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INTERPOLATION ON COUNTABLY MANY ALGEBRAIC SUBSETS FOR WEIGHTED ENTIRE FUNCTIONS

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1. Introduction

Let X_ν ($\nu \in \mathbb{N}$, the set of positive integers) be k_ν -codimensional complex affine subspaces of \mathbb{C}^n ($1 \leq k_\nu \leq n$). Assume that $X_\nu \cap X_{\nu'} = \emptyset$ for $\nu \neq \nu'$. Let N_ν be the orthogonal complement of X_ν , where we use the canonical inner product $\langle z, w \rangle = \sum_{l=1}^n z_l \bar{w}_l$ on \mathbb{C}^n . Set $S_\nu = N_\nu \cap S^{2n-1}$, where $S^{2n-1} = \{u \in \mathbb{C}^n : |u| = 1\}$. Then Oh'uchi [10] proved the following result:

Theorem A. *Let $X = \bigcup_{\nu \in \mathbb{N}} X_\nu$ be an analytic subset of \mathbb{C}^n consisting of disjoint complex affine subspaces X_ν . Let p be a weight function on \mathbb{C}^n . Then X is interpolating for $A_p(\mathbb{C}^n)$ if and only if there exist $f_1, \dots, f_m \in A_p(\mathbb{C}^n)$ ($m \geq \sup_{\nu \in \mathbb{N}} k_\nu$) and constants $\varepsilon, C > 0$ such that*

$$(1.1) \quad X \subset Z(f_1, \dots, f_m) = \{z \in \mathbb{C}^n : f_1(z) = \dots = f_m(z) = 0\}$$

and

$$(1.2) \quad \sum_{j=1}^m |D_u f_j(\zeta)| \geq \varepsilon \exp(-Cp(\zeta))$$

for all $u \in S_\nu$, $\zeta \in X_\nu$ and $\nu \in \mathbb{N}$.

Here the directional derivative $D_u f$ with a vector $u = (u_1, \dots, u_n) \in S^{2n-1}$ is defined by

$$D_u f = \sum_{l=1}^n \frac{\partial f}{\partial z_l} \cdot u_l.$$

Note that by the proof of Theorem A in [10] the above m may be set equal to $\sup_{\nu \in \mathbb{N}} k_\nu$ when X is interpolating for $A_p(\mathbb{C}^n)$. For the terminologies, see §2. It ex-

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tends the result of Berenstein and Li [2, Theorem 2.5], which deals with the case of $k_\nu = n$ for all $\nu \in \mathbb{N}$.

In the present paper, we would like to discuss the case where X_ν are algebraic subsets, not necessarily affine linear. Because of the difficulties to deal with in general, we formulate this problem as follows. It is first noted that Theorem A implies the following corollary:

Corollary 1.1. *Let $p_m(z_1, \dots, z_m) = q(|z|)$ be a radial weight function on \mathbb{C}^m and set $p_n(z_1, \dots, z_n) = q(|z|)$, which is a radial weight function on \mathbb{C}^n ($m < n$). Let $X = \{\zeta_\nu\}_{\nu \in \mathbb{N}}$ be a discrete variety in \mathbb{C}^n . Then $X \times \mathbb{C}^{n-m}$ is interpolating for $A_{p_n}(\mathbb{C}^n)$ if and only if X is interpolating for $A_{p_m}(\mathbb{C}^m)$.*

Corollary 1.1 can be restated as follows: Define a mapping $F = (F_1, \dots, F_m) : \mathbb{C}^n \rightarrow \mathbb{C}^m$ by $F_j(z) = z_j$ ($j = 1, \dots, m$). Then $F^{-1}(X)$ is interpolating for $A_p(\mathbb{C}^n)$ if and only if X is interpolating for $A_p(\mathbb{C}^m)$. Conversely, when F is a linear mapping from \mathbb{C}^n onto \mathbb{C}^m with $\text{rank } F = m$, we can reduce the interpolation problem for $F^{-1}(X)$ to that for $X' \times \mathbb{C}^{n-m}$, where X' is the image of X by some linear mapping determined by F and X . By [2, Theorem 2.5], X' is interpolating for $A_p(\mathbb{C}^m)$ if and only if X is interpolating for $A_p(\mathbb{C}^n)$. The main result of this paper is as follows:

Main Theorem. *Suppose that $m \leq n$. Let $X = \{\zeta_\nu\}_{\nu \in \mathbb{N}}$ be a discrete variety in \mathbb{C}^m and let $F = (F_1, \dots, F_m) \in \mathbb{C}[z_1, \dots, z_n]^m$. Put $d = \max_{j=1, \dots, m} \deg F_j$. For $a > 0$, we assume that*

- (1) X is interpolating for $A_{|\cdot|^a}(\mathbb{C}^m)$;
- (2) there exist constants $\varepsilon, C > 0$ and a finite subset E of \mathbb{N} such that

$$\sum_{\kappa=1}^{\binom{n}{m}} |\Delta_\kappa^F(z)| \geq \varepsilon \exp(-C|z|^{ad})$$

for all $z \in F^{-1}(\zeta_\nu)$, $\nu \in \mathbb{N} \setminus E$.

Here the sum is taken over all $m \times m$ minors Δ_κ^F of Jacobian matrix JF . Then $F^{-1}(X)$ is interpolating for $A_{|\cdot|^b}(\mathbb{C}^n)$ for every $b \geq ad$.

REMARK. If $F : \mathbb{C}^n \rightarrow \mathbb{C}^m$ is the standard projection with $\text{rank } F = m$ and $p(z) = |z|^a$, then the sufficiency part of Corollary 1.1 is deduced from the main theorem, where $d = 1$ and $b = ad = a$.

2. Preliminaries

We fix the notation. A plurisubharmonic function $p : \mathbb{C}^n \rightarrow [0, \infty)$ is called a *weight function* if it satisfies

$$(2.1) \quad \log(1 + |z|^2) = O(p(z))$$

and there exist constants $C_1, C_2 > 0$ such that for all z, z' with $|z - z'| \leq 1$

$$(2.2) \quad p(z') \leq C_1 p(z) + C_2.$$

A weight function p is said to be *radial* if

$$(2.3) \quad p(z) = p(|z|).$$

DEFINITION 2.1. Let $\mathcal{O}(\mathbb{C}^n)$ be the ring of all entire functions on \mathbb{C}^n and let p be a weight function on \mathbb{C}^n . Set

$$A_p(\mathbb{C}^n) = \{f \in \mathcal{O}(\mathbb{C}^n) : \text{There exist constants } A, B > 0 \text{ such that} \\ |f(z)| \leq A \exp(Bp(z)) \text{ for all } z \in \mathbb{C}^n.\}$$

Then $A_p(\mathbb{C}^n)$ is a subring of $\mathcal{O}(\mathbb{C}^n)$. The following lemma is easily deduced from (2.1) and (2.2):

Lemma 2.2. *Let p be a weight function on \mathbb{C}^n . Then the following hold:*

- (1) $\mathbb{C}[z_1, \dots, z_n] \subset A_p(\mathbb{C}^n)$.
- (2) If $f \in A_p(\mathbb{C}^n)$, then $\partial f / \partial z_j \in A_p(\mathbb{C}^n)$ for $j = 1, \dots, n$.
- (3) $f \in \mathcal{O}(\mathbb{C}^n)$ belongs to $A_p(\mathbb{C}^n)$ if and only if there exists a constant $K > 0$ such that

$$\int_{\mathbb{C}^n} |f|^2 \exp(-Kp) d\lambda < \infty,$$

where $d\lambda$ denotes the Lebesgue measure on \mathbb{C}^n .

For the proof, see e.g. [8].

- EXAMPLE 2.3. (1) If $p(z) = \log(1 + |z|^2)$, then $A_p(\mathbb{C}^n) = \mathbb{C}[z_1, \dots, z_n]$.
- (2) If $p(z) = |z|^a$ ($a > 0$), then $A_p(\mathbb{C}^n)$ is the space of entire functions which are of order $= a$ and of finite type, or which are of order $< a$.
- (3) If $p(z) = |\operatorname{Im} z| + \log(1 + |z|^2)$, then $A_p(\mathbb{C}^n) = \hat{\mathcal{E}}'(\mathbb{R}^n)$, that is, the space of Fourier transforms of distributions with compact support on \mathbb{R}^n (see e.g. [7]).
- (4) When $p(z) = \exp |z|^a$ ($a > 0$), p is a weight function if and only if $a \leq 1$.

In the rest of this paper, p will always represent a weight function.

DEFINITION 2.4. Let X be an analytic subset of \mathbb{C}^n , and let $\mathcal{O}(X)$ be the space of analytic functions on X . Then we define

$$A_p(X) = \{f \in \mathcal{O}(X) : \text{There exist constants } A, B > 0 \text{ such that} \\ |f(z)| \leq A \exp(Bp(z)) \text{ for all } z \in X.\}.$$

DEFINITION 2.5. An analytic subset X of \mathbb{C}^n is said to be *interpolating for* $A_p(\mathbb{C}^n)$ if the restriction map $R_X : A_p(\mathbb{C}^n) \rightarrow A_p(X)$ defined by $R_X(f) = f|_X$ is surjective.

The semilocal interpolation theorem by [4] is useful to show an analytic subset to be interpolating. Let X be given by

$$X = Z(f_1, \dots, f_N) = \{z \in \mathbb{C}^n : f_1(z) = \dots = f_N(z) = 0\}$$

with $f_1, \dots, f_N \in A_p(\mathbb{C}^n)$. Then for $\varepsilon, C > 0$, we define

$$S_p(f; \varepsilon, C) = \left\{ z \in \mathbb{C}^n : |f(z)| = \left(\sum_{j=1}^N |f_j(z)|^2 \right)^{1/2} < \varepsilon \exp(-Cp(z)) \right\},$$

which is an open neighborhood of X . We recall the semilocal interpolation theorem of [4].

Semilocal Interpolation Theorem. *Let h be a holomorphic function in $S_p(f; \varepsilon, C)$ such that*

$$|h(z)| \leq A_1 \exp(B_1 p(z))$$

for all $z \in S_p(f; \varepsilon, C)$, where $\varepsilon, C > 0$. Then there exist an entire function $H \in A_p(\mathbb{C}^n)$, constants $\varepsilon_1, C_1, A, B > 0$ and holomorphic functions g_1, \dots, g_N in $S_p(f; \varepsilon_1, C_1)$ such that

$$H(z) - h(z) = \sum_{j=1}^N g_j(z) f_j(z)$$

and

$$|g_j(z)| \leq A \exp(Bp(z))$$

for all $z \in S_p(f; \varepsilon_1, C_1)$ and $j = 1, \dots, N$. In particular, $H = h$ on the variety $X = Z(f_1, \dots, f_N)$.

