the circumstance where the two-cycle lives in a $2 + (8 - 6) = 4$-dimensional curved manifold X. The existence of eight scalar supercharges indicates that the four-manifold preserves $\frac{8}{16} = \frac{1}{2}$ of the supersymmetries. This is the case where a holomorphic Riemann surface Σ_g is embedded in a K3 surface.

Locally the K3 geometry is the cotangent bundle $T^*\Sigma_g$. The remaining two scalar fields combine to yield one-forms on the Riemann surface. They represent the motion of the M2-branes along the non-trivial normal bundle N_Σ over the Riemann surface inside the K3 surface. Under the $SO(6)$ rotational group of the six uncompactified dimensions, the six scalars transform as vector representations $\mathbf{6}_v$ and the one-forms are just singlets. We take the eleven-dimensional space-time configuration as

$$\begin{array}{cccccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \mathrm{K3} & \times & \circ & \circ & \times & \times & \times & \times & \times & \times & \circ & \circ \\ \mathrm{M2} & \circ & \circ & \circ & \times & \times & \times & \times & \times & \times & \times & \times \\ \Sigma_g & \times & \circ & \circ & \times & \times & \times & \times & \times & \times & \times & \times \end{array} \tag{10.1.2}$$

where $\circ$ denotes the direction in which the geometrical objects extend, while $\times$ denotes the direction in which they localize. Note that the projection (9.2.11) for the K3 surface encodes the configuration (10.1.2). The world-volume of the M2-branes extend to a time direction x^0 and spacial directions x^1, x^2. The spacial directions x^1, x^2 are tangent to the compact Riemann surface in the K3 surface. The normal geometry of the M2-branes is divided into two parts; one is the normal bundle N_Σ inside the K3 surface, extending to two directions x^9, x^{10} and the other is the flat Euclidean space transverse to the K3 surface, labeled by $x^3, \cdots, x^8$.

	$SO(2)_E$	$SO(2)_1$	$SO(2)'_E$	$\mathcal{L}$
ϕ^1	0	0	0	$\mathcal{O}$
ϕ^2	0	0	0	$\mathcal{O}$
ϕ^3	0	0	0	$\mathcal{O}$
ϕ^4	0	0	0	$\mathcal{O}$
ϕ^5	0	0	0	$\mathcal{O}$
ϕ^6	0	0	0	$\mathcal{O}$
$\Phi_{\bar{z}}$	0	2	-2	K^{-1}
Φ_z	0	-2	2	K
Ψ_{z1}	1	-1	2	K
Ψ_{z2}	1	-1	2	K
Ψ_{z3}	1	-1	2	K
Ψ_{z4}	1	-1	2	K
$\tilde{\lambda}_1$	1	1	0	$\mathcal{O}$
$\tilde{\lambda}_2$	1	1	0	$\mathcal{O}$
$\tilde{\lambda}_3$	1	1	0	$\mathcal{O}$
$\tilde{\lambda}_4$	1	1	0	$\mathcal{O}$
$\tilde{\Psi}_{\bar{z}1}$	-1	1	-2	K^{-1}
$\tilde{\Psi}_{\bar{z}2}$	-1	1	-2	K^{-1}
$\tilde{\Psi}_{\bar{z}3}$	-1	1	-2	K^{-1}
$\tilde{\Psi}_{\bar{z}4}$	-1	1	-2	K^{-1}
ψ_1	-1	-1	0	$\mathcal{O}$
ψ_2	-1	-1	0	$\mathcal{O}$
ψ_3	-1	-1	0	$\mathcal{O}$
ψ_4	-1	-1	0	$\mathcal{O}$

Table 10.1: The twisting for bosonic scalar fields X^I's and fermionic fields Ψ's of the BLG-model when the Riemann surface of genus $g > 1$ is embedded in a K3 surface. $\mathcal{L}$ is the complex line bundle over Σ_g in which the fields take values. $\mathcal{O}$ and K are the trivial bundle and the canonical bundle respectively.

	$SO(2)_E$	$SO(2)_1$	$SO(2)'_E$	$\mathcal{L}$
ϵ_1	1	1	0	$\mathcal{O}$
ϵ_2	1	1	0	$\mathcal{O}$
ϵ_3	1	1	0	$\mathcal{O}$
ϵ_4	1	1	0	$\mathcal{O}$
$\tilde{\epsilon}_{z1}$	1	-1	2	K
$\tilde{\epsilon}_{z2}$	1	-1	2	K
$\tilde{\epsilon}_{z3}$	1	-1	2	K
$\tilde{\epsilon}_{z4}$	1	-1	2	K
$\tilde{\epsilon}_1$	-1	-1	0	$\mathcal{O}$
$\tilde{\epsilon}_2$	-1	-1	0	$\mathcal{O}$
$\tilde{\epsilon}_3$	-1	-1	0	$\mathcal{O}$
$\tilde{\epsilon}_4$	-1	-1	0	$\mathcal{O}$
$\epsilon_{\bar{z}1}$	-1	1	-2	K^{-1}
$\epsilon_{\bar{z}2}$	-1	1	-2	K^{-1}
$\epsilon_{\bar{z}3}$	-1	1	-2	K^{-1}
$\epsilon_{\bar{z}4}$	-1	1	-2	K^{-1}

Table 10.2: The twisted supersymmetry parameters of the BLG-model probing a K3 surface. The eight covariant constant spinors play the role of BRST generators in the twisted theory. The result is consistent to the fact that a holomorphic curve inside a K3 surface can preserve a half of the supersymmetries (see Table 9.1).

10.2 Twisted theory

The space-time configuration (10.1.2) breaks down the space-time symmetry group $SO(1,10)$ to $SO(2)_E \times SO(6)_R \times SO(2)_1$. Then the $SO(1,10)$ gamma matrix can be decomposed as

$$\begin{cases} \Gamma^\mu = \gamma^\mu \otimes \hat{\Gamma}^7 \otimes \sigma_2 & \mu = 0,1,2 \\ \Gamma^{I+2} = \mathbb{I}_2 \otimes \hat{\Gamma}^I \otimes \sigma_2 & I = 1,\cdots,6 \\ \Gamma^{i+8} = \mathbb{I}_2 \otimes \mathbb{I}_8 \otimes \gamma^i & i = 1,2 \end{cases} \tag{10.2.1}$$

where [1] $\hat{\Gamma}^I$ is the $SO(6)$ gamma matrix obeying

$$\{\hat{\Gamma}^I, \hat{\Gamma}^J\} = 2\delta^{IJ}, \quad (\hat{\Gamma}^I)^\dagger = \Gamma^I \tag{10.2.2}$$

$$\hat{\Gamma}^7 = -i\hat{\Gamma}^{12\cdots 6} = \begin{pmatrix} \mathbb{I}_4 & 0 \\ 0 & -\mathbb{I}_4 \end{pmatrix}. \tag{10.2.3}$$

