Asymptotics of solutions to the generalized reduced Ostrovsky equation

Tomoyuki Niizato

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Contents

CHAPTER 1

Introduction

In this thesis, we consider the Cauchy problem for the generalized reduced Ostrovsky equation

$$\begin{cases} u_{tx} = u + f(u)_{xx}, & x \in \mathbb{R},\ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where u_0 is a real valued function, $f(u) = u^\rho$ if ρ is an integer and $f(u) = |u|^{\rho-1}u$ if ρ is not an integer. Equation (1.1) is known as a reduced version of Ostrovsky equations [**26**] which models small-amplitude long waves in rotating fluids of finite depth, under the assumption of no-high frequency dispersion. Especially, when $\rho = 2$ equation (1.1) is called the reduced Ostrovsky equation [**2**] or the short-wave equation [**19**], and when $\rho = 3$ equation (1.1) is called the short pulse equation [**28**]. The short pulse equation is also known as approximate solutions of Maxwell's equations describing the propagation of ultra short pulses in nonlinear media. For the details of physical background, see [**2**], [**19**], [**26**], [**28**] and references therein.

Recently, the Cauchy problem for (1.1) was studied by many authors. In paper [**29**], the local well-posedness in $\mathbf{H}^s$ for $s > 3/2$ was obtained for the generalized reduced Ostrovsky equation with integer $\rho \geq 2$. Global existence of small solutions to the Cauchy problem (1.1) with integer $\rho \geq 4$ was also established in [**29**] in $\mathbf{H}^5 \cap \mathbf{H}^3_1$ with a time decay estimate of solutions $\|u(t)\|_{\mathbf{L}^r} \sim t^{-(1/2-1/r)}$ where $2 \leq r < \infty$. For $\rho = 3$, global well-posedness of small solutions was proved in [**27**] by using the conservation laws. Suitable smallness conditions are necessary for obtaining the global in time existence of solutions since Liu, Pelinovsky and Sakovich [**21**], [**22**] showed blow-up results for the short pulse equation ($\rho = 3$) and the reduced Ostrovsky equation ($\rho = 2$) for large data.

The purpose of this thesis is to consider the scattering problem for the generalized reduced Ostrovsky equation. This thesis is based on [**17**], [**18**] and [**25**].

Notation and Function Spaces. Let $\mathbf{S}$ and $\mathbf{S}'$ denote the Schwartz space and its dual space. As usually, we denote the Lebesgue space by $\mathbf{L}^p = \{\phi \in \mathbf{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\}$, where the norm $\|\phi\|_{\mathbf{L}^p} = \left(\int_{\mathbb{R}} |\phi(x)|^p \, dx\right)^{1/p}$ for $1 \leq p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \sup_{x\in\mathbb{R}} |\phi(x)|$ for $p = \infty$. The weighted Sobolev space is

$$\mathbf{H}^{m,s}_p = \left\{\phi \in \mathbf{S}'; \|\phi\|_{\mathbf{H}^{m,s}_p} = \left\|\langle x\rangle^{\frac{s}{2}} \langle i\partial_x\rangle^{\frac{m}{2}} \phi\right\|_{\mathbf{L}^p} < \infty\right\},$$

where $m, s \in \mathbb{R}, 1 \leq p \leq \infty$ and $\langle x\rangle = \sqrt{1+x^2}, \langle i\partial_x\rangle = \sqrt{1-\partial_x^2}$. We also use the following notations $\mathbf{H}^{m,s} = \mathbf{H}^{m,s}_2$ and $\mathbf{H}^m = \mathbf{H}^{m,0}$, unless it causes any confusion. The homogeneous Sobolev space is given by

$$\dot{\mathbf{H}}^m = \left\{\phi \in \mathbf{S}'; \|\phi\|_{\dot{\mathbf{H}}^m} = \left\|\left(-\partial_x^2\right)^{\frac{m}{2}} \phi\right\|_{\mathbf{L}^2} < \infty\right\}.$$

Let $\mathbf{C}(\mathbf{I};\mathbf{B})$ be the space of continuous functions from an interval $\mathbf{I}$ to a Banach space $\mathbf{B}$. Different positive constants might be denoted by the same letter C. We define the Fourier transform by

$$\mathcal{F}\phi = \hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}\phi(x)dx.$$

Also the inverse Fourier transform is defined by

$$\mathcal{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi}\phi(\xi)d\xi.$$

Denote the free evolution group related with equation (1.1) by

$$\mathcal{U}(t) = \mathcal{F}^{-1}\exp\left(-\frac{it}{\xi}\right)\mathcal{F}.$$

We introduce the following operator

$$\mathcal{J} = \mathcal{U}(t)x\mathcal{U}(-t) = x - t\partial_x^{-2}$$

where $\partial_x^{-m} = \mathcal{F}^{-1}(i\xi)^{-m}\mathcal{F}$. It is known that the operator $\mathcal{J}$ is a useful tool to obtain the $\mathbf{L}^{\infty}$ - time decay estimate. However, the operator $\mathcal{J}$ does not act on the nonlinear term like a first order differential operator. For this reason, we use the operator

$$\mathcal{P} = \mathcal{J}\partial_x - t\mathcal{L} = x\partial_x - t\partial_t$$

instead of $\mathcal{J}$, where $\mathcal{L} = \partial_t - \partial_x^{-1}$ is the linear part of the reduced Ostrovsky equation. Note that $\mathcal{P}$ acts well on the nonlinear term like a first order differential operator.

The thesis is organized as follows. In Section 2, we give preliminary estimates which are needed to prove the main results. In Section 3, we show the local existence of solutions to the generalized reduced Ostrovsky equation in the weighted Sobolev space. In Section 4, we consider the generalized reduced Ostrovsky equation with super critical case ($\rho > 3$), and we give the existence of the usual scattering states for this case. In Section 5, we consider the sub critical case ($1 < \rho < 3$) and we prove the nonexistence of the usual scattering states for this case. In Section 6, we consider the critical case ($\rho = 3$), and we give the asymptotic behavior of solutions for this case.

CHAPTER 2

Preliminary estimates

1. Linear estimates for the free evolution group of the reduced Ostrovsky equation

We first give the $\mathbf{L}^p - \mathbf{L}^q$ estimates for the free evolution group of the reduced Ostrovsky equation. The proof of the lemma was given by P. I. Naumkin.

LEMMA 2.1. *The estimate*

$$\|\mathcal{U}(t)\phi(x)\|_{\mathbf{L}^p} \le Ct^{-\frac{1}{2}\left(1-\frac{2}{p}\right)} \left\| \left(-\partial_x^2\right)^{\frac{3}{4}\left(1-\frac{2}{p}\right)} \phi \right\|_{\mathbf{L}^q}$$

is true for $t > 0$, where $1/p + 1/q = 1$ and $2 \le p \le \infty$.

PROOF. In paper [**29**], Stefanov, Shen and Kevrekidis prove the case $2 \le p < \infty$. Thus, we only consider the case $p = \infty$. We have

$$\begin{aligned} \mathcal{U}(t)\phi &= \mathcal{F}^{-1} e^{-it\frac{1}{\xi}} \mathcal{F}\phi = \frac{1}{\sqrt{2\pi}} \int e^{ix\xi} e^{-it\frac{1}{\xi}} |\xi|^{-\frac{3}{2}} |\xi|^{\frac{3}{2}} \mathcal{F}\phi d\xi \\ &= \frac{1}{\sqrt{2\pi}} \lim_{\delta\to 0} \int_{|\xi|\ge\delta} e^{ix\xi} e^{-it\frac{1}{\xi}} |\xi|^{-\frac{3}{2}} |\xi|^{\frac{3}{2}} \mathcal{F}\phi d\xi. \end{aligned}$$

Hence changing the order of integration we get

$$\begin{aligned} \mathcal{U}(t)\phi &= \frac{1}{2\pi} \lim_{\delta\to 0} \int_{|\xi|\ge\delta} e^{ix\xi} e^{-it\frac{1}{\xi}} |\xi|^{-\frac{3}{2}} \int_{\mathbb{R}} e^{-iy\xi} \left(-\partial_y^2\right)^{\frac{3}{4}} \phi(y) dy d\xi \\ &= \frac{1}{2\pi} \lim_{\delta\to 0} \int_{\mathbb{R}} \left(-\partial_y^2\right)^{\frac{3}{4}} \phi(y) dy \int_{|\xi|\ge\delta} e^{i(x-y)\xi} e^{-it\frac{1}{\xi}} |\xi|^{-\frac{3}{2}} d\xi \\ &= \lim_{\delta\to 0} \int_{\mathbb{R}} G_\delta(t, x-y) \left(-\partial_y^2\right)^{\frac{3}{4}} \phi(y)\, dy, \end{aligned}$$

where the kernel

$$G_\delta(t,x) = \frac{1}{2\pi} \int_{|\xi|\ge\delta} e^{ix\xi - it\frac{1}{\xi}} |\xi|^{-\frac{3}{2}} d\xi = \frac{1}{\pi} Re \int_\delta^\infty e^{ix\xi - it\frac{1}{\xi}} \xi^{-\frac{3}{2}} d\xi.$$

Let us estimate the integral

$$I_\delta(t,x) = \int_\delta^\infty e^{ix\xi - it\frac{1}{\xi}} \xi^{-\frac{3}{2}} d\xi$$

for $\delta \in (0,1)$. Changing the variable of integration $\xi = t\eta^{-2}$, we find

$$I_\delta(t,x) = 2t^{-\frac{1}{2}} \int_0^{\sqrt{\frac{t}{\delta}}} e^{i\frac{x}{|x|}\mu^2\eta^{-2} - i\eta^2} d\eta,$$

where $\mu=\sqrt[4]{|x|\,t}$. First we consider the case of $x\geq 0$. We integrate by parts via the identity

$$e^{i\mu^4\eta^{-2}-i\eta^2}=H_1\partial_\eta\left(\eta e^{i\mu^4\eta^{-2}-i\eta^2}\right),$$

where $H_1=\left(1-2i\left(\eta^2+\mu^4\eta^{-2}\right)\right)^{-1}$. Then we get

$$\begin{aligned}I_\delta\left(t,x\right)&=2t^{-\frac{1}{2}}\int_0^{\sqrt{\frac{t}{\delta}}}e^{i\mu^4\eta^{-2}-i\eta^2}d\eta=2t^{-\frac{1}{2}}\int_0^{\sqrt{\frac{t}{\delta}}}H_1\partial_\eta\left(\eta e^{i\mu^4\eta^{-2}-i\eta^2}\right)d\eta\\&=2t^{-\frac{1}{2}}e^{i\mu^4\eta^{-2}-i\eta^2}\eta H_1\Big|_{\eta=\sqrt{\frac{t}{\delta}}}+8it^{-\frac{1}{2}}\int_0^{\sqrt{\frac{t}{\delta}}}e^{i\mu^4\eta^{-2}-i\eta^2}\left(\eta^2-\mu^4\eta^{-2}\right)H_1^2d\eta,\end{aligned}$$

from which it follows

$$\begin{aligned}\left|I_\delta\left(t,x\right)\right|&\leq\frac{Ct^{-\frac{1}{2}}\sqrt{\frac{t}{\delta}}}{1+\frac{t}{\delta}+\mu^4\frac{\delta}{t}}+Ct^{-\frac{1}{2}}\int_0^{\sqrt{\frac{t}{\delta}}}\frac{\eta^2+\mu^4\eta^{-2}}{\left(1+\eta^2+\mu^4\eta^{-2}\right)^2}d\eta\\&\leq Ct^{-\frac{1}{2}}\sqrt{\frac{t}{\delta}}\left\langle\frac{t}{\delta}\right\rangle^{-1}+Ct^{-\frac{1}{2}}\int_0^\infty\langle\eta\rangle^{-2}d\eta\leq Ct^{-\frac{1}{2}}\end{aligned}$$

for all $x\geq 0,t\geq 1,\ \delta\in(0,1)$.

Next we consider the case of $x<0$. Denote $b=\min\left(\frac{\mu}{2},\sqrt{\frac{t}{\delta}}\right)$ and represent

$$I_\delta\left(t,x\right)=2t^{-\frac{1}{2}}\int_0^b e^{-i\mu^4\eta^{-2}-i\eta^2}d\eta+2t^{-\frac{1}{2}}\int_b^{\sqrt{\frac{t}{\delta}}}e^{-i\mu^4\eta^{-2}-i\eta^2}d\eta=I_1+I_2.$$

In the first integral I_1 we integrate by parts using the identity

$$e^{-i\mu^4\eta^{-2}-i\eta^2}=H_2\partial_\eta\left(\eta e^{-i\mu^4\eta^{-2}-i\eta^2}\right),$$

where $H_2=\left(1-2i\left(\eta^2-\mu^4\eta^{-2}\right)\right)^{-1}$. Then

$$\begin{aligned}I_1&=2t^{-\frac{1}{2}}\int_0^b e^{-i\mu^4\eta^{-2}-i\eta^2}d\eta=2t^{-\frac{1}{2}}\int_0^b H_2\partial_\eta\left(\eta e^{-i\mu^4\eta^{-2}-i\eta^2}\right)d\eta\\&=2t^{-\frac{1}{2}}e^{-i\mu^4\eta^{-2}-i\eta^2}\eta H_2\Big|_{\eta=b}+8it^{-\frac{1}{2}}\int_0^b e^{-i\mu^4\eta^{-2}-i\eta^2}\left(\eta^2+\mu^4\eta^{-2}\right)H_2^2d\eta.\end{aligned}$$

Since $\mu^4\eta^{-2}-\eta^2\geq\frac{1}{2}\mu^4\eta^{-2}\geq\eta^2$ for $\eta\in[0,b]$, we get

$$\begin{aligned}\left|I_1\right|&\leq\frac{Ct^{-\frac{1}{2}}\sqrt{b}}{1+\mu^4b^{-2}-b^2}+Ct^{-\frac{1}{2}}\int_0^b\frac{\eta^2+\mu^4\eta^{-2}}{\left(1+\mu^4\eta^{-2}-\eta^2\right)^2}d\eta\\&\leq Ct^{-\frac{1}{2}}\sqrt{b}\,\langle b\rangle^{-1}+Ct^{-\frac{1}{2}}\int_0^\infty\langle\eta\rangle^{-2}d\eta\leq Ct^{-\frac{1}{2}}\end{aligned}$$

for all $x<0,t\geq 1,\ \delta\in(0,1)$. In the second integral I_2 over the domain $b<\eta<\sqrt{\frac{t}{\delta}}$ we integrate by parts using the following identity

$$e^{-i\mu^4\eta^{-2}-i\eta^2}=H_3\partial_\eta\left(\left(\eta-\mu\right)e^{-i\mu^4\eta^{-2}-i\eta^2}\right),$$

where

$$H_3=\left(1-2i\frac{\left(\eta+\mu\right)\left(\eta^2+\mu^2\right)}{\eta^3}\left(\eta-\mu\right)^2\right)^{-1}.$$

Then we obtain

$$I_2 = 2t^{-\frac{1}{2}} \int_b^{\sqrt{\frac{t}{\delta}}} e^{-i\mu^4\eta^{-2}-i\eta^2} d\eta = 2t^{-\frac{1}{2}} \int_b^{\sqrt{\frac{t}{\delta}}} H_3 \partial_\eta \left((\eta-\mu) e^{-i\mu^4\eta^{-2}-i\eta^2} \right) d\eta$$

$$= \quad 2t^{-\frac{1}{2}} e^{-i\mu^4\eta^{-2}-i\eta^2} (\eta-\mu) H_3 \Big|_{\eta=b}^{\eta=\sqrt{\frac{t}{\delta}}}$$

$$-4it^{-\frac{1}{2}} \int_b^{\sqrt{\frac{t}{\delta}}} e^{-i\mu^4\eta^{-2}-i\eta^2} H_3^2 (\eta-\mu) \partial_\eta \left((\eta+\mu) \left(\eta^2+\mu^2\right) \eta^{-3} (\eta-\mu)^2 \right) d\eta.$$

Since $|H_3| \leq \langle \eta - \mu \rangle^{-2}$ and

$$\left| (\eta-\mu) \partial_\eta \left((\eta+\mu) \left(\eta^2+\mu^2\right) \eta^{-3} (\eta-\mu)^2 \right) \right| \leq C (\eta-\mu)^2$$

for $b < \eta < \sqrt{\frac{t}{\delta}}$, we get

$$|I_2| \leq Ct^{-\frac{1}{2}} + Ct^{-\frac{1}{2}} \int_b^{\sqrt{\frac{t}{\delta}}} \langle \eta - \mu \rangle^{-2} d\eta \leq Ct^{-\frac{1}{2}}$$

for all $x < 0, t \geq 1,\ \delta \in (0,1)$. Thus we find

$$|G_\delta (t,x)| \leq Ct^{-\frac{1}{2}}$$

for all $x \in \mathbb{R}, t \geq 1,\ \delta \in (0,1)$. Therefore

$$\begin{aligned} |\mathcal{U}(t)\phi| &\leq \lim_{\delta\to 0} \int_{\mathbb{R}} \left| G_\delta (t, x-y) \left(-\partial_y^2\right)^{\frac{3}{4}} \phi(y) \right| dy \\ &\leq Ct^{-\frac{1}{2}} \int_{\mathbb{R}} \left| \left(-\partial_y^2\right)^{\frac{3}{4}} \phi(y) \right| dy = Ct^{-\frac{1}{2}} \left\| \left(-\partial_y^2\right)^{\frac{3}{4}} \phi \right\|_{\mathbf{L}^1}. \end{aligned}$$

This completes the proof of Lemma 2.1. □

In the next lemma, we show the large time asymptotics of $\mathcal{U}(t)\phi$ in $\mathbf{L}^2$. To state the lemma, we define the main term $\Phi(t)$ by

$$\Phi(t) = \frac{\chi^{\frac{3}{2}}}{\sqrt{2\pi}} \left(e^{-i\left(\frac{2t}{\chi}+\frac{\pi}{4}\right)} \hat{\phi}(\chi) + e^{i\left(\frac{2t}{\chi}+\frac{\pi}{4}\right)} \hat{\phi}(-\chi) \right)$$

where $\chi = \sqrt{t/|x|}$. When ϕ is a real valued function, then $\Phi(t)$ is represented as

$$\Phi(t) = 2\frac{\chi^{\frac{3}{2}}}{\sqrt{2\pi}} \Re e^{-i\left(\frac{2t}{\chi}+\frac{\pi}{4}\right)} \hat{\phi}(\chi).$$

The next lemma says that solutions of the linear problem decay rapidly in the region $x > -1$, which implies that $\mathcal{U}(t)\phi$ is remainder term for $x > -1$ and the main term of the large time asymptotics of $\mathcal{U}(t)\phi$ lies in the domain $(-\infty, -1)$.

Lemma 2.2. ([**18**]) *Let $\phi, x\phi_x \in \mathbf{H}^1$. Then the estimates*

$$\|\mathcal{U}(t)\phi\|_{\mathbf{L}^2(0,\infty)} \leq CAt^{-\frac{1}{2}},$$

$$\|\mathcal{U}(t)\phi\|_{\mathbf{L}^2(-1,0)} \leq CAt^{-\frac{1}{3}},$$

and

$$\|\mathcal{U}(t)\phi - \Phi\|_{\mathbf{L}^2(-\infty,-1)} \leq CAt^{-\frac{1-\alpha}{4}}$$

are true for $t \geq 1$, where $A = \|x\phi_x\|_{\mathbf{H}^1} + \|\phi\|_{\mathbf{H}^1}$ and $\alpha \in (0, 1/2)$.

PROOF. The free evolution group is represented as

$$\mathcal{U}(t)\phi = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{ix\xi - i\frac{t}{\xi}} \hat{\phi}(\xi) d\xi.$$

We first consider the case $x \geq 0$. We note that there is no stationary point in this case. Then, by integration by parts we have

$$\sqrt{(2\pi)}\mathcal{U}(t)\phi = \int_{\mathbb{R}} \left(1 + ix\xi + i\frac{t}{\xi}\right)^{-1} \partial_\xi \left(\xi e^{ix\xi - i\frac{t}{\xi}}\right) \hat{\phi}(\xi) d\xi$$

$$= \int_{\mathbb{R}} \frac{\left(ix\xi - \frac{it}{\xi}\right)}{\left(1 + ix\xi + i\frac{t}{\xi}\right)^2} e^{ix\xi - i\frac{t}{\xi}} \hat{\phi}(\xi) d\xi - \int_{\mathbb{R}} \frac{\xi \partial_\xi \hat{\phi}(\xi)}{1 + ix\xi + i\frac{t}{\xi}} e^{ix\xi - i\frac{t}{\xi}} d\xi.$$

Applying the Cauchy-Schwarz inequality to the right hand side, we find for $tx \geq 0$,

$$|\mathcal{U}(t)\phi| \leq C \int_{\mathbb{R}} \frac{|\hat{\phi}(\xi)| + |\xi \partial_\xi \hat{\phi}(\xi)|}{1 + t/|\xi| + x|\xi|} d\xi$$

$$\leq C(\left\| \langle \xi \rangle \hat{\phi} \right\|_{\mathbf{L}^2} + \left\| \langle \xi \rangle \xi \partial_\xi \hat{\phi} \right\|_{\mathbf{L}^2}) \left(\int_{\mathbb{R}} \langle \xi \rangle^{-2} \left(1 + t/|\xi| + x|\xi|\right)^{-2} d\xi \right)^{1/2}.$$

Since $t^\beta x^{2-\beta} |\xi|^{2-2\beta} \leq Ct^2/\xi^2 + Cx^2\xi^2$ for $\beta \in (0, 2)$, we have

$$\int_{\mathbb{R}} \langle \xi \rangle^{-2} \left(1 + t/|\xi| + x|\xi|\right)^{-2} d\xi \leq \frac{C}{t^\beta x^{2-\beta}} \int_{\mathbb{R}} |\xi|^{2\beta - 2} \langle \xi \rangle^{-2} d\xi \leq Ct^{-2} \left(\frac{x}{t}\right)^{-2+\beta},$$

where $\beta \in (1/2, 3/2)$. Hence, choosing $\beta = 4/3$ for $0 < x < t$ and $\beta = 2/3$ for $x > t$ we obtain

$$\int_{\mathbb{R}} \langle \xi \rangle^{-2} \left(1 + t/|\xi| + x|\xi|\right)^{-2} d\xi \leq Ct^{-2} \left(\frac{x}{t}\right)^{-2/3} \langle \frac{x}{t} \rangle^{-2/3},$$

Therefore, we get the first estimate of the lemma

$$\|\mathcal{U}(t)\phi\|_{\mathbf{L}^2(0,\infty)} \leq Ct^{-1} \left\| \left(\frac{x}{t}\right)^{-1/3} \langle \frac{x}{t} \rangle^{-1/3} \right\|_{\mathbf{L}^2(0,\infty)} \leq Ct^{-1/2}.$$

To get the second estimate of the lemma, we just apply Lemma 2.1. By Lemma 2.1 with $p = 6$, we obtain

$$\|\mathcal{U}(t)\phi\|_{\mathbf{L}^2(-1,0)} \leq C \|\mathcal{U}(t)\phi\|_{\mathbf{L}^6(-1,0)} \leq Ct^{-1/3} \|\phi_x\|_{\mathbf{L}^{6/5}} \leq CAt^{-1/3}.$$

Finally, we consider the last estimate of the lemma. Consider $x < -1$. Denote $S(\xi, \chi) = \xi/\chi^2 + 1/\xi$, $\chi = \sqrt{t/-x}$, then we write

$$\mathcal{U}(t)\phi = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-itS} \hat{\phi}(\xi) d\xi + \frac{1}{\sqrt{2\pi}} \overline{\int_0^\infty e^{-itS} \overline{\hat{\phi}(-\xi)} d\xi}. \tag{2.1}$$

Consider the first integral on the right hand side of (2.1). Note that $S(\chi, \chi) = 2/\chi$ and

$$S(\xi, \chi) - S(\chi, \chi) = (\xi - \chi)^2/\chi^2 \xi$$

for $\xi > 0$. To estimate the integral, we define a new variable of integration $z = (\xi - \chi)/\chi\sqrt{\xi}$. Since $z'(\xi) = (\xi + \chi)/(2\xi^{3/2}\chi) > 0$ for all $\xi > 0$, there exists the inverse function $\xi(z) = (z\chi + \sqrt{4\chi + z^2\chi^2})^2/4$. Denote

$$\psi(z) = \frac{\xi^{3/2}(z) \hat{\phi}(\xi(z))}{\xi(z) + \chi}.$$

Since $\xi(0)=\chi$, $\psi(0)=\hat{\phi}(\chi)/2\chi$ and $\int_{-\infty}^{\infty}e^{-itz^2}dz=\sqrt{\pi/it}$, we get

$$\begin{aligned}(2.2)\qquad &\int_0^\infty e^{-itS}\hat{\phi}(\xi)d\xi = 2\chi e^{-itS(\chi,\chi)}\int_{-\infty}^{\infty}e^{-itz^2}\psi(z)dz\\ = \;& e^{-i\left(\frac{2t}{\chi}+\frac{\pi}{4}\right)}t^{-\frac12}\sqrt{\pi}\chi^{3/2}\hat{\phi}(\chi)+2\chi e^{-i\frac{2t}{\chi}}\int_{-\infty}^{\infty}e^{-itz^2}(\psi(z)-\psi(0))dz.\end{aligned}$$

By integration by parts, the second summand of (2.2) can be rewritten as

$$\begin{aligned}&\int_{-\infty}^{\infty}e^{-itz^2}(\psi(z)-\psi(0))dz=\int_{-\infty}^{\infty}(1-2itz^2)^{-1}\partial_z(ze^{-itz^2})(\psi(z)-\psi(0))dz\\ = \;& -\int_{-\infty}^{\infty}e^{-itz^2}4itz^2\frac{\psi(z)-\psi(0)}{(1-2itz^2)^2}dz-\int_{-\infty}^{\infty}e^{-itz^2}\frac{z\psi_z(z)}{(1-2itz^2)}dz.\end{aligned}$$

By a direct calculation, we get

$$\begin{aligned}&|\psi_z(z)|=\left|\frac{\frac32\xi^{1/2}\hat{\phi}(\xi)+\xi^{3/2}\hat{\phi}_\xi(\xi)}{\xi+\chi}-\frac{\xi^{3/2}\hat{\phi}(\xi)}{(\xi+\chi)^2}\right||\xi_z(z)|\\ = \;& \chi^{\alpha/2}\left|\frac{\frac32\xi^{\frac54-\frac{\alpha}{2}}\hat{\phi}(\xi)+\xi^{\frac94-\frac{\alpha}{2}}\hat{\phi}_\xi(\xi)}{\xi+\chi}-\frac{\xi^{3/2}\hat{\phi}(\xi)}{(\xi+\chi)^2}\right|\sqrt{\frac{2\xi^\alpha\chi^{1-\alpha}}{\xi+\chi}}\sqrt{\xi_z(z)}\\ \le \;& C\chi^{\frac{\alpha}{2}}\left(\xi^{\frac14-\frac{\alpha}{2}}|\xi\hat{\phi_\xi}(\xi)|+\xi^{\frac14-\frac{\alpha}{2}}|\hat{\phi}(\xi)|\right)\sqrt{\xi_z(z)}.\end{aligned}$$

