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ON MULTIPLY TRANSITIVE GROUPS XI

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1. Introduction

In this paper we shall prove the following

Theorem. Let $G$ be a 4-fold transitive group on $\Omega = \{1, 2, \ldots, n\}$. If a Sylow 2-subgroup of a stabilizer of four points in $G$ fixes exactly seven points, then $G$ must be $A_7$ or $M_{23}$.

To prove the theorem we need the following Lemma 1, which will be proved in the section 3.

Lemma 1. Let $G$ be a permutation group on $\Omega = \{1, 2, \ldots, 23\}$ and assume the following:

1. To any two distinct points $i, j$ of $\Omega$ there corresponds a subset $\Delta(i, j)$ of $\Omega$.
2. $|\Delta(i, j)| \geq 1$, $\Delta(i, j) = \Delta(j, i)$ and $\Delta(i, j) \ni i, j$.
3. If $\Delta(i, j) \ni k$, then $\Delta(i, k) \ni j$.
4. $G$ is a subgroup of $M_{23}$.
5. Let $a \in G$ and $\Delta(i, j) = \{k_1, k_2, \ldots, k_t\}$. Then $\Delta(i, j)^a = \Delta(i^a, j^a) = \{k_1^a, k_2^a, \ldots, k_t^a\}$.
6. If $\Delta(i, j) \ni k$, then there is an involution $a$ of $G$ such that $I(a) \ni \{i, j, k\}$.
7. Let $a$ be an involution of $G$. If $I(a) \ni \{i, j\}$, then $|\Delta(i, j) \cap I(a)| = 1$.

Then $G$ is isomorphic to a subgroup of $M_{11}$, which is one of the following groups:

(i) $G \cong M_{11}$ and the lengths of the $G$-orbits are 11, 12.
(ii) $G \cong M_{10}$ and the lengths of the $G$-orbits are 1, 10, 12.
(iii) $G \cong M_{10}^*$, where $M_{10}^*$ is a subgroup of $M_{10}$ of index 2, and the lengths of the $G$-orbits are 1, 10, 6, 6.

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(iv) \( G \cong N_{M_{31}}(M_a) \) and the lengths of the \( G \)-orbits are 2, 9, 12.
(v) \( G \cong N_{M_{31}}(M_b)^* \), where \( N_{M_{31}}(M_a)^* \) is a subgroup of \( N_{M_{31}}(M_a) \) of index 2, and the lengths of the \( G \)-orbits are 2, 9, 6, 6.
(vi) \( G \cong PSL(2, 11) \) and the lengths of the \( G \)-orbits are 11, 1, 11.
(vii) \( G \cong S_5 \) and the lengths of the \( G \)-orbits are 5, 6, 2, 10.

We shall use the same notations as in [3].

2. Proof of the theorem

The proof of the theorem is obtained in the following way: If a Sylow 2-subgroup of the stabilizer of four points in \( G \) is semiregular, then \( G \) is \( A_7 \) or \( M_{23} \) by [3]. Here we say that \( P \) is semiregular when \( P \) is “semiregular” on \( \Omega - I(P) \) in the usual sense. Hence, by way of contradiction, we assume that \( P \) is not semiregular. Let \( R \) be a subgroup of \( P \) such that its order is maximal among all subgroups of \( P \) which fix more than twenty-three points and set \( N = N_G(R)^{I(Q)} \). Then \( N \) has an involution \( a \) which fixes exactly twenty-three points and then \( C_N(a)^{I(Q)} \) is a subgroup of \( M_{23} \). Let \( (i, j) \) be a 2-cycle of \( a \), and for \( \{i, j\} \subseteq I(a) \), let \( \Delta(i, j) \) be the intersection of \( I(a) \) and the \( G_{i, j} \)-orbit of odd length \((>1)\) which is uniquely determined by \( \{i, i_1, i_2, j\} \). Then \( (C_N(a)_{i, i_2})^{I(Q)} \) and \( \{\Delta(i, j)\} \) satisfy the assumption of Lemma 1, and hence \( (C_N(a)_{i_1, i_2})^{I(Q)} \) is one of the permutation groups listed in Lemma 1. Thus we see that \( N \) satisfies the conditions of Lemma 3 and we have a contradiction. The above result on \( (C_N(a)_{i_1, i_2})^{I(Q)} \) is useful in determining \( C_N(a)^{I(Q)} \).

Let \( P \) be a Sylow 2-subgroup of the stabilizer \( G_{1234} \). If \( P \) is semiregular, then \( G \) is \( A_7 \) or \( M_{23} \) by [3]. Hence from now on we assume that \( P \) is not semiregular and proceed by way of contradiction.

2.1. For any point \( t \) of \( \Omega - I(P) \) \( |I(P_t)| \geq 23 \). In particular if \( |I(P_t)| = 23 \), then \( N_G(G_{I(P_t)})^{I(P_t)} = M_{23} \).

Proof. Let \( |I(P_t)| \) be the smallest number such that \( t \in \Omega - I(P) \) and \( Q \) a Sylow 2-subgroup of \( G_{I(P_t)} \). Then \( I(Q) = I(P_t) \). For any four points \( i, j, k, l \) of \( I(P_t) \) let \( P' \) be a Sylow 2-subgroup of \( G_{i, j, k, l} \) containing \( Q \). Then \( P' \) is conjugate to \( P \). Hence by minimality of \( |I(P_t)| \), \( N_{P'}(Q)^{I(Q)} \) is a semiregular 2-group fixing \( I(P') \) pointwise. Thus \( N_G(Q)^{I(Q)} = M_{23} \) and \( |I(Q)| = 23 \) by Theorem 1 in [4].

In particular if \( |I(P_t)| = 23 \), then by the same argument as above \( N_G(Q)^{I(Q)} = M_{23} \) where \( Q \) is a Sylow 2-subgroup of \( G_{I(P_t)} \). Hence by the Frattini argument, \( N_G(G_{I(P_t)}) = G_{I(P_t)} \cdot N_G(Q) \) and so \( N_G(G_{I(P_t)})^{I(P_t)} = N_G(Q)^{I(P_t)} = M_{23} \).

2.2. Let \( t \in \Omega - I(P) \). If \( |I(P_t)| = 23 \), then \( t \) is a point of a \( G_{1234} \)-orbit of even length.
Proof. By (2.1), \( N_G(G_{I(P)})^I(P) = M_{23} \). Hence a Sylow 2-subgroup of \( (N_G(G_{I(P)})^I(P))_{1234} \) is a regular normal subgroup of \( (N_G(G_{I(P)})^I(P))_{1234} \). On the other hand since \( t \in \Omega - I(P) \), \( N_p(P)_{1(P)}^I(P) \) is a nonidentity 2-subgroup of \( (N_G(G_{I(P)})^I(P))_{1234} \). Hence any nonidentity 2-subgroup of \( (N_G(G_{I(P)})^I(P))_{1234} \) fixes exactly \( I(P) \) pointwise.

Suppose by way of contradiction that \( t \) is a point of a \( G_{1234} \)-orbit of odd length. Let \( Q \) be a Sylow 2-subgroup of \( G_{I(P)} \) containing \( P_t \) and \( P' \) be a Sylow 2-subgroup of \( G_{1234} \) containing \( Q \). Since the index of \( G_{1234} \) in \( G_{1234} \) is odd, \( P' \) is a Sylow 2-subgroup of \( G_{1234} \). Hence \( N_{P'}(Q)_{1234} \) is a nonidentity 2-subgroup of \( (N_G(G_{I(P)})^I(P))_{1234} \). Thus \( I(P') = I(P) \). This shows that \( t \in I(P) \), contrary to the assumption.

2.3. Since \( P \) is not semiregular, \( |\Omega| > 23 \). Hence there is a subgroup of \( P \) which has more than twenty-three fixed points. Let \( R \) be a subgroup of \( P \) such that its order is maximal among all subgroups of \( P \) which have more than twenty-three fixed points. Here \( R \) may be the identity. Then \( R \) satisfies the following conditions:

1. \( |I(R)| > 22 \).
2. Any involution of \( N_G(R)^I(R) \) fixes exactly seven or twenty-three points. In particular any central involution of a Sylow 2-subgroup in \( N_G(R)^I(R) \) fixes exactly twenty-three points.
3. The stabilizer of any four points of \( I(R) \) in \( N_G(R)^I(R) \) has an involution fixing exactly twenty-three points.

Proof. Since Sylow 2-subgroups of the stabilizer of four points in \( G \) are conjugate in \( G \), \( R \) is a Sylow 2-subgroup of \( G_{I(R)} \). Clearly \( |I(R)| > 23 \).

For any four points \( i, j, k, l \) of \( I(R) \) let \( P' \) be a Sylow 2-subgroup of \( G_{i, j, k, l} \) containing \( R \). Since \( R \) is a Sylow 2-subgroup of \( G_{I(R)} \), \( N_p(R)^I(R) \) is a Sylow 2-subgroup of \( G_{i, j, k, l} \) containing \( R \). Thus \( (N_G(R)^I(R))_{i, j, k, l} \) has a nonidentity 2-subgroup. Let \( Q \) be a Sylow 2-subgroup of \( (N_G(R)^I(R))_{i, j, k, l} \) and \( x \) an involution of \( Q \). Then by assumption \( |I(Q)| \leq 23 \) and \( |I(x)| \leq 23 \). Hence by assumption and (2.1), \( |I(Q)| = 7 \) or \( 23 \) and \( N_{N_G(R)^I(R)}(Q)^I(Q) \leq A_7 \) or \( M_{23} \) respectively and \( |I(x)| = 7 \) or \( 23 \).

Now if any involution of \( N_G(R)^I(R) \) fixing at least four points fixes exactly seven points, then \( |I(R)| = 23 \) by Theorem 1 in [4], which is a contradiction. Thus there is an involution fixing exactly twenty-three points in \( N_G(R)^I(R) \). To complete the proof of (2.3) it is sufficient to prove the following lemma.

**Lemma 2.** Let \( G \) be a permutation group on \( \Omega = \{1, 2, \cdots, n\} \), \( n > 23 \). Let \( P \) be a Sylow 2-subgroup of the stabilizer of any four points in \( G \). Assume that \( P \) satisfies the following conditions:

1. \( P \) fixes exactly seven or twenty-three points.
2. If \( |I(P)| = 7 \), then \( N_G(P)^I(P) \leq A_7 \), and if \( |I(P)| = 23 \), then \( N_G(P)^I(P) \leq M_{23} \).
(iii) Let \( x \) be any involution of \( P \). Then \( |I(x)| = 7 \) or 23. Moreover if \( |I(x)| = 23 \), then \( N_G(G_{I(x)})^{I(x)} \leq M_{23} \).

Then

1. any involution of \( G \) fixes exactly seven or twenty-three points and any central involution of a Sylow 2-subgroup of \( G \) fixes exactly twenty-three points,
2. the stabilizer of four points in \( G \) has an involution fixing exactly twenty-three points.

Proof. If any involution of \( G \) fixing at least four points fixes exactly seven points, then \( |\Omega| = 23 \) by Theorem 1 in [4], contrary to the assumption. Thus there is an involution fixing twenty-three points.

Let \( y \) be an involution. Since \( |I(P)| = 7 \) or 23, \( |\Omega| \) is odd and so \( |I(y)| \) is odd. Suppose that \( |I(y)| = 1 \) and \( y = (t_1, t_2, \ldots) \). Since \( y \in N_G(G_{t_1, t_2, \ldots}) \), there is a Sylow 2-subgroup \( P' \) of \( G_{t_1, t_2, \ldots} \) such that \( y \in N_G(P') \). By assumption, \( N_G(P')^{I(P')} \leq A_7 \) or \( M_{23} \). Hence \( |I(y')^{I(P')}| = 3 \) or 7, which is a contradiction. Thus \( |I(y)| = 1 \). If \( |I(y)| \geq 4 \), then by assumption, \( |I(y)| = 7 \) or 23. Thus \( |I(y)| = 3, 7 \) or 23.

Now we remark that if \( y \) and \( y' \) are involutions such that \( |I(y)| = 23 \) and \( yy' = y'y \), then \( |I(y')^{I(P')}| = 7 \) or 23 since \( y'^{I(P')} \in N_G(G_{I(x)})^{I(x)} \leq M_{23} \).

Since there is an involution fixing exactly twenty-three points, let \( a \) be an involution such that \( |I(a)| = 23 \). We may assume that

\[
a = (1) (2) \cdots (23) (24 25) \cdots.
\]

Since \( a \in N_G(G_{t_1, t_2, \ldots}) \), there is an involution \( b \) of \( G_{t_1, t_2, \ldots} \) commuting with \( a \). Then by the remark above \( |I(b^{I(x)})| = 7 \) or 23. Furthermore \( b \) fixes 24, 25 which are not contained in \( I(a) \). Thus by assumption, \( |I(b)| = 23 \). Hence we may assume that

\[
b = (1) (2) \cdots (7) (8 9) (10 11) \cdots (22 23) (24) (25) \cdots (39) \cdots.
\]

Since \( ab = ba \), we may assume that

\[
a = (1) (2) \cdots (23) (24 25) (26 27) \cdots (38 39) \cdots.
\]

Since \( \langle a, b \rangle \leq N_G(G_{t_1, t_2, \ldots}) \), there is an involution \( c \) of \( G_{t_1, t_2, \ldots} \) commuting with \( a \) and \( b \). Similarly \( |I(c)| = 23 \). Since \( N_G(G_{I(a)})^{I(a)}, N_G(G_{I(b)})^{I(b)} \) and \( N_G(G_{I(c)})^{I(c)} \) are subgroups of \( M_{23} \), we may assume that

\[
\]

Consequently we may assume that

\[
a = (1) (2) \cdots (23) (24 25) (26 27) \cdots (50 51) \cdots.
\]
Now let \( u \) be a central involution of a Sylow 2-subgroup of \( G \) containing \( \langle a, b, c \rangle \). Then by the remark above \( |I(u)| = 7 \) or 23. Suppose by way of contradiction that \( |I(u)| = 7 \). Since \( u \) commutes with \( a, b \) and \( c \), \( I(u) \) is contained in \( I(a), I(b) \) and \( I(c) \). This is a contradiction since \( |I(a) \cap I(b) \cap I(c)| = 3 \) and \( |I(u)| = 7 \). Thus \( |I(u)| = 23 \). By the conjugacy of Sylow 2-subgroups of \( G \), any central involution of a Sylow 2-subgroup of \( G \) fixes exactly twenty-three points. Furthermore this shows that there is no involution fixing three points by the remark above. Thus (1) holds.

Next by assumption, for any four points there is an involution \( x \) fixing these four points. Then \( |I(x)| = 7 \) or 23. If \( |I(x)| = 7 \), then by what we have proved above, there is an involution \( x' \) fixing exactly twenty-three points and commuting with \( x \). Since \( |I(x)| = 7 \), \( I(x') \supseteq I(x) \). Thus (2) holds.

From now on \( R \) denotes the 2-group as in (2.3).

2.4. Let \( i_1, i_2, i_3, i_4 \) be any four points of \( \Omega \). Then \( G_{i_1 i_2 i_3 i_4} \) has exactly one orbit of odd length \( (\pm 1) \).

Proof. Let \( P' \) be a Sylow 2-subgroup of \( G_{i_1 i_2 i_3 i_4} \). Since \( |I(P')| = 7 \), we may assume that \( I(P') = \{i_1, i_2, i_3, i_4, i_5, i_6, i_7\} \). Then from \( N_G(P')I(P') = A_7 \), there is an element

\[ x = (i_1)(i_2)(i_3)(i_4)i_5i_6i_7 \cdots . \]

Since \( x \in G_{i_1 i_2 i_3 i_4} \), three points \( i_5, i_6, i_7 \) belong to the same \( G_{i_1 i_2 i_3 i_4} \)-orbit. Since \( P' \) is a Sylow 2-subgroup of \( G_{i_1 i_2 i_3 i_4} \), this orbit is of odd length. Furthermore by the conjugacy of Sylow 2-subgroups of \( G_{i_1 i_2 i_3 i_4} \), this orbit is only one \( G_{i_1 i_2 i_3 i_4} \)-orbit of odd length \( (\pm 1) \).

2.5. Let \( a \) be a 2-element of \( N_G(R) \) such that \( a^{I(R)} \) is an involution fixing twenty-three points and let \( \delta(i, j) \) be a 2-cycle of \( a^{I(R)} \). Set \( \Gamma = I(a^{I(R)}) \). For any two points \( i, j \) of \( \Gamma G_{i_1 i_2} \) has exactly one orbit of odd length \( (\pm 1) \) by (2.5). Hence we denote the intersection of the \( G_{i_1} i_1 i_2 \)-orbit of odd length \( (\pm 1) \) and \( \Gamma \) by \( \Delta_{i_1 i_2}(i, j) \) or merely \( \Delta(i, j) \). Then we have

\[
\begin{align*}
(1) \quad & |\Delta(i, j)| \geq 1 \text{ and } \Delta(i, j) = \Delta(j, i). \\
(2) \quad & \text{If } \Delta(i, j) \ni k, \text{ then } \Delta(i, k) \ni j. \\
(3) \quad & \text{If } \Delta(i, j) \ni k, \text{ then there is a 2-element } x \text{ of } N_G(R) \text{ such that } x^{I(R)} \\
& \text{ is an involution commuting with } a^{I(R)}, \quad |I(x^{I(R)})| = 23 \text{ and } I(x^{I(R)}) \supseteq \{i, j, k, i_1, i_2\}. \\
(4) \quad & \text{Let } x \text{ be a 2-element of } N_G(R) \text{ such that } x^{I(R)} \text{ is an involution fixing}
\end{align*}
\]
twenty-three points and commuting with \( a^{I(R)} \). If \( x \) fixes \( i, j, i_1, i_2 \), then \( |\Delta(i, j) \cap I(x^y)| = 1 \).

Proof. Since \( \langle a, R \rangle < N_G(G_{ij} i_1 i_2) \) and \( G_{ij} i_1 i_2 \) has exactly one orbit of odd length \((\pm 1)\), \( \langle a, R \rangle \) fixes this \( G_{ij} i_1 i_2 \) -orbit as a set. Furthermore since \( \langle a, R \rangle \) is a 2-group, \( \langle a, R \rangle \) has fixed points in this \( G_{ij} i_1 i_2 \) -orbit. Thus \( |\Delta(i, j)| \geq 1 \). Clearly \( \Delta(i, j) = \Delta(j, i) \). Thus (1) holds.

By the 4-fold transitivity of \( G \), \( |G_{ij} i_1 i_2 j: G_{ij} i_1 i_2 j k| = |G_{ij} i_1 i_2 k: G_{ij} i_1 i_2 j k| \), where \( k \in \{ i_1, i_2, i, j \} \). Thus (2) holds.

Let \( k \) be a point of \( \Delta(i, j) \). Since \( a \in N_G(G_{ij} i_1 i_2 j k) \), \( a \) normalizes some Sylow 2-subgroup \( P' \) of \( G_{ij} i_1 i_2 j k \) containing \( R \). Since \( k \) is a point of the \( G_{ij} i_1 i_2 j k \) -orbit of odd length, \( P' \) is also a Sylow 2-subgroup of \( G_{ij} i_1 i_2 j k \). Set \( Q = N_{P'}(R) \). Since \( R \) is a Sylow 2-subgroup of \( G_{ij} i_1 i_2 j k \) and \( a \in N_G(Q) \) and \( I(R)^a = I(R) \), there is an element \( x \) in \( Q \) such that \( x^{I(R)} \) is an involution and commutes with \( a^{I(R)} \). Since \( I(Q) \supseteq I(P') \supseteq \{ i, j, k, i_1, i_2 \} \), \( I(x) \supseteq \{ i, j, k, i_1, i_2 \} \). Furthermore since \( N_G(a)^r \subseteq M_{23}, \ |I(x^y)| = 7 \) or 23. On the other hand \( x^{I(R)} \) fixes two points \( k, l \) which are not in \( G \). Hence by (2.3), \( |I(x^{I(R)})| = 23 \). Thus (3) holds.

Let \( x \) be a 2-element of \( N_G(R) \) satisfying the assumption of (4). Since \( a \in N_G(G_{ij} i_1 i_2 j k) \), \( a \) normalizes some Sylow 2-subgroup \( P' \) of \( G_{ij} i_1 i_2 j k \) containing \( R \). Since \( k \) is a point of the \( G_{ij} i_1 i_2 j k \) -orbit of odd length, \( P' \) is also a Sylow 2-subgroup of \( G_{ij} i_1 i_2 j k \). Set \( Q = N_{P'}(R) \). Since \( R \) is a Sylow 2-subgroup of \( G_{ij} i_1 i_2 j k \) and \( a \in N_G(Q) \) and \( I(R)^a = I(R) \), there is an element \( x \) in \( Q \) such that \( x^{I(R)} \) is an involution and commutes with \( a^{I(R)} \). Since \( I(Q) \supseteq I(P') \supseteq \{ i, j, k, i_1, i_2 \} \), \( I(x) \supseteq \{ i, j, k, i_1, i_2 \} \). Furthermore since \( N_G(a)^r \subseteq M_{23}, \ |I(x^y)| = 7 \) or 23. On the other hand \( x^{I(R)} \) fixes two points \( k, l \) which are not in \( G \). Hence by (2.3), \( |I(x^{I(R)})| = 23 \). Thus (3) holds.

By the assumption of the theorem, (2.3) and (2.6), \( N = N_{G(R)}^{I(R)} \) satisfies the following conditions:

(1) Let \( P \) be a Sylow 2-subgroup of the stabilizer of any four points in \( N \). Then \( |I(P)| = 7 \) and \( N_N(P)^{I(P)} \subseteq M_{23} \), or \( |I(P)| = 23 \) and \( N_N(P)^{I(P)} \subseteq M_{23} \).

(2) Any involution \( u \) of \( P \) fixes exactly seven or twenty-three points. If \( |I(u)| = 23 \), then \( N_N(u)^{I(u)} \subseteq M_{23} \) and for any 2-cycle \( (i, j) \) of \( u \) \((N_N(u)^{I(u)})_{i, j} \) is one of the groups as in Lemma 1.
In Lemma 3 we shall show that there is no group satisfying the above conditions (1) and (2). Then we shall complete the proof of the theorem.

**Lemma 3.** Let $G$ be a permutation group on $\Omega = \{1, 2, \ldots, n\}$, $n \geq 23$. Then there is no group which satisfies the following conditions:

1. Let $P$ be a Sylow 2-subgroup of the stabilizer of any four points in $G$. Then $|I(P)| = 7$ and $N_G(P)^I(P) \leq A_n$ or $|I(P)| = 23$ and $N_G(P)^I(P) \leq M_{23}$.

2. Any involution $u$ of $P$ fixes exactly seven or twenty-three points. If $|I(u)| = 23$, then $C_G(u)^I(u) \leq M_{23}$ and for any 2-cycle $(i j)$ of $u$, $(C_G(u)_{i j})^I(u)$ is one of the groups as in Lemma 1.

Proof. The proof will be given in various steps. Suppose by way of contradiction that $G$ is a counter-example to Lemma 3. The following (i) and (ii) follow immediately from Lemma 2.

1. Any involution of $G$ fixes exactly seven or twenty-three points and any central involution of a Sylow 2-subgroup of $G$ fixes exactly twenty-three points.

2. The stabilizer of any four points in $G$ has an involution fixing exactly twenty-three points.

(iii) Let $Q$ be a 2-group fixing exactly twenty-three points. Then $Q$ is semiregular and elementary abelian.

Proof. By assumption, $Q$ is semiregular. Suppose that $Q$ has an element $x=(i j k l) \cdots$. Then $x \in N_G(G_{i j k l})$. Hence $x$ normalizes a Sylow 2-subgroup $P'$ of $G_{i j k l}$. By assumption, $N_G(P')^I(P') \leq A_n$ or $M_{23}$. Hence $x$ has a 2-cycle, which is a contradiction. Thus $Q$ is elementary abelian.

(iv) Let $a$ be an involution fixing twenty-three points and let $(i j)$ be a 2-cycle of $a$. Then any involution of $C_G(a)_{i j}$ fixes exactly twenty-three points. If $x$ and $y$ are two distinct involutions of $C_G(a)_{i j}$ such that $xy=yx$, then $|I(x) \cap I(y) \cap I(a)| = 3$.

Proof. Since $C_G(a)^I(a)$ is a subgroup of $M_{23}$, two commuting involutions with distinct fixed points have three fixed points in common. On the other hand $(C_G(a)_{i j})^I(a)$ is one of the groups as in Lemma 1 which is isomorphic to a subgroup of $M_{11}$, where $M_{11}$ is embedded in $M_{23}$ and has orbits of lengths 11 and 12, and two commuting involutions in $M_{11}$ of degree 11 have distinct fixed points. Hence two commuting distinct involutions of $(C_G(a)_{i j})^I(a)$ have exactly three fixed points in common. Thus $x^I(a) = y^I(a)$ or $|I(x) \cap I(y) \cap I(a)| = 3$. If $x^I(a) = y^I(a)$, then $xy$ is an involution fixing $I(a) \cap \{i, j\}$ pointwise. Thus $|I(xy)| > 23$, contrary to the assumption (2). Hence $|I(x) \cap I(y) \cap I(a)| = 3$. 


Since any involution of $C_G(a)_{ij}$ fixes seven or twenty-three points of $I(a)$ and at least two points $i, j$ which are not in $I(a)$, by the assumption (2), any involution of $C_G(a)_{ij}$ fixes exactly twenty-three points.

(v) We may assume that there are three involutions $a, b$ and $c$ of the following forms such that $|I(a)| = |I(b)| = |I(c)| = 23$ and $\langle a, b, c \rangle$ is elementary abelian.

\[ a = (1) (2) \cdots (23) (24 25) (26 27) \cdots (50 51) \cdots, \]
\[ b = (1) (2) \cdots (7) (8 9) (10 11) \cdots (22 23) (24) (25) \cdots (39) (40 41) (42 43) (44 46) (45 47) (48 50) (49 51) \cdots, \]

Proof. By (ii), there is an involution $a$ fixing twenty-three points. We may assume that

\[ a = (1) (2) \cdots (23) (24 25) \cdots. \]

By (iv), there is an involution $b$ of $C_G(a)_{24, 25}$. We may assume that

\[ b = (1) (2) \cdots (7) (8 9) (10 11) \cdots (22 23) (24) (25) \cdots (39). \]

Since $\langle a, b \rangle \leq N_G(G_{24, 25})$, there is an involution $c$ commuting with $a$ and $b$. By (iv), $c$ fixes exactly twenty-three points. Hence we may assume that


Hence we may assume that

\[ a = (1) (2) \cdots (23) (24 25) (26 27) \cdots (50 51) \cdots, \]
\[ b = (1) (2) \cdots (7) (8 9) (10 11) \cdots (22 23) (24) (25) \cdots (39) (40 41) (42 43) (44 46) (45 47) (48 50) (49 51) \cdots. \]

From now on $a, b$ and $c$ denote the elements as in (v).

(vi) Set $T = (C_G(a)_{24, 25})^{T(a)}$. Let

\[ \bar{b} = b^{T(a)}, \]
\[ \bar{c} = c^{T(a)}, \]
\[ x_1 = (1) (2) (3 9) (10 12) (11 15) (13 14) (4) (18 19) (16) (5 17) (20) (6 21) (23) (7 22), \]
\[ x_2 = (1) (2) (10) (3 11) (8 12) (9 14) (13 15) (18) (4 19) (5) (16 17) (20) (6 21) (22) (7 23), \]
MULTIPLY TRANSITIVE GROUPS

\[
x_3 = (1 \ 2) (3 \ 8) (9 \ 12 \ 13 \ 10 \ 14) (11 \ 15) (4 \ 5 \ 6 \ 7) (16 \ 20) (17 \ 21) (18 \ 19) (22 \ 23),
\]

\[
x_4 = (1 \ 2) (3 \ 8 \ 13) (9 \ 12 \ 10 \ 11) (14 \ 15) (5 \ 4 \ 7) (6 \ 16) (17 \ 20 \ 21) (18 \ 23) (19 \ 22),
\]

\[
x_5 = (2) (8) (10) (1 \ 12) (9 \ 13) (3 \ 15) (14 \ 11) (4) (7) (17) (18) (5) (23) (6) (21) (16 \ 22) (19 \ 20),
\]

\[
x_6 = (3) (13) (10) (2 \ 14) (1 \ 8) (9 \ 11) (12 \ 15) (4) (5) (17) (22) (6) (18) (7) (20) (16 \ 21) (19 \ 23),
\]

\[
y_1 = x_1^{z_5} = (2) (3 \ 8) (1 \ 9) (10 \ 15) (11 \ 13) (12 \ 14) (20) (22) (4 \ 21) (5 \ 23) (17) (18 \ 19) (7) (16),
\]

\[
y_2 = x_2^{z_5} = (2) (3) (15) (14 \ 8) (10 \ 9 \ 13) (11 \ 12) (4) (22) (20 \ 21) (5 \ 23) (18) (16) (6) (19) (7) (17),
\]

\[
y_3 = x_3^{z_5} = (2 \ 3) (12) (13 \ 8) (11 \ 9 \ 10) (14 \ 15) (4) (5) (20) (21) (22 \ 23) (6) (7) (16) (19) (17) (18).
\]

Then \( T \) is isomorphic to one of the following groups, and hence we may identify \( T \) with one of these groups:

1. \(<b, c, x_1, x_2, x_3>, which is isomorphic to \( M_{11} \) and has orbits \{1, 2, 3, 8, 9, \ldots, 15\} and \{4, 5, 6, 7, 16, 17, \ldots, 23\}.

2. \(<b, c, x_1, x_2, x_3>, which is isomorphic to \( M_{10} \) and has orbits \{1, 2, 3, 8, 9, \ldots, 15\} and \{4, 5, 6, 7, 16, 17, \ldots, 23\}.

3. \(<b, c, x_1, x_2>, which is isomorphic to \( M_{10}^* \) and has orbits \{1, 2, 3, 8, 9, \ldots, 15\}, \{4, 5, 6, 7, 16, 17, 18, 19\} and \{6, 7, 20, 21, 22, 23\}.