Similarly the $SO(1,10)$ charge conjugation matrix $\mathcal{C}$ is expressed as

$$\mathcal{C} = \epsilon \otimes \hat{C} \otimes \epsilon \tag{10.2.4}$$

where $\epsilon := i\sigma_2$ is introduced as the charge conjugation matrix with the relations

$$\epsilon^T = -\epsilon, \quad \epsilon\gamma^\mu\epsilon^{-1} = -(\gamma^\mu)^T \tag{10.2.5}$$

while $\hat{C}$ is the $SO(6)$ charge conjugation matrix satisfying [2]

$$\hat{C}^T = -\hat{C}, \qquad \hat{C}\hat{\Gamma}^I\hat{C}^{-1} = (\hat{\Gamma}^I)^T, \qquad \hat{C}\hat{\Gamma}^7\hat{C}^{-1} = -(\hat{\Gamma}^7)^T. \tag{10.2.7}$$

Under the decomposition (10.2.1), the $SO(8)$ chiral matrix becomes

$$\Gamma^{012} = \Gamma^{34\cdots 10} = \mathbb{I}_2 \otimes \hat{\Gamma}^7 \otimes \sigma_2. \tag{10.2.8}$$

[1] $(d+1)$-th component of $d = t + s$ dimensional gamma matrices can be defined by [320]

$$\Gamma^{d+1} := \sqrt{(-1)^{\frac{s-t}{2}}}\Gamma^{12\cdots d}$$

where s and t are correspond to the dimension of space and time respectively. In the above case $s = 6$ and $t = 0$. Note that minus sign should be included in (10.2.3) since we are now considering the decomposition of (4.1.6).

[2] In even dimensional space-time, a charge conjugation matrix can be defined in two ways. Instead of (10.2.7), we may define

$$\hat{C}^T = \hat{C}, \quad \hat{C}\hat{\Gamma}^I\hat{C}^{-1} = -(\hat{\Gamma}^I)^T. \tag{10.2.6}$$

However, Majorana spinors are only allowed for (10.2.7).

For the twisted bosonic fields we set

$$\phi^I := X^{I+2}, \tag{10.2.9}$$

$$\Phi_z := \frac{1}{\sqrt{2}}(X^9 - iX^{10}), \qquad \Phi_{\bar{z}} := \frac{1}{\sqrt{2}}(X^9 + iX^{10}), \tag{10.2.10}$$

$$A_z := \frac{1}{\sqrt{2}}(A_1 - iA_2), \qquad A_{\bar{z}} := \frac{1}{\sqrt{2}}(A_1 + iA_2) \tag{10.2.11}$$

where the bosonic scalar fields ϕ^I's transform as the vector representations $\mathbf{6}_v$ of the $SO(6)$ global symmetry and the indices $I = 1, \cdots, 6$ label the flat transverse directions. The bosonic one-froms, Φ_z and $\Phi_{\bar{z}}$ are the $SO(6)$-singlets and they describe the motion in the normal geometry N_Σ of the Riemann surface inside the K3 surface. These Higgs fields ϕ^I, Φ_z and $\Phi_{\bar{z}}$ are the 3-algebra valued.

Next, consider the twisted fermionic objects. Originally the fermionic fields Ψ are $SL(2, \mathbb{R})$ spinors that transform as the spinor representations $\mathbf{8}_c$ of the $SO(8)_R$ R-symmetry. After the decompositions $Spin(1, 10) \to Spin(2) \times Spin(6) \times Spin(2)$, as seen from (10.1.1), the fermionic fields Ψ are split into the representations $\mathbf{4}_2$, $\bar{\mathbf{4}}_0$, $\mathbf{4}_0$ and $\bar{\mathbf{4}}_{-2}$, whose component fields are denoted by Ψ_z, $\tilde{\lambda}$, ψ and $\tilde{\Psi}_{\bar{z}}$ respectively. Accordingly they can be expanded as

$$\Psi_A^{\alpha\beta} = \frac{i}{\sqrt{2}}\psi_A(\gamma_+\epsilon^{-1})^{\alpha\beta} + i\tilde{\Psi}_{\bar{z}A}(\gamma^{\bar{z}}\epsilon^{-1})^{\alpha\beta} - \frac{i}{\sqrt{2}}\tilde{\lambda}_A(\gamma_-\epsilon^{-1})^{\alpha\beta} - i\Psi_{zA}(\gamma^z\epsilon^{-1})^{\alpha\beta} \tag{10.2.12}$$

where the three indices α, A and β denote the $SO(2)_E$ spinor, the $SO(6)_R$ spinor and the $SO(2)_1$ spinor respectively. Here we have introduced the matrices $\gamma_\pm, \gamma^z$ and $\gamma^{\bar{z}}$ defined by

$$\gamma_+ := \frac{1}{\sqrt{2}}(\mathbb{I}_2 + \sigma_2), \qquad \gamma_- := \frac{1}{\sqrt{2}}(\mathbb{I}_2 - \sigma_2), \tag{10.2.13}$$

$$\gamma^z := \frac{1}{\sqrt{2}}(\gamma^1 + i\gamma^2) = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}, \tag{10.2.14}$$

$$\gamma^{\bar{z}} := \frac{1}{\sqrt{2}}(\gamma^1 - i\gamma^2) = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}. \tag{10.2.15}$$

As seen from (10.2.12), the above matrices enable us to carry out the topological twisting, or in other words the identification of the index α with the index β. The matrices γ_+ and $\gamma^{\bar{z}}$ are associated with the conjugate spinor representations $\mathbf{8}_{c-}$ and yield $\mathbf{4}_0$ and $\bar{\mathbf{4}}_{-2}$, while the other pair of matrices γ_- and γ^z are associated with $\mathbf{8}_{c+}$ and give rise to $\mathbf{4}_2$ and $\bar{\mathbf{4}}_0$. Together with the decomposition (10.2.8) and the chirality condition (4.1.8) for Ψ, one can check that the expansion (10.2.12) leads to the relations; $\hat{\Gamma}^7\psi = \psi$, $\hat{\Gamma}^7\tilde{\Psi}_{\bar{z}} = -\tilde{\Psi}_{\bar{z}}$, $\hat{\Gamma}^7\tilde{\lambda} = -\tilde{\lambda}$ and $\hat{\Gamma}^7\Psi_z = \Psi_z$. For the $\mathcal{A}_4$ algebra all of these fermionic fields are the fundamental representations of the $SO(4)$ gauge group. We define the conjugate of the $SO(6)$ spinors as

$$\overline{\psi} := \psi^T\hat{C}, \quad \overline{\tilde{\lambda}} := \tilde{\lambda}^T\hat{C}, \quad \overline{\Psi}_z := \Psi_z^T\hat{C}, \quad \overline{\tilde{\Psi}}_{\bar{z}} := \tilde{\Psi}_{\bar{z}}^T\hat{C}. \tag{10.2.16}$$