Hence

$$\begin{aligned}&\left|\int_{-\infty}^{\infty}\psi_z^2(z)dz\right|\le C\chi^\alpha\int_0^\infty\left(\xi^{\frac14-\frac{\alpha}{2}}|\xi\hat{\phi_\xi}(\xi)|+\xi^{\frac14-\frac{\alpha}{2}}|\hat{\phi}(\xi)|\right)d\xi\\ \le \;& C\chi^\alpha\left(\left\|\xi^{\frac14-\frac{\alpha}{2}}\xi\hat{\phi_\xi}\right\|^2_{\mathbf{L}^2(0,\infty)}+\left\|\xi^{\frac14-\frac{\alpha}{2}}\hat{\phi}\right\|^2_{\mathbf{L}^2(0,\infty)}\right)\le CA^2\chi^\alpha.\end{aligned}$$

Also, by the Cauchy-Schwarz inequality we have

$$|\psi(z)-\psi(0)|=\left|\int_0^z\phi_z(z)dz\right|\le C\sqrt{|z|}A\chi^{\frac{\alpha}{2}}.$$

Therefore,

$$\left|\int_{-\infty}^{\infty}e^{-itz^2}4itz^2\frac{\psi(z)-\psi(0)}{(1-2itz^2)^2}dz\right|\le C|\psi(z)-\psi(0)||z|^{-1/2}\int_{-\infty}^{\infty}\frac{|z|^{1/2}}{1+tz^2}dz\le CA\chi^{\alpha/2}t^{-3/4}$$

and

$$\left|\int_{-\infty}^{\infty}e^{-itz^2}\frac{z\psi_z(z)}{(1-2itz^2)}dz\right|\le C\left(\int_{-\infty}^{\infty}\frac{z^2dz}{1+tz^2}\right)^{1/2}\left(\int_{-\infty}^{\infty}\psi_z^2(z)dz\right)^{1/2}\le CA\chi^{\frac{\alpha}{2}}t^{-\frac34}.$$

Hence, the second summand on the right hand side of (2.2) can be estimated as

$$(2.3)\qquad \left|2\chi e^{-i\frac{2t}{\chi}}\int_{-\infty}^{\infty}e^{-itz^2}(\psi(z)-\psi(0))dz\right|\le CA\chi^{1+\frac{\alpha}{2}}t^{-3/4}.$$

In the same manner, the second integral on the right hand side of (2.1) can be represented as

$$\overline{\int_0^\infty e^{-itS}\overline{\hat{\phi}(-\xi)}d\xi} = \overline{2\chi e^{-itS(\chi,\chi)}\int_{-\infty}^\infty e^{-itz^2}\psi_1(z)dz}$$
$$= e^{i\left(\frac{2t}{\chi}+\frac{\pi}{4}\right)}t^{-1/2}\sqrt{\pi}\chi^{\frac{3}{2}}\hat{\phi}(-\chi) + \overline{2\chi e^{-\frac{2it}{\chi}}\int_{-\infty}^\infty e^{-itz^2}(\psi_1(z)-\psi(0))dz},$$

where

$$\psi_1(z) = \frac{\xi^{3/2}(z)\overline{\hat{\phi}(-\xi(z))}}{\xi(z)+\chi}.$$

In the same manner as in the proof of (2.3), we also have

$$2\chi e^{-\frac{2it}{\chi}}\int_{-\infty}^\infty e^{-itz^2}(\psi_1(z)-\psi(0))dz \le CA\chi^{1+\frac{\alpha}{2}}t^{-\frac{3}{4}}.$$

Thus, we obtain

$$\|\mathcal{U}(t)\phi - \Phi\|_{\mathbf{L}^2(-\infty,0)} \le CAt^{-\frac{3}{4}}\left\|\chi^{1+\frac{\alpha}{2}}\right\|_{\mathbf{L}^2(-\infty,-1)}.$$

Since

$$\left\|\chi^{1+\frac{\alpha}{2}}\right\|^2_{\mathbf{L}^2(-\infty,-1)} = t^{1+\frac{\alpha}{2}}\int_{-\infty}^{-1}|x|^{-1-\frac{\alpha}{2}}dx = Ct^{1+\frac{\alpha}{2}},$$

we arrive at the third estimate of the lemma. This completes the proof of Lemma 2.2 □

To show the lower bound of the free evolution group, we show the following lemma.

LEMMA 2.3. ([**18**]) *Let $\phi \in \mathbf{H}^1$ be such that $x\phi_x \in \mathbf{H}^1$. Then, the estimate*

$$\|\mathcal{U}(t)\phi\|_{\mathbf{L}^2(-kt,-1)} \ge \frac{1}{2}\left\|\hat{\phi}\right\|_{\mathbf{L}^2(k^{-\frac{1}{2}},t^{\frac{1}{2}})} + \frac{1}{2}\left\|\hat{\phi}\right\|_{\mathbf{L}^2(-t^{\frac{1}{2}},-k^{-\frac{1}{2}})} - CAt^{-\frac{1}{4}+\frac{\alpha}{4}}$$

is true for all $k \ge 1$, $t \ge 1$, where $A = \|x\phi_x\|_{\mathbf{H}^1} + \|\phi\|_{\mathbf{H}^1}$, $\alpha \in (0,1/2)$.

PROOF. By Lemma 2.2, we get

$$\|\mathcal{U}(t)\phi\|_{\mathbf{L}^2(-kt,-1)} \ge \|\Phi\|_{\mathbf{L}^2(-kt,-1)} - \|\mathcal{U}(t)\phi - \Phi(t)\|_{\mathbf{L}^2(-kt,-1)}$$
$$\ge (2\pi)^{-\frac{1}{2}}\left\|\chi^{\frac{3}{2}}(e^{-i\left(\frac{2t}{\chi}+\frac{\pi}{4}\right)}\hat{\phi}(\chi) + e^{i\left(\frac{2t}{\chi}+\frac{\pi}{4}\right)}\hat{\phi}(-\chi))\right\|_{\mathbf{L}^2(-kt,-1)} - CAt^{-\frac{1}{4}+\frac{\alpha}{4}}$$

for all $k \ge 1$, $t \ge 1$. Changing the variable of integration $-x = t\chi^{-2}$, $dx = 2d\chi$, we find

$$\|\mathcal{U}(t)\phi\|^2_{\mathbf{L}^2(-kt,-1)} \ge \frac{1}{2t}\left\|\chi^{\frac{3}{2}}\hat{\phi}(\chi)\right\|^2_{\mathbf{L}^2(-kt,-1)} + \frac{1}{2t}\left\|\chi^{\frac{3}{2}}\hat{\phi}(-\chi)\right\|^2_{\mathbf{L}^2(-kt,-1)} \tag{2.4}$$
$$+\Re\int_{-kt}^{-1} e^{-i\left(\frac{4t}{\chi}+\frac{\pi}{2}\right)}\hat{\phi}(\chi)\overline{\hat{\phi}(-\chi)}\chi^3 t^{-1}dx - CA^2t^{-\frac{1}{2}+\frac{\alpha}{2}} = \left\|\hat{\phi}\right\|^2_{\mathbf{L}^2(k^{-\frac{1}{2}},t^{\frac{1}{2}})}$$
$$+\left\|\hat{\phi}\right\|^2_{\mathbf{L}^2(-t^{\frac{1}{2}},-k^{-\frac{1}{2}})} + \Re\int_{k^{-\frac{1}{2}}}^{t^{\frac{1}{2}}} e^{-i\left(\frac{4t}{\chi}+\frac{\pi}{2}\right)}\hat{\phi}(\chi)\overline{\hat{\phi}(-\chi)}d\chi - CA^2t^{-\frac{1}{2}+\frac{\alpha}{2}}.$$

We now consider the second summand of the last line of (2.4). By integration by parts we have

$$\int_{k^{-\frac{1}{2}}}^{t^{\frac{1}{2}}} e^{-i\frac{4t}{\chi}}\hat{\phi}(\chi)\overline{\hat{\phi}(-\chi)}d\chi = \int_{k^{-\frac{1}{2}}}^{t^{\frac{1}{2}}} \left(\frac{\chi^2}{4it}\partial_\chi e^{-i\frac{4t}{\chi}}\right)\hat{\phi}(\chi)\overline{\hat{\phi}(-\chi)}d\chi$$

$$= \left[\frac{\chi^2}{4it}e^{-i\frac{4t}{\chi}}\right]_{k^{-\frac{1}{2}}}^{t^{\frac{1}{2}}} - \int_{k^{-\frac{1}{2}}}^{t^{\frac{1}{2}}} e^{-i\frac{4t}{\chi}}\partial_\chi\left(\frac{\chi^2}{4it}\hat{\phi}(\chi)\overline{\hat{\phi}(-\chi)}\right)d\chi,$$

from which it follows that

$$\begin{aligned}&\left|\int_{k^{-\frac{1}{2}}}^{t^{\frac{1}{2}}} e^{-i\frac{4t}{\chi}}\hat{\phi}(\chi)\overline{\hat{\phi}(-\chi)}d\chi\right| \\ \leq\ & Ct^{-1}\left|t^{\frac{1}{2}}\hat{\phi}(t^{\frac{1}{2}})t^{\frac{1}{2}}\overline{\hat{\phi}(-t^{\frac{1}{2}})}\right| + Ct^{-1}\left|k^{-\frac{1}{2}}\hat{\phi}(k^{-\frac{1}{2}})t^{\frac{1}{2}}\overline{\hat{\phi}(-k^{-\frac{1}{2}})}\right| \\ &+Ct^{-1}\left\||\xi|^{\frac{1}{2}}\hat{\phi}\right\|_{\mathbf{L}^2}^2 + Ct^{-1}\left\|\xi\hat{\phi}\right\|_{\mathbf{L}^2}\left\|\xi\hat{\phi}_\xi\right\|_{\mathbf{L}^2} \\ \leq\ & Ct^{-1}\|\phi\|_{\mathbf{H}^1}^2 + Ct^{-1}\|x\phi_x\|_{\mathbf{H}^1}^2 = CA^2t^{-1}.\end{aligned}$$

Applying this estimate to (2.4), we obtain the desired estimate. □

Using Lemma 2.3, we give the lower bound of the free evolution group.

LEMMA 2.4. ([**18**]) *Let $\phi, x\phi_x \in \mathbf{H}^1$. Then the estimate*

$$\|\mathcal{U}(t)\phi\|_{\mathbf{L}^r(-t,0)} \geq \frac{1}{2}t^{-\frac{1}{2}(1-\frac{2}{r})}\left(\left\|\hat{\phi}\right\|_{\mathbf{L}^2(1,\sqrt{T})} + \left\|\hat{\phi}\right\|_{\mathbf{L}^2(-\sqrt{T},-1)}\right) - CAt^{-\frac{1}{4}-\frac{1}{2}(1-\frac{2}{r})+\frac{\alpha}{4}}$$

is true for all $t \geq T > 1$, where $2 \leq r \leq \infty$, $\alpha \in (0, 1/2)$ and $A = \|\phi\|_{\mathbf{H}^1} + \|x\phi_x\|_{\mathbf{H}^1}$.

PROOF. By the Hölder's inequality we obtain

$$\|\mathcal{U}(t)\phi\|_{\mathbf{L}^2(-t,-1)} \leq \|\mathcal{U}(t)\phi\|_{\mathbf{L}^r(-t,-1)}\, t^{\frac{1}{2}(1-\frac{2}{r})} \leq \|\mathcal{U}(t)\phi\|_{\mathbf{L}^r(-t,0)}\, t^{\frac{1}{2}(1-\frac{2}{r})},$$

where $2 \leq r \leq \infty$. Applying Lemma 2.3 with $k = 1$, we obtain

$$\|\mathcal{U}(t)\phi\|_{\mathbf{L}^r(-t,0)} \geq \frac{1}{2}t^{-\frac{1}{2}(1-\frac{2}{r})}\left(\left\|\hat{\phi}\right\|_{\mathbf{L}^2(1,\sqrt{t})} + \left\|\hat{\phi}\right\|_{\mathbf{L}^2(-\sqrt{t},-1)}\right) - CAt^{-\frac{1}{4}-\frac{1}{2}(1-\frac{2}{r})+\frac{\alpha}{4}}$$

for all $t \geq 1$. This completes the proof of Lemma 2.4. □

In the next lemma, we give the asymptotic expansion for the free evolution group.

LEMMA 2.5. ([**25**]) *The following asymptotic expansion is valid for the large time $t \geq 1$ uniformly with respect to $x \in \mathbb{R}$:*

(2.5)

$$\mathcal{U}(t)\phi = \Re\sqrt{\frac{2}{t}}\theta(x)e^{-i\left(\frac{2t}{\chi}+\frac{\pi}{4}\right)}\chi^{\frac{3}{2}}\hat{\phi}(\chi) + O\left(t^{-\frac{1}{2}-\delta}\left(\|x\phi_x\|_{\mathbf{H}^{1+2\delta}} + \|\phi\|_{\mathbf{H}^{\frac{3}{2}+\delta}}\right)\right),$$

where $\chi = \sqrt{t/-x}$, $\delta \in (0, 1/4)$ $\theta(x) = 1$ when $x < 0$ and $\theta(x) = 0$ when $0 \leq x$.

PROOF. By the definition of $\mathcal{U}(t)$, we have

$$\mathcal{U}(t)\phi = \mathcal{F}^{-1}e^{-i\frac{t}{\xi}}\mathcal{F}\phi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-it\left(\frac{1}{\xi}-\frac{x\xi}{t}\right)}\hat{\phi}\left(\xi\right)d\xi.$$

We first consider the non-stationary contributions. For the case $x \geq 0$, it is easy to see that there is no stationary point. Then, integration by parts yields

$$
\begin{aligned}
(2.6) \qquad \mathcal{U}(t)\,\phi(x) &= \sqrt{\frac{2}{\pi}}\Re \int_0^\infty \frac{\frac{d}{d\xi}\left(\xi e^{-it\left(\frac{1}{\xi}-\frac{\xi}{\chi^2}\right)}\right)\hat{\phi}(\xi)}{1+it\left(\frac{1}{\xi}+\frac{\xi}{\chi^2}\right)}d\xi \\
&= -\Re\sqrt{\frac{2}{\pi}}\int_0^\infty \left[\frac{e^{-it\left(\frac{1}{\xi}-\frac{\xi}{\chi^2}\right)}\xi\hat{\phi}'(\xi)}{1+it\left(\frac{1}{\xi}+\frac{\xi}{\chi^2}\right)}d\xi + \frac{e^{-it\left(\frac{1}{\xi}-\frac{\xi}{\chi^2}\right)}it\xi\left(\frac{1}{\chi^2}-\frac{1}{\xi^2}\right)\hat{\phi}(\xi)}{\left(1+it\left(\frac{1}{\xi}+\frac{\xi}{\chi^2}\right)\right)^2}\right]d\xi,
\end{aligned}
$$

where $\chi = \sqrt{t/x}$. Since

$$
(2.7) \qquad \left|\left(1+it\left(\frac{1}{\xi}+\frac{\xi}{\chi^2}\right)\right)^{-1}\right| \leq \frac{\chi^{2\gamma}\,|\xi|^\gamma}{t^\gamma\,|\chi^2+\xi^2|^\gamma} \leq C\frac{|\xi|^\gamma}{t^\gamma}
$$

for $\gamma = 1/2+\delta$ and $\delta \in (0,1/2)$, we can estimate the first term on the right hand side of (2.6) as

$$
\begin{aligned}
& Ct^{-\frac{1}{2}-\delta}\int_0^\infty |\xi|^{\frac{3}{2}+\delta}\left|\hat{\phi}'(\xi)\right|d\xi \leq Ct^{-\frac{1}{2}-\delta}\left\|\langle\xi\rangle^{-\frac{1}{2}-\delta}\right\|_{\mathbf{L}^2}\left\|\langle\xi\rangle^{\frac{1}{2}+\delta}\,|\xi|^{\frac{3}{2}+\delta}\,\hat{\phi}'\right\|_{\mathbf{L}^2} \\
\leq\; & Ct^{-\frac{1}{2}-\delta}\left(\|\phi\|_{\mathbf{H}^{2+2\delta}} + \|x\phi_x\|_{\mathbf{H}^{1+2\delta}}\right).
\end{aligned}
$$

Also, using the estimate

$$
\left|\left(1+it\left(\frac{1}{\xi}+\frac{\xi}{\chi^2}\right)\right)^{-1}\right| \leq C\frac{|\chi|^{\frac{3}{2}+\delta}}{t^{\frac{3}{4}+\frac{\delta}{2}}|\xi|^{\frac{3}{4}+\frac{\delta}{2}}}
$$

on $[2\chi,\infty)$ and (2.7) with $\gamma = 3/4+\delta/2$, one can see that the last term on the right hand side of (2.6) can be estimated as

$$
\begin{aligned}
& Ct^{-\frac{1}{2}-\delta}\int_0^{2\chi}|\xi|^{\frac{1}{2}+\delta}\left|\hat{\phi}(\xi)\right|d\xi + Ct^{-\frac{1}{2}-\delta}\int_{2\chi}^\infty |\chi|^{1+2\delta}\,|\xi|^{-\frac{1}{2}-\delta}\left|\hat{\phi}(\xi)\right|d\xi \\
\leq\; & Ct^{-\frac{1}{2}-\delta}\left\|\langle\xi\rangle^{-\frac{1}{2}-\delta}\right\|_{\mathbf{L}^2}\left\|\langle\xi\rangle^{\frac{1}{2}+\delta}|\xi|^{\frac{1}{2}+\delta}\hat{\phi}\right\|_{\mathbf{L}^2} \leq Ct^{-\frac{1}{2}-\delta}\,\|\phi\|_{\mathbf{H}^{1+2\delta}}.
\end{aligned}
$$

Therefore we get the asymptotic expansion (2.5) for $x \geq 0$.

We next consider the case $x < 0$. We decompose the integral into the non-stationary contributions and the stationary contribution. Put $\chi = \sqrt{t/-x}$, then, we have

$$
\mathcal{U}(t)\phi(x) = \Re\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-it\left(\frac{1}{\xi}+\frac{\xi}{\chi^2}\right)}\hat{\phi}(\xi)d\xi = \Re\sqrt{\frac{2}{\pi}}\left(\mathrm{I}+\mathrm{II}+\mathrm{III}\right),
$$

where $\mathrm{I} = \int_0^{\frac{\chi}{2}} e^{-it\left(\frac{1}{\xi}+\frac{\xi}{\chi^2}\right)}\hat{\phi}(\xi)\,d\xi$, $\mathrm{II} = \int_{\frac{\chi}{2}}^{2\chi} e^{-it\left(\frac{1}{\xi}+\frac{\xi}{\chi^2}\right)}\hat{\phi}(\xi)\,d\xi$ and $\mathrm{III} = \int_{2\chi}^{\infty} e^{-it\left(\frac{1}{\xi}+\frac{\xi}{\chi^2}\right)}\hat{\phi}(\xi)\,d\xi$. We first consider the non-stationary contributions I and III. By integration by parts, we get

$$\mathrm{I}=\int_0^{\frac{\chi}{2}}\frac{d}{d\xi}\left(\xi e^{-it\left(\frac{\xi}{\chi^2}+\frac{1}{\xi}\right)}\right)\frac{\hat{\phi}(\xi)d\xi}{1-it\left(\frac{\xi}{\chi^2}-\frac{1}{\xi}\right)}=\left[\frac{e^{-it\left(\frac{\xi}{\chi^2}+\frac{1}{\xi}\right)}\xi\hat{\phi}(\xi)}{1-it\left(\frac{\xi}{\chi^2}-\frac{1}{\xi}\right)}\right]_{\xi=0}^{\frac{\chi}{2}}$$

$$-\int_0^{\frac{\chi}{2}}\frac{e^{-it\left(\frac{\xi}{\chi^2}+\frac{1}{\xi}\right)}\xi\hat{\phi}'(\xi)d\xi}{1-it\left(\frac{\xi}{\chi^2}-\frac{1}{\xi}\right)}-\int_0^{\frac{\chi}{2}}\frac{e^{-it\left(\frac{\xi}{\chi^2}+\frac{1}{\xi}\right)}\xi\hat{\phi}(\xi)it\left(\frac{1}{\chi^2}+\frac{1}{\xi^2}\right)d\xi}{\left(1-it\left(\frac{\xi}{\chi^2}-\frac{1}{\xi}\right)\right)^2}.$$

Since

$$\left|\left(1-it\left(\frac{\xi}{\chi^2}-\frac{1}{\xi}\right)\right)^{-1}\right|\le\frac{|\chi|^{2\gamma}|\xi|^{\gamma}}{t^{\gamma}|\chi-\xi|^{\gamma}|\chi+\xi|^{\gamma}}\le C\frac{|\xi|^{\gamma}}{t^{\gamma}}\tag{2.8}$$

for $\gamma=1/2+\delta,3/4+\delta/2$ and $\delta\in(0,1/2)$, it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}(2.9)\quad |\mathrm{I}| &\le \frac{C}{t^{\frac{1}{2}+\delta}}\left(\left\|\xi^{\frac{3}{2}+\delta}\hat{\phi}\right\|_{\mathbf{L}^\infty}+\int_0^{\frac{\chi}{2}}|\xi|^{\frac{3}{2}+\delta}|\hat{\phi}'(\xi)|+|\xi|^{\frac{1}{2}+\delta}|\hat{\phi}(\xi)|d\xi\right)\\ &\le Ct^{-\frac{1}{2}-\delta}\left(\|\phi\|_{\mathbf{H}^{\frac{3}{2}+\delta}}+\|x\phi_x\|_{\mathbf{H}^{\frac{1}{2}+\delta}}\right)+Ct^{-\frac{1}{2}-\delta}\left\|\langle\xi\rangle^{-\frac{1}{2}-\delta}\right\|_{\mathbf{L}^2}\\ &\quad\times\left(\left\|\langle\xi\rangle^{\frac{1}{2}+\delta}|\xi|^{\frac{3}{2}+\delta}\hat{\phi}'\right\|_{\mathbf{L}^2}+\left\|\langle\xi\rangle^{\frac{1}{2}+\delta}|\xi|^{\frac{1}{2}+\delta}\hat{\phi}\right\|_{\mathbf{L}^2}\right)\\ &\le Ct^{-\frac{1}{2}-\delta}\left(\|\phi\|_{\mathbf{H}^{\frac{3}{2}+\delta}}+\|x\phi_x\|_{\mathbf{H}^{1+2\delta}}\right).\end{aligned}$$

Similarly, by integration by parts, we have

$$\mathrm{III}=\int_{2\chi}^{\infty}\frac{d}{d\xi}\left(\xi e^{-it\left(\frac{\xi}{\chi^2}+\frac{1}{\xi}\right)}\right)\frac{\hat{\phi}(\xi)d\xi}{1-it\left(\frac{\xi}{\chi^2}-\frac{1}{\xi}\right)}=\left[\frac{e^{-it\left(\frac{\xi}{\chi^2}+\frac{1}{\xi}\right)}\xi\hat{\phi}(\xi)}{1-it\left(\frac{\xi}{\chi^2}-\frac{1}{\xi}\right)}\right]_{\xi=2\chi}^{\infty}\tag{2.10}$$

$$-\int_{2\chi}^{\infty}\frac{e^{-it\left(\frac{\xi}{\chi^2}+\frac{1}{\xi}\right)}\xi\hat{\phi}'(\xi)d\xi}{1-it\left(\frac{\xi}{\chi^2}-\frac{1}{\xi}\right)}-\int_{2\chi}^{\infty}\frac{e^{-it\left(\frac{\xi}{\chi^2}+\frac{1}{\xi}\right)}\xi\hat{\phi}(\xi)it\left(\frac{1}{\chi^2}+\frac{1}{\xi^2}\right)d\xi}{\left(1-it\left(\frac{\xi}{\chi^2}-\frac{1}{\xi}\right)\right)^2}.$$

Since estimate (2.8) is also valid on $[2\chi,\infty)$, the first and the second term on the right hand side of (2.10) is bounded by $Ct^{-\frac{1}{2}-\delta}\left(\|\phi\|_{\mathbf{H}^{3/2+\delta}}+\|x\phi_x\|_{\mathbf{H}^{1+2\delta}}\right)$. Since

$$\left|\left(1-it\left(\frac{\xi}{\chi^2}-\frac{1}{\xi}\right)\right)^{-1}\right|\le\frac{C|\chi|^{\frac{3}{2}+\delta}}{t^{\frac{3}{4}+\frac{\delta}{2}}|\xi|^{\frac{3}{4}+\frac{\delta}{2}}}$$

on $[2\chi,\infty)$ for $\delta\in(0,1/2)$, the last term on the right hand side of (2.10) is bounded by

$$\begin{aligned}\frac{C}{t^{\frac{1}{2}+\delta}}\int_{2\chi}^{\infty}|\chi|^{1+2\delta}|\xi|^{-\frac{1}{2}-\delta}\left|\hat{\phi}(\xi)\right|d\xi &\le Ct^{-\frac{1}{2}-\delta}\left\|\langle\xi\rangle^{-\frac{1}{2}-\delta}\right\|_{\mathbf{L}^2}\left\|\langle\xi\rangle^{\frac{1}{2}+\delta}|\xi|^{\frac{1}{2}+\delta}\hat{\phi}\right\|_{\mathbf{L}^2}\\ &\le Ct^{-\frac{1}{2}-\delta}\|\phi\|_{\mathbf{H}^{1+2\delta}}.\end{aligned}$$

Therefore, collecting these estimates, we obtain

$$|\mathrm{III}|\le Ct^{-\frac{1}{2}-\delta}\left(\|\phi\|_{\mathbf{H}^{3/2+\delta}}+\|x\phi_x\|_{\mathbf{H}^{1+2\delta}}\right).\tag{2.11}$$

We finally consider the stationary contribution II. We decompose the integral II into the leading term and the remainder term. That is,

$$\mathrm{II} = \hat{\phi}(\chi)\int_{\frac{\chi}{2}}^{2\chi} e^{-it\left(\frac{1}{\xi}+\frac{\xi}{\chi^2}\right)}d\xi + \int_{\frac{\chi}{2}}^{2\chi} e^{-it\left(\frac{1}{\xi}+\frac{\xi}{\chi^2}\right)}\left(\hat{\phi}(\xi)-\hat{\phi}(\chi)\right)d\xi = \mathrm{II}' + \mathrm{R}.$$

