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<th><strong>Title</strong></th>
<th>Homology localizations after applying some right adjoint functors</th>
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Osaka University
0. Introduction

Each homology theory \( E_* \) determines a natural \( E_* \)-localization \( \eta: X \to L_\infty X \) in the homotopy category \( hCW \) of \( CW \)-complexes or \( hCW S \) of \( CW \)-spectra. It is full of interest to study the behavior of \( E_* \)-localizations after application of various functors \( T \) to the category \( hCWS \) or \( hCWS^0 \). Consider as \( T \) the 0-th space functor \( \Omega^0: hC \to hC(W) \) which is right adjoint to the suspension spectrum functor \( \Sigma^0 \). Bousfield [4] showed that the \( E_* \)-localization of an infinite loop space \( \Omega^\omega X \) is still an infinite loop space. More precisely, he proved

**Theorem 0.1** ([4, Theorem 1.1]). There exists an idempotent monad \( L: hCWS_0 \to hCWS_0 \) and \( \eta: 1 \to L \) such that the map \( \Omega^\omega \eta: \Omega^\omega X \to \Omega^\omega LX \) is an \( E_* \)-localization in \( hCW \). Here \( hCWS_0 \) denotes the full subcategory of \( hCW S \) consisting of \((-1)\)-connected \( CW \)-spectra.

As remarked by Bousfield [4], this implies

**Proposition 0.2.** If \( f: A \to B \) is an \( E_* \)-equivalence in \( hCW \), then so is \( \Omega^\omega \Sigma^0 f: \Omega^\omega \Sigma^0 A \to \Omega^\omega \Sigma^0 B \).

On the other hand, Kuhn [7, Proposition 2.4] gave recently a simple proof of Proposition 0.2 using the stable decompositions of \( \Omega^\omega \Sigma^0 A \) and \( \Omega^\omega \Sigma^0 B \) (see [9]).

In this note we will show that Proposition 0.2 is essential to the existence theorem 0.1. Thus, by use of only Proposition 0.2 we give a direct proof of the existence theorem 0.1 along the primary line of Bousfield [1, 2 and 3]. In our proof we don't need the knowledge of very special \( \Gamma \)-spaces although Bousfield did in [4].

Let \( T: \mathcal{C} \to \mathcal{B} \) be a functor with a left adjoint \( S \) and \( \mathcal{W} \) be a morphism class in \( \mathcal{B} \). In §1 we introduce \( T^*\mathcal{W} \)- and \( (\mathcal{W}, T) \)-localizations in \( \mathcal{C} \) and discuss a relation between them. Following our notation Theorem 0.1 says that there exists an \( (E_*, \Omega^\omega) \)-localization in \( hCWS_0 \) where \( E_* \) stands for the morphism class of \( E_* \)-equivalences in \( hCW \). Don't confuse our notation with Bousfield's [4]. We next give three conditions (C.1)–(C.3) under which we can construct
a \((\mathcal{W}, T)\)-localization \(\eta: X \to LX\) for each \(X \in \mathcal{C}\) where \(\mathcal{C}=hC\mathcal{W}\) or \(hC\mathcal{W}S\), by the same method as Bousfield used in constructing \(E^*_\ast\)-localizations in [1, 3].

It might be indistinctly known that the 0-th space functor \(\Omega^\ast\) converts generally a cofiber sequence in \(hC\mathcal{W}S\) to a fiber sequence in \(hC\mathcal{W}\). Nevertheless we prove this fact in §2 by making use of secondary operations on mappings [10]. This result yields a key lemma for proving the existence theorem of \((E^*_\ast, \Omega^\ast)\)-localization.

In §3 we first check that the conditions (C.1)-(C.3) are satisfied for the triple \((\mathcal{W}, T, S)=(E^*_\ast, \Omega^\ast, \Sigma^\ast)\). As a result we can give a new proof of the existence theorem of \((E^*_\ast, \Omega^\ast)\)-localization in \(hC\mathcal{W}S\). Since the equivariant version of Proposition 0.2 is valid when \(G\) is a finite group (use [8, V]), we obtain the equivariant version of Theorem 0.1. Of course we may prove it by using very special \(G\)-\(\Gamma\) spaces following Bousfield's approach. Let \(G\) be a compact Lie group and \(\phi_K\) be the \(K\)-fixed point functors. Applying our method to \(T=\prod \phi_K\) we also obtain the existence theorem of \((\prod E^*_K, \prod \phi_K)\)-localization which was studied in [11, Theorem 2.1].

1. \((\mathcal{W}, T)\)- and \(T^*\mathcal{W}\)-localizations

1.1. Let \(\mathcal{B}\) be a category. We call a functor and transformation \(L: \mathcal{B} \to \mathcal{B}\), \(\eta: 1 \to L\) idempotent if \(\eta_L = L\eta\): \(LA \to L^2A\) and it is an equivalence for each \(A \in \mathcal{B}\). It is easy to show

\[(1.1) \text{ A functor } L: \mathcal{B} \to \mathcal{B} \text{ and transformation } \eta: 1 \to L \text{ is idempotent if and only if } \eta_A: A \to LA \text{ induces a bijection } \eta^*_A: \mathcal{B}(LA, LB) \to \mathcal{B}(A, LB) \text{ for any } A, B \in \mathcal{B}.\]

Given a morphism class \(\mathcal{W}\) in a category \(\mathcal{B}\), an object \(D \in \mathcal{B}\) is called \(\mathcal{W}\)-local if each \(f: A \to B\) in \(\mathcal{W}\) induces a bijection \(f^*: \mathcal{B}(B, D) \to \mathcal{B}(A, D)\). For each \(A \in \mathcal{B}\) a morphism \(g: A \to D\) is called a \(\mathcal{W}\)-localization of \(A\) if \(g\) belongs to \(\mathcal{W}\) and \(D\) is \(\mathcal{W}\)-local. If all objects of \(\mathcal{B}\) admit \(\mathcal{W}\)-localizations, then there exists a functor \(L: \mathcal{B} \to \mathcal{B}\) and transformation \(\eta: 1 \to L\) such that \(\eta_A: A \to LA\) is a \(\mathcal{W}\)-localization for each \(A \in \mathcal{B}\). Such an \((L, \eta)\) is unique up to natural equivalence, so it is called the \(\mathcal{W}\)-localization in \(\mathcal{B}\). It follows from (1.1) that the \(\mathcal{W}\)-localization is idempotent [1].