Likewise, the supersymmetry parameters originally transform as the $SL(2,\mathbb{R})$ spinor representations of the rotational group of the world-volume and $\mathbf{8}_s$ of the $SO(8)$ R-symmetry in the BLG-model, while in the twisted theory they reduce to the four distinct representations $\mathbf{4}_0$, $\overline{\mathbf{4}}_2$, $\mathbf{4}_{-2}$ and $\overline{\mathbf{4}}_0$. Thus we can write supersymmetry parameters as

$$\epsilon_A^{\alpha\beta} = \frac{i}{\sqrt{2}}\tilde{\epsilon}_A(\gamma_+\epsilon^{-1})^{\alpha\beta} + i\epsilon_{\bar{z}A}(\gamma^{\bar{z}}\epsilon^{-1})^{\alpha\beta} - \frac{i}{\sqrt{2}}\epsilon_A(\gamma_-\epsilon^{-1})^{\alpha\beta} - i\tilde{\epsilon}_{zA}(\gamma^z\epsilon^{-1})^{\alpha\beta}. \tag{10.2.17}$$

Here again the indices α, A and β label $SO(2)_E$, $SO(6)_R$ and $SO(2)_1$ respectively. Since ϵ and $\tilde{\epsilon}$ are fermionic scalars on an arbitrary Riemann surface, they are identified with supercharges and hence the effective theory will be endowed with the corresponding eight supercharges.

Plugging the expressions (10.2.1), (10.2.9), (10.2.10), (10.2.11) and (10.2.12) into the original BLG Lagrangian (4.1.31), we find the topologically twisted BLG Lagrangian

$$\begin{aligned}
\mathcal{L} = &\frac{1}{2}(D_0\phi^I, D_0\phi^I) - (D_z\phi^I, D_{\bar{z}}\phi^I) + (D_0\Phi_z, D_0\Phi_{\bar{z}}) - 2(D_z\Phi_{\bar{w}}, D_{\bar{z}}\Phi_w) \\
&+ (\overline{\tilde{\lambda}}, D_0\psi) + (\overline{\Psi}_z, D_0\tilde{\Psi}_{\bar{z}}) - (\overline{\tilde{\Psi}}_{\bar{z}}, D_0\Psi_z) - 2i(\overline{\tilde{\Psi}}_{\bar{z}}, D_z\psi) + 2i(\overline{\tilde{\lambda}}, D_{\bar{z}}\Psi_z) \\
&+ \frac{i}{2}(\overline{\tilde{\lambda}}\hat{\Gamma}^{IJ}, [\phi^I, \phi^J, \psi]) - i(\overline{\tilde{\Psi}}_{\bar{z}}\hat{\Gamma}^{IJ}, [\phi^I, \phi^J, \Psi_z]) \\
&+ 2i(\overline{\psi}\hat{\Gamma}^I, [\Phi_{\bar{z}}, \phi^I, \Psi_z]) - 2i(\overline{\tilde{\lambda}}\hat{\Gamma}^I, [\Phi_z, \phi^I, \tilde{\Psi}_{\bar{z}}]) \\
&+ i(\overline{\tilde{\lambda}}, [\Phi_z, \Phi_{\bar{z}}, \psi]) - 2i(\overline{\tilde{\Psi}}_{\bar{w}}, [\Phi_z, \Phi_{\bar{z}}, \Psi_w]) \\
&- \frac{1}{12}\Big([\phi^I, \phi^J, \phi^K], [\phi^I, \phi^J, \phi^K]\Big) - \frac{1}{2}\Big([\Phi_z, \phi^I, \phi^J], [\Phi_{\bar{z}}, \phi^I, \phi^J]\Big) \\
&- \frac{1}{2}\Big([\Phi_z, \Phi_w, \phi^I], [\Phi_{\bar{z}}, \Phi_{\bar{w}}, \phi^I]\Big) - \frac{1}{2}\Big([\Phi_z, \Phi_{\bar{w}}, \phi^I], [\Phi_{\bar{z}}, \Phi_w, \phi^I]\Big) \\
&+ \frac{1}{6}\left([\Phi_z, \Phi_w, \Phi_v], [\Phi_{\bar{z}}, \Phi_{\bar{w}}, \Phi_{\bar{v}}]\right) + \frac{1}{2}\left([\Phi_z, \Phi_w, \Phi_{\bar{v}}], [\Phi_{\bar{z}}, \Phi_{\bar{w}}, \Phi_v]\right) + \mathcal{L}_{\text{TCS}}.
\end{aligned} \tag{10.2.18}$$

Here we have introduced $(\ ,\ )$ as the trace form on the 3-algebra introduced in (4.1.13) and we have defined the covariant derivatives $D_z := \frac{1}{\sqrt{2}}(D_1 - iD_2)$ and $D_{\bar{z}} := \frac{1}{\sqrt{2}}(D_1 + iD_2)$.

Substituting the expressions (10.2.1), (10.2.9), (10.2.10), (10.2.11), (10.2.12) and (10.2.17) into the supersymmetry transformations (4.1.49)-(4.1.51) for BLG theory, we can read off the following BRST transformations

$$\delta\phi_a^I = i\bar{\tilde{\epsilon}}\hat{\Gamma}^I\tilde{\lambda}_a - i\bar{\epsilon}\hat{\Gamma}^I\psi_a, \tag{10.2.19}$$

$$\delta\Phi_{za} = -i\bar{\tilde{\epsilon}}\Psi_{za}, \tag{10.2.20}$$

$$\delta\Phi_{\bar{z}a} = -i\bar{\epsilon}\tilde{\Psi}_{\bar{z}a}, \tag{10.2.21}$$

$$\delta\psi_a = iD_0\phi_a^I\hat{\Gamma}\tilde{\epsilon} - 2D_{\bar{z}}\Phi_{za}\epsilon + \frac{1}{6}[\phi^I,\phi^J,\phi^K]_a\hat{\Gamma}^{IJK}\tilde{\epsilon} + [\Phi_z,\Phi_{\bar{z}},\phi^I]_a\hat{\Gamma}^I\tilde{\epsilon}, \tag{10.2.22}$$

$$\delta\tilde{\lambda}_a = iD_0\phi_a^I\hat{\Gamma}^I\epsilon - 2D_z\Phi_{\bar{z}a}\tilde{\epsilon} - \frac{1}{6}[\phi^I,\phi^J,\phi^K]_a\hat{\Gamma}^{IJK}\epsilon + [\Phi_z,\Phi_{\bar{z}},\phi^I]_a\hat{\Gamma}^I\epsilon, \tag{10.2.23}$$