We first consider II′. Making the change of the integral variable $\xi = \chi\xi'$ (we omit the prime) and applying the stationary phase method (see [**31**]), we have for large $t/\chi \geq 1$,

$$\begin{aligned}\mathrm{II}' &= \chi\hat{\phi}(\chi)\int_{\frac{1}{2}}^{2} e^{-\frac{it}{\chi}\left(\xi+\frac{1}{\xi}\right)}d\xi = \sqrt{\frac{\pi}{t}}e^{-i\left(\frac{2t}{\chi}+\frac{\pi}{4}\right)}\chi^{\frac{3}{2}}\hat{\phi}(\chi) + O\left(\frac{\chi^2}{t}\hat{\phi}(\chi)\right) \\ &= \sqrt{\pi}t^{-\frac{1}{2}}e^{-i\left(\frac{2t}{\chi}+\frac{\pi}{4}\right)}\chi^{\frac{3}{2}}\hat{\phi}(\chi) + O\left(t^{-\frac{1}{2}-\delta}\left\|\chi^{\frac{3}{2}+\delta}\hat{\phi}\right\|_{\mathbf{L}^\infty}\right).\end{aligned} \tag{2.12}$$

For small t/χ, we also have $|\mathrm{II}'| \leq Ct^{-\frac{1}{2}-\delta}\left\|\chi^{\frac{3}{2}+\delta}\hat{\phi}\right\|_{\mathbf{L}^\infty}$. We next consider the remainder term R. By integration by parts, we have

$$\begin{aligned}\mathrm{R} &= \int_{\frac{\chi}{2}}^{2\chi}\frac{d}{d\xi}\left((\xi-\chi)e^{-it\left(\frac{1}{\xi}+\frac{\xi}{\chi^2}\right)}\right)\frac{\hat{\phi}(\xi)-\hat{\phi}(\chi)}{1+it(\xi-\chi)\left(\frac{1}{\xi^2}-\frac{1}{\chi^2}\right)}d\xi \\ &= \left[(\xi-\chi)e^{-it\left(\frac{1}{\xi}+\frac{\xi}{\chi^2}\right)}\frac{\hat{\phi}(\xi)-\hat{\phi}(\chi)}{1+it(\xi-\chi)\left(\frac{1}{\xi^2}-\frac{1}{\chi^2}\right)}\right]_{\xi=\frac{\chi}{2}}^{2\chi} \\ &\quad - \int_{\frac{\chi}{2}}^{2\chi} e^{-it\left(\frac{1}{\xi}+\frac{\xi}{\chi^2}\right)}\frac{(\xi-\chi)\hat{\phi}'(\xi)}{1+it(\xi-\chi)\left(\frac{1}{\xi^2}-\frac{1}{\chi^2}\right)}d\xi \\ &\quad + \int_{\frac{\chi}{2}}^{2\chi} e^{-it\left(\frac{1}{\xi}+\frac{\xi}{\chi^2}\right)}\left(\hat{\phi}(\xi)-\hat{\phi}(\chi)\right)\frac{it(\xi-\chi)^2\left(\frac{\xi+\chi}{\xi^2\chi^2}-\frac{2}{\xi^3}\right)}{\left(1+it(\xi-\chi)\left(\frac{1}{\xi^2}-\frac{1}{\chi^2}\right)\right)^2}d\xi.\end{aligned} \tag{2.13}$$

Since

$$\left|\left(1+it(\xi-\chi)\left(\frac{1}{\xi^2}-\frac{1}{\chi^2}\right)\right)^{-1}\right| \leq C\frac{|\chi|^{3\gamma}}{t^\gamma|\xi-\chi|^{2\gamma}} \tag{2.14}$$

on $[\chi/2, 2\chi]$ for $\gamma = 1/2+\delta$ and $\delta \in (0, 1/2)$, it follows from the Cauchy-Schwarz inequality that the first and the second term on the right hand side of (2.13) is bounded by

$$\begin{aligned}& Ct^{-\frac{1}{2}-\delta}\left\|\xi^{\frac{3}{2}+\delta}\hat{\phi}\right\|_{\mathbf{L}^\infty} + \frac{C}{t^{\frac{1}{2}+\delta}}\int_{\frac{\chi}{2}}^{2\chi}\frac{|\chi|^{\frac{3}{2}+3\delta}\left|\hat{\phi}'(\xi)\right|d\xi}{|\xi-\chi|^{2\delta}} \\ \leq\ & Ct^{-\frac{1}{2}-\delta}\left(\|\phi\|_{\mathbf{H}^{\frac{3}{2}+\delta}} + \|x\phi_x\|_{\mathbf{H}^{\frac{1}{2}+\delta}}\right) + \frac{C\left\||\xi|^{2+\delta}\hat{\phi}'\right\|_{\mathbf{L}^2}}{t^{\frac{1}{2}+\delta}|\chi|^{\frac{1}{2}-2\delta}}\left(\int_{\frac{\chi}{2}}^{2\chi}\frac{d\xi}{|\xi-\chi|^{4\delta}}\right)^{\frac{1}{2}} \\ \leq\ & Ct^{-\frac{1}{2}-\delta}\left(\|\phi\|_{\mathbf{H}^{\frac{3}{2}+\delta}} + \|x\phi_x\|_{\mathbf{H}^{1+2\delta}}\right).\end{aligned} \tag{2.15}$$

By using the estimate

$$\left|\hat{\phi}(\xi)-\hat{\phi}(\chi)\right| \le \left|\int_{\chi}^{\xi}\left|\hat{\phi}'(\eta)\right|d\eta\right| \le C\left|\xi-\chi\right|^{\frac{1}{2}}\left|\chi\right|^{-2-\delta}\left\|\left|\xi\right|^{2+\delta}\hat{\phi}'\right\|_{\mathbf{L}^2}$$

on $\xi \in [\chi/2, 2\chi]$ and (2.14) with $\gamma = 3/4 + \delta/2$, the last term on the right hand side of (2.13) is bounded by

$$\begin{aligned}(2.16)\qquad & \frac{C}{t^{\frac{1}{2}+\delta}}\int_{\frac{\chi}{2}}^{2\chi}\frac{|\chi|^{\frac{3}{2}+3\delta}}{|\xi-\chi|^{1+2\delta}}\left|\hat{\phi}(\xi)-\hat{\phi}(\chi)\right|d\xi \\ \le\ & \frac{C\left\||\xi|^{2+\delta}\hat{\phi}'\right\|_{\mathbf{L}^2}}{t^{\frac{1}{2}+\delta}|\chi|^{\frac{1}{2}-2\delta}}\int_{\frac{\chi}{2}}^{2\chi}\frac{d\xi}{|\xi-\chi|^{\frac{1}{2}+2\delta}} \le \frac{C}{t^{\frac{1}{2}+\delta}}\left(\|x\phi_x\|_{\mathbf{H}^{1+\delta}}+\|\phi\|_{\mathbf{H}^{1+\delta}}\right),\end{aligned}$$

where we chose $\delta \in (0, 1/4)$ so that $1/2 + 2\delta < 1$. Therefore, by (2.15) and (2.16), we obtain

$$|R| \le Ct^{-\frac{1}{2}-\delta}\left(\|\phi\|_{\mathbf{H}^{3/2+\delta}}+\|x\phi_x\|_{\mathbf{H}^{1+2\delta}}\right). \tag{2.17}$$

Collecting estimates (2.9), (2.11), (2.12) and (2.17), we obtain (2.5) for $x < 0$. This completes the proof of Lemma 2.6. □

2. Time decay estimate

In this section, we give the $\mathbf{L}^\infty$ time decay estimate. To prove the time decay estimate, we need following interpolation property.

Lemma 2.6. ([**17**]) *The estimate*

$$\left\|\left(-\partial_x^2\right)^{\frac{\mu}{2}}\phi\right\|_{\mathbf{H}^{0,\alpha}} \le C\left(\|\phi\|_{\mathbf{H}^{\frac{\mu-\beta}{1-\beta}}}+\|x\phi_x\|_{\mathbf{L}^2}\right)$$

is true, provided that the right-hand side is finite, where $\mu \ge 1$, $0 < \alpha < \beta < 1$.

Proof. By the definition of the norm and the Plancherel theorem we get

$$\begin{aligned}\left\|\left(-\partial_x^2\right)^{\frac{\mu}{2}}\phi\right\|_{\mathbf{H}^{0,\alpha}} &\le C\left\|\left(-\partial_x^2\right)^{\frac{\mu}{2}}\phi\right\|_{\mathbf{L}^2}+C\left\||x|^{\alpha}\left(-\partial_x^2\right)^{\frac{\mu}{2}}\phi\right\|_{\mathbf{L}^2} \\ &\le C\|\phi\|_{\mathbf{H}^{\mu}}+C\left\||\partial_\xi|^{\alpha}|\xi|^{\mu-2}\xi\hat{\phi_1}\right\|_{\mathbf{L}^2},\end{aligned}$$

where we denote $\phi_1 = \phi_x$. Note that the fractional derivative is represented as

$$|\partial_\xi|^{\alpha}\phi(\xi) = C\int_{\mathbb{R}}(\phi(\xi,\eta)-\phi(\xi))\frac{d\eta}{|\eta|^{1+\alpha}}$$

where $C = -\alpha/(2\Gamma(1-\alpha)\cos(\pi\alpha/2))$ and Γ is the Euler gamma function. Then, we have

$$\begin{aligned}(2.18)\qquad \left\||\partial_\xi|^{\alpha}|\xi|^{\mu-2}\xi\hat{\phi_1}\right\|_{\mathbf{L}^2} &\le C\left\|\int_{|\eta|\le 1}|\xi|^{\mu-2}\xi(\hat{\phi_1}(\xi+\eta)-\hat{\phi_1}(\xi))\frac{d\eta}{|\eta|^{1+\alpha}}\right\|_{\mathbf{L}^2} \\ &+C\left\|\int_{|\eta|\le 1}(|\xi+\eta|^{\mu-2}(\xi+\eta)-|\xi|^{\mu-2}\xi)\hat{\phi_1}(\xi+\eta)\frac{d\eta}{|\eta|^{1+\alpha}}\right\|_{\mathbf{L}^2} \\ &+C\left\|\int_{|\eta|\ge 1}(|\xi+\eta|^{\mu-2}(\xi+\eta)\hat{\phi_1}(\xi+\eta)-|\xi|^{\mu-2}\xi\hat{\phi_1}(\xi))\frac{d\eta}{|\eta|^{1+\alpha}}\right\|_{\mathbf{L}^2}.\end{aligned}$$

Note that $|a-b|^{1-\beta} \leq |a|^{1-\beta}+|b|^{1-\beta}$ for $\beta \in (0,1)$. Hence $|a-b| \leq |a-b|^{\beta}(|a|^{1-\beta}+|b|^{1-\beta})$. Therefore, by the Hölder inequality, we find that

$$\begin{aligned} \left\| |\xi|^{\mu-2}\xi(\hat{\phi_1}(\xi+\eta)-\hat{\phi_1}(\xi)) \right\|_{\mathbf{L}^2} &\leq \left\| \hat{\phi_1}(\xi+\eta)-\hat{\phi_1}(\xi) \right\|_{\mathbf{L}^2}^{\beta} \left\| \langle\xi\rangle^{\frac{\mu-1}{1-\beta}} \hat{\phi_1} \right\|_{\mathbf{L}^2}^{1-\beta} \\ &\leq C|\eta|^{\beta} \left\| \partial_\xi \hat{\phi_1} \right\|_{\mathbf{L}^2}^{\beta} \left\| \langle\xi\rangle^{\frac{\mu-1}{1-\beta}} \hat{\phi_1} \right\|_{\mathbf{L}^2}^{1-\beta} \end{aligned}$$

and

$$\left\| (|\xi+\eta|^{\mu-2}(\xi+\eta)-|\xi|^{\mu-2}\xi)\hat{\phi_1}(\xi+\eta) \right\|_{\mathbf{L}^2} \leq C|\eta| \left\| \langle\xi\rangle^{\mu-2}\hat{\phi_1} \right\|_{\mathbf{L}^2}$$

for $|\eta| \leq 1$. Thus, substituting above estimates into (2.18), we obtain

$$\begin{aligned} &\left\| |\partial_\xi|^{\alpha}|\xi|^{\mu-2}\xi\hat{\phi_1} \right\|_{\mathbf{L}^2} \leq C \left\| \partial_\xi \hat{\phi_1} \right\|_{\mathbf{L}^2}^{\beta} \left\| \langle\xi\rangle^{\frac{\mu-1}{1-\beta}} \hat{\phi_1} \right\|_{\mathbf{L}^2}^{1-\beta} \int_{|\eta|\leq 1} \frac{d\eta}{|\eta|^{1+\alpha-\beta}} \\ &+C \left\| \langle\xi\rangle^{\mu-2}\hat{\phi_1} \right\|_{\mathbf{L}^2} \int_{|\eta|\leq 1} \frac{d\eta}{|\eta|^{\alpha}} + C \left\| |\xi|^{\mu-1}\hat{\phi_1} \right\|_{\mathbf{L}^2} \int_{|\eta|\geq 1} \frac{d\eta}{|\eta|^{1+\alpha}} \\ \leq\ & C\|\phi_1\|_{\mathbf{H}^{\frac{\mu-1}{1-\beta}}} + C\|x\phi_1\|_{\mathbf{L}^2} \leq C\|\phi\|_{\mathbf{H}^{\frac{\mu-\beta}{1-\beta}}} + C\|x\phi_x\|_{\mathbf{L}^2}. \end{aligned}$$

This completes the proof of Lemma 2.6. □

Applying the above lemma, we can show the $\mathbf{L}^{\infty}$ time decay estimate. The estimate says that the $\mathbf{L}^{\infty}$ - norm of solutions in higher order Sobolev spaces can be estimated through the $\mathbf{L}^2$ - norm of $\mathcal{J}\partial_x u \simeq \mathcal{P}u$.

LEMMA 2.7. ([**17**]) *Let* $\varepsilon \in (0,1/2)$ *and* $l \geq 0$. *Then the estimate*

$$\left\| \left(-\partial_x^2\right)^{\frac{l}{2}} \phi \right\|_{\mathbf{L}^{\infty}} \leq Ct^{-\frac{1}{2}} \left(\|\phi\|_{\mathbf{H}^{\frac{2l+2-2\varepsilon}{1-2\varepsilon}}} + \|\mathcal{J}\phi_x\|_{\mathbf{L}^2} \right)$$

is true, provided that the right-hand side is finite.

PROOF. Since

$$\|\phi\|_{\mathbf{L}^1} \leq C\|\phi\|_{\mathbf{H}^{0,\frac{1+\varepsilon}{2}}}$$

we have by Lemma 2.1 and Lemma 2.6

$$\begin{aligned} \left\| \left(-\partial_x^2\right)^{\frac{l}{2}} \phi \right\|_{\mathbf{L}^{\infty}} &= \left\| \mathcal{U}(t)\,\mathcal{U}(-t) \left(-\partial_x^2\right)^{\frac{l}{2}} \phi \right\|_{\mathbf{L}^{\infty}} \\ \leq\ & Ct^{-\frac{1}{2}} \left\| \left(-\partial_x^2\right)^{\frac{3}{4}+\frac{l}{2}} \mathcal{U}(-t)\,\phi \right\|_{\mathbf{L}^1} \\ \leq\ & Ct^{-\frac{1}{2}} \left\| \left(-\partial_x^2\right)^{\frac{3}{4}+\frac{l}{2}} \mathcal{U}(-t)\,\phi \right\|_{\mathbf{H}^{0,\frac{1+\varepsilon}{2}}} \\ \leq\ & Ct^{-\frac{1}{2}} \left(\|\mathcal{U}(-t)\,\phi\|_{\mathbf{H}^{\frac{2l+2-2\varepsilon}{1-2\varepsilon}}} + \|x\partial_x\mathcal{U}(-t)\,\phi\|_{\mathbf{L}^2} \right). \end{aligned}$$

This completes the proof of Lemma 2.7. □

3. A priori energy estimate

The following estimate was shown in [**29**].

LEMMA 2.8. ([**29**]) *Let* u *be a smooth solution of*

$$u_{tx} = u + F(t,x)\,u_{xx} + G(t,x).$$

Then for any $s>1$*, there exists a constant* $C_s \simeq 1/(s-1)$*, and a positive constant* C *such that*

$$\frac{d}{dt}\left\|\left(-\partial_x^2\right)^{\frac{s}{2}} u(t)\right\|_{\mathbf{L}^2}^2 \leq C_s \left\|\partial_x F(t)\right\|_{\mathbf{L}^\infty} \left\|\left(-\partial_x^2\right)^{\frac{s}{2}} u(t)\right\|_{\mathbf{L}^2}^2$$
$$+2\left\|\left(-\partial_x^2\right)^{\frac{s}{2}} u(t)\right\|_{\mathbf{L}^2}\left(\left\|\left(-\partial_x^2\right)^{\frac{s-1}{2}} G(t)\right\|_{\mathbf{L}^2} + C\left\|\partial_x u(t)\right\|_{\mathbf{L}^\infty}\left\|\left(-\partial_x^2\right)^{\frac{s}{2}} F(t)\right\|_{\mathbf{L}^2}\right).$$

CHAPTER 3

Local existence theorem for the generalized reduced Ostrovsky equation

In the function space $\mathbf{H}^m \cap \dot{\mathbf{H}}^{-1}$ local well posedness was treated in papers [**21**], [**22**]. However we do not know local well posedness for (1.1) in the weighted Sobolev spaces. For the convenience of the readers, we give a local existence theorem for (1.1) which is needed in later chapters.

PROPOSITION 3.1. ([**17**]) *Let the initial data* $u_0 \in \mathbf{X}_0^m, m > 3/2$, *and the order of nonlinearity* ρ *satisfies* $\rho > m+1$, *or is an integer. Then there exists a time* $T(u_0) > 0$ *and a unique solution*

$$u \in \mathbf{C}([0,T]; \mathbf{H}^m \cap \dot{\mathbf{H}}^{-1}),\ xu_x \in \mathbf{C}\left([0,T]; \mathbf{L}^2\right)$$

of the Cauchy problem (1.1).

PROOF. We consider the parabolic regularization

$$\left\{ \begin{array}{c} u_{tx} - u - \nu u_{xxx} = (f(u))_{xx},\ x \in \mathbb{R},\ t > 0, \\ u(0,x) = u_0(x),\ x \in \mathbb{R}, \end{array} \right. \tag{3.1}$$

with $\nu > 0$. By the contraction mapping principle, we first prove the existence of solutions in $\|u\|_{\mathbf{H}^m} + \|u_x\|_{\mathbf{H}^{0,1}} + \|u\|_{\dot{\mathbf{H}}^{-1}}$. Let us consider the linearized integral equation associated with (3.1)

$$u(t) = \mathcal{U}_\nu(t) u_0 + \int_0^t \mathcal{U}_\nu(t-s)(f(v(s)))_x ds,$$

where $\mathcal{U}_\nu(t) = \mathcal{F}^{-1} \exp\left(-it/\xi - \nu t\xi^2\right) \mathcal{F}$ and $\|v\|_{\mathbf{X}_T^m} \le M$. We have

$$\|u\|_{\mathbf{H}^m} \le \|u_0\|_{\mathbf{H}^m} + C(\nu) \int_0^t (t-s)^{-\frac{1}{2}} \|v\|_{\mathbf{H}^m}^\rho ds \le \|u_0\|_{\mathbf{H}^m} + C(\nu) T^{\frac{1}{2}} M^\rho \tag{3.2}$$

and

$$\|u\|_{\dot{\mathbf{H}}^{-1}} \le \|u_0\|_{\dot{\mathbf{H}}^{-1}} + C \int_0^t \|v\|_{\mathbf{L}^\infty}^{\rho-1} \|v\|_{\mathbf{L}^2} ds \le \|u_0\|_{\dot{\mathbf{H}}^{-1}} + CT^{\frac{1}{2}} M^\rho, \tag{3.3}$$

since the estimate $\|\mathcal{U}_\nu(t)\partial_j \phi\|_{\mathbf{L}^2} \le C(\nu t)^{-\frac{j}{2}} \|\phi\|_{\mathbf{L}^2}$ is valid for $j \in \{0\} \cup \mathbb{N}$. Multiplying integral equation by $x\partial_x$, using

$$[x\partial_x, \mathcal{U}_\nu(t)] = \mathcal{F}^{-1} \left(\frac{it}{\xi} - 2\nu t\xi^2\right) \mathcal{F}\mathcal{U}_\nu(t) = -\mathcal{U}_\nu(t)\left(t\left(\partial_x^{-1} - 2\nu\partial_x^2\right)\right) \tag{3.4}$$

we obtain

$$\begin{aligned} xu_x(t) &= \mathcal{U}_\nu(t)\left(x\partial_x - t\left(\partial_x^{-1} - 2\nu\partial_x^2\right)\right)u_0 \\ &+ \int_0^t \mathcal{U}_\nu(t-s)\left(x\partial_x^2 - (t-\tau)\left(1-2\nu\partial_x^3\right)\right)f(v(s))ds. \end{aligned}$$

Taking the $\mathbf{L}^2$ - norm, we get

$$\begin{aligned} &\|xu_x\|_{\mathbf{L}^2} \le \|x\partial_x u_0\|_{\mathbf{L}^2} + CT\|u_0\|_{\dot{\mathbf{H}}^{-1}} + C(\nu)\|u_0\|_{\mathbf{L}^2} \\ &+C(\nu)\int_0^T (t-s)^{-\frac{1}{2}}\|v\|_{\mathbf{L}^\infty}^{\rho-1}\left(\|xv_x\|_{\mathbf{L}^2} + \|v\|_{\mathbf{L}^2}\right)ds \\ &+C(\nu)\int_0^T T\left(\|v\|_{\mathbf{L}^\infty}^{\rho-1}\|v\|_{\mathbf{L}^2} + \|v\|_{\mathbf{L}^\infty}^{\rho-1}\|v_x\|_{\mathbf{L}^2}\right)ds \\ \le\ &\|x\partial_x u_0\|_{\mathbf{L}^2} + CT\|u_0\|_{\dot{\mathbf{H}}^{-1}} + C(\nu)\|u_0\|_{\mathbf{L}^2} + C(\nu)M^\rho\left(T^2 + T^{\frac{1}{2}}\right). \end{aligned} \tag{3.5}$$

By virtue of (3.2), (3.3), (3.5), we find that there exists a time T_ν such that (3.1) has a unique solution $u = u^{(\nu)}$. To prove Proposition 3.1 we need a-priori estimates in the norms $\|u\|_{\mathbf{H}^m} + \|\partial_x u\|_{\mathbf{H}^{0,1}} + \|u\|_{\dot{\mathbf{H}}^{-1}}$ uniformly with respect to $\nu > 0$. A-priori estimates in the norms $\|u\|_{\mathbf{H}^m} + \|u\|_{\dot{\mathbf{H}}^{-1}}$ can be obtained by integration by parts. We next prove a-priori estimate of $\|xu_x\|_{\mathbf{L}^2}$. Multiplying equation (3.1) by x^2u_x and integrating with respect to x we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} x^2u_x^2dx = \int_{\mathbb{R}} x^2uu_xdx + \nu\int_{\mathbb{R}} x^2u_xu_{xxx}dx + \int_{\mathbb{R}} x^2(f(u))_{xx}u_xdx. \tag{3.6}$$

Integration by parts in view of the Schwarz inequality yields

$$\begin{aligned} &\left|\int_{\mathbb{R}} x^2uu_xdx\right| = \left|\int_{\mathbb{R}} xu^2dx\right| = \left|\int_{\mathbb{R}}(u+xu_x)\partial_x^{-1}udx\right| \\ \le\ &\left(\|u\|_{\mathbf{L}^2} + \|x\partial_x u\|_{\mathbf{L}^2}\right)\left\|\partial_x^{-1}u\right\|_{\mathbf{L}^2}. \end{aligned} \tag{3.7}$$

In the same manner

$$\nu\int_{\mathbb{R}} x^2u_xu_{xxx}dx = \nu\|u_x\|_{\mathbf{L}^2}^2 - \nu\|xu_{xx}\|_{\mathbf{L}^2}^2 \tag{3.8}$$

and

$$\left|\int_{\mathbb{R}} x^2(f(u))_{xx}u_xdx\right| \le C\|u\|_{\mathbf{H}^1_\infty}^{\rho-1}\left(\|u_x\|_{\mathbf{L}^2} + \|xu_x\|_{\mathbf{L}^2}\right)\|xu_x\|_{\mathbf{L}^2}. \tag{3.9}$$

Substituting (3.7), (3.8) and (3.9) into (3.6) we get

$$\begin{aligned} &\frac{d}{dt}\|xu_x\|_{\mathbf{L}^2}^2 \le C\left(\|u\|_{\mathbf{L}^2} + \|xu_x\|_{\mathbf{L}^2}\right)\left\|\partial_x^{-1}u\right\|_{\mathbf{L}^2} \\ &+C\|u_x\|_{\mathbf{L}^2}^2 + C\|u\|_{\mathbf{H}^1_\infty}^{\rho-1}\left(\|u_x\|_{\mathbf{L}^2} + \|xu_x\|_{\mathbf{L}^2}\right)\|x\partial_x u\|_{\mathbf{L}^2} \\ &\le C + C\|xu_x\|_{\mathbf{L}^2}^2 \end{aligned}$$

from which it follows that

$$\frac{d}{dt}e^{-Ct}\|xu_x\|_{\mathbf{L}^2}^2 \le Ce^{-Ct}.$$

Integrating in time, we obtain

$$\|xu_x\|_{\mathbf{L}^2}^2 \le \|x\partial_x u_0\|_{\mathbf{L}^2}^2 e^{CT} + C$$

for $0 \leq t \leq T$. Therefore we have a desired a-priori estimates of solutions uniformly with respect to $\nu > 0$. Letting $\nu \to 0$, we have $\|u\|_{\mathbf{X}_T^m} \leq C$. This completes the proof of Proposition 3.1. □

REMARK 3.1. *When we consider (1.1) in the weighted Sobolev space, the problem becomes delicate. By (3.4) we have*

$$x\partial_x \mathcal{U}_\nu (t) - \mathcal{U}_\nu (t) x\partial_x = \mathcal{F}^{-1} \left(\frac{it}{\xi} - 2\nu t\xi^2 \right) \mathcal{F}\mathcal{U}_\nu (t) .$$

Therefore

$$\begin{aligned} & (x\partial_x)^2 \mathcal{U}_\nu (t) - \mathcal{U}_\nu (t) (x\partial_x)^2 \\ = \; & 2\mathcal{F}^{-1} \left(\frac{it}{\xi} - 2\nu t\xi^2 \right) \mathcal{F}\mathcal{U}_\nu (t) x\partial_x \\ & + \mathcal{F}^{-1} \left(-\frac{it}{\xi} - 4\nu t\xi^2 \right) \mathcal{F}\mathcal{U}_\nu (t) + \mathcal{F}^{-1} \left(\frac{it}{\xi} - 2\nu t\xi^2 \right)^2 \mathcal{F}\mathcal{U}_\nu (t) \\ = \; & \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \end{aligned}$$

In order to estimate $\| (x\partial_x)^2 u\|_{\mathbf{L}^2}$ *we find*

$$\begin{aligned} \left\| (x\partial_x)^2 u \right\|_{\mathbf{L}^2} \leq \; & \left\| (x\partial_x)^2 u_0 \right\|_{\mathbf{L}^2} + C \|x\partial_x u_0\|_{\mathbf{L}^2} + C \|u_0\|_{\mathbf{L}^2} \\ & + CT \|x\partial_x u_0\|_{\dot{\mathbf{H}}^{-1}} + CT \|u_0\|_{\dot{\mathbf{H}}^{-1}} + CT^2 \|u_0\|_{\dot{\mathbf{H}}^{-2}} \\ & + C \sum_{j=1}^{3} \int_0^t \|\mathcal{I}_j (t-s) \partial_x f(v(s))\|_{\mathbf{L}^2} ds. \end{aligned}$$

Then, $\mathcal{I}_3$ *can be estimated as*

$$\begin{aligned} & \int_0^t \|\mathcal{I}_3 (t-s) \partial_x v^\rho (s)\|_{\mathbf{L}^2} ds \\ = \; & \int_0^t \left\| \mathcal{F}^{-1} \left(\frac{i (t-s)}{\xi} - 2\nu (t-s) \xi^2 \right)^2 \mathcal{F}\mathcal{U}_\nu (t-s) \partial_x v^\rho (s) \right\|_{\mathbf{L}^2} ds \\ \leq \; & C \int_0^t \left(\|f(v(s))\|_{\dot{\mathbf{H}}^{-1}} + (t-s)^{-\frac{1}{2}} \|f(v(s))\|_{\mathbf{L}^2} \right) ds. \end{aligned}$$

The first term of the right-hand side of the above inequality is difficult to treat. This is the reason why we avoid to use the operator $\mathcal{P}^2$.