Let \(T: \mathcal{C} \to \mathcal{B}\) be a functor and \(\mathcal{W}\) be a morphism class in \(\mathcal{B}\). An idempotent monad \(L: \mathcal{C} \to \mathcal{C}\) and \(\eta: 1 \to L\) is called the \((\mathcal{W}, T)\)-localization in \(\mathcal{C}\) if \(T\eta_X: TX \to TLX\) is a \(\mathcal{W}\)-localization for each \(X \in \mathcal{C}\).

We here restrict to a morphism class \(\mathcal{W}\) in \(\mathcal{B}\) satisfying the condition:

\[(C.0) \text{ i) Each equivalence } f: A \to B \text{ is contained in } \mathcal{W}. \]
\[\text{ ii) If two of } f: A \to B, g: B \to C \text{ and } gf: A \to C \text{ are in } \mathcal{W}, \text{ so is the third.}\]

**Lemma 1.1.** Let \(T: \mathcal{C} \to \mathcal{B}\) be a functor with a left adjoint \(S\), and \(\mathcal{W}\) be
a morphism class in $\mathcal{B}$ satisfying the condition (C.0). Assume that there exists a $(\mathcal{W}, T)$-localization $(L, \eta)$ in $\mathcal{C}$. If $f: A \to B$ is contained in $\mathcal{W}$, then so is $TSf: TSA \to TSB$. (Cf., [4, Remark following Proposition 1.2]).

Proof. Each $f: A \to B$ in $\mathcal{W}$ induces a bijection $f^*: \mathcal{B}(B, TLX) \to \mathcal{B}(A, TLX)$ for any $X \in \mathcal{C}$ since $TLX$ is $\mathcal{W}$-local. By adjointness $Sf^*: \mathcal{C}(SB, LX) \to \mathcal{C}(SA, LX)$ is bijective, too. Making use of (1.1) we easily verify that $LSf: LSA \to LSB$ is an equivalence. It is now immediate that $TSf: TSA \to TSB$ is in $\mathcal{W}$ because $\mathcal{W}$ satisfies the condition (C.0).

Given a functor $T: \mathcal{C} \to \mathcal{B}$ and a morphism class $\mathcal{W}$ in $\mathcal{B}$ we denote by $T^*\mathcal{W}$ the morphism class in $\mathcal{C}$ which consists of all $u: X \to Y$ with $Tu \in \mathcal{W}$. We here study a relation between the $T^*\mathcal{W}$-localization and the $(\mathcal{W}, T)$-localization.

**Proposition 1.2.** Let $T: \mathcal{C} \to \mathcal{B}$ be a functor with a left adjoint $S$, and $\mathcal{W}$ be a morphism class in $\mathcal{B}$ satisfying the condition (C.0). Assume that $u: X \to Y \in \mathcal{C}$ is an equivalence whenever so is $Tu: TX \to TY$. Then an idempotent monad $(L, \eta)$ is the $(\mathcal{W}, T)$-localization in $\mathcal{C}$ if and only if it is the $T^*\mathcal{W}$-localization in $\mathcal{C}$ and moreover $TSf: TSA \to TSB$ is in $\mathcal{W}$ when so is $f: A \to B$.

Proof. The "if" part: It is sufficient to show that $TLZ$ is $\mathcal{W}$-local for each $Z \in \mathcal{C}$. Given any $f: A \to B$ in $\mathcal{W}$, $Sf^*: \mathcal{C}(SB, LZ) \to \mathcal{C}(SA, LZ)$ is bijective since $LZ$ is $T^*\mathcal{W}$-local. By adjointness this means that $TLZ$ is $\mathcal{W}$-local.

The "only if" part: The latter part follows from Lemma 1.1. So we only have to show that $LZ$ is $T^*\mathcal{W}$-local for each $Z \in \mathcal{C}$. Taking any $u: X \to Y$ in $T^*\mathcal{W}$, $Tu: TLX \to TLY$ is an equivalence since it is in $\mathcal{W}$ and $TLX$, $TLY$ are both $\mathcal{W}$-local. Under our assumption $Lu: LX \to LY$ is also an equivalence. It is immediate from (1.1) that $u^*: \mathcal{C}(Y, LZ) \to \mathcal{C}(X, LZ)$ is bijective, thus $LZ$ is $T^*\mathcal{W}$-local.

1.2. Let $G$ be a compact Lie group. Let $G\mathcal{W}$ denote the category of based $G$-spaces with $G$-fixed basepoint, and $GS\mathcal{A}$ the category of $G$-spectra indexed on an indexing set $\mathcal{A}$ in a $G$-universe $U$. Let us write $GSU$ for $GS\mathcal{A}$ when $\mathcal{A}$ is the standard indexing set in $U$. The category $GS\mathcal{A}$ is equivalent to $GSU$ for any indexing set $\mathcal{A}$ in $U$. The suspension spectrum functor $\Sigma^\infty: G\mathcal{W} \to GS\mathcal{A}$ has a right adjoint functor $\Omega^\infty: GS\mathcal{A} \to G\mathcal{W}$ called the 0-th space functor [8, Proposition II. 2.3].