$$\delta\Psi_{za} = -D_z\phi^I\hat{\Gamma}^I\tilde{\epsilon} - iD_0\Phi_z\epsilon + \frac{1}{2}[\Phi_z,\phi^I,\phi^J]_a\hat{\Gamma}^{IJ}\epsilon + \frac{1}{3}[\Phi_w,\Phi_{\overline{w}},\Phi_z]_a\epsilon, \tag{10.2.24}$$

$$\delta\tilde{\Psi}_{\bar{z}a} = D_{\bar{z}}\phi_a^I\hat{\Gamma}^I\epsilon + iD_0\Phi_{\bar{z}a}\tilde{\epsilon} + \frac{1}{2}[\Phi_{\bar{z}},\phi^I,\phi^J]_a\hat{\Gamma}^{IJ}\tilde{\epsilon} + \frac{1}{3}[\Phi_{\overline{w}},\Phi_w,\Phi_{\bar{z}}]_a\tilde{\epsilon}, \tag{10.2.25}$$

$$\delta\tilde{A}_{0a}^b = -\bar{\epsilon}\hat{\Gamma}^I\phi_c^I\psi_d f^{cdb}{}_a - \bar{\tilde{\epsilon}}\hat{\Gamma}^I\phi_c^I\tilde{\lambda}_d f^{cdb}{}_a - 2\bar{\epsilon}\Phi_{zc}\tilde{\Psi}_{\bar{z}d}f^{cdb}{}_a + 2\bar{\tilde{\epsilon}}\Phi_{\bar{z}c}\Psi_{zd}f^{cdb}{}_a, \tag{10.2.26}$$

$$\delta\tilde{A}_{za}^b = 2i\bar{\epsilon}\hat{\Gamma}^I\phi_c^I\Psi_{zd}f^{cdb}{}_a + 2i\bar{\epsilon}\Phi_{zc}\tilde{\lambda}_d f^{cdb}{}_a, \tag{10.2.27}$$

$$\delta\tilde{A}_{\bar{z}a}^b = -2i\bar{\tilde{\epsilon}}\hat{\Gamma}^I\phi_c^I\tilde{\Psi}_{\bar{z}d}f^{cdb}{}_a + 2i\bar{\tilde{\epsilon}}\Phi_{\bar{z}c}\psi_d f^{cdb}{}_a. \tag{10.2.28}$$

10.3 Derivation of quantum mechanics

In the previous section we have derived the topologically twisted BLG-model as the low-energy effective field theories on the curved M2-branes. Now we attempt to reduce the theory further to a low-energy effective one-dimensional field theory on $\mathbb{R}$. As mentioned in the analysis for the membranes wrapped around a torus, when the size of the Riemann surface shrinks, only the light degrees of freedom are relevant. To keep track of them we have to find the static configurations that minimize the energy, that is the zero-energy conditions. We can replace the zero-energy conditions by a set of BPS equations. In addition, we set all the fermionic fields to zero because we are interested in bosonic BPS configurations. Then the BPS equations, which correspond to the vanishing conditions of the BRST transformations (10.2.22)-(10.2.25) for the fermionic fields, are

$$D_z\phi^I = 0, \quad D_{\bar{z}}\phi^I = 0, \tag{10.3.1}$$

$$D_z\Phi_{\bar{z}} = 0, \quad D_{\bar{z}}\Phi_z = 0, \tag{10.3.2}$$

$$[\phi^I,\phi^J,\phi^K] = 0, \tag{10.3.3}$$

$$[\Phi_z,\Phi_{\bar{z}},\phi^I] = 0, \quad [\Phi_z,\phi^I,\phi^J] = 0, \quad [\Phi_{\bar{z}},\phi^I,\phi^J] = 0, \tag{10.3.4}$$

$$[\Phi_w,\Phi_{\overline{w}},\Phi_z] = 0, \quad [\Phi_{\overline{w}},\Phi_w,\Phi_{\bar{z}}] = 0. \tag{10.3.5}$$

Note that due to the algebraic equations (10.3.3), (10.3.4) and (10.3.5), all the bosonic Higgs fields have to lie in the same plane in the $SO(4)$ gauge group. Thus we can write them as

$$\phi^I = (\phi^{I1}, \phi^{I2}, 0, 0)^T, \quad \Phi_z = (\Phi_z^1, \Phi_z^2, 0, 0)^T, \quad \Phi_{\bar{z}} = (\Phi_{\bar{z}}^1, \Phi_{\bar{z}}^2, 0, 0)^T. \tag{10.3.6}$$

From the supersymmetry we can write the corresponding fermionic partners as

$$\psi = (\psi^1, \psi^2, 0, 0)^T, \qquad \tilde{\lambda} = (\tilde{\lambda}^1, \tilde{\lambda}^2, 0, 0)^T, \tag{10.3.7}$$
$$\Psi_z = (\Psi_z^1, \Psi_z^2, 0, 0)^T, \qquad \tilde{\Psi}_{\bar{z}} = (\tilde{\Psi}_{\bar{z}}^1, \tilde{\Psi}_{\bar{z}}^2, 0, 0)^T. \tag{10.3.8}$$

The configurations (10.3.6)-(10.3.8) generically break the original $SO(4)$ gauge group down to $U(1) \times U(1)$. Taking into account these solutions and the BPS equations (10.3.1), (10.3.2) we find that $\tilde{A}^1_{z3} = \tilde{A}^2_{z3} = \tilde{A}^1_{z4} = \tilde{A}^2_{z4} = 0$. This implies that these components of the gauge field now become massive by the Higgs mechanism. Then we should follow the time evolution for remaining degrees of freedom in the low-energy effective theory.

To achieve this consistently we further need to impose the Gauss law constraint. This requires that the gauge field is flat; $\tilde{F}_{z\bar{z}} = 0$. Recall that we are now considering the case where the genus of the Riemann surface is greater than one. In that case the generic flat connections are irreducible. As long as we only consider irreducible flat connections, the Laplacian has no zero modes. Accordingly it is not allowed for scalar fields to have non-trivial values and it is required that $\phi^I = 0$ [3]. To sum up, the above set of equations over the compact Riemann surface of genus $g > 1$ reduces to

$$\tilde{F}^1_{z\bar{z}2} = 0, \tag{10.3.9}$$
$$\partial_{\bar{z}}\Phi_{z1} + \tilde{A}^1_{\bar{z}2}\Phi_{z2} = 0, \tag{10.3.10}$$
$$\partial_{\bar{z}}\Phi_{z2} - \tilde{A}^1_{\bar{z}2}\Phi_{z1} = 0. \tag{10.3.11}$$

We now want to determine the generic BPS configuration obeying (10.3.9)-(10.3.11). Since we are now considering a compact Riemann surface of genus g, there are g holomorphic $(1,0)$-forms ω_i, $i = 1, \cdots, g$ and g anti-holomorphic $(0,1)$-forms $\overline{\omega}_i$. Let us normalize them as

$$\int_{a_i} \omega_j = \delta_{ij}, \quad \int_{b_i} \omega_j = \Omega_{ij} \tag{10.3.12}$$

with a_i, b_i being canonical homology basis for $H_1(\Sigma_g)$ (see Figure 10.1). The matrix