CHAPTER 4

Asymptotics of solutions to the generalized reduced Ostrovsky equation with supercritical nonlinearity

1. Main results

Our aim in this section is to consider the asymptotic behavior of solutions to the generalized reduced Ostrovsky equation with supercritical nonlinearity. In Theorem 4.1, we show the existence of the usual scattering states if the order of nonlinearity $\rho > 3 + 1/2$. In Theorem 4.2, an almost global existence of solutions to the short pulse equation ($\rho = 3$) is given. More precisely, we show that the lower bound for the maximal existence time T can be estimated as $T \geq \exp(B/\epsilon^2)$, where ϵ is the size of initial data. In Theorem 4.3, we consider the short pulse equation with time dependent coefficient. Under the suitable assumption on the time dependent coefficient, we show the existence of the usual scattering states. To state our results precisely, we introduce the function spaces

$$\mathbf{X}_T^m = \left\{ u(t) \in \mathbf{C}\left([0,T);\mathbf{L}^2\right); \|u\|_{\mathbf{X}_T^m} < \infty \right\},\ \mathbf{X}_0^m = \left\{ \phi \in \mathbf{L}^2; \|\phi\|_{\mathbf{X}_0^m} < \infty \right\},$$

where

$$\|u\|_{\mathbf{X}_T^m} = \sup_{t\in[0,T)} \|u(t)\|_{\mathbf{H}^m} + \sup_{t\in[0,T)} \|\mathcal{J}\partial_x u(t)\|_{\mathbf{L}^2} + \sup_{t\in[0,T)} \|u(t)\|_{\dot{\mathbf{H}}^{-1}}$$

and

$$\|\phi\|_{\mathbf{X}_0^m} = \|\phi\|_{\mathbf{H}^m} + \|\partial_x\phi\|_{\mathbf{H}^{0,1}} + \|\phi\|_{\dot{\mathbf{H}}^{-1}}.$$

Theorem 4.1. ([**17**]) *Assume that the initial data $u_0 \in \mathbf{X}_0^m$, where $m = 2+\epsilon$, $\epsilon > 0$ and the order ρ of the nonlinearity satisfies $\rho > \max\{3 + 2/(3+2\epsilon), 3+\epsilon\}$, or is an integer, such that $\rho \geq 4$. Then there exists a positive constant $\widetilde{\varepsilon}$ such that (1.1) has a unique global solution $u \in \mathbf{X}_\infty^m$ with the time decay*

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C\langle t\rangle^{-\frac{1}{2}}$$

for any u_0 satisfying $\|u_0\|_{\mathbf{X}_0^m} \leq \widetilde{\varepsilon}$. Moreover for any $u_0 \in \mathbf{X}_0^m$ such that $\|u_0\|_{\mathbf{X}_0^m} \leq \widetilde{\varepsilon}$, there exists a unique scattering state $u_+ \in \mathbf{H}^{m-\delta}\cap\dot{\mathbf{H}}^{-1}, \partial_x u_+ \in \mathbf{H}^{0,1-\delta}$ satisfying

$$\begin{aligned}&\|\mathcal{U}(-t)u(t) - u_+\|_{\mathbf{H}^{m-\delta}} + \|\mathcal{U}(-t)u(t) - u_+\|_{\dot{\mathbf{H}}^{-1}}\\ &+\|\mathcal{U}(-t)\partial_x u(t) - \partial_x u_+\|_{\mathbf{H}^{0,1-\delta}} \to 0\end{aligned} \tag{4.1}$$

as $t \to \infty$ for small $\delta > 0$.

Next result states an almost global existence of small solutions to (1.1) with $\rho = 3$. To state the result, we define a maximal existence time T^* by

$$T^* = \sup \left\{ T > 0; \|u\|_{\mathbf{X}_T^m} < \infty \right\}.$$

THEOREM 4.2. ([**17**]) *Assume that $\rho = 3$, the initial data $u_0 \in \mathbf{X}_0^m$, where $m > 4$ and $\|u_0\|_{\mathbf{X}_0^m} = \widetilde{\varepsilon}$. Then there exist positive constants ε_0 and B such that*

$$T^* \geq \exp\left(\frac{B}{\widetilde{\varepsilon}^2}\right)$$

for all $0 < \widetilde{\varepsilon} \leq \varepsilon_0$.

The proof of Theorem 4.2 works also for the Cauchy problem

$$\begin{cases} u_{tx} = u + a(t)(u^3)_{xx} \\ u(0) = u_0 \end{cases}, \tag{4.2}$$

if the coefficient $a(t) \in \mathbf{C}^1(\mathbb{R})$ satisfies the following time decay estimate

$$\left|\partial_t^j a(t)\right| \leq C(1+|t|)^{-j} (\log(2+|t|))^{-1-\gamma}$$

for $j = 0, 1$ and $t > 0$, where $\gamma > 0$. Therefore, we obtain following theorem.

THEOREM 4.3. ([**17**]) *Let the initial data $u_0 \in \mathbf{X}_0^m$, where $m > 4$ and $\|u_0\|_{\mathbf{X}_0^m} = \widetilde{\varepsilon}$. Then there exists a positive constant $\widetilde{\varepsilon}$ such that (4.2) has a unique global solution $u \in \mathbf{X}_\infty^m$ with the time decay*

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C \langle t \rangle^{-\frac{1}{2}}$$

for any u_0 satisfying $\|u_0\|_{\mathbf{X}_0^m} \leq \widetilde{\varepsilon}$. Moreover for any $u_0 \in \mathbf{X}_0^m$ such that $\|u_0\|_{\mathbf{X}_0^m} \leq \widetilde{\varepsilon}$, there exists a unique scattering state $u_+ \in \mathbf{H}^{m-\delta} \cap \dot{\mathbf{H}}^{-1}, \partial_x u_+ \in \mathbf{H}^{0,1-\delta}$ satisfying (4.1) for small $\delta > 0$.

For the convenience of the reader, we explain our strategy of the proof. The operator $\mathcal{J}$ was introduced by [**5**] first to study the scattering problem for the nonlinear Schrödinger equations and was used by many authors, see, e.g., [**3**]. However, the operator $\mathcal{J}$ does not work well on the nonlinear term. To overcome this difficulty, we introduce the operator $\mathcal{P}$, which was used in [**8**] for studying the global existence of solutions to the Schrödinger equations in three space dimensions. After that the operator $\mathcal{P}$ was used often for several equations appeared in fluid mechanics such as the modified Korteweg-de Vries equation [**9**], [**10**], the generalized Benjamin-Ono equation [**11**] and the generalized Kadomtsev-Petviashvili equation [**16**]. According to these papers, we use the set of operators $(\mathcal{I}, \partial_x, \mathcal{P})$ in order to get desired time decay estimates of solutions. While, we already know that local solutions of the reduced Ostrovsky equation is local well-posed in $\mathbf{H}^s$ for $s > 3/2$. Hence, it is reasonable to define our function space through the operators $(\mathcal{I}, \partial_x, \mathcal{P}, \partial_x^2, \partial_x\mathcal{P}, \mathcal{P}^2)$. However the operator $\mathcal{P}^2$ is not acceptable for the case since $\mathcal{P} = x\partial_x - t\partial_x^{-1} - t\mathcal{L}$ and $\mathcal{P}^2 \sim (x\partial_x - t\partial_x^{-1})^2$ is equivalent to use of ∂_x^{-2} (See also remark 3.1). On the other hand, our equation is $u_t = \partial_x^{-1}u + (f(u))_x$ and we can not apply ∂_x^{-2} to the nonlinear term. Therefore, we need to use the set of operators $(\mathcal{I}, \partial_x^m, \mathcal{P})$ instead of $(\mathcal{I}, \partial_x, \mathcal{P}, \partial_x^2, \partial_x\mathcal{P}, \mathcal{P}^2)$. Thanks to Lemma 2.7, the set of operators works well even if the orders of $\mathcal{P}$ and ∂_x are different each other.

2. Proof of Theorem 4.1

Proof. We prove that for any $T > 0$

$$\|u\|_{\mathbf{X}_T^{2+\varepsilon}} < \sqrt{\widehat{\varepsilon}}$$

by a contradiction argument. We assume that there exists a time T such that

$$\|u\|_{\mathbf{X}_T^{2+\varepsilon}} = \sqrt{\widehat{\varepsilon}}.$$

Taking $m = 2+\varepsilon$, $F = f'(u)$, $G = f''(u)u_x^2$ in Lemma 2.8 and using the Sobolev inequality such that

$$\|u_x\|_{\mathbf{L}^\infty} \le C\|u\|_{\mathbf{L}^\infty}^{\frac{1+2\varepsilon}{3+2\varepsilon}} \|u\|_{\mathbf{H}^{2+\varepsilon}}^{\frac{2}{3+2\varepsilon}}, \tag{4.3}$$

we find that

$$\begin{aligned}\frac{d}{dt}\left\|(-\partial_x)^{\frac{s}{2}}u(t)\right\|_{\mathbf{L}^2}^2 &\le C\|u(t)\|_{\mathbf{L}^\infty}^{\rho-2+\frac{1+2\varepsilon}{3+2\varepsilon}}\|u(t)\|_{\mathbf{H}^s}^{\frac{2}{3+2\varepsilon}}\|u(t)\|_{\mathbf{H}^s}^2 \\ &\le C\langle t\rangle^{-\frac{1}{2}\left(\rho-2+\frac{1+2\varepsilon}{3+2\varepsilon}\right)}\left(\|u\|_{\mathbf{H}^s}+\|\mathcal{J}\partial_x u\|_{\mathbf{L}^2}\right)^{\rho-2+\frac{1+2\varepsilon}{3+2\varepsilon}}\|u(t)\|_{\mathbf{H}^s}^{\frac{2}{3+2\varepsilon}}\|u(t)\|_{\mathbf{H}^s}^2,\end{aligned} \tag{4.4}$$

because of Lemma 2.7. Therefore

$$\|u(t)\|_{\mathbf{H}^m}^2 \le \widehat{\varepsilon}^2 + C\widehat{\varepsilon}^{\frac{\rho+1}{2}}\int_0^t \langle\tau\rangle^{-\frac{1}{2}\left(\rho-2+\frac{1+2\varepsilon}{3+2\varepsilon}\right)}d\tau \le \widehat{\varepsilon}^2 + C\widehat{\varepsilon}^{\frac{\rho+1}{2}} \le 2\widehat{\varepsilon}^2 \tag{4.5}$$

since $\rho > 3 + 2/(3+2\epsilon)$ and $\varepsilon > 0$ is small. To get the a-priori estimate of $\left\|\partial_x^{-1}u(t)\right\|_{\mathbf{L}^2}$, we define $\chi_n \in \mathbf{S}$ such that $0 \le \widehat{\chi_n}(\xi) \le 1$ and

$$\widehat{\chi_n}(\xi) = \begin{cases} 1 & 2^{-n+1} \le |\xi| \le 2^n \\ 0 & 0 \le |\xi| \le 2^{-n},\ 2^{n+1} \le |\xi| \end{cases}.$$

Multiplying (1.1) by $\chi_n *$, we get

$$(\chi_n * u_t + \chi_n * (f(u))_x)_x = \chi_n * u, \tag{4.6}$$

where $*$ is the convolution. Multiplying ∂_x^{-2} both side of (4.6) and taking the dot product with $\partial_x^{-1}(\chi_n * u)$, we have

$$(\partial_x^{-1}(\chi_n * u_t), \partial_x^{-1}(\chi_n * u)) + (\chi_n * f(u), \partial_x^{-1}(\chi_n * u)) = (\partial_x^{-2}(\chi_n * u), \partial_x^{-1}(\chi_n * u)).$$

Note that $(\partial_x^{-2}(\chi_n * u), \partial_x^{-1}(\chi_n * u)) = 0$. Then, letting $n \to \infty$ and integrating with respect to t, we have

$$\begin{aligned}\left\|\partial_x^{-1}u(t)\right\|_{\mathbf{L}^2} &\le \left\|\partial_x^{-1}u_0\right\|_{\mathbf{L}^2} + \int_0^t \|u(t)\|_{\mathbf{L}^\infty}^{\rho-1}\|u(t)\|_{\mathbf{L}^2}d\tau \\ &\le \tilde{\epsilon} + C\tilde{\epsilon}^{\frac{\rho-1}{2}}\int_0^t \left(\|u\|_{\mathbf{H}^s} + \|\mathcal{J}\partial_x u\|_{\mathbf{L}^2}\right)^{\rho-1}\langle\tau\rangle^{\rho-1}d\tau \le C\tilde{\epsilon}.\end{aligned} \tag{4.7}$$

We next consider the estimate of $\|\mathcal{P}u(t)\|_{\mathbf{L}^2}$. Multiplying $\mathcal{P} = x\partial_x - t\partial_t$ both side of (1.1), we have

$$\mathcal{P}u = \left((\mathcal{P}u)_t + (f(u))_x - \mathcal{P}\left(f(u)\right)_x\right)_x. \tag{4.8}$$

In the same manner of the above, we multiply (4.8) by $\partial_x^{-1}\chi_n *$ and take the dot product with $\chi_n * \mathcal{P}u$, then we have Letting $n \to \infty$, we have

$$\begin{aligned}\frac{d}{dt}\|\mathcal{P}u(t)\|_{\mathbf{L}^2} &\le C\|u(t)\|_{\mathbf{L}^\infty}^{\rho-1}\|u_x(t)\|_{\mathbf{L}^2} + C\|u(t)\|_{\mathbf{H}_\infty^1}^{\rho-1}\|\mathcal{P}u(t)\|_{\mathbf{L}^2} \\ &\le C\widehat{\varepsilon}^{\frac{\rho}{2}}\langle t\rangle^{\frac{\rho-1}{2}} + C\widehat{\varepsilon}^{\frac{\rho-1}{2}}\langle t\rangle^{-\frac{1}{2}\left(\rho-2+\frac{1+2\varepsilon}{3+2\varepsilon}\right)}\|\mathcal{P}u(t)\|_{\mathbf{L}^2}.\end{aligned} \tag{4.9}$$

Hence,

$$\|\mathcal{P}u\|_{\mathbf{L}^2} \le C\tilde{\epsilon}.$$

By the identity

$$(\mathcal{P}-\mathcal{J}\partial_x)\,u = -t\left(u_t - \int_{-\infty}^{x} udy\right) = -t\,(f(u))_x\,,$$

we obtain

$$\begin{aligned}(4.10)\qquad \|\mathcal{J}\partial_x u\|_{\mathbf{L}^2} &\le \|\mathcal{P}u\|_{\mathbf{L}^2} + t\,\|u\|_{\mathbf{L}^\infty}^{\rho-1}\,\|\partial_x u\|_{\mathbf{L}^2}\\ &\le \|\mathcal{P}u\|_{\mathbf{L}^2} + C\,\langle t\rangle^{1-\frac{1}{2}(\rho-1)}\,(\|u\|_{\mathbf{H}^s} + \|\mathcal{J}\partial_x u\|_{\mathbf{L}^2})^{\rho-1}\,\|\partial_x u\|_{\mathbf{L}^2}\\ &\le C\widetilde{\varepsilon} + C\widetilde{\varepsilon}^{\frac{\rho-1}{2}} \le C\widetilde{\varepsilon}.\end{aligned}$$

By (4.5), (4.7) and (4.10)

$$\|u\|_{\mathbf{X}_T^m} \le C\widetilde{\varepsilon} < \sqrt{\widetilde{\varepsilon}}.$$

This is the desired contradiction. Hence we have a global in time solution satisfying the estimate

$$\|u\|_{\mathbf{X}_\infty^m} \le \sqrt{\widetilde{\varepsilon}}.$$

This completes the proof of the first part of Theorem 4.1. We now consider the scattering problem. It is sufficient to prove that $\{\mathcal{U}\,(-t)\,u\,(t)\}$ is a Cauchy sequence in $\mathbf{C}([0,\infty)\,;\mathbf{L}^2)\cap\mathbf{C}([0,\infty)\,;\dot{\mathbf{H}}^{-1})$ since by the Sobolev inequality

$$\begin{aligned}(4.11)\qquad &\|\mathcal{U}\,(-t)\,u\,(t) - \mathcal{U}\,(-s)\,u\,(s)\|_{\mathbf{H}^{m-\delta}}\\ &\le C\,\|\mathcal{U}\,(-t)\,u\,(t) - \mathcal{U}\,(-s)\,u\,(s)\|_{\mathbf{L}^2}^{\frac{\delta}{m}}\,\|\mathcal{U}\,(-t)\,u\,(t) - \mathcal{U}\,(-s)\,u\,(s)\|_{\mathbf{H}^m}^{1-\frac{\delta}{m}}\\ &\le C\,\|\mathcal{U}\,(-t)\,u\,(t) - \mathcal{U}\,(-s)\,u\,(s)\|_{\mathbf{L}^2}^{\frac{\delta}{m}}\,,\end{aligned}$$

and

$$\begin{aligned}(4.12)\quad &\|\mathcal{U}\,(-t)\,\partial_x u\,(t) - \mathcal{U}\,(-s)\,\partial_x u\,(s)\|_{\mathbf{H}^{0,1-\delta}}\\ &\le C\,\|\mathcal{U}\,(-t)\,\partial_x u\,(t) - \mathcal{U}\,(-s)\,\partial_x u\,(s)\|_{\mathbf{L}^2}^{\delta}\,\|\mathcal{U}\,(-t)\,\partial_x u\,(t) - \mathcal{U}\,(-s)\,\partial_x u\,(s)\|_{\mathbf{H}^{0,1}}^{1-\delta}\\ &\le C\,\|\mathcal{U}\,(-t)\,u\,(t) - \mathcal{U}\,(-s)\,u\,(s)\|_{\mathbf{L}^2}^{\delta\left(1-\frac{1}{m}\right)}\,\|\mathcal{U}\,(-t)\,u\,(t) - \mathcal{U}\,(-s)\,u\,(s)\|_{\mathbf{H}^m}^{\frac{\delta}{m}}\\ &\le C\,\|\mathcal{U}\,(-t)\,u\,(t) - \mathcal{U}\,(-s)\,u\,(s)\|_{\mathbf{L}^2}^{\delta\left(1-\frac{1}{m}\right)}\end{aligned}$$

with $\delta\in(0,1)$. By the integral equation associated with (1.1), we get

$$(4.13)\qquad \|\mathcal{U}\,(-t)\,u\,(t) - \mathcal{U}\,(-s)\,u\,(s)\|_{\dot{\mathbf{H}}^{-1}\cap\mathbf{L}^2} \le C\int_s^t \langle\tau\rangle^{-\frac{1}{2}(\rho-1)}\,d\tau$$

from which it follows that $\{\mathcal{U}\,(-t)\,u\,(t)\}$ is a Cauchy sequence in $\mathbf{C}\left([0,\infty)\,;\mathbf{L}^2\right)\cap\mathbf{C}([0,\infty)\,;\dot{\mathbf{H}}^{-1})$. By applying (4.13) to (4.11) and (4.12), we obtain the result of Theorem 4.1. □

3. Proof of Theorem 4.2

PROOF. We next consider the space $\mathbf{X}_T^{4+\varepsilon}$. We take $m = 4+\varepsilon$, $F = f'(u)$, $G = f''(u)u_x^2$ in Lemma 2.8, where $\rho = 3, 4$ or $\rho > 5$

$$\frac{d}{dt}\left\|(-\partial_x)^{\frac{m}{2}} u(t)\right\|_{\mathbf{L}^2}^2 \leq C_m \|\partial_x F(t)\|_{\mathbf{L}^\infty} \left\|(-\partial_x)^{\frac{m}{2}} u(t)\right\|_{\mathbf{L}^2}^2$$
$$+2\left\|(-\partial_x)^{\frac{m}{2}} u(t)\right\|_{\mathbf{L}^2}\left(\left\|(-\partial_x)^{\frac{m-1}{2}} G(t)\right\|_{\mathbf{L}^2} + C\|\partial_x u(t)\|_{\mathbf{L}^\infty}\left\|(-\partial_x)^{\frac{m}{2}} F(t)\right\|_{\mathbf{L}^2}\right)$$
$$\leq \quad C\|u\|_{\mathbf{L}^\infty}^{\rho-2}\|\partial_x u\|_{\mathbf{L}^\infty}\|u\|_{\mathbf{H}^s}^2.$$

We apply the estimate of Lemma 2.7 to find

$$\frac{d}{dt}\left\|(-\partial_x)^{\frac{m}{2}} u(t)\right\|_{\mathbf{L}^2}^2 \leq C\langle t\rangle^{-\frac{\rho-1}{2}}\left(\|u\|_{\mathbf{H}^m} + \|\mathcal{J}\partial_x u\|_{\mathbf{L}^2}\right)^{\rho-1}\|u\|_{\mathbf{H}^m}^2$$

from which it follows that

$$\|u\|_{\mathbf{H}^m}^2 \leq \widetilde{\varepsilon}^2 + C\int_0^t \langle\tau\rangle^{-\frac{\rho-1}{2}}\left(\|u\|_{\mathbf{H}^m} + \|\mathcal{J}\partial_x u\|_{\mathbf{L}^2}\right)^{\rho-1}\|u\|_{\mathbf{H}^m}^2\, d\tau. \tag{4.14}$$

By the first line of (4.9) and Lemma 2.7, we have

$$\|\mathcal{P}u\|_{\mathbf{L}^2}^2 \leq \widetilde{\varepsilon}^2 + C\int_0^t \langle\tau\rangle^{-\frac{\rho-1}{2}}\left(\|u\|_{\mathbf{H}^m} + \|\mathcal{J}\partial_x u\|_{\mathbf{L}^2}\right)^{\rho-1} \times\left(\|\mathcal{P}u\|_{\mathbf{L}^2} + \|u\|_{\mathbf{L}^2}\right)\|\mathcal{P}u\|_{\mathbf{L}^2}\, d\tau. \tag{4.15}$$

By virtue of (4.14), (4.15), the first line of (4.10) and (4.7), we get

$$\|u\|_{\mathbf{X}_T^m} \leq C\widetilde{\varepsilon} + \|u\|_{\mathbf{X}_T^m}^3 \int_0^t \langle\tau\rangle^{\frac{\rho-1}{2}} d\tau \leq C\widetilde{\varepsilon} + \|u\|_{\mathbf{X}_T^m}^3 \log\langle T\rangle,$$

where we put $\rho = 3$. From this estimate and a contradiction argument used in [**24**], Theorem 4.2 follows. □

4. Proof of Theorem 4.3

PROOF. We now prove that

$$\|u\|_{\mathbf{X}_T^{4+\varepsilon}} < (\widetilde{\varepsilon})^{\frac{2}{3}}$$

for any $T > 0$. By the contradiction we assume that there exists a time T such that

$$\|u\|_{\mathbf{X}_T^{4+\varepsilon}} = (\widetilde{\varepsilon})^{\frac{2}{3}}.$$

In the same way as in the proof of (4.14) and (4.15), we have with $m = 4+\varepsilon$

$$\begin{aligned}\|u\|_{\mathbf{H}^m}^2 &\leq \widetilde{\varepsilon}^2 + C\int_0^t \langle\tau\rangle^{-1}\left(\log\langle\tau\rangle\right)^{-1-\gamma}\left(\|u\|_{\mathbf{H}^m} + \|\mathcal{J}\partial_x u\|_{\mathbf{L}^2}\right)^2\|u\|_{\mathbf{H}^m}^2\, d\tau \\ &\leq \widetilde{\varepsilon}^2 + C(\widetilde{\varepsilon})^{\frac{8}{3}} \leq 2\widetilde{\varepsilon}^2\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{P}u\|_{\mathbf{L}^2}^2 &\leq \widetilde{\varepsilon}^2 + C\int_0^t \langle\tau\rangle^{-1}\left(\log\langle\tau\rangle\right)^{-1-\gamma}\left(\|u\|_{\mathbf{H}^m} + \|\mathcal{J}\partial_x u\|_{\mathbf{L}^2}\right)^2 \\ &\quad\times\left(\|\mathcal{P}u\|_{\mathbf{L}^2} + \|u\|_{\mathbf{L}^2}\right)\|\mathcal{P}u\|_{\mathbf{L}^2}\, d\tau \\ &\leq \widetilde{\varepsilon}^2 + C(\widetilde{\varepsilon})^{\frac{4}{3}}\int_0^t \langle\tau\rangle^{-1}\left(\log\langle\tau\rangle\right)^{-1-\gamma}\|\mathcal{P}u\|_{\mathbf{L}^2}^2\, d\tau\end{aligned}$$

from which with the Gronwall inequality it follows that

$$\|\mathcal{P}u\|_{\mathbf{L}^2} \leq \sqrt{2}\widetilde{\varepsilon}.$$

By the identity

$$(\mathcal{P} - \mathcal{J}\partial_x)\, u = -t\left(u_t - \int_{-\infty}^{x} u dy\right) = -ta\,(t)\,\left(u^3\right)_x$$

we obtain

$$\begin{aligned} \|\mathcal{J}\partial_x u\|_{\mathbf{L}^2} &\leq \|\mathcal{P}u\|_{\mathbf{L}^2} + t\,|a\,(t)|\,\|u\|_{\mathbf{L}^\infty}^2\,\|\partial_x u\|_{\mathbf{L}^2} \\ &\leq \|\mathcal{P}u\|_{\mathbf{L}^2} + C\,(\log\langle\tau\rangle)^{-1-\gamma}\,(\|u\|_{\mathbf{H}^{2+\varepsilon}} + \|\mathcal{J}\partial_x u\|_{\mathbf{L}^2})^2\,\|\partial_x u\|_{\mathbf{L}^2} \\ &\leq \sqrt{2}\widetilde{\varepsilon} + C\widetilde{\varepsilon}^{\frac{4}{3}} \leq 2\widetilde{\varepsilon}. \end{aligned}$$

Therefore we have the desired contradiction. This completes the proof of Theorem 4.3. □

CHAPTER 5

Nonexistence result of the usual scattering states

1. Main result

In this section,we consider the Cauchy problem (1.1) with $f(u) = |u|^{\rho-1}u$. That is, we consider the Cauchy problem

$$\begin{cases} u_{tx} = u + (|u|^{\rho-1}u)_{xx}, & x \in \mathbb{R},\ t > 0 \\ u(0,x) = u_0(x), & x \in \mathbb{R} \end{cases}. \tag{5.1}$$

Our aim in this section is to prove the nonexistence of the usual scattering states for the Cauchy problem (5.1) with $1 < \rho \leq 3$. That is, we prove that it is impossible to find a solution of (5.1) with $1 < \rho \leq 3$ in the neighborhood of the free solution as $t \to \infty$. Note that the lower bounds were not shown previously which are important for proving the nonexistence of the usual scattering states (See [**6**] and [**23**]). In these papers, finite propagation speed was used to get the lower bound time decay estimate. Since (1.1) does not have finite propagation speed property, we need to apply the sharp asymptotic behavior of solutions. Asymptotic behavior of solutions to the free Schrödinger evolution group is well known and the lower bounds for the solutions can be obtained easily (See [**1**]). The lower bound for the linear combination of different Schrödinger evolution groups was obtained in [**15**] which was used to prove the nonexistence of the usual scattering states for a system of nonlinear Schrödinger equations.