Let $hG\mathcal{W}$ or $hGS\mathcal{A}$ be the category obtained from the homotopy category $hG\mathcal{W}$ or $hGS\mathcal{A}$ by formally inverting the weak equivalences respectively. The category $hG\mathcal{W}$ is equivalent to the homotopy category $hGC\mathcal{W}$ of $G$-$CW$ complexes and cellular maps. Similarly the stable category $hGS\mathcal{A}$ is equivalent to the homotopy category $hGC\mathcal{W}\mathcal{S}\mathcal{A}$ of $G$-$CW$ spectra and cellular maps.
Let us abbreviate by $GC$ the category $G CW$ of $G$-CW complexes or the category $GCW,J$ of $G$-CW spectra indexed on $J$, and by $hGC$ its homotopy category. Let $S:B \rightarrow hGC$ be a functor and $W$ be a morphism class in $B$. For a fixed infinite cardinal number $\sigma$ we consider the subclass $W_{\sigma} = \{ f_{a}: A_{a} \rightarrow B_{a} \}_{a \in I}$ consisting of morphisms in $W$ with $\#A_{a} \leq \sigma$ and $\#B_{a} \leq \sigma$, where $\#X$ denotes the number of $G$-cells in $X \in GC$. Note that $Sf_{a}: SA_{a} \rightarrow SB_{a}$ may be represented by an inclusion $i_{a}$, when replacing $SB_{a}$ by the mapping cylinder of $Sf_{a}$ if necessary.

We say an inclusion map $u: X \rightarrow Y \in GC$ admits an $(S, W_{\sigma})$-decomposition if there exists a transfinite sequence

$$X = X_{0} \subset X_{1} \subset \cdots \subset X_{s} \subset X_{s+1} \subset \cdots \subset X_{\gamma} = Y$$

in $GC$ such that $X_{\lambda} = \bigcup_{\lambda < \lambda} X_{\lambda}$ when $\lambda$ is a limit ordinal and $X_{s} \subset X_{s+1}$ is obtained from a pushout square

$$\begin{array}{ccc}
\vee SA_{a} & \rightarrow & X_{s} \\
\vee i_{a} \downarrow & & \downarrow \\
\vee SB_{a} & \rightarrow & X_{s+1}
\end{array}$$

(1.2)

in $GC$ where the inclusion $i_{a}$ is a representative of $Sf_{a}$ for $f_{a}: A_{a} \rightarrow B_{a}$ in $W_{\sigma}$.

Let $\gamma$ be the first infinite ordinal of cardinality greater than $\sigma$. For each $X \in GC$ we inductively construct a transfinite sequence

$$X = X_{0} \subset X_{1} \subset \cdots \subset X_{s} \subset X_{s+1} \subset \cdots$$

in $GC$ where $X_{\lambda} = \bigcup_{\lambda < \lambda} X_{\lambda}$ for each limit ordinal $\lambda$ and $X_{s} \subset X_{s+1}$ is given by the pushout square

$$\begin{array}{ccc}
\vee_{a \in I} \vee gSA_{a} & \rightarrow & X_{s} \\
\downarrow & & \downarrow \\
\vee_{a \in I} \vee gSB_{a} & \rightarrow & X_{s+1}
\end{array}$$

(1.3)

in which $g$ ranges over all representative cellular maps $SA_{a} \rightarrow X_{s}$ (cf., [2]). Putting $LX = X_{\gamma}$, we see immediately

(1.4) The inclusion map $\eta_{X}: X \rightarrow LX$ admits an $(S, W_{\sigma})$-decomposition.

Each cellular map $k: SA_{a} \rightarrow LX$ passes through $SB_{a}$ because the image of $k$ is contained in $X_{s}$ for some $s < \gamma$. Therefore any $f_{a}: A_{a} \rightarrow B_{a}$ in $W_{\sigma}$ induces a surjection $Sf_{a}^{*}: hGC(SB_{a}, LX) \rightarrow hGC(SA_{a}, LX)$. This implies

(1.4) If an inclusion map $v: Y \rightarrow Z$ admits an $(S, W_{\sigma})$-decomposition, then $v^{*}: hGC(Z, LX) \rightarrow hGC(Y, LX)$ is surjective.
each of which is represented by some inclusion having an \((S, W_\sigma)\)-decomposition. We now assume that \(S W_\sigma\) satisfies the condition:

(C.1) Given \(u: X \to Y\) in \(S W_\sigma\) and \(f, g: Y \to Z\) such that \(fu = gu\) in \(hGC\), there exists \(w: Z \to W\) in \(S W_\sigma\) such that \(wf = wg\) in \(hGC\).

Under the condition (C.1) it is easy to show

(1.5) Each \(v: Y \to Z\) in \(S_1 W_\sigma\) induces a bijection \(v^*: hGC(Z, LX) \to hGC(Y, LX)\) (see [1, Lemma 2.5]).

By use of (1.1), (1.3) and (1.5) we obtain

**Lemma 1.3.** Let \(S: B \to hGC\) be a functor and \(W\) be a morphism class in \(B\). Fix an infinite cardinal number \(\sigma\) and assume that the morphism class \(S W_\sigma\) satisfies the condition (C.1). Then the inclusion map \(\eta_X: X \to LX\) give rise to an idempotent monad \((L, \eta)\) in \(hGC\).

Let \(S: B \to hGC\) be a functor with a right adjoint \(T\) and \(W\) be a morphism class in \(B\). We moreover assume that the following conditions are satisfied:

(C.2) For each \(f: A \to B\) in \(W\) the morphism \(Sf: SA \to SB\) is in \(S W_\sigma\).

(C.3) If \(u: X \to Y\) is in \(S W_\sigma\), then the morphism \(Tu: TX \to TY\) is in \(W\).