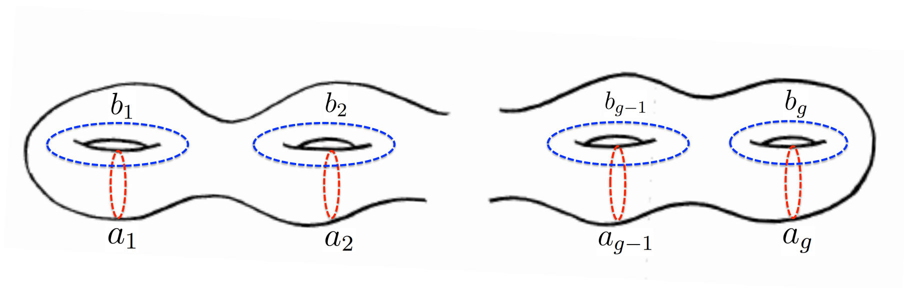


Figure 10.1: A Riemann surface Σ_g of genus g. a_i and b_i generate $H_1(\Sigma_g)$.

Ω is the period matrix of the Riemann surface. It is a $g \times g$ complex symmetric matrix with positive imaginary part. The equation (10.3.9) imposes the flatness condition for the $U(1)$ gauge field $\tilde{A}^1_{z2}$. The space of the $U(1)$ flat connection on a compact Riemann surface is the torus known as the Jacobi variety denoted by $\mathrm{Jac}(\Sigma_g)$. The flat gauge fields can be expressed in the form [284]

$$\tilde{A}^1_{z2} = -2\pi \sum_{i,j=1}^{g} \left(\Omega - \overline{\Omega}\right)^{-1}_{ij} \Theta^i \omega_j, \qquad \tilde{A}^1_{\bar{z}2} = 2\pi \sum_{i,j=1}^{g} \left(\Omega - \overline{\Omega}\right)^{-1}_{ij} \overline{\Theta}^i \overline{\omega}_j \tag{10.3.13}$$

where $\Theta^i := \zeta^i + \overline{\Omega}_{ij}\xi^j$ represents the complex coordinate of $\mathrm{Jac}(\Sigma_g)$ which characterizes the twists $e^{2\pi i \xi^i}$ and $e^{-2\pi i \zeta^i}$ around the i-th homology cycles a_i and b_i. Notice that $\xi^i \to \xi^i + m^i$, $\zeta^i \to \zeta^i + n^i$ for $n^i, m^i \in \mathbb{Z}$ gives rise to the same point on $\mathrm{Jac}(\Sigma_g)$. This implies that $\mathrm{Jac}(\Sigma_g) = \mathbb{C}^g / L_\Omega$ where L_Ω is the lattice generated by $\mathbb{Z}^g + \Omega\mathbb{Z}^g$. We define a function

$$\varphi := -2\pi \sum_{i,j=1}^{g} \left(\Omega - \overline{\Omega}\right)^{-1}_{ij} \left(\Theta^i f_j(z) - \overline{\Theta}^i \overline{f}_j(\overline{z})\right) \tag{10.3.14}$$

where $f_i(z) := \int^z \omega_i$ is the holomorphic function of z that obeys the relations $f_i|_{a_j} = \delta_{ij}$ and $f_i|_{b_j} = \Omega_{ij}$. We then can express the flat gauge fields as

$$\tilde{A}^1_{z2} = \partial_z \varphi, \qquad \tilde{A}^1_{\bar{z}2} = \partial_{\bar{z}} \varphi. \tag{10.3.15}$$

The above expressions (10.3.13) for the $U(1)$ flat connection allows us to write the

[3] Such BPS solutions with the irreducible connections have been considered in the four-dimensional topologically twisted Yang-Mills theories defined on the product of two Riemann surfaces [321, 36, 293] and the corresponding decoupling limit for the brane description has been argued in [40].

generic solutions to the equation (10.3.10) and (10.3.11) in the following forms:

$$\begin{aligned}
\Phi_{z1}(z,\bar{z}) - i\Phi_{z2}(z,\bar{z}) =& e^{-i\varphi(z,\bar{z})}\sum_{i=1}^{g} x_A^i \omega_i, \\
\Phi_{z1}(z,\bar{z}) + i\Phi_{z2}(z,\bar{z}) =& e^{i\varphi(z,\bar{z})}\sum_{i=1}^{g} x_B^i \omega_i
\end{aligned} \qquad (10.3.16)$$

where x_A^i, $x_B^i \in \mathbb{C}$ are constant on the Riemann surface. Since we take the limit where the Riemann surface Σ_g shrinks to zero size, the space-time configurations of the membranes should be expressed as single-valued functions of z and $\bar{z}$ in the low-energy effective quantum mechanics. In other words, ξ^i and ζ^i can only be integers and therefore the $U(1)$ flat gauge fields $\tilde{A}^1_{z2}$ and $\tilde{A}^1_{\bar{z}2}$ are quantized. The single-valuedness condition requires that the point of the $\mathrm{Jac}(\Sigma_g)$ is fixed.

Putting all together, the general bosonic BPS configurations are given by

$$\begin{aligned}
&\phi^I = 0 \\
&\Phi_z = \sum_{i=1}^{g} \begin{pmatrix} \frac{1}{2}\left(e^{-i\varphi}x_A^i + e^{i\varphi}x_B^i\right) \\ \frac{i}{2}\left(e^{-i\varphi}x_A^i - e^{i\varphi}x_B^i\right) \\ 0 \\ 0 \end{pmatrix} \omega_i, \quad \Phi_{\bar{z}} = \sum_{i=1}^{g} \begin{pmatrix} \frac{1}{2}\left(e^{i\varphi}\bar{x}_A^i + e^{-i\varphi}\bar{x}_B^i\right) \\ -\frac{i}{2}\left(e^{i\varphi}\bar{x}_A^i - e^{-i\varphi}\bar{x}_B^i\right) \\ 0 \\ 0 \end{pmatrix} \overline{\omega}_i, \\
&\tilde{A}_z = \begin{pmatrix} 0 & \partial_z\varphi(z,\bar{z}) & 0 & 0 \\ -\partial_z\varphi(z,\bar{z}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{A}^3_{z4}(z,\bar{z}) \\ 0 & 0 & -\tilde{A}^3_{z4}(z,\bar{z}) & 0 \end{pmatrix}
\end{aligned} \qquad (10.3.17)$$

where $\tilde{A}^3_{z4}$ and $\tilde{A}^3_{\bar{z}4}$ are the Abelian gauge fields associated with preserved $U(1)$ symmetry and they do not receive any constraints from the BPS conditions.