3. A_p -interpolation on algebraic subsets

To prove the main theorem, we first show the following result:

Theorem 3.1. *Every algebraic subset $V \subset \mathbb{C}^n$ is interpolating for $A_p(\mathbb{C}^n)$.*

We assume that V is irreducible until we begin the proof of Theorem 3.1 after Lemma 3.17. Then we have the prime ideal $I_V \subset \mathbb{C}[z_1, \dots, z_n]$ such that $V = I_V^{-1}(0) = \{z \in \mathbb{C}^n : P(z) = 0 \text{ for all } P \in I_V\}$. Defining the terminology, we state the normalization theorem.

Normalization Theorem. *After a suitable linear change of coordinates, the following conditions hold:*

- (1) *There exists $k \in \{0, 1, \dots, n-1\}$ such that $I_V \cap \mathbb{C}[z_1, \dots, z_k] = \{0\}$ and the factor ring $\mathbb{C}[z_1, \dots, z_n]/I_V$ is a finitely generated $\mathbb{C}[z_1, \dots, z_k]$ -module. Here we set $z' = (z_1, \dots, z_k) \in \mathbb{C}^k$ and $z'' = (z_{k+1}, \dots, z_n) \in \mathbb{C}^{n-k}$.*
- (2) *There exists $C_0 > 0$ such that $|z_{k+j}| \leq C_0(1+|z'|)$ for all $z \in V$ and $j = 1, \dots, n-k$.*
- (3) *I_V contains irreducible polynomials*

$$Q_j(z', z_{k+j}) = z_{k+j}^\mu + q_{j,1}(z')z_{k+j}^{\mu-1} + \dots + q_{j,\mu}(z')$$

of degree μ , where $q_{j,\nu} \in \mathbb{C}[z_1, \dots, z_k]$.

Let $\alpha_1(z'), \dots, \alpha_\mu(z')$ be the roots of $Q_1(z', z_{k+1})$ as a polynomial in z_{k+1} . Then we denote by $\Delta(z')$ the discriminant of Q_1 as a polynomial in z_{k+1} , that is,

$$\Delta(z') = \prod_{\nu \neq \nu'} (\alpha_\nu(z') - \alpha_{\nu'}(z')).$$

- (4) *We have polynomials $T_j \in \mathbb{C}[z_1, \dots, z_k, z_{k+j}]$ ($j = 2, \dots, n-k$) with $\Delta(z')z_{k+j} - T_j(z', z_{k+j}) \in I_V$.*

Put $V_0 = V \setminus \Delta^{-1}(0)$.

- (5) *V_0 is an open dense subset of V and a μ -dimensional complex submanifold of $\mathbb{C}^n \setminus \Delta^{-1}(0)$.*

Let $\pi_V : \mathbb{C}^n \ni z = (z', z'') \mapsto z' \in \mathbb{C}^k$ be the projection.

- (6) *π_V is a finite μ -fold covering map from V_0 onto $\mathbb{C}^k \setminus \Delta^{-1}(0)$.*

For the proof, see e.g. [6, Theorem A.1.1 in Chapter 3], [9, Proposition 7.7.3].

For $\varepsilon > 0$, $N > 0$ and $\xi \in \mathbb{C}^n$, we define the polydisc

$$D_{\varepsilon, N}(\xi) = \{z \in \mathbb{C}^n : |z_j - \xi_j| < \varepsilon(1 + |\xi|)^{-N} \ (\forall j = 1, \dots, n)\}.$$

For the given $f \in A_p(V)$, we take $A, B > 0$ such that

$$|f(z)| \leq A \exp(Bp(z)), \quad \forall z \in V.$$

Lemma 3.2. *We have $\varepsilon, N, A_1, B_1 > 0$ satisfying: for all $\xi \in V$ there exists $F \in \mathcal{O}(D_{\varepsilon, N}(\xi))$ such that $\Delta f - F = 0$ on $V \cap D_{\varepsilon, N}$ and*

$$|F(z)| \leq A_1 \exp(B_1 p(z)), \quad z \in D_{\varepsilon, N}(\xi).$$

Proof. If $\dim V = k = 0$, V consists of only one point, so the lemma is trivial. Then we assume that $1 \leq k \leq n - 1$. To apply the normalization theorem, we give a suitable linear change of coordinates. Set

$$D'_{\varepsilon, N}(\xi) = \{z' \in \mathbb{C}^k : |z_j - \xi_j| < \varepsilon(1 + |\xi|)^{-N} \ (\forall j = 1, \dots, k)\}.$$

Here we need the following lemma:

Lemma 3.3. *There exists $\varepsilon > 0$ such that for all $\xi \in V$ we have $v_j(\xi) \in \{1, \dots, 2\mu - 1\}$ ($j = 1, \dots, n - k$) satisfying that if*

$$(3.1) \quad z = (z', z'') \in D'_{\varepsilon, 2\mu-2}(\xi) \times \mathbb{C}^{n-k} \text{ and } |z_{k+j} - \xi_{k+j}| = v_j(\xi)$$

for some $j = 1, \dots, n - k$, then $z \notin V$.

Proof. It is sufficient to prove that $|Q_1(z)| \geq 1/2$ for z satisfying (3.1). Factorizing Q_1 , we have

$$Q_1(\xi_1, \dots, \xi_k, z_{k+1}) = (z_{k+1} - \alpha_1(\xi')) \cdots (z_{k+1} - \alpha_\mu(\xi')).$$

Then there exists $v_1(\xi) \in \{1, \dots, 2\mu - 1\}$ such that for $|z_{k+1} - \xi_{k+1}| = v_1(\xi)$ we have $|z_{k+1} - \alpha_1(\xi')|, \dots, |z_{k+1} - \alpha_\mu(\xi')| \geq 1$, and hence $|Q_1(\xi_1, \dots, \xi_k, z_{k+1})| \geq 1$. In fact, we set $\{|\alpha_1(\xi') - \xi_{k+1}|, \dots, |\alpha_\mu(\xi') - \xi_{k+1}|\} = \{\gamma_1, \dots, \gamma_{\hat{\mu}}\}$ ($\hat{\mu} \leq \mu$) as sets, and we assume that $\gamma_1 < \gamma_2 < \dots < \gamma_{\hat{\mu}}$. Since $Q_1(\xi) = 0$, we have $\gamma_1 = 0$. Here we would like to find the minimal positive integer $v_1(\xi)$ satisfying $\gamma_\nu \leq v_1(\xi) - 1$ and $\gamma_{\nu+1} \geq v_1(\xi) + 1$ for some ν . For example, if $\gamma_2 \geq 2$, then we can take $v_1(\xi) = 1$. In the case where we have such ν , $v_1(\xi)$ is maximal if and only if $\gamma_2 \in (1, 2)$, $\gamma_3 \in (3, 4)$, \dots , $\gamma_{\hat{\mu}-1} \in (2\hat{\mu} - 5, 2\hat{\mu} - 4)$ and $\gamma_{\hat{\mu}} \geq 2\hat{\mu} - 2$. In this case, we can take $v_1(\xi) = 2\hat{\mu} - 3$. If there exists no such ν , that is, $\gamma_2 \in (1, 2)$, $\gamma_3 \in (3, 4)$, \dots , $\gamma_{\hat{\mu}-1} \in (2\hat{\mu} - 5, 2\hat{\mu} - 4)$ and $\gamma_{\hat{\mu}} \in (2\hat{\mu} - 3, 2\hat{\mu} - 2)$, then we take $v_1(\xi) = 2\hat{\mu} - 1$. Hence we can take $v_1(\xi) \in \{1, \dots, 2\mu - 1\}$ satisfying the above condition.

Here we would like to take $\varepsilon \in (0, 1)$ so that if $|z_1 - \xi_1|, \dots, |z_k - \xi_k| < \varepsilon(1 + |\xi|)^{-2\mu+2}$ and $|z_{k+1} - \xi_{k+1}| = v_1(\xi)$, then $|Q_1(z_1, \dots, z_k, z_{k+1}) - Q_1(\xi_1, \dots, \xi_k, z_{k+1})| \leq 1/2$. Let M be the maximum of moduli of all coefficients in $q_{1,1}, \dots, q_{1,\mu}$. We can

write

$$(3.2) \quad \begin{aligned} & |Q_1(z_1, \dots, z_k, z_{k+1}) - Q_1(\xi_1, \dots, \xi_k, z_{k+1})| \\ & \leq M \sum_{|\beta| \leq 1} |z_1^{\beta_1} \cdots z_k^{\beta_k} - \xi_1^{\beta_1} \cdots \xi_k^{\beta_k}| |z_{k+1}|^{\mu-1} \\ & \quad + \cdots + M \sum_{|\beta| \leq \mu} |z_1^{\beta_1} \cdots z_k^{\beta_k} - \xi_1^{\beta_1} \cdots \xi_k^{\beta_k}|, \end{aligned}$$

where $\beta = (\beta_1, \dots, \beta_k)$ is a multi-index and $|\beta| = \beta_1 + \cdots + \beta_k$. Here we have the following estimates:

(1) Since $|z_{k+1} - \xi_{k+1}| = v_1(\xi)$,

$$|z_{k+1}| \leq |\xi_{k+1}| + v_1(\xi) \leq |\xi| + 2\mu - 1 \leq (2\mu - 1)(1 + |\xi|).$$

(2) Since $|z_j|, |\xi| < |\xi| + \varepsilon(1 + |\xi|)^{-2\mu+2} \leq 1 + |\xi|$, we obtain

$$(3.3) \quad \begin{aligned} & |z_1^{\beta_1} \cdots z_k^{\beta_k} - \xi_1^{\beta_1} \cdots \xi_k^{\beta_k}| \\ & \leq |z_1^{\beta_1} \cdots z_k^{\beta_k} - z_1^{\beta_1} \cdots z_{k-1}^{\beta_{k-1}} z_k^{\beta_k-1} \xi_k| \\ & \quad + \cdots + |z_1 \xi_1^{\beta_1-1} \xi_2^{\beta_2} \cdots \xi_k^{\beta_k} - \xi_1^{\beta_1} \cdots \xi_k^{\beta_k}| \\ & = |z_k - \xi_k| |z_1^{\beta_1} \cdots z_{k-1}^{\beta_{k-1}} z_k^{\beta_k-1}| + \cdots + |z_1 - \xi_1| |\xi_1^{\beta_1-1} \xi_2^{\beta_2} \cdots \xi_k^{\beta_k}| \\ & \leq |\beta| \varepsilon (1 + |\xi|)^{-2\mu+2} (1 + |\xi|)^{|\beta|-1} \\ & \leq \mu \varepsilon (1 + |\xi|)^{-\mu+1}, \end{aligned}$$

where the number of terms in (3.3) is $|\beta|$.