Note that both (C.2) and (C.3) imply

(C.4) If \(f: A \to B\) is in \(W\), then so is \(TSf: TSA \to TSB\).

**Proposition 1.4.** Let \(T: hGC \to B\) be a functor with a left adjoint \(S\) and \(W\) be a morphism class in \(B\). Fix an infinite cardinal number \(\sigma\) and assume that the three conditions (C.1), (C.2) and (C.3) are all satisfied. Then there exists a \((W, T)\)-localization \((L, \eta)\) in \(hGC\).

Proof. Under our assumptions it follows from (1.3) and (1.5) that the morphism \(T \eta_X: TX \to TLX\) is a \(W\)-localization. The result is now immediate from Lemma 1.3.

2. Homotopy theoretic fiber sequences

Given maps \(d_1, d_2: K \land I^+ \to N\) in \(G \mathcal{G}\) such that \(d_1| K \times \{1\} = d_2| K \times \{0\}\) we define a \(G\)-map \(d_1 \land d_2: K \land I^+ \to N\) as \(d_1 \land d_2(x, t) = d_1(x, 2t)\) if \(0 \leq t \leq 1/2\) and to \(d_2(x, 2-2t)\) if \(1/2 \leq t \leq 1\). Consider a sequence \(K \xrightarrow{f} L \xrightarrow{g} M \xrightarrow{h} N\) in \(G \mathcal{G}\) such that the two composite \(gf, hg\) are both \(G\)-null homotopic. Then there are \(G\)-maps \(F: CK \to M\) and \(H: CL \to N\) such that \(F| K \times \{1\} = gf\) and \(H| L \times \{1\} = hg\) where \(C\) denotes the reduced cone functor. Two maps \(hF, H(Cf)\) give
rise to a $G$-map $d(hF, H(Cf)): \Sigma K \to N$ obtained as $d(hF, H(Cf)) = hF \downarrow H(f \land \tau)$ where $\Sigma$ denotes the reduced suspension functor and $\tau: I^+ \to I^+$ is the twisting map. The bracket $\langle f, g, h \rangle$ is defined to be the double coset of $h_\ast[\Sigma K, M]_G$ and $\Sigma f^* [\Sigma L, N]_G$ in $[\Sigma K, N]_G$ determined by $[d(hF, H(Cf))]$.

Consider the mapping cocylinder

$$E_h = \{(z, \omega) \in M \times F(I, N); h(z) = \omega(0)\}$$

of $h: M \to N$. The $G$-map $p: E_h \to N$ defined to be $p(z, \omega) = \omega(1)$ is a $G$-fibration. Let us denote by $F_h$ the fiber of $p$ over the basepoint of $N$, which is called the mapping fiber of $h$. The $G$-map $q: F_h \to M$ defined to be $q(z, \omega) = z$ is a $G$-fibration, too. Notice that the fiber of $q$ is just the loop space $\Omega N$.

Assume that there exist $G$-maps $b: C_f \to M$, $a: \Sigma K \to N$ making the diagram below $G$-homotopy commutative

$$\begin{array}{ccc}
L & \to & C_f \\
\downarrow b & & \downarrow a \\
M & \to & N
\end{array}$$

where we write $C_f$ for the mapping cone of $f: K \to L$. According to [10, Theorem 3.3] the bracket $\langle f, g, h \rangle$ is represented by the map $a$. So we may choose $G$-maps $F: CK \to M$ and $H: CL \to N$ such as $F \downarrow K \times \{1\} = gf$, $H \downarrow L \times \{1\} = hg$ and $[d(hF, H(Cf))] = [a] \in [\Sigma K, N]_G$.

Using such a map $H$ we define a $G$-map $\beta: L \to F_h$ to be

$$\beta(y) = (g(y), H(1 \land \tau) \downarrow \{y\} \times I) \in M \times F(I, N).$$

As is easily seen, the following diagram

$$\begin{array}{ccc}
K & \to & L \\
\downarrow f & & \downarrow g \\
L & \to & M \\
\downarrow a & & \downarrow h \\
\Omega N & \to & F_h \\
\downarrow q & & \\
\Omega N & \to & M
\end{array}$$

is $G$-homotopy commutative where $a$ is the adjoint of $a$.

A sequence $K \to L \to M \to N$ in $G\Xi$ is said to be a fiber sequence in $hG\Xi$ if there exist weak equivalences $\beta: L \to F_h$, $\alpha: K \to \Omega N$ such that the diagram below is $G$-homotopy commutative:

$$\begin{array}{ccc}
K & \to & L \\
\downarrow \alpha & & \downarrow \beta \\
\Omega N & \to & F_h \\
\downarrow & & \downarrow \\
\Omega N & \to & M
\end{array}$$

**Proposition 2.1.** Let $X \to Y \to Z \to \Sigma X$ be a cofiber sequence in $hG\Sigma A$. 
Then the sequence $\Omega^\infty X \to \Omega^\infty Y \to \Omega^\infty Z \to \Omega^\infty \Sigma X$ is a fiber sequence in $hG\Sigma$.