By virtue of the supersymmetry we can write the corresponding fermionic fields from the bosonic configurations (10.3.17) as

$$\begin{aligned}
&\psi = 0, && \tilde{\lambda} = 0, \\
&\Psi_z = \sum_{i=1}^{g} \begin{pmatrix} \frac{1}{2}\left(\Psi_A^i + \Psi_B^i\right) \\ \frac{i}{2}\left(\Psi_A^i - \Psi_B^i\right) \\ 0 \\ 0 \end{pmatrix} \omega_i, && \tilde{\Psi}_{\bar{z}} = \sum_{i=1}^{g} \begin{pmatrix} \frac{1}{2}\left(\tilde{\Psi}_A^i + \tilde{\Psi}_B^i\right) \\ -\frac{i}{2}\left(\tilde{\Psi}_A^i - \tilde{\Psi}_B^i\right) \\ 0 \\ 0 \end{pmatrix} \overline{\omega}_i.
\end{aligned} \qquad (10.3.18)$$

By inserting the BPS configuration (10.3.17) and (10.3.18) into the twisted action

(10.2.18), we find

$$S = \int_{\mathbb{R}} dt \int_{\Sigma_g} d^2z \Bigg[(D_0\Phi^a_z, D_0\Phi_{\bar{z}a}) + \left(\overline{\Psi}^a_z, D_0\tilde{\Psi}_{\bar{z}a}\right) - \left(\overline{\tilde{\Psi}}^a_{\bar{z}}, D_0\Psi_{za}\right)$$

$$- \frac{k}{2\pi}\tilde{A}^1_{02}\tilde{F}^3_{z\bar{z}4} - \frac{k}{4\pi}\left(\tilde{A}^1_{z2}\dot{\tilde{A}}^3_{\bar{z}4} - \tilde{A}^1_{\bar{z}2}\dot{\tilde{A}}^3_{z4}\right)\Bigg]. \qquad (10.3.19)$$

Since the gauge fields $\tilde{A}^1_{z2}$, $\tilde{A}^1_{\bar{z}2}$ are quantized and there are no their time derivatives in the effective action, we can integrate them out as the auxiliary fields. They give rise to the constraints $\dot{\tilde{A}}^3_{z4} = \dot{\tilde{A}}^3_{\bar{z}4} = 0$.

In order to perform the integration over the Riemann surface, we use the Riemann bilinear relation [322]

$$\int_{\Sigma_g} \omega \wedge \eta = \sum_{i=1}^{g} \left[\int_{a_i} \omega \int_{b_i} \eta - \int_{b_i} \omega \int_{a_i} \eta \right]. \qquad (10.3.20)$$

By carrying out the integration over the Riemann surface Σ_g we find the gauged quantum mechanical action

$$S = \int_{\mathbb{R}} dt \left[\sum_{i,j} (\mathrm{Im}\,\Omega)_{ij} \left(D_0 x^{ia} D_0 \overline{x}^j_a + \overline{\Psi}^{ia} D_0 \tilde{\Psi}^j_a - \overline{\tilde{\Psi}}^{ia} D_0 \Psi^j_a \right) - kC_1(E)\tilde{A}^1_{02} \right]. \qquad (10.3.21)$$

Here the indices $a = A, B$ stand for the two internal degrees of freedom for the two M2-branes. The covariant derivatives are defined by

$$D_0 x^i_A = \dot{x}^i_A + i\tilde{A}^1_{02} x^i_A, \qquad D_0 x^i_B = \dot{x}^i_B - i\tilde{A}^1_{02} x^i_B, \qquad (10.3.22)$$

$$D_0 \Psi^i_A = \dot{\Psi}^i_A + i\tilde{A}^1_{02} \Psi^i_A, \qquad D_0 \Psi^i_B = \dot{\Psi}^i_B - i\tilde{A}^1_{02} \Psi^i_B, \qquad (10.3.23)$$

$$D_0 \tilde{\Psi}^i_A = \dot{\tilde{\Psi}}^i_A - i\tilde{A}^1_{02} \tilde{\Psi}^i_A, \qquad D_0 \Psi^i_B = \dot{\tilde{\Psi}}^i_B + i\tilde{A}^1_{02} \tilde{\Psi}^i_B \qquad (10.3.24)$$

and the Chern number $C_1(E) \in \mathbb{Z}$ is associated to the $U(1)$ principal bundle $E \to \Sigma_g$ over the Riemann surface

$$C_1(E) = \int_{\Sigma_g} c_1(E) = \frac{1}{2\pi} \int_{\Sigma_g} d^2z \tilde{F}^3_{z\bar{z}4}. \qquad (10.3.25)$$

The action (10.3.21) has the invariance under the one-dimensional $SL(2, \mathbb{R})$ conformal transformations

$$\delta t = f(t) = a + bt + ct^2, \qquad \delta\partial_0 = -\dot{f}\partial_0, \qquad (10.3.26)$$

$$\delta x^i_a = \frac{1}{2}\dot{f} x^i_a, \qquad \delta\tilde{A}^1_{02} = -\dot{f}\tilde{A}^1_{02}, \qquad (10.3.27)$$

$$\delta\Psi^i_a = 0, \qquad \delta\tilde{\Psi}^i_a = 0. \qquad (10.3.28)$$

The action (10.3.21) is also invariant under the $\mathcal{N} = 8$ supersymmetry transformation laws

$$\delta x^i_a = 2i\bar{\tilde{\epsilon}}\Psi^i_a, \qquad \delta \overline{x}^i_a = 2i\bar{\epsilon}\tilde{\Psi}^i_a, \tag{10.3.29}$$

$$\delta \Psi^i_a = -iD_0 x^i_a \epsilon, \qquad \delta \tilde{\Psi}^i_a = iD_0 \overline{x}^i_a \tilde{\epsilon}, \tag{10.3.30}$$

$$\delta \tilde{A}^1_{02} = 0. \tag{10.3.31}$$

We thus conclude that the $\mathcal{N} = 8$ superconformal gauged quantum mechanics (10.3.21) may describe the low-energy effective motion of the two wrapped M2-branes around Σ_g probing a K3 surface.