(3) The number of terms in $\sum_{|\beta| \leq \nu} |z_1^{\beta_1} \cdots z_k^{\beta_k} - \xi_1^{\beta_1} \cdots \xi_k^{\beta_k}| |z_{k+1}|^{\mu-\nu}$ is bounded from above by

$$1 + k + \cdots + k^j \leq 1 + k + \cdots + k^\mu \leq (\mu + 1)k^\mu.$$

It follows from (3.2) and these estimates that

$$|Q_1(z_1, \dots, z_k, z_{k+1}) - Q_1(\xi_1, \dots, \xi_k, z_{k+1})| \leq M \mu^2 (\mu + 1)^{\mu-1} k^\mu \varepsilon.$$

Hence, we set

$$\varepsilon = \frac{1}{2M \mu^2 (\mu + 1)^{\mu-1} k^\mu},$$

and then the lemma holds for all $\xi \in V$. □

For simplification, we fix $\xi \in V$ and put $D' = D'_{\varepsilon, 2\mu-2}(\xi)$, $D'' = \{z'' \in \mathbb{C}^{n-k} : |z_{k+j} - \xi_{k+j}| < v_j(\xi) \text{ for all } j = 1, \dots, n-k\}$ and $D = D' \times D''$. By Lemma 3.2, $\pi|_{D \cap V}$:

$D \cap V \rightarrow D'$ is proper. It follows from the normalization theorem that $D' \setminus \Delta^{-1}(0)$ is connected and

$$\pi|_{(V \cap D) \setminus \Delta^{-1}(0)} : (V \cap D) \setminus \Delta^{-1}(0) \rightarrow D' \setminus \Delta^{-1}(0)$$

is a $\tilde{\mu}$ -fold covering mapping with $1 \leq \tilde{\mu} \leq \mu$. For $z' \in D' \setminus \Delta^{-1}(0)$, by renumbering $\alpha_1(z'), \dots, \alpha_\mu(z')$ we have $\alpha_1(z'), \dots, \alpha_{\tilde{\mu}}(z') \in \{z_{k+1} \in \mathbb{C} : |z_{k+1} - \xi_{k+1}| < v_1(\xi)\}$. Since symmetric polynomials of $\alpha_1, \dots, \alpha_{\tilde{\mu}}$ are bounded holomorphic functions in $D' \setminus \Delta^{-1}(0)$, it follows from Riemann's Extension Theorem that they extend to holomorphic functions in D' . Hence

$$\Delta'(z') = \prod_{1 \leq j < j' \leq \tilde{\mu}} (\alpha_j(z') - \alpha_{j'}(z'))^2$$

is holomorphic in D' .

Let $\pi^{-1}(z') \cap V \cap D = \{\tau_1(z'), \dots, \tau_{\tilde{\mu}}(z')\}$ as sets such that

$$\{(\tau_1(z'))_{k+1}, \dots, (\tau_{\tilde{\mu}}(z'))_{k+1}\} = \{\alpha_1(z'), \dots, \alpha_{\tilde{\mu}}(z')\}$$

for $z' \in D' \setminus \Delta^{-1}(0)$, where $(\tau_j(z'))_{k+1}$ ($1 \leq j \leq \tilde{\mu}$) denote the $(k+1)$ -th coordinate of $\tau_j(z')$. Then there exist $\varphi_0(z'), \dots, \varphi_{\tilde{\mu}-1}(z') \in \mathbb{C}$ uniquely such that

$$(3.4) \quad f(\tau_j(z')) = \varphi_0(z') + \varphi_1(z')\alpha_j(z') + \dots + \varphi_{\tilde{\mu}-1}(z')\alpha_j(z')^{\tilde{\mu}-1}$$

for all $j = 1, \dots, \tilde{\mu}$ and $z' \in D' \setminus \Delta^{-1}(0)$. In fact, if we think (3.4) to be a system of linear equations in $\varphi_0(z'), \dots, \varphi_{\tilde{\mu}-1}(z')$, the determinant $W(z')$ of its coefficient matrix \mathcal{A} is given by

$$\begin{aligned} W(z') &= \det \begin{pmatrix} 1 & \dots & 1 \\ \alpha_1(z') & \dots & \alpha_{\tilde{\mu}}(z') \\ \vdots & & \vdots \\ \alpha_1(z')^{\tilde{\mu}-1} & \dots & \alpha_{\tilde{\mu}}(z')^{\tilde{\mu}-1} \end{pmatrix} \\ &= \prod_{1 \leq j < j' \leq \tilde{\mu}} (\alpha_j(z') - \alpha_{j'}(z')) \neq 0, \quad \forall z' \in D' \setminus \Delta^{-1}(0). \end{aligned}$$

Then $W(z')^2 = \Delta'(z')$ and Cramer's rule gives

$$W(z')\varphi_j(z') = \sum_{l=1}^{\tilde{\mu}} T_{l,j}(z')f(\tau_l(z'))$$

for all $j = 0, \dots, \tilde{\mu} - 1$, where $(T_{l,j})_{l=1, \dots, \tilde{\mu}; j=0, \dots, \tilde{\mu}-1}$ is the cofactor matrix of \mathcal{A} . It follows from the normalization theorem (2) that

$$(3.5) \quad |\alpha_l(z')| \leq C_0(1 + |z'|)$$

for all $z' \in D'$ and $l = 1, \dots, \tilde{\mu}$. Thus, we have

Lemma 3.4. *There exist $C_3 > 0$ and $\omega \in \mathbb{N}$ depending only on Q_1 such that*

$$|\Delta'(z')\varphi_j(z')| \leq C_3 M_D(f)(1 + |z'|)^\omega$$

for all $z' \in D' \setminus \Delta^{-1}(0)$ and $j = 0, \dots, \tilde{\mu} - 1$, where $M_D(f) = \sup\{|f(z)| : z \in V \cap D\}$.

For the other roots $\alpha_{\tilde{\mu}+1}(z'), \dots, \alpha_\mu(z')$ of Q_1 , setting

$$\Delta''(z') = \prod_{\substack{1 \leq j \leq \mu \\ \tilde{\mu}+1 \leq j' \leq \mu}} (\alpha_j(z') - \alpha_{j'}(z'))^2,$$

we have $\Delta = \Delta' \Delta''$. Since (3.5) hold for $l = \tilde{\mu} + 1, \dots, \mu$, we obtain $C_4 > 0$ satisfying

$$\begin{aligned} |\Delta''(z')| &\leq C_4(1 + |z'|)^{\mu(\mu-1) - \tilde{\mu}(\tilde{\mu}-1)} \\ &\leq C_4(1 + |z'|)^{\mu(\mu-1)}. \end{aligned}$$

Hence there exist $C_5 > 0$ and $\omega' \in \mathbb{N}$ independent of $\tilde{\mu}$ such that

$$|\Delta(z')\varphi_j(z')| \leq C_5 M_D(f)(1 + |z'|)^{\omega'}$$

for all $z' \in D' \setminus \Delta^{-1}(0)$. In particular, all $\Delta\varphi_j$ are bounded holomorphic functions. By Riemann's extension theorem, they extend to holomorphic functions in D' .

Since p is a weight function, we have $A', B' > 0$ independent of ξ satisfying

$$M_D(f) \leq A' \exp(B' p(\xi)).$$

Set

$$F(z) = \Delta(z')\varphi_0(z') + \Delta(z')\varphi_1(z')z_{k+1} + \dots + \Delta(z')\varphi_{\tilde{\mu}-1}(z')z_{k+1}^{\tilde{\mu}-1}.$$

By the definition of weight functions, there exist $A_1, B_1 > 0$ independent of ξ such that

$$\begin{aligned} |F(z)| &\leq |\Delta(z')\varphi_0(z')| + |\Delta(z')\varphi_1(z')||z_{k+1}| + \dots + |\Delta(z')\varphi_{\tilde{\mu}-1}(z')||z_{k+1}|^{\tilde{\mu}-1} \\ &\leq \tilde{\mu} C_5 A' \exp(B' p(\xi))(1 + |z|)^{\omega' + \tilde{\mu} - 1} \\ &\leq A_1 \exp(B_1 p(z)) \end{aligned}$$

for all $z \in D_{\varepsilon, 2\mu-2}(\xi)$. Finally, it follows from (3.4) that

$$(3.6) \quad F = \Delta f$$

in $(V \setminus \Delta^{-1}(0)) \cap D_{\varepsilon, 2\mu-2}(\xi)$. Since $V \setminus \Delta^{-1}(0)$ is dense in V , (3.6) holds on $V \cap D_{\varepsilon, 2\mu-2}(\xi)$. The proof of Lemma 3.2 is completed. \square

We next solve the Cousin first problem with estimates. We shall use some results from [9].

Lemma 3.5 ([9, Lemma 7.6.1]). *Let $d : \mathbb{R}^{2n} \rightarrow (0, 1]$ be a function such that*

$$(3.7) \quad d(x+y) \leq 2d(x), \quad \text{if } |y|_\infty = \max_{j=1, \dots, 2n} |y_j| \leq 1.$$

Then there exist an open covering $\mathcal{U}^d = \{U_j^d\}_{j \in I(d)}$ of \mathbb{R}^{2n} with open cubes U_j^d , a partition of unity $\chi_j^d \in C_0^\infty(U_j^d)$ and $C_6 > 0$ such that

- (1) $|x-y|_\infty \leq d(x)$ for all $x, y \in U_j^d$ and $j \in I(d)$;
- (2) $\#\{j' \in I(d) : U_{j'}^d \cap U_j^d \neq \emptyset\} \leq 2^{8n}$ for all $j \in I(d)$.
- (3) $|(\partial\chi_j/\partial x_\nu)(x)| \leq C_6/d(x)$ for all $j \in I(d)$, $\nu = 1, \dots, 2n$ and $x \in \mathbb{R}^{2n}$.
- (4) *Let d' be another function satisfying (3.7) and $0 < d' \leq d$. There exists a refinement $\mathcal{U}^{d'}$ of \mathcal{U}^d defined by a mapping $\rho_{d,d'} : I(d') \rightarrow I(d)$ with $\rho_{d,d''} = \rho_{d,d'} \circ \rho_{d',d''}$ satisfying (1), (2) and (3). Moreover, if $d' \leq \tilde{\varepsilon}d$, $\tilde{\varepsilon} < 1/64$, $j' \in I(d')$, $j = \rho_{d,d'}(j')$ and $x \in U_{j'}^{d'}$, then*

$$U_{j'}^{d'} \subset \{y \in \mathbb{R}^{2n} : |y-x|_\infty < \tilde{\varepsilon}d(x)\}$$

and

$$U_j^d \supset \left\{ y \in \mathbb{R}^{2n} : |y-x|_\infty < \left(\frac{1}{64} - \tilde{\varepsilon} \right) d(x) \right\}.$$

For $J = (j_0, \dots, j_\sigma) \in I(d)^{\sigma+1}$ we denote $U_J^d = U_{j_0}^d \cap \dots \cap U_{j_\sigma}^d$. Let c be a cochain in $C^\sigma(\mathcal{U}^d, \mathcal{O})$ and let φ be a plurisubharmonic function in \mathbb{C}^n . Then we write

$$\|c\|_\varphi^2 = \sum_{J \in I(d)^{\sigma+1}} \int_{U_J^d} |c_J|^2 \exp(-\varphi) d\lambda.$$

We also define a coboundary operator $\delta : C^\sigma(\mathcal{U}^d, \mathcal{O}) \rightarrow C^{\sigma+1}(\mathcal{U}^d, \mathcal{O})$ by

$$(\delta c)_{J \in I(d)^{\sigma+2}} = \sum_{\nu=0}^{\sigma+1} (-1)^\nu c_{(j_0, \dots, \check{j}_\nu, \dots, j_{\sigma+1})}.$$

Lemma 3.6 ([9, Proposition 7.6.2]). *Let $-\log d$ be a plurisubharmonic function on \mathbb{C}^n . For every $c \in C^\sigma(\mathcal{U}^d, \mathcal{O})$ ($\sigma > 0$) with $\delta c = 0$ and $\|c\|_\varphi < \infty$, we can find a cochain $c' \in C^{\sigma-1}(\mathcal{U}^d, \mathcal{O})$ such that $\delta c' = c$ and $\|c'\|_\psi \leq K_1 \|c\|_\varphi$, where ψ is a plurisubharmonic function in \mathbb{C}^n defined by*

$$\psi(z) = \varphi(z) - \sigma \log d(z) + 2 \log(1 + |z|^2),$$

and K_1 is a constant independent of φ , d and c .