**Proof.** Consider the following diagram

$$
\begin{array}{ccc}
\Sigma \Omega^\infty X & \to & \Sigma \Omega^\infty Y \\
\varepsilon \uparrow & & \downarrow \varepsilon \\
X & \to & Y \to Z \to \Sigma X
\end{array}
$$

in $hG\Sigma$ where $\varepsilon$'s are the adjunction maps. Both of horizontal rows are cofiber sequences in $hG\Sigma$ and the left square is commutative. So there exists a $G$-map $b: \Sigma \Omega^\infty w \to Z$ such that the remaining squares become $G$-homotopy commutative. Taking the adjoint situation the maps $b: C_{\Omega^\infty w} \to \Omega^\infty Z$ and $a: \Sigma \Omega^\infty X \to \Omega^\infty \Sigma X$ give a $G$-homotopy commutative diagram such as (2.1). From (2.2) and (2.3) we obtain a $G$-map $\beta: \Omega^\infty Y \to F_{\Omega^\infty w}$ such that the following diagram is $G$-homotopy commutative:

$$
\begin{array}{ccc}
\Omega^\infty X & \to & \Omega^\infty Y \\
\downarrow a & & \downarrow \beta \\
\Omega \Sigma \Sigma X & \to & F_{\Omega^\infty w} \to \Omega^\infty Z.
\end{array}
$$

By use of the desuspension theorem [8, Theorem II. 6.1] we observe that the adjoint $a$ of $a$ is a weak equivalence. Applying Five lemma we moreover verify that $\beta$ is also a weak equivalence.

**2.2.** Given two sequences $\Phi: K \to L \to M \to N$, $\Phi': K' \to L' \to M' \to N'$ in $G\Sigma$ we consider a morphism $\xi = (k, l, m, n): \Phi \to \Phi'$ such that the induced diagram is $G$-homotopy commutative. Choose a $G$-homotopy $P: K \wedge I^+ \to L'$ from $f'k$ to $fL$ and define a $G$-map $\mu: C_f \to C_{f'}$ by $\mu|CK = Ck|P$ and $\mu|L = f$. We here assume that there are four $G$-maps $b, b', a$ and $a'$ making the diagram below $G$-homotopy commutative:

$$
\begin{array}{ccc}
M & & h \\
\uparrow g & \uparrow b & \uparrow a \\
L & \to & C_f \to \Sigma K \\
\downarrow i & \downarrow \mu & \downarrow m \\
L' & \to & C_{f'} \to \Sigma K' \\
\uparrow g' & \downarrow b' & \downarrow a' \\
M' & \to & N'
\end{array}
$$

Choose $G$-homotopies $U: L \wedge I^+ \to M$ from $bi$ to $g$, $U': L' \wedge I^+ \to M'$ from $b'i'$ to $g'$ and $V: C_f \wedge I^+ \to M'$ from $mb$ to $b'\mu$, and then define a $G$-map $b_1: C_f \to M$ by $b_1|CK = b|CK \wedge U(f \wedge 1)$ and $b_1|L = g$, and similarly a $G$-map $b'_1$:}
$C'_f \to M'$ using the homotopy $U'$. Combine $U$, $U'$ and $V$ to obtain a $G$-homotopy $Q: L \wedge I^+ \to M'$ from $m g$ to $g' l$ defined to be $Q = m U (1 \wedge \tau) \wedge V (l \wedge 1) \wedge U' (l \wedge 1)$. Putting $F = b_1 | CK$ and $F' = b_1' | CK'$ we have

**Claim 2.2.** $m F \wedge Q (f \wedge 1)$ is $G$-homotopic rel $K \wedge \partial I^+$ to $F' (CK) \wedge g' P$.

Proof. $b' \mu | CK$ is $G$-homotopic rel $K \wedge \partial I^+$ to $mb | CK \wedge V (f \wedge 1)$ and also $b' \nu | U (f \wedge 1)$ is so to $U' (f' k \wedge 1) \wedge g' P$. Hence the result is easily shown.

Since $[b] = [b_1] \in [C_f, M]_G$ we get a $G$-map $H: CL \to N$ such that $[d (h F, H (C f))] = [a] \in [\Sigma K, N]_G$ (see [10, Lemma 3.2 and Theorem 3.3]), and similarly a $G$-map $H': CL' \to N'$ such that $[d (h' F', H' (C f'))] = [a'] \in [\Sigma K', N']_G$. Choose a $G$-homotopy $R: M \wedge I^+ \to N'$ from $h' m$ to $n h$. Then we have

**Claim 2.3.** There exists a $G$-map $W: \Sigma M \to N'$ such that $R (g \wedge 1) \wedge n H (1 \wedge \tau) \wedge W (\Sigma g)$ is $G$-homotopic rel $L \wedge \partial I^+$ to $h' Q (f \wedge 1) \wedge H' (l \wedge \tau)$.

Proof. $n h F$ is $G$-homotopic rel $K \wedge \partial I^+$ to $h' m F \wedge R (g f \wedge 1)$ and similarly $H'(f' k \wedge \tau)$ is so to $h' g' P \wedge H'(f' k \wedge \tau)$. By means of Claim 2.2 the equality $[d (n h F, n H (C f))] = [d (h' F', H' (C f'))] \in [\Sigma K', N']_G$ implies that $R (g \wedge 1) \wedge n H (f \wedge \tau)$ is $G$-homotopic rel $K \wedge \partial I^+$ to $h' Q (f \wedge 1) \wedge H' (l \wedge \tau)$. The result is now immediate.

Using the maps $R$ and $W$ we define a $G$-map $\lambda: F_h \to F_{h'}$ to be

$$\lambda (x, \omega) = (m z, R \{z\} \times I \wedge n \omega \wedge W \{z\} \times I).$$

By means of Claim 2.3 we see easily that the following diagrams are $G$-homotopy commutative:

$$\begin{array}{ccc}
\Omega N \to F_h \\
\Omega n \downarrow \lambda \downarrow m \downarrow \lambda \\
\Omega N' \to F_{h'}
\end{array} \quad \begin{array}{ccc}
L \to F_h \\
L' \downarrow \lambda \downarrow m \downarrow \lambda \\
L' \to F_{h'}
\end{array}$$

where $\beta$ and $\beta'$ are defined as (2.2).