We see from the action (10.3.21) that the $U(1)$ gauge field $\tilde{A}^1_{02}$, due to the absence of the kinetic term, is regarded as an auxiliary field. In consequence the gauge field has no contribution to the Hamiltonian. Hence the corresponding gauge symmetry yields an integral of motion as a moment map $\mu : \mathcal{M} \to \mathfrak{u}(1)^*$ and we can reduce the phase space $\mathcal{M}$ to $\mathcal{M}_c = \mu^{-1}(c)$ by fixing the inverse of the moment map at a point $c \in \mathfrak{u}(1)^*$. Choosing a temporal gauge $\tilde{A}^1_{02} = 0$, we find the action

$$S = \int_{\mathbb{R}} dt \sum_{i,j} (\mathrm{Im}\Omega)_{ij} \left(\dot{x}^{ia}\dot{\overline{x}}^j_a + \overline{\Psi}^{ia}\dot{\tilde{\Psi}}^j_a - \overline{\tilde{\Psi}}^{ia}\dot{\Psi}^j_a \right) \tag{10.3.32}$$

and the Gauss law constraint

$$\phi_0 := kC_1(E) + i\sum_{i,j} (\mathrm{Im}\Omega)_{ij} \left[K_{ij} + 2\left(\overline{\Psi}^i_A \tilde{\Psi}^j_A - \overline{\Psi}^i_B \tilde{\Psi}^j_B \right) \right] = 0 \tag{10.3.33}$$

where

$$K_{ij} := \left(\dot{x}^i_A \overline{x}^j_A - x^i_A \dot{\overline{x}}^j_A \right) - \left(\dot{x}^i_B \overline{x}^j_B - x^i_B \dot{\overline{x}}^j_B \right). \tag{10.3.34}$$

The constraint equation (10.3.33) requires that all states in the Hilbert space are gauge invariant. In this case the symmetry of the system is not so large as in the previous superconformal gauged quantum mechanical models (6.1.13) and (7.1.17). It is curious to know whether the superconformal gauged quantum mechanics (10.3.21) (or (10.3.32) together with (10.3.33)) have a reduced Lagrangian description with an inverse-square type potential. However, our result may drop a hint on the obstructed construction of SCQM that a large class of SCQM could be formulated as "gauged quantum mechanics" with the help of auxiliary gauge fields as in [150, 151, 152].

It might be helpful to determine the corresponding supermultiplet for our $\mathcal{N} = 8$ superconformal quantum mechanics (10.3.21). We, however, do not fully understand it because our derivation does not rely on the superfield formulation and the reduced quantum mechanical description has not been acquired so far. Judging from the representations (10.1.1) of the physical variables under the remaining R-symmetry $SO(6)$, the corresponding supermultiplet may be inferred as the g sets of $(\mathbf{2}, \mathbf{8}, \mathbf{6})$ multiplet. However, after integrating out the single auxiliary gauge field $\tilde{A}^1_{02}$, the physical degrees of freedom may be reduced and thus the supermultiplets may be modified.

Chapter 11

Conclusion and Discussion

11.1 Conclusion

In this thesis we have established the new connection between two subjects; the superconformal quantum mechanics and the M2-branes by examining the IR superconformal quantum mechanics resulting from the multiple M2-branes wrapped around a compact Riemann surface Σ_g after shrinking the size of the Riemann surface.

We have seen that conformal symmetry and supersymmetry in quantum mechanics, i.e. one-dimensional field theory are rather out of the way in that they contain numerous unfamiliar features which are not observed in higher dimensional field theories.

Instead of the morbid Hamiltonian, one can label the state in terms of the eigenstate of the compact operator $L_0 = \frac{1}{2}(H + K)$ and the second Casimir operator of the $SL(2, \mathbb{R})$ conformal symmetry group. Although one cannot assume the existence of both normalizable conformally invariant states and invariant primary operators due to the fact that the quantum mechanics is based on the Hilbert space not on the Fock space, the 2-point, 3-point and 4-point functions which satisfy the conformal constraints can be constructed by using those two defects [86, 87]. We have also discussed the interesting observations [65, 66] that the motion of the particle near the horizon of the extreme Reissner-Nordström black hole is described by the (super)conformal mechanics. This indicates that (super)conformal quantum mechanics may caputure the information of the dual AdS_2 gravity. Obviously further surveys are needed to understand $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence.

Due to the reduced Poincaré symmetry, one-dimensional supersymmetry has the special properties that (i) the number of the component fields in the supermultiplet is larger than the number $\mathcal{N}$ of supersymmetry if $\mathcal{N}$ is greater thatn eight and that (ii) the number n of physical bosonic component fields is not necessarily same as that of the fermions. These facts allow us to construct various supermultiplets $(\boldsymbol{n}, \mathcal{N}, \mathcal{N} - \boldsymbol{n})$ only for $\mathcal{N} = 1, 2, 4$ and 8 supersymmetric quantum mechanics. Indeed we have argued that for such supersymmetric quantum mechanics there have been continuous attempts to construct superconformal mechanical models by appealing the superspace and superfield formalism.

We have shown that the IR quantum mechanics arising from the BLG-model and the ABJM-model wrapped on a torus are the $\mathcal{N} = 16$ and $\mathcal{N} = 12$ superconformal gauged quantum mechanical models respectively. Furthermore after the integration of the auxiliary gauge fields, we found that the $OSp(16|2)$ quantum mechanics (6.3.1) and $SU(1,1|6)$ quantum mechanics (7.3.1) emerge from the reduced theories. Both of them are $\mathcal{N} > 8$ superconformal quantum mechanical models which have not been available by the superspace and superfield formalism so far. It is interesting to investigate their spectrums, wavefunctions and correlation functions for those new superconformal mechanical models.

We have also surveyed the membranes wrapped around a genus $g \neq 1$ Riemann surface. In this case the surface is singled out as a calibrated holomorphic curve in a Calabi-Yau manifold to preserve supersymmetry. We have found that the IR quantum mechanical models have $\mathcal{N} = 8$, 4, 2 and 2 supersymmetries for K3, CY_3, CY_4 and CY_5 respectively. Especially when the Calabi-Yau manifolds are constructed via decomposable line bundles over the Riemann surface, the K3 surface essentially allows for a unique topological twist while for the other Calabi-Yau manifolds there are infinitely many topological twists which are specified by the degrees of the line bundles.

We have especially analyzed the two membranes wrapping a holomorphic genus $g > 1$ curve embedded in a K3 surface based on the topologically twisted BLG-model. We have found the new $\mathcal{N} = 8$ superconformal gauged quantum mechanics (10.3.21) that may describe the low-energy dynamics of the wrapped M2-branes in a K3 surface. It is known that [150, 151, 152] there are the connections of the gauged quantum mechanics to the conformal mechanical models, the Calogero model and their generalizations. An interesting question is what type of interaction potential, if it exists, may characterize our superconformal "gauged" quantum mehcanics (10.3.21). The structure of the resulting theory may indicate that generic SCQM takes the form of superconformal gauged quantum mechanics along with auxiliary gauge fields.

11.2 Future directions

There may be a number of future aspects of the present work. In the following we will briefly discuss the possible three applications.

11.2.1 $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence

$\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ correspondence [17] is an important example of the holographic principle [323].