Let

$$P = \begin{pmatrix} P_{1,1} & \cdots & P_{1,T} \\ \vdots & & \vdots \\ P_{\Lambda,1} & \cdots & P_{\Lambda,T} \end{pmatrix}$$

be a matrix with polynomial elements. Then P defines the sheaf homomorphism

$$(3.8) \quad P : \mathcal{O}^T \ni g \mapsto Pg \in \mathcal{O}^\Lambda.$$

Lemma 3.7 ([9, Lemma 7.6.3]). *The kernel $\ker P$ of the sheaf homomorphism (3.8) is generated by the germs of a finite number of $Q_s = (Q_{1,s}, \dots, Q_{T,s}) \in \mathbb{C}[z_1, \dots, z_n]^T$ ($s = 1, \dots, S$) satisfying*

$$\sum_{t=1}^T P_{\lambda,t} Q_{t,s} = 0$$

for all $\lambda = 1, \dots, \Lambda$ and $s = 1, \dots, S$.

Lemma 3.8 ([9, Lemma 7.6.4]). *Let Ω be a pseudoconvex domain and let P and Q be matrixes in Lemma 3.7. Then if $g = (g_1, \dots, g_T) \in \mathcal{O}(\Omega)^T$ satisfies*

$$\sum_{t=1}^T P_{\lambda,t} g_t = 0$$

for all $\lambda = 1, \dots, \Lambda$, there exists $h = (h_1, \dots, h_S) \in \mathcal{O}(\Omega)^S$ such that

$$g_t = \sum_{s=1}^S Q_{t,s} h_s$$

for all $t = 1, \dots, T$. In particular, $\ker P = \text{Im } Q$ holds.

By putting $\Lambda = 1$, Lemmas 3.7 and 3.8 imply that $\mathcal{O}(\Omega)$ is a flat $\mathbb{C}[z_1, \dots, z_n]$ -module. This fact will play an important role later.

The following lemma gives estimates of solutions of the equation $Pv = u$ for $u \in \text{Im } P$:

Lemma 3.9 ([9, Lemma 7.6.5]). *Let Ω be a neighborhood of $0 \in \mathbb{C}^n$. Then we have a neighborhood Ω' of $0 \in \mathbb{C}^n$ and constants C_7, N_1 satisfying that for all $\eta \in (0, 1)$, $z \in \mathbb{C}^n$ and $u \in \mathcal{O}(\eta\Omega + \{z\})^T$, there exists $v \in \mathcal{O}(\eta\Omega' + \{z\})^T$ such that $Pv = Pu$*

and

$$\sup_{\eta\Omega' + \{z\}} |v| \leq C_7(1 + |z|)^{N_1} \eta^{-N_1} \sup_{\eta\Omega + \{z\}} |Pu|.$$

Here $\eta\Omega + \{z\} = \{\eta w + z : w \in \Omega\}$.

We now prove a lemma important to solve the Cousin first problem with estimates. Let $P : \mathcal{O}^T \rightarrow \mathcal{O}^\Lambda$ be the sheaf homomorphism as above. Then $\mathcal{M}_P = \text{Im } P$ is a subsheaf of \mathcal{O}^Λ generated by $(P_{1,t}, \dots, P_{\Lambda,t})$ for $t = 1, \dots, T$. We denote by $C^\sigma(\mathcal{U}^d, \mathcal{M}_P, p)$ the set of cochains $c = \{c_J\}_{J \in I(d)^{\sigma+1}} \in C^\sigma(\mathcal{U}^d, \mathcal{M}_P)$ satisfying

$$\|c\|_p^2 = \sum_{J \in I(d)^{\sigma+1}} \int_{U_J^d} |c_J|^2 \exp(-p) d\lambda < \infty.$$

Lemma 3.10 (cf. [9, Lemma 7.6.10]). *We assume that $-\log d$ is a plurisubharmonic function. Then we have $N_2, K_2 > 0$ and $\varepsilon_0 < 1/192$ satisfying that for all $c \in C^\sigma(\mathcal{U}^d, \mathcal{M}_P, p)$ ($\sigma > 0$) with $\delta c = 0$, there exists $c' \in C^{\sigma-1}(\mathcal{U}^{\varepsilon_0 d}, \mathcal{M}_P, p_{N_2})$ such that $\delta c' = \rho_{d, \varepsilon_0 d}^* c$ and*

$$\|c'\|_{p_{N_2}} \leq K_2 \|c\|_p,$$

where $p_{N_2}(z) = N_2(p(z) - \log d(z) + \log(1 + |z|^2))$.

Proof. Applying Lemma 3.9 for $\Omega := \{z \in \mathbb{C}^n : |z|_\infty < 1\}$, we have $r \in (0, 1)$ and constants C_7, N_1 satisfying for all $\eta \in (0, 1)$, $\xi \in \mathbb{C}^n$ and $u \in \mathcal{O}(\eta\Omega + \{\xi\})^T$, there exists $v \in \mathcal{O}(\eta\Omega' + \{\xi\})^T$ such that $Pv = Pu$ and

$$(3.9) \quad \sup_{\eta\Omega' + \{\xi\}} |v| \leq C_7(1 + |\xi|)^{N_1} \eta^{-N_1} \sup_{\eta\Omega + \{\xi\}} |Pu|,$$

where $\Omega' = \{z \in \mathbb{C}^n : |z|_\infty < r\}$. For $\tilde{\varepsilon} < 1/128$, it follows from Lemma 3.5 (4) that if $j' \in I(\tilde{\varepsilon}d)$, $j = \rho_{d, \tilde{\varepsilon}d}(j')$ and $\xi \in U_{j'}^{\tilde{\varepsilon}d}$, then

$$(3.10) \quad U_{j'}^{\tilde{\varepsilon}d} \subset \tilde{\varepsilon}d(\xi)\Omega + \{\xi\} \subset \left(\frac{1}{64} - \tilde{\varepsilon}\right)\Omega + \{\xi\} \subset U_j^d.$$

Here defining $\tilde{\varepsilon} := r/(128(2+r)) (\leq 1/384)$ and $\eta := (1/128 - \tilde{\varepsilon}/2)d(\xi)$, we have $\tilde{\varepsilon}d(\xi) < r\eta$, hence (3.10) implies that

$$(3.11) \quad U_{j'}^{\tilde{\varepsilon}d} \subset r\eta\Omega + \{\xi\} = \eta\Omega' + \{\xi\}.$$

On the other hand, we have $\eta < (1/96)d(\xi) < ((1/64) - \tilde{\varepsilon})d(\xi)$, that is,

$$(3.12) \quad \eta\Omega + \{\xi\} \subset \subset \frac{1}{96}d(\xi)\Omega \subset \left(\frac{1}{64} - \tilde{\varepsilon}\right)d(\xi)\Omega \subset U_j^d.$$

Then for $J' = (j'_0, \dots, j'_\sigma) \in I(\tilde{\varepsilon}d)^{\sigma+1}$ $J = \rho_{d, \tilde{\varepsilon}d}(J') := (\rho_{d, \tilde{\varepsilon}d}(j'_0), \dots, \rho_{d, \tilde{\varepsilon}d}(j'_\sigma))$ and $\xi \in U_{J'}^{\tilde{\varepsilon}d}$, we obtain from (3.11) and (3.12)

$$(3.13) \quad U_{J'}^{\tilde{\varepsilon}d} \subset \eta\Omega' + \{\xi\} \subset \eta\Omega + \{\xi\} \subset \frac{1}{96}d(\xi)\Omega + \{\xi\} \subset U_J^d.$$

Hence it follows from (3.9) that for all $u \in \mathcal{O}(U_J^d)^T$ ($\subset \mathcal{O}(\eta\Omega + \{\xi\})^T$) there exists $v \in \mathcal{O}(\eta\Omega' + \{\xi\})^T$ such that $Pv = Pu$ and

$$(3.14) \quad \sup_{U_{J'}^{\tilde{\varepsilon}d}} |v| \leq C_7(1 + |\xi|)^{N_1} \eta^{-N_1} \sup_{\eta\Omega + \{\xi\}} |Pu|.$$

By [9, Theorem 2.2.3], (3.12) implies that there exists $C_8 > 0$ independent of ξ such that

$$\sup_{\eta\Omega + \{\xi\}} |g| \leq C_8 \|g\|_{L^1((1/96)d(\xi)\Omega + \{\xi\})}$$

for all $g \in \mathcal{O}(U_J^d)$. It follows from Schwarz's inequality that

$$\begin{aligned} \sup_{\eta\Omega + \{\xi\}} |Pu| &\leq C_8 (\|(Pu)_1\|_{L^1((1/96)d(\xi)\Omega + \{\xi\})} + \dots + \|(Pu)_\Lambda\|_{L^1((1/96)d(\xi)\Omega + \{\xi\})}) \\ &\leq \Lambda C_8 \left(\int_{(1/96)d(\xi)\Omega + \{\xi\}} |Pu|^2 d\lambda \right)^{1/2} \leq \Lambda C_8 \left(\int_{U_J^d} |Pu|^2 d\lambda \right)^{1/2}. \end{aligned}$$