THEOREM 5.1. ([**18**]) *Assume that there exists a solution* $u \in \mathbf{C}(\mathbb{R}; \dot{\mathbf{H}}^{-1} \cap \mathbf{L}^2)$ *of the Cauchy problem (5.1) with* $1 < \rho \leq 3$. *Furthermore, we assume that if* $2 < \rho \leq 3$ *then the time decay estimate*

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C\langle t\rangle^{-\frac{1}{2}}$$

holds. Then, these does not exist any free solution $w(t)$ *of the linear Cauchy problem (5.1) with initial data*

$$\phi \in \mathbf{H}^2 \cap \dot{\mathbf{H}}^{-1}, x\phi_x \in \mathbf{H}^1$$

and some $T > 1$,

$$\left\|\hat{\phi}\right\|_{\mathbf{L}^2(1,T)} + \left\|\hat{\phi}\right\|_{\mathbf{L}^2(-T,-1)} \neq 0$$

such that

$$\lim_{t\to\infty} \|u(t) - w(t)\|_{\dot{\mathbf{H}}^{-1}\cap\mathbf{L}^2} = 0,$$

where $w(t) = \mathcal{U}(t)\phi$.

2. Proof of Theorem 5.1

PROOF. We prove the theorem by a contradiction argument. Suppose that there exists a free solution $w(t) = \mathcal{U}(t)\phi$ of the linear Cauchy problem (5.1) with initial data ϕ and satisfying

$$\lim_{t\to\infty} \|u(t) - w(t)\|_{\mathbf{L}^2\cap\dot{\mathbf{H}}^{-1}} = 0. \tag{5.2}$$

Define the functional

$$H_u(t) = \int_{\mathbb{R}} w(t,x)\partial_x^{-1}u(t,x)dx.$$

as in the papers [**6**] and [**23**]. In view of equation (5.1), we have $\partial_t\mathcal{U}(-t)w(t) = 0$ and $\partial_t\mathcal{U}(-t)\partial_x^{-1}u(t) = \mathcal{U}(-t)\left(|u|^{\rho-1}u\right)$. Also we can represent

$$H_u(t) = \int_{\mathbb{R}} (\mathcal{U}(-t)w(t))(\mathcal{U}(-t)\partial_x^{-1}u(t))dx.$$

Then by a direct calculation we find

$$\begin{aligned}\frac{d}{dt}H_u(t) &= \int_{\mathbb{R}} (\mathcal{U}(-t)w(t))\left(\mathcal{U}(-t)(|u|^{\rho-1}u)\right)dx = \int_{\mathbb{R}} w|u|^{\rho-1}udx \\ &= \int_{\mathbb{R}} |w|^{\rho+1}dx + \int_{\mathbb{R}} (w|u|^{\rho-1}u - |w|^{\rho+1})dx.\end{aligned}$$

For $3 \geq \rho > 2$, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}} (w|u|^{\rho-1}u - w|w|^{\rho-1}w)dx\right| \\ \leq\ & C\|w\|_{\mathbf{L}^\infty}(\|u\|_{\mathbf{L}^2} + \|w\|_{\mathbf{L}^2})(\|w\|_{\mathbf{L}^\infty} + \|u\|_{\mathbf{L}^\infty})^{\rho-2}\|u-w\|_{\mathbf{L}^2} \\ \leq\ & C(A+1)^\rho t^{-\frac{\rho-1}{2}}\|u-w\|_{\mathbf{L}^2},\end{aligned}$$

where $A = \|\phi\|_{\mathbf{H}^1} + \|x\phi_x\|_{\mathbf{H}^1}$. Here we used the estimate $\|w\|_{\mathbf{L}^\infty} \leq Ct^{-1/2}$ and the assumption $\|u\|_{\mathbf{L}^\infty} \leq Ct^{-1/2}$. Next we consider the case $1 < \rho \leq 2$. By the Hölder inequality, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}} (w|u|^{\rho-1}u - w|w|^{\rho-1}w)dx\right| \leq C\|w\|_{\mathbf{L}^{\frac{2}{2-\rho}}}\left\||u|^{\rho-1}u - w|w|^{\rho-1}w\right\|_{\mathbf{L}^{\frac{2}{\rho}}} \\ \leq\ & C\|w\|_{\mathbf{L}^{\frac{2}{2-\rho}}}(\|u\|_{\mathbf{L}^2} + \|w\|_{\mathbf{L}^2})^{\rho-1}\|u-w\|_{\mathbf{L}^2} \leq C(A+1)^\rho t^{-\frac{\rho-1}{2}}\|u-w\|_{\mathbf{L}^2}.\end{aligned}$$

Then by Lemma 2.4, we estimate

$$\begin{aligned}\frac{d}{dt}H_u(t) &\geq \int_{\mathbb{R}} |w|^{\rho+1}dx - C(A+1)^\rho t^{-\frac{\rho-1}{2}}\|u-w\|_{\mathbf{L}^2} \\ &\geq 2^{-\rho-1}t^{-\frac{\rho-1}{2}}(\|\hat{\phi}\|_{\mathbf{L}^2(1,\sqrt{T})} + \|\hat{\phi}\|_{\mathbf{L}^2(-\sqrt{T},-1)})^{\rho+1} \\ &\quad -CA^{\rho+1}t^{-\frac{\rho-1}{2}-\frac{1-\alpha}{4}(\rho+1)} - C(A+1)^\rho t^{-\frac{\rho-1}{2}}\|u-w\|_{\mathbf{L}^2}.\end{aligned}$$

By the assumption of Theorem 5.1, there exists $T > 1$ such that $\|u(t) - w(t)\|_{\mathbf{L}^2} < \epsilon$ for all $t \geq T$ with $\epsilon > 0$ such that

$$C(A+1)^\rho \leq 2^{-\rho-1}(\|\hat{\phi}\|_{\mathbf{L}^2(1,\sqrt{T})} + \|\hat{\phi}\|_{\mathbf{L}^2(-\sqrt{T},-1)})^{\rho+1}.$$

Hence, by integration with respect to t, we have

$$|H_u(2T) - H_u(T)| \geq C \int_T^{2T} t^{-\frac{\rho-1}{2}} dt \geq CT^{\frac{3-\rho}{2}}, \tag{5.3}$$

for large T. On the other hand, by the definition of $H_u(t)$ and (5.2), we find

$$\begin{aligned} H_u(t) &= \int_{\mathbb{R}} w \partial_x^{-1}(u-w) dx \leq C \|w(t)\|_{\mathbf{L}^2} \left\| \partial_x^{-1}(u(t) - w(t)) \right\|_{\mathbf{L}^2} \\ &\leq C \|u_0\|_{\mathbf{L}^2} \left\| \partial_x^{-1}(u(t) - w(t)) \right\|_{\mathbf{L}^2} \to 0, \end{aligned} \tag{5.4}$$

for $t \to \infty$. From (5.3) and (5.4), we obtain a desired contradiction. This completes the proof of Theorem 5.1. □

CHAPTER 6

Asymptotics of solutions to the short pulse equation with critical nonlinearity

1. Main result

In this section, we consider the Cauchy problem

$$\left\{ \begin{array}{ll} u_{tx} = u + (u^3)_{xx}, & x \in \mathbb{R},\ t > 0 \\ u(0,x) = u_0(x), & x \in \mathbb{R} \end{array} \right. . \tag{6.1}$$

The result of section 5 says that there is no asymptotically free solution for the short pulse equation. However, it is not clear how solutions behave in large time. So, our aim in this section is to show the asymptotic behavior of solutions to (6.1) under the smallness condition on the initial data.

Scattering problems for nonlinear dispersive equations in one dimension with critical nonlinearity were intensively studied by many authors. Hayashi and Naumkin showed the asymptotic behavior of solutions to cubic nonlinear Schrödinger equation [**12**], Hartree equations [**13**], modified KdV equation [**9**], and cubic Benjamin-Ono equation [**11**]. Delort studies the scattering problem for the quadratic and cubic Klein-Gordon equation [**4**]. Recently, Ionescu and Pusateri consider the scattering problem for nonlinear fractional Schrödinger equations [**20**]. Even if the initial data is sufficiently small, these equations are not tend to free solutions as $t \to \infty$. In fact, to find their asymptotics, we need a suitable phase correction on free solutions.

To state our result, we introduce the function spaces

$$\mathbf{X}_0^m = \left\{ \phi \in \mathbf{L}^2; \|\phi\|_{\mathbf{X}_0^m} = \|\phi\|_{\mathbf{H}^m} + \|x\phi_x\|_{\mathbf{H}^5} + \|\phi\|_{\dot{\mathbf{H}}^{-1}} < \infty \right\}$$

and

$$\mathbf{X}_T^m = \left\{ u(t) \in \mathbf{C}\left([0,T); \mathbf{L}^2\right); \|u\|_{\mathbf{X}_T^m} < \infty \right\},$$

equipped with the norm

$$\|u\|_{\mathbf{X}_T^m} = \sup_{t\in[0,T)} \langle t\rangle^{-\epsilon^{\frac{1}{7}}} \left(\|u(t)\|_{\mathbf{H}^m} + \|\mathcal{J}u_x(t)\|_{\mathbf{H}^5} + \|u(t)\|_{\dot{\mathbf{H}}^{-1}}\right) + \sup_{t\in[0,T)} \langle t\rangle^{\frac{1}{2}} \|u(t)\|_{\mathbf{H}^2_\infty},$$

where $\epsilon > 0$ is small. We are now in a position to state our result.

THEOREM 6.1. ([**25**]) *Let the initial data $u_0 \in \mathbf{X}_0^m$ and $m > 10$. Assume that $\|u_0\|_{\mathbf{X}_0^m} \le \epsilon$ and $\epsilon > 0$ is sufficiently small. Then there exists a unique global solution $u \in \mathbf{X}_\infty^m$ of (6.1) such that*

$$\|u(t)\|_{\mathbf{H}^2_\infty} \le C\langle t\rangle^{-\frac{1}{2}}.$$

Moreover for any $u_0 \in \mathbf{X}_0^m$ *there exists a unique function* $W \in \mathbf{H}_\infty^{0,2}$ *such that the following asymptotics is valid for large* $t \geq 1$ *uniformly with respect to* $x \in \mathbb{R}$*:*

$$(6.2) \quad u(t) = \Re\sqrt{\frac{2}{t}}\theta(x)W(\chi)\exp\left(-i\left(\frac{2t}{\chi}+\frac{\pi}{4}+\frac{3\chi}{\sqrt{2}}|W(\chi)|^2\log t\right)\right) +O\left(t^{-\frac{1}{2}-\delta}\right),$$

where $\delta \in (0,1/12)$*,* $\chi = \sqrt{t/-x}$*,* $\theta(x) = 1$ *when* $x < 0$ *and* $\theta(x) = 0$ *when* $0 \leq x$*.*

REMARK 6.1. *The assumption* $\partial_x^{-1}u_0 \in \mathbf{L}^2$ *is crucial to our proof since we use the operator* $\mathcal{J}\partial_x = x\partial_x - t\partial_x^{-1}$*. The author do not know whether the assumption can be removed or not. This problem is more challenging.*

We now state our strategy of the proof of Theorem 6.1. According to Proposition 3.1, there exists local solutions to (6.1) in the function space $\mathbf{X}_T^m$. Thus, to obtain the global existence theorem, we need to show the a-priori estimate of local solutions in the norm $\|\cdot\|_{\mathbf{X}_T^m}$. However, it seems to be difficult to get the estimate $\langle t\rangle^{1/2}\|u(t)\|_{\mathbf{L}^\infty} \leq C$ as in the same manner of [**17**], since the order of the nonlinearity of (6.1) is critical. So we need another approach to get the $\mathbf{L}^\infty$ estimate. In this paper, we adopt the method closely related in papers [**9**], [**11**] and [**12**] by Hayashi and Naumkin. Papers [**9**] and [**11**] deal with the scattering problem for dispersive equations with critical nonlinearity :

$$\begin{cases} u_t + (-\partial_x^2)^{\frac{p-1}{2}}u_x = (u^3)_x, & x \in \mathbb{R}, \quad t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where $\left(-\partial_x^2\right)^{m/2} = \mathcal{F}^{-1}|\xi|^m\mathcal{F}$. More precisely, it was shown the scattering result for modified KdV equation ($p=3$) for small initial data $u_0 \in \mathbf{H}^{1,1}$ in [**9**]. While it was shown the scattering result for Benjamin-Ono equation ($p=2$) for small initial data $u_0 \in \mathbf{H}^{1,2} \cap \mathbf{H}^3$ in [**11**]. From these results, if p becomes smaller, then we need more regularity on the initial data to obtain the scattering result by using the method [**9**], [**11**] and [**12**] directly. Since our equation (6.1) is the case of $p=-1$, we need more regularity on the initial data than [**9**] and [**11**], and we need to construct solutions in corresponding weighted Sobolev space. However, as pointed out in [**17**], it seems to be difficult to construct solutions in Sobolev space with $\langle x\rangle^2$. Hence, we cannot apply the operator $\mathcal{P} = t\partial_t - x\partial_x$ twice to the equation (6.1) since $\mathcal{P}^2 = (x\partial_x - t\partial_t)^2$ and ∂_t^2 is similar to the anti-derivatives ∂_x^{-2} in the short pulse case. Note that this difficulty does not arise in previous works such as [**11**] and [**12**] . That is, there is no difficulty to use the operator $\mathcal{P}$ twice in these cases since the linear part of these equations do not have an anti-derivative. To avoid the difficulty, we use Lemma 2.7 which was the main tool in the proof of Theorem 4.1. In the same reason, we also need to avoid using two weights in our proof (See Lemma 6.5). Thanks to these lemmas, we can avoid using the operator $\mathcal{P}$ twice.

2. A priori estimate

In order to obtain global solutions to (6.1), we should show the a-priori estimate of solutions to (6.1) in the norm $\|\cdot\|_{\mathbf{X}_T^m}$. In the following lemma, we give the a-priori estimate of $\|u(t)\|_{\mathbf{H}^m}$, $\|u(t)\|_{\dot{\mathbf{H}}^{-1}}$ and $\|\mathcal{J}u_x(t)\|_{\mathbf{H}^5}$.

Lemma 6.1. ([**25**]) *Let u be a solution obtained by Proposition 3.1 with $m > 10$. Then, the following estimate*

$$\langle t\rangle^{-\epsilon^{\frac{1}{7}}}\left(\|u(t)\|_{\mathbf{H}^m}+\|u(t)\|_{\dot{\mathbf{H}}^{-1}}+\|\mathcal{J}u_x(t)\|_{\mathbf{H}^5}\right)\le C\epsilon,$$

is valid for any $t\in[0,T]$.

Proof. By Lemma 2.7, we have

$$\frac{d}{dt}\|u(t)\|_{\mathbf{H}^m}\le C\left(\|u(t)\|_{\mathbf{L}^\infty}+\|u_x(t)\|_{\mathbf{L}^\infty}\right)^2\|u(t)\|_{\mathbf{H}^m}\le C\epsilon\langle t\rangle^{-1}\|u(t)\|_{\mathbf{H}^m}.$$

Whence, by the Gronwall inequality, we get

$$\|u(t)\|_{\mathbf{H}^m}\le C\epsilon\langle t\rangle^{C\epsilon}\le C\epsilon\langle t\rangle^{\epsilon^{\frac{1}{2}}}.$$

To get the a-priori estimate of $\left\|\partial_x^{-1}u(t)\right\|_{\mathbf{L}^2}$, we define $\chi_n\in\mathbf{S}$ such that $0\le\widehat{\chi_n}(\xi)\le 1$ and $\widehat{\chi_n}(\xi)=\begin{cases}1 & 2^{-n+1}\le|\xi|\le 2^n\\ 0 & 0\le|\xi|\le 2^{-n},\ 2^{n+1}\le|\xi|\end{cases}$. Multiplying (6.1) by $\chi_n *$, we get

$$(\chi_n * u_t+\chi_n*(u^3)_x)_x=\chi_n*u, \tag{6.3}$$

where $*$ is the convolution. Multiplying ∂_x^{-2} both side of (6.3) and taking the dot product with $\partial_x^{-1}(\chi_n*u)$, we have

$$(\partial_x^{-1}(\chi_n*u_t),\partial_x^{-1}(\chi_n*u))+(\chi_n*u^3,\partial_x^{-1}(\chi_n*u))=(\partial_x^{-2}(\chi_n*u),\partial_x^{-1}(\chi_n*u)).$$

Note that $(\partial_x^{-2}(\chi_n*u),\partial_x^{-1}(\chi_n*u))=0$. Then, letting $n\to\infty$ and integrating with respect to t, we have

$$\left\|\partial_x^{-1}u(t)\right\|_{\mathbf{L}^2}\le\left\|\partial_x^{-1}u_0\right\|_{\mathbf{L}^2}+\int_0^t\|u(t)\|_{\mathbf{L}^\infty}^2\|u(t)\|_{\mathbf{L}^2}\,d\tau\le\epsilon+C\epsilon^3\int_0^t\langle\tau\rangle^{-1+\epsilon^{\frac{1}{2}}}d\tau\le C\epsilon\langle t\rangle^{\epsilon^{\frac{1}{2}}}.$$

We next consider the estimate of $\|\mathcal{P}u(t)\|_{\mathbf{H}^1}$. Multiplying $\mathcal{P}=x\partial_x-t\partial_t$ both side of (1.1), we have

$$\mathcal{P}u=\left((\mathcal{P}u)_t+\left(u^3\right)_x-\mathcal{P}\left(u^3\right)_x\right)_x. \tag{6.4}$$

Multiplying (6.4) by $\partial_x^{-1}\chi_n*$ and taking the dot product with $\chi_n*\mathcal{P}u$, we have

$$\frac{d}{dt}\|\chi_n*\mathcal{P}u\|_{\mathbf{L}^2}\le C\left\|\chi_n*(u^3)_x\right\|_{\mathbf{L}^2}+C\left\|\chi_n*\mathcal{P}(u^3)_x\right\|_{\mathbf{L}^2},$$

since $(\partial_x^{-1}(\chi_n*\mathcal{P}u),\chi_n*\mathcal{P}u)=0$. Letting $n\to\infty$, we have

$$\begin{aligned}\frac{d}{dt}\|\mathcal{P}u(t)\|_{\mathbf{L}^2}&\le C\|u(t)\|_{\mathbf{L}^\infty}^2\|u_x(t)\|_{\mathbf{L}^2}+C\|u(t)\|_{\mathbf{H}^1_\infty}^2\|\mathcal{P}u(t)\|_{\mathbf{H}^1}\\&\le C\epsilon\langle t\rangle^{-1+\epsilon^{\frac{1}{2}}}+C\epsilon\langle t\rangle^{-1}\|\mathcal{P}u(t)\|_{\mathbf{H}^1}.\end{aligned} \tag{6.5}$$

We next consider the norm $\|(\mathcal{P}u(t))_x\|_{\mathbf{L}^2}$. Applying the classical energy method to (6.4), we have

$$\begin{aligned}\frac{d}{dt}\|(\mathcal{P}u(t))_x\|_{\mathbf{L}^2}&\le C\|u(t)\|_{\mathbf{L}^\infty}^2\|u(t)\|_{\mathbf{H}^2}+C\|u(t)\|_{\mathbf{H}^2_\infty}^2\|\mathcal{P}u(t)\|_{\mathbf{H}^1}\\&\le C\epsilon\langle t\rangle^{-1+\epsilon^{\frac{1}{2}}}+C\epsilon\langle t\rangle^{-1}\|\mathcal{P}u(t)\|_{\mathbf{H}^1}.\end{aligned} \tag{6.6}$$

By (6.5) and (6.6), we have

$$\frac{d}{dt}\|\mathcal{P}u(t)\|_{\mathbf{H}^1}\le C\epsilon\langle t\rangle^{-1+\epsilon^{\frac{1}{2}}}+C\epsilon\langle t\rangle^{-1}\|\mathcal{P}u(t)\|_{\mathbf{H}^1}.$$

Whence, by the Gronwall inequality, we obtain

$$\|\mathcal{P}u(t)\|_{\mathbf{H}^1} \leq C\epsilon\langle t\rangle^{C\epsilon} + C\epsilon\langle t\rangle^{\epsilon^{\frac{1}{2}}+C\epsilon} \leq C\epsilon\langle t\rangle^{\epsilon^{\frac{1}{3}}}.$$

It follows from the identity $\mathcal{P} = \mathcal{J}\partial_x + t\mathcal{L}$ that

$$\|\mathcal{J}u_x(t)\|_{\mathbf{H}^1} \leq \|\mathcal{P}u(t)\|_{\mathbf{H}^1} + t\,\|\mathcal{L}u(t)\|_{\mathbf{H}^1} \leq C\epsilon\langle t\rangle^{\epsilon^{\frac{1}{3}}}.$$

We next give the estimate of $\|(\mathcal{J}u_x)_{xx}(t)\|_{\mathbf{L}^2}$. Note that the following estimate

$$\|u(t)\|_{\mathbf{H}^{k+1}_\infty} \leq Ct^{-\frac{1}{2}}\left(\|u(t)\|_{\mathbf{H}^{m+k-5}} + \|\mathcal{J}u_x(t)\|_{\mathbf{H}^{k-1}}\right) \tag{6.7}$$

is valid for $1 \leq k$, because of Lemma 2.7 with $l = 2$, $\delta = m - 10$ and $\phi = \langle i\partial_x\rangle^{k-1}u$. Then, applying the energy method to (6.4), we have

$$\begin{aligned}\frac{d}{dt}\|(\mathcal{P}u(t))_{xx}\|_{\mathbf{L}^2} &\leq C\,\|u(t)\|^2_{\mathbf{H}^3_\infty}\,\|\mathcal{P}u(t)\|_{\mathbf{H}^1} + C\,\|u(t)\|^2_{\mathbf{H}^1_\infty}\,\|(\mathcal{P}u(t))_{xx}\|_{\mathbf{L}^2}\\ &+C\,\|u(t)\|^2_{\mathbf{L}^\infty}\,\|u(t)\|_{\mathbf{H}^3} \leq C\epsilon\langle t\rangle^{-1+3\epsilon^{\frac{1}{3}}} + C\epsilon\langle t\rangle^{-1}\,\|(\mathcal{P}u(t))_{xx}\|_{\mathbf{L}^2},\end{aligned}$$

by (6.7) with $k = 2$. Whence, by the Gronwall inequality, we have

$$\|(\mathcal{P}u(t))_{xx}\|_{\mathbf{L}^2} \leq C\epsilon\langle t\rangle^{C\epsilon} + C\epsilon\langle t\rangle^{3\epsilon^{\frac{1}{3}}+C\epsilon} \leq C\epsilon\langle t\rangle^{\epsilon^{\frac{1}{4}}}.$$

Thus,

$$\|(\mathcal{J}u_x(t))_{xx}\|_{\mathbf{L}^2} \leq \|(\mathcal{P}u(t))_{xx}\|_{\mathbf{L}^2} + t\,\|(\mathcal{L}u(t))_{xx}\|_{\mathbf{L}^2} \leq C\epsilon\langle t\rangle^{\epsilon^{\frac{1}{4}}}.$$

By using the above argument repeatedly, we can get $\|(\mathcal{J}u_x(t))_{xxx}\|_{\mathbf{L}^2} \leq C\epsilon\langle t\rangle^{\epsilon^{1/5}}$ because of (6.7) with $k = 3$, and then, $\|(\mathcal{J}u_x(t))_{xxxx}\|_{\mathbf{L}^2} \leq C\epsilon\langle t\rangle^{\epsilon^{1/6}}$ because of (6.7) with $k = 4$. Finally, we get $\|(\mathcal{J}u_x(t))_{xxxxx}\|_{\mathbf{L}^2} \leq C\epsilon\langle t\rangle^{\epsilon^{1/7}}$ because of (6.7) with $k = 5$. Therefore, we obtain the desired estimate. □

We next give the $\mathbf{L}^\infty$ a-priori estimate of solutions to (1.1).