4. \(<b, c, y_1, y_2, x_2, x_3>, which is isomorphic to \( N(M_9) \) and has orbits \{2, 3\}, \{1, 8, 9, 10, 11\}, \{4, 5, 6, 7, 16, 17, 18, 19\} and \{6, 7, 16, 17, 18, 19\}.

5. \(<b, c, y_1, y_2, y_3, x_3>, which is isomorphic to \( N(M_9)^* \) and has orbits \{2, 3\}, \{1, 8, 9, 10, 11\}, \{4, 5, 6, 7, 16, 17, 18, 19\}.

6. \(<b, c, x_3>, which is isomorphic to \( PSL(2, 11) \) and has orbits \{1, 2, 3, 8, 9, \ldots, 15\}, \{7\} and \{4, 5, 6, 16, 17, \ldots, 23\}.

7. \(<b, c, x_3>, which is isomorphic to \( S_6 \) and has orbits \{1, 8, 9, 10, 11\}, \{2, 3, 12, 13, 14, 15\}, \{4, 5\} and \{6, 7, 16, 17, 18, 19\}.

Proof. By the assumption (2), \( T \) is isomorphic to one of the following groups: \( M_{11}, M_{10}, M_{10}^*, N(M_9), N(M_9)^*, PSL(2, 11) \) or \( S_6 \). In (3.16) and (3.17) of the section 3 we shall show that these groups have the generators and the orbits as in (vi).

(vii) Let \( Q \) be a 2-subgroup fixing exactly twenty-three points. If \( Q \) is a Sylow 2-subgroup of \( G \), then \( Q \) is of order two and \( N_G(Q) \leq M_{23} \).

Proof. Let \( I(Q) = \{1, 2, \ldots, 23\} \). Suppose that the order of \( Q \) is at least four. Then by (iii), \( Q \) has two involutions \( x \) and \( y \). We may assume that
x = (1) (2) ...(23) (24 25) (26 27) ... ,
y = (1) (2) ...(23) (24 26) (25 27) ... .

Since \( \langle x, y \rangle < N_G(G_{24 25 26 27}) \), there is an involution \( z \) of \( G_{24 25 26 27} \) fixing twenty-three points and commuting with \( x \) and \( y \). By the assumption (2), we may assume that

\[
z = (1) (2) ... (7) (8 9) (10 11) ... (22 23) (24) (25) (26) (27) ...
\]

Then \( \langle x, y \rangle < C_G(z) \) and \( |I(x) \cap I(y) \cap I(z)| = 7 \), contrary to (iv). Thus the order of \( Q \) is two.

By the assumption (2), \( N_G(Q) \leq M_{23} \).

From now on set \( T = \langle C_G(a)_{24 25} \rangle / \langle a \rangle \) and \( C = C_G(a) \).

Then \( \langle a \rangle \) is a Sylow 2-subgroup of \( G \).

(viii) Let \( x \) and \( y \) be involutions such that both \( x \) and \( y \) belong to \( C_G(a) \cap C_G(b) \) and \( x^{I(a)} = y^{I(a)} \). Then \( x^{I(b)} = y^{I(b)} \) or \( (ya)^{I(b)} \.

Proof. Since \( (xy)^{I(a)} = 1 \) and \( xy \in C_G(b) \), \( |I(xy) \cap I(b)| = 7 \) or \( 23 \). Hence \( (xy)^{I(b)} \) is an involution or the identity. If \( (xy)^{I(b)} \) is the identity, then \( x^{I(b)} = y^{I(b)} \). Next suppose that \( (xy)^{I(b)} \) is an involution. Assume that \( a^{I(b)} \neq (xy)^{I(b)} \).

Since \( xy \in C, \langle a, xy \rangle^{I(b)} \) is of order four. Hence \( G^{I(a)} \) has a subgroup \( \langle a, xy \rangle \) of order divisible by four, contrary to (vii). Thus \( a^{I(b)} = (xy)^{I(b)} \) and so \( x^{I(b)} = (ya)^{I(b)} \).

(ix) Let \( (i j) \) be any 2-cycle of \( a \). If \( (i j)^{I(a)} \) is not isomorphic to \( M_{11} \), then \( C^{I(a)} \) contains \((i j)^{I(a)} \) properly.

Proof. We may assume that \( (i j) = (24 25) \). Then \( C_{24 25} > \langle b, c \rangle \). By assumption, \( T = \langle C_{24 25} \rangle^{I(a)} \) is one of the groups other than \( (1) \) in (vi). Suppose by way of contradiction that \( T = C^{I(a)} \). Then \( (C_{32 33})^{I(a)} \) is a subgroup of \( T \) and isomorphic to one of the groups other than \( (i) \) in Lemma 1, where \( 32 \) and \( 33 \) are fixed by \( b \) and \( a \) has a 2-cycle \( (32 33) \).

Now we show that \( C_{32 33} \) has an element \( x \) such that \( x^{I(a)} = e^{I(a)} \). Since none of groups \( M_{10}^*, N(M_4)^*, PSL(2, 11) \) or \( S_5 \) as Lemma 1 contains any group as Lemma 1 properly, if \( T \) is isomorphic to \( M_{10}^*, N(M_4)^*, PSL(2, 11) \) or \( S_5 \), then \( (C_{32 33})^{I(a)} = T \). Hence \( C_{32 33} \) has an element \( x \) such that \( x^{I(a)} = e^{I(a)} \). Next suppose that \( T \) is isomorphic to \( M_{10} \). Then \( (C_{32 33})^{I(a)} \) is isomorphic to \( M_{10} \) or \( M_{10}^* \). On the other hand if a subgroup of \( T \) is isomorphic to \( M_{10}^* \), then this subgroup is the group \( (3) \) in (vi). Thus \( (C_{32 33})^{I(a)} = T \) or \( (C_{32 33})^{I(a)} \) is the group \( (3) \) in (vi), and so \( C_{32 33} \) has an element \( x \) such that \( x^{I(a)} = e^{I(a)} \). Finally suppose that \( T \) is isomorphic to \( N(M_4) \). Then \( (C_{32 33})^{I(a)} \) is isomorphic to \( N(M_4) \) or \( N(M_4)^* \). Similarly if a subgroup of \( T \) is isomorphic to \( N(M_4)^* \), then this subgroup is the group \( (5) \) in (vi). Thus \( (C_{32 33})^{I(a)} = T \) or \( (C_{32 33})^{I(a)} \) is the group \( (5) \) in (vi), and so \( C_{32 33} \)
has an element $x$ such that $x^{I(a)} = c^{I(a)}$. Thus in any case $C_{32 \cdot 33}$ has an element $x$ such that $x^{I(a)} = c^{I(a)}$.

Then $x$ is an involution on $I(a) \cup \{32, 33\}$. Since $x^s$ fixes $I(a) \cup \{32, 33\}$ pointwise and $|I(a) \cup \{32, 33\}| = 25$, $x^s$ is odd order. Hence $x^s$ is an involution, where $s$ is a suitable odd integer. Set $c' = x^s$. Then $C_{32 \cdot 33}$ has an involution $c'$ such that $c'^{I(a)} = c^{I(a)}$. Then $bc'$ is an involution on $I(a) \cup \{32, 33\}$. Since $(bc')^6$ fixes $I(a) \cup \{32, 33\}$ pointwise and $|I(a) \cup \{32, 33\}| = 25$, $(bc')^6$ is of odd order. Hence $bc'$ is of order $2r$, where $r$ is an odd integer. Set $c'' = b(bc')^r$. Then $c''$ is an involution commuting with $b$ and $c''^{I(a)} = c^{I(a)}$. Since $c''^{I(b)} = (32 \cdot 34) (33 \cdot 35) \cdots$, $c''^{I(b)} = (32 \cdot 33) \cdots$ and $a^{I(b)} = (32 \cdot 33) \cdots$, $(cc')^{I(b)} \neq 1$, $a^{I(b)}$, contrary to (viii).

(x) Let $x$ be a 2-element of $C$ such that $x^{I(a)}$ is an involution. If there is an involution $y$ in $C$ such that $|I(y)| = 23$ and $x^{I(a)}$ and $y^{I(a)}$ are conjugate in $C^{I(a)}$, then $x$ is an involution fixing twenty-three points.

Proof. By assumption, there is an element $z$ of $C$ such that $(x^s)^{I(a)} = y^{I(a)}$. Let $R$ be a Sylow 2-subgroup of $\langle \langle x^s, y, a \rangle \rangle$. Since $R^{I(a)} = \langle y \rangle^{I(a)}$, the order of $R^{I(a)}$ is two. By (vii), $R^{I(a)} = \langle a \rangle$. Thus the order of $R$ is four. This implies that both $\langle x^s, a \rangle$ and $\langle y, a \rangle$ are Sylow 2-subgroups of $\langle x^s, y, a \rangle$. Hence $x^s$ is conjugate to $y$ or $ya$. Thus $x$ is conjugate to the involution $y$ or $ya$ which fixes exactly twenty-three points by (iv). Hence $x$ is an involution fixing twenty-three points.

(xii) $C^{I(a)}$ is intransitive and has no orbit of length twenty-two. Furthermore it is impossible that $C^{I(a)}$ has two orbits $\Gamma$ and $\Delta$ such that $|\Gamma| = 21$ and $(C^{I(a)})_{\Gamma}$ is doubly transitive on $\Delta$.

Proof. Assume by way of contradiction that (xii) is false. Then we show first that there is a 2-element

$$x = (1 \ 2 \ \cdots \ 7 \ 8 \ 16 \ 9 \ 17 \ \cdots)$$

in $C$ such that $x^{I(a)}$ commutes with $b$ and $\overline{c}$. First suppose that $C^{I(a)}$ is transi-
tive. Then $C^{(a)}$ is $M_{23}$ or contained in a group of order 23.11 (see [1], p. 235). The latter case does not occur since $C^{(a)}$ has an involution. Thus $C^{(a)}$ is $M_{23}$.

Since
\[ \bar{b} = (1) (2) \ldots (7) (8 \ 9) (16 \ 17) \ldots \]
and
\[ \bar{c} = (1) (2 \ 3) (4 \ 5) (6) (7) (8) (9) (16) (17) \ldots , \]
$C$ has a 2-element $x$ as above. Next suppose that $C^{(a)}$ has an orbit of length twenty-two. Then $C^{(a)}$ is a subgroup of $M_{22}$. Since $M_{22}$ has no proper transitive subgroup on twenty-two points (see [1], p. 235), $C^{(a)}$ is $M_{22}$. Since $T = (C_{24}^{(a)})^{1/3}$ is contained in $C^{(a)}$ and $C^{(a)}$ has an orbit of length one, $T$ is isomorphic to $M_{10}$, $M_{10} \ast$ or $PSL(2, 11)$ by (vi). Hence by (vi), the $C^{(a)}$-orbit of length one is \{1\} or \{7\}. Hence similarly to the case above $C^{(a)}$ has a 2-element $x$ as above. Finally suppose that $C^{(a)}$ has two orbits $F$ and $A$ such that $|F| = 1$, $|A| = 21$ and $(C^{(a)})^\Gamma$ is doubly transitive on $A$. Since $T$ is contained in $C^{(a)}$ and $C^{(a)}$ has an orbit of length two, $T$ is isomorphic to $N(M_{23}), N(M_{23}) \ast$ or $S_5$ by (vi). Hence by (vi), $(C^{(a)})^\Gamma$ is a Sylow 2-subgroup of $C^{(a)}$. Since $T$ is contained in $C^{(a)}$ and $C^{(a)}$ has an orbit of length two, $T$ is isomorphic to $N(M_{9}), N(M_{9}) \ast$ or $S_5$ by (vi). Hence by (vi), $\Gamma$ is \{2, 3\} or \{4, 5\} and contained in $I(\bar{b})$. Let $\bar{M}$ be a normalizer of a Sylow 2-subgroup of $(C^{(a)})_{I(\bar{b})}$ in $(C^{(a)})_I$. Since a Sylow 2-subgroup of $(C^{(a)})_{I(\bar{b})}$ is a Sylow 2-subgroup of a stabilizer of two points of $\Delta$ in $(C^{(a)})_I$, by a lemma of E. Witt ([6], Theorem 4.4), $\bar{M}$ is doubly transitive on $I(\bar{b}) - \Gamma$. Hence $\bar{M}$ has an element $\bar{u}$ of order five and fixing exactly one point of $I(\bar{a}) - I(\bar{b})$. Since a Sylow 2-subgroup of $(C^{(a)})_I$ is of order divisible by $3 \cdot 16$. Hence $C^{(a)} = N_M_{23}(M_{23})$, and so similarly to the case above $C$ has a 2-element $x$ as above.

Let $R$ be a Sylow 2-subgroup of $\langle a, b, c, x \rangle$ containing $\langle a, b, c \rangle$. Then $R^{(a)} = \langle a, b, c, x \rangle^{(a)}$. Hence there is an element $x'$ of $R$ such that $x'^{(a)} = x^{(a)}$. Since $C^{(a)} = M_{23}, M_{22} \ast$ or $N_{M_{23}}(M_{23})$, any involution of $C^{(a)}$ is conjugate to $\bar{b}$ or $\bar{c}$ in $C^{(a)}$. Hence by (vi), $R$ is elementary abelian. Since $I(\bar{b}) = I(x'^{(a)})$, from (xi) we have that $I(\bar{b}) = I(\bar{c})$, which is a contradiction.

(viii) It is impossible that $C^{(a)}$ has an orbit of length fourteen.

Proof. Assume by way of contradiction that $C^{(a)}$ has an orbit of length fourteen. Then by (vi), $T$ is isomorphic to $N(M_9)$ or $N(M_9) \ast$, which has orbits of lengths 2, 9, 12 or 2, 9, 6, 6 respectively. Hence $C^{(a)}$ has orbits of lengths 9, 14. On the other hand $M_{23}$ has not a maximal subgroup having orbits of lengths 9, 14 (see [1], p. 235). Hence if $C^{(a)}$ has orbits of lengths 9, 14, then the maximal
subgroup of $M_{23}$ containing $C^{I(a)}$ is transitive and so of order 23. 11 (see [1], p. 235). This is a contradiction since the order of $C^{I(a)}$ is even. Thus (xiii) holds.

(xiv) It is impossible that $C^{I(a)}$ has an orbit of length fifteen.

Proof. Assume by way of contradiction that $C^{I(a)}$ has an orbit of length fifteen. Then by (vi), $T$ is isomorphic to $N(M_9)^* \circ S_5$, which has orbits of lengths 2, 9, 6, 6 or 5, 6, 2, 10 respectively. Hence $C^{I(a)}$ has orbits of lengths 2, 6, 15 or 8, 15.

First we show that if $C^{I(a)}$ has orbits of lengths 2, 6, 15 then $C^\Delta$ is $S_5$, where $\Delta$ is the $C^{I(a)}$-orbit of length six. In the case $T=N(M_9)^*$, we may assume that $\Delta=\{4, 5, 20, 21, 22, 23\}$. In the case $T=S_5$, $\Delta=\{2, 3, 12, 13, 14, 15\}$. Suppose that $(C^{I(a)})_\Delta \neq 1$. Since $|\Delta|=6$ and $C^{I(a)} \leq M_{23}$, $(C^{I(a)})_\Delta$ is an elementary abelian 2-group and so $(C^{I(a)})_\Delta$ has exactly one orbit of length one in $I(a)-\Delta$. On the other hand $(C^{I(a)})_\Delta$ is a normal subgroup of $C^{I(a)}$. Hence $C^{I(a)}$ has an orbit of length one in $I(a)-\Delta$, which is a contradiction. Thus $(C^{I(a)})_\Delta=1$. Hence $C^\Delta$ is isomorphic to $C^{I(a)}$ and so contains $T^\Delta$ properly. First assume that $T$ is isomorphic to $N(M_9)^*$. Since $C^{I(a)}$ has an orbit of length fifteen, $C^\Delta$ has an element of order five. Since $|\Delta|=6$, $C^\Delta$ is doubly transitive. Furthermore $C^\Delta$ is of order divisible by $5 \cdot |T^\Delta|$ and has an odd permutation $\epsilon$. Hence by [5], $C^\Delta$ is isomorphic to $S_5$. Next assume that $T$ is isomorphic to $S_5$. Since $C^\Delta$ contains $T^\Delta$ properly, $C^\Delta$ is $S_5$ by [5].

Next we show that if $C^{I(a)}$ has orbits of lengths 8, 15, then $C^\Delta$ contains $A_4$, where $\Delta$ is the $C^{I(a)}$-orbit of length eight. In the case $T=N(M_9)^*$ we may assume that $\Delta'=\{2, 3, 4, 5, 20, 21, 22, 23\}$. In the case $T=S_5$, $\Delta'=\{4, 5, 2, 3, 12, 13, 14, 15\}$. Take two points 2, 4 of $\Delta'$. In the case $T=N(M_9)^*$, $T_2=\langle y_1, \bar{y}_2 \rangle$ has orbits of lengths 1, 6 on $\Delta'-\{2\}$ and $T_4=\langle b, y_3, y_4 \rangle$ has no orbit of length one in $\Delta'-\{4\}$. In the case $T=S_5$, $T_4=\langle \bar{b}, x_4, x_6, \bar{c} \rangle$ has orbits of lengths 1, 6 on $\Delta'-\{4\}$ and $T_2=\langle \bar{b}, (b x_6 \bar{c})^s \rangle$ has no orbit of length one in $\Delta'-\{2\}$. Since $\Delta'$ is an orbit of $C$, $C_2$ is conjugate to $C_4$. Hence $C_2$ is transitive on $\Delta'-\{2\}$ and so $C^\Delta'$ is doubly transitive on $\Delta'$. Furthermore since $C^{I(a)}$ has an orbit of length fifteen, $C^{I(a)}$ has an element of order five. Since an element of order five fixes exactly three points and $|\Delta'|=8$, $C^{\Delta'}$ has an element of order five. Thus by [5], $C^{\Delta'}$ contains $A_4$.

Hence there is an involution $d$ in $C$ such that $d$ is conjugate to $b$ and has the following form: If $T$ is isomorphic to $N(M_9)^*$, then
\[ d = (2) (3) (4) (5) (20 23) (21 22) \cdots, \]
and if $T$ is isomorphic to $S_5$, then
\[ d = (2) (3) (4) (5) (12 15) (13 14) \cdots. \]
Since $\langle \bar{b}, d^{I(a)} \rangle < (C^{I(a)})_{2, 3, 4, 5}$, $I(\bar{b})=I(d^{I(a)})$ and $(bd)^{I(a)}$ is an involution. Hence $(bd)^s$ is a 2-element ($\pm 1$), where $s$ is a suitable odd integer. Since $((bd)^s)^{I(a)}$ is
conjugate to $\bar{b}$ and of order two, $(bd)^{r}$ is an involution by (x). Thus $(bd)^{r}$ is a central involution of a dihedral group $\langle b, d \rangle$. Set $d'=b(bd)^{r}$. Then $d'$ is an involution commuting with $b$ and has the same form as $d$ on $I(a)$. Furthermore $(cd)^{I(a)}=(1)\ (6)\ (7)\ (2\ 3)\ (4\ 5)\ (20\ 21)\ \ldots$ or $(1)\ (6)\ (7)\ (2\ 3)\ (4\ 5)\ (12\ 13)\ (14\ 15)\ \ldots$ Thus $(cd)^{I(a)}$ is an involution. Hence by the same argument as above, $d''=(cd')^{r}$ is an involution commuting with $c$ and $d''t_{I(a)}=d''I(a)=dI(a)$, where $r$ is a suitable odd integer. Since $b$ commutes with $c$ and $d'$, $b$ commutes with $d''$. Furthermore $I(\bar{b})=I(d''I(a))$ and $d''ba$. On the other hand $c$ is an involution commuting with $b$ and $d''$ and fixing twenty-three points. Hence by (xi) $I(\bar{b})=I(c)$, which is a contradiction.

(xv) It is impossible that $C^{I(a)}$ has an orbit of length eight.

Proof. Assume by way of contradiction that $C^{I(a)}$ has an orbit of length eight. Then by (vi), $T$ is isomorphic to $N(M_{10})^{*}$ or $S_{5}$, which has orbits of lengths $2, 9, 6, 6$ or $5, 6, 2, 10$ respectively. Since $C^{I(a)}$ has no orbit of length fifteen by (xiv), $C^{I(a)}$ has orbits of lengths $8, 9, 6$ or $5, 8, 10$. Since the $C^{I(a)}$-orbit of length eight is $\Delta'$ as in the proof of (xiv), by the same argument as is used in (xiv) $C^{\Delta'}$ is doubly transitive on $\Delta'$. Hence $C^{I(a)}$ has an element of order seven which fixes exactly two points. On the other hand $C^{I(a)}$ has an orbit of length six or five, which must be fixed by an element of order seven pointwise. Thus we have a contradiction.

(xvi) It is impossible that $C^{I(a)}$ has an orbit of length sixteen.

Proof. Assume by way of contradiction that $C^{I(a)}$ has an orbit of length sixteen. Then by (vi), $T$ is isomorphic to $M_{10}^{*}$ or $S_{5}$, which has orbits of lengths $1, 10, 6, 6$ or $5, 6, 2, 10$ respectively. Hence $C^{I(a)}$ has orbits of lengths $1, 6, 16$ or $2, 5, 16$, or $7, 16$. Let $\Delta$ be the $C^{I(a)}$-orbit of length sixteen. Set $\Gamma=I(a)-\Delta$ and $E=C(I(a))$. Since $|\Gamma|=7$, $C^{\Gamma}$ is a subgroup of $S_{5}$, and so a Sylow $2$-subgroup of $C^{I(a)}$ is of order at most sixteen. On the other hand since $(C^{I(a)})_{2}$ has an involution $\bar{b}$, where $2\in\Delta$, and $|\Delta|=16$, the order of $C^{I(a)}$ is divisible by $16\cdot2$. Hence $C^{\Gamma}$ is not isomorphic to $C^{I(a)}$. Since $C^{\Gamma}$ is isomorphic to $C^{I(a)}/E$, $E\neq1$.

Now we show that the order of $E$ is sixteen and any two involutions of $E$ are conjugate. Since $T$ is isomorphic to $M_{10}^{*}$ or $S_{5}$, $\Gamma$ contains a $T$-orbit of length six or five on which $T$ is $A_{6}$ or $S_{5}$ respectively. Hence $C^{I(a)}$ has an element $\bar{u}$ of order three and fixing four points of $\Gamma$ and an element $\bar{v}$ of order five and fixing two points of $\Gamma$. Since $|I(\bar{u})|=5$ and $|I(\bar{v})|=3$, $\bar{u}$ and $\bar{v}$ fix exactly one point of $\Delta$. Since $\Delta$ is a $C^{I(a)}$-orbit, we may assume that $\bar{u}$ and $\bar{v}$ fix the same point $i$ of $\Delta$. Then $\langle \bar{u}, \bar{v} \rangle$ is transitive on $\Delta-\{i\}$ and so $C^{I(a)}$ is doubly transitive on $\Delta$. On the other hand $E$ is a normal subgroup of $C^{I(a)}$ and semiregular on $\Delta$. Hence $E$ is regular on $\Delta$. Thus the order of $E$ is sixteen. Furthermore since $\langle \bar{u}, \bar{v} \rangle$ acts transitively on $E-\{1\}$ by conjugation, any two involutions of $E$ are conjugate.
Let \( x \) and \( x' \) be 2-elements of \( C \) such that \( x^{I(a)} = x'^{I(a)} \in E - \{1\} \). Then \( x^2 \) and \( x'^2 \) are contained in \( \langle a \rangle \) which is of order two. Hence the orders of \( x \) and \( x' \) are two or four. Assume that the order of \( x \) is four and the order of \( x' \) is two. Then \( \langle x, x' \rangle^{I(a)} = \langle x^2 \rangle^{I(a)} \) and \( \langle x, x' \rangle^I \geq \langle x^2 \rangle = \langle a \rangle \) by (vii). Hence a Sylow 2-subgroup of \( \langle x, x' \rangle \) is of order four. This implies that both \( \langle x \rangle \) and \( \langle x', a \rangle \) are Sylow 2-subgroups of \( \langle x, x' \rangle \) and not conjugate, which is a contradiction. Thus \( x \) and \( x' \) are of the same order. Then since any two involutions of \( E \) are conjugate, every 2-elements of \( C - \langle a \rangle \) have the same order. Let \( Q \) be a Sylow 2-subgroup of \( C \). Then \( Q^{I(a)} = E \) and \( Q_{I(a)} = \langle a \rangle \). Hence \( Q \) is of order 16·2. If every elements of \( Q - \langle a \rangle \) are of order four, then \( a \) is the only one involution of \( Q \). Hence \( Q \) is a cyclic or generalized quaternion group. This is impossible since \( Q^{I(a)} \) is an elementary abelian group of order sixteen and isomorphic to \( Q/I(a) \). Hence \( Q \) is elementary abelian. Since \( b \) normalizes \( \langle a \rangle \), we may assume that \( b \) normalizes \( Q \). Since \( b \) fixes exactly four points of \( \Delta \) and \( Q^{I(a)} \) is regular on \( \Delta \), \( C_\Delta^{(a)}(b) \) is of order four. Hence there are two involutions \( y \) and \( z \) in \( Q - \langle a \rangle \) such that both \( y^{I(a)} \) and \( z^{I(a)} \) commute with \( b \) and \( y^{I(a)} + z^{I(a)} \). Suppose that \( b \) commutes with neither \( y \) nor \( z \). Then since \( (y \cdot y^b)^{I(a)} = (z \cdot z^b)^{I(a)} = 1 \), \( y^b = yz \) and \( z^b = za \). Hence \( (yz)^b = yza = yz \) and \( yz = a \). Hence we may assume that \( b \) commutes with \( y \). Then \( y \) fixes exactly three points of \( I(b) \cap I(a) \) and four points of \( I(b) - (I(b) \cap I(a)) \). Hence there is a 2-cycle \( (i \ j) \) of \( a \) such that \( C_{i \ j} \geq \langle b, y \rangle \). Since \( (C_{i \ j})^{I(a)} \) is a subgroup of \( C^{I(a)} \), \( \Gamma \) is unions of \( C_{i \ j} \)-orbits. Since \( |\Gamma| = 7 \), by the assumption (2) \( (C_{i \ j})^{I(a)} \) is isomorphic to \( M_{10}^* \) or \( S_5 \) and has orbits of lengths 1, 6 or 2, 5 on \( \Gamma \) respectively. Therefore \( ((C_{i \ j})^{I(a)})_{\Gamma} = 1 \). However \( y^{I(a)} \) fixes \( \Gamma \) pointwise. Thus we have a contradiction.

**(xvii)** It is impossible that \( C^{I(a)} \) has an orbit of length seven.

Proof. Assume by way of contradiction that \( C^{I(a)} \) has an orbit of length seven. Then by (vi) \( T \) is isomorphic to \( M_{10}^* \) or \( S_5 \), which has orbits of lengths 1, 10, 5, 6 or 5, 6, 2, 10 respectively. Hence \( C^{I(a)} \) has orbits of lengths 7, 10, 6 or 7, 16. Then \( C^{I(a)} \) has an element of order seven. Since an element of order seven fixes exactly two points, it is impossible that \( C^{I(a)} \) has orbits of lengths 7, 10, 6. By (xvi), the second case does not occur. Thus (xvii) holds.

**(xviii)** It is impossible that \( C^{I(a)} \) has an orbit of length nine.

Proof. Assume by way of contradiction that \( C^{I(a)} \) has an orbit of length nine. Then by (vi), \( T \) is isomorphic to \( N(M_4) \) or \( N(M_6)^* \), which has orbits of lengths 2, 9, 12 or 2, 9, 6, 6 respectively. By (xiii) and (xv), \( C^{I(a)} \) has no orbit of length fourteen or eight. Hence \( C^{I(a)} \) has orbits of lengths 2, 9, 12 or 2, 9, 6, 6, where the orbit of length two is \{2, 3\} and the orbit of length nine is \{1, 8, 9, \ldots, 15\}. Since \( C^{I(a)} \) is a subgroup of \( M_{23} \), the order of \( (C^{I(a)})_2, 1 \) is a divisor of \( 20 \cdot 48 \). Since an element of order five fixes exactly three points and
there are $C^{j(a)}$-orbits of lengths 2, 12 or 2, 6, 6, $C^{j(a)}$ has no element of order five. Suppose that $(C^{j(a)})_{3, 3}$, has an element of order three. Since $(C^{j(a)})_{3, 3}$ and \{1, 8, 9, ..., 15\} is a $C^{j(a)}$-orbit of length nine, $C^{j(a)}$ is of order divisible by 3. This is a contradiction because $M_{23}$ has no subgroup of order 3. Thus $(C^{j(a)})_{3, 3}$ is a 2-group. Furthermore $T$ is primitive on \{1, 8, 9, ..., 15\}. Hence by [5], the order of $C^{j(a)}$ is 144 and so $C^{j(a)}$ is isomorphic to $N(M_{9})$. Hence by (ix), $T$ is not isomorphic to $N(M_{9})$. Hence $T$ is isomorphic to $N(M_{9})$. Take $(C^{j(a)})_{3, 3}$, where 32, 33 are fixed by $b$ and $a$ has a 2-cycle (32 33). Then by (ix), $(C^{j(a)})_{3, 3}$ is isomorphic to $N(M_{9})$ and so $(C^{j(a)})_{3, 3}=T$. Thus $C^{j(a)}$ is isomorphic to $M_{9}$. 

(xix) *It is impossible that $C^{j(a)}$ has an orbit of length thirteen or seventeen.*

Proof. $M_{23}$ has no element of order thirteen or seventeen. Hence (xix) holds.