Since p is a weight, by Lemma 3.5 (1) there exist $C'_1, C'_2 > 0$ independent of d and J such that $p(z') \leq C'_1 p(z) + C'_2$ for $z, z' \in U_J^d$. Then we obtain

$$\exp(-C'_1 p(\xi)) \int_{U_J^d} |Pu(z)|^2 d\lambda(z) \leq e^{C'_2} \int_{U_J^d} |Pu(z)|^2 \exp(-p(z)) d\lambda(z).$$

Hence it follows from (3.7) that

$$|v(\xi)|^2 (1 + |\xi|^2)^{-2N_1} d(\xi)^{2N_1} \exp(-C'_1 p(\xi)) \leq C_9 \int_{U_J^d} |Pu(z)|^2 \exp(-p(z)) d\lambda(z),$$

where $C_9 = \Lambda C_7 C_8 2^{2N_1} (1/128 - \tilde{\varepsilon}/2)^{-2N_1} e^{C'_2}$. Therefore putting $N'_2 = \max\{N_1, C'_1\}$, we obtain

$$(3.15) \quad \int_{U_{J'}^{\tilde{\varepsilon}d}} |v(\xi)|^2 \exp(-p_{N'_2}(\xi)) d\lambda(\xi) \leq C_9 \int_{U_J^d} |Pu(z)|^2 \exp(-p(z)) d\lambda(z).$$

We prove this lemma by induction for decreasing σ . Note that it is valid when $\sigma = 2^{8n} + 1$, since $C^\sigma(\mathcal{U}, \cdot) = \{0\}$ by Lemma 3.5 (2). We assume that it have been proved for all P when σ is replaced by $\sigma + 1$. By [9, Lemma 7.2.9], there exists $\gamma \in$

$C^\sigma(\mathcal{U}^d, \mathcal{O}^T)$ such that $c_J = P\gamma_J$ for all $J \in I(d)^{\sigma+1}$. To obtain control of γ_J we pass to the refinement $\mathcal{U}^{\tilde{\varepsilon}d}$ for which (3.15) is applicable. Then we can choose $\gamma'_{J'} \in \mathcal{O}(U_{J'}^{\tilde{\varepsilon}d})^T$ ($J' \in I(\tilde{\varepsilon}d)^{\sigma+1}$) so that with $J = \rho_{d, \tilde{\varepsilon}d} J'$ we have

$$(3.16) \quad P\gamma'_{J'} = P\gamma_J = c_J$$

in $U_{J'}^{\tilde{\varepsilon}d}$ and

$$\int_{U_{J'}^{\tilde{\varepsilon}d}} |\gamma'_{J'}|^2 \exp(-p_{N'_2}) d\lambda \leq C_9 \int_{U_J^d} |c_J|^2 \exp(-p) d\lambda.$$

Here we need to calculate $\sharp\rho_{d, \tilde{\varepsilon}d}^{-1}(J)$ to give the estimate of $\|\gamma'\|_{p_{N'_2}}$. For the refinement $\mathcal{U}^{\tilde{\varepsilon}^2d}$ of $\mathcal{U}^{\tilde{\varepsilon}d}$, it follows from Lemma 3.5 (4) that

$$U_{J'}^{\tilde{\varepsilon}d} \supset \left\{ z \in \mathbb{C}^n : |z - \xi|_\infty < \tilde{\varepsilon} \left(\frac{1}{64} - \tilde{\varepsilon} \right) d(\xi) \right\}$$

for $\xi \in U_{J''}^{\tilde{\varepsilon}^2d}$ and $J' = \rho_{\tilde{\varepsilon}d, \tilde{\varepsilon}^2d}(J'')$. On the other hand, we know

$$U_J^d \subset \{z \in \mathbb{C}^n : |z - \xi|_\infty < d(\xi)\}.$$

Hence it follows from Lemma 3.5 (2) that

$$\sharp\rho_{d, \tilde{\varepsilon}d}^{-1}(J) \leq 2^{8n} \left(\frac{\tilde{\varepsilon}}{32} - 2\tilde{\varepsilon}^2 \right)^{-2n} =: C_{10}.$$

Thus we obtain

$$(3.17) \quad \begin{aligned} \|\gamma'\|_{p_{N'_2}}^2 &= \sum_{J' \in I(\tilde{\varepsilon}d)^{\sigma+1}} \int_{U_{J'}^{\tilde{\varepsilon}d}} |\gamma'_{J'}|^2 \exp(-p_{N'_2}) d\lambda \\ &\leq C_{10} \sum_{J \in I(d)^{\sigma+1}} C_9 \int_{U_J^d} |c_J|^2 \exp(-p) d\lambda \\ &= C_{10} C_9 \|c\|_p^2. \end{aligned}$$

It also follows from (3.16) that $P\gamma' = \rho_{d, \tilde{\varepsilon}d}^* c$. Since $\delta c = 0$ and P is defined globally, we have $P\delta\gamma' = \delta P\gamma' = 0$. Thus $\delta\gamma' = \gamma''$ belongs to $C^{\sigma+1}(\mathcal{U}^{\tilde{\varepsilon}d}, \ker P, p_{N'_2})$, and $\delta\gamma'' = 0$. If we choose a $T \times S$ matrix Q as in Lemma 3.8, it follows that $\ker P = \text{Im } Q = \mathcal{M}_Q$, so the inductive hypothesis can be applied. It shows that we can find $\hat{\varepsilon} < \tilde{\varepsilon}$, $N'_2 > N'_2$ and $K'_2 > 0$ such that for some $\gamma''' \in C^\sigma(\mathcal{U}^{\hat{\varepsilon}d}, \ker P, p_{N'_2})$ we have $\|\gamma'''\|_{p_{N'_2}} \leq K'_2 \|\gamma''\|_{p_{N'_2}}$ and $\delta\gamma''' = \rho_{\tilde{\varepsilon}d, \hat{\varepsilon}d}^* \gamma''$.

Setting $\tilde{\gamma} = \rho_{\tilde{\varepsilon}d, \hat{\varepsilon}d}^* \gamma' - \gamma''' \in C^\sigma(\mathcal{U}^{\hat{\varepsilon}d}, \mathcal{O}^T)$, we have $\delta\tilde{\gamma} = \rho_{\tilde{\varepsilon}d, \hat{\varepsilon}d}^* \gamma'' - \delta\gamma''' = 0$, and for some C_{11} independent of c we have $\|\tilde{\gamma}\|_{p_{N'_2}} \leq C_{11} \|c\|_p$ by the same method

that we have proved (3.17). Hence Lemma 3.6 shows that for some $N_2''' > 0$ we can find $\hat{\gamma} \in C^{\sigma-1}(\mathcal{U}^{\varepsilon d}, \mathcal{O}^T)$ so that $\tilde{\gamma} = \delta\hat{\gamma}$ and $\|\hat{\gamma}\|_{p_{N_2'''}} \leq K_1\|\tilde{\gamma}\|_{p_{N_2'''}} \leq K_1C_{11}\|c\|_p$. If we set $c' = P\hat{\gamma}$, it follows that

$$\delta c' = P\delta\hat{\gamma} = P\tilde{\gamma} = P\rho_{\varepsilon d, \varepsilon d}^* \gamma' - P\gamma''' = \rho_{\varepsilon d, \varepsilon d}^* P\gamma' = \rho_{\varepsilon d, \varepsilon d}^* \rho_{d, \varepsilon d}^* c = \rho_{d, \varepsilon d}^* c.$$

Finally, it is clear that there exists $N_2, K_2 > 0$ such that $\|c'\|_{p_{N_2}} \leq K_2\|c\|_p$, because it is sufficient to consider the estimate about P . The proof of the lemma is finished. \square

We shall apply Lemma 3.10 to the following settings. Put

$$d_V(z) = \frac{\varepsilon}{2\sqrt{2}}(2\sqrt{2n}(2\mu - 2) + |z|)^{2-2\mu},$$

where ε and μ are decided before.

Lemma 3.11. *d_V has the following properties:*

- (1) *If $w \in \mathbb{C}^n$ and $|w|_\infty \leq 1$, then we have $d_V(z+w) \leq 2d_V(z)$ for all $z \in \mathbb{C}^n$. Hence there exists an open covering $\mathcal{U}^{d_V} = \{U_j^{d_V}\}_{j \in I(d_V)}$ satisfying Lemma 3.5.*
- (2) *$-\log d_V(z)$ is a plurisubharmonic function.*
- (3) *If $U_j^{d_V} \in \mathcal{U}^{d_V}$ and $U_j^{d_V} \cap V \neq \emptyset$, then $U_j^{d_V} \subset D_{\varepsilon, 2\mu-2}(\xi)$ holds for every $\xi \in U_j^{d_V} \cap V$.*

Proof. The lemma is clear when $\mu = 1$, so we assume that $\mu \geq 2$.

- (1) If $w \in \mathbb{C}^n$ and $|w|_\infty \leq 1$, then $|z| \leq |z+w| + |w| \leq |z+w| + \sqrt{2n}$. Hence we have

$$\begin{aligned} d_V(z) &= \frac{\varepsilon}{2\sqrt{2}}(2\sqrt{2n}(2\mu - 2))^{2-2\mu} \left(1 + \frac{|z|}{2\sqrt{2n}(2\mu - 2)}\right)^{2-2\mu} \\ &\geq \frac{\varepsilon}{2\sqrt{2}}(2\sqrt{2n}(2\mu - 2))^{2-2\mu} \left(1 + \frac{|z+w|}{2\sqrt{2n}(2\mu - 2)} + \frac{1}{2(2\mu - 2)}\right)^{2-2\mu} \\ &\geq \frac{\varepsilon}{2\sqrt{2}}(2\sqrt{2n}(2\mu - 2))^{2-2\mu} \left(1 + \frac{|z+w|}{2\sqrt{2n}(2\mu - 2)}\right)^{2-2\mu} \left(1 + \frac{1}{2(2\mu - 2)}\right)^{2-2\mu} \\ &\geq \frac{1}{2} \cdot \frac{\varepsilon}{2\sqrt{2}}(2\sqrt{2n}(2\mu - 2))^{2-2\mu} \left(1 + \frac{|z+w|}{2\sqrt{2n}(2\mu - 2)}\right)^{2-2\mu} \\ &\geq \frac{1}{2}d_V(z+w), \end{aligned}$$

since $(1 + 1/2\nu)^{-\nu} \searrow \exp(-1/2) > 1/2$ as $\nu \rightarrow \infty$.

(2) is clear.

- (3) Fix $\xi \in U_j^{d_V} \cap V$. It follows from Lemma 3.5 (1) that $|z - \xi|_\infty \leq d_V(\xi)$ for all

$z \in U_j^{dv}$. Hence we obtain

$$|z_j - \xi_j| \leq \frac{\varepsilon(2\sqrt{2n}(2\mu - 2) + |\xi|)^{2-2\mu}}{2} < \varepsilon(1 + |\xi|)^{2-2\mu}$$

for all $j = 1, \dots, n$, so $z \in D_{\varepsilon, 2\mu-2}(\xi)$. \square

Since the polynomial ring $\mathbb{C}[z_1, \dots, z_n]$ is Noetherian, the prime ideal I_V is finitely generated by $P_1, \dots, P_T \in \mathbb{C}[z_1, \dots, z_n]$. Let $P = (P_1, \dots, P_T)$ be a $1 \times T$ matrix.