Let $\Phi: K \to L \to M \to N$, $\Phi': K' \to L' \to M' \to N'$ be fiber sequences in $h G \mathcal{D}$. A morphism $\xi = (k, l, m, n): \Phi \to \Phi'$ is said to be a morphism between fiber sequences in $h G \mathcal{D}$ if there are four weak equivalences $\beta$, $\beta'$, $\alpha$ and $\alpha'$ and a $G$-map $\lambda$ such that the diagram below is $G$-homotopy commutative:

$$\begin{array}{ccc}
K \to L \\
\downarrow \alpha \downarrow \beta \\
\Omega N \to F_h \to M \to N \\
\Omega n \downarrow \lambda \downarrow m \downarrow n \\
\Omega N' \to F_{h'} \to M' \to N'
\end{array} \quad \begin{array}{ccc}
K' \to L' \\
\uparrow \alpha' \uparrow \beta' \\
\Omega N \to F_h \to M \to N \\
\Omega n \downarrow \lambda \downarrow m \downarrow n \\
\Omega N' \to F_{h'} \to M' \to N'
\end{array}$$
Proposition 2.4. Let $\psi: X \to Y \to Z \to \Sigma X$, $\psi': X' \to Y' \to Z' \to \Sigma X'$ be cofiber sequences in $hGSA$ and $\zeta = (r, s, t, \Sigma r): \psi \to \psi'$ be a morphism between cofiber sequences in $hGSA$. Then $\Omega^\infty \psi \to \Omega^\infty \psi'$ is a morphism between fiber sequences in $hG\Sigma$.

Proof. Pick up a $G$-homotopy $P: X \land I^+ \to Y'$ from $u'\tau$ to $su$ and consider the $G$-maps $\mu: C_{\Omega^\infty u} \to C_{\Omega^\infty u'}$ given by $\mu|C_{\Omega^\infty X}=C_{\Omega^\infty r}|C_{\Omega^\infty P}$ and $\mu|\Omega^\infty Y=\Omega^\infty s$. By observing standard cofiber sequences in $GSA$ we can easily find $G$-maps $b: \Sigma C_{\Omega^\infty u} \to Z$ and $b': \Sigma C_{\Omega^\infty u'} \to Z'$ in the proof of Proposition 2.1 such as $tb$ is $G$-homotopic to $b'(\Sigma \mu)$. Hence we get four $G$-maps $b: C_{\Omega^\infty u} \to \Omega^\infty Z$, $b': C_{\Omega^\infty u'} \to \Omega^\infty Z'$, $a: \Sigma \Omega^\infty X \to \Omega^\infty \Sigma X$ and $\alpha': \Sigma \Omega^\infty X' \to \Omega^\infty \Sigma X'$ such that the diagram (2.5) is $G$-homotopy commutative. Making use of Proposition 2.1, (2.6) and (2.7) we immediately obtain four weak equivalences $\beta: \Omega^\infty Y \to \Omega^\infty X$, $\beta': \Omega^\infty Y' \to \Omega^\infty X'$, $\alpha=\alpha': \Omega^\infty X \to \Omega^\infty X'$ and a $G$-map $\lambda: F_{\Omega^\infty u'} \to F_{\Omega^\infty u}$ making the diagram (2.8) $G$-homotopy commutative.

3. $(E^*_*, \Omega^\infty)$- and $(\{E_K\}, \Pi_\phi K)$-localizations

3.1. Let $E_\ast$ be an $RO(G; U)$-graded homology theory defined on the stable homotopy category $hGCWSU$. A map $u: X \to Y$ in $hGCWSU$ is called an $E^*_\ast$-equivalence if $u^*: U_\ast X \to E^*_\ast Y$ is an isomorphism, and also a map $f: A \to B$ in $hGCWSU$ is called an $E^*_\ast$-equivalence if so is $\Sigma^\infty f: \Sigma^\infty A \to \Sigma^\infty B$. Let us denote by $\mathcal{W}E$ the morphism class consisting of all $E^*_\ast$-equivalences in $hGCWSU$. We simply write $\mathcal{W}E$ for the class $\Sigma^\ast \mathcal{W}E$ consisting of all $E^*_\ast$-equivalences in $hGCWSU$. As usual we adopt the terms of $E^*_\ast T$- and $(E^*_\ast, T)$-localizations in place of those of $T^*\mathcal{W}$- and $(\mathcal{W}, T)$-localizations when $\mathcal{W}=\mathcal{W}E$. Obviously the morphism class $\mathcal{W}E$ in $hGC$ satisfies the condition (C.0), where $hGC=hGCWSU$.

Lemma 3.1. Let $\sigma$ be an infinite cardinal number which is at least equal to the cardinality of $E_\ast$. Then

$$\mathcal{W}E = Id_\ast \mathcal{W}_\ast E$$

where $Id$ denotes the identity functor.

Proof. Trivially $Id_\ast \mathcal{W}_\ast E \subset \mathcal{W}E$. Taking an $E_\ast$-equivalence $u: X \to Y$ in $hGC$, it may be regarded as an inclusion $X \subset Y$. Let $\gamma$ be an infinite cardinal number of cardinality greater than $\#Y-\#X$. As in the non-equivariant case (see [3, Lemma 1.13]) we can construct a transfinite sequence $X=X_0 \subset X_1 \subset \cdots \subset X_\gamma \subset X_{\gamma+1} \subset \cdots$ in $GC$ such that i) if $\lambda$ is a limit ordinal then $X_\lambda = \bigcup_{\gamma \lt \lambda} X_\gamma$, ii) if $X_\gamma = Y$ then $X_{\gamma+1} = Y$, and iii) if $X_\gamma \neq Y$ then $X_{\gamma+1} = X_\gamma \cup W$ for some $W \subset Y$ where $\#W \leq \sigma$, $W \subset X_\gamma$, and the inclusion $W \cap X_\gamma \to W$ is an $E_\ast$-equivalence. Clearly $Y=X_\gamma$. Hence we observe that the inclusion $u: X \to Y$ admits
an \((\text{Id}, \mathcal{W}_W^E)\)-decomposition.