For $d = 2$ it has been shown [324] that the Hilbert space of the any quantum gravity on an asymptotically AdS_3 space-time is a representation of the two-dimensional conformal group and that the central charge of the $d = 2$ CFT is given by

$$c = \frac{3l}{2G} \tag{11.2.1}$$

where l is the AdS_3 radius and G is Newton constant. The relationship between the BTZ black hole and the state in the two-dimensional CFT indicates that the entropy of the black may be defined as the logarism of the degeneracy of the corresponding states in the CFT. In this perspective the entropy of the $d = 3$ Baãdos-Teitelboim-Zanelli (BTZ) black hole is computed by counting the states of the $d = 2$ conformal field theory on the boudnary of AdS_3 [325]

$$S = 2\pi\sqrt{\frac{cn_R}{6}} + 2\pi\sqrt{\frac{cn_L}{6}} \tag{11.2.2}$$

where n_R and n_L are the eigenvalues of the Virasoro generators L_0 and $\overline{L}_0$ respectively. For large L_0 one can use the Cardy formula to evaluate the degeneracy of the states and it has been shown [326, 327, 328, 329] that the result agrees with the one obtained by Wald's formula [330, 331, 332, 333].

The case of $d = 1$, i.e. $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence [78, 334, 80, 79, 81, 83, 82, 335, 336, 337, 75, 84, 85, 338, 339, 340, 341, 76, 342, 343, 344] is less understood, however, it is extremely significant case of $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ correspondence in that all known extremal black holes contain the AdS_2 factor in their near horizon geometries [345, 346]. The two candidates for the CFT_1 have been proposed

(i) conformal quantum mechanics

(ii) a chiral half of a $d = 2$ CFT.

For the former only the global $SL(2, \mathbb{R})$ acts nontrivially on the Hilbert space, while in the in latter case one copy of the Virasoro generators acts nontrivially on the Hilbert space. In [341] the central charge for the CFT_1 which corresponds to the quantum gravity with a $U(1)$ gauge field on AdS_2 has been given by

$$c = \frac{3kE^2 l^4}{4} \tag{11.2.3}$$

where l is the AdS_2 radius, E is the electric field and k is the level of the $U(1)$ current. The expression is similar to (11.2.1) for $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. It has been discussed [78, 341] that the latter idea of the non-trivial action of the Virasoro could be consistent and $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence reduces to the $\mathrm{CFT}_2/\mathrm{CFT}_2$ duality on the strip. As discussed in [78, 341], this idea could be true when AdS_2 is generated as a S^1 compactification of AdS_3, however, there may be other types of the AdS_2 which do not arise as a S^1 compactification of AdS_3 and therefore the former possibility could still be a good candidate of the CFT_1. In the former perspective, it has been proposed [343] that the logarithm of the ground state degeneracy in a conformal quantum mechanics living on the boundary of AdS_2 yields the definition of the entropy of the extremal black hole in the quantum theory. Furthermore it has been pointed out in [86, 87] that the correlation functions of the conformal quantum mechanics [54] have the expected scaling behaviors from $\mathrm{AdS}_2/\mathrm{CFT}_1$ correpondence although one cannot assume the existence of the normalized and conformal invariant vacuum states in conformal quantum mechanics as in other higher dimensional conformal field theories. It is interesting to investigate whether our superconformal quantum mechanics resulting from the wrapped M2-branes around a compact Riemann surface in M-theory could provide some examples of the $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence in the former perspective.

11.2.2 Indices and the reduced Gromov-Witten invariants

Another topic is the computation of the indices and their applications. For instance, the BPS partition function which gives rise to the counding of the BPS states may be related to the number of the supersymmetric two-cycles of genus g in our setup. Indeed, in the setup where the curved D3-branes wrapping supersymmetric two-cycles embedded in K3 surface, the formula for the numbers of rational curves with g double points on a K3 surface, the so-called reduced Gromov-Witten invariants [347] has been conjectured by Yau and Zaslow [39] in the computation of the BPS partition function by appealing the string duality. Closely related to their setup, our $\mathcal{N} = 8$ superconformal gauged quantum mechanics (10.3.21) appears from the wrapped M2-branes instead of the D3-brane. It would be interesting to compute the indices and to extract enumerative information and structure from our model.

In order to compute the indices we take a trace over the eigenstates. As discussed in section 2.2, it is difficult to calculate a trace over the eigenstate of the Hamiltonian H for the superconformal quantum mechanics because there is no normalizable ground state and its spectrum is continuous. As proposed in [73], the indices in superconformal quantum mechanics can be defined by taking a trace over the eigenstates of the compact operator $L_0 = \frac{1}{2}(H + K)$ which has a normalizable ground state and the discrete eigenvalues with equal spacing as

$$\mathcal{I}(\mathcal{O}) = \mathrm{Tr}_{L_0}(-1)^{2J}\mathcal{O}e^{-\beta(L_0 - J)} \tag{11.2.4}$$

where J is the R-symmetry generator and $\mathcal{O}$ is some operator in the theory. It is an open problem to evaluate indices and understand their physical and mathematical implication for our superconformal quantum mechanics.

11.2.3 1d-2d relation

Finally we want to comment on the "1d-2d relation", which is motivated by the fascinating stories arising from the compactification of M5-branes, for example, the AGT-relation [348], the DGG-relation [349] and the 2d-4d relation [350]. It has been argued that the world-volume theories of multiple M5-branes can be described by the six-dimensional superconformal field theories labeled by a simply-laced Lie algebra $\mathfrak{g}$, the so-called $(2,0)$ theories. Via compactification, such theories leads to a family $T[M_{6-d}, \mathfrak{g}]$ of d-dimensional superconformal field theories which can be labeled by a choice of a specific manifold M_{6-d} and a Lie algebra $\mathfrak{g}$. From this perspective the AGT-relation, the DGG-relation and 2d-4d relation are regarded as the decomposition of the six-dimensional world-volume of M5-branes as $6 = 4 + 2$, $3 + 3$ and $2 + 4$ respectively.

On the other hand, the world-volume theories of multiple M2-branes can be described by the three-dimensional superconformal field theories. Unlike the M5-branes we know the explicit Lagrangian for such world-volume theories as the BLG-model and the ABJ(M) model. It would be attractive to find the new relationship between the superconformal field theories and the geometries or relevant dualities from M2-branes, i.e. "1d-2d relation" arising from the decomposition of the three-dimensional world-volume of M2-branes as $3 = 1 + 2$. As an exchange of the order we may have two ways of the compactification

$$\begin{tikzcd} & \text{3d SCFT on } \mathbb{R}\times\Sigma_g \arrow[dl] \arrow[dr] & \\ \text{1d SCQM on } \mathbb{R} & & \text{2d TQFT on } \Sigma_g, \end{tikzcd} \tag{11.2.5}$$

which suggests a new set of dualities in the sense that the partition functions or indices on both sides yield the same result. As we discussed in section 3.5, the WDVV equation [230, 231] and the twisted periods [232, 233] which are relevant to two-dimensional geometries and topological field theories appear from the constraint conditions for the constructions of $\mathcal{N} = 4$ superconformal mechanics. It would be interesting to investigate whether our M-theoretical construction of superconformal quantum mechanics could help to understand and generalize such relations as the "1d-2d relation".

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