(xx) *If $C^{j(a)}$ has an orbit of length eleven, then $C^{j(a)}$ is isomorphic to $M_{11}$.*

Proof. An element of order eleven has two 11-cycles and fixes one point. Hence by (xii), $C^{j(a)}$ has orbits of lengths 11, 1, 11 or 11, 12. First assume that $C^{j(a)}$ has orbits of lengths 11, 1, 11. Then by (vi), $T$ is isomorphic to $PSL(2, 11)$. Hence by (ix), $C^{j(a)}$ contains $T$ properly. This is a contradiction since $PSL(2, 11)$ is a maximal subgroup of $M_{22}$ (see [1], p. 235). Next assume that $C^{j(a)}$ has orbits of lengths 11, 12. Then the maximal subgroup of $M_{23}$ containing $C^{j(a)}$ is isomorphic to $M_{11}$ or of order 23·11 (see [1], p. 235). However the latter case does not occur since $C^{j(a)}$ is of even order. Suppose that $C^{j(a)}$ is isomorphic to a proper subgroup of $M_{11}$. Since the maximal subgroup of $M_{11}$ whose order is divisible by 11 is $PSL(2, 11)$, $C^{j(a)}$ is isomorphic to a subgroup of $PSL(2, 11)$. Hence $T$ is isomorphic to $PSL(2, 11)$ and so $C^{j(a)}=T$, contrary to (ix). Hence $C^{j(a)}$ is isomorphic to $M_{11}$.

(xxii) *$C^{j(a)}$ is isomorphic to $M_{11}$.*

Proof. $T$ is one of the seven groups of (vi). In the following we treat seven cases separately.

(1) Let $T$ be isomorphic to $M_{11}$. Then $T$ is a maximal subgroup of $M_{23}$ (see [1], p. 235). Hence if $C^{j(a)}$ contains $T$ properly, then $C^{j(a)}$ is $M_{23}$, contrary to (xii). Thus $C^{j(a)}=T$ and so $C^{j(a)}$ is isomorphic to $M_{11}$.

(2) Let $T$ be isomorphic to $M_{10}$. Then the lengths of the $T$-orbits are 1, 10, 12. By (xii) and (xix), $C^{j(a)}$ has no orbit of length 22, 23 or 13. Hence $C^{j(a)}$ has orbits of lengths 1, 10, 12 or 11, 12. By (ix), $C^{j(a)}$ contains $T$ properly. Hence the first case does not occur because $M_{10}$ is a maximal subgroup of $M_{22}$ (see [1], p. 235). In the second case $C^{j(a)}$ is isomorphic to $M_{11}$ by (xix).
(3) Let $T$ be isomorphic to $M_{10}^*$. Then the lengths of the $T$-orbits are 1, 10, 6, 6. By (xii), (xvi), (xvii) and (xix), $C^{I(a)}$ has no orbit of length 23, 22, 16, 7, 13 or 17. Hence the lengths of the $C^{I(a)}$-orbits are one of the following:

(a) 1, 10, 6, 6, (b) 1, 10, 12, (c) 11, 6, 6, (d) 11, 12.

First consider the case (a). Since $C^{I(a)}$ contains $T$ properly, $C^{I(a)}$ is contained in $N_{M_{22}}(E)$ (see [1], p. 235), where $E$ is an elementary abelian 2-group of order sixteen and fixing seven points of $I(a)$. Set $\Delta = I(E)$. Since $C^{I(a)}$ contains $T$ properly and $|\Delta| = 7$, $(C^{I(a)})_{\Delta}$ is a nonidentity elementary abelian 2-group. Since $C^{I(a)}/(C^{I(a)})_{\Delta} = A_6$, $(C^{I(a)})_{\Delta}$ has an automorphism group which is isomorphic to $A_6$. Hence $(C^{I(a)})_{\Delta}$ is of order sixteen and has an orbit of length sixteen. This is a contradiction. Thus the case (a) does not occur.

Next consider the case (b). Then $C^{I(a)}$ is isomorphic to $M_{10}$. Take $(C_{32,33})^{I(a)}$, where 32, 33 are fixed by $b$ and $a$ has a 2-cycle $(32\ 33)$. Then by (ix), $(C_{32,33})^{I(a)}$ is isomorphic to $M_{10}^*$ and so $(C_{32,33})^{I(a)} = T$. Thus $C_{32,33}$ has an element $x$ such that $\bar{e} = x^{I(a)}$. Then by the same argument as in (ix), we have a contradiction. Thus the case (b) does not occur.

Finally consider the cases (c) and (d). Since $C^{I(a)}$ has an orbit of length eleven, $C^{I(a)}$ is isomorphic to $M_9$, by (xx). Thus the lengths of the $C^{I(a)}$-orbits are 11, 12 and the case (c) does not occur.

(4) Let $T$ be isomorphic to $N(M)_2$. Then the lengths of the $T$-orbits are 2, 9, 12. By (xii), (xiii) and (xviii), $C^{I(a)}$ has no orbit of length 23, 14 or 9. Hence the lengths of the $C^{I(a)}$-orbits are one of the following:

(a) 2, 21, (b) 11, 12.

First consider the case (a). Since $T_1 = \langle \bar{b}, \bar{e}, x_3, x_4 \rangle$, the $T_1$-orbits on $I(a) - \{1, 2, 3\}$ are $\{8, 9, \ldots, 15\}$, $\{4, 5, 6, 7\}$ and $\{16, 17, \ldots, 23\}$, whose lengths are 8, 4 and 8 respectively. Since $T_1 = \langle \bar{b}, \bar{e}, x_3 \rangle$, the $T_1$-orbits on $I(a) - \{6, 2, 3\}$ are $\{1, 8, 9\}$, $\{10, 11, \ldots, 15\}$, $\{4, 5, 20, 21, 22, 23\}$, $\{7, 16, 17\}$ and $\{18, 19\}$, whose lengths are 3, 6, 6, 3, and 2 respectively. Since 1 and 6 are contained in the same $C^{I(a)}$-orbit, $(C^{I(a)})_1$ and $(C^{I(a)})_6$ are conjugate in $C^{I(a)}$ and so the $(C^{I(a)})_1$-orbits and $(C^{I(a)})_6$-orbits have the same lengths. Thus $(C^{I(a)})_1$ is transitive or has orbits of lengths eight and twelve on $I(a) - \{1, 2, 3\}$.

Suppose that $(C^{I(a)})_2$ has two orbits on $I(a) - \{1, 2, 3\}$. Then $\{8, 9, \ldots, 15\}$ and $\{4, 5, 6, 7, 16, 17, \ldots, 23\}$ or $\{4, 5, \ldots, 15\}$ and $\{16, 17, \ldots, 23\}$ are the $(C^{I(a)})_1$-orbits. In the first case $\langle (C^{I(a)})_2, T \rangle$ has an orbit $\{1, 8, 9, \ldots, 15\}$ of length nine and $\langle (C^{I(a)})_2, T \rangle = (C^{I(a)})$, has an orbit of length twelve. Hence $\langle (C^{I(a)})_2, T \rangle$ has a subgroup of order 3⁴. This is a contradiction since $M_{22}$ has no subgroup of order 3³. In the second case set $\Delta = \{4, 5, \ldots, 15\}$. Let $\bar{x}$ be an involution of $C^{I(a)}$ and conjugate to $\bar{b}$ in $(C^{I(a)})_2$. Since $\bar{b}$ fixes three points 2, 3, 1 and exactly four points of the $(C^{I(a)})$-orbit $\Delta$, $\bar{x}$ fixes three points 2, 3, 1
and exactly four points of $\Delta$. We may assume that $x$ fixes a point 9 of $\Delta$ which is not fixed by $\bar{b}$. Then $I(\bar{b}) \neq I(x)$ and so $|I(\bar{b}) \cap I(x)| \leq 3$. Hence $I(\bar{b}) \cap I(x) = \{1, 2, 3\}$. Since $\bar{b}$ fixes four points 4, 5, 6, 7 of $\Delta$ and $x$ fixes a point 9 of $\Delta$, $x$ fixes exactly three points of $\Delta - \{4, 5, 6, 7, 9\} = \{8, 10, 11, \ldots, 15\}$. On the other hand $y_1$ is an element of $T$ and fixes $\{1, 2, 3, 9\}$ as a set which is contained in $I(x)$. Hence $y_1$, $x$ fixes $I(x)$ as a set. Then since $y_1 = (8 \ 10 \ 15 \ 11 \ 13 \ 12 \ 14)$ on $\{8, 10, 11, \ldots, 15\}$ and $|I(x) \cap \{8, 10, 11, \ldots, 15\}| = 3$, $I(x) \supset 8$ and so $I(x) \supset \{1, 2, 3, 8, 9\}$. Thus $y_1$ fixes $I(x)$ as a set. Then since $y_1 = (8 \ 10 \ 15 \ 11 \ 13 \ 12 \ 14)$ on $\{8, 10, 11, \ldots, 15\}$ and $|I(x) \cap \{8, 10, 11, \ldots, 15\}| = 3$, $I(x) \supset 8$ and so $I(x) \supset \{1, 2, 3, 8, 9\}$. Hence $\langle \bar{b}, y_1 \rangle$ fixes $I(x)$ as a set. However since $x$ has fixed points in the $\langle \bar{b}, y_1 \rangle$-orbits $\{10, 11, \ldots, 15\}$, $I(x) \supset \{10, 11, \ldots, 15\}$, which is a contradiction.

Thus $(C_{\alpha}^{\{\alpha\}})$, is transitive on $I(a) - \{1, 2, 3\}$ and so $C_{\alpha}^{\{\alpha\}}$ is doubly transitive on $I(a) - \{2, 3\}$. Since $(C_{\alpha}^{\{\alpha\}})_{2,3}$ is a normal subgroup of $C_{\alpha}^{\{\alpha\}}$, $(C_{\alpha}^{\{\alpha\}})_{2,3}$ is transitive on $I(a) - \{2, 3\}$. Furthermore the order of $C_{\alpha}^{\{\alpha\}}$ is divisible by 21 \cdot 20. Hence $(C_{\alpha}^{\{\alpha\}})$ has an element of five fixing exactly three points. Hence $(C_{\alpha}^{\{\alpha\}})_{2,3}$ has an element of order five which has no fixed points in $I(a) - \{1, 2, 3\}$. On the other hand $T_{1,2,3} = \langle \bar{b}, x_1, x_0, (x_2 x_0)^{-1} \rangle$ has three orbits $\{8, 9, \ldots, 15\}, \{4, 5, 6, 7\}$ and $\{16, 17, \ldots, 23\}$, whose lengths are 8, 4 and 8 respectively. Hence $(C_{\alpha}^{\{\alpha\}})_{2,3}$ is transitive on $I(a) - \{1, 2, 3\}$. Thus $(C_{\alpha}^{\{\alpha\}})_{2,3}$ is doubly transitive on $I(a) - \{2, 3\}$, contrary to (xii).

In the case (b) by (xix), $C_{\alpha}^{\{\alpha\}}$ is isomorphic to $M_{21}$. Let $T$ be isomorphic to $N(M_9)*$. Then the lengths of the $T$-orbits are 2, 9, 6, 6. By (xii), (xiii), (xiv), (xv), (xviii) and (xix), $C_{\alpha}^{\{\alpha\}}$ has no orbit of length 23, 14, 15, 8, 9 or 17. Hence the lengths of the $C_{\alpha}^{\{\alpha\}}$-orbits are one of the following: (a) 2, 21, (b) 11, 6, 6, (c) 11, 12.

First consider the case (a). Set $\Delta = \{1, 4, 5, \ldots, 23\}$ which is the $C_{\alpha}^{\{\alpha\}}$-orbit of length twenty-one. Since $T_1 = \langle \bar{b}, \bar{c}, y_3 \rangle$, the $T_1$-orbits on $I(a) - \{2, 3\}$ are $\{8, 9, 10, 11\}, \{12, 13, 14, 15\}, \{4, 5, 6, 7\}, \{16, 17, 18, 19\}$ and $\{20, 21, 22, 23\}$, whose lengths are 4, 4, 2, 2 and 4 respectively. Since $T_1 = \langle \bar{b}, y_3, y_3 \rangle$, the $T_1$-orbits on $I(a) - \{2, 3\}$ are $\{1, 14, 15\}, \{8, 9, \ldots, 13\}, \{5, 22, 23\}, \{6, 7, 16, 17, 18, 19\}$ and $\{20, 21\}$, whose lengths are 3, 6, 3, 6 and 2 respectively. Since 1 and 4 belong to the $C_{\alpha}^{\{\alpha\}}$-orbit of $\Delta$, $(C_{\alpha}^{\{\alpha\}})$ is conjugate in $C_{\alpha}^{\{\alpha\}}$ and so the $(C_{\alpha}^{\{\alpha\}})$-orbits and the $(C_{\alpha}^{\{\alpha\}})$-orbits have the same lengths. Thus the lengths of the $(C_{\alpha}^{\{\alpha\}})$-orbits on $\Delta - \{4\}$ are one of the following:

(a.1) 2, 6, 12, (a.2) 2, 18, (a.3) 8, 6, 6, (a.4) 8, 12, (a.5) 14, 6, (a.6) 20.

First consider the cases (a.1) and (a.2). Since a point 4 belongs to the $C_{\alpha}^{\{\alpha\}}$-orbit of length twenty-one and $\{2, 3\}$ is a $C_{\alpha}^{\{\alpha\}}$-orbit of length two, $|C_{\alpha}^{\{\alpha\}}|: |C_{\alpha}^{\{\alpha\}}|_{2,3} = 2 \cdot 21$. Hence $|C_{\alpha}^{\{\alpha\}}|: |C_{\alpha}^{\{\alpha\}}|_{2,3} \leq 21$. Where $(i, j)$ is the $(C_{\alpha}^{\{\alpha\}})$-orbit of length two in $\Delta$. On the other hand $T$ is of order 9 \cdot 8. Hence
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(C/W) 2 3 4 1 j is of order divisible by two and so has an involution x. Then since (C/W) 2 3 4 1 j fixes a subset \{2, 3, 4, i, j\} of I(x) with cardinality five, (C/W) 2 3 4 1 j fixes I(x). Hence (C/W) 2 3 4 1 j fixes I(x) - \{2, 3, 4, i, j\}. Thus I(x) - \{2, 3, 4, i, j\} is a union of two (C/W) 2 3 4 1 j-orbits of length one or a (C/W) 2 3 4 1 j-orbit of length two in \(\Delta - \{4, i, j\}\), which is a contradiction.

Next consider the cases (a.3) and (a.4). Let i be a point of \(\Delta \cap I(\bar{b})\) and \(\Gamma\) be the (C/W) 2 3 4 1 j-orbit of length eight. Then we show that \(|\Gamma \cap I(\bar{b})| \neq 2\). Suppose by way of contradiction that \(|\Gamma \cap I(\bar{b})| = 2\). Set \(\{j, k\} = \Gamma \cap I(\bar{b})\).

Since \(|C^\alpha(\bar{b})| = 2\cdot 21\) and \(T\) is of order 9 \(\cdot 8\), (C/W) 2 3 4 1 j is of order divisible by 3 and so has an element \(u\) of order three. Since \(u\) fixes exactly five points, \(u\) fixes exactly two points of \(\Gamma\). Hence we may assume that \(\bar{u}\) fixes the point \(j\) of \(\Gamma \cap I(\bar{b})\). Then \(\bar{u}\) fixes a subset \(\{2, 3, i, j\}\) of \(I(\bar{b})\) with cardinality four. Hence \(\bar{u}\) fixes \(I(\bar{b})\). Since \(I(\bar{b})\) contains \(\{2, 3, 4, i, j\}\), \(\bar{u}\) has a 3-cycle \((k k' k'')\) in \(I(\bar{b})\). On the other hand \(\bar{u}\) is an element of (C/W) 2 3 4 1 j and \(k\) is a point of the (C/W) 2 3 4 1 j-orbit \(\Gamma\). Hence \(\{k, k', k''\}\) is contained in \(\Gamma\) and so contained in \(\Gamma \cap I(\bar{b})\). This is a contradiction since \(\Gamma \cap I(\bar{b}) = \{j, k\}\). Thus \(|\Gamma \cap I(\bar{b})| \neq 2\).

Since \(T\)-orbits on \(\Delta - \{4\}\) are \(\{1, 14, 15\}, \{8, 9, \ldots, 13\}, \{5, 22, 23\}, \{6, 7, 16, 17, 18, 19\}\) and \(\{20, 21\}\), the (C/W) 2 3 4 1 j-orbit of length eight is \(\{20, 21, 1, 8, 9, 10, 11, 12, 13\}\). Hence by what we have proved above, the (C/W) 2 3 4 1 j-orbit of length eight is \(\{20, 21, 8, 9, 10, 11, 12, 13\}\).

Since \(T\)-orbits on \(\Delta - \{4\}\) are \(\{1, 14, 15\}, \{8, 9, \ldots, 13\}, \{5, 22, 23\}, \{6, 7, 16, 17, 18, 19\}\) and \(\{20, 21\}\), the (C/W) 2 3 4 1 j-orbit of length eight is \(\{20, 21, 1, 8, 9, 10, 11, 12, 13\}\). Similarly since \(T_4 = \langle \bar{y}, \bar{z}, y, z \rangle\) has orbits \(\{1, 8, 9, 10, 11, 12, 13, 14, 15\}\), \{4, 5, 20, 21, 22, 23\}, \{18, 19\}, and \(\{7, 16, 17\}\) on \(\Delta - \{6\}\), by what we have proved above the (C/W) 2 3 4 1 j-orbit of length eight is \(\{18, 19, 10, 11, 12, 13, 14, 15\}\). Therefore \(|(C/W)_{4 12}| = |(C/W)_{8 12}| = 8\).

Since \(|(C/W)_{4 12}| = |(C/W)_{8 12}| = |(C/W)_{4 12}|\), 4 and 6 belong to the (C/W) 2 3 4 1 j-orbit of length eight. On the other hand \(T_{12} = T_{12}^2 \Gamma\) has orbits \(\{4, 20, 22, 23\}\) and \(\{6, 7, 16, 19\}\). Hence the (C/W) 2 3 4 1 j-orbit of length eight is \(\{4, 20, 22, 23, 6, 7, 16, 19\}\). Since \(\Gamma = \{20, 21, 8, 9, 10, 11, 12, 13\}\) is a (C/W) 2 3 4 1 j-orbit of length eight, by the same argument as is used above for \(\bar{u}\), (C/W) 2 3 4 1 j has an element \(\bar{v}\) which is of order three and fixes 12 and exactly one more point of \(\Gamma\). Since \(\bar{v}\) is also an element of (C/W) 2 3 4 1 j, \(\bar{v}\) fixes exactly two points of the (C/W) 2 3 4 1 j-orbit of length eight. Hence \(\bar{v}\) fixes exactly five points 2, 3, 4, 12, 20 because the (C/W) 2 3 4 1 j-orbit of length eight and (C/W) 2 3 4 1 j-orbit of length eight have exactly one point 20 in common. On the other hand \(T\) has an element \(\bar{y} = (2) (3) (4) (20) (21) (9 10 12) (8 13 11)\). ... Thus \(\langle \bar{v}, \bar{y} \rangle = (C/W)_{2 3 4 20}\). Since \(\bar{v}\) fixes one point 12 and have two 3-cycles and \(\bar{y} = (21) (9 10 12) (18 13 11)\) on \(\Gamma - \{20\}\), (C/W) 2 3 4 20 is transitive or has two orbits \(\{21, 9, 10, 12\}\) and \(\{8, 11, 13\}\) on \(\Gamma - \{20\}\). In the first case since \(|\Gamma - \{20\}| = 7\), (C/W) 2 3 4 20 has an element of order seven. This is a contradiction since an element of order seven fixes exactly two points. In the second case since \(|\{21, 9, 10, 12\}| = 4\) and \(|\{8, 11, 13\}| = 3\),
there is an involution $\bar{x}$ fixing $\{2, 3, 4, 20, 8, 11, 13\}$ pointwise. This is a contradiction since $y$ fixes four points $2, 3, 4, 8$ of $I(x)$ but does not fix $I(\bar{x})$.

In the case (a.5), there is an element of order seven which fixes a $(C^{(a)})_{23}$-orbit of length six pointwise. This is a contradiction.

Finally consider the case (a.6). Then $C^{(a)}$ is doubly transitive on $\Delta$. Since $(C^{(a)})_{23}$ is normal in $C^{(a)}$, $(C^{(a)})_{23}$ is isomorphic to $M_{23}$. Thus $(C^{(a)})_{23}$ has an element of order five, which fixes the $(C^{(a)})_{23}$-orbit $\{2,3\}$ pointwise. Thus $(C^{(a)})_{123}$ has an element of order five. On the other hand since $T_{123} = \langle \bar{b}, \bar{c}, \bar{y}_1 \rangle$, the lengths of the $T_{123}$-orbits on $\Delta \setminus \{1\}$ are $4, 4, 2, 4, 4$. Hence $(C^{(a)})_{123}$ is transitive or has two orbits of length ten on $\Delta \setminus \{1\}$. Furthermore since $T_{234} = \langle \bar{b}, \bar{y}_2 \bar{y}_3 \rangle$, the lengths of the $T_{234}$-orbits are $3, 6, 2, 3, 3$ on $\Delta \setminus \{4\}$. Since $(C^{(a)})_{23}$ is conjugate to $(C^{(a)})_{123}$ in $C^{(a)}$, $(C^{(a)})_{23}$ has no orbit of length ten in $\Delta \setminus \{4\}$. Thus $(C^{(a)})_{23}$ is doubly transitive on $\Delta$. Contrary to (xii).

(6) Let $T$ be isomorphic to $PSL(2, 11)$. Then the lengths of the $T$-orbits are $11, 1, 11$. By (xii), $C^{(a)}$ has no orbit of length twenty-three or twenty-two. Hence $C^{(a)}$ has an orbit of length eleven. Thus by (xx), $C^{(a)}$ is isomorphic to $M_{23}$.

(7) Let $T$ be isomorphic to $S_5$. Then the lengths of the $T$-orbits are $5, 6, 2, 10$. By (xii), (xiii), (xiv), (xv), (xvi), and (xvii), $C^{(a)}$ has no orbit of length $23, 15, 8, 16, 7, 13$ or $17$. Hence the lengths of the $C^{(a)}$-orbits are one of the following:

(a) $5, 6, 2, 10$,  
(b) $5, 6, 12$,  
(c) $5, 18$,  
(d) $11, 2, 10$,  
(e) $11, 12$,  
(f) $2, 21$.

First consider the cases (a) and (b). Let $\Delta$ be the $C^{(a)}$-orbit of length six. Suppose that $(C^{(a)})_{\Delta} = 1$. Since $|\Delta| = 6$, $(C^{(a)})_{\Delta}$ is a 2-group and fixes exactly one more point $i$ which is not contained in $\Delta$. Since $(C^{(a)})_{\Delta}$ is normal in $C^{(a)}$, $C^{(a)}$ fixes the point $i$, which is a contradiction. Hence $(C^{(a)})_{\Delta} = 1$ and so $C^{(a)}$ isomorphic to $C^\Delta$. By (iv), $C^{(a)}$ contains $T$ properly. Hence $C^\Delta$ contains $T^\Delta$ properly. Since $T^\Delta$ is isomorphic to $S_5$, $C^\Delta$ is $S_5$. Let $j$ be a point of the $C^{(a)}$-orbit of length five. Then $|C^{(a)} : (C^{(a)})_j| = 5$. This is a contradiction since $S_5$ has no subgroup of index five.

Next consider the case (c). Since the $C^{(a)}$-orbit of length eighteen contains $4$ and the order of $T_4$ is $|S_4|/2$, the order of $C^{(a)}$ is divisible by $3^3$. This is a contradiction since the order of $M_{23}$ is not divisible by $3^3$.

In the cases (d) and (e) by (xx), $C^{(a)}$ is isomorphic to $M_{23}$. Thus the lengths of the $C^{(a)}$-orbits are $11, 12$ and the case (d) does not occur.

Finally consider the case (f). Then the $C^{(a)}$-orbit of length two is $\{4, 5\}$. Set $\Delta = I(\bar{a}) = \{4, 5\}$. Since $T_1 = \langle \bar{b}, \bar{c}, \bar{e}^{s}, \bar{e}^{x} \rangle$, the $T_1$-orbits on $\Delta \setminus \{1\}$ are $\{8, 9, 10, 11\}$, $\{2, 3, 12, 13, 14, 15\}$, $\{6, 7, 20, 21, 22, 23\}$ and $\{16, 17, 18, 19\}$, whose lengths are $4, 6, 6$ and $4$ respectively. Since $T_1 = \langle \bar{b}, \bar{c}^{s}, \bar{e}^{x} \rangle$, the $T_1$-orbits on
\[\Delta - \{3\} \text{ are } \{1, 8, 9, 10, 11\}, \{2, 12, 13, 14, 15\} \text{ and } \{6, 7, 16, 17, 18, 19, 20, 21, 22, 23\}, \text{ whose lengths are } 5, 5 \text{ and } 10 \text{ respectively. Since } (C^{(a)})_3 \text{ is conjugate to } (C^{(a)})_h, (C^{(a)})_3 \text{ is transitive or has two orbits of length ten on } \Delta - \{3\}. \text{ Suppose that } (C^{(a)})_3 \text{ has two orbits of length ten. Then } \{1, 2, 8, 9, \ldots , 15\} \text{ and } \{6, 7, 16, 17, \ldots , 23\} \text{ are } (C^{(a)})_3\text{-orbits of length ten. Hence the } <(C^{(a)})_3, T>_\text{orbits are } \{1, 2, 3, 8, 9, \ldots , 15\}, \{4, 5\} \text{ and } \{6, 7, 16, 17, \ldots , 23\}, \text{ whose lengths are } 11, 2 \text{ and } 10 \text{ respectively. This is a contradiction since an element of order eleven has two 11-cycles. Hence } (C^{(a)})_3 \text{ is transitive on } \Delta - \{3\} \text{ and so } C^{(a)} \text{ is doubly transitive on } \Delta. \text{ Since } (C^{(a)})_{45} \text{ is normal in } C^{(a)}, (C^{(a)})_{45} \text{ is transitive on } \Delta. \text{ Since } T_{45} = <5, c, c^x_2> \text{ and } T_{453} = <5, (c^x_2c^x_3), T_{451} \text{ and } T_{453} \text{ has the same orbits as } T_1 \text{ and } T_2 \text{ respectively. Hence in the same way above } (C^{(a)})_{45} \text{ is doubly transitve on } \Delta, \text{ contrary to (xii).}

(xxii) If } C^{(a)} \text{ is isomorphic to } M_{11}, \text{ then we have a contradiction.}

\text{Proof. Suppose by way of contradiction that } C^{(a)} \text{ is isomorphic to } M_{11}. \text{ If } T \text{ is one of the groups of (vi) other than } PSL(2, 11), \text{ then } T \text{ has an element of order four. Hence first we assume that } T \text{ is not isomorphic to } PSL(2, 11). \text{ Then } C_{24,25} \text{ has a 2-element }


Then \langle u, b \rangle^{I(a) \cup (24,25)} \text{ is a cyclic group of order four and } \langle u, b \rangle^{I(a) \cup (24,25)} \text{ is of odd order because } |I(a) \cup \{24, 25\}| = 25. \text{ Hence a Sylow 2-subgroup of } \langle u, b \rangle \text{ containing } b \text{ is a cyclic group of order four and has a generator which has the same form as } u \text{ on } I(a) \cup \{24, 25\}. \text{ Hence we may assume that } u^2 = b. \text{ Then } u^{(b)} = (1)(2)(3)(4 5)(6 7)(24)(25)\ldots \text{ and so } u^{(b)} \text{ is of order two. On the other hand } c^{(b)} \text{ is an involution of } C_2(b)^{(b)} \text{ and } (cu)^{(b)} \text{ is also an involution of } (C_2(b)^{(b)} \text{ because } (cu)^{(b)} = (1)(2)(3)(4 5)(6 7)(24)(25)\ldots. \text{ Thus } C_2(b)^{(b)} \text{ has an elementary abelian group } \langle a, u, c^{(b)} \rangle \text{ of order eight. On the other hand by the same argument as is used for } a C_2(b)^{(b)} \text{ is isomorphic to } M_{11}. \text{ This is a contradiction since } M_{11} \text{ has no elementary abelian group of order eight. Thus } T \text{ must be isomorphic to } PSL(2, 11). \text{ In the same way we have that for any 2-cycle } (i j) \text{ of } a \text{ or } (C_i)\text{ is isomorphic to } PSL(2, 11). \text{ Since } C^{(a)} \text{ is isomorphic to } M_{11}, C \text{ has a 2-element}


Then \langle v^2 \rangle^{I(a)} = b. \text{ Hence by (x), } v^2 \text{ is an involution fixing twenty-three points. Let } Q \text{ be a Sylow 2-subgroup of } \langle b, v \rangle \text{ containing } \langle b \rangle. \text{ Since } Q^{I(a)} = \langle v \rangle^{I(a)}, \text{ there is an element } w \text{ in } Q \text{ such that } w^{I(a)} = v^{I(a)}. \text{ Then } w^2 b = \text{ a 2-element of } C \text{ and fixes at least twenty-three points. Thus } w^2 b = 1 \text{ or } a \text{ and so } w^2 = b \text{ or } ab,
Thus \( w^{t(a)} \) or \( w^{t(ab)} \) is an involution fixing seven points. Since \( I(a) \cap I(b) \cap I(w) = I(a) \cap I(ab) \cap I(w) = \{1, 2, 3\} \), there are two points \( i, j \) in \( I(b) \) or \( I(ab) \) such that \( a \) has a 2-cycle \((i \, j)\) and \( w \) fixes \( i, j \). Thus \( w \in C_{i, j} \). Hence \( w^{t(a)} \) is of order four and contained in \( (C_{i, j})^{t(a)} \). However since \( (C_{i, j})^{t(a)} \) is isomorphic to \( PSL(2, 11) \), \( (C_{i, j})^{t(a)} \) has no element of order four. Thus we have a contradiction.

Thus we complete the proof of Lemma 3. Hence this completes the proof of the theorem.

3. Proof of Lemma 1

In this section we assume that \( G \) is a permutation group as in Lemma 1.