As is easily shown, we have

**Corollary 3.2.** Let \(\sigma\) be an infinite cardinal number which is at least equal to the cardinality of \(E_\ast\). Then \(\Sigma_\eta^E\mathcal{W}_W^E\) satisfies the condition \((C.2)\).

It is known that \(\mathcal{W}_W^E\) admits a calculus of left fractions in \(hGC\) (see [1, Lemma 3.6]). In particular, \(\mathcal{W}_W^E=\text{Id}_W\mathcal{W}_W^E\) satisfies the condition \((C.1)\).

**Lemma 3.3.** Fix an infinite cardinal number \(\sigma\). The morphism class \(\Sigma_\eta^E\mathcal{W}_W^E\) admits a calculus of left fractions in \(hGC\mathcal{W}_W^E\). In particular, it satisfies the condition \((C.1)\).

**Proof.** We only show that \(\Sigma_\eta^E\mathcal{W}_W^E\) satisfies the condition \((C.1)\) because the remainders are easy. Represent \(u: X \to Y\) in \(\Sigma_\eta^E\mathcal{W}_W^E\) by a transfinite sequence \(X=X_0 \subset X_1 \subset \cdots \subset X_\eta \subset X_{\eta+1} \subset \cdots \subset X_\pi = Y\) in \(GC\mathcal{W}_W^E\), where \(X_\eta \subset X_{\eta+1}\) is given by a pushout square as \((1.2)\). Put \(V_\eta = Y \times \{0\} \cup X_\eta \land I^+ \cup Y \times \{1\}\) and consider the square

\[
\begin{array}{ccc}
V_\eta (\Sigma^\omega (B_\sigma \times \{0\} \cup A_\sigma \land I^+ \cup B_\sigma \times \{1\})) & \to & V_\eta \\
\downarrow & & \downarrow \\
\Sigma^\omega (B_\sigma \land I^+) & \to & V_{\eta+1},
\end{array}
\]

which is also pushout. The transfinite sequence

\[
Y \times \{0\} \cup X \land I^+ \cup Y \times \{1\} = V_0 \subset V_1 \subset \cdots \subset V_\eta \subset V_{\eta+1} \subset \cdots \subset V_\pi = Y \land I^+
\]
gives a \((\Sigma^\omega, \mathcal{W}_W^E)\)-decomposition for the inclusion \(\nu: V_0 \to V_\pi\). Given \(f, g: Y \to Z\) such that \(fu=gu\) in \(hGC\mathcal{W}_W^E\), there is a map \(k: V_0 \to Z\) with \(k|Y \times \{0\} =f\) and \(k|Y \times \{1\} =g\). Take the double mapping cylinder \(W\) of \(\nu\) and \(k\), then it follows immediately that the inclusion \(\nu: Z \to W\) has a \((\Sigma^\omega, \mathcal{W}_W^E)\)-decomposition and \(\nu f=\nu g\) in \(hGC\mathcal{W}_W^E\).

Without use of the existence theorem of \((E_\ast, \Omega^\omega)\)-localization Kuhn [7, Proposition 2.4] proved that \((\mathcal{W}_W^E, \Omega^\omega \Sigma^\omega)\) satisfies the condition \((C.4)\) in the non-equivariant case. By virtue of [8, Theorem V. 5.6] we can apply the method of Kuhn in the finite groups case to show

**Proposition 3.4.** Assume that \(G\) is a finite group. If a map \(f: A \to B\) in \(GC\mathcal{W}\) is an \(E_\ast\)-equivalence, then so is \(\Omega^\omega \Sigma^\omega f: \Omega^\omega \Sigma^\omega A \to \Omega^\omega \Sigma^\omega B\). (Cf., [7] and [5]).

**Proposition 3.5.** Given a homotopy pushout square
in $G\mathcal{W}SU$ such that $\Omega^\infty s: \Omega^\infty Y \to \Omega^\infty Y'$ is an $E_\infty$-equivalence, then $\Omega^\infty t: \Omega^\infty Z \to \Omega^\infty Z'$ is an $E_\infty$-equivalence, too.

Proof. Let $\Sigma X$ be the cofiber of $\nu: Y \to Z$. Then there is a $G$-homotopy commutative diagram

\[
\begin{array}{ccc}
\Omega^\infty X & \to & \Omega^\infty Y \\
\downarrow & & \downarrow \\
\Omega^\infty Y' & \to & \Omega^\infty Z' \\
\end{array}
\]

Propositions 2.1 and 2.4 assert that the horizontal rows may be regarded as fiber sequences of $G$-CW complexes. Compare the Atiyah–Hirzebruch spectral sequences (see [6, Theorem 1]). Since the base space $\Omega^\infty \Sigma X$ is a $G$-homotopy commutative $H$-space and $\pi_0^c(\Omega^\infty \Sigma X)$ is an abelian group for each closed subgroup $K$ of $G$, the result is now easily shown.

Making use of Propositions 3.4 and 3.5 we have

**Corollary 3.6.** Assume that $G$ is a finite group and fix an infinite cardinal number $\sigma$. The morphism class $\Sigma^\infty \mathcal{W}\sigma^\infty$ satisfies the condition (C.3).

Let $\sigma$ be an infinite cardinal number which is at least equal to the cardinality of $E_\infty$. Lemma 3.3 and Corollaries 3.2 and 3.6 say that the morphism class $\Sigma_1^\infty \mathcal{W}\sigma^\infty$ satisfies the conditions (C.1), (C.2) and (C.3) when $G$ is finite. So we can apply Proposition 1.4 to show the existence theorem of $(E_\infty, \Omega^\infty)$-localization.