LEMMA 6.2. ([**25**]) *Let u be a solution obtained by Proposition 3.1 with $m > 10$. Then, the following estimate*

$$\langle t\rangle^{\frac{1}{2}}\,\|u(t)\|_{\mathbf{H}^2_\infty} \leq C\epsilon \tag{6.8}$$

is valid for any $t \in [0, T]$.

PROOF. By Lemma 2.6, we have for $j = 0, 1, 2$,

$$\begin{aligned}\left\|\partial_x^j u(t)\right\|_{\mathbf{L}^\infty} &= \left\|\mathcal{U}(t)\mathcal{U}(-t)\partial_x^j u(t)\right\|_{\mathbf{L}^\infty}\\ &\leq t^{-\frac{1}{2}}\left\||\xi|^{\frac{3}{2}+j}\,\mathcal{F}\mathcal{U}(-t)u(t)\right\|_{\mathbf{L}^\infty} + t^{-\frac{1}{2}-\delta}\left(\|\mathcal{J}u_x(t)\|_{\mathbf{H}^{1+2\delta+j}} + \|u(t)\|_{\mathbf{H}^{\frac{3}{2}+\delta+j}}\right).\end{aligned} \tag{6.9}$$

Thus, to get the a-priori estimate (6.8), we should show the estimate of $\left\||\xi|^{\frac{3}{2}+j}\,\mathcal{F}\mathcal{U}(-t)u(t)\right\|_{\mathbf{L}^\infty}$. Multiplying $\mathcal{U}(-t)$ both side of (6.1), we have

$$(\mathcal{U}(-t)u)_t = \mathcal{U}(-t)\left(u^3\right)_x. \tag{6.10}$$

Put $v = \mathcal{U}(-t)u$, and then, apply Fourier transform both side of (6.10) to get

$$\hat{v}_t(t,\xi) = \frac{i\xi}{2\pi}\iint_{\mathbb{R}^2} e^{-it\left(\frac{1}{\xi}-\frac{1}{\xi_1}-\frac{1}{\xi_2}-\frac{1}{\xi-\xi_1-\xi_2}\right)}\hat{v}(t,\xi_1)\hat{v}(t,\xi_2)\hat{v}(t,\xi-\xi_1-\xi_2)d\xi_1 d\xi_2.$$

Changing the variables of integration $\xi_1 = \xi\xi_1'$ and $\xi_2 = \xi\xi_2'$ (we omit the prime) and multiplying $|\xi|^{3/2+j}$, we have

$$|\xi|^{\frac{3}{2}+j}\,\hat{v}_t(t,\xi) = \frac{i\xi^3}{2\pi}\,|\xi|^{\frac{3}{2}+j}\iint_{\mathbb{R}^2} e^{-i\frac{t}{\xi}S}\hat{v}(t,\xi\xi_1)\hat{v}(t,\xi\xi_2)\hat{v}(t,\xi\xi_3)d\xi_1 d\xi_2,$$

where $\xi_3 = 1-\xi_1-\xi_2$, $S = 1-\frac{1}{\xi_1}-\frac{1}{\xi_2}-\frac{1}{\xi_3}$ and $j=0,1,2$. Here we use the fact that the following asymptotic expansion of the above integral is valid :

$$\begin{aligned} |\xi|^{\frac{3}{2}+j}\hat{v}_t(t,\xi) &= i\frac{3\xi^4|\xi|^{\frac{3}{2}+j}}{\sqrt{2}t}\,|\hat{v}(t,\xi)|^2\,\hat{v}(t,\xi) + \frac{\xi^4|\xi|^{\frac{3}{2}+j}}{3^3\sqrt{6}t}e^{i\frac{11t}{\xi}}\hat{v}\left(t,\frac{\xi}{3}\right)^3 \\ &\quad +O\left(t^{-1-\delta}\left(\|u(t)\|_{\mathbf{H}^6}+\|u(t)\|_{\dot{\mathbf{H}}^{-1}}+\|\mathcal{J}u_x(t)\|_{\mathbf{H}^5}\right)^3\right), \end{aligned} \tag{6.11}$$

for $0<\delta<1/12$. The proof of the asymptotic expansion stated in above is slightly technical. So we now continue the proof of Lemma 6.2, and we will prove the asymptotic expansion (6.11) in Section 6.3. To eliminate the first term, put

$$w_j(t,\xi) = |\xi|^{\frac{3}{2}+j}\hat{v}(t,\xi)e^{-i\frac{3\xi^4}{\sqrt{2}}\int_1^t\frac{|\hat{v}(\tau,\xi)|^2}{\tau}d\tau} \tag{6.12}$$

and $A(\tau) = \exp(-i\frac{3\xi^4}{\sqrt{2}}\int_1^t\frac{|\hat{v}(\tau,\xi)|^2}{\tau}d\tau)$. Then, we have

$$(w_j(t,\xi))_t = \frac{\xi^4|\xi|^{\frac{3}{2}+j}}{3^3\sqrt{6}t}e^{i\frac{11t}{\xi}}A(t)\hat{v}\left(t,\frac{\xi}{3}\right)^3 + O\left(\epsilon t^{-1-\delta+3\epsilon^{\frac{1}{7}}}\right). \tag{6.13}$$

Integrating (6.13) with respect to t, we get

(6.14)

$$|w_j(t)| \le |w_j(1)| + \left|\int_1^t \frac{\xi^4|\xi|^{\frac{3}{2}+j}}{3^3\sqrt{6}t}e^{i\frac{11t}{\xi}}A(\tau)\hat{v}\left(t,\frac{\xi}{3}\right)^3 d\tau\right| + C\epsilon\int_1^t\tau^{-1-\delta+3\epsilon^{\frac{1}{7}}}d\tau,$$

by Lemma 6.1. Note that

$$e^{i\frac{11t}{\xi}} = \frac{1}{1+i\frac{11t}{\xi}}\frac{d}{dt}\left(te^{i\frac{11t}{\xi}}\right),$$

then, by integration by parts, the second term on the right hand side of (6.14) is bounded by

$$\begin{aligned} &(6.15)|\xi|^{\frac{11}{2}+j}\left|\int_1^t\frac{d}{d\tau}\left(\tau e^{i\frac{11\tau}{\xi}}\right)\left(\frac{A(\tau)\hat{v}\left(\tau,\frac{\xi}{3}\right)^3}{\tau\left(1+i\frac{11\tau}{\xi}\right)}\right)d\tau\right| \\ \le\ & C|\xi|^{\frac{11}{2}+j}\left|\left[e^{i\frac{11\tau}{\xi}}\frac{A(\tau)\hat{v}\left(\tau,\frac{\xi}{3}\right)^3}{1+i\frac{11\tau}{\xi}}\right]_{\tau=1}^t\right| + C|\xi|^{\frac{11}{2}+j}\left|\int_1^t e^{i\frac{11\tau}{\xi}}\frac{A(\tau)\hat{v}\left(\tau,\frac{\xi}{3}\right)^3}{1+i\frac{11\tau}{\xi}}\frac{d\tau}{\tau}\right| \\ &+C|\xi|^{\frac{9}{2}+j}\left|\int_1^t e^{i\frac{11\tau}{\xi}}\frac{A(\tau)\hat{v}\left(\tau,\frac{\xi}{3}\right)^3}{\left(1+i\frac{11\tau}{\xi}\right)^2}d\tau\right| + C|\xi|^{\frac{11}{2}+j}\left|\int_1^t e^{i\frac{11\tau}{\xi}}\frac{A'(\tau)\hat{v}\left(\tau,\frac{\xi}{3}\right)^3}{1+i\frac{11\tau}{\xi}}d\tau\right| \\ &+C|\xi|^{\frac{11}{2}+j}\left|\int_1^t e^{i\frac{11\tau}{\xi}}\frac{A(\tau)\hat{v}\left(\tau,\frac{\xi}{3}\right)^2\hat{v}_t\left(\tau,\frac{\xi}{3}\right)}{1+i\frac{11\tau}{\xi}}d\tau\right|. \end{aligned}$$

To estimate the right hand side of (6.15), we note that

$$\|\xi^a \hat{v}(\tau)\|_{\mathbf{L}^\infty} \le \|u(\tau)\|_{\mathbf{H}^a} + \|\mathcal{J} u_x(\tau)\|_{\mathbf{H}^{a-1}} \le C\epsilon \langle\tau\rangle^{\epsilon^{\frac{1}{7}}}, \tag{6.16}$$

by (6.35) and Lemma 6.1. Also, we note that

$$\begin{aligned} \left\| |\xi|^{\frac{3}{2}+j} \hat{v}_t(\tau) \right\|_{\mathbf{L}^\infty} &\le C\tau^{-1} \left\| |\xi|^{\frac{11}{6}+\frac{j}{3}} \hat{v}(\tau) \right\|^3_{\mathbf{L}^\infty} + C\epsilon\tau^{-1-\delta+3\epsilon^{\frac{1}{7}}} \\ &\le C\tau^{-1}(\|u(\tau)\|_{\mathbf{H}^{\frac{11}{6}+\frac{j}{3}}} + \|\mathcal{J}u_x(\tau)\|_{\mathbf{H}^{\frac{5}{6}+\frac{j}{3}}})^3 + C\epsilon\tau^{-1-\delta+3\epsilon^{\frac{1}{7}}} \le C\epsilon\tau^{-1+3\epsilon^{\frac{1}{7}}}, \end{aligned}$$

by (6.11), Lemma 6.1 and (6.16). Since $|(1+i11\tau/\xi)|^{-1} \le C|\tau|^{-\delta}|\xi|^\delta$ and $|A'(\tau)| \le C\tau^{-1}|\xi|^4|\hat{v}(\tau)|^2$, the right hand side of (6.15) is bounded by

$$\begin{aligned} & t^{-1} \left\| |\xi|^{\frac{13}{6}+\frac{j}{3}} \hat{v}(t) \right\|^3_{\mathbf{L}^\infty} + \left\| |\xi|^{\frac{11}{6}+\frac{j}{3}} \hat{v}(1) \right\|^3_{\mathbf{L}^\infty} + \int_1^t \tau^{-\delta-1} \left\| |\xi|^{\frac{11}{6}+\frac{\delta+j}{3}} \hat{v}(\tau) \right\|^3_{\mathbf{L}^\infty} d\tau \\ &+ \int_1^t \tau^{-2\delta-1} \left\| |\xi|^{\frac{3}{2}+\frac{2\delta+j}{3}} \hat{v}(\tau) \right\|^3_{\mathbf{L}^\infty} d\tau + \int_1^t \tau^{-\delta-1} \left\| |\xi|^{\frac{19}{10}+\frac{\delta+j}{5}} \hat{v}(\tau) \right\|^5_{\mathbf{L}^\infty} d\tau \\ &+ \int_1^t \tau^{-\delta} \left\| |\xi|^{2+\frac{\delta}{2}} \hat{v}(\tau) \right\|^2_{\mathbf{L}^\infty} \left\| |\xi|^{\frac{3}{2}+j} \hat{v}_t(\tau) \right\|_{\mathbf{L}^\infty} d\tau \\ \le\ & C\epsilon + C\epsilon \int_1^t t^{-1-\delta+5\epsilon^{\frac{1}{7}}} d\tau \le C\epsilon, \end{aligned} \tag{6.17}$$

where we chose $5\epsilon^{1/7} < \delta < 1/12$. Therefore, by (6.14) and (6.17), we have

$$\left\| |\xi|^{\frac{3}{2}+j} \mathcal{F}\mathcal{U}(-t)u(t) \right\|_{\mathbf{L}^\infty} = \|w_j(t)\|_{\mathbf{L}^\infty} \le C\epsilon. \tag{6.18}$$

Applying (6.18) to (6.9), we obtain

$$\left\| \partial_x^j u(t) \right\|_{\mathbf{L}^\infty} \le C\epsilon t^{-\frac{1}{2}} + C\epsilon t^{-\frac{1}{2}-\delta+\epsilon^{\frac{1}{7}}} \le C\epsilon\langle t\rangle^{-\frac{1}{2}}.$$

This completes the proof of Lemma 6.2. □

3. Proof of Theorem 6.1

PROOF. By a contradiction argument, we shall prove that $\|u\|_{\mathbf{X}^m_T} < \sqrt{\epsilon}$ for any $T > 0$. Assume that there exists $T_0 > 0$ such that $\|u\|_{\mathbf{X}^m_{T_0}} = \sqrt{\epsilon}$. On the other hand, we have $\|u\|_{\mathbf{X}^m_{T_0}} \le C\epsilon$ because of Lemma 6.1 and Lemma 6.2. If ϵ is sufficiently small so that $C\epsilon < \sqrt{\epsilon}$, then we arrive at the contradiction. Thus, we obtain a unique global solution $u \in \mathbf{X}^m_\infty$ satisfying $\|u(t)\|_{\mathbf{H}^2_\infty} \le C\langle t\rangle^{-\frac{1}{2}}$. We next prove the last statement of Theorem 6.1. By Lemma 2.6 and (6.12), the solution u can be represented as

$$\begin{aligned} u(t) &= \mathcal{U}(t)\mathcal{U}(-t)u(t) = \Re t^{-\frac{1}{2}}\sqrt{2}\theta(x) e^{-i\left(\frac{2t}{\chi}+\frac{\pi}{4}\right)} \chi^{\frac{3}{2}} v(t,\chi) \\ &\quad + O\left(t^{-\frac{1}{2}-\delta}\left(\|\mathcal{J}u_x(t)\|_{\mathbf{H}^{1+2\delta}} + \|u(t)\|_{\mathbf{H}^{\frac{3}{2}+\delta}}\right)\right) \\ &= t^{-\frac{1}{2}}\sqrt{2}\Re\theta(x) e^{-i\left(\frac{2t}{\chi}+\frac{\pi}{4}-\frac{3\chi}{\sqrt{2}}\int_1^t \frac{|w_0(\tau,\chi)|^2}{\tau} d\tau\right)} w_0(t,\chi) \\ &\quad + O\left(t^{-\frac{1}{2}-\delta}\left(\|\mathcal{J}u_x(t)\|_{\mathbf{H}^{1+2\delta}} + \|u(t)\|_{\mathbf{H}^{\frac{3}{2}+\delta}}\right)\right). \end{aligned} \tag{6.19}$$

Thus, we need to consider w_0 to get the asymptotic expansion (6.2). Integrating (6.13) from s to t and applying the same manner as in the proof of (6.17), we get

$$\|w_j(t) - w_j(s)\|_{\mathbf{L}^\infty} \le C\epsilon s^{-1+3\epsilon^{\frac{1}{7}}} + C\epsilon \int_s^t \tau^{-1-\delta+5\epsilon^{\frac{1}{7}}} d\tau \le C\epsilon s^{-\delta+5\epsilon^{\frac{1}{7}}}, \tag{6.20}$$

for $j = 0,1,2$ where $5\epsilon^{1/7} < \delta < 1/12$. Thus, we find that $\{w_0(t)\}$ is a Cauchy sequence in $\mathbf{H}^{0,2}_\infty$. Hence, there exists $\widetilde{W} \in \mathbf{H}^{0,2}_\infty$ such that $\left\|\widetilde{W} - w_0(t)\right\|_{\mathbf{H}^{0,2}_\infty} \to 0$ as $t \to \infty$. Put

$$\psi(t) = \xi \int_1^t \left(|w_0(t,\xi)|^2 - |w_0(\tau,\xi)|^2\right) \frac{d\tau}{\tau},$$

then, we get for $t > s$,

$$\begin{aligned}
&\|\psi(t) - \psi(s)\|_{\mathbf{L}^\infty} \\
= &\left\|\xi \int_s^t \left(|w_0(t)|^2 - |w_0(\tau)|^2\right) \frac{d\tau}{\tau} + \xi \left(|w_0(t)|^2 - |w_0(s)|^2\right) \int_1^s \frac{d\tau}{\tau}\right\|_{\mathbf{L}^\infty} \\
\le &\int_s^t \|w_0(t) - w_0(\tau)\|_{\mathbf{L}^\infty} \left(\|w_1(t)\|_{\mathbf{L}^\infty} + \|w_1(\tau)\|_{\mathbf{L}^\infty}\right) \frac{d\tau}{\tau} \\
&+ \|w_0(t) - w_0(s)\|_{\mathbf{L}^\infty} \left(\|w_1(t)\|_{\mathbf{L}^\infty} + \|w_1(s)\|_{\mathbf{L}^\infty}\right) \log s \\
\le &C\epsilon \int_s^t \tau^{-1-\delta+5\epsilon^{\frac{1}{7}}} d\tau + C\epsilon s^{-\delta+5\epsilon^{\frac{1}{7}}} \log s \le C\epsilon s^{-\delta+5\epsilon^{\frac{1}{7}}} \log s,
\end{aligned} \tag{6.21}$$

because of (6.20). It follows that $\|\psi(t) - \psi(s)\|_{\mathbf{L}^\infty} \to 0$ as $s \to \infty$. Thus, there exists the real valued function $\Psi \in \mathbf{L}^\infty$ such that $\|\Psi - \psi(t)\|_{\mathbf{L}^\infty} \to 0$ as $t \to \infty$. Since

$$\begin{aligned}
\xi \int_1^t \frac{|w_0(\tau,\xi)|^2}{\tau} d\tau &= \xi \left|\widetilde{W}(\xi)\right|^2 \int_1^t \frac{d\tau}{\tau} + \xi \int_1^t \left(|w_0(t,\xi)|^2 - \left|\widetilde{W}(\xi)\right|^2\right) \frac{d\tau}{\tau} \\
&\quad - (\psi(t,\xi) - \Psi(\xi)) - \Psi(\xi),
\end{aligned}$$

we have

$$\begin{aligned}
&\left\|\xi \int_1^t \frac{|w_0(\tau)|^2}{\tau} d\tau - \xi \left|\widetilde{W}\right|^2 \log t + \Psi\right\|_{\mathbf{L}^\infty} \\
\le &\|\psi(t) - \Psi\|_{\mathbf{L}^\infty} + \left\|w_0(t) - \widetilde{W}\right\|_{\mathbf{L}^\infty} \left(\|w_1(t)\|_{\mathbf{L}^\infty} + \left\|\xi\widetilde{W}\right\|_{\mathbf{L}^\infty}\right) \int_1^t \frac{d\tau}{\tau} \\
\le &C\epsilon t^{-\delta+5\epsilon^{\frac{1}{7}}} \log t,
\end{aligned} \tag{6.22}$$

because of (6.20) and (6.21). Therefore, by (6.19) and (6.22), we have

$$\begin{aligned}
&\left\|u(t) - \sqrt{2}\Re\theta(x)\widetilde{W}(\chi)\, e^{\left(-i\left(\frac{2t}{\chi} + \frac{\pi}{4} + \frac{3\chi}{\sqrt{2}}\left|\widetilde{W}(\chi)\right|^2 \log t - \frac{3}{\sqrt{2}}\Psi(\chi)\right)\right)}\right\|_{\mathbf{L}^\infty} \\
\le\ &Ct^{-\frac{1}{2}} \left\|w_0 - \widetilde{W}\right\|_{\mathbf{L}^\infty} + Ct^{-\frac{1}{2}} \left\|\widetilde{W}\left(e^{i\frac{3\chi}{\sqrt{2}}\int_1^t \frac{|w(\tau)|^2}{\tau} d\tau} - e^{-i\frac{3\chi}{\sqrt{2}}\left|\widetilde{W}\right|^2 \log t + i\frac{3}{\sqrt{2}}\Psi}\right)\right\|_{\mathbf{L}^\infty} \\
&+ Ct^{-\frac{1}{2}-\delta}\left(\|\mathcal{J}u_x(t)\|_{\mathbf{H}^{1+2\delta}} + \|u(t)\|_{\mathbf{H}^{\frac{3}{2}+\delta}}\right) \\
\le\ &C\epsilon t^{-\frac{1}{2}-\delta+5\epsilon^{\frac{1}{7}}} + C\epsilon t^{-\frac{1}{2}-\delta+5\epsilon^{\frac{1}{7}}} \log t + Ct^{-\frac{1}{2}-\delta+\epsilon^{\frac{1}{7}}} \le C\epsilon t^{-\frac{1}{2}-\delta+5\epsilon^{\frac{1}{7}}} \log t.
\end{aligned}$$

Put $W(\xi) = \widetilde{W}(\xi)e^{i3\Psi(\xi)/\sqrt{2}}$, then we obtain the desired asymptotics. This completes the proof of Theorem 6.1. □

4. Proof of (6.11)

In this section, we will give the proof of (6.11). Consider the integral

$$\mathrm{I} = i\xi^3 |\xi|^{\frac{3}{2}+j} \iint_{\mathbb{R}^2} e^{-i\frac{t}{\xi}S}\hat{v}(t,\xi\xi_1)\hat{v}(t,\xi\xi_2)\hat{v}(t,\xi\xi_3)d\xi_1 d\xi_2,$$

where $\xi_3 = 1-\xi_1-\xi_2$, $S = 1-\frac{1}{\xi_1}-\frac{1}{\xi_2}-\frac{1}{\xi_3}$ and $j = 0,1,2$. Since the stationary points for the integral I are $(\xi_1,\xi_2) = (1,1),(1,-1),(-1,1),(\frac{1}{3},\frac{1}{3})$, we decompose the integral I into three parts : $\mathrm{I} = \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3$, where

$$\begin{aligned}\mathrm{I}_1 &= i\xi^3 |\xi|^{\frac{3}{2}+j} \iint_{\mathbb{R}^2} e^{-i\frac{t}{\xi}S}\hat{v}(t,\xi\xi_1)\hat{v}(t,\xi\xi_2)\hat{v}(t,\xi\xi_3)\Phi_1'(\xi_1,\xi_2)d\xi_1 d\xi_2,\\ \mathrm{I}_2 &= i\xi^3 |\xi|^{\frac{3}{2}+j} \iint_{\mathbb{R}^2} e^{-i\frac{t}{\xi}S}\hat{v}(t,\xi\xi_1)\hat{v}(t,\xi\xi_2)\hat{v}(t,\xi\xi_3)\Phi_2(\xi_1,\xi_2)\,d\xi_1 d\xi_2\end{aligned}$$

and

$$\mathrm{I}_3 = i\xi^3 |\xi|^{\frac{3}{2}+j} \iint_{\mathbb{R}^2} e^{-i\frac{t}{\xi}S}\hat{v}(t,\xi\xi_1)\hat{v}(t,\xi\xi_2)\hat{v}(t,\xi\xi_3)\Phi_3(\xi_1,\xi_2)\,d\xi_1 d\xi_2.$$

Here, $\Phi_1,\Phi_2 \in \mathbf{C}^\infty(\mathbb{R}^2)$ are cut-off functions such that $0 \le \Phi_1,\Phi_2 \le 1$,

$$\Phi_1(\xi_1,\xi_2) = \begin{cases} 1 & if\,|\xi_1-1|+|\xi_2-1| \le 10^{-2} \\ 0 & if\,|\xi_1-1|+|\xi_2-1| \ge 10^{-1}\end{cases},$$

$$\Phi_2(\xi_1,\xi_2) = \begin{cases} 1 & if\,|\xi_1-1/3|+|\xi_2-1/3| \le 10^{-2} \\ 0 & if\,|\xi_1-1/3|+|\xi_2-1/3| \ge 10^{-1}\end{cases}.$$

Φ_1' and Φ_3 are defined as follows : $\Phi_1'(\xi_1,\xi_2) = \Phi_1(\xi_1,\xi_2)+\Phi_1(\xi_2,\xi_3)+\Phi_1(\xi_3,\xi_1)$ and $\Phi_3(\xi_1,\xi_2) = 1-\Phi_1'(\xi_1,\xi_2)-\Phi_2(\xi_1,\xi_2)$. To estimate I_1 and I_2, we state the next lemma which was essentially shown in [**9**].

Lemma 6.3. ([**25**]) *Consider the integral*

$$\mathrm{I}'_l = \iint_{\mathbb{R}^2} e^{-(itS+ix\xi_1+iy\xi_2+iz\xi_3)}|\xi_1\xi_2\xi_3|^{-\mu}\,\Phi_l(\xi_1,\xi_2)\,d\xi_1 d\xi_2,$$

where $l = 1,2$. Then, the following asymptotics

$$\mathrm{I}_1' = \frac{\pi\sqrt{2}}{t}e^{-i(x+y-z)} + O\left(\frac{1}{t^{1+\alpha}}\right) + O\left(\frac{|x|^{2\beta}+|y|^{2\beta}+|z|^{2\beta}}{t^{1+\beta}}\right) \tag{6.23}$$

and

$$\mathrm{I}_2' = i\frac{\sqrt{2}\cdot 3^{3(\mu-1)}\pi}{t\sqrt{3}}e^{11it-i\left(\frac{x+y+z}{3}\right)} + O\left(\frac{1}{t^{1+\alpha}}\right) + O\left(\frac{|x|^{2\beta}+|y|^{2\beta}+|z|^{2\beta}}{t^{1+\beta}}\right) \tag{6.24}$$

are valid for large time $t \ge 1$ uniformly with respect to $x,y,z \in \mathbb{R}$, where $0 < \alpha \le 1, 0 < \beta \le 1$ and $\mu \ge 0$.