3.1. Let \( x \) be an element of \( G \).

(i) If \( x \) is an involution, then \( |I(x)| = 7 \).

(ii) If \( x \) is of order three, then \( |I(x)| = 5 \).

(iii) If \( x \) is of order four, then \( |I(x)| = 3 \) and \( x \) has two 2-cycles.

(iv) If \( x \) is of order five, then \( |I(x)| = 3 \).

(v) If \( x \) is of order six, then \( |I(x)| = 1 \) and \( x \) has two 2-cycles and two 3-cycles.

(vi) Let \( y \) be an element of \( G \). If \( x \) and \( y \) are of order two and \( |I(x) \cap I(y)| \geq 4 \), then \( I(x) = I(y) \).

Proof. This follows immediately from the assumption (4).

3.2. Let \( a \) be an involution of \( G \). Then we have a projective plane \( P(a) \) of order two with the following incidence structure:

(i) The set of points of \( P(a) \) is \( I(a) \).

(ii) For two distinct points \( i, j \) of \( I(a) \) the line containing \( i, j \) is \( \{\Delta(i, j) \cup \{i, j\}\} \cap I(a) \).

Proof. By (i) of (3.1), \( |I(a)| = 7 \). By the assumption (7), each line contains three points. Hence if \( \Delta(i, j) \cap I(a) = \{k\} \), then a line containing any two points of \( \{i, j, k\} \) contains the remaining one point of \( \{i, j, k\} \) by the assumption (3). Thus any two distinct points are contained in one and only one line.

Next let \( L_1 \) and \( L_2 \) be distinct lines. Suppose that \( L_1 \) has no point in common with \( L_2 \). Then there is exactly one point \( i \) neither on \( L_1 \) nor on \( L_2 \). Let \( j \) be a point on \( L_1 \) and let \( k_1, k_2 \) be two distinct points on \( L_2 \). Then the line \( L \) containing \( j \) and \( k_1 \) is distinct from \( L_1 \) and \( L_2 \). Hence \( L \) contains \( i \). In the same way the line \( L' \) containing \( j \) and \( k_2 \) contains \( i \). Thus two distinct lines \( L \) and \( L' \) have two points \( i, j \) in common, which is a contradiction. Hence any two distinct lines contain one and only one point in common. Thus (3.2) is proved.
From now on for an involution \( a \) a projective plane defined in (3.2) is called a plane \( P(a) \) or merely a plane.

3.3. If \( \Delta(i, j) \ni k \), then \( i, j, k \) are collinear in every plane containing \( i, j, k \).

Proof. Let \( a \) be any involution such that \( I(a) \ni \{i, j, k\} \). By the assumption (7), \( I(a) \cap \Delta(i, j) = \{k\} \). Hence \( i, j, k \) are collinear in the plane \( P(a) \).

3.4. For an involution \( a \) \( N(G_{I(a)}^{I(a)}) \leq PSL(3, 2) \).

Proof. Let \( x \) be any element of \( N(G_{I(a)}) \) and let \( i, j \) be any two distinct points of \( I(a) \). Then by the assumption (5), \( \{\Delta(i, j) \cup \{i, j\} \cap I(a)\}^x = \{\Delta(i^x, j^x) \cup \{i^x, j^x\} \cap I(a)\} \). Thus \( x^{I(a)} \) is a collineation of the plane \( P(a) \). Hence \( N(G_{I(a)})^{I(a)} \leq PSL(3, 2) \).

3.5. If an involution \( a \) has a 2-cycle \((i, j)\), then there is an involution of \( G_{i, j} \) commuting with \( a \).

Proof. By the assumptions (1) and (6), \( G_{i, j} \) has an involution. Since \( a \in N_{G_{i, j}} \), there is an involution of \( G_{i, j} \) commuting with \( a \).

3.6. Let \( a \) and \( b \) be involutions such that \( I(a) \cap I(b) = \{i, j, k\} \) and \( I(a) \ni \{i, j, k\} \). If \( \Delta(i, j) \ni k \), then \( ab \) is of odd order.

Proof. Suppose by way of contradiction that \( ab \) is of even order. Then there is an involution \( c \) of \( G_{i, j} \) commuting with \( a \) and \( b \). Clearly \( c \) fixes \( I(a) \). Since \( i, j, k \) are noncollinear in \( P(a) \) and fixed by \( c \), \( c \) fixes \( I(a) \) pointwise by (3.4). Thus \( I(a) = I(c) \). In the same way \( I(b) = I(c) \). Hence \( I(a) = I(b) \), which is a contradiction.

3.7. Let \( a \) and \( b \) be involutions such that \( I(a) \ni \{i, j, k\} \) and \( I(b) \ni \{i, j, l\} \). If \( \Delta(i, j) \ni k, l \) and \( k \neq l \), then \( ab \) is of odd order.

Proof. Suppose by way of contradiction that \( ab \) is of even order. Then there is an involution \( c \) of \( G_{i, j} \) commuting with \( a \) and \( b \). Clearly \( c \) fixes \( I(a) \). Since \( i, j, k \) are collinear in \( P(a) \) and \( c \) fixes \( i, j, k \) fixes \( k \). Similarly \( c \) fixes \( l \). Thus \( \Delta(i, j) \cap I(c) \ni \{k, l\} \), contrary to the assumption (7).

3.8. \( \Delta(i, j) \) is an orbit of \( G_{i, j} \).

Proof. Let \( k \) be a point of \( \Delta(i, j) \) and \( l \) a point of a \( G_{i, j} \)-orbit containing \( k \). Then there is an element \( x \) of \( G_{i, j} \) such that \( l = k^x \). Since \( \Delta(i, j)^x = \Delta(i, j) \), \( \Delta(i, j) \ni k^x = l \). Conversely let \( \Delta(i, j) \ni k, l \) and \( k \neq l \). By the assumption (6), there are involutions \( a \) and \( b \) such that \( I(a) \supseteq \{i, j, k\} \) and \( I(b) \supseteq \{i, j, l\} \). By (3.7), \( ab \) is of odd order. Hence \( a^x = b \) for some element \( x \) of \( G_{i, j} \). On the other hand \( \Delta(i, j) \cap I(a) = \{k\} \) and \( \Delta(i, j) \cap I(b) = \{l\} \). Hence \( k^x = l \). Thus \( \Delta(i, j) \) is an orbit of \( G_{i, j} \).
3.9. $|\Delta(i, j)| > 1$ for some $i, j$ of $\Omega$.

Proof. Suppose by way of contradiction that $|\Delta(i, j)| = 1$ for any two points $i, j$. Let $\Delta(i, j) = \{k\}$ and set $\Delta'(i, j) = \Delta(i, j) \cup \{i, j\}$. Then $\Delta'(i, j) = \Delta'(j, k) = \Delta'(k, i) = \{i, j, k\}$. Conversely if $\Delta'(i', j') = \Delta'(i, j)$, then $\{i', j'\} \subset \{i, j, k\}$. Hence $\Delta(k', j')$ is one of the sets $\Delta(i, j)$, $\Delta(j, k)$ or $\Delta(k, i)$. Thus the number of combinations of twenty-three points taken two at a time is divisible by three, which is a contradiction. Thus $|\Delta(i, j)| > 1$ for some $i, j$ of $\Omega$.

3.10. $|\Delta(i, j)|$ is odd and at most nine.

Proof. By the assumption (6), there is an involution $x$ fixing $i$ and $j$. Then $x$ fixes $\Delta(i, j)$. Since $|\Delta(i, j) \cap I(x)| = 1$, $|\Delta(i, j)|$ is odd.

Now let $|\Delta(i, j)| > 1$. Then for distinct points $k_1, l_1$ of $\Delta(i, j)$ there are involutions $a$ and $b$ such that $I(a) \supset \{i, j, k_1\}$ and $I(b) \supset \{i, j, l_1\}$. By (i) of (3.1) $I(a) = \{i, j, k_1, k_2, \ldots, k_3\}$ and $I(b) = \{i, j, l_1, l_2, \ldots, l_4\}$. Then by the assumption (7) $k_t, l_t \in \Delta(i, j)$, $2 \leq t \leq 5$. Since $|I(a) \cap I(b)| \leq 3$ by (vi) of (3.1). Hence $|\{k_2, k_3, k_4, k_5\} \cap \{l_2, l_3, l_4, l_5\}| \leq 1$. Furthermore $I(a^a) \supset \{i, j\}$. If $I(a^a) = I(a)$, then $a \in \langle a \rangle$. Consequently $aba = a \cdot aba$, and so $[a^a, a] = 1$, contrary to (3.7). Thus $I(a^a) \neq I(a)$. Moreover from $k_t^a = k_t \pm l_t$, $l_t^a = k_t$. Hence in the same way $|\{k_2, k_3, k_4, k_5\} \cap \{l_2, l_3, l_4, l_5\}| \leq 1$. Next if $I(a^a) = I(a^b)$, then similarly $aba = b \cdot aba$, and so $[a^a, a^b] = 1$, contrary to (3.7). Thus $I(a^a) = I(a)$. Since $l_t^a = l_t$, in the same way $|\{l_2, l_3, l_4, l_5\} \cap \{l_2, l_3, l_4, l_5\}| \leq 1$. Thus $|\{k_2, k_3, k_4, k_5\} \cap \{l_2, l_3, l_4, l_5\}| \geq 12 - 3 = 9$. Since $i, j, k_t, l_t$, where $2 \leq t \leq 5$, do not belong to $\Delta(i, j)$, $|\Delta(i, j)| \leq 23 - (2 + 9) = 12$. Since $|\Delta(i, j)|$ is odd, $|\Delta(i, j)| \leq 11$. If $|\Delta(i, j)| = 11$, then $G_{i, j}$ has an element of order eleven by (3.8). This is a contradiction since an element of order eleven in $M_{23}$ fixes exactly one point. Thus (3.10) is proved.

3.11. $G$ has no element of order seven.

Proof. Suppose by way of contradiction that $G$ has an element $x$ of order seven. Then we may assume that

$$x = (1) (2) (3 \ 4 \ldots 9) (10 \ 11 \ldots 16) (17 \ 18 \ldots 23).$$

Set $\Gamma_1 = \{3, 4, \ldots, 9\}$, $\Gamma_2 = \{10, 11, \ldots, 16\}$ and $\Gamma_3 = \{17, 18, \ldots, 23\}$. Since $1 \leq |\Delta(1, 2)| \leq 9$, we may assume that $|\Delta(1, 2)| = \Gamma_1$ by (3.8). By the assumption (6), there is an involution $a$ fixing $1, 2, 3$. Since $|I(a)| = 7$ and $|I(a) \cap \Gamma_1| = 1$, $a$ fixes exactly four points of $\Gamma_1 \cup \Gamma_3$, say $i, i, i', i''$. Set $H = \langle a, x \rangle$. Then $\{1\}$, $\{2\}$ and $\Gamma_1$ are $H$-orbits.

Suppose that $\Gamma_2 \cap \Gamma_3$ is an $H$-orbit. Since $|\Gamma_2 \cup \Gamma_3| = 14$ and $a \in H_{i, i'}$, $|H| = |a| H_{i, i'} |H_{i, i'}| = 14 \cdot 2^r$, where $r$ is some integer. On the other hand $|H| = |3^H| \cdot |H_{i, i'}| = |\Gamma_1| \cdot |H_{i, i'}|$. Hence $|H_{i, i'}| = 4^r$. Thus there is an element $b$ different
from $a$ and $1$ in a Sylow 2-subgroup of $H_8$ containing $a$. From $I(b) \supseteq \{1, 2, 3\}$, we get that the order of $b$ is four or two. First assume that $b$ is of order four. Then by (iii) of (3.1), $b$ fixes exactly three points and has two 2-cycles. Since $b$ fixes $\Gamma_1 - \{3\}$ and $|\Gamma_1 - \{3\}| = 6$, $b$ has a 2-cycle on $\Gamma_1$. Thus $b^s$ is an involution such that $I(b^s) \supseteq \{1, 2\}$ and $|\Delta(1, 2) \cap I(b^s)| > 1$, contrary to the assumption (7). Next assume that $b$ is of order two. By the same reason, $ab$ is also of order two. Thus $a$ and $b$ have the same 2-cycle on $\Gamma_1$. Hence $|\Delta(1, 2) \cap I(ab)| > 1$, contrary to the assumption (7).

Therefore $\Gamma_2$ and $\Gamma_3$ are two distinct $H$-orbits. Since $I(a) \cap \{\Gamma_2 \cup \Gamma_3\} = \{i, i_2, i_3, i_4\}$ and $|\Gamma_2| = |\Gamma_3| = 7$, we may assume that $I(a) \cap \Gamma_i = \{i, i_2, i_3\}$. Then for some integer $s = a^s$ fixes $i$ and does not fix $3$. By (3.7), $ac$ is of odd order. Since $I(ac) \supseteq \{1, 2, i_3\}$, the order of $ac$ is three or five. First assume that $ac$ is of order three. Then by (ii) of (3.1), $|I(ac)| = 5$. Since $|\Gamma_i| = 7$, $|I(ac) \cap \Gamma_i| = 1$, $1 \leq i \leq 3$. However $|I(a) \cap I(c) \cap \Gamma_i| = 1$. Hence $|I(ac) \cap \Gamma_i| \neq 1$. Thus we have a contradiction. Next assume that $ac$ is of order five. Since $|\Gamma_i| = 7$, $|I(ac) \cap \Gamma_i| \geq 2$, $1 \leq i \leq 3$. This is a contradiction since $|I(ac)| = 3$. Thus $G$ has no element of order seven.

3.12. $|\Delta(i, j)| = 1, 3, 5$ or $9$.

Proof. This follows immediately from (3.10) and (3.11).

3.13. If there is a plane containing three distinct points $i, j, k$, then the number of planes containing $i, j, k$ is one, three or five.

Proof. We may assume that $\{i, j, k\} = \{1, 2, 3\}$. For a point $t$ of $\Omega - \{1, 2, 3\}$ the number of planes containing $\{1, 2, 3, t\}$ is at most one by (vi) of (3.1). Since $|\Omega - \{1, 2, 3\}| = 20$ and there are seven points in a plane, the number of planes containing $1, 2, 3$ is at most $20/4 = 5$.

Suppose by way of contradiction that the number of planes containing $1, 2, 3$ is even. Then we may assume that there is an involution $x$ of the following form

$$x = (1) (2) \cdots (7) (8 9) (10 11) \cdots.$$  

We denote a set of planes containing $1, 2, 3$ by $S$. Then $x$ is regarded as a permutation on $S$. Since $|S|$ is even and $x$ fixes $P(x)$, $x$ fixes at least one more plane of $S$ different from $P(x)$. Let $P(y) \neq P(x)$ be a plane of $S$ fixed by $x$. We may assume that $I(y) = \{1, 2, 3, 8, 9, 10, 11\}$. Then

$$xy = (1) (2) (3) (8 9) (10 11) \cdots.$$  

Hence the order of $xy$ is two or four.

First assume that the order of $xy$ is two. Then $P(xy)$ is a plane of $S$ different from both $P(x)$ and $P(y)$. Since $|S|$ is even and at most five, $|S| = 4$, namely...
\[ S = \{P(x), P(y), P(xy), P(z)\}, \] where \( z \) is an involution. We may set \( I(xy) = \{1, 2, 3, 12, 13, 14, 15\} \) and \( I(z) = \{1, 2, 3, 16, 17, 18, 19\} \). Then the group \( \langle x, y, z \rangle \) fixes \( \{I(x), I(y), I(xy), I(z)\} \) as a set. Hence \( \langle x, y, z \rangle \) fixes \( \{20, 21, 22, 23\} \) as a set. Then \( x, y, xy \) and \( z \) have two 2-cycles on \( \{21, 21, 22, 23\} \) respectively. Hence there is an involution \( u \) of \( \langle x, y, z \rangle \) such that \( u \) and \( z \) are the same form on \( \{20, 21, 22, 23\} \). This implies that \( uz \) is an involution fixing \( \{1, 2, 3, 20, 21, 22, 23\} \) pointwise. Thus \( P(uz) \subseteq S \), which is a contradiction.

Next assume that \( xy \) is of order four. If \( xy \) fixes some plane of \( S \) different from \( P(y) \), then \( xy \) fixes three points \( 1, 2, 3 \) and has one 4-cycle on the set of points of this plane. This contradicts (3.4). Thus \( xy \) fixes only \( P(y) \) of \( S \). Hence \( |S| \) is odd, which is a contradiction. Thus \( |S| = 1, 3 \) or 5.

**3.14. G is intransitive and has no orbit of length twenty-two.**

**Proof.** First suppose by way of contradiction that \( G \) is transitive. Then \( G \) is \( M_{23} \) or contained in a group of order \( 23 \cdot 11 \) (see [1], p. 235). Since \( G \) has an involution, \( G \) must be \( M_{23} \). Then by (3.8) for any two points \( i, j \) of \( \Omega \) \( |\Delta(i, j)| = 21 \), contrary to (3.12).

Next suppose by way of contradiction that \( G \) has orbits of lengths one and twenty-two. Then \( G \) is a subgroup of \( M_{23} \). Since \( M_{23} \) has no proper subgroup which is transitive on twenty-two points (see [1], p. 235), \( G \) is \( M_{23} \). Let \( \{i\} \) be the \( G \)-orbit of length one. Then by (3.8) for any point \( j \) of \( \Omega - \{i\} \) \( |\Delta(i, j)| = 21 \), which is also a contradiction.

**3.15. If G has an orbit of length eleven, then G is (i) or (vi) of Lemma 1.**

**Proof.** By assumption \( G \) has an element of order eleven consisting of one fixed point and two 11-cycles. Hence the lengths of \( G \)-orbits are 11, 1, 11 or 11, 12.

First assume that \( G \) has three orbits \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \), where \( |
\Gamma_1| = 1 \) and \( |
\Gamma_2| = |
\Gamma_3| = 11 \). By the assumption (4), \( G^{\Gamma_2} \) is isomorphic to \( G \). Since for any two points \( i, j \) of \( \Gamma_2 \) there is an involution fixing \( i, j \), \( \langle G^{\Gamma_2} \rangle = 1 \). Hence \( G^{\Gamma_2} \) is doubly transitive on \( \Gamma_2 \) by Theorem 11.6 and Theorem 11.7 in [6]. By [5], \( G^{\Gamma_2} \) is isomorphic to \( PSL(2, 11) \) or contains \( M_{11} \). Since the lengths of \( G \)-orbits are 11, 1, 11, \( G \) is isomorphic to \( PSL(2, 11) \) (see [1], p. 235).

Next assume that \( G \) has two orbits \( \Gamma_1 \) and \( \Gamma_2 \), where \( |
\Gamma_1| = 11 \) and \( |
\Gamma_2| = 12 \). By the similar reason above, \( G^{\Gamma_1} \) is isomorphic to \( G \) and \( G \) is isomorphic to \( PSL(2, 11) \) or contains \( M_{11} \). Since the lengths of \( G \)-orbits are 11, 12, \( G \) is isomorphic to \( M_{11} \) (see [1], p. 235).

**3.16. If there are two points \( i, j \) such that \( |\Delta(i, j)| = 9 \), then \( G \) is one of the groups (i), (ii), (iii), (iv) or (v) of Lemma 1, and we have the following:**
(i) If $G$ is isomorphic to $M_{11}$, then $G$ is isomorphic to $\langle a, b, c, y_1, x_1 \rangle$ as a permutation group.
(ii) If $G$ is isomorphic to $M_{10}$, then $G$ is isomorphic to $\langle a, b, c, y_1, x_1, y_2 \rangle$ as a permutation group.
(iii) If $G$ is isomorphic to $M_{10}^*$, then $G$ is isomorphic to $\langle a, b, c, y_1, (x_1, y_2)^{x_1} \rangle$ as a permutation group.
(iv) If $G$ is isomorphic to $N(M_9)$, then $G$ is isomorphic to $\langle a, b, c, y_1, (x_1, y_2)^{x_1}, \gamma_1, y_2^x, x \rangle$ as a permutation group.
(v) If $G$ is isomorphic to $N(M_9)^*$, then $G$ is isomorphic to $\langle a, b, c, y_1, (x_1, y_2)^{x_1}, \gamma_1, y_2^x, x \rangle$ as a permutation group.

Here

\[ a = (1) (2) (3) (8 9) (10 11) (12 13) (14 15) (4) (18 19) (5) (16 17) (6) (20 21) (7) (22 23), \]
\[ b = (1) (2) (8) (3 9) (10 12) (11 15) (13 14) (4) (18 19) (16) (5 17) (20) (6 21) (23) (7 22), \]
\[ c = (1) (2) (10) (3 11) (8 12) (9 14) (13 15) (18) (4 19) (5) (16 17) (20) (6 21) (22) (7 23), \]
\[ x_1 = (1) (2) (3) (8) (9) (12 13) (10 14) (11 15) (4) (5 6) (7) (16 20) (17 21) (18 19) (22) (23), \]
\[ y_1 = (1) (2) (3) (8) (9) (10 11) (12 14) (13 15) (4 5) (6) (7) (16) (17) (18 19) (20 22) (21 23), \]

Proof. Let $|\Delta(1, 2)| = 9$ and $\Delta(1, 2) = \{3, 8, 9, \ldots, 15\}$. Then we may assume that there is an involution $a$ of the form

\[ a = (1) (2) \cdots (7) (8 9) (10 11) (12 13) (14 15) (16 17) (18 19) (20 21) (22 23). \]

For any point $i$ of $\Delta(1, 2)$ there is a plane containing points $1, 2, i$, in which there is no point of $\Delta(1, 2) - \{i\}$ by the assumptions (6) and (7). Hence this plane has four points in $\Omega - \{\Delta(1, 2) \cup \{1, 2\}\}$. Since $|\Omega - \{\Delta(1, 2) \cup \{1, 2\}\}| = 12$ the number of planes containing $1, 2, i$ is at most three. Suppose that there are exactly three planes containing $1, 2, i$, say $P_1, P_2$ and $P_3$. Let $P'$ be a plane containing $1, 2, j$, where $j \in \Delta(1, 2)$ and $j \neq i$. Then $P'$ also has four points in $\Omega - \{\Delta(1, 2) \cup \{1, 2\}\}$. Hence $P'$ and $P$, where $P$ is one of $P_1, P_2, P_3$, have at least two points of $\Omega - \{\Delta(1, 2) \cup \{1, 2\}\}$ in common. Since $1, 2$ are the points of both $P'$ and $P$, $P'$ and $P$ have at least four points in common. Thus $P' = P$, which is a contradiction. Therefore for any point $i$ of $\Delta(1, 2)$ there is exactly one plane containing $1, 2, i$ by (3.13).

Let $b$ be an involution fixing $1, 2, 8$. Then $ab$ is of odd order by (3.7). Since $ab$ fixes $\Delta(1, 2)$ with length nine, the order of $ab$ is not five by (vi) of (3.1).
Hence the order of $ab$ is three. Since $b$ fixes $\Delta(1, 2)$ as a set and $ab$ fixes exactly five points by (ii) of (3.1), we may assume that $b$ is of the form


Then


Since an element of order three of $PSL(3, 2)$ fixes exactly one point, $ab$ does not fix a plane containing 1, 2, 4. Thus the lengths of $\langle ab \rangle$-orbits on the set of planes containing 1, 2, 4 are three. Hence by (3.13) the planes containing 1, 2, 4 are $P(a)$, $P(a^3b^3) = P(bab)$ and $P(a^{ab}) = P(b)$. Let $c$ be an involution fixing 1, 2, 10. Then in the same way $ac$ and $bc$ are of order three. Hence

$$c = (1) (2) (10) (3 11) (8 12) (9 14) (13 15) \cdots.$$

Then

$$abc = (1) (2) (3 14) (8 11) (9 12) (10 13) (15) \cdots.$$
order three and \( ac \) fixes 1, 2, 5, 16, 17, \( ab^c \) fixes 1, 2, 6, 20, 21 and \( ab^c \) fixes 1, 2, 7, 22, 23, by the same argument as is used for the planes containing 1, 2, 4, we have that for any point \( i \) of \( \{4, 5, 6, 7, 16, 17, \ldots, 23\} \) the number of planes containing 1, 2, \( i \) is three.

Set \( \Delta = \Delta(1, 2) \), \( \Gamma_a = \{4, 18, 19\} \), \( \Gamma_b = \{5, 16, 17\} \), \( \Gamma_c = \{6, 20, 21\} \) and \( \Gamma_i = \{7, 22, 23\} \). In the plane \( P(a) \) 1, 2, 3 are collinear, Hence from now on we may assume that \( \Delta(1, 4) \geq 5 \), \( \Delta(1, 6) \geq 7 \) and \( (2, 4) \geq 6 \) in \( I(a) \).

Now we show that for any point \( i \) of \( \{1, 2\} \) and \( j \) of \( \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \) \( |\Delta(i, j)| \geq 3 \) or 5. If we prove that \( |\Delta(1, 4)| = 3 \) or 5, then since \( \Delta(1, 18) = \Delta(1, 4)^a \) and \( \Delta(1, 19) = \Delta(1, 4)^b \), \( |\Delta(1, 18)| = |\Delta(1, 19)| = |\Delta(1, 4)| = 3 \) or 5, and in the remaining cases the proofs are similar. By (3.12), \( |\Delta(1, 4)| \geq 1, 3, 5 \) or 9. Since \( \Delta(1, 4) \geq 5 \), \( \Delta(1, 4) \geq \{5\} \). On the other hand \( \Delta(1, 4) \geq 2, 3, 6, 7 \) in \( I(a) \). Hence any point of \( \{2, 3, 6, 7\} \) does not belong to \( \Delta(1, 4) \). Thus \( \Delta(1, 4) = \Gamma_2 \cup \Gamma_3 \cup \{18, 19\} \) or \( \Gamma_1 \cup \Gamma_2 \cup \{10, 11, \ldots, 15\} \). Suppose that \( \Delta(1, 4) = \Gamma_2 \cup \{10, 11, \ldots, 15\} \). Then there is an involution \( a \) fixing 1, 4, 10 and the form of \( a \) is determined by the same argument as is used for \( c \). Hence the \( \langle a, b, x \rangle \)-orbits have the same lengths as the lengths of the \( \langle a, b, c \rangle \)-orbits. Furthermore \( \Delta(1, 4) \) is a \( G_1 \)-orbit and \( \langle a, b \rangle \) has orbits \( \{1, 2\} \), \( \{3, 9, 8\} \), \( \{10, 11, \ldots, 15\} \), \( \{4\} \), \( \{18, 19\} \), \( \Gamma_2 \) and \( \Gamma_3 \). Hence the \( \langle a, b, x \rangle \)-orbits are \( \{1\}, \{4\}, \Gamma_2 \cup \{10, 11, \ldots, 15\}, \{2, 18, 19\}, \{3, 8, 9\}, \Gamma_3 \) and \( \Gamma_4 \). Hence the \( \langle a, b, c, x \rangle \)-orbits are \( \{1\}, \Delta \cup \Gamma_2, \{2\} \cup \Gamma_3, \Gamma_4 \) and \( \Gamma_5 \). Since \( |\Delta(1, 4)| = 4 \), the order of \( \langle a, b, c \rangle \) is divisible by four. On the other hand \( |\Delta(1, 2)| = 3 \). Hence \( \langle a, b, c, x \rangle \) has an involution \( y \). Since \( bc \) fixes four points 1, 6, 20, 21 of \( I(y) \), \( bc \) fixes \( I(y) \) pointwise by (3.4). Thus \( |I(bc)| \geq 5 \), which is a contradiction. Hence \( \Delta(1, 4) = \Gamma_1 \cup \{18, 19\} \). Thus for any point \( i \) of \( \{1, 2\} \) and \( j \) of \( \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \) \( |\Delta(i, j)| \geq 3 \) or 5. Since \( \Delta(1, 5) = \Delta(1, 5) \) and \( \Delta(1, 5) \geq 4 \), \( \Delta(1, 5) = \Gamma_5 \) or \( \Gamma_1 \cup \{16, 17\} \). Since \( \Delta(1, 6) \geq 5 \) and \( \Delta(1, 6) \geq 7 \), \( \Delta(1, 6) = \Gamma_1 \) or \( \Gamma_4 \cup \{20, 21\} \). Since \( \Delta(1, 7) \geq 5 \) and \( \Delta(1, 7) \geq 6 \) \( \Delta(1, 7) = \Gamma_2 \) or \( \Gamma_4 \cup \{22, 23\} \). In the same way \( \Delta(2, 4) = \Gamma_4 \) or \( \Gamma_1 \cup \{18, 19\} \), \( \Delta(2, 5) = \Gamma_3 \) or \( \Gamma_4 \cup \{16, 17\} \), \( \Delta(2, 6) = \Gamma_5 \) or \( \Gamma_1 \cup \{20, 21\} \) and \( \Delta(2, 7) = \Gamma_2 \) or \( \Gamma_4 \cup \{22, 23\} \).