**Theorem 3.7.** Assume that $G$ is a finite group. Then there exists an $(E_\infty, \Omega^\infty)$-localization $(L, \eta)$ in $hG\mathcal{W}SU$. (Cf., [4, Theorem 1.1]).

Let $hG\mathcal{W}SU_0$ denote the full subcategory of $hG\mathcal{W}SU$ consisting of ($-1$)-connected $G$-CW spectra. The 0-th space functor $\Omega^\infty: hG\mathcal{W}SU_0 \to hG\mathcal{W}$ satisfies the assumption in Proposition 1.2. So we get

**Corollary 3.8.** Assume that $G$ is a finite group. Then there exists an $E_\infty \Omega^\infty$-localization $(L, \eta)$ in $hG\mathcal{W}SU_0$. (See [4]).

3.2. Let $G$ be a compact Lie group and $\mathcal{F}$ be a collection of closed subgroups of $G$ which are not conjugate subgroups each other. We partially order a list $\mathcal{F}$ by writing $H \leq K$ if $H$ is subconjugate to $K$. Let $\mathcal{E}_\mathcal{F} = \{E_K\}_{K \in \mathcal{F}}$ be a family of homology theories defined on $hG\mathcal{W}SU$. A family $\mathcal{E}_\mathcal{F}$ is said...
to be order preserving if \( E_{K^*}X = 0 \) implies \( E_{H^*}X = 0 \) for each pair \( H \leq K \) in \( \mathcal{F} \). Write \( \mathcal{W}^{E\mathcal{F}} \) for the morphism class \( \prod_{K \in \mathcal{F}} \mathcal{W}^{E_K} \) in \( \prod_{K \in \mathcal{F}} hCGW \) or in \( \prod_{K \in \mathcal{F}} hCGWSU \).

For each closed subgroup \( K \) of \( G \) the \( K \)-fixed point functor \( \phi_K: G\mathcal{F} \to \mathcal{F} \) or \( G\mathcal{A} = S\mathcal{A} \) has a left adjoint functor \( (G/K)^+ \wedge - \) (see [8, Proposition II. 4.6]). Abbreviate by \( \mathcal{C} \) the category \( C\mathcal{W} \) or \( C\mathcal{WSU} \) and similarly by \( GC \). The fixed points functor \( \phi_{\mathcal{F}} = \prod_{K \in \mathcal{F}} \phi_K: GC \to \prod_{K \in \mathcal{F}} \mathcal{C} \) has a left adjoint \( \psi_{\mathcal{F}}: \prod_{K \in \mathcal{F}} \mathcal{C} \to GC \) defined to be \( \psi_{\mathcal{F}}(\{X_K\}) = \bigvee_K (G/K)^+ \wedge X_K \). We here show that \( (\mathcal{W}^{E\mathcal{F}}, \phi_{\mathcal{F}} \psi_{\mathcal{F}}) \) satisfies the condition (C.4).

**Lemma 3.9.** Assume that a family \( \mathcal{E}_\mathcal{F} = \{E_K\} \) is order preserving. Given \( E_K^\ast \)-equivalences \( f_K: X_K \to Y_K \) in \( h\mathcal{C} \) for all \( K \in \mathcal{F} \), then \( \phi_{\mathcal{F}} \psi_{\mathcal{F}}(\{f_K\}): (\bigvee_K (G/K)^+ \wedge X_K) \ast \to (\bigvee_K (G/K)^+ \wedge Y_K) \ast \) is also an \( E_K^\ast \)-equivalence for each \( H \in \mathcal{F} \).

(Cf., [11, Lemma 2.2]).

Proof. Under the hypothesis on \( \mathcal{E}_\mathcal{F} \) it follows that \( 1 \wedge f_K: (G/K)^H \wedge X_K \to (G/K)^H \wedge Y_K \) is an \( E_K^\ast \)-equivalence since \( (G/K)^H = \phi \) unless \( H \leq K \).

Let \( \mathcal{E}_\mathcal{F} = \{E_K\} \) be an order preserving family and \( \sigma \) be an infinite cardinal number which is at least equal to the cardinality of \( \bigoplus_{K \in \mathcal{F}} E_{K^*} \). By similar arguments to Lemma 3.3 and Corollaries 3.2 and 3.6 involving Lemma 3.9 we easily verify that \( \psi_{\mathcal{F}} \mathcal{W}^{E\mathcal{F}} \) in \( hGC \) satisfies the conditions (C.1), (C.2) and (C.3). Applying Proposition 1.4 we obtain

**Theorem 3.10.** Let \( G \) be a compact Lie group and \( \mathcal{E}_\mathcal{F} = \{E_K\} \) be a family of homology theories defined on \( hCGWSU \). Assume that \( \mathcal{E}_\mathcal{F} \) is order preserving. Then there exists an \( (E_{\mathcal{F}}, \phi_{\mathcal{F}}) \)-localization \( (L, \eta) \) in \( hCGW \) or in \( hCGWSU \) where \( \phi_{\mathcal{F}} = \prod_{K \in \mathcal{F}} \phi_K \) denotes the fixed points functor.

If a list \( \mathcal{F} \) contains precisely one subgroup from every conjugacy class of closed subgroups of \( G \), then it is said to be complete. As is well known, the fixed points functor \( \phi_{\mathcal{F}} \) satisfies the assumption in Proposition 1.2 when \( \mathcal{F} \) is complete. Hence we have

**Corollary 3.11.** Assume that a list \( \mathcal{F} \) is complete and a family \( \mathcal{E}_\mathcal{F} = \{E_K\} \) is order preserving. Then there exists an \( \mathcal{E}_{\mathcal{F}} \phi_{\mathcal{F}} \)-localization \( (L, \eta) \) in \( hCGW \) or in \( hCGWSU \). (Cf., [12, Theorem 2.1]).

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**References**


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