PROOF. First we consider the integral I_2'. Note that

$$\nabla S-\left(\frac{x-z}{t},\frac{y-z}{t}\right)=\left(\left(\frac{1}{\xi_1^2}-\frac{1}{\xi_3^2}\right)-\frac{x-z}{t},\left(\frac{1}{\xi_2^2}-\frac{1}{\xi_3^2}\right)-\frac{y-z}{t}\right)$$

and

$$H(S)=-2\begin{pmatrix}\frac{1}{\xi_1^3}+\frac{1}{\xi_3^3} & \frac{1}{\xi_3^3}\\ \frac{1}{\xi_3^3} & \frac{1}{\xi_2^3}+\frac{1}{\xi_3^3}\end{pmatrix},$$

where $H(S)$ is the Hessian matrix of S. Since $detH(S)$ does not have zero point in the support of Φ_2, we can use the stationary phase method. If $|x-z|/t+|y-z|/t=\varepsilon\le A$ and A is sufficiently small, the integral I_2' has the only one stationary point $(\xi_1',\xi_2')=(1/3,1/3)+O(\varepsilon)$ in the support of Φ_2 since the other zero points are not included in the support of Φ_2. Then, by the stationary phase method (see [**31**]), we have for large $t\ge1$,

$$\begin{aligned}\mathrm{I}_2' &= \frac{2\pi e^{-i\left(\frac{\pi}{4}sgnH(S)(\xi_1',\xi_2')+tS(\xi_1',\xi_2')+x\xi_1'+y\xi_2'+z(1-\xi_1'-\xi_2')\right)}}{t\left|\xi_1'\xi_2'(1-\xi_1'-\xi_2')\right|^{-\mu}\sqrt{|detH(S)(\xi_1',\xi_2')|}}+O(t^{-2})\\ &= i\frac{2\cdot3^{3\mu}\pi e^{i\left(11t-\frac{x+y+z}{3}+O(\varepsilon)+tO(\varepsilon^2)\right)}}{t\sqrt{2\cdot3^7+O(\varepsilon)}}\left(1+O\left(\varepsilon\right)\right)+O\left(t^{-2}\right)\\ &= i\frac{\sqrt{2}\cdot3^{3(\mu-1)}\pi}{t\sqrt{3}}e^{11it-i\left(\frac{x+y+z}{3}\right)}+O\left(\frac{1}{t^{1+\alpha}}\right)+O\left(\frac{|x|^{2\beta}+|y|^{2\beta}+|z|^{2\beta}}{t^{1+\beta}}\right),\end{aligned}$$

since $S(\xi_1',\xi_2')=-11+O(\varepsilon^2)$, $detH(S)(\xi_1',\xi_2')=2\cdot3^7+O(\varepsilon)$ and $sgnH(S)=-2$ where $sgnH(S)$ is the number of positive eigenvalues of $H(S)$ minus the number of negative eigenvalues of $H(S)$. If $A\le|x-z|/t+|y-z|/t$, then we get $\mathrm{I}_2'=O(t^{-1})$ by the stationary phase method when there exists the stationary point in the support of Φ_2 or integration by parts when there is no stationary point in the support of Φ_2. Because of the assumption $A\le|x-z|/t+|y-z|/t$, we have $|x/t|\ge A/4$ or $|y/t|\ge A/4$ or $|z/t|\ge A/4$. Without loss of generality, we assume $|x/t|\ge A/4$. Then, it follows that $|\mathrm{I}_2'|\le Ct^{-1}\le CA^{-\beta}t^{-1-\beta}|x|^{\beta}$. Therefore, collecting these estimates, the asymptotic expansion (6.24) is valid. In the same manner, we obtain the asymptotic expansion (6.23) since $detH(S)\,(1,1)=2$ and $sgnH(S)\,(1,1)=0$. □

By virtue of Lemma 6.3, we give the asymptotic expansion of the integral I_1 and I_2.

LEMMA 6.4. ([**25**]) *The following asymptotics*

(6.25)
$$\mathrm{I}_1=i\frac{3\sqrt{2}\pi\xi^4\left|\xi\right|^{\frac{3}{2}+j}}{t}\left|\hat{v}(t,\xi)\right|^2\hat{v}(t,\xi)+O\left(\frac{1}{t^{1+\delta}}\left(\|v(t)\|_{\mathbf{H}^{\frac{11}{6}+\delta+\frac{j}{3}}}+\|xv_x(t)\|_{\mathbf{H}^{\frac{5}{6}+\delta+\frac{j}{3}}}\right)^3\right)$$

and

(6.26)
$$\mathrm{I}_2=\frac{\sqrt{2}\pi\xi^4|\xi|^{\frac{3}{2}+j}}{t3^3\sqrt{3}}e^{i\frac{11t}{\xi}}\hat{v}\left(t,\frac{\xi}{3}\right)^3+O\left(\frac{1}{t^{1+\delta}}\left(\|v(t)\|_{\mathbf{H}^{\frac{11}{6}+\delta+\frac{j}{3}}}+\|xv_x(t)\|_{\mathbf{H}^{\frac{5}{6}+\delta+\frac{j}{3}}}\right)^3\right)$$

are true for large time $t\ge1$ uniformly with respect to $\xi\in\mathbb{R}$ where $0<\delta<1/4$ and $j=0,1,2$.

PROOF. By the symmetry of ξ_1, ξ_2 and ξ_3, we can rewrite the integral I_1 in

$$\mathrm{I}_1 = 3i\xi^3 |\xi|^{\frac{3}{2}+j} \iint_{\mathbb{R}^2} e^{-i\frac{t}{\xi}S} \hat{v}(t,\xi\xi_1)\hat{v}(t,\xi\xi_2)\hat{v}(t,\xi\xi_3)\Phi_1(\xi_1,\xi_2)\, d\xi_1 d\xi_2.$$

By the identity $\hat{v}(t,\xi\xi_1) = |\xi\xi_1|^{-\mu}\mathcal{F}\left(-\partial_x^2\right)^{\mu/2} v$ with $\mu = 11/6 + \delta + j/3$, we have

$$\begin{aligned}(6.27)\quad \mathrm{I}_1 = i\frac{3\xi^3|\xi|^{\frac{3}{2}+j}}{(2\pi)^{\frac{3}{2}}|\xi|^{3\mu}} \iint_{\mathbb{R}^2}\iiint_{\mathbb{R}^3} d\xi_1 d\xi_2 dxdydz \times \\ \frac{\Phi_1(\xi_1,\xi_2)}{|\xi_1\xi_2\xi_3|^{\mu}} e^{-i\frac{t}{\xi}S - ix\xi\xi_1 - iy\xi\xi_2 - iz\xi\xi_3}\left(-\partial_x^2\right)^{\frac{\mu}{2}} v(x)\left(-\partial_y^2\right)^{\frac{\mu}{2}} v(y)\left(-\partial_z^2\right)^{\frac{\mu}{2}} v(z).\end{aligned}$$

Applying (6.23) with $\alpha = 3\delta$, $\beta = \delta$ and $\delta \in (0, 1/4)$ to (6.27), we get

$$\begin{aligned} & i\frac{3\sqrt{2}\pi\xi^4|\xi|^{\frac{3}{2}+j}}{t(2\pi)^{\frac{3}{2}}|\xi|^{3\mu}} \iiint_{\mathbb{R}^3} dxdydz e^{-i\xi(x+y-z)}\left(-\partial_x^2\right)^{\frac{\mu}{2}} v(x)\left(-\partial_y^2\right)^{\frac{\mu}{2}} v(y)\left(-\partial_z^2\right)^{\frac{\mu}{2}} v(z) \\ & + i\frac{3\sqrt{2}\pi\xi^3|\xi|^{\frac{3}{2}+j}}{(2\pi)^{\frac{3}{2}}|\xi|^{3\mu}} \iiint_{\mathbb{R}^3} dxdydz\left(-\partial_x^2\right)^{\frac{\mu}{2}} v(x)\left(-\partial_y^2\right)^{\frac{\mu}{2}} v(y)\left(-\partial_z^2\right)^{\frac{\mu}{2}} v(z) \times \\ & \left(O\left(\frac{\xi^{1+3\delta}}{t^{1+3\delta}}\right) + O\left(\frac{\xi^{1+3\delta}\left(|x|^{2\delta} + |y|^{2\delta} + |z|^{2\delta}\right)}{t^{1+\delta}}\right)\right) \\ = \; & i\frac{3\sqrt{2}\pi\xi^4|\xi|^{\frac{3}{2}+j}}{t}|\hat{v}(t,\xi)|^2\hat{v}(t,\xi) + O\left(\frac{|\xi|^{\frac{11}{2}+j+3\delta-3\mu}}{t^{1+\delta}}\left(\|v(t)\|_{\mathbf{H}_1^{\mu}} + \|v(t)\|_{\mathbf{H}_1^{\mu,2\delta}}\right)^3\right) \\ = \; & i\frac{3\sqrt{2}\pi\xi^4|\xi|^{\frac{3}{2}+j}}{t}|\hat{v}(t,\xi)|^2\hat{v}(t,\xi) + O\left(t^{-1-\delta}\left(\|v(t)\|_{\mathbf{H}^{\frac{11}{6}+\delta+\frac{j}{3}}} + \|xv_x(t)\|_{\mathbf{H}^{\frac{5}{6}+\delta+\frac{j}{3}}}\right)^3\right),\end{aligned}$$

for large $t/\xi \geq 1$. If t/ξ is small, that is, $t \leq C\xi$, then we have

$$\begin{aligned}|\mathrm{I}_1| &\leq C|\xi|^{\frac{9}{2}+j-3\mu}\|v(t)\|_{\mathbf{H}_1^{\mu}}^3 \leq C|\xi|^{-1-3\delta}\|v(t)\|^3_{\mathbf{H}_1^{\frac{11}{6}+\delta+\frac{j}{3}}} \\ &\leq Ct^{-1-3\delta}\left(\|v(t)\|_{\mathbf{H}^{\frac{11}{6}+\delta+\frac{j}{3}}} + \|xv_x(t)\|_{\mathbf{H}^{\frac{5}{6}+\delta+\frac{j}{3}}}\right)^3\end{aligned}$$

where we chose $\mu = 11/6 + \delta + j/3$. Thus, we obtain the asymptotic expansion (6.25). Similarly, thanks to (6.24), we can get (6.26). This completes the proof of Lemma 6.4. □

We finally give the estimate of the non-stationary contribution I_3. As mentioned above, we cannot use the operator $\mathcal{P}^2$. Namely, we cannot use two weights with respect to x. The next lemma says that we only need one weight with respect to x to bound the non-stationary contribution I_3.

LEMMA 6.5. ([**25**]) *The estimate*

$$(6.28)\qquad |\mathrm{I}_3| \leq Ct^{-1-\delta}\left(\|xv_x(t)\|_{\mathbf{H}^5} + \|v(t)\|_{\mathbf{H}^6} + \|v(t)\|_{\dot{\mathbf{H}}^{-1}}\right)^3$$

is true for $\delta \in (0, 1/12)$ and $t \geq 1$, provided that the right-hand side is finite.

PROOF. In order to estimate I_3, we introduce a cut off function $\Phi_{3,1}(\xi_1,\xi_2) \in \mathbf{C}^\infty(\mathbb{R}^2)$ such that $0 \le \Phi_{3,1} \le 1$ and

$$\Phi_{3,1}(\xi_1,\xi_2) = \begin{cases} 1 & if \quad |\xi_2 - 1| \le 10^{-2}, \\ 1 & if \quad |\xi_1 - \xi_3| \le 10^{-2}, \\ 0 & if \quad |\xi_2 - 1| \ge 10^{-1}, \quad |\xi_1 - \xi_3| \ge 10^{-1} \end{cases}.$$

Also define $\Phi_{3,2}(\xi_1,\xi_2) = 1 - \Phi_{3,1}(\xi_1,\xi_2)$. In addition, to denote shortly, we use following notations:

$$F_0(\xi_1,\xi_2) = \hat{v}(t,\xi\xi_1)\hat{v}(t,\xi\xi_2)\hat{v}(t,\xi\xi_3), \quad F_1(\xi_1,\xi_2) = \hat{v}_\xi(t,\xi\xi_1)\hat{v}(t,\xi\xi_2)\hat{v}(t,\xi\xi_3),$$
$$F_2(\xi_1,\xi_2) = \hat{v}(t,\xi\xi_1)\hat{v}_\xi(t,\xi\xi_2)\hat{v}(t,\xi\xi_3), \quad F_3(\xi_1,\xi_2) = \hat{v}(t,\xi\xi_1)\hat{v}(t,\xi\xi_2)\hat{v}_\xi(t,\xi\xi_3),$$
$$F_{1,2}(\xi_1,\xi_2) = \hat{v}_\xi(t,\xi\xi_1)\hat{v}_\xi(t,\xi\xi_2)\hat{v}(t,\xi\xi_3), \quad F_{1,3}(\xi_1,\xi_2) = \hat{v}_\xi(t,\xi\xi_1)\hat{v}(t,\xi\xi_2)\hat{v}_\xi(t,\xi\xi_3).$$

We now consider the integral I_3. Using the cut off functions $\Phi_{3,1}$ and $\Phi_{3,2}$, we decompose the integral I_3 into two parts : $\mathrm{I}_3 = \mathrm{I}_{3,1} + \mathrm{I}_{3,2}$, where

$$\mathrm{I}_{3,1} = i\xi^{3+j}|\xi|^{\frac{3}{2}} \iint_{\mathbb{R}^2} e^{-i\frac{t}{\xi}S} F_0(\xi_1,\xi_2)\Phi_3(\xi_1,\xi_2)\Phi_{3,1}(\xi_1,\xi_2)d\xi_1 d\xi_2$$

and

$$\mathrm{I}_{3,2} = i\xi^{3+j}|\xi|^{\frac{3}{2}} \iint_{\mathbb{R}^2} e^{-i\frac{t}{\xi}S} F_0(\xi_1,\xi_2)\Phi_3(\xi_1,\xi_2)\Phi_{3,2}(\xi_1,\xi_2)d\xi_1 d\xi_2.$$

Let us consider the integral $\mathrm{I}_{3,2}$. Integration by parts yields

$$\begin{aligned} \mathrm{I}_{3,2} &= i\xi^{3+j}|\xi|^{\frac{3}{2}} \iint_{\mathbb{R}^2} \left(\frac{d}{d\xi_1} e^{-i\frac{t}{\xi}S}\right) H_1(\xi_1,\xi_2) F_0(\xi_1,\xi_2)d\xi_1 d\xi_2 \\ &= -i\xi^{4+j}|\xi|^{\frac{3}{2}} \iint_{\mathbb{R}^2} e^{-i\frac{t}{\xi}S} H_1(\xi_1,\xi_2) F_1(\xi_1,\xi_2)d\xi_1 d\xi_2 \\ &\quad + i\xi^{4+j}|\xi|^{\frac{3}{2}} \iint_{\mathbb{R}^2} e^{-i\frac{t}{\xi}S} H_1(\xi_1,\xi_2) F_3(\xi_1,\xi_2)d\xi_1 d\xi_2 \\ &\quad - i\xi^{3+j}|\xi|^{\frac{3}{2}} \iint_{\mathbb{R}^2} e^{-i\frac{t}{\xi}S} \partial_{\xi_1} H_1(\xi_1,\xi_2) F_0(\xi_1,\xi_2)d\xi_1 d\xi_2, \end{aligned} \tag{6.29}$$

where

$$H_1(\xi_1,\xi_2) = -\frac{\xi}{t} \frac{\Phi_3(\xi_1,\xi_2)\Phi_{3,2}(\xi_1,\xi_2)}{i\left(\frac{1}{\xi_1^2} - \frac{1}{\xi_3^2}\right)}.$$

To avoid two derivatives falling on the same profile, we consider the third line of (6.29). Changing the integral variables $\xi_1' = \xi_3$, $\xi_2' = \xi_2$, and putting $\xi_3' = 1 - \xi_1' - \xi_2'$, we have

$$\begin{aligned} & -i\xi^{4+j}|\xi|^{\frac{3}{2}} \iint_{\mathbb{R}^2} e^{-i\frac{t}{\xi}S(\xi_3',\xi_2')} H_1(\xi_3',\xi_2') F_3(\xi_3',\xi_2')d\xi_1' d\xi_2' \\ &= -i\xi^{4+j}|\xi|^{\frac{3}{2}} \iint_{\mathbb{R}^2} e^{-i\frac{t}{\xi}S(\xi_1',\xi_2')} H_1(\xi_3',\xi_2') F_1(\xi_1',\xi_2')d\xi_1' d\xi_2', \end{aligned} \tag{6.30}$$

since $F_3(\xi_3',\xi_2') = \hat{v}(t,\xi\xi_3')\hat{v}(t,\xi\xi_2')\hat{v}_\xi(t,\xi\xi_1') = F_1(\xi_1',\xi_2')$ and $S(\xi_1,\xi_2) = S(\xi_3,\xi_2)$. Hereafter we omit the prime. Substituting (6.30) into (6.29), we have

$$\begin{aligned} \mathrm{I}_{3,2} &= -i\xi^{4+j}|\xi|^{\frac{3}{2}} \iint_{\mathbb{R}^2} e^{-i\frac{t}{\xi}S} \tilde{H}_1(\xi_1,\xi_2) F_1(\xi_1,\xi_2)d\xi_1 d\xi_2 \\ &\quad - i\xi^{3+j}|\xi|^{\frac{3}{2}} \iint_{\mathbb{R}^2} e^{-i\frac{t}{\xi}S} \partial_{\xi_1} H_1(\xi_1,\xi_2) F_0(\xi_1,\xi_2)d\xi_1 d\xi_2, \end{aligned}$$

where $\tilde{H}_1(\xi_1,\xi_2) = H_1(\xi_1,\xi_2) + H_1(\xi_3,\xi_2)$. Note that performing integration by parts with respect to ξ_2 on the above equation does not cause two derivatives falling on the same profile. Hence, applying integration by parts with respect to ξ_2, we get

$$\begin{aligned}
\mathrm{I}_{3,2} &= -i\xi^{4+j}|\xi|^{\frac{3}{2}}\iint_{\mathbb{R}^2}\frac{d}{d\xi_2}\left((\xi_2-\xi_3)\,e^{-i\frac{t}{\xi}S}\right)\tilde{H}_1(\xi_1,\xi_2)H_2(\xi_1,\xi_2)F_1 d\xi_1 d\xi_2 \\
&\quad -i\xi^{3+j}|\xi|^{\frac{3}{2}}\iint_{\mathbb{R}^2}\frac{d}{d\xi_2}\left((\xi_2-\xi_3)\,e^{-i\frac{t}{\xi}S}\right)\partial_{\xi_1}\tilde{H}_1(\xi_1,\xi_2)H_2(\xi_1,\xi_2)F_0 d\xi_1 d\xi_2 \qquad (6.31)\\
&= \mathrm{I}_{3,2,1}+\mathrm{I}_{3,2,2}+\mathrm{I}_{3,2,3}+\mathrm{I}_{3,2,4}+\mathrm{I}_{3,2,5}+\mathrm{I}_{3,2,6},
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{I}_{3,2,1} &= i\xi^{5+j}|\xi|^{\frac{3}{2}}\iint_{\mathbb{R}^2} e^{-i\frac{t}{\xi}S}(\xi_2-\xi_3)\,\tilde{H}_1(\xi_1,\xi_2)H_2(\xi_1,\xi_2)\left(F_{1,2}-F_{1,3}\right)d\xi_1 d\xi_2,\\
\mathrm{I}_{3,2,2} &= i\xi^{4+j}|\xi|^{\frac{3}{2}}\iint_{\mathbb{R}^2} e^{-i\frac{t}{\xi}S}(\xi_2-\xi_3)\,\tilde{H}_1(\xi_1,\xi_2)\partial_{\xi_2}H_2(\xi_1,\xi_2)F_1 d\xi_1 d\xi_2,\\
\mathrm{I}_{3,2,3} &= i\xi^{4+j}|\xi|^{\frac{3}{2}}\iint_{\mathbb{R}^2} e^{-i\frac{t}{\xi}S}(\xi_2-\xi_3)\,\partial_{\xi_2}\tilde{H}_1(\xi_1,\xi_2)H_2(\xi_1,\xi_2)F_{1,2} d\xi_1 d\xi_2,\\
\mathrm{I}_{3,2,4} &= i\xi^{4+j}|\xi|^{\frac{3}{2}}\iint_{\mathbb{R}^2} e^{-i\frac{t}{\xi}S}(\xi_2-\xi_3)\,\partial_{\xi_1}\tilde{H}_1(\xi_1,\xi_2)H_2(\xi_1,\xi_2)\left(F_2-F_3\right)d\xi_1 d\xi_2,\\
\mathrm{I}_{3,2,5} &= i\xi^{3+j}|\xi|^{\frac{3}{2}}\iint_{\mathbb{R}^2} e^{-i\frac{t}{\xi}S}(\xi_2-\xi_3)\,\partial_{\xi_1}\tilde{H}_1(\xi_1,\xi_2)\partial_{\xi_2}H_2(\xi_1,\xi_2)F_0 d\xi_1 d\xi_2,\\
\mathrm{I}_{3,2,6} &= i\xi^{3+j}|\xi|^{\frac{3}{2}}\iint_{\mathbb{R}^2} e^{-i\frac{t}{\xi}S}(\xi_2-\xi_3)\,\partial_{\xi_1}\partial_{\xi_2}\tilde{H}_1(\xi_1,\xi_2)H_2(\xi_1,\xi_2)F_0 d\xi_1 d\xi_2,
\end{aligned}$$

and

$$H_2(\xi_1,\xi_2) = \left(2 - i\frac{t}{\xi}(\xi_2-\xi_3)\left(\frac{1}{\xi_2^2}-\frac{1}{\xi_3^2}\right)\right)^{-1}.$$

Thus, we need to estimate $\mathrm{I}_{3,2,1}, \mathrm{I}_{3,2,2}, \mathrm{I}_{3,2,3}, \mathrm{I}_{3,2,4}, \mathrm{I}_{3,2,5}$ and $\mathrm{I}_{3,2,6}$.

Estimate of $\mathrm{I}_{3,2,1}$. In order to estimate $\mathrm{I}_{3,2,1}$, we first give the estimate of $\tilde{H}_1$ and H_2. Since $|1-\xi_2| = |\xi_1+\xi_3|$, we get

$$\left|\tilde{H}_1(\xi_1,\xi_2)\right| \le \frac{|\xi|}{t}\frac{|\xi_1|^2|\xi_3|^2}{|\xi_1-\xi_3||1-\xi_2|} \le \frac{|\xi|}{t}\frac{|\xi_1|^2|\xi_3|^2\left(|\xi_1|+|\xi_3|\right)^{2\gamma}}{|\xi_1-\xi_3|^{1+\gamma}|1-\xi_2|^{1+\gamma}}, \qquad (6.32)$$

for $\gamma > 0$. Also, by the Young inequality, we get

$$|H_2(\xi_1,\xi_2)| \le C\frac{|\xi|^\alpha}{t^\alpha}\frac{|\xi_2|^{2\alpha}|\xi_3|^{2\alpha}}{|\xi_2-\xi_3|^{2\alpha}|1-\xi_1|^\alpha}, \qquad (6.33)$$

for $0 \le \alpha \le 1$. Then, by (6.32) with $\gamma = 3/4-\alpha/2+j/2$ and (6.33), we have

$$|\mathrm{I}_{3,2,1}| \le C\frac{|\xi|^{\frac{15}{2}+\alpha+j}}{t^{1+\alpha}}\iint_D \frac{|\xi_1|^2|\xi_2||\xi_3|^{2\alpha+2}\left(|\xi_1|+|\xi_3|\right)^{\frac{3}{2}-\alpha+j}\left(|F_{1,2}|+|F_{1,3}|\right)d\xi_1 d\xi_2}{|\xi_1-\xi_3|^{\frac{7}{4}-\frac{\alpha}{2}+\frac{j}{2}}|1-\xi_2|^{\frac{7}{4}-\frac{\alpha}{2}+\frac{j}{2}}|\xi_2-\xi_3|^{2\alpha-1}|1-\xi_1|^\alpha|\xi_2|^{1-2\alpha}},$$

where $D = supp\Phi_3 \cap \Phi_{3,2}$. It follows from the Cauchy-Schwarz inequality that the right hand side of the above inequality can be estimated as

$$\frac{C|\xi|^{\frac{15}{2}+\alpha+j}}{t^{1+\alpha}}K(\alpha)\left\|\xi_1^2\xi_2\xi_3^{2+2\alpha}(|\xi_1|+|\xi_3|)^{\frac{3}{2}-\alpha+j}\left(|F_{1,2}|+|F_{1,3}|\right)\right\|_{\mathbf{L}^2_{\xi_1,\xi_2}}, \qquad (6.34)$$

where

$$K_1(\alpha) = \left(\iint_D \frac{|\xi_2 - \xi_3|^{2-4\alpha} d\xi_1 d\xi_2}{|\xi_1 - \xi_3|^{\frac{7}{2}-\alpha+j} |1 - \xi_2|^{\frac{7}{2}-\alpha+j} |1 - \xi_1|^{2\alpha} |\xi_2|^{2(1-2\alpha)}} \right)^{\frac{1}{2}}.$$

Since $|\xi_1 - \xi_3| \geq 10^{-2}$, $|1 - \xi_2| \geq 10^{-2}$ in D and $|\xi_2 - \xi_3| \leq |\xi_2| + |\xi_3| \leq 1 + 3|\xi_2 - 1|/2 + |\xi_1 - \xi_3|$, we have $K_1(\alpha) \leq C$ for $1/4 < \alpha < 1/2$. Whence, by making the change of the integral variables $\xi_1' = \xi\xi_1$ and $\xi_2' = \xi\xi_2$ (we omit the prime) and a direct calculation, (6.34) is bounded by

$$\begin{aligned} & \frac{C}{t^{1+\alpha}} \left\| \xi_1^2 \xi_2 \xi_3^{2+2\alpha} (|\xi_1| + |\xi_3|)^{\frac{3}{2}-\alpha+j} \hat{v}_\xi(\xi_1) \left(|\hat{v}_\xi(\xi_2)\hat{v}(\xi_3)| + |\hat{v}(\xi_2)\hat{v}_\xi(\xi_3)| \right) \right\|_{\mathbf{L}^2_{\xi_1,\xi_2}} \\ \leq\ & \frac{C}{t^{1+\alpha}} \left\| \xi^{\frac{7}{2}-\alpha+j} \hat{v}_\xi \right\|_{\mathbf{L}^2} \left(\|\xi\hat{v}_\xi\|_{\mathbf{L}^2} \left\|\xi^{2+2\alpha}\hat{v}\right\|_{\mathbf{L}^\infty} + \|\xi\hat{v}\|_{\mathbf{L}^\infty} \left\|\xi^{2+2\alpha}\hat{v}_\xi\right\|_{\mathbf{L}^2} \right) \\ & + \frac{C}{t^{1+\alpha}} \left\|\xi^2\hat{v}_\xi\right\|_{\mathbf{L}^2} \left(\|\xi\hat{v}_\xi\|_{\mathbf{L}^2} \left\|\xi^{\frac{7}{2}+\alpha+j}\hat{v}\right\|_{\mathbf{L}^\infty} + \|\xi\hat{v}\|_{\mathbf{L}^\infty} \left\|\xi^{\frac{7}{2}+\alpha+j}\hat{v}_\xi\right\|_{\mathbf{L}^2} \right). \end{aligned}$$

Since

$$\|\xi^a \hat{v}\|_{\mathbf{L}^\infty} \leq C \left\| \left(-\partial_x^2\right)^{\frac{a}{2}} v \right\|_{\mathbf{L}^1} \leq C \|v\|_{\mathbf{H}^a} + C \|xv_x\|_{\mathbf{H}^{a-1}} \tag{6.35}$$

for $a \geq 1$, we have for $1/4 < \alpha < 1/2$,

$$|\mathrm{I}_{3,2,1}| \leq \frac{C}{t^{1+\alpha}} \left(\|v(t)\|_{\mathbf{H}^{\frac{7}{2}+\alpha+j}} + \|xv_x(t)\|_{\mathbf{H}^{\frac{5}{2}+\alpha+j}} \right)^3.$$