Furthermore since for \( j \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \) \( |\Delta(j)| \geq 3 \) or 5 and \( |\Delta(1, 2)| = 9 \), the points 2 and \( j \) of \( \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \) do not belong to the same \( G_i \)-orbit. Hence \( G_i \) has an orbit \( \{2\} \) or \( \{2\} \cup \Delta \). Similarly \( G_2 \) has an orbit \( \{1\} \) or \( \{1\} \cup \Delta \). Using this result we prove that any involution fixing \( I(a) \) fixes \( \{1, 2, 3\} \). Let \( x \) be an involution fixing \( I(a) \). Then \( x \) is one of the following forms: \( x = (1) (2) \cdots, (1 2) \cdots, (1 \ i \ i) (2) \cdots \) or \( (1 \ i \ i) (2) \cdots \), where \( i, j \in \{3, 4, \ldots, 7\} \). By the incidence structure of \( P(a) \), if \( x = (1) (2) \cdots \) or \( (1 2) \cdots \), then \( x \) fixes \( \{1, 2, 3\} \). Assume that \( x = (1) (2) \cdots \), where \( i \in \{3, 4, \ldots, 7\} \). Then \( x \in G_i \). Since the \( G_i \)-orbit containing 2 is \( \{2\} \) or \( \{2\} \cup \Delta(1, 2) \), \( x = (1) (2 3) \cdots \). In the same way if \( x = (1) \ i \ i \cdot \cdot \cdot \), where \( i \in \{3, 4, \ldots, 7\} \), then \( i = 3 \). Next assume that \( x = (1) \ i \ i \cdot \cdot \cdot \), where \( i, j \in \{3, 4, \ldots, 7\} \). Since 1, 2, 3 are collinear in \( P(a) \), \( i, j \neq 3 \).
Let $i=4$. Then by the incidence structure of $P(a)$, $x=(1\ 4\ 2\ 7\ 3\ 5\ 6)\cdots$. Since $G_i$ has an orbit $\{2\}$ or $\{2\} \cup \Delta(1,2)$, $G_{i}=G_i^x$ has an orbit $\{7\}$ or $\{7\} \cup \Delta(4,7)$. Since $G_i^x \triangleleft \langle a, b \rangle$, $G_i$-orbits consist of unions of some $\langle a, b \rangle$-orbits. Thus $\{7\}$ is not a $G_i$-orbit. Since $|\{7\} \cup \Delta(4,7)|=10$, if $\{7\} \cup \Delta(4,7)$ is a $G_i$-orbit, then $\{7\} \cup \Delta(4,7)$ contains at least one $\langle a, b \rangle$-orbit of length one. However the $\langle a, b \rangle$-orbits of length one are $\{1\}$, $\{2\}$ and $\{4\}$. Since $1, 4, 7 \in \{2\} \cup \Delta(1,2)$, $\{1, 4, 7\}^x = \{1, 4, 2\}$ and $\{2\} \cup \Delta(1,2))^x = \{7\} \cup \Delta(4,7)$, any point of $\{1, 2, 4\}$ does not belong to $\{7\} \cup \Delta(4,7)$. This contradiction shows that $x=(1\ 4\ 2\ 7)\cdots$. Finally by the incidence structure of $P(a)$, for $i=5, 6$ or $7$, $x=(1\ 5\ 2\ 6\ 3\ 4\ 7)\cdots$, $(1\ 6\ 2\ 5\ 4\ 7\ 3\ 2)\cdots$ or $(1\ 7\ 2\ 4)\cdots$ respectively. Then similarly to the case $i=4$, we have a contradiction and so $i=5, 6, 7$. Thus any involution fixing $I(a)$ fixes $\{1, 2, 3\}$.

By (3.5), there is an involution fixing $8, 9$ and commuting with $a$. Hence from now on let $x$ be an involution fixing $8, 9$ and commuting with $a$. Then by the assertion above, $x$ fixes $\{1, 2, 3\}$. Hence $x$ is one of the following forms:

1. $x=(1\ 2\ 3\ 8\ 9)\cdots$,
2. $x=(1\ 2\ 3\ 8\ 9)\cdots$,
3. $x=(1\ 2\ 3\ 8\ 9)\cdots$,
4. $x=(2\ 3\ 8\ 9)\cdots$,
5. $x=(1\ 3\ 8\ 9)\cdots$.

(i) Assume that $x=(1\ 2\ 3\ 8\ 9)\cdots$. Then $P(x)$ and $P(a)$ are distinct planes containing $\{1, 2, 3\}$. This is a contradiction since there is no plane containing $1, 2, 3$ except $P(a)$.

(ii) Assume that $x=(1\ 2\ 3)\cdots$. Then by the incidence structure of $P(a)$, $x=(1\ 2\ 3)\cdots$. or $(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)\cdots$. Let $x=(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)\cdots$. Then $\Delta(1,4)^x=\Delta(2,7)$. Since $\Delta(1,4)^x=\Gamma_2$ or $\Gamma_2 \cup \{18, 19\}$ and $\Delta(2,7)^x=\Gamma_2$ or $\Gamma_2 \cup \{22, 23\}$, $\Gamma_2^x=\Gamma_2$. Thus $x=(16\ 17\cdots$ or $(16\ 17\cdots$. First assume that

$$x=(1\ 2\ 3\ 4\ 7\ 5\ 6\ 8)\cdots$$

Then

$$bx=(1\ 2\ 3\ 9\ 8\ 16\ 5\ 17\ 22\ 4\ 7)\cdots$$

Thus $(bx)^2$ is not the identity element fixing eight points, which is a contradiction. Next assume that

$$x=(1\ 2\ 3\ 4\ 7\ 5\ 6)\cdots$$

Then $\Delta(1, 5)^x=\Delta(2, 5)$. Since $\Delta(1, 5)^x=\Gamma_2$ or $\Gamma_2 \cup \{16, 17\}$ and $\Delta(2, 5)^x=\Gamma_4$ or $\Gamma_4 \cup \{16, 17\}$, $\Gamma_2^x=\Gamma_4$. Thus $x=(18\ 22)\cdots$ or $(18\ 23\ 19\ 22)\cdots$. If $x=(1\ 2\ 3)\cdots$ then

$$cx=(1\ 2\ 5\ 16\ 17\ 4\ 23)\cdots$$
Hence $cx$ is an involution and so $c$ commutes with $x$. Since $I(x) \supseteq \{3, 6, 8, 9\}$, $I(x) \supset \{3, 6, 8, 9\} << = \{3, 11, 6, 21, 8, 12, 9, 14\}$, contrary to (i) of (3.1). Thus $x = (1 2) (3) (4 7) (5) (6) (8) (9) (16 17) (18 23) (19 22) \cdots$.

Hence $cx = (1 2) (5) (16) (17) (4 22 19 7 18 23) \cdots$, contrary to (v) of (3.1). Therefore case (ii.i) does not occur.

(ii.ii) Let $x = (1 2) (3) (4) (5 6) (7) (8) (9) \cdots$.

Then $bx = (1 2) (3) (5 6) (7) (8) (9) (17 21) (16 20) (22) (23) (18 19) \cdots$, and on $\{10, 11, \ldots, 15\}$ $x = (10 11) (12 15) (13 14), (12 13) (10 14) (11 15)$ or $(14 15) (11 13) (10 12)$. If $x = (10 11) (12 15) (13 14) \cdots$, then $I(ax) = \{3, 4, 7, 18, 19, 10, 11\}$. Hence $I(ax) \cap I((ax)c) = \{3, 4, 19, 18, 10\}$, contrary to the assumption (7). Thus $bx$ is of order two. Hence

$$x = (1 2) (3) (4) (5 6) (7) (8) (9) (12 13) (10 14) (11 15) (4) (5 6) (7) (16 20) (17 21) (18 19) \cdots.$$

From now on the element $x$ of this form is denoted by $x_i$. Then the $\langle a, b, c, x_i \rangle$-orbits are $\{1, 2\}, \Delta, \Gamma_i, \Gamma_i \cup \Gamma_s$ and $\Gamma_s$.

By (3.5), $G$ has an involution $y$ fixing 16, 17 and commuting with $a$. Suppose that $y \in \langle a, b, c, x_i \rangle$. Since $\{a, b, c\}^s = \{a, b, a bc\}$ and $x_i$ is of order two, the index of $\langle a, b, c \rangle$ in $\langle a, b, c, x_i \rangle$ is two. Hence $|\langle a, b, c, x_i \rangle| = 2 \cdot |\langle a, b, c \rangle| = 2 \cdot 9 \cdot 2$. Thus both $\langle a, x_i \rangle$ and $\langle a, y \rangle$ are Sylow 2-subgroups of $\langle a, b, c, x_i \rangle$.

On the other hand $a$ is only one involution of $\langle a, x_i \rangle$ having fixed points in a $\langle a, b, c, x_i \rangle$-orbit $\Gamma_i \cup \Gamma_s$ and $\langle a, y \rangle$ has at least two involutions $a$ and $y$ having fixed points in $\Gamma_i \cup \Gamma_s$. This is a contradiction since $\langle a, x_i \rangle$ and $\langle a, y \rangle$ are conjugate in $\langle a, b, c, x_i \rangle$. Thus $y$ is not an element of $\langle a, b, c, x_i \rangle$. By the same argument as is used above for $x, y = (1 2) (3) (16) (17) \cdots$, (2) (1 3) (16) (17) \cdots or (1 2) (3) (16) (17) \cdots. Set $H = \langle a, b, c, x_i, y \rangle$.

(ii.iii) Let $y = (1 2) (3) (16) (17) \cdots$. The proof in the case $y = (2) (3) (16) (17) \cdots$ is similar. Then $\{1, 2\}$ and $\Delta$ are contained in the same $H$-orbit. By (3.14), $H$ is intransitive. Hence the $H$-orbit containing $\{1, 2\} \cup \Delta$ is $\{1, 2\} \cup \Delta, \{1, 2\} \cup \Delta \cup \Gamma_i, \{1, 2\} \cup \Delta \cup \Gamma_s, \{1, 2\} \cup \Delta \cup \Gamma_i \cup \Gamma_s, \{1, 2\} \cup \Delta \cup \Gamma_s \cup \Gamma_s, \{1, 2\}$.
$\Delta \cup \Gamma_i \cup \Gamma_j \cup \Gamma_k$ or \{1, 2\} $\cup \Delta \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5$. Since $|\{1, 2\} \cup \Delta \cup \Gamma_1| = 14$ ($i = 1, 4$), \{1, 2\} $\cup \Delta \cup \Gamma_i$ is not an $H$-orbit by (3.11). Since $|\{1, 2\} \cup \Delta \cup \Gamma_i \cup \Gamma_j| = 17$ ($i, j = \{1, 4\}$ or \{2, 3\}), \{1, 2\} $\cup \Delta \cup \Gamma_i \cup \Gamma_j$ is also not an $H$-orbit by the assumption (4).

Next suppose that \{1, 2\} $\cup \Delta \cup \Gamma_i \cup \Gamma_j \cup \Gamma_k$ ($i = 1, 4$), is an $H$-orbit. Let $i = 1$. Then $\Gamma_i$ is an $H$-orbit. Since $x_i$ fixes $\Gamma_i$ pointwise and four points of \{1, 2\} $\cup \Delta \cup \Gamma_i \cup \Gamma_j \cup \Gamma_k$, there is an involution $z$ which is conjugate to $x_i$ and fixes $1$ and $\Gamma_i$ pointwise. Since $ab$ fixes \{1\} $\cup \Gamma_i$ pointwise, $ab$ fixes $I(z)$ pointwise by (3.4). This is a contradiction since $|I(ab)| = 5$. Hence \{1, 2\} $\cup \Delta \cup \Gamma_i \cup \Gamma_j \cup \Gamma_k$ is not an $H$-orbit. Since $ax_i$ is an involution fixing $\Gamma_i$ pointwise, in the same way as above \{1, 2\} $\cup \Delta \cup \Gamma_i \cup \Gamma_j \cup \Gamma_k$ is not an $H$-orbit. Therefore \{1, 2\} $\cup \Delta$ is an $H$-orbit. Since $|\{1, 2\} \cup \Delta| = 11$, $H$ has an element of order eleven, which fixes exactly one point and has two 11-cycles. Hence $\Gamma_i \cup \Gamma_j \cup \Gamma_k \cup \Gamma_i$ is also an $H$-orbit.

We shall determine the form of the involution $y$. By the incidence structure of $P(a)$, $y = (1) (2 3) (4) (5) (6 7) (16) (17)$ or $y = (1) (2 3) (4 5) (6) (7) (16) (17)$ ... Suppose that $y$ is of the first form. Since $ab$ fixes \{1, 4, 5, 16, 17\} which is contained in $I(y)$, $ab$ fixes $I(y)$. This contradicts (3.4) since the order of $ab$ is three and $|I(ab) \cap I(y)| \geq 2$. Hence $y$ must be of the second form. Then $y$ fixes $\Delta(1, 6)$ which is $\Gamma_4$ or $\Gamma_1 \cup \{20, 21\}$. Since $\Gamma_i$ is a $<a, b, c, x_i>$-orbit and $\Gamma_i \cup \Gamma_j \cup \Gamma_k \cup \Gamma_i$ is an $H$-orbit, $\Gamma_i \neq \Gamma_i$. Thus $\{20, 21\} = \{22, 23\}$ and so $y = (20 23) (21 22)$ or $\{20 22\} (21 23)$. ... If $y = (1) (2 3) (4 5) (6) (7) (16) (17) (20 23) (21 22)$, then $by = (1) (16) (4 5 17) (6 22 7 21)$, contrary to the assumption (4). Thus $y = (1) (2 3) (4 5) (6) (7) (16) (17) (20 22) (21 23)$ ...

Hence $by = (1) (4 5 17) (16) (23 21) (7 20 22) (9 2 3)$ ... This shows that $by$ is of order three. Hence $y$ fixes 9 and so 8 because $ay = ya$. On the other hand $\Gamma_i \cup \Gamma_j \cup \Gamma_k \cup \Gamma_i$ is an $H$-orbit. Therefore $y = (18 19)$ ... Thus $y = (1) (2 3) (8) (9) (4 5) (6) (7) (16) (17) (18 19) (20 22) (21 23)$ ...

Then $cy = (1) (4 18 19 5) (16 17) (6 23 7 21) (20 22)$ ...

Hence $cy$ is of order four. By (iii) of (3.1), $|I(cy)| = 3$. This implies that $c$ and $y$ have exactly one 2-cycle in common on \{10, 11, ..., 15\}. Thus $y = (1) (2 3) (8) (9) (13 15) (12 14) (10 11) (4 5) (6) (7) (16) (17) (18 19) (20 22) (21 23)$.
From now on the element $y$ of this form is denoted by $y$. Since $H = \langle a, b, c , x_1, y_1 \rangle$ is 3-fold transitive on $\{1, 2, 8, 9, \ldots, 15\}$, $H$ is isomorphic to $M_{11}$, $A_{11}$ or $S_{11}$ by (5). On the other hand $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ is an $H$-orbit of length twelve. Since $A_{11}$ and $S_{11}$ have no permutation representation of degree twelve, $H$ is isomorphic to $M_{11}$. Furthermore by (3.14), there is no group containing $H$ as a proper subgroup.

(iii.iii) Let $y = (1\ 2)\ (3)\ (16)\ (17)\ \cdots$. Then $y = (1\ 2)\ (3)\ (4)\ (5\ 6)\ (7)\ (16)\ (17)\ \cdots$ or $(1\ 2)\ (3)\ (5)\ (6)\ (4\ 7)\ (16)\ (17)\ \cdots$. Suppose that $y$ is of the first form. Then $x_1 y$ is not the identity element and fixes $\{1, 2, \ldots, 7\}$ pointwise. Hence $x_1 y$ is of order two and so $y$ commutes with $x_1$. Since $x_1 = (16\ 20)\ (17\ 21)\ \cdots$ and $y$ fixes 16, 17, $y$ fixes also 20, 21. On the other hand $y$ fixes $I(x_1)$ and $A(1, 2)$. Hence $y$ fixes $I(x_1) \cap A(1, 2) = \{3, 8, 9\}$ and so $y$ has a 2-cycle $(8\ 9)$ because $|I(y)| = 7$. Thus $a x_1 y$ is an involution fixing $\{1, 2, \ldots, 9\}$ pointwise, contrary to (i) of (3.1). Therefore $y$ must be of the second form. Then

$$x_1 y = (1\ 2)\ (3)\ (4\ 7)\ (5\ 6)\ \cdots.$$  

Hence $x_1 y$ is of order two or four. If $x_1 y$ is an involution, then $P(x_1 y)$ is the plane containing 1, 2, 3. This is a contradiction since there is no plane containing 1, 2, 3 and different from $P(a)$. Thus $x_1 y$ is of order four. Since $\Delta(1, 7) = \Gamma_2$ or $\Gamma_8 \cup \{22, 23\}$, $\Delta(2, 4) = \Gamma_9$ or $\Gamma_5 \cup \{18, 19\}$ and $\Delta(1, 7) = \Delta(2, 4)$, $\Gamma_2 = \Gamma_9$. By (iii) of (3.1), $x_1 y$ has exactly two 2-cycles. Hence $y$ has a 2-cycle $(20\ 21)$ on $\Gamma_2$. Since $\Delta(1, 5) = \Gamma_3$ or $\Gamma_1 \cup \{16, 17\}$, $\Delta(2, 5) = \Gamma_8$ or $\Gamma_4 \cup \{16, 17\}$ and $\Delta(1, 5) = \Delta(2, 5)$, $\Gamma_1 = \Gamma_9$. Thus $y = (18\ 22)\ (19\ 23)\ \cdots$ or $(18\ 23)\ (19\ 22)\ \cdots$. If $y$ is of the first form, then $cy = (1\ 2)\ (4\ 23)\ (18\ 22)\ (6\ 20\ 21)\ \cdots$, which contradicts (v) of (3.1). Thus

$$y = (1\ 2)\ (3)\ (4\ 7)\ (5)\ (6)\ (16)\ (17)\ (20\ 21)\ (18\ 23)\ (19\ 22)\ \cdots.$$  

This implies that $(bc)^y = bc$. Hence $y$ fixes 14, 15, and so $y$ has a 2-cycle $(10\ 11)$ because the order of $x_1 y$ is four. Then $y$ has 2-cycles $(8\ 13)$ and $(9\ 12)$ since $bc = (8\ 12\ 10)\ (9\ 11\ 13)\ \cdots$ and $(bc)^y = bc$. Thus

$$y = (1\ 2)\ (3)\ (8\ 13)\ (9\ 12)\ (10\ 11)\ (14)\ (15)\ (4\ 7)\ (5)\ (6)\ (16)\ (17)\ (20\ 21)\ (18\ 23)\ (19\ 22)\ \cdots.$$  

From now on the element $y$ of this form is denoted by $y_2$. Set $H = \langle a, b , c , x_1, y_2 \rangle$. Then $H$ is isomorphic to a subgroup of $N_{M_{11}}(M_8)$ of index 2, which we denote by $N(M_8)^*$, and the $H$-orbits are $\{1, 2\}$, $\Delta$, $\Gamma_1 \cup \Gamma_8$ and $\Gamma_2 \cup \Gamma_3$.

Suppose that $H$ is a proper subgroup of $G$. First assume that $\{1,2\}$ is a $G$-orbit. Then $\Delta$ is a $G$-orbit. By the assumption (4), the order of $G_{1, 2, 3}$ is a divisor of 20·48. Since $|\Delta| = 9$ and $3 \subseteq \Delta$, if $G_{1, 2, 3}$ has an element of order three, then the order of $G$ is divisible by $3^3$. This is a contradiction since the order of $M_{23}$ is not divisible by $3^3$. If $G_{1, 2, 3}$ has an element of order five, then this element fixes three points 1, 2, 3 and at least three points of $\Delta \setminus \{3\}$ since $|\Delta \setminus \{3\}| = 8$, 

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\]
contrary to (iv) of (3.1). Thus $G_{123}$ is a semiregular 2-group on $\Delta - \{3\}$ by the assumption (7). Since $G$ contains $H$ as a proper subgroup, $|G: G_{123}| = |H: H_{123}| = 2 \cdot 9$ and $|H_{123}| = 4$, $G_{123}$ is of order 8. Thus $G$ is isomorphic to $N_{M_{11}}(M_9)$ by [5]. Therefore $G = \langle a, b, c, x_1, (x_1 y_2)^x \rangle$ and the $G$-orbits are $\{1, 2\}, \Delta, \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$.

Next assume that $\{1, 2\}$ is contained in some $G$-orbit properly. If $\{1, 2\} \cup \Delta$ is a $G$-orbit, then $G$ is isomorphic to $M_{11}$ by (3.15). If $\{1, 2\} \cup \Gamma_1 \cup \Gamma_4$ is a $G$-orbit, then the remaining $G$-orbits are $\Delta$ and $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ or $\Delta \cup \Gamma_2 \cup \Gamma_4$. Suppose that $\Gamma_1 \cup \Gamma_3$ is a $G$-orbit. Since $|\Gamma_2 \cup \Gamma_3| = 6$, $G_{123}$ has an involution. Hence for any point $t$ of $\{1, 2\} \cup \Gamma_1 \cup \Gamma_4$ or $\Delta$, there is an involution fixing $t$ and $\Gamma_2 \cup \Gamma_3$ pointwise. This is a contradiction since every involution fixing $\Gamma_2 \cup \Gamma_3$ pointwise fixes the same seven points by (vi) of (3.1).

Next suppose that $\Delta \cup \Gamma_2 \cup \Gamma_3$ is a $G$-orbit. Since $|\Delta \cup \Gamma_2 \cup \Gamma_3| = 15$, $G$ has an element of order five. Hence $G$ has an element $u$ of order five. On the other hand we have already proved that the $G_1$-orbit containing 2 is $\{2\}$ or $\{2\} \cup \Delta$. Since $\{2\}$ and $\Delta$ are contained in the different $G$-orbit, $\{2\}$ and $\Delta$ are the $G_1$-orbit. Hence $u$ fixes 1, 2 and four points of $\Delta$, contrary to (3.1). In the same way it is impossible that $\{1, 2\} \cup \Gamma_1 \cup \Gamma_4$ is a $G$-orbit.

Next since $|\{1, 2\} \cup \Delta \cup \Gamma_1 \cup \Gamma_4| = 17$ and $M_{23}$ has no element of order seventeen, $\{1, 2\} \cup \Delta \cup \Gamma_1 \cup \Gamma_4$ is not a $G$-orbit. In the same way $\{1, 2\} \cup \Delta \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ is not a $G$-orbit.

Since $\{1, 2\} \cup \Delta \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 = 14$, $\{1, 2\} \cup \Delta \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_4$ is not a $G$-orbit by (3.11).

Thus if $G$ contains $x = (1 2) (3) (8) (9) \cdots$, then $G$ is isomorphic to $M_{11}, N_{M_{11}} (M_9)$ or $N_{M_{11}} (M_9) \ast$.

(iii) Assume that $x = (1) (2 3) (8) (9) \cdots$. The proof is similar. By the incidence structure of $P(a)$, $x = (1) (2 3) (4) (5) (6 7) (8) (9) \cdots$ or $(1) (2 3) (4 5) (6) (7) (8) (9) \cdots$. Suppose that $x$ is of the first form. Since $\Delta(1, 4) = \Gamma_2$ or $\Gamma_2 \cup \{18, 19\}$ and $\Delta(1, 4)^x = \Delta(1, 4)$, $\{16, 17\}^x = \{16, 17\}$ or $\{18, 19\}$. If $\{16, 17\}^x = \{16, 17\}$, then $x$ or $ax$ fixes $\{1, 2, 5, 16, 17\}$ pointwise. This is a contradiction since $\Delta(1, 4) \supseteq \{5, 16, 17\}$. Hence $\{16, 17\}^x = \{18, 19\}$, and so $x = (16 18) (17 19) \cdots$ or $(16 19) (17 18) \cdots$. Then $bx = (1) (2 3 9) (8) (4) (5 19 18 17) \cdots$ or $(1) (2 3 9) (8) (4) (5 18 16 19) \cdots$. Hence $|I((bx)^x)| \geq 8$, contrary to (ii) of (3.1). Thus $x$ is of the second form. Since $\Delta(1, 4) = \Gamma_4$ or $\Gamma_4 \cup \{20, 21\}$ and $\Delta(1, 6)^x = \Delta(1, 6)$, $\{22, 23\}^x = \{22, 23\}$ or $\{20, 21\}$. If $\{22, 23\}^x = \{22, 23\}$, then $x$ or $ax$ fixes $\{1, 6, 7, 22, 23\}$ pointwise. This is a contradiction since $\Delta(1, 6) \supseteq \{7, 22, 23\}$. If $x = (20 23) (21 22) \cdots$, then $bx = (1) (2 3 9) (8) (20 23) \cdots$, contrary to (v) of (3.1). Thus $x = (20 22) (21 23) \cdots$. Furthermore $\Delta(1, 5) = \Gamma_2$, or $\Gamma_2 \cup \{16, 17\}$ and $\Delta(1, 4)^x = \Delta(1, 5)$. Hence similarly $x = (16) (17) (18 19) \cdots$. Thus
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\[ x = (1\ 2\ 3\ 8\ 9\ 4\ 5\ 6\ 7\ 16\ 17\ 18\ 19\ 20\ 22\ 21\ 23) \ldots \]

Hence \( x = y \). Set \( H = \langle a, b, c, y \rangle \). Then the \( H \)-orbits are \{1\}, \{2\} \cup \Delta, \Gamma_1 \cup \Gamma_2\) and \( \Gamma_3 \cup \Gamma_4 \). Since \( H \leq \langle a, b, c, x_1, y_1 \rangle \) and \( \langle a, b, c, x_1, y_1 \rangle = M_{10} \), \( H \) is isomorphic to a subgroup of \( M_{10} \). Furthermore since \( M_{15} \) has orbits of lengths 1, 10, 12 as a subgroup of \( M_{16} \) (see [1], p. 235), \( H \) is isomorphic to a proper subgroup of \( M_{10} \).

On the other hand \( H \) is doubly transitive on \{2\} \cup \Delta and has an element \( c y_1 = (1\ 13\ 15\ 2\ 3\ 10\ 11\ 8\ 14\ 9\ 12) \ldots \), the order of \( H \) is a multiple of \( 10 \cdot 9 \cdot 4 \).

Thus \( H \) is isomorphic to a subgroup of \( M_{10} \) of index two, which is denoted by \( M_{10}' \).

Suppose that \( H \) is a proper subgroup of \( G \). If \{2\} \cup \Delta is a \( G \)-orbit, then \( G \) is isomorphic to \( M_{10} \) by [5]. In this case \( G = \langle a, b, c, y_1 \rangle \) and the \( G \)-orbits are \{1\}, \{2\} \cup \Delta and \( \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \). Next assume that \( G \) has an orbit containing \{2\} \cup \Delta properly. If \{1\} \cup \{2\} \cup \Delta is a \( G \)-orbit, then \( G \) must be isomorphic to \( M_{16} \) by (3.15). If \{2\} \cup \Delta \cup \Gamma_1 \cup \Gamma_2 \) is a \( G \)-orbit, then \( | \{2\} \cup \Delta \cup \Gamma_1 \cup \Gamma_2 | = 16 \). Since \( G_2 \) is of even order, the order of \( G \) is divisible by \( 16 \cdot 2 \). On the other hand \{1\} \cup \Gamma_3 \cup \Gamma_4 \) is fixed by \( G \) as a set and \( G_{\{1\}} \) is isomorphic to a subgroup of \( M_{10} \). Hence \( G \) has a subgroup of index two, which is denoted by \( M_{10}' \).

Thus if \( G \) contains \( x = (1\ 2\ 3\ 8\ 9) \ldots \) or \( (2\ 1\ 3)\ (8\ 9) \ldots \), then \( G \) is isomorphic to \( M_{10}' \), \( M_{10} \) or \( M_{15} \).

When \( G \) is isomorphic to \( N(M_4) \) or \( N(M_6)' \), we may assume that \( G \) is isomorphic to \( \langle a, b, c, x_1, y_1 \rangle \) or \( \langle a, b, c, x_1, y_2 \rangle \) respectively, and these generators were used in the proof of Lemma 3.

3.17. Assume that \( |\Delta(i, j)| \leq 5 \) for any two points \( i, j \) of \( \Omega \). If there are two points \( i', j' \) of \( \Omega \) such that \( |\Delta(i', j')| = 5 \), then \( G \) is (vi) or (vii) of Lemma 1, and we have the following:

(vi) If \( G \) is isomorphic to \( PSL(2, 11) \), then \( G \) is isomorphic to \( \langle a, x_1, x_2 \rangle \) as a permutation group.

(vii) If \( G \) is isomorphic to \( S_6 \), then \( G \) is isomorphic to \( \langle a, x_1, x_2 \rangle \) as a permutation group.

Here

\[ a = (1\ 2\ 3\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 4\ 5\ 6\ 7\ 16\ 17\ 18\ 19\ 20\ 21\ 22\ 23) \]
\[ x_1 = (1\ 2\ 3\ 8\ 9\ 10\ 11\ 12\ 14\ 13\ 15\ 4\ 5\ 6\ 7\ 16\ 17\ 18\ 19\ 20\ 22\ 21\ 23) \]
Proof. Let $|\Delta(1, 2)|=5$ and $\Delta(1, 2)=\{3, 8, 9, 10, 11\}$. Then we may assume that there is an involution $a$ of the form


Let $b$ be an involution fixing $1, 2, 8$. By (3.7), $ab$ is of order three or five. If the order of $ab$ is three, then the order of $(G_{12})_{A}$ is a multiple of $5\cdot3\cdot2$. Hence $(G_{12})_{A}$ contains $A_{5}$ by [5]. Thus $G$ has an element $b'$ such that $b'$ is conjugate to $b$ and $ab'$ is of order five. Therefore we may assume that the order of $ab$ is five and $b$ is of the form


Hence


Set $\Gamma_1 = \{3, 8, 9, 10, 11\}, \Gamma_2 = \{5, 12, 13, 14, 15\}, \Gamma_3 = \{6, 16, 17, 18, 19\}$, and $\Gamma_4 = \{7, 20, 21, 22, 23\}$. Similarly to (3.16) we may assume that $\Delta(1, 4) \equiv 5$, $\Delta(1, 6) \equiv 7$ and $\Delta(2, 4) \equiv 6$. Then since $\{1, 2\}^{ab}=\{1, 2\}, \Delta(1, 2) \equiv 3$ and $|\Delta(1, 2)| \leq 5, \Delta(1, 2) \equiv \Gamma_1$. Similarly $\Delta(1, 4) = \Gamma_2$ and $\Delta(2, 4) = \Gamma_3$. Since $G_{12}$ contains $\langle a, b \rangle$ and $\Delta(1, 2), \Delta(1, 4)$ and $\Delta(2, 4)$ are orbits of $G_{12}, G_{14}$ and $G_{24}$ respectively by (3.8), the $G_{12}$-orbits are $\{1\}, \{2\}, \{4\}, \Gamma_1, \Gamma_2, \Gamma_3$ and $\Gamma_4$.