Lemma 3.12. *There exists $\tilde{F} \in A_p(\mathbb{C}^n)$ such that $\tilde{F}|_V \equiv \Delta f$.*

Proof. It follows from Lemma 3.2 and Lemma 3.11 (3) that if $U_j^{dv} \in \mathcal{U}^{dv}$ and $U_j^{dv} \cap V \neq \emptyset$, then there exist $\xi \in V$ and $F^j \in \mathcal{O}(D_{\varepsilon, 2\mu-2}(\xi))$ such that $\Delta f - F^j = 0$ on $V \cap D_{\varepsilon, 2\mu-2}(\xi)$ and

$$(3.18) \quad |F^j(z)| \leq A_1 \exp(B_1 p(z))$$

for every $z \in D_{\varepsilon, 2\mu-2}(\xi)$. We also put $F^j = 0$, when $U_j^{dv} \cap V = \emptyset$. We would like to apply Lemma 3.10 for $\sigma = 1$. Defining $c \in C^1(\mathcal{U}^{dv}, \mathcal{O})$ by $c_{(j_0, j_1)} = F^{j_0} - F^{j_1}$, we have $F^{j_0} - F^{j_1} = \Delta f - \Delta f = 0$ on $V \cap U_{j_0}^{dv} \cap U_{j_1}^{dv}$. It follows from (3.18) and Lemma 3.5 (2) that there exists $C_{12} > 0$ such that $\|c\|_{C_{12}p} < \infty$. On the other hand, it is clear that $\delta c = 0$, that is, $c \in C^1(\mathcal{U}^{dv}, \mathcal{M}_p, C_{12}p)$. Hence Lemma 3.10 gives $\varepsilon_0 < 1/384$, $N_2, K_2 > 0$ and $c' \in C^0(\mathcal{U}^{\varepsilon_0 dv}, \mathcal{M}_p, p_{N_2})$ so that $\delta c' = \rho_{dv, \varepsilon_0 dv}^* c$ and $\|c'\|_{p_{N_2}} \leq K_2 \|c\|_{C_{12}p}$. It follows from the definition of weight functions that there exists $N_3 > 0$ such that $\|c'\|_{N_3 p} \leq K_2 \|c\|_{C_{12}p}$. Here we put $\tilde{F} = F^j + c'_j$ in $U_{j'}^{\varepsilon_0 dv}$, where $j = \rho_{dv, \varepsilon_0 dv}(j')$. Then \tilde{F} belongs to $\mathcal{O}(\mathbb{C}^n)$ and Lemma 2.2 (3) gives $\tilde{F} \in A_p(\mathbb{C}^n)$. \square

Here we make $\hat{F} \in A_p(\mathbb{C}^n)$ with the required properties from $\tilde{F} \in A_p(\mathbb{C}^n)$ made in Lemma 3.12. We shall use some result in the ring theory. For an ideal $I \subset \mathbb{C}[z_1, \dots, z_n]$, we set $\tilde{I} = \mathcal{O}(\mathbb{C}^n) \otimes_{\mathbb{C}[z_1, \dots, z_n]} I = \mathcal{O}(\mathbb{C}^n)I$.

Lemma 3.13 (cf. [6, Lemma 3.5 in Chapter 8]). *For two ideals I_1 and I_2 in $\mathbb{C}[z_1, \dots, z_n]$, $\widehat{(I_1 \cap I_2)} = \tilde{I}_1 \cap \tilde{I}_2$.*

For $R \in \mathbb{C}[z_1, \dots, z_n]$, set $(I : R) = \{g \in \mathbb{C}[z_1, \dots, z_n] : Rg \in I\}$ and $(\tilde{I} : R) = \{\tilde{g} \in \mathcal{O}(\mathbb{C}^n) : R\tilde{g} \in \tilde{I}\}$.

Lemma 3.14 (cf. [6, Lemma 3.6 in Chapter 9]). *For an ideal $I \subset \mathbb{C}[z_1, \dots, z_n]$, $\widehat{(\tilde{I} : R)} = (\tilde{I} : R)$.*

Note that Lemmas 3.13 and 3.14 follow from the flatness of $\mathcal{O}(\mathbb{C}^n)$.

Lemma 3.15 (cf. [6, Lemma 3.13 in Chapter 8]). *Let $I \subset \mathbb{C}[z_1, \dots, z_n]$ be a primary ideal. Set $V_I = \{z \in \mathbb{C}^n : P(z) = 0 \text{ for all } P \in I\}$. Then we have $(I : R) = I$, if $R|_{V_I} \not\equiv 0$.*

Proof. Since it is obvious that $I \subset (I : R)$, we have only to prove that $I \supset (I : R)$. For $P \in (I : R)$, it follows $RP \in I$. Assuming that $P \notin I$, we have $R^\nu \in I$ for some $\nu \in \mathbb{N}$, since I is a primary ideal. Hence it follows that $R|_{V_I} \equiv 0$, which is a contradiction. \square

Here we can prove the following lemma by an argument similar to the proof of Lemma 3.12:

Lemma 3.16 (cf. [9, Theorem 7.6.11]). *Let $I \subset \mathbb{C}[z_1, \dots, z_n]$ be an ideal generated by Q_1, \dots, Q_T . If $g \in \tilde{I} \cap A_p(\mathbb{C}^n)$, then there exist $a_1, \dots, a_T \in A_p(\mathbb{C}^n)$ such that*

$$g = a_1 Q_1 + \dots + a_T Q_T.$$

[9, Theorem 7.4.8] also implies that there exists $\tilde{f} \in \mathcal{O}(\mathbb{C}^n)$ with no growth conditions such that $\tilde{f}|_V \equiv f$.

Lemma 3.17. *We have $\hat{F} \in A_p(\mathbb{C}^n)$ satisfying that $\hat{F} - \tilde{f} \in \tilde{I}_V$, that is, $\hat{F}|_V \equiv f$.*

Proof. Let $J \in \mathbb{C}[z_1, \dots, z_n]$ be the ideal generated by P_1, \dots, P_T and Δ . By Lemma 3.12, it follows that $\tilde{F} - \Delta \tilde{f} \in \tilde{I}_V$, that is, $\tilde{F} \in \tilde{J}$. Applying Lemma 3.16 to J , we have $a_1, \dots, a_T, b \in A_p(\mathbb{C}^n)$ satisfying that

$$\tilde{F} = a_1 P_1 + \dots + a_T P_T + b \Delta.$$

Here if we set $\hat{F} = b$, then $\Delta \hat{F} - \Delta \tilde{f} = \Delta(\hat{F} - \tilde{f}) \in \tilde{I}_V$. Hence it follows from Lemmas 3.14 and 3.15 that

$$\hat{F} - \tilde{f} \in (\tilde{I}_V : \Delta) = \widetilde{(\tilde{I}_V : \Delta)} = \tilde{I}_V,$$

so that $\hat{F}|_V \equiv f$. \square

Proof of Theorem 3.1. Let $V \subset \mathbb{C}^n$ be an algebraic subset. Then there exist a finite number of irreducible algebraic varieties V_1, \dots, V_S such that $V = V_1 \cup \dots \cup V_S$. We shall prove Theorem 3.1 by induction on S . When $S = 1$, we have already proved in Lemma 3.17. Here we can assume that $S \geq 2$, since the proofs for $S \geq 3$ are the same as for $S = 2$. Then we have $V = V_1 \cup V_2$ and $I_V = I_{V_1} \cap I_{V_2}$. For

$f \in A_p(V)$, it follows from [9, Theorem 7.4.8] that there exists $\tilde{f} \in \mathcal{O}(\mathbb{C}^n)$ with no growth conditions such that $\tilde{f}|_V \equiv f$. Since the theorem is valid for V_1 (resp. V_2), we have $\hat{F}_1 \in A_p(\mathbb{C}^n)$ (resp. $\hat{F}_2 \in A_p(\mathbb{C}^n)$) such that $\hat{F}_1|_{V_1} \equiv f$ (resp. $\hat{F}_2|_{V_2} \equiv f$). Let P_1, \dots, P_{T_1} (resp. Q_1, \dots, Q_{T_2}) generate I_{V_1} (resp. I_{V_2}). If $J \subset \mathbb{C}[z_1, \dots, z_n]$ is the ideal generated by $P_1, \dots, P_{T_1}, Q_1, \dots, Q_{T_2}$, we have $I_{V_1} \cap I_{V_2} \subset J$. Since $\hat{F}_1 - \tilde{f} \in \tilde{I}_{V_1}$ and $\hat{F}_2 - \tilde{f} \in \tilde{I}_{V_2}$, it follows that

$$\hat{F}_1 - \hat{F}_2 = (\hat{F}_1 - \tilde{f}) - (\hat{F}_2 - \tilde{f}) \in \tilde{I}_{V_1} - \tilde{I}_{V_2} \subset \tilde{J}.$$

Applying Lemma 3.16 to J , we have $a_1, \dots, a_{T_1}, b_1, \dots, b_{T_2} \in A_p(\mathbb{C}^n)$ satisfying

$$\hat{F}_1 - \hat{F}_2 = a_1 P_1 + \dots + a_{T_1} P_{T_1} + b_1 Q_1 + \dots + b_{T_2} Q_{T_2}.$$

Here we set

$$\hat{F} = \hat{F}_1 - (a_1 P_1 + \dots + a_{T_1} P_{T_1}) = \hat{F}_2 + (b_1 Q_1 + \dots + b_{T_2} Q_{T_2}).$$

Then since $\hat{F}_1 - \tilde{f} \in \tilde{I}_{V_1}$ and $\hat{F}_2 - \tilde{f} \in \tilde{I}_{V_2}$, it follows from Lemma 3.13 that

$$\hat{F} - \tilde{f} \in \tilde{I}_{V_1} \cap \tilde{I}_{V_2} = \widetilde{(I_{V_1} \cap I_{V_2})} = \tilde{I}_V,$$

so that $\hat{F}|_V \equiv f$. Thus the proof of Theorem 3.1 is finished. \square