Estimate of $\mathrm{I}_{3,2,2}$. By the Young inequality, we have

$$|\partial_{\xi_2} H_2(\xi_1,\xi_2)| = \left| \frac{t}{\xi} \frac{2\left(\frac{1}{\xi_2^2} - \frac{1}{\xi_3^2}\right) - 2(\xi_2 - \xi_3)\left(\frac{1}{\xi_2^3} + \frac{1}{\xi_3^3}\right)}{\left(2 - i\frac{t}{\xi}(\xi_2 - \xi_3)\left(\frac{1}{\xi_2^2} - \frac{1}{\xi_3^2}\right)\right)^2} \right| \leq C \frac{|\xi|^{2\beta}}{t^{2\beta}} \frac{|\xi_2\xi_3|^{4\beta-1} (|\xi_2| + |\xi_3|)^2}{|\xi_2 - \xi_3|^{4\beta+1} |1 - \xi_1|^{2\beta}} \tag{6.36}$$

for $0 \leq \beta \leq 1/2$. It follows from (6.32) with $\gamma = 1/2 - 2\beta + j/2$ and (6.36) that

$$|\mathrm{I}_{3,2,2}| \leq C \frac{|\xi|^{\frac{13}{2}+2\beta+j}}{t^{1+2\beta}} \iint_D \frac{|\xi_1|^2 |\xi_2|^{4\beta-1} |\xi_3|^{4\beta+1} (|\xi_2| + |\xi_3|)^2 (|\xi_1| + |\xi_3|)^{1-4\beta+j} |F_1| d\xi_1 d\xi_2}{|\xi_1 - \xi_3|^{\frac{3}{2}-2\beta+\frac{j}{2}} |1 - \xi_2|^{\frac{3}{2}-2\beta+\frac{j}{2}} |1 - \xi_1|^{2\beta} |\xi_2 - \xi_3|^{4\beta}}.$$

As in the same manner in the proof of the estimate of $\mathrm{I}_{3,2,1}$, the right hand side of the above inequality is bounded by

$$\begin{aligned} & \frac{CK_2(\beta)}{t^{1+2\beta}} \left\| \xi_1^{\frac{5}{2}-2\beta} \xi_2^{4\beta-1} \xi_3^{4\beta+1} (|\xi_2| + |\xi_3|)^2 (|\xi_1| + |\xi_3|)^{1-4\beta+j} \hat{v}_\xi(\xi_1)\hat{v}(\xi_2)\hat{v}(\xi_3) \right\|_{\mathbf{L}^2_{\xi_1,\xi_2}} \\ \leq\ & \frac{CK_2(\beta)}{t^{1+2\beta}} \left\| \xi^{\frac{7}{2}-6\beta+j} \hat{v}_\xi \right\|_{\mathbf{L}^2} \left(\left\|\xi^{4\beta+1}\hat{v}\right\|_{\mathbf{L}^\infty} \left\|\xi^{4\beta+1}\hat{v}\right\|_{\mathbf{L}^2} + \left\|\xi^{4\beta-1}\hat{v}\right\|_{\mathbf{L}^2} \left\|\xi^{4\beta+3}\hat{v}\right\|_{\mathbf{L}^\infty} \right) \\ & + \frac{CK_2(\beta)}{t^{1+2\beta}} \left\| \xi^{\frac{5}{2}-2\beta} \hat{v}_\xi \right\|_{\mathbf{L}^2} \left(\left\|\xi^{4\beta+1}\hat{v}\right\|_{\mathbf{L}^\infty} \left\|\xi^{2+j}\hat{v}\right\|_{\mathbf{H}^2} + \left\|\xi^{4\beta-1}\hat{v}\right\|_{\mathbf{L}^2} \left\|\xi^{4+j}\hat{v}\right\|_{\mathbf{L}^\infty} \right), \end{aligned}$$

where

$$K_2(\beta) = \left(\iint_D \frac{d\xi_1 d\xi_2}{|\xi_1 - \xi_3|^{3-4\beta+j} |1 - \xi_2|^{3-4\beta+j} |\xi_2 - \xi_3|^{8\beta} |1 - \xi_1|^{4\beta} |\xi_1|^{1-4\beta}} \right)^{\frac{1}{2}}.$$

Since $K_2(\beta) \leq C$ for $0 < \beta < 1/8$, it follows from (6.35) that

$$|\mathrm{I}_{3,2,2}| \leq \frac{C}{t^{1+2\beta}} \left(\|v(t)\|_{\dot{\mathbf{H}}^{4\beta-1}} + \|v\|_{\mathbf{H}^{4+j}} + \|xv_x(t)\|_{\mathbf{H}^{3+j}} \right)^3,$$

for $0<\beta<1/8$.

Estimate of $\mathrm{I}_{3,2,3}$. Since $|\xi_1|\le|\xi_1-\xi_3|+|1-\xi_2|$, $|\xi_1-\xi_3|\ge 10^{-2}$ and $|1-\xi_2|\ge 10^{-2}$ in D, we have for $\gamma\ge 0$,

$$\left|\partial_{\xi_2}\tilde{H}_1(\xi_1,\xi_2)\right|\le C\frac{|\xi|}{t}\left|\frac{1}{\xi_1^2}-\frac{1}{\xi_3^2}\right|^{-2}\left|\frac{1}{\xi_3^3}\right|+C|\tilde{H}_1(\xi_1,\xi_2)| \tag{6.37}$$
$$\le C\frac{|\xi|}{t}\frac{|\xi_1|^4|\xi_3|}{|\xi_1-\xi_3|^2|1-\xi_2|^2}+C|\tilde{H}_1(\xi_1,\xi_2)|\le C\frac{|\xi|}{t}\frac{|\xi_1|^3|\xi_3|}{|\xi_1-\xi_3||1-\xi_2|}$$
$$+C|\tilde{H}_1(\xi_1,\xi_2)|\le C\frac{|\xi|}{t}\frac{|\xi_1|^2|\xi_3|(|\xi_1|+|\xi_3|)^{1+2\gamma}}{|\xi_1-\xi_3|^{1+\gamma}|1-\xi_2|^{1+\gamma}}.$$

Thanks to (6.33) and (6.37) with $\gamma=1/4+j/2$, $\mathrm{I}_{3,2,3}$ is bounded by

$$\frac{|\xi|^{\frac{13}{2}+\alpha+j}}{t^{1+\alpha}}\iint_D\frac{|\xi_1|^2|\xi_2||\xi_3|^{1+2\alpha}(|\xi_1|+|\xi_3|)^{\frac{3}{2}+j}|F_{1,2}|\,d\xi_1d\xi_2}{|\xi_1-\xi_3|^{\frac{5}{4}+\frac{j}{2}}|1-\xi_2|^{\frac{5}{4}+\frac{j}{2}}|\xi_2-\xi_3|^{2\alpha-1}|1-\xi_1|^{\alpha}|\xi_2|^{1-2\alpha}}$$
$$\le\frac{CK_3(\alpha)}{t^{1+\alpha}}\left\|\xi_1^{2-\alpha}\xi_2\xi_3^{1+2\alpha}(|\xi_1|+|\xi_3|)^{\frac{3}{2}+j}\hat{v}_\xi(\xi_1)\hat{v}_\xi(\xi_2)\hat{v}(\xi_3)\right\|_{\mathbf{L}^2_{\xi_1,\xi_2}}$$
$$\le\frac{CK_3(\alpha)}{t^{1+\alpha}}\|\xi\hat{v}_\xi\|_{\mathbf{L}^2}\left(\left\|\xi^{\frac{7}{2}-\alpha+j}\hat{v}_\xi\right\|_{\mathbf{L}^2}\left\|\xi^{1+2\alpha}\hat{v}\right\|_{\mathbf{L}^\infty}+\left\|\xi^{2-\alpha}\hat{v}_\xi\right\|_{\mathbf{L}^2}\left\|\xi^{\frac{5}{2}+2\alpha+j}\hat{v}\right\|_{\mathbf{L}^\infty}\right),$$

where

$$K_3(\alpha)=\left(\iint_D\frac{|\xi_1|^{2\alpha}|\xi_2-\xi_3|^{2-4\alpha}d\xi_1d\xi_2}{|\xi_1-\xi_3|^{\frac{5}{2}+j}|1-\xi_2|^{\frac{5}{2}+j}|1-\xi_1|^{2\alpha}|\xi_2|^{2(1-2\alpha)}}\right)^{\frac{1}{2}}.$$

Since $|\xi_2-\xi_3|\le 1+3|\xi_2-1|/2+|\xi_1-\xi_3|$, we have $K_3(\alpha)\le C$ for $1/4<\alpha<1/2$. Thus, by (6.35), we obtain for $1/4<\alpha<1/2$,

$$|\mathrm{I}_{3,2,3}|\le\frac{C}{t^{1+\alpha}}\left(\|v(t)\|_{\mathbf{H}^{\frac{5}{2}+2\alpha+j}}+\|xv_x(t)\|_{\mathbf{H}^{\frac{3}{2}+2\alpha+j}}\right)^3.$$

Estimate of $\mathrm{I}_{3,2,4}$. Note that

$$\left|\partial_{\xi_1}\tilde{H}_1(\xi_1,\xi_2)\right|\le C\frac{|\xi|}{t}\left|\frac{1}{\xi_1^2}-\frac{1}{\xi_3^2}\right|^{-2}\left|\frac{1}{\xi_1^3}+\frac{1}{\xi_3^3}\right|+C|\tilde{H}_1(\xi_1,\xi_2)| \tag{6.38}$$
$$\le C\frac{|\xi|}{t}\frac{|\xi_1||\xi_3|(|\xi_1|+|\xi_3|)^2}{|\xi_1-\xi_3|^2|1-\xi_2|}+C|\tilde{H}_1(\xi_1,\xi_2)|\le C\frac{|\xi|}{t}\frac{|\xi_1||\xi_3|(|\xi_1|+|\xi_3|)^{2+2\gamma}}{|\xi_1-\xi_3|^{1+\gamma}|1-\xi_2|^{1+\gamma}}$$

in D. Then, by (6.33) and (6.38) with $\gamma=1/4+(j-\alpha)/2$, $\mathrm{I}_{3,2,4}$ is bounded by

$$\frac{|\xi|^{\frac{13}{2}+\alpha+j}}{t^{1+\alpha}}\iint_D\frac{|\xi_1||\xi_2||\xi_3|^{1+2\alpha}(|\xi_1|+|\xi_3|)^{\frac{5}{2}-\alpha+j}(|F_2|+|F_3|)\,d\xi_1d\xi_2}{|\xi_1-\xi_3|^{\frac{5}{4}+\frac{j-\alpha}{2}}|1-\xi_2|^{\frac{5}{4}+\frac{j-\alpha}{2}}|\xi_2-\xi_3|^{2\alpha-1}|1-\xi_1|^{\alpha}|\xi_2|^{1-2\alpha}}$$
$$\le\frac{CK_4(\alpha)}{t^{1+\alpha}}\left\|\xi_1\xi_2\xi_3^{1+2\alpha}(|\xi_1|+|\xi_3|)^{\frac{5}{2}-\alpha+j}(|\hat{v}(\xi_1)\hat{v}_\xi(\xi_2)\hat{v}(\xi_3)|+|\hat{v}(\xi_1)\hat{v}(\xi_2)\hat{v}_\xi(\xi_3)|)\right\|_{\mathbf{L}^2_{\xi_1,\xi_2}}$$
$$\le\frac{CK_4(\alpha)}{t^{1+\alpha}}\|\xi\hat{v}_\xi\|_{\mathbf{L}^2}\left(\left\|\xi^{\frac{7}{2}-\alpha+j}\hat{v}\right\|_{\mathbf{L}^2}\left\|\xi^{1+2\alpha}\hat{v}\right\|_{\mathbf{L}^\infty}+\|\xi\hat{v}\|_{\mathbf{L}^\infty}\left\|\xi^{\frac{7}{2}+\alpha+j}\hat{v}\right\|_{\mathbf{L}^2}\right)$$
$$+\frac{CK_4(\alpha)}{t^{1+\alpha}}\|\xi\hat{v}\|_{\mathbf{L}^\infty}\left(\left\|\xi^{\frac{7}{2}-\alpha+j}\hat{v}\right\|_{\mathbf{L}^2}\left\|\xi^{1+2\alpha}\hat{v}_\xi\right\|_{\mathbf{L}^2}+\|\xi\hat{v}\|_{\mathbf{L}^2}\left\|\xi^{\frac{7}{2}+\alpha+j}\hat{v}_\xi\right\|_{\mathbf{L}^2}\right),$$

where

$$K_4(\alpha) = \left(\iint_D \frac{|\xi_2 - \xi_3|^{2-4\alpha} d\xi_1 d\xi_2}{|\xi_1 - \xi_3|^{\frac{5}{2}-\alpha+j} |1 - \xi_2|^{\frac{5}{2}-\alpha+j} |1 - \xi_1|^{2\alpha} |\xi_2|^{2(1-2\alpha)}} \right)^{\frac{1}{2}}.$$

Since $K_4(\alpha) \le C$ for $1/4 < \alpha < 1/2$, it follows from (6.35) that

$$|\mathrm{I}_{3,2,4}| \le \frac{C}{t^{1+\alpha}} \left(\|xv_x(t)\|_{\mathbf{H}^{\frac{5}{2}+\alpha+j}} + \|v(t)\|_{\mathbf{H}^{\frac{7}{2}+\alpha+j}} \right)^3,$$

for $1/4 < \alpha < 1/2$.

Estimate of $\mathrm{I}_{3,2,5}$. By (6.36) and (6.38) with $\gamma = 1/4 - 3\beta + j/2$ where $0 < \beta < 1/12$, we have

$$\begin{aligned}
|\mathrm{I}_{3,2,5}| &\le \frac{|\xi|^{\frac{11}{2}+2\beta+j}}{t^{1+2\beta}} \iint_D \frac{|\xi_1| |\xi_2|^{4\beta-1} |\xi_3|^{4\beta} (|\xi_2| + |\xi_3|)^2 (|\xi_1| + |\xi_3|)^{\frac{5}{2}-6\beta+j} |F_0| d\xi_1 d\xi_2}{|\xi_1 - \xi_3|^{\frac{5}{4}-3\beta+\frac{j}{2}} |1 - \xi_2|^{\frac{5}{4}-3\beta+\frac{j}{2}} |\xi_2 - \xi_3|^{4\beta} |1 - \xi_1|^{2\beta}} \\
&\le \frac{CK_5(\beta)}{t^{1+2\beta}} \left\| \xi_1 \xi_2^{4\beta-1} \xi_3^{4\beta} (|\xi_2| + |\xi_3|)^2 (|\xi_1| + |\xi_3|)^{\frac{5}{2}-6\beta+j} \hat{v}(\xi_1) \hat{v}(\xi_2) \hat{v}(\xi_3) \right\|_{\mathbf{L}^2_{\xi_1,\xi_2}} \\
&\le \frac{CK_5(\beta)}{t^{1+2\beta}} \left\| \xi^{\frac{7}{2}-6\beta+j} \hat{v} \right\|_{\mathbf{L}^2} \left(\|\xi^{1+4\beta} \hat{v}\|_{\mathbf{L}^\infty} \|\xi^{4\beta} \hat{v}\|_{\mathbf{L}^2} + \|\xi^{4\beta-1} \hat{v}\|_{\mathbf{L}^2} \|\xi^{4\beta+2} \hat{v}\|_{\mathbf{L}^\infty} \right) \\
&\quad + \frac{CK_5(\beta)}{t^{1+2\beta}} \|\xi \hat{v}\|_{\mathbf{L}^\infty} \left(\|\xi^{4\beta+1} \hat{v}\|_{\mathbf{L}^2} \left\| \xi^{\frac{5}{2}-2\beta+j} \hat{v} \right\|_{\mathbf{L}^2} + \|\xi^{4\beta-1} \hat{v}\|_{\mathbf{L}^2} \left\| \xi^{\frac{9}{2}-2\beta+j} \hat{v} \right\|_{\mathbf{L}^2} \right),
\end{aligned}$$

where

$$K_5(\beta) = \left(\iint_D \frac{d\xi_1 d\xi_2}{|\xi_1 - \xi_3|^{\frac{5}{2}-6\beta+j} |1 - \xi_2|^{\frac{5}{2}-6\beta+j} |\xi_2 - \xi_3|^{8\beta} |1 - \xi_1|^{4\beta}} \right)^{\frac{1}{2}}.$$

Since $K_5(\beta) \le C$, it follows from (6.35) that

$$|\mathrm{I}_{3,2,5}| \le \frac{C}{t^{1+2\beta}} \left(\|xv_x(t)\|_{\mathbf{H}^{1+4\beta}} + \|v(t)\|_{\dot{\mathbf{H}}^{4\beta-1}} + \|v(t)\|_{\mathbf{H}^{\frac{9}{2}-2\beta+j}} \right)^3,$$

for $0 < \beta < 1/12$.

Estimate of $\mathrm{I}_{3,2,6}$. Since $|\xi_1| \le |\xi_1 - \xi_3| + |1 - \xi_2|$, $|\xi_1 - \xi_3| \ge 10^{-2}$ and $|1 - \xi_2| \ge 10^{-2}$ in D, we get for $0 \le \gamma$,

$$\begin{aligned}
&\left| \partial_{\xi_1} \partial_{\xi_2} \tilde{H}_1(\xi_1, \xi_2) \right| \le C \frac{|\xi|}{t} \left| \frac{1}{\xi_1^2} - \frac{1}{\xi_3^2} \right|^{-3} \left| \frac{1}{\xi_1^3} + \frac{1}{\xi_3^3} \right| \left| \frac{1}{\xi_3^3} \right| + C \frac{|\xi|}{t} \left| \frac{1}{\xi_1^2} - \frac{1}{\xi_3^2} \right|^{-2} \left| \frac{1}{\xi_3^4} \right| \\
&+ C|\tilde{H}_1(\xi_1, \xi_2)| + C|\partial_{\xi_1} \tilde{H}_1(\xi_1, \xi_2)| + C|\partial_{\xi_2} \tilde{H}_1(\xi_1, \xi_2)| \le C \frac{|\xi|}{t} \frac{|\xi_1|^3 (|\xi_1| + |\xi_3|)^2}{|\xi_1 - \xi_3|^3 |1 - \xi_2|^2} \\
&+ C \frac{|\xi|}{t} \frac{|\xi_1| (|\xi_1| + |\xi_3|)^{3+2\gamma}}{|\xi_1 - \xi_3|^{1+\gamma} |1 - \xi_2|^{1+\gamma}} \le C \frac{|\xi|}{t} \frac{|\xi_1| (|\xi_1| + |\xi_3|)^{3+2\gamma}}{|\xi_1 - \xi_3|^{1+\gamma} |1 - \xi_2|^{1+\gamma}}.
\end{aligned}$$

By (6.33) and the above inequality with $\gamma = j/2$, we have

$$|\mathrm{I}_{3,2,6}| \le \frac{|\xi|^{\frac{11}{2}+\alpha+j}}{t^{1+\alpha}} \iint_D \frac{|\xi_1| |\xi_2|^{2\alpha} |\xi_3|^{2\alpha} |\xi_2 - \xi_3|^{1-2\alpha} (|\xi_1| + |\xi_3|)^{3+j} |F_0| d\xi_1 d\xi_2}{|\xi_1 - \xi_3|^{1+\frac{j}{2}} |1 - \xi_2|^{1+\frac{j}{2}} |1 - \xi_1|^{\alpha}}.$$

It follows from the Hölder inequality that

$$
\begin{aligned}
|\mathrm{I}_{3,2,6}| &\le \frac{CK_6(\alpha)}{t^{1+\alpha}} \left\| \xi_1^{1-\alpha}\xi_2^{2\alpha}\xi_3^{2\alpha} |\xi_2-\xi_3|^{1-2\alpha} (|\xi_1|+|\xi_3|)^{3+j} \hat{v}(\xi_1)\hat{v}(\xi_2)\hat{v}(\xi_3) \right\|_{\mathbf{L}^4_{\xi_1,\xi_2}} \\
&\le \frac{CK_6(\alpha)}{t^{1+\alpha}} \left\|\xi^{4-\alpha+j}\hat{v}\right\|_{\mathbf{L}^\infty} \left\|\xi\hat{v}\right\|_{\mathbf{L}^4} \left\|\xi^{2\alpha}\hat{v}\right\|_{\mathbf{L}^4} \\
&\quad + \frac{CK_6(\alpha)}{t^{1+\alpha}} \left\|\xi^{1-\alpha}\hat{v}\right\|_{\mathbf{L}^4} \left(\left\|\xi\hat{v}\right\|_{\mathbf{L}^4} \left\|\xi^{3+2\alpha+j}\hat{v}\right\|_{\mathbf{L}^\infty} + \left\|\xi^{1-\alpha}\hat{v}\right\|_{\mathbf{L}^4} \left\|\xi^{4+j}\hat{v}\right\|_{\mathbf{L}^\infty} \right),
\end{aligned}
$$

for $0 < \alpha < 1/2$, where

$$
K_6(\alpha) = \left(\iint_D \frac{|\xi_1|^{\frac{4\alpha}{3}} \, d\xi_1 d\xi_2}{|\xi_1-\xi_3|^{\frac{4}{3}\left(1+\frac{j}{2}\right)} |1-\xi_2|^{\frac{4}{3}\left(1+\frac{j}{2}\right)} |1-\xi_1|^{\frac{4\alpha}{3}}} \right)^{\frac{3}{4}}.
$$

Note that $K_6(\alpha) \le C$ for $0 < \alpha < 3/4$ and

$$
\begin{aligned}
& \left\| |x|^{1/2} \left(-\partial_x^2\right)^{\frac{a}{2}} v \right\|_{\mathbf{L}^2}^2 = \left(|x|^{1/2} \left(-\partial_x^2\right)^{\frac{a}{2}} v, |x|^{1/2} \left(-\partial_x^2\right)^{\frac{a}{2}} v \right) \\
= & -\left(\partial_x |x| \left(-\partial_x^2\right)^{\frac{a}{2}} v, \partial_x^{-1} \left(-\partial_x^2\right)^{\frac{a}{2}} v \right) \le C \left(\|v\|_{\mathbf{H}^a} + \|xv_x\|_{\mathbf{H}^a} \right) \|v\|_{\dot{\mathbf{H}}^{-1+a}}
\end{aligned}
$$

from which

$$
\begin{aligned}
\text{(6.39)} \quad \left\|\xi^a \hat{v}\right\|_{\mathbf{L}^4} &\le C \left\| \left(-\partial_x^2\right)^{\frac{a}{2}} v \right\|_{\mathbf{L}^{\frac{4}{3}}} \le C\|v\|_{\mathbf{H}^a} + C \left\| |x|^{1/2} \left(-\partial_x^2\right)^{\frac{a}{2}} v \right\|_{\mathbf{L}^2} \\
&\le C\|v\|_{\mathbf{H}^a} + C\|v\|_{\dot{\mathbf{H}}^{-1+a}} + C\|xv_x\|_{\mathbf{H}^a}.
\end{aligned}
$$

Then, by (6.35) and (6.39) with $a = 2\alpha, 1-\alpha, 1$, we obtain

$$
|\mathrm{I}_{3,2,6}| \le \frac{C}{t^{1+\alpha}} \left(\|v(t)\|_{\mathbf{H}^{4+j}} + \|v(t)\|_{\dot{\mathbf{H}}^{-1+2\alpha}} + \|v(t)\|_{\dot{\mathbf{H}}^{-\alpha}} + \|xv_x(t)\|_{\mathbf{H}^{3+j}} \right)^3,
$$

for $0 < \alpha < 1/2$.

Collecting these estimates, we obtain

$$
\text{(6.40)} \qquad |\mathrm{I}_{3,2}| \le \frac{C}{t^{1+\delta}} \left(\|xv_x(t)\|_{\mathbf{H}^5} + \|v(t)\|_{\mathbf{H}^6} + \|v(t)\|_{\dot{\mathbf{H}}^{-1}} \right)^3,
$$

for $0 < \delta < 1/12$. In the same manner as in the proof of (6.40), we can obtain the estimate of $\mathrm{I}_{3,1}$ since there exists $C > 0$ such that $|\xi_2 - \xi_3| \ge C$ and $|\xi_1 - 1| \ge C$ in the support of $\Phi_3 \cap \Phi_{3,1}$. More precisely, we integrate by parts with respect to ξ_2 instead of ξ_1 in (6.29), and then, we integrate by parts with respect to ξ_1 instead of ξ_2 in (6.31). After that, by using the same manner in the case of $\mathrm{I}_{3,2,1}$, $\mathrm{I}_{3,2,2}$, $\mathrm{I}_{3,2,3}$, $\mathrm{I}_{3,2,4}$, $\mathrm{I}_{3,2,5}$ and $\mathrm{I}_{3,2,6}$, we obtain

$$
\text{(6.41)} \qquad |\mathrm{I}_{3,1}| \le \frac{C}{t^{1+\delta}} \left(\|xv_x(t)\|_{\mathbf{H}^5} + \|v(t)\|_{\mathbf{H}^6} + \|v(t)\|_{\dot{\mathbf{H}}^{-1}} \right)^3,
$$

for $0 < \delta < 1/12$. Because of (6.40) and (6.41), we get (6.28). This completes the proof of Lemma 6.5. □

We finally give the proof of (6.11).

PROOF OF (6.11). By Lemma 6.4 and 6.5, we have

$$\begin{aligned} \mathrm{I} &= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 \\ &= i\frac{3\xi^4|\xi|^{\frac{3}{2}+j}}{\sqrt{2}t}\left|\hat{v}(t,\xi)\right|^2\hat{v}(t,\xi) + \frac{\xi^4|\xi|^{\frac{3}{2}+j}}{3^3\sqrt{6}t}e^{i\frac{11t}{\xi}}\hat{v}\left(t,\frac{\xi}{3}\right)^3 \\ &\quad + O\left(t^{-1-\delta}\left(\|v(t)\|_{\mathbf{H}^6} + \|v(t)\|_{\dot{\mathbf{H}}^{-1}} + \|xv_x(t)\|_{\mathbf{H}^5}\right)^3\right). \end{aligned}$$

Since $v = \mathcal{U}(-t)u$ and $\|xv_x(t)\|_{\mathbf{H}^5} = \|x\mathcal{U}(-t)u_x\|_{\mathbf{H}^5} = \|\mathcal{J}u_x\|_{\mathbf{H}^5}$, we obtain desired asymptotic expansion. □

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