Now we have one of the following cases and treat these cases separately.

(i) 1, 2, 4 belong to the same $G$-orbit.

(ii) Exactly two points of $\{1, 2, 4\}$ belong to the same $G$-orbit.

(iii) 1, 2, 4 belong to distinct $G$-orbits.

(i) Suppose that 1, 2, 4 belong to the same $G$-orbit. Then the $G$-orbit containing 1, 2, 4 is $\{1, 2, 4\}$, $\{1, 2, 4\} \cup \Gamma_1, \{1, 2, 4\} \cup \Gamma_2, \{1, 2, 4\} \cup \Gamma_3, \{1, 2, 4\} \cup \Gamma_4$. First assume that $\{1, 2, 4\}$ is a $G$-orbit. Then $G$ fixes and is transitive on $\Delta(1, 2) \cup \Delta(1, 4) \cup \Delta(2, 4)$. Thus the $G$-orbits are $\{1, 2, 4\}, \Gamma_1 \cup \Gamma_2, \Gamma_3$ and $\Gamma_4$. By (3.5), there is an involution $x$ fixing 20 and commuting with $a$. If $x = (1)(2)(4)\cdots$, then $x$ fixes $I(a)$ pointwise by the incidence structure of $P(a)$. Thus $I(x) \subseteq I(a) \cup \{20, 21\}$, which is a contradiction. Hence we may assume that $x = (1)(2)(4)\cdots$. By the incidence structure of $P(a), x = (1)(2)(4)(3)(5)(6)(7)(20)(21)\cdots$. Since $\Gamma_4$ is a $G$-orbit, $\{22, 23\}^x = \{22, 23\}$. If $x$ fixes $\{22, 23\}$ pointwise,
then since $G$ is transitive on \{1, 2, 4\} for any point $i$ of \{1, 2, 4\} some involution conjugate to $x$ fixes $\Gamma_i \cup \{i\}$ pointwise, contrary to (vi) of (3.1). Thus $x$ has a 2-cycle (22 23). Let $y$ be an involution fixing 7, 22 and commuting with $b$. In the same way $y=(7) (22) (20) (21 23)\cdots$ and so $xy=(7) (20) (21 23 22)\cdots$ and so $xy$ is of order three. Hence $xy$ fixes \{1, 2, 4\} pointwise or has a 3-cycle on \{1, 2, 4\}. If $xy=(1) (2) (4) (7) (20) (21 23 22)\cdots$, then $|I(xy) \cap I(a)| \geq 4$. Thus $xy$ fixes noncollinear four points of $P(a)$ and hence it fixes $I(a)$ pointwise. This contradicts (ii) of (3.1). Hence $xy$ has a 3-cycle on \{1, 2, 4\}. Then $xy$ permutes $\Delta(1, 2) \Delta(1, 4)$ and $\Delta(2, 4)$ cyclically. Since $\Delta(1, 2)=\Gamma_1$, $\Delta(1, 4)=\Gamma_2$ and $\Delta(2, 4)=\Gamma_3$, $xy$ has no fixed point in $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Thus $|I(xy)|=2$, contrary to (ii) of (3.1).

Next assume that $\{1, 2, 4\} \cup \Gamma_i$ is a $G$-orbit, where $i \in \{1, 2, 3, 4\}$. Set $\Gamma=\{1, 2, 4\} \cup \Gamma_i$. Since $|\{2, 4\} \cup \Gamma_i|=7$, the $G_i$-orbit containing 2 is $\{2\}$, $\{2, 4\}$ or $\{2\} \cup \Gamma_i$ by (3.11). If $\{2, 4\}$ is a $G_i$-orbit, then $(G_i)^{(i)}$ has orbits of lengths two and five. Hence $G_i^3$ is primitive. Since a primitive group of degree eight is doubly transitive by [5], $G_i^3$ is doubly transitive. This is a contradiction. Next suppose that $\{2\}$ is a $G_i$-orbit. Then $(G_i^3)^{(i)}=(G_i^3)_2$. If $\{4\}$ and $\Gamma_i$ are $G_i$-orbits, then $(G_i^3)^{(i)}=(G_i^3)_2=(G_i^3)_\Gamma$. Hence $|\Gamma|$ is divisible by 3, which is a contradiction. If $\{4\} \cup \Gamma_i$ is a $G_i$-orbit, then $(G_i^3)_\Gamma=(G_i^3)_\Gamma$. Since $G_i$ is conjugate to $G_i^3$, $G_i^3$ has exactly one orbit $\{t\}$ of length one, where $t \in \Gamma-\{1, 2, 4\}$. On the other hand $G_i$ contains $\langle ab \rangle$ which fixes exactly three points 1, 2, 4 of $\Gamma$. Hence $t=1$ or 2, which is a contradiction. If $\{2\} \cup \Gamma_i$ is a $G_i$-orbit, then $\{4\}$ is a $G_i$-orbit. Hence in the same way we have a contradiction.

Next assume that $\{1, 2, 4\} \subset \Gamma_i \cup \Gamma_j$ is a $G$-orbit, where $i, j \in \{1, 2, 3, 4\}$. Then $|\{1, 2, 4\} \cup \Gamma_i \cup \Gamma_j|=13$ and so $G$ has an element of order thirteen, contrary to the assumption (4).

Finally assume that the $G$-orbits are $\{1, 2, 4\} \cup \Gamma_i \cup \Gamma_j \cup \Gamma_k$ and $\Gamma_i$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Then $G$ has orbits of lengths five and eighteen. This is a contradiction since $M_{23}$ has no subgroup having orbits of lengths five and eighteen (see [1], p. 235).

Thus it is impossible that 1, 2, 4 belong to the same $G$-orbit.

(ii) Suppose that exactly two points of \{1, 2, 4\} belong to the same $G$-orbit. We may assume that 1, 2 belong to the same $G$-orbit. Then the $G$-orbit containing \{1, 2\} is \{1, 2\}, \{1, 2\} \cup \Gamma_i, \{1, 2\} \cup \Gamma_i \cup \Gamma_j, \{1, 2\} \cup \Gamma_i \cup \Gamma_j \cup \Gamma_k$ or \{1, 2\} \cup \Gamma_i \cup \Gamma_j \cup \Gamma_k \cup \Gamma_l$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. By (3.11), $\{1, 2\} \cup \Gamma_i$ is not a $G$-orbit. Since $M_{23}$ has no element of order seventeen, $\{1, 2\} \cup \Gamma_i \cup \Gamma_j \cup \Gamma_k$ is not a $G$-orbit. Furthermore by (3.14), $\{1, 2\} \cup \Gamma_i \cup \Gamma_j \cup \Gamma_k \cup \Gamma_l$ is not a $G$-orbit. Thus the $G$-orbit containing \{1, 2\} is \{1, 2\} or $\{1, 2\} \cup \Gamma_i \cup \Gamma_j$. We treat these cases separately.

(ii.i) Assume that \{1, 2\} is a $G$-orbit. Then $\Delta(1, 2)=\Gamma_1$ is a $G$-orbit. By (3.5), there is an involution $x$ fixing 8, 9 and commuting with $a$. By the assumption (7), $x=\Delta(1, 2)(8)(9)\cdots$ Thus $x$ has a 2-cycle (1 2) and so $x$ fixes a
point 3 by the incidence structure of $P(a)$. Thus $x= (1\ 2\ (3\ (8\ (9\ (10\ (11)\ ...

or (1\ 2\ (3\ (8\ (9\ (10\ 11)\ ...

If $x$ is of the first form, then $I(x)\supseteq \Gamma_1$. Now

$I(x^{ab})=I(x)^{ab}\supseteq \Gamma_1^{ab}$. Thus by (vi) of (3.1) $I(x)^{ab}=I(x)$. This contradicts

(3.4) since $PSL(3, 2)$ has no element of order five. Hence $x$ is of the second form.

By the incidence structure of $P(a)$, $x= (1\ 2\ (3\ (8\ (9\ (10\ 11)\ ...

or (1\ 2\ (3\ (8\ (9\ (10\ 11)\ ...

(ii.i.i) Let $x= (1\ 2\ (3\ (4\ (5\ 6)\ (7)\ (8)\ (9)\ (10\ 11)) \ ...

Now $I(x^{ab})=I(x)^{ab}=I(x)$. This contradicts

(3.4) since $PSL(3, 2)$ has no element of order five. Hence $x$ is of the second form.

Then $\{1, 2\}$ and $\Gamma_1$ are $H$-orbits and the remaining $H$-orbits consist of unions of

$\Gamma_2, \Gamma_3$ and $\{4\} \cup \Gamma_4$. Suppose that $\Gamma_2, \Gamma_3$ and $\{4\} \cup \Gamma_4$ are $H$-orbits. Since

$|\{4\} \cup \Gamma_4|=6$, $H$ has an element $u$ of order three. Then $u$ has at least two fixed

points in each orbits of length five. Thus $|I(u)| \geq 6$, contrary to (ii) of (3.1). Next suppose that

$\{4\} \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ is an $H$-orbit. Then $|\{4\} \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4|=16$. Since $H$ has an involution $a$, $H$ is of order divisible by $16 \cdot 2$. On the other

hand since $|\{1, 2\} \cup \Gamma_1|=7$, $H: H_{(1, 2) \cup \Gamma_1}$ is a divisor of $|S_5|=7 \cdot 5 \cdot 9 \cdot 16$. Hence

$H_{(1, 2) \cup \Gamma_1}$ has an involution, which fixes $\{1, 2\}$ and $\Delta(1, 2)=\Gamma_1$ pointwise, contrary to the assumption (7). Next suppose that $\Gamma_2, \Gamma_3 \cup \{4\} \cup \Gamma_4$ are $H$-orbits, where

$\{i, j\}=\{2, 3\}$. Then $|\Gamma_2 \cup \{4\} \cup \Gamma_4|=11$. Hence $H$ has an element of order

eleven. Since an element of order eleven has two 11-cycles and one fixed point, it is impossible that $\{1, 2\}$ is an $H$-orbit of length two. Thus we have a contradiction. Therefore the $H$-orbits must be $\{1, 2\}, \Gamma_1, \Gamma_2 \cup \Gamma_3$ and $\{4\} \cup \Gamma_4$. Using this result we shall determine the form of $x$. Since

$bx= (1\ 2\ (8)\ (3\ 11\ 9\ 10)\ (22\ 4\ 7)\ ...

$bx$ is of order four. Moreover $\{4\} \cup \Gamma_4$ is an $H$-orbit. Hence

$x= (1\ 2\ (3\ (4\ 7)\ (5)\ (6)\ (8)\ (9)\ (10\ 11)\ (20\ 22)\ (21\ 23)\ ...

and so

$bx= (1\ 2\ (8)\ (3\ 11\ 9\ 10)\ (22\ 4\ 7\ 20)\ (21\ 23)\ ...

Hence $bx$ has exactly two 2-cycles $(1\ 2)$ and $(i\ j)$ by (iii) of (3.1), where $i, j \subseteq \Gamma_2 \cup \Gamma_3$. Thus $I((bx)^g)=\{1, 2, 8, 21, 23, i, j\}$. On the other hand $(bx)^g$ is a central involution of a dihedral group $\langle b, x \rangle$. Hence $b^i(x^{ab})=(1\ 2) \ (8)\ (21\ 23)\ (i\ j)$ and $x^i(x^{ab})=(1\ 2) \ (8)\ (21\ 23)\ (i\ j)$. Thus $\{i, j\}=\{5, 14\}, \{13, 15\}, \{6, 18\}$ or $\{17, 19\}$.

Suppose that $\{i, j\}=\{13, 15\}$ or $\{17, 19\}$. Then $I(x)\supseteq \cup \{13, 15\}$ or $\{17, 19\}$. Since $x$ commutes with $a$ and $a^{\Gamma_2 \cup \Gamma_3}=\{5\} \ {12, 13\} \ {14, 15\} \ {6\} \ {16, 17\} \ {18, 19\}$, $x$ fixes $\{3, 8, 9\} \cup \Gamma_2$ or $\{3, 8, 9\} \cup \Gamma_3$ pointwise. Thus $|I(x)| \geq 8$, contrary to (i) of (3.1). Hence $\{i, j\}=\{5, 14\}$ or $\{6, 18\}$ and so $x$ fixes $\{5, 14, 15\}$ or $\{6, 18, 19\}$
pointwise because \( x \) commutes with \( a \).

Suppose that \( x \) fixes 5, 14, 15. Since \( \langle b, x \rangle \) is a dihedral group of order eight and not semiregular on \( \Gamma_2 \cup \Gamma_3 - \{5, 14\} \) of length eight, \( \langle b, x \rangle \) is intransitive on \( \Gamma_2 \cup \Gamma_3 - \{5, 14\} \). Furthermore \( bx \) has two 4-cycles on \( \Gamma_2 \cup \Gamma_3 - \{5, 14\} \). Hence \( \langle b, x \rangle \) has two orbits of length four on \( \Gamma_2 \cup \Gamma_3 - \{5, 14\} \). On the other hand \( x \) has exactly two fixed points 15, 6 and three 2-cycles on \( \Gamma_2 \cup \Gamma_3 - \{5, 14\} \). Hence 15, 6 are contained in the same \( \langle b, x \rangle \)-orbit. Thus \( \{15, 6, 13, 18\} \) and \( \{12, 16, 17, 19\} \) are \( \langle b, x \rangle \)-orbits on \( \Gamma_2 \cup \Gamma_3 - \{5, 14\} \). Hence \( x= (15) (6) (13 18) \cdots \) and so \( x= (15) (6) (13 18) (12 19) (16 17) \cdots \) because \( x \) commutes with \( a \).

Thus
\[
 x = (1 \ 2) (3 \ 8) (9 \ 10 \ 11) (4 \ 7) (5 \ 6) (20 \ 22) (21 \ 23) (14 \ 15) (12 \ 19) (16 \ 17) (13 \ 18).\]

From now on the element \( x \) of this form is denoted by \( x_1 \).

Similarly if \( x \) fixes 6, 18, 19, then
\[
 x = (1 \ 2) (3 \ 8) (9 \ 10 \ 11) (4 \ 7) (5 \ 6) (20 \ 22) (21 \ 23) (18 \ 19) (14 \ 17) (15 \ 16) (12 \ 13).\]

From now on the element \( x \) of this form is denoted by \( x_1' \). Then both \( \langle a, b, x_1 \rangle \) and \( \langle a, b, x_1' \rangle \) are isomorphic to \( S_5 \).

Set \( H = \langle a, b, x_1 \rangle \) and suppose that \( H \) is a proper subgroup of \( G \). The proof in the case \( H = \langle a, b, x_1 \rangle \) is similar. Since \( H^r_1 = S_5 \) and \( \Gamma_1 \) is a \( G \)-orbit, \( H^r_1 = G^r_1 \). Moreover \( H_{\Gamma_1} = 1 \) and \( G \) contains \( H \) properly. Hence \( G_{\Gamma_1} \neq 1 \) and so \( G \) has a nonidentity element \( u \) fixing \( \Gamma_1 \) pointwise. Then \( |I(u)| \geq 5 \). Hence \( u \) is of order three or two. If \( u \) is of order three, then \( u \) fixes the \( G \)-orbit \( \{1, 2\} \) pointwise. Thus \( |I(u)| \geq 7 \), contrary to (ii) of (3.1). Thus \( u \) is of order two. Then since \( ab \) fixes \( \Gamma_1 \), \( ab \) fixes \( I(u) \). Thus \( (ab)^r \) is of order five, contrary to (3.4). Thus there is no group which contains \( H \) properly and satisfies the assumption of Lemma 1.

(iii) Assume that \( \{1, 2\} \cup \Gamma_i \cup \Gamma_j \) is a \( G \)-orbit, where \( i, j \in \{1, 2, 3, 4\} \).

Then the remaining \( G \)-orbits are unions of \( \{4\}, \Gamma_k \) and \( \Gamma_l \), where \( k, l \in \{1, 2, 3, 4\} \) \( \not\in \{i, j\} \). Hence the \( G \)-orbit containing \( \{4\} \) is of length one, six or eleven. Since \( |\{1, 2\} \cup \Gamma_i \cup \Gamma_j| = 12 \), if \( G \) has an orbit of length six or eleven, then \( G \) is a subgroup of a group which is isomorphic to \( M_{11} \) (see [1], p. 235) and if \( G \) has an orbit of length one, then \( G \) is a subgroup of \( M_{22} \). However it is impossible that \( G \) has orbits of lengths six, five and twelve and isomorphic to a subgroup of \( M_{11} \) (see [1], p. 235). Next suppose that \( G \) has an orbit of length eleven. Then by (3.15), \( G \) is isomorphic to \( M_{11} \). Let \( i_1, i_2 \) be two points of the \( G \)-orbit of length eleven. Then \( G_{i_1, i_2} \) has orbits of lengths nine and twelve on \( \Omega - \{i_1, i_2\} \). By (3.8), \( |\Delta(i_1, i_2)| \geq 9 \), contrary to the assumption. Hence \( G \) is a proper subgroup of \( M_{22} \). Thus the lengths of the \( G \)-orbits are 1, 12, 5, 5 or 1, 12, 10. Hence \( G \) is a subgroup of \( M_{10} \) (see [1], p. 235) and has orbits \( \{4\}, \{1, 2\} \cup \Gamma_i \cup \Gamma_j, \Gamma_k, \Gamma_l, \Gamma_i \) or \( \{4\}, \{1, 2\} \cup \Gamma_i \cup \Gamma_j, \Gamma_k \cup \Gamma_l \). Since \( |\{1, 2\} \cup \Gamma_i \cup \Gamma_j| = 12 \), \( G \) has an element
of order three. On the other hand since \( G^{\Gamma_9} \) is a subgroup of \( M_{10} \), hence it is impossible that \( G \) has two orbits \( \Gamma_9 \) and \( \Gamma_j \), of length five. Thus \( \Gamma_9 \cup \Gamma_j \) is a \( G \)-orbit. Since \( G \) fixes a point 4 and two points 1, 2 belong to the same \( G \)-orbit, \( \Delta(4, 1) = \Gamma_2 \) and \( \Delta(4, 2) = \Gamma_3 \) are contained in the same \( G \)-orbit. Thus the \( G \)-orbits are \( \{4\} \cup \Gamma_2 \cup \Gamma_3 \) and \( \Gamma_1 \cup \Gamma_4 \) or \( \{4\} \cup \Gamma_2 \cup \Gamma_4 \) and \( \Gamma_1 \cup \Gamma_3 \). First assume that \( \{4\} \cup \Gamma_2 \cup \Gamma_3 \) and \( \Gamma_1 \cup \Gamma_4 \) are \( G \)-orbits. Since \( |\{4\} \cup \Gamma_2 \cup \Gamma_3| = 12 \) and \( G^{\Gamma_2} \) is a subgroup of \( M_{10} \) of degree ten, \( G \) has an element of order three which has no fixed point on \( \Gamma_2 \cup \Gamma_3 \). Hence the lengths of the \( G \)-orbits in \( \Gamma_1 \cup \Gamma_4 \) are multiples of three. Thus \( |\Delta(4, 3)| = 3 \) because \( \Delta(4, 3) \equiv 7 \mod 3 \) and \( |\Delta(4, 3)| \leq 5 \). On the other hand \( |\{1, 2\} \cup \Gamma_2 \cup \Gamma_3| = 12 \) and \( G \) is of even order. Hence \( G \) is of order divisible by 4. Then since \( |G_4: G_{ss}| = 10 \) and \( \Delta(4, 3) \) is a \( G_4 \)-orbit of length three, \( G \) has an involution fixing \( \{4\} \cup \Delta(4, 3) \) pointwise, contrary to the assumption (7). Next assume that \( \{4\} \cup \{1, 2\} \cup \Gamma_2 \cup \Gamma_3 \) and \( \Gamma_1 \cup \Gamma_4 \) are \( G \)-orbits. Then for two points 1 and 3 of the \( G \)-orbit \( \{1, 2\} \cup \Gamma_2 \cup \Gamma_3 \) and \( \Gamma_1 \cup \Gamma_4 \) are contained in the same \( G \)-orbit. However this is a contradiction since \( \Delta(4, 1) = \Gamma_4 \) and \( \Delta(4, 3) \) contains a point 7 of \( \Gamma_4 \).

(iii) Suppose that 1, 2, 4 belong to distinct \( G \)-orbits. First assume that \( G \) has no orbit of length one. Then the lengths of the \( G \)-orbits are 6, 6, 6, 5 or 6, 6, 11. By (3.15), the latter case does not occur. Hence \( G \) has orbits of lengths 6, 6, 6, 5. But there is no maximal subgroup of \( M_{23} \) containing \( G \) (see [1], p. 235). Thus \( G \) has no orbit of length one.

Next assume that \( G \) has at least two orbits of length one. Then we may assume that \( \{1\} \) and \( \{2\} \) are \( G \)-orbits. By (3.5), there is an involution \( x \) fixing 8, 9 and commuting with \( a \). Then \( x \) fixes 1, 2 and two points 8, 9 of \( \Delta(1, 2) \), contrary to the assumption (7).

Thus \( G \) has exactly one orbit of length one. Hence we may assume that \( \{1\} \) is a \( G \)-orbit. Then the remaining \( G \)-orbits are unions of \( \{2\} \cup \Gamma_9 \), \( \{4\} \cup \Gamma_j \), \( \Gamma_8 \) and \( \Gamma_i \), where \( \{2\} \cup \Gamma_9 \) and \( \{4\} \cup \Gamma_j \) are not contained in the same \( G \)-orbit and \( \{i, j, k, l\} \) is \( \{1, 2, 3, 4\} \). Thus the length of the \( G \)-orbit containing \( \{2\} \) or \( \{4\} \) is 16, 11 or 6. We treat these cases separately.

(iii.i) Suppose that there is a \( G \)-orbit of length sixteen. Then the \( G \)-orbits are \( \{1\} \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_9 \) and \( \{2\} \cup \Gamma_9 \cup \Gamma_i \cup \Gamma_j \), where \( \{2\} \cup \Gamma_9 \cup \Gamma_i \cup \Gamma_j \) is of order divisible by 16. Then the other hand since \( |\{t_3\} \cup \Gamma_9| = 6 \), a Sylow 2-subgroup of \( G^{(\Gamma_8)} \) is of order at most sixteen. Hence \( G^{(\Gamma_8)} \) is of even order. Thus there is an involution \( x \) fixing \( \{t_3\} \cup \Gamma_9 \) pointwise. Since \( |I(x)| = 7 \), \( I(x) = \{1, t_3\} \cup \Gamma_9 \). Then \( ab \) fixes \( I(x) \) and \( (ab)^{I(x)} \) is of order five, contrary to (3.4). Thus \( G \) has no orbit of length sixteen.

(iii.ii) Suppose that there is a \( G \)-orbit of length eleven. Since \( \{1\} \) is a \( G \)-orbit of length one, \( G \) is isomorphic to \( \text{PSL}(2, 11) \) and the \( G \)-orbits are \( \{1\}, \{2, 4\} \cup \Gamma_9 \cup \Gamma_i \cup \Gamma_j \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_9 \cup \Gamma_i \cup \Gamma_j \), where \( \{2\} \cup \Gamma_9 \cup \Gamma_i \cup \Gamma_j \) is of order divisible by 16. Then the other hand since \( |\{t_3\} \cup \Gamma_9| = 6 \), a Sylow 2-subgroup of \( G^{(\Gamma_8)} \) is of order at most sixteen. Hence \( G^{(\Gamma_8)} \) is of even order. Thus there is an involution \( x \) fixing \( \{t_3\} \cup \Gamma_9 \) pointwise. Since \( |I(x)| = 7 \), \( I(x) = \{1, t_3\} \cup \Gamma_9 \). Then \( ab \) fixes \( I(x) \) and \( (ab)^{I(x)} \) is of order five, contrary to (3.4). Thus \( G \) has no orbit of length sixteen.
{2} \cup \Gamma_i \cup \Gamma_j \text{ and } \{4\} \cup \Gamma_k \cup \Gamma_l, \text{ where } \{i, j, k, l\} = \{1, 2, 3, 4\}. \text{ Now we determine the generators of } G. \text{ Assume that } \{2\} \text{ and } \Gamma_i = \Delta(1, 2) \text{ are contained in the same } G\text{-orbit. Then } G \text{ has an orbit } \{2\} \cup \Gamma_i \cup \Gamma_k \text{ where } \{i, k\} = \{1, 2, 3, 4\}. \text{ Let } \{t\} = \Gamma_i \cap I(a). \text{ Then } \Delta(1, 2) = \Gamma_i \text{ and } \Delta(1, t) \text{ are contained in the same } G\text{-orbit. On the other hand by the incidence structure of } P(a), I(a) \cap \Delta(1, t) \text{ is different from } \{2\}, \{3\} \text{ and } \{t\}. \text{ Thus } \{\{2\} \cup \Gamma_i \cup \Gamma_k\} \cap I(a) \supseteq \{2, 3, t\}, \text{ which is a contradiction. Similarly } \{4\} \text{ and } \Gamma_2 = \Delta(1, 2) \text{ are not contained in the same } G\text{-orbit. Thus the } G\text{-orbits are } \{1\}, \{2\} \cup \Gamma_2 \cup \Gamma_3 \text{ and } \{4\} \cup \Gamma_4 \cup \Gamma_5, \text{ where } \{i, j\} = \{3, 4\}. \text{ Now we assume that the } G\text{-orbits are } \{1\}, \{2\} \cup \Gamma_2 \cup \Gamma_3 \text{ and } \{4\} \cup \Gamma_4 \cup \Gamma_5, \text{ the generators are determined in the similar way and this group is isomorphic as a permutation group whose orbits are } \{1\}, \{2\} \cup \Gamma_2 \cup \Gamma_3 \text{ and } \{4\} \cup \Gamma_4 \cup \Gamma_5. \text{ Let } x \text{ be an involution fixing } 8, 9 \text{ and commuting with } a. \text{ Since } \Delta(1, 2) \ni 8, 9 \text{ and } I(x) \ni 1, x \text{ does not fix } 2. \text{ Furthermore since } \{2\} \cup \Gamma_2 \cup \Gamma_3 \text{ is a } G\text{-orbit, } x = (1) (2 5) (6) \cdots \text{ or } (1) (2 6) (5) \cdots. \text{ Now we assume that } x = (1) (2 5) (6) (8) (9) \cdots. \text{ Then by the incidence structure of } P(a), x = (1) (2 5) (3 4) (6) (7) (8) (9) \cdots. \text{ Hence }

bx = (1) (8) (14 2 5 \cdots) (10 4 3 \cdots).

Since } b \text{ and } x \text{ fix } 1, 8, \Delta(1, 8) \cap I(b) = \{2\} \text{ and } \Delta(1, 8) \cap I(x) \neq \{2\}, \text{ the order of } bx \text{ is odd by (3.7). Furthermore since } \Delta(1, 8) \text{ is a } G_i \text{-orbit and } |\Delta(1, 8)| \leq 5, \text{ the order of } bx \text{ is three or five. If } bx \text{ is of order three, then } x \text{ fixes } 14, 10 \text{ and so fixes } 15, 11 \text{ because } ax = xa. \text{ Thus } |I(x)| > 7, \text{ contrary to (i) of (3.1). Therefore } bx \text{ is of order five and so } |I(bx)| = 3. \text{ Hence } b \text{ and } x \text{ fix exactly one more point other than } 1, 8 \text{ in common. Therefore } x \text{ fixes exactly one point of } \{12, 16, 20\}. \text{ Assume that } x \text{ fixes } 16. \text{ Then } x \text{ fixes } 17 \text{ because } ax = xa. \text{ Thus } I(x) = \{1, 6, 7, 8, 9, 16, 17\}. \text{ Since } \Delta(1, 8) \ni 2 \text{ and } \{2\} \cup \Gamma_2 \cup \Gamma_3 \text{ is a } G\text{-orbit, } \Delta(1, 8) \subseteq \{2\} \cup \Gamma_2 \cup \Gamma_3 \text{ by (3.8). Since three points } 1, 6, 7 \text{ are collinear, } \Delta(1, 8) \ni 6. \text{ Since three points } 1, 2, 8 \text{ are collinear in } P(b), \Delta(1, 8) \ni 16. \text{ Thus } \Delta(1, 8) \ni 17 \text{ in } P(x). \text{ Furthermore since } \Delta(1, 8) < b, x > = \Delta(1, 8) \text{ and } |\Delta(1, 8)| \leq 5, \Delta(1, 8) = \{2, 5, 14, 17, 19\}. \text{ Therefore }

x = (1) (2 5) (3 4) (6) (7) (8) (9) (16) (17) (14 19) (15 18) \cdots.

Hence

\begin{align*}
ax & = (1) (2 5) (3 4) (6) (7) (8 9) (16 17) (14 18) (15 19) \cdots \\
\text{and } bx & = (1) (2 18 14 13 19 16 17 15) \cdots.
\end{align*}

This shows that } bx \text{ is of order five. Hence } ax \text{ fixes } 13 \text{ and so fixes } 12. \text{ Furthermore since } |I(bx)| = 3, b \text{ and } ax \text{ fix exactly one more point other than } 1, 12 \text{ in common. Hence } ax \text{ fixes } 20 \text{ and so fixes } 22. \text{ Thus }

Therefore \(x = (10\ 23) (11\ 22)\) or \((10\ 22) (11\ 23)\) on \(\{10, 11, 22, 23\}\). If \(x\) is of the first form, then \(bx = (3\ 23\ 20\ 21\ 10\ 4)\cdots\). This is a contradiction since \(bx\) is of order five. Thus

\[
x = (1) (2\ 5) (3\ 4) (6) (7) (8) (9) (16) (17) (14\ 19) (15\ 18) (12\ 13) (20\ 21)
(10\ 22) (11\ 23).
\]

The element \(x\) of this form is denoted by \(x_2\). Then \(\langle a, b, x_2 \rangle\) is isomorphic to \(PSL(2, 11)\). Since any subgroup of \(M_{23}\) which is isomorphic to \(PSL(2, 11)\) and has orbits of lengths \(1, 11, 11\) is isomorphic as a permutation group to \(\langle a, b, x_2 \rangle\), we need not show the form of \(x\) when \(x = (1) (2\ 6) (5) (8) (9)\cdots\) or \((1) (2\ 5) (6) (8) (9)\cdots\) and \(x\) fixes \(16\) or \(20\).