4. Proof of the main theorem

Applying Theorem A for $X = \{\zeta_\nu\}$, we have $f_1, \dots, f_m \in \mathcal{O}(\mathbb{C}^m)$ and constants $\varepsilon_1, C_3, A, B > 0$ with

$$(4.1) \quad |f_j(\zeta)| \leq A \exp(B|\zeta|^a)$$

for all $\zeta \in \mathbb{C}^m$ and $j = 1, \dots, m$,

$$(4.2) \quad Z(f_1, \dots, f_m) \supset X$$

and

$$(4.3) \quad \sum_{j=1}^m |D_u f_j(\zeta_\nu)| \geq \varepsilon_1 \exp(-C_3 |\zeta_\nu|^a)$$

for all $\nu \in \mathbb{N}$ and $u \in S^{2m-1}$. Fix $\nu \in \mathbb{N}$ and $u \in S^{2m-1}$. Set $\tilde{f}_{j,\nu,u}(w) = f_j(\zeta_\nu + wu)$, which is an entire function on \mathbb{C} . It follows from the chain rule that

$$\tilde{f}'_{j,\nu,u}(0) = \sum_{l=1}^m \frac{\partial f_j}{\partial \xi_l}(\zeta_\nu) \cdot u_l = D_u f_j(\zeta_\nu).$$

Hence from (4.3), there exists $j(\nu, u) \in \{1, \dots, m\}$ such that

$$(4.4) \quad |\tilde{f}'_{j(\nu,u),\nu,u}(0)| \geq \frac{\varepsilon_1}{m} \exp(-C_3|\zeta_\nu|^a).$$

In the rest of the proof, we denote $\tilde{f}_{j(\nu,u),\nu,u}$ by $\tilde{f}_{j(\nu,u)}$. Put $Z_{\nu,u} = \{w \in \mathbb{C} : \tilde{f}_{j(\nu,u)}(w) = 0\}$, which contains 0 by (4.2), and $d_{\nu,u} = \min\{1, \text{dist}(0, Z_{\nu,u} \setminus \{0\})\}$. From (4.1), we have $|\tilde{f}_{j(\nu,u),\nu,u}(\zeta_\nu + wu)| \leq A \exp(B|\zeta_\nu + wu|^a)$ for $|w| \leq 1$. Since $|(\zeta_\nu + wu) - \zeta_\nu| = |wu| = |w| \leq 1$ and $|\cdot|^a$ is a weight function, there exists $A_1, B_1 > 0$ independent of ν and u such that

$$(4.5) \quad |\tilde{f}_{j(\nu,u)}(w)| \leq A_1 \exp(B_1|\zeta_\nu|^a),$$

Set $g_{\nu,u}(w) = \tilde{f}_{j(\nu,u)}(w)/w$. Since $\tilde{f}_{j(\nu,u)}$ has zero of order only one at $w = 0$ by (4.2) and (4.4), we obtain $g_{\nu,u} \in A(\mathbb{C})$ and

$$(4.6) \quad g_{\nu,u}(0) = \tilde{f}'_{j(\nu,u)}(0) \neq 0.$$

It is satisfied for $|w| = 1$ that

$$|g_{\nu,u}(w)| = \frac{|\tilde{f}_{j(\nu,u)}(w)|}{|w|} = |\tilde{f}_{j(\nu,u)}(w)| \leq A_1 \exp(B_1|\zeta_\nu|^a).$$

Hence it follows from the Maximal Modulus Theorem that

$$(4.7) \quad |g_{\nu,u}(w)| \leq A_1 \exp(B_1|\zeta_\nu|^a)$$

for $|w| \leq 1$. We denote $G_{\nu,u} \in A(\mathbb{C})$ by

$$G_{\nu,u}(w) = \frac{g_{\nu,u}(w) - g_{\nu,u}(0)}{3A_1 \exp(B_1|\zeta_\nu|^a)}.$$

Then we have $G_{\nu,u}(0) = 0$ and (4.7) gives that $|G_{\nu,u}(w)| < 1$ for $|w| \leq 1$. Hence the Schwarz lemma implies that $|G_{\nu,u}(w)| \leq |w|$ for $|w| \leq 1$. In particular, for $\tilde{w} \in (Z_{\nu,u} \setminus \{0\}) \cap \{w \in \mathbb{C} : |w| < 1\}$, which is a zero of $g_{\nu,u}$ in $\{w \in \mathbb{C} : |w| < 1\}$, we have from (4.4) and (4.6)

$$|\tilde{w}| \geq |G_{\nu,u}(\tilde{w})| = \frac{|g_{\nu,u}(0)|}{3A_1 \exp(B_1|\zeta_\nu|^a)} = \frac{|\tilde{f}'_{j(\nu,u)}(0)|}{3A_1 \exp(B_1|\zeta_\nu|^a)} \geq \varepsilon_2 \exp(-C_4|\zeta_\nu|^a),$$

where ε_2 and C_4 are independent of ν and u . Thus the definition of $d_{\nu,u}$ gives that

$$(4.8) \quad d_{\nu,u} \geq \varepsilon_2 \exp(-C_4|\zeta_\nu|^a).$$

Now, we need the Borel-Carathéodory inequality. (For the proof, see e.g. [1, Corollary 4.5.10].)

Borel-Carathéodory inequality. *Let h be a function which is holomorphic in a neighborhood of $|w| \leq R$ and has no zero in $|w| < R$. If $h(0) = 1$ and $0 \leq |z| \leq r < R$, then the following estimate holds:*

$$\log |h(z)| \geq -\frac{2r}{R-r} \log \max_{|\omega|=R} |h(\omega)|.$$

Since $g_{\nu,u}(0) \neq 0$ from (4.6), we apply this inequality to $h(w) = g_{\nu,u}(w)/g_{\nu,u}(0)$, $R = d_{\nu,u}$ and $r = d_{\nu,u}/2$ to obtain

$$\begin{aligned} \log \left| \frac{g_{\nu,u}(w)}{g_{\nu,u}(0)} \right| &\geq -\frac{2 \cdot d_{\nu,u}/2}{d_{\nu,u} - d_{\nu,u}/2} \log \max_{|\omega|=d_{\nu,u}} \left| \frac{g_{\nu,u}(\omega)}{g_{\nu,u}(0)} \right| \\ &= -2 \log \max_{|\omega|=d_{\nu,u}} \left| \frac{g_{\nu,u}(\omega)}{g_{\nu,u}(0)} \right| \end{aligned}$$

for $|w| \leq d_{\nu,u}/2$. Then it follows from (4.4), (4.6) and (4.7) that

$$\begin{aligned} (4.9) \quad |g_{\nu,u}(w)| &\geq |g_{\nu,u}(0)| \left(\max_{|\omega|=d_{\nu,u}} \left| \frac{g_{\nu,u}(\omega)}{g_{\nu,u}(0)} \right| \right)^{-2} \\ &= |g_{\nu,u}(0)|^3 \left(\max_{|\omega|=d_{\nu,u}} |g_{\nu,u}(\omega)| \right)^{-2} \\ &\geq \varepsilon_3 \exp(-C_5 |\zeta_\nu|^a), \end{aligned}$$

where ε_3 and C_5 is independent of ν and u . Put $\hat{d}_\nu = \varepsilon_2 \exp(-C_4 |\zeta_\nu|^a)$, where ε_2 and C_4 are given in (4.8). Since $\hat{d}_\nu \leq d_{\nu,u}$ by (4.8), it follows from (4.9) that for $|w| = \hat{d}_\nu/2$ $|\tilde{f}_{j(\nu,u)}(w)| = |w \cdot g_{\nu,u}(w)| \geq \varepsilon_4 \exp(-C_6 |\zeta_\nu|^a)$, where ε_4 and C_6 is independent of ν and u . Thus we have proved that for every $u \in S^{2m-1}$, there exists $j(\nu, u) \in \{1, \dots, m\}$ such that $|f_{j(\nu,u)}(\zeta_\nu + wu)| \geq \varepsilon_4 \exp(-C_6 |\zeta_\nu|^a)$ for $|w| = \hat{d}_\nu/2$. Hence we have

$$\begin{aligned} (4.10) \quad |f(\zeta_\nu + wu)| &= \left(\sum_{j=1}^m |f_j(\zeta_\nu + wu)|^2 \right)^{1/2} \geq |f_{j(\nu,u)}(\zeta_\nu + wu)| \\ &\geq \varepsilon_4 \exp(-C_6 |\zeta_\nu|^a) \end{aligned}$$

for $|w| = \hat{d}_\nu/2$.

We now consider $f \circ F : \mathbb{C}^n \rightarrow \mathbb{C}^m$. Since $\max_{j=1, \dots, m} \deg F_j = d$ and $b \geq ad$, there exist $\alpha, \beta > 0$ such that

$$(4.11) \quad |F(z)|^a \leq \alpha |z|^b + \beta$$

for all $z \in \mathbb{C}^n$. Then we have from (4.1) and (4.11)

$$|f_j \circ F(z)| \leq A \exp(B |F(z)|^a) \leq A e^{\beta B} \exp(\alpha B |z|^b)$$

for all $z \in \mathbb{C}^n$ and $j = 1, \dots, m$, that is, $f \circ F \in A_{|\cdot|, b}(\mathbb{C}^n)^m$.

Set $U_\nu = \{\xi \in \mathbb{C}^m : |\xi - \zeta_\nu| \leq \hat{d}_\nu/2\}$. Denote by V_ν the connected component of $S_{|\cdot|, a}(f; \varepsilon_4, C_6)$ including ζ_ν . Then (4.10) implies that $V_\nu \subset U_\nu$. We also have $U_\nu \cap (Z(f_1, \dots, f_m) \setminus \{\zeta_\nu\}) = \emptyset$. Namely, for $\xi \in Z(f_1, \dots, f_m) \setminus \{\zeta_\nu\}$ there exists $u \in S^{2m-1}$ such that $\xi = \zeta_\nu + |\xi - \zeta_\nu|u$. It follows from the definition of $d_{\nu, u}$ and (4.8) that $|\xi - \zeta_\nu| \geq d_{\nu, u} \geq \varepsilon_2 \exp(-C_4|\zeta_\nu|^a) = \hat{d}_\nu$, so that $\xi \notin U_\nu$. Now setting $\varepsilon_5 = \varepsilon_4 \exp(-\beta C_6)$ and $C_7 = \alpha C_6$, we claim that the union \hat{V}_ν of the connected components of $S_{|\cdot|, b}(f \circ F; \varepsilon_5, C_7)$ including $F^{-1}(\zeta_\nu)$ is contained in $F^{-1}(V_\nu)$. In fact, it follows from (4.11) that for $z \in \hat{V}_\nu$

$$\begin{aligned} |f \circ F(z)| &< \varepsilon_4 \exp(-\beta C_6) \exp(-\alpha C_6 |z|^b) \\ &\leq \varepsilon_4 \exp\left(-\beta C_6 - \alpha C_6 \cdot \frac{|F(z)|^a - \beta}{\alpha}\right) \\ &= \varepsilon_4 \exp(-C_6 |F(z)|^a), \end{aligned}$$

which implies that $F(z) \in S_{|\cdot|, a}(f; \varepsilon_4, C_6)$. For $z' \in F^{-1}(\zeta_\nu)$, the above inequality holds on every curve through z and z' in \hat{V}_ν . The connectedness of V_ν proves that $z \in F^{-1}(V_\nu)$. It is clear that $\hat{V}_\nu \cap F^{-1}(Z(f_1, \dots, f_m) \setminus \{\zeta_\nu\}) = \emptyset$ for all $\nu \in \mathbb{N}$ by the above argument.