(iii. iii) Suppose that there is a \(G\)-orbit of length six. Then by (iii. i) and (iii. ii), the lengths of the \(G\)-orbits are \(1, 6, 6, 5, 5\) or \(1, 6, 6, 10\).

Assume that \(\{t\} \cup \Gamma_i\) is a \(G\)-orbit, where \(t \in \{2, 4\}\) and \(i \in \{3, 4\}\). Let \(t = 2\) and \(i = 3\). The proofs in the remaining cases are similar. For two points \(2\) and \(6\) of \(\{2\} \cup \Gamma_3\), \(\Delta(1, 2)\) and \(\Delta(1, 6)\) are contained in the same \(G\)-orbit. Since \(\Delta(1, 2) = \Gamma_3\) and \(\Delta(1, 6) \supseteq 7\), \(\Gamma_3\) and \(\Gamma_1\) are contained in the same \(G\)-orbit. Thus \(G\)-orbits are \(\{1\}, \{2\} \cup \Gamma_3, \Gamma_1 \cup \Gamma_3, \) and \(\{4\} \cup \Gamma_3\). Let \(x\) be an involution fixing \(12, 13\) and commuting with \(a\). Since \(\Delta(1, 4) \supseteq 12, 13\), \(x\) does not fix the point \(4\) by the assumption (7). Furthermore \(\{4\} \cup \Gamma_3\) is a \(G\)-orbit. Hence by the incidence structure of \(P(a)\), \(x = (1) (4\ 5) (2\ 3) (6) (7)\cdots\) or \((1) (4\ 5) (2) (3) (6\ 7)\cdots\). If \(x\) is of the first form, then \(2\) and \(3\) belong to the same \(G\)-orbit, which is a contradiction. If \(x\) is of the second form, then \(6\) and \(7\) belong to the same \(G\)-orbit, which is also a contradiction.

Thus the \(G\)-orbit containing \(\{2\}\) is \(\{2\} \cup \Gamma_1\) or \(\{2\} \cup \Gamma_2\) and the \(G\)-orbit containing \(\{4\}\) is \(\{4\} \cup \Gamma_1\) or \(\{4\} \cup \Gamma_2\). Suppose that the stabilizer of two points in \(G^{(\{4\} \cup \Gamma_1)}\) contains a four group, where \(\{t\} \cup \Gamma_i\) is a \(G\)-orbit, \(t \in \{2, 4\}\) and \(i \in \{1, 2\}\). Since \(G^{(\{4\} \cup \Gamma_1)}\) is doubly transitive on \(\{t\} \cup \Gamma_i\), \(G^{(\{4\} \cup \Gamma_1)}\) contains \(A_4\) by [5]. Then since \(A_4\) has no permutation representation of degree five, \(\Gamma_3 \cup \Gamma_4\) is a \(G\)-orbit. Furthermore since \(A_4\) is isomorphic to \(M_{10}^*\) of degree ten, \(G^{\{4\} \cup \Gamma_1}\) is doubly transitive on \(\Gamma_3 \cup \Gamma_4\). On the other hand since \(\Delta(1, 6) \supseteq 7\), \(\Delta(1, 6)\) is a \(G_{1,4}\)-orbit containing \(7\) by (3, 8). Thus \(\Delta(1, 6) = \Gamma_3 \cup \Gamma_4 \cup \{6\}\) and so \(|\{\Delta(1, 6)\}| = 9\). This is a contradiction. Thus the stabilizer of two points in \(G^{(\{4\} \cup \Gamma_1)}\) does not contain a four group. We use this result in the following proof.

First assume that \(\{2\} \cup \Gamma_1\) and \(\{4\} \cup \Gamma_3\) are \(G\)-orbits. Let \(x\) be an involution fixing \(16, 17\) and commuting with \(a\). Since \(\{2\} \cup \Gamma_3\) is a \(G\)-orbit, \(\{2, 3\}^* = \{2, 3\}\). Suppose that \(x\) fixes \(\{2, 3\}\) pointwise. Since \(\Delta(1, 2) = \Gamma_3, I(x) \supseteq \{1, 2, 3\}\) and \(I(ax) \supseteq \{1, 2, 3\}\), \(x\) and \(ax\) have no fixed point in \(\Gamma_3\), \(\{3\}\) by the assumption (7). Hence \(x = (1) (2\ 3) (8\ 10) (9\ 11)\cdots\) or \((1) (2\ 3) (8\ 11) (9\ 10)\cdots\). Thus \(\langle a, x \rangle^{\{3\} \cup \Gamma_1}\) is a four group contained in \(G^{(\{3\} \cup \Gamma_1)}\), contrary to the remark above. Similarly \(x\) does not fix \(\{4, 5\}\) pointwise. Hence \(x = (1) (2\ 3) (4\ 5) (6) (7) (16)\).
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(17) ««« . Then \( x \) fixes two more points other than \( 1, 6, 7, 16 \) and \( 17 \). Since \( \{2\} \cup \Gamma_1 \) and \( \{4\} \cup \Gamma_2 \) are \( G \)-orbits of even length, it is impossible that \( x \) has a fixed point in both \( \{2\} \cup \Gamma_1 \) and \( \{4\} \cup \Gamma_2 \). Hence now we assume that \( x \) has no fixed point in \( \{2\} \cup \Gamma_1 \). The proof in the case \( |I(x) \cap \{4\} \cup \Gamma_2| = 0 \) is similar.

Then \( x = (1) (2 3) (4 5) (6) (7) (8 10) (9 11) \ldots \) or \( (1) (2 3) (4 5) (6) (7) (8 10) (9 11) \ldots \). If \( x \) is of the first form, then \( axb = (1) (2 10) (3) (8) (9 11) \ldots \), contrary to (v) or (3.1). Hence \( x \) is of the second or third form. If \( x \) is of the third form, then \( ax = (1) (2 3) (4 5) (6) (7) (10 11) \ldots \). Hence set \( x' = x \) or \( ax \) in the second or third case respectively. Then \( x'b = (1) (2 10) (3) (8) (9) (11) (5 4 14) \ldots \). Hence \( x'b \) is of order four and so \( x'b \) has exactly two 2-cycles and three fixed points by (ii) of (3.1). Hence \( x'b = (1) (2 10) (3) (8) (9) (11) (5 4 14 12) (13 15) \ldots \). Then \( \langle x', x'b \rangle \cong \{4\} \cap \Gamma_2 \) is a four group contained in \( \langle G(a) \cap \Gamma \rangle \rangle_{1 \leq 18} \), contrary to the remark above.

Next assume that \( \{2\} \cup \Gamma_2 \) and \( \{4\} \cup \Gamma_4 \) are \( G \)-orbits. Let \( x \) be an involution fixing \( 16, 17 \) and commuting with \( a \). Since \( \{2\} \cup \Gamma_1 \) is a \( G \)-orbit, \( \{2, 5\} = \{2, 5\} \). If \( x \) fixes \( \{2, 5\} \) pointwise, then by the incidence structure of \( P(a) \) \( x \) fixes \( I(a) \) pointwise. Thus \( I(x) \supseteq I(a) \cup \{16, 17\} \), contrary to (i) of (3.1). Hence \( x \) has a 2-cycle (2 5). Then by the incidence structure of \( P(a) \), \( x = (1) (2 5) (3 4) (6) (7) (16) (17) \ldots \). Then by the same argument as above we have a contradiction. Thus we complete the proof of (3.17).

Let

\[ \sigma_1 = (1 4 3) (2 5 6 7) (8 10) (9 11) (12 22 14 18 20 13 23 15 19 21) \]
\[ (16 17) \]

and

\[ \sigma_2 = (1 7 6) (2) (3 5) (4) (18 17 9 16) (10 23 20 18 12) (11 22 19 16) \]
\[ (14 15) \]

Then

\[ a = a^\sigma_1 = a^\sigma_2, \]
\[ (ax)^{\sigma_1} = x_{\sigma_2} = (1) (2 3) (8) (9) (10 11) (12 14) (13 15) (4 5) (6) (7) (16) \]
\[ (17) (18 19) (20 22) (21 23), \]
\[ b^{\sigma_1} = (3) (13) (10) (2 14) (1 8) (9 11) (12 15) (4) (5) (17) (22) (6 18) \]
\[ (7 20) (16 21) (19 23), \]
\[ b^{\sigma_2} = (2) (8) (10) (1 12) (9 13) (3 15) (14 11) (4) (7) (17) (18) (5 23) \]
\[ (6 21) (16 22) (19 20). \]

Then these elements belong to the group of (i) of (3.16), \( \langle a, (ax)^{\sigma_1}, b^{\sigma_1} \rangle \supseteq S_t \) and \( \langle a, x_{\sigma_2}^+, b^{\sigma_2} \rangle \supseteq PSL(2, 11) \). We used in the proof of Lemma 3 these generators.

Next we need the following (3.18) and (3.19), which are frequently used in the proof in (3.20).
3.18. Let
\[ a = (i_1) (i_2) (j_1, j_2) \cdots. \]
and let \( b \) be an involution of the form
\[ b = (i_1) (i_2) (j_1, j_2) \cdots. \]
If \( \Delta(i_1, i_2) \ni j_t \) or \( \Delta(i_1, i_2) \ni i_t \) \((t = 1, 2, 3)\), then \( (ab)^2 = 1 \) and \( a^b = a^{-1} \).

Proof. If \( \Delta(i_1, i_2) \ni i_t \), then \( \Delta(i_1, i_2) \ni j_t \) by the assumption (3), \( t = 1, 2, 3 \). Hence we may assume that \( \Delta(i_1, i_2) \ni j_t, t = 1, 2, 3 \). Then since \( \Delta(i_1, i_2)^a = \Delta(i_1, i_2), \Delta(i_1, i_2) \ni \{j_1, j_2, j_3\} \). Since
\[ ab = (i_1) (i_2) (j_1, j_2) (j_3) \cdots, \]
\[ |I((ab)^2)| \geq 5. \]
Hence the order of \( (ab)^2 \) is 1, 2 or 3 by the assumption (4). If \( (ab)^2 \) is of order two, then \( I((ab)^2) \ni \{i_1, i_2\} \) and \( I((ab)^2) \cap \Delta(i_1, i_2) \ni \{j_1, j_2, j_3\} \), contrary to the assumption (7). If \( (ab)^2 \) is of order three, then \( ab \) is of order six. However \( |I(ab)| \geq 3 \), contrary to (v) of (3.1). Hence \( (ab)^2 = 1 \) and so \( bab = a^{-1} \). Since \( b \) is an involution, \( a^b = a^{-1} \).

3.19. Assume that \( |\Delta(i, j)| \leq 3 \) for any two points \( i, j \) of \( \Omega \). Let \( x, y \) be involutions such that \( xy = yx \) and \( I(x) \neq I(y) \). Then \( I(x) \cap I(y) \) consists of three points, say \( i_1, i_2, i_3 \), and \( \Delta(i_1, i_2) = \{i_3\} \).

Proof. Since \( I(x) \neq I(y), |I(x) \cap I(y)| = 3 \). Set \( I(x) \cap I(y) = \{i_1, i_2, i_3\} \). Then \( I(\langle x, y \rangle) = \{i_1, i_2, i_3\} \). Thus \( \langle x, y \rangle \) is of order four and has orbits of lengths one, two and four. Since \( \langle x, y \rangle \) fixes \( \{i_1, i_2, i_3\} \), \( \Delta(i_1, i_2) \) consists of unions of \( \langle x, y \rangle \)-orbits. Since \( |\Delta(i_1, i_2)| \leq 3, \Delta(i_1, i_2) \) does not contain the \( \langle x, y \rangle \)-orbit of length four. Furthermore for any \( \langle x, y \rangle \)-orbit of length two there is an involution in \( \langle x, y \rangle \) fixing the \( \langle x, y \rangle \)-orbit of length two pointwise. Hence by the assumption (7), \( \Delta(i_1, i_2) \) does not contain the \( \langle x, y \rangle \)-orbit of length two. Thus \( \Delta(i_1, i_2) \) is the \( \langle a, x \rangle \)-orbit \( \{i_3\} \) of length one.

3.20. It is impossible that \( |\Delta(i, j)| \leq 3 \) for any two points \( i, j \) of \( \Omega \).

Proof. Suppose by way of contradiction that \( |\Delta(i, j)| \leq 3 \) for any two points \( i, j \) of \( \Omega \). By (3. 9), we may assume that \( \Delta(1, 2) = \{3, 8, 9\} \). Then by the assumption (6), there are two involutions \( a \) and \( b \) fixing 1, 2, 3 and 1, 2, 8 respectively. Then by (3. 7), \( ab \) is of odd order. Since \( |\Delta(1, 2)| = 3, ab \) is of order three. Thus we may assume that
and
Then

\[ ab = (1) (2) (4) (3 \ 9 \ 8) (5 \ 11 \ 10) (6 \ 13 \ 12) (7 \ 15 \ 14) (16) (17) (18 \ 22 \ 21) (19 \ 20 \ 23) \]

Set \( \Gamma_1 = \{3, 8, 9\}, \Gamma_2 = \{5, 10, 11\}, \Gamma_3 = \{6, 12, 13\}, \Gamma_4 = \{7, 14, 15\}, \Lambda_1 = \{18, 21, 22\}, \) and \( \Lambda_2 = \{19, 20, 23\}. \) Then \( \langle a, b \rangle \)-orbits are \( \{1\}, \{2\}, \{4\}, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Lambda_1 \cup \Lambda_2 \) and \( \{16, 17\}. \) Similarly to (3.16) we may assume that \( \Delta(1, 4) \ni 5, \Delta(1, 6) \ni 7 \) and \( \Delta(2, 4) \ni 6. \) Then \( \Delta(1, 2) = \Gamma_1, \Delta(1, 4) = \Gamma_2 \) and \( \Delta(2, 4) = \Gamma_3. \)

Since \( \Delta(16, 1)^{ab} = \Delta(16, 1), \Delta(16, 1) = \{2\}, \{4\}, \{17\}, \{2, 4, 17\}, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Lambda_1 \) or \( \Lambda_2. \) If \( \Delta(16, 1) \ni 2, \) then \( \Delta(1, 2) \ni 16, \) which is a contradiction. Similarly \( \Delta(16, 1) \ni 4. \) Suppose that \( \Delta(16, 1) = \Gamma_3. \) Then \( \Delta(17, 1) = \Delta(16, 1)^a = \Gamma_3. \) Hence \( \Delta(1, 6) = \{7, 16, 17\}. \) This implies that there is an involution \( x \) fixing 1, 6, 16. By the assumption (7),

\[ x = (1) (6) (16) (7 \ 17 \cdots) \]

Hence

\[ abx = (1) (16) (14 \ 17 \ 7 \cdots) \cdots. \]

Thus \( abx \) is not an involution. On the other hand since \( \Delta(16, 1) = \{6, 12, 13\} \) and \( x \) fixes 16, 1, 6,

\[ x = (16) (1) (6) (12 \ 13) \cdots. \]

Since \( ab = (16) (1) (6 \ 13 \ 12) \cdots, \) \( abx \) is of order two by (3.18), which is a contradiction. Thus \( \Delta(16, 1) \ni \Gamma_3. \) Similarly \( \Delta(16, 1) \ni \Gamma_4. \) Hence \( \Delta(16, 1) = \{17\}, \Gamma_1, \Gamma_2, \Lambda_1 \) or \( \Lambda_2. \) In the same way \( \Delta(16, 2) = \{17\}, \Gamma_1, \Gamma_2, \Lambda_1 \) or \( \Lambda_2, \Delta(17, 2) = \{16\}, \Gamma_1, \Gamma_2, \Lambda_1 \) or \( \Lambda_2. \)

Thus \( \Delta(16, 17) = \Delta(16, 17)^{ab} = \Delta(16, 17). \) Hence \( \Delta(16, 17) = \{1\}, \{2\}, \{4\}, \{1, 2, 4\}, \Gamma_1, \Gamma_2, \Gamma_3 \) or \( \Gamma_4. \) In the following we treat these cases for \( \Delta(16, 17) \) separately.

(i) Assume that \( \Delta(16, 17) = \Gamma_1. \) The proof in the case \( \Delta(16, 17) = \Gamma_2 \) or \( \Gamma_3 \) is similar. Then by the assumption (6), there is an involution \( x \) of the form

\[ x = (16) (17) (3 \ 8 \ 9) \cdots. \]

Since \( ab = (16) (17) (3 \ 9 \ 8) \cdots, (ab)^x = (ab)^{-1} \) by (3.18). Since \( I(ab) = \{1, 2, 4, 16, 17\}, x \) fixes \( \{1, 2, 4\} \) and so \( x = (1) (2) (4) \cdots, (1) (2) (4) \cdots, (2) (1) (4) \cdots (4) \cdots. \) Suppose that \( x = (1) (2) (4) \cdots. \) Then \( I(a) \cap I(x) = \{1, 2, 3, 4\} \) and \( I(a) \ni I(x), \) contrary to (vi) of (3.1). Next suppose that \( x = (1) (2) (4) \cdots. \) Then \( \Delta(1, 3)^x = \Delta(1, 3). \) Since \( \Delta(1, 3) \ni 2, \Delta(1, 3) \ni 2^x = 4. \) Thus \( \Delta(1, 3) \cap I(a) = \{2, 4\}, \) which is a contradiction since \( a \) is an involution fixing 1, 3. Similarly \( x = (2) (1) (4) \cdots. \) Thus
Since $ax = (1 2) (3) (4) (8 9) (16 17)\cdots$, $x$ commutes with $a$. Therefore by the incidence structure of $P(a)$,

$$x = (1 2) (3) (4) (5 6) (7) (8 9) (16) (17)\cdots.$$ 

Then because $(ab)^* = (ab)^{-1}$

$$x = (1 2) (4) (3) (8 9) (5 6) (10 13) (11 12) (7) (14 15) (16) (17)\cdots.$$ 

Since $|I(x)| = 7$, $x$ fixes exactly two points of $\{18, 19, 20, 21, 22, 23\}$. Without loss we may assume that $x$ fixes 18, 19. Then


Since $I(<a, x>) = \{3, 7, 4\}, \Delta(3, 4) = \{7\}, \Delta(3, 7) = \{4\}$ and $\Delta(4, 7) = \{3\}$ by (3. 19).

Now $\Delta(16, 4) = \{17\}, \Gamma_2, \Gamma_3, \Lambda_1$ or $\Lambda_2$ and $\Delta(16, 1) = \{17\}, \Gamma_1, \Gamma_3, \Lambda_1$ or $\Lambda_2$.

Since $\Delta(16, 17) = \Gamma_2, \Delta(16, 4) = \{17\}$ and $\Delta(16, 1) = \{17\}$. Since $\Delta(16, 4)^* = \Delta(16, 4)$ and $\Gamma_2^* = \Gamma_3, \Delta(16, 4) = \Gamma_2, \Gamma_3$. Thus $\Gamma(16, 4) = \Lambda_1$ or $\Lambda_2$. On the other hand $\Delta(16, 1) = \{17\}, \Gamma_1, \Gamma_2, \Lambda_1$ or $\Lambda_2$. Since $\Delta(16, 17) = \Gamma_1, \Delta(16, 1) = \{17\}$. Suppose that $\Delta(16, 1) = \Gamma_1$. Then $\Delta(17, 1) = \Delta(16, 1)^* = \Gamma_3$. Hence $\Delta(1, 3) = \{2, 16, 17\}$. Then by the assumption (6), there is an involution

$$u = (1) (16) (3) (8 9) (2 17)\cdots.$$ 

Since $\Delta(16, 1) = \Gamma_1$ and $ab = (16) (1) (3 9 8)\cdots$, $(ab)^* = (ab)^{-1}$ by (3. 18). Hence

$$u = (1) (16) (2 17) (3) (4) (8 9)\cdots.$$ 

Since $\Delta(3, 4) = \{7\}$, $u$ fixes 7 and so

$$au = (1) (3) (4) (2 17 16) (7) (8) (9)\cdots,$$

contrary to (ii) of (3. 1). Thus $\Delta(16, 1) = \Gamma_1$. Next suppose that $\Delta(16, 1) = \Gamma_2$. Since $\Delta(17, 1) = \Delta(16, 1)^* = \Gamma_3, \Delta(1, 5) = \{4, 16, 17\}$. Then by the assumption (6) there is an involution

$$u = (1) (16) (5) (10 11) (4 17)\cdots.$$ 

By (3. 18), $(ab)^* = (ab)^{-1}$. Hence

$$u = (1) (16) (4 17) (2) (5) (10 11)\cdots.$$ 

This implies that $\Delta(16, 4) = \Delta(16, 17)^* = \Gamma_3$. However $u$ fixes 1, 2. Hence $\Gamma_3^* = \Gamma_2$, which is a contradiction since $\Delta(16, 4) = \Lambda_1$ or $\Lambda_2$. Thus $\Delta(16, 1) = \Lambda_1$ or $\Lambda_2$. Since $\Delta(16, 1) = \Delta(16, 2)^* = \Delta(16, 1) = \Lambda_1$ or $\Lambda_2$.

By (3. 5), there is an involution $y$ fixing 10, 11 and commuting with $a$. Since $y$ fixes $I(a), y = (1) (4)\cdots, (1 4)\cdots, (1 (4 i)\cdots, (4) (1 i)\cdots$ or $(1 i) (4 j)\cdots$, 

$$y = (1) (4) (1 2) (3) (4) (8 9) (16) (17)\cdots.$$ 

Since $\Delta(16, 4) = \Delta(16, 17)^* = \Gamma_3$. However $u$ fixes 1, 2. Hence $\Gamma_3^* = \Gamma_2$, which is a contradiction since $\Delta(16, 4) = \Lambda_1$ or $\Lambda_2$. Thus $\Delta(16, 1) = \Lambda_1$ or $\Lambda_2$. Since $\Delta(16, 2) = \Delta(16, 1)^* = \Delta(16, 2) = \Lambda_1$ or $\Lambda_2$.
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\[ i, j \in \{2, 3, 5, 6, 7\}. \text{ Since } \Delta(1, 4) \supseteq \{10, 11\}, \text{ the first case does not occur. In}
\]

the remaining cases \( y^{(1, 4)_a} \) is determined by the incidence structure of \( P(a) \).

(i) Assume that \( y=(1 4) (10) (11)\). Then \( y=(1 4) (5) (2) (7) (3 6) (10) (11)\) or \( (1 4) (5) (2 7) (3) (6) (10) (11)\).

(i.i) Suppose that \( y=(1 4) (5) (2 7) (3 6) (10) (11)\). Then

\[ by = (1 4) (2 5) (11) (10)\].

This shows that \( by \) is of order two or four. Since \( (by)^2 \) fixes \( \{1, 4\} \cup \Delta(1, 4) \)

pointwise, the order of \( by \) is not four. Hence \( by \) is of order two and so \( y \) commutes with \( b \) and so \( ab \). Then \( y \) fixes \( \{16, 17\} \). However \( \Delta(16, 17)=\Gamma_1 \)

and \( \Gamma_1 \neq \Gamma_1 \). Thus we have a contradiction.

(i.i.i) Let \( i=2 \). Then \( \Delta(1, 4)^2=\Delta(1, 2) \). Since \( \Delta(1, 4)=\Gamma_2 \equiv 10 \), \n
\( \Delta(1, 2) \equiv 10^2=10 \), which is a contradiction.

(iii.i) Let \( i=3 \). Then \( y=(1) (4 3) (2 5) (6) (7) (10) (11)\). Since \( \Delta(1, 10)^{<b,y>} = \Delta(1, 10) \) and \( \Delta(1, 10) \equiv 10 \), \n
\( \Delta(1, 10) = \{4, 3, 9\} \). Hence \( y \) fixes 9 and so 8. This implies that \( y \) fixes the subset \{3, 4, 7, 8, 9\} of \( I(ax) \). Hence by (vi) of (3.1), \( y \) fixes \( I(ax) \) and consequently \( y \) has a 2-cycle (14 15). Thus

\[ by = (1) (2 5) (11) (3 9) (4) (10) (7 14) (15) (13 6)\].

This shows that \( by \) is of order three. Hence \( y \) has a 2-cycle (12 13) and \( by \) fixes

eactly two points of \{16, 17, \ldots, 23\}. Therefore \( y \) has a common 2-cycle \((i j)\) in \{16, 17, \ldots, 23\} with \( b \). If \((i j)=(16 17)\), then \(|I(ay)| \geq 9\), which is a

contradiction. Next suppose that \((i j)=(21 23)\). Since \( by \) is of order three and \( ay \)

is of order two, \( y=(21 23)(20 22) (16 18) (17 19)\) or \( (21 23) (20 22) (16 19) (17 18)\)

and so \( xy=(1 5 6 2) (3 4) (8 9) (16 18)\) or \( (1 5 6 2) (3 4) (8 9) (16 19)\) respectively, contrary to (iii) of (3.1). Thus \((i j)=(18 20)\) or \( (19 22)\).

Since \( by \) is of order three and \( ay \) is of order two, \( y \) is one of the following form:

\[ y_1=(1) (4 3) (2 5) (6) (7) (8) (9) (10) (11) (12 13) (14 15) (18 20) (19 21) (16 22) (17 23), \]

\[ y_2=(1) (4 3) (2 5) (6) (7) (8) (9) (10) (11) (12 13) (14 15) (19 22) (18 23) (16 20) (17 21) \]

or \( y_i'=y_i^\sigma \), where \( i=1 \) or 2 and \( \sigma=(16 17) \). We assume that \( y=y_1 \) or \( y_2 \). The

proof in the case \( y=y_1' \) or \( y_2' \) is similar. Then

\[ aby_1 = (1) (2 5 11 10) (3 9 8 4) (6 12) (13) (7 14) (15) (18 16 22 19) (17 23 21 20), \]
aby2 = (1) (2 5 11 10) (3 9 8 4) (6 12) (13) (7 14) (15) (18 19 16 20) (17 21 23 22).

Set $u_1 = x^{aby_1}$ and $u_2 = x^{aby_2}$. Then

$u_1 = (1 5) (3) (9) (4 8) (11 12) (10 6) (2 13) (14) (7 15) (22) (23) (16) (20 19) (18) (17 21),$


Now $\Delta(16, 1) = \Lambda_1$ or $\Lambda_2$. Assume that $\Delta(16, 1) = \Lambda_1$. The proof in the case $\Delta(16, 1) = \Lambda_2$ is similar. Then $\Delta(16, 5) = \Delta(16, 1)^* = \{18, 17, 22\}$. Hence $\Delta(16, 17) \ni 5$. However $\Delta(16, 17) = \Gamma_1$. Thus we have a contradiction. Next $\Delta(16, 5) = \Delta(16, 1)^* = \Lambda_1$. Hence $\Delta(16, 11) = \Delta(16, 5)^{a_2} = \Lambda_1$ and $\Delta(16, 10) = \Delta(16, 11)^{a_2} = \Lambda_1$. Thus $\Delta(16, 1) = \Delta(16, 5) = \Delta(16, 11) = \Delta(16, 10) = \Lambda_1$. This implies that $\Delta(16, 18) \ni \{1, 5, 10, 11\}$, which is a contradiction. Thus $y \equiv \pm (1) (4 3)\ldots$

(i.ii.iii) Let $i = 5$. Then $y = (1) (4 5) (2) (3) (6 7) (10) (11)\ldots$ or $y = (1) (4 5) (2 3) (6) (7) (10) (11)\ldots$. If $y$ is of the first form, then $I(\langle a, y \rangle) = \{1, 2, 3\}$. Hence by (3.19) $\Delta(1, 2) = \{3\}$, which is a contradiction. Hence $y$ is of the second form. Then

$by = (1) (4 5 11) (10) (9 2 3)\ldots (13 6)\ldots (15 7)\ldots\ldots.$

This shows that $by$ is of order three. Hence $y$ fixes 9 and so 8. Furthermore $y = (12 13) (14 15)\ldots$, or $(12 13) (14 15)\ldots$, then $xy = (13 2) (8 9) (14) (15)\ldots$, contrary to (v) of (3.1). Next suppose that $y = (12 15) (13 14)\ldots$. Since $|I(ay)| = 7$ and $|I(by)| = 5$, $ay$ and $by$ fix exactly four and two points of $\{16, 17, 19, 20\}$ respectively. Thus $y = (16 17) (18 19)\ldots$, or $(16 17) (20 21)\ldots$, or $(16 17) (22 23)\ldots$. However in any case $\{16, 17\}^y = \{16, 17\}$ and $\Delta^*_y = \Gamma_1$, which is a contradiction.

(i.ii.iv) Let $i = 6$ or 7. Then $y = (1) (4 6) (5 7) (2) (3) (10) (11)\ldots$ or $y = (1) (4 7) (5 6) (2) (3) (10) (11)\ldots$. Then $I(\langle a, y \rangle) = \{1, 2, 3\}$. Hence by (3.19) $\Delta(1, 2) = \{3\}$, which is a contradiction.

(i.iii) Assume that $y = (4) (1) (10) (11)\ldots$, $i \in \{2, 3, 5, 6, 7\}$.

(i.iii.i) Let $i = 2$. Then $\Delta(1, 4)^y = \Delta(1, 2)$. Since $\Delta(1, 4) = \Gamma_2 \ni 10$, $\Delta(1, 2) \ni 10^y = 10$. This is a contradiction.

(i.iii.ii) Let $i = 3$. Then $y = (4) (1 3) (10) (11)\ldots$. Hence $\Delta(4, 3) = \Delta(4, 1)^y$. However $|\Delta(4, 3)| = 1$ and $|\Delta(4, 1)| = 3$. Thus we have a contradiction.