Here we need the following lemma:

Lemma 4.1 (cf. [2, Lemma 3.2]). *Let $f_1, \dots, f_m \in A_p(\mathbb{C}^m)$. Then there exist constants $\varepsilon, C > 0$ such that*

$$\sum_{j=1}^m |D_u f_j(\zeta_\nu)| \geq \varepsilon \exp(-Cp(\zeta_\nu))$$

for all $\nu \in \mathbb{N} \setminus E$ and $u \in S^{2m-1}$ if and only if we have constants $\varepsilon', C' > 0$ satisfying

$$|\det Jf(\zeta_\nu)| \geq \varepsilon' \exp(-C'p(\zeta_\nu))$$

for all $\nu \in \mathbb{N}$, where Jf is the Jacobian matrix of $f = (f_1, \dots, f_m)$.

We apply this lemma to obtain $\varepsilon_6, C_8 > 0$ such that

$$(4.12) \quad |\det Jf(\zeta_\nu)| \geq \varepsilon_6 \exp(-C_8|\zeta_\nu|^a)$$

for all $\nu \in \mathbb{N}$. Calculating a sum of the moduli of all $m \times m$ minors of $J(f \circ F)$, we have from (2) of Main Theorem, (4.11) and (4.12)

$$\sum_{\kappa=1}^{\binom{n}{m}} |\Delta_{\kappa}^{f \circ F}(z)| = |\det Jf(F(z))| \cdot \sum_{\kappa=1}^{\binom{n}{m}} |\Delta_{\kappa}^F(z)|$$

$$\begin{aligned} &\geq \varepsilon_6 \exp(-C_8 |F(z)|^a) \cdot \varepsilon \exp(-C|z|^b) \\ &\geq (\varepsilon \varepsilon_6 e^{-\beta C_8}) \exp(-(\alpha C_8 + C)|z|^b) \end{aligned}$$

for all $z \in \bigcup_{\nu \in \mathbb{N} \setminus E} F^{-1}(\zeta_\nu)$.

Here the proof of [5, Theorem 1] implies the following:

Lemma 4.2. *For $f_1, \dots, f_N \in A_p(\mathbb{C}^n)$, let Z' be a union of connected components of $Z(f_1, \dots, f_N)$ which are k -codimensional manifolds, so that*

$$\sum_{\vartheta=1}^{\binom{N}{k}} \sum_{\kappa=1}^{\binom{n}{k}} |\Delta_{\vartheta, \kappa}^f(z)| \geq \varepsilon \exp(-Cp(z))$$

for all $z \in Z'$, where the sum is taken over all $k \times k$ minors of the Jacobian matrix Jf . If we can choose constants $\varepsilon'', C'' > 0$ such that every connected component of $S_p(f; \varepsilon'', C'')$ including a connected component of Z' does not intersect the other connected components of $Z(f_1, \dots, f_N)$, then we have constants $\varepsilon''' < \varepsilon'', C''' > C''$ satisfying: Let Y be a connected component of Z' and let V_Y be the connected component of $S_p(f; \varepsilon''', C''')$ including Y . Then there exists a holomorphic retract Φ_Y from V_Y onto Y such that $|z - \Phi_Y(z)| \leq 1$ for all $z \in V_Y$.

By setting $\varepsilon'' = \varepsilon_5$, $C'' = C_7$ and $Z' = \bigcup_{\nu \in \mathbb{N} \setminus E} F^{-1}(\zeta_\nu)$, we can apply this lemma to obtain $\varepsilon_7, C_9 > 0$ and a holomorphic retract Φ_ν from \tilde{V}_ν onto $F^{-1}(\zeta_\nu)$ such that

$$(4.13) \quad |z - \Phi_\nu(z)| \leq 1$$

for all $\nu \in \mathbb{N} \setminus E$, where \tilde{V}_ν ($\nu \in \mathbb{N}$) is the union of the connected components of $S_{|\cdot|^b}(f \circ F; \varepsilon_7, C_9)$ including $F^{-1}(\zeta_\nu)$. It is clear that $\tilde{V}_\nu \cap \tilde{V}_{\nu'} = \emptyset$ for $\nu \neq \nu'$.

For $h \in A_{|\cdot|^b}(F^{-1}(X))$, it follows from Theorem 3.1 that there exists $\tilde{H} \in A_{|\cdot|^b}(\mathbb{C}^n)$ such that $\tilde{H}|_{\bigcup_{\nu \in E} F^{-1}(\zeta_\nu)} \equiv h|_{\bigcup_{\nu \in E} F^{-1}(\zeta_\nu)}$. Then we define

$$\tilde{h}(z) = \begin{cases} \Phi_\nu^* h(z), & \text{if } z \in \tilde{V}_\nu \text{ and } \nu \in \mathbb{N} \setminus E, \\ \tilde{H}(z), & \text{if } z \in \tilde{V}_\nu \text{ and } \nu \in E, \\ 0, & \text{if } z \in S_{|\cdot|^b}(f \circ F; \varepsilon_7, C_9) \setminus \bigcup_{\nu \in \mathbb{N}} \tilde{V}_\nu. \end{cases}$$

Since $|\cdot|^b$ is a weight function, (4.13) implies that there exist $A_2, B_2 > 0$ such that $\tilde{h}(z) \leq A_2 \exp(B_2 |z|^b)$ for all $z \in S_{|\cdot|^b}(f \circ F; \varepsilon_7, C_9)$. Hence it follows from the semiloal interpolation theorem that we obtain $H \in A_{|\cdot|^b}(\mathbb{C}^n)$ with $H|_{F^{-1}(X)} \equiv h$. Thus $F^{-1}(X)$ is interpolating for $A_{|\cdot|^b}(\mathbb{C}^n)$.

5. Examples and remarks

The following is an easy example for the main theorem:

EXAMPLE 5.1. Set $X = \{\nu\}_{\nu \in \mathbb{Z}} \subset \mathbb{C}$. By applying Theorem A (or [3, Corollary 3.5]) to $f(z) = \sin 2\pi z \in A_{|\cdot|}(\mathbb{C})$, we know that V is interpolating for $A_{|\cdot|}(\mathbb{C})$. Put $F(z_1, \dots, z_n) = z_1^2 + \dots + z_n^2$, which satisfies

$$\sum_{j=1}^n \left| \frac{\partial F(z)}{\partial z_j} \right| \geq |\text{grad } F(z)| = 2|z|.$$

If $z \in F^{-1}(\nu)$, we have $|z|^2 \geq |\nu|$. In particular, for $\nu \in \mathbb{Z} \setminus \{0\}$, it follows that

$$\sum_{j=1}^n \left| \frac{\partial F(z)}{\partial z_j} \right| \geq 2\sqrt{|\nu|} \geq 2.$$

Hence the main theorem implies that $F^{-1}(X)$ is interpolating for $A_{|\cdot|^b}(\mathbb{C}^n)$ for all $b \geq 2$. (In this case, $E = \{0\}$.)

In the case where $n = m = 1$, we can improve the main theorem as follows:

Corollary 5.2. *Let $X = \{\zeta_\nu\}_{\nu \in \mathbb{N}}$ be a discrete variety in \mathbb{C} and let $F \in \mathbb{C}[z]$. Put $d = \deg F$. For $a > 0$, we assume that X is interpolating for $A_{|\cdot|^a}(\mathbb{C})$. Then $F^{-1}(X)$ is interpolating for $A_{|\cdot|^b}(\mathbb{C})$ for every $b \geq ad$.*

Finally, we remark that the term ‘ $b \geq ad$ ’ in the main theorem is sharp in the sense of the following open problem:

Open Problem ([5, Problem 1]). Let q be another weight function on \mathbb{C}^n satisfying $q \geq p$ everywhere. Assume that an analytic subset X of \mathbb{C}^n is interpolating for $A_p(\mathbb{C}^n)$. Then is V interpolating for $A_q(\mathbb{C}^n)$?

We prove this remark by giving an example for which $F^{-1}(X)$ is not interpolating for $A_{|\cdot|^b}(\mathbb{C}^n)$ for $b < ad$. Let $X = \{\zeta_\nu\}_{\nu \in \mathbb{N}}$ be a discrete variety in \mathbb{C} . Then Nevanlinna’s counting function is defined as follows: $n(r, \zeta, X) = \#\{\nu \in \mathbb{N} : |\zeta_\nu - \zeta| \leq r\}$ and

$$N(r, \zeta, X) = \int_0^r \frac{n(t, \zeta, X) - n(0, \zeta, X)}{t} dt + n(0, \zeta, X) \log r.$$

EXAMPLE 5.3. Assume that $n = m = 1$. Put $X = \{\nu\}_{\nu \in \mathbb{N}} \subset \mathbb{C}$. As in Example 5.1, it follows that X is interpolating for $A_{|\cdot|}(\mathbb{C})$. Set $F(z) = z^4$, so $\deg F = 4$. Then we have

$$F^{-1}(X) = \left\{ \sqrt[4]{\nu} \cdot \exp\left(\frac{l\pi i}{2}\right) \right\}_{\nu \in \mathbb{N}; l=0,1,2,3}.$$

Corollary 5.2 implies that $F^{-1}(X)$ is interpolating for $A_{|\cdot|^b}(\mathbb{C})$ for every $b \geq 4$.

Here we claim that $F^{-1}(X)$ is not interpolating for $A_{|\cdot|^b}(\mathbb{C})$ for any $b < 4$. In fact, we have $n(r, 0, F^{-1}(X)) = 4\nu$ when $\sqrt[4]{\nu} \leq r < \sqrt[4]{\nu+1}$, so $n(r, 0, F^{-1}(X)) = 4[r^4]$, where $[x] = \max\{y \in \mathbb{Z} : y \leq x\}$. Since $n(s, 0, F^{-1}(X)) = 0$ for all $s \in [0, 1)$ and $[t^4] \geq t^4 - 1$ for all $t \in \mathbb{R}$, we obtain

$$\begin{aligned} N(r, 0, F^{-1}(X)) &= \int_0^r \frac{4[t^4]}{t} dt \\ &\geq \int_1^r \frac{4(t^4 - 1)}{t} dt \\ &= r^4 - 4 \log r - 1. \end{aligned}$$

Hence for every $b < 4$ there do not exist two constants $A, B > 0$ such that

$$N(r, 0, F^{-1}(X)) \leq Ar^b + B$$

for all $r \geq 0$. Then it follows from [3, Corollary 4.8] that $F^{-1}(X)$ is not interpolating for $A_{|\cdot|^b}(\mathbb{C})$ for any $b < 4$.

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