(i.iii.iii) Let $i = 5$. Then $y = (4) (1 5) (2) (6) (3 7) (10) (11)\ldots$, or $y = (4) (1 5) (2 6) (3) (7) (10) (11)\ldots$. If $y$ is of the first form, then $I(\langle a, y \rangle) = \{2, 4, 6\}$. Hence by (3.19) $\Delta(2, 4) = \{6\}$, which is a contradiction. Hence $y$ is of the second form. Then
by = (4) (1 5 11) (10) (13 2 6...) (9 3...) (15 7...)...

This shows that by is of order three. Hence y = (12) (13) (8 15) (9 14)... or (12) (13) (8 9) (14 15)...

Assume that y = (12) (13) (8 15) (9 14)...

Then ay fixes exactly three points 3, 4, 7 of \{1, 2, ..., 15\} and by fixes exactly three points 4, 10, 12 of \{1, 2, ..., 15\}. Since \(|I(ay)| = 7\) and \(|I(by)| = 5\), ay and by fix exactly four and two points of \{16, 17, ..., 23\} respectively. Thus y = (16 17) (18 19)..., (16 17) (20 21)... or (16 17) (22 23)...

However in any case \(\Gamma_y \neq \Gamma_y\), which is a contradiction since \(\Delta(16, 17) = \Gamma_1\). Hence y = (12) (13) (8 9) (14 15)...

Then

\[xy = (1 6) (2 5) (4) (3) (8) (9) (10 13) (11 12) (7) (14) (15)\ldots\]

This shows that xy is of order two. Hence

\[y = (4) (1 5) (2 6) (3) (7) (10) (11) (12) (13) (8 9) (14 15) (16 18) (17 19) (20 22) (21 23)\]

or the same permutation with 16 and 17 interchanged. We assume that y is of the first form. The proof in the second case is similar. Since \(\Delta(16, 1) = \Lambda_1\) or \(\Lambda_2\), first assume that \(\Delta(16, 1) = \Lambda_1\). Then \(\Delta(16, 18) \supseteq 1\). Since \(\langle x, y \rangle\) fixes \{16, 18\}, \(\Delta(16, 18) \supseteq \{1\} \subset \langle x, y \rangle =\{1, 5, 6, 2\}\), which is a contradiction. Next assume that \(\Delta(16, 1) = \Lambda_2\). Then \(\Delta(16, 19) \supseteq 1\). Since \(\langle x, ay \rangle\) fixes \{16, 19\}, \(\Delta(16, 19) = \{1\} \subset \langle x, ay \rangle = \{2, 1, 5, 6\}\), which is also a contradiction.

(i.iii.iv) Let \(i = 6\). Then y = (4) (1 6) (2 5) (3) (7) (10) (11)...

Then y is of the same form as y = (4) (1 5) (2 6) (3) (7) (10) (11)...

with 1 and 2 interchanged. Hence in the same way we have a contradiction.

(i.iii.v) Let \(i = 7\). Then y = (4) (1 7) (5 3) (2) (6) (10) (11)...

Then \(I(\langle a, x \rangle) = \{2, 4, 6\}\). Hence by (3.19) \(\Delta(2, 4) = \{6\}\), which is a contradiction.

(i.iv) Assume that y = (1 i) (4 j)i, \(i, j \in \{2, 3, 5, 6, 7\}\).

(i.iv.i) Let \(i = 2\). Then y = (1 2) (4 7) (3) (5) (6) (10) (11)...

This shows that \(\Delta(1, 2)^y = \Delta(1, 2)\). Hence y = (8) (9)...

If y is of the first form, then

\[by = (1 2) (3 9) (8) (5 11) (15 4 7\ldots)\ldots\]

Then \(|I(by)| \geq 7\) and \((by)^2 \neq 1\). Hence \((by)^2\) is of order two and fixes \{1, 2\} \(\cup\Delta(1, 2)\) pointwise, contrary to the assumption (7). Next if y is of the second form, then

\[xy = (1) (2) (4 7) (8) (9) (3\ldots)\]

Hence xy is an involution fixing \{1, 2\} \(\cup\Delta(1, 2)\) pointwise, contrary to the assumption (7).

(i.iv.ii) Let \(i = 3\). Then y = (1 3) (4 6) (2) (5) (7) (10) (11)...

Since b fixes a subset \{2, 5, 10, 11\} of \(I(y)\), b fixes \(I(y)\) by (vi) of (3.1). Hence \(I(y) \supseteq 7b = 15\) and so \(I(y) \supseteq 14\) because \(ay = ya\).

Thus
\[ y = (1 \ 3 \ 4 \ 6 \ 2 \ 5 \ 7 \ 10 \ 11 \ 14 \ 15) \ldots \]

Since \( I(ax) = \{4, 3, 8, 9, 7, 14, 15\} \), \( I(ax)^y = \{6, 1, 8^y, 9^y, 7, 14, 15\} \). Then \( <ab> \) fixes the subset \( \{1, 7, 14, 15\} \) of \( I(ax)^y \). Hence \( <ab> \) fixes \( I(ax)^y \). Since \( \{6\} <ab> = \{6, 12, 13\} \subset I(ax)^y \). Thus \( \{8, 9\}^y = \{12, 13\} \). Furthermore since \( \Delta(16, 17) = \Gamma_1 \) and \( \Gamma_1^y = \Gamma_4 \), \( y \) does not fix \( \{16, 17\} \). Thus the \( \langle a, b, x, y \rangle \)-orbits are \( \{1, 2, 4\} \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_4 \), \( \Gamma_4 \) and \( \{16, 17\} \cup \Lambda_1 \cup \Lambda_2 \). Since \( |\{1, 2, 4\} \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_4| = 12 \) and \( \langle a, b, x, y \rangle \) has an element \( ab \) of order three, the order of \( \langle a, b, x, y \rangle \) is a multiple of 3. By the assumption (4), a Sylow 3-subgroup of \( \langle a, b, x, y \rangle \) is an elementary abelian group of order 3. Since \( \Gamma_4 \) is a \( \langle a, b, x, y \rangle \)-orbit of length three, the order of \( \langle a, b, x, y \rangle \) is a multiple of three. Furthermore \( \langle a, b, x, y \rangle \Gamma_4 \) is normalized by \( ab \). Hence \( \langle a, b, x, y \rangle \Gamma_4 \) has an element \( u \) of order three and commuting with \( ab \). Since \( \Lambda_1 \cup \Lambda_2 \cup \{16, 17\} \) is a \( \langle a, b, x, y \rangle \)-orbit of length eight and \( |I(u)| = 5 \), \( u \) fixes exactly two points of \( \Lambda_1 \cup \Lambda_2 \cup \{16, 17\} \). Thus \( u \) fixes 16, 17 because \( (ab)u = u(ab) \). On the other hand by the assumption (4), \( \langle ab, u \rangle \) has an orbit \( \Gamma \) of length nine. Since \( \{1, 2, 4\} \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_4 \) is the only one \( \langle a, b, x, y \rangle \)-orbit of length at least nine, \( \Gamma \) is contained in \( \{1, 2, 4\} \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_4 \). Since \( \langle ab, u \rangle \) is of order nine, \( \langle ab, u \rangle \) is regular on \( \Gamma \). Hence \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_4 \). Thus \( \langle ab, u \rangle \) fixes \( \{16, 17\} \) and does not fix \( \Gamma_4 = \Delta(16, 17) \), which is a contradiction.

(i.iv.iii) Let \( i = 5 \). Then \( y = (1 \ 5) (4 \ j) (10) (11) \ldots \). Since 1, 5, 4 are collinear in \( P(a) \), this is a contradiction.

(i.iv.iv) Let \( i = 6 \). Then \( y = (1 \ 6) (4 \ 3) (2 \ 5) (7 \ 10) (11) \ldots \). Since \( b \) fixes a subset \( \{2, 5, 10, 11\} \) of \( I(y) \), \( b \) fixes \( I(y) \) by (vi) of (3.1). Hence \( I(y) \supseteq 7^y = 15 \), and so \( I(y) \supseteq 14 \) because \( ay = ya \). Then \( y \) fixes a subset \( \{3, 4, 7, 14, 15\} \) of \( I(ax) = \{3, 4, 7, 14, 15, 8, 9\} \). Hence by (vi) of (3.1), \( y \) fixes \( I(ax) \) and so \( y \) has a 2-cycle \( (8 \ 9) \). Then

\[
\begin{align*}
&\text{by } = (2 \ 10) (5 \ 11) (14) (3 \ 8 \ 9 \ 4) (13 \ 1 \ 6 \ldots) \\
&\text{This shows that } by \text{ is of order four. Hence } y \text{ has a 2-cycle } (12 \ 13). \quad \text{Thus}
\end{align*}
\]

\[
\begin{align*}
y &= (1 \ 6) (4 \ 3) (2 \ 5) (7 \ 10) (11) (14) (15) (8 \ 9) (12 \ 13) \ldots, \\
y^y &= (2 \ 5) (4 \ 3) (1 \ 6) (7 \ 12) (13) (14) (15) (8 \ 9) (10 \ 11) \ldots, \\
ay &= (2 \ 5) (4 \ 3) (1 \ 6) (7 \ 12) (13) (14) (15) (8 \ 9) (10 \ 11) \ldots
\end{align*}
\]

Then \( ay^y \) is in (i.iv.ii). Hence \( y \) is not of this form.

(i.iv.v) Let \( i = 7 \). Then \( y = (1 \ 7) (4 \ 2) (3 \ 5) (6) (10) (11) \ldots \). Then \( \Delta(1, 2)^y = \Delta(4, 7) \), which is a contradiction since \( |\Delta(1, 2)| = 3 \) and \( |\Delta(4, 7)| = 1 \).

(ii) Assume that \( \Delta(16, 17) = \{1, 2, 4\} \). Then there is an involution \( x = (16) (17) (1) (2 \ 4) \ldots \). Then by (3.18), \( (ab)^y = (ab)^{-1} \). Furthermore \( \Delta(2, 4)^y = \Delta(2, 4) \). Hence \( x = (6) (12 \ 13) \ldots, (12) (6 \ 13) \ldots \). Since in the second and third cases we have the first form by transforming \( x \) by \( (ab)^y \) and \( ab \) respectively, we may assume that
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\[ x = (16)(17)(1)(24)(6)(1213) \ldots \]

Then

\[ ax = (1)(24)(6)(12)(13)(1617) \ldots \]

Thus \( ax \) is of order two. Hence from \((ab)^* = (ab)^{-1}\), we have that


Since \(|I(x)| = 7\), \( x \) fixes two points of \{18, 19, \ldots, 23\}. Without loss we may assume that \( x \) fixes 18, 19. Thus

   \hspace{1cm} (19)(2122)(2023), \]

Next there is an involution \( y \) fixing 16, 17, 2. Then by the same argument as is used above for \( x \)

\[ y = (14)(2)(5)(36)(7)(813)(912)(1011)(1415)(16)(17) \ldots \]

Then

\[ xy = (142)(356)(7)(81012)(91113)(1415)(16)(17) \ldots \]

Thus \( xy \) is of order three. Hence \( y \) is one of the following elements:

   \hspace{1cm} (21)(1822)(1923), \]

   \hspace{1cm} (23)(1821)(1920). \]

Suppose that \( y = y_1 \). The proof in the case \( y = y_2 \) is similar. Then

   \hspace{1cm} (2122)(23)(1920), \]

\[ xy_1^{ab} = (142)(31013)(5129)(6811)(71415)(16)(17)(18)(21)
   \hspace{1cm} (22)(192023). \]

Since \( \Delta(1, 6) \ni 7 \) and \( \Delta(1, 16) \ni 17, \Delta(6, 18) \ni 16 \) or 17 in the plane \( P(x) \). Without loss we may assume that \( \Delta(6, 18) \ni 16 \). Then \( \Delta(7, 18) \ni 17 \). Thus

\[ \Delta(16, 18) \ni 6 \text{ and } \Delta(17, 18) \ni 7. \]

Since \( \Delta(16, 18)^{xy} = \Delta(16, 18) \) and \( \Delta(17, 18)^{xy} = \Delta(17, 18), \Delta(16, 18) = \{6, 8, 11\} \) and \( \Delta(17, 18) = \{7, 14, 15\}. \) Furthermore since \( \Delta(21, 22)^{<x, y>} = \Delta(21, 22), \Delta(21, 22) = \{16\}, \{17\}, \{18\}, \{16, 17, 18\}, \{1, 2, 4\}, \{6, 8, 11\}, \{7, 14, 15\} \text{ or } \{19, 20, 23\}. \) On the other hand we have shown that for \( I(<a, b>) = \{1, 2, 4\} \) and \( I(ab) - I(<a, b>) = \{16, 17\}, \Delta(16, 1) = \{17\}, \Delta(1, 2) \Delta(1, 4), \Lambda_1 \text{ or } \Lambda_2, \) where \( \Lambda_1 \cup \Lambda_2 \) is a \( <a, b>-\)orbit of length six. Furthermore in the case (i) we have proved that \( \Delta(16, 17) = \Delta(1, 2), \Delta(1, 4) \text{ and } \Delta(2, 4). \) Since \( I(<x, y^{ab}>) = \{16, 17, 18\} \) and \( I(xy, y^{ab}) - I(<x, y^{ab}>) = \{21, 22\}, \Delta(18, 16), \Delta(18, 17), \{3, 10, 13\} \text{ or } \{5, 12, 9\} \) and \( \Delta(21, 22) = \Delta(16, 17), \Delta(16, 18), \)

\[ \Delta(16, 18), \]
Δ(17, 18). Thus Δ(21, 18) = {22}, {6, 8, 11}, {7, 14, 15}, {3, 10, 13} or {5, 12, 9}, and Δ(20, 21) = {16}, {17}, {16, 17, 18} or {19, 20, 23}. On the other hand since Δ(21, 22) = Δ(21, 28)^* = {18}, {3, 13, 10}, {7, 14, 15}, {6, 11, 8} or {5, 9, 12}, Δ(21, 22) = {18}. Hence there is an involution u fixing 21, 22 and commuting with x. Thus u = (21) (22) (18) (16) ... or (21) (22) (18) (16) i ..., i ∈ {1, 6, 7, 17, 19}. We consider u^i(x) in terms of the incidence structure of P(x).

(ii.i) Assume that u = (21) (22) (18) (16) ... Then I(⟨x, u⟩) ⊃ {16, 18}. Hence by (3.19) |Δ(16, 18)| = 1, which is a contradiction since Δ(16, 18) = {6, 8, 11}.

(ii.ii) Assume that u = (21) (22) (18) (16) i ..., i ∈ {1, 6, 7, 17, 19}.

(ii.ii.i) Let i = 1. Then u = (21) (22) (16) (16 1) (6 19) (7) (17) ... Then I(⟨x, u⟩) = {17, 18, 7}. Hence by (3.19) Δ(17, 18) = {7}, which is a contradiction since Δ(17, 18) = {7, 14, 15}.

(ii.ii.ii) Let i = 6. Then u = (21) (22) (18) (16 6) (17 7) (1 19) ... or (21) (22) (18) (16 6) (17 7) (1 19) .... If u is of the first form, then I(⟨x, u⟩) = {17, 18, 7}. Hence by (3.19) Δ(17, 18) = {7}, which is a contradiction. Next suppose that u is of the second form. Then Δ(16, 17)^* = Δ(6, 7). Hence |Δ(6, 7)| = 3. On the other hand since I(⟨a, x⟩) = {1, 6, 7}, Δ(6, 7) = {1} by (3.19). Thus we have a contradiction.

(ii.ii.iii) Let i = 7. Then u = (21) (22) (18) (16 7) (17 6) (1 19) ... Then Δ(16, 17)^* = Δ(6, 7). Hence |Δ(6, 7)| = 3, which is a contradiction since Δ(6, 7) = {1}.

(ii.ii.iv) Let i = 17. Then u = (21) (22) (16) (17 6) (17 7) (1 19) ... Thus u fixes Δ(16, 17) = {1, 2, 4}. Hence {2, 4}^* = {2, 4}. However this is a contradiction since Δ(2, 4) = {6, 12, 13} and 6^* = 7 ∉ Δ(2, 4)

(ii.ii.v) Let i = 19. Then u = (21) (22) (16) (19 6) (7) (17) ... Then I(⟨x, u⟩) = {17, 18, 7}. Hence by (3.19) Δ(17, 18) = {7}, which is a contradiction.

(iii) Assume that Δ(16, 17) = Γ_x. By (3.5), there is an involution x fixing 16, 17 and commuting with a. Since Δ(16, 17) = Γ_x, we may assume that x = (16) (17) (7) (14 15) ... Then by (3.18), (ab)^* = (ab)^{-1}. Hence x fixes {1, 2, 4}. If x fixes {1, 2, 4} pointwise, then I(a) ∩ I(x) ⊇ {1, 2, 4, 7} and I(a) ≠ I(x), which is a contradiction by (vi) of (3.1). Thus x = (1 2) (4) ... or (2 4) (1) ... Assume that x = (1 2) (4) ... The proof in the case x = (1 4) (2) ... or (2 4) (1) ... is similar. Since ax = xa and (ab)^* = (ab)^{-1},

x = (1 2) (4) (3) (5 6) (7) (8 9) (10 13) (11 12) (14 15) (16) (17) ...

This shows that x fixes exactly two points of {18, 19, ..., 23}. Hence we may assume that

Thus this element \( x \) is the element \( x \) in the proof of the case (i). Moreover since 
\[ I(\langle a, x \rangle) = \{3, 4, 7\}, \]
by (3.19) \( \Delta(i, j) = \{k\} \), where \( \{i, j, k\} = \{3, 4, 7\} \).

Now \( \Delta(16, 1) = \{17\}, \Gamma_1, \Gamma_2, \Lambda_1 \) or \( \Lambda_2 \) and \( \Delta(16, 4) = \{17\}, \Gamma_3, \Gamma_4, \Lambda_1 \) or \( \Lambda_2 \).

Since \( \Delta(16, 1) = \Gamma_4, \Delta(16, 1) \neq \{17\} \) and \( \Delta(16, 4) \neq \{17\} \). Furthermore in the proof of the case (i) to show that \( \Delta(16, 4) \neq \Gamma_1, \Gamma_4 \) and \( \Delta(16, 1, 4) \neq \Gamma_1 \); we did not require that \( \Delta(16, 17) = \Gamma_1 \). Hence by the same argument as in the case (i) \( \Delta(16, 4) = \Lambda_1 \) or \( \Lambda_2 \) and \( \Delta(16, 1) = \Gamma_2, \Lambda_1 \) or \( \Lambda_2 \). We shall treat these cases separately.

(iii.i) Assume that \( \Delta(16, 1) = \Gamma_2 \) and \( \Delta(16, 4) = \Lambda_1 \). The proof in the case \( \Delta(16, 1) = \Gamma_2 \) and \( \Delta(16, 4) = \Lambda_2 \) is similar. Since \( \Delta(17, 1) = \Delta(16, 1) = \Gamma_2 \), \( \Delta(1, 5) = \{4, 16, 17\} \). Thus there is an involution \( y \) fixing 1, 5, 16. Then since \( \Delta(16, 1) = \{5, 10, 11\} \),
\[
y = (1) (5) (16) (4 17) (10 11)\ldots.
\]
Since \( \Delta(16, 1) = \{5, 10, 11\} \), \( (ab)^y = (ab)^{-1} \) by (3.18). Thus
\[
y = (1) (2) (5) (10 11) (16) (4 17)\ldots.
\]
Hence \( \Delta(16, 4)^y = \Delta(16, 17) \) and so \( \Lambda_1^y = \Gamma_4 \). Furthermore \( y \)-fixes \( \{1, 2\} \) point-wise. Since \( \Delta(1, 2) = \Gamma_4, y = (3) (8 9)\ldots, (8) (3 9)\ldots \) or \( (9) (3 8)\ldots \).

(iii.i.ii) Suppose that \( y = (3) (8 9) (1) (2) (5) (10 11) (16) (4 17)\ldots \). Then
\[
ay = (1) (2) (3) (8) (9) (5) (10) (11) (4 17 16)\ldots.
\]
which is a contradiction by (ii) of (3.1).

(iii.i.ii) Suppose that \( y = (8) (3 9) (1) (2) (5) (10 11) (16) (4 17)\ldots \) or \( (9) (3 8) (1) (2) (5) (10 11) (16)\ldots \).

First assume that \( y = (8) (3 9) (1) (2) (5) (10 11) (16) (4 17)\ldots \). Since
\[
x^{(ab)^{-1}} = (1 2) (4) (8) (3 9) (10 12) (5 13) (11 6) (14) (7 15) (16) (17) (21) (18 22) (23) (19 20),
\]
\[
x^{(ab)^{-1}}y = (1 2) (4 17) (16) (8) (3) (9)\ldots.
\]
Then \( x^{(ab)^{-1}}y \) is of order two. Hence form \( (ab)^y = (ab)^{-1} \) and \( \Lambda_1^y = \Gamma_4 \), it follows that
\[
\]
which is denoted by \( y_1 \).

Next assume that \( y = (9) (3 8) (1) (2) (5) (10 11) (16)\ldots \). Since
\[
x^ab = (1 2) (4) (9) (3 8) (11 13) (6 10) (5 12) (15) (7 14) (16) (17) (22) (21 18) (20) (23 19),
\]
\[
x^aby = (1 2) (4 17) (16) (9) (3) (8)\ldots.
\]
Then \( x^aby \) is of order two. Hence form \( \Lambda_1^y = \Gamma_4 \) and \( (ab)^y = (ab)^{-1} \), we have that
where is denoted by $y_2$.

Then

$$ay_1 = (1) (2) (5) (10) (11) (3 8 9) (4 17 16) (7 22 23) (6 12 13) (18 20 14) (19 15 21),$$


First suppose that $y=y_1$. Since $\Delta(10, 11)^{<a, y_1>} = \Delta(10, 11)$ and $|\Delta(10, 11)|=1$ or 3, $\Delta(10, 11) = \{1\}, \{2\}, \{5\}, \{1, 2, 5\}, \{3, 8, 9\}, \{4, 16, 17\}, \{7, 22, 23\}$ or $\{6, 12, 13\}$.

In the cases (i) and (ii) we have proved that for $I(\langle a, b\rangle) = \{1, 2, 4\}$ and $I(ab) - I(\langle a, b\rangle) = \{16, 17\}$, $\Delta(16, 17) = \Delta(1, 2), \Delta(1, 4), \Delta(2, 4)$ and $\{1, 2, 4\}$. Since $I(\langle a, y_1\rangle) = \{1, 2, 5\}$ and $I(ay_1) - I(\langle a, y_1\rangle) = \{10, 11\}$, by the same argument for $\langle a, y_1\rangle$ as is used for $\langle a, b\rangle$ in the cases (i) and (ii), $\Delta(10, 11) = \{1, 2, 5\}$, $\Delta(1, 2), \Delta(1, 5), \Delta(2, 5)$ where $\Delta(1, 2) = \{3, 8, 9\}$ and $\Delta(1, 5) = \{4, 16, 17\}$. Since $\Delta(2, 5) \ni 7$ and $\Delta(2, 5)^{y_1} = \Delta(2, 5)$, $\Delta(2, 5) = \{7, 22, 23\}$. Hence $\Delta(10, 11) = \{1\}, \{2\}, \{5\}$ or $\{6, 12, 13\}$. Furthermore since $\Delta(10, 11) = \Delta(1, 5)^{ab}, F \in \{4, 16, 17\}, \Delta(10, 11) = \{1\}$. Similarly $\Delta(10, 11) = \{2\}$. Thus $\Delta(10, 11) = \{5\}$ or $\{6, 12, 13\}$. Suppose that $\Delta(10, 11) = \{6, 12, 13\}$. Then $\Delta(10, 5) = 0(10, 11)^{y_1} = \{6, 12, 13\}$. Thus $\Delta(10, 12) \ni \{11, 5\}$. Furthermore in the plane $P(b) \Delta(12, 10) \ni 8$. Thus $\Delta(12, 10) = \{8, 11, 15\}$. Hence $\Delta(11, 13) = 0(12, 10) = \{9, 12, 6\}$. However since $\Delta(10, 11) = \{6, 12, 13\}, \Delta(10, 13) \ni 11$. Thus we have a contradiction. Hence $\Delta(10, 11) = \{5\}$.

Next suppose that $y = y_2$. Since $\Delta(10, 11)^{<a, y_2>} = \Delta(10, 11), \Delta(10, 11) = \{1\}, \{2\}, \{5\}, \{1, 2, 5\}, \{3, 8, 9\}, \{4, 16, 17\}, \{7, 20, 21\}$ or $\{6, 12, 13\}$. Since $\Delta(2, 5) \ni 7$ and $\Delta(2, 5)^{y_2} = \Delta(2, 5)$, $\Delta(2, 5) = \{7, 12, 20\}$. Hence by the same argument as is used above, $\Delta(10, 11) = \{5\}$.

Now we assume that $\Delta(10, 11) = \{5\}$ and $y = y_1$. In the following if we interchage 8 and 9, then we have the proof for the case $\Delta(10, 11) = \{5\}$ and $y = y_2$. By (3.5), there is an involution $u$ fixing 10, 11 and commuting with $a$. Since $\Delta(10, 11) = \{5\}, u$ fixes 5. If $u$ fixes 1 or 2, then $ay_1$ fixes $I(u)$, contrary to (3.4). Hence $u = (10) (11) (5) (1 2) \cdots$ or $(10) (11) (5) (1 i) (2 j) \cdots, i, j \in \{3, 4, 6, 7\}$. If $u = (10) (11) (5) (1 2) \cdots$, then by the incidence structure of $P(a)$,

$$u = (10) (11) (5) (1 2) (4 7) (3) (6) \cdots.$$ 

Since $u$ fixes $\Delta(1, 2), u = (8) (9) \cdots$ or $(8 9) \cdots$. If $u = (8) (9) \cdots$, then

$$ay_1 u = (1 2) (5) (10) (11) (3 8 9) \cdots,$$

contrary to (v) of (3.1). Hence $u = (8 9) \cdots$ and
\[ ay, u = (1\ 2\ 5\ 10\ 11\ 3\ 8\ 9) \ldots \]

Then \( ay, u \) is of order two and so \( (ay)^2 = (ay)^{-1} \). Hence

\[ u = (10\ 11\ 5\ 1\ 2\ 4\ 7\ 3\ 6\ 8\ 9\ 12\ 13) \ldots \]

Then

\[ xu = (1\ 2\ 3\ 8\ 9\ 10\ 12\ 11\ 13) \ldots \]

contrary to (iii) of (3.1). Next suppose that \( u = (10\ 11\ 5\ 1\ i\ 2\ j) \ldots, i, j \in \{3, 4, 6, 7\} \). Then by the incidence structure of \( P(a) \), \( u = (10\ 11\ 5\ 1\ 4\ 2\ 7\ 3\ 6) \ldots \) or \( (10\ 11\ 5\ 1\ 7\ 2\ 4\ 3\ 6) \ldots \). Then \( \Delta(1, 2)^* = \Delta(4, 7) \) and so \( |\Delta(4, 7)| = 3 \). This is a contradiction since \( \Delta(4, 7) = \{3\} \).

(iii.ii) Assume that \( \Delta(16, 1) = \Lambda_1 \) and \( \Delta(16, 4) = \Lambda_2 \). The proof in the case \( \Delta(16, 1) = \Lambda_2 \) and \( \Delta(16, 4) = \Lambda_1 \) is similar. Since \( \Delta(16, 2) = \Delta(16, 1)^* = \Lambda_1^* = \Lambda_2 \), \( \Delta(16, 18) \equiv 1, 2 \). Moreover by the incidence structure of \( P(x) \), \( \Delta(16, 18) \equiv 3 \). Thus \( \Delta(16, 18) = \{1, 2, 3\} \). Hence \( \Delta(16, 22) = \Delta(16, 18)^a = \{1, 2, 9\} \) and \( \Delta(16, 21) = \Delta(16, 22)^a = \{1, 2, 8\} \). On the other hand since \( \Delta(16, 1) = \Lambda_1 \) and \( \Delta(16, 18) = \{1, 2, 3\} \), there is an involution

\[ u = (16\ 18\ 1) (2\ 3) (21\ 22) \ldots \]

Then \( \Delta(16, 22) = \Delta(16, 21)^a \equiv 2^a = 3 \), which is a contradiction.

(iii.iii) Assume that \( \Delta(16, 1) = \Lambda_1 \) and \( \Delta(16, 4) = \Lambda_2 \). The proof in the case \( \Delta(16, 1) = \Lambda_2 \) and \( \Delta(16, 4) = \Lambda_1 \) is similar. Since \( \Delta(16, 2) = \Delta(16, 1)^* = \Lambda_1^* = \Lambda_2 \), \( \Delta(16, 18) = \{1, 2, 4\} \). Hence there is an involution \( y \) fixing 16, 1, 18. Then since \( \Delta(16, 1) = \Lambda_1 \),

\[ y = (1\ 16\ 18\ 21\ 22\ 2) \]

Hence by (3.18), \( (ab)^x = (ab)^{-1} \). Hence

\[ y = (1\ 2\ 4\ 16) (17) (18) (21\ 22) \ldots \]

Thus \( y \) fixes 16, 17. Hence \( y \) fixes \( \Delta(16, 17) \) and so \( y = (7\ 14\ 15) \ldots, (14\ 7\ 15) \ldots \) or \( (15\ 7\ 14) \ldots \). If \( y = (7\ 14\ 15) \ldots \), then \( I(xy) \supseteq \{16, 17, 18, 21, 22, 7, 14, 15\} \) and \( x \neq y \), which is a contradiction. Hence \( y = (14\ 7\ 15) \ldots \) or \( (15\ 7\ 14) \ldots \).

First assume that \( y = (14\ 7\ 15) \ldots \). Then

\[ by = (1\ 2\ 4\ 16\ 17\ 14) (7) (15) \ldots \]

Thus \( by \) is of order two. Furthermore \( \Delta(1, 2)^x = \Delta(1, 4) \) and \( \Delta(2, 4)^x = \Delta(2, 4) \).

Hence

\[ y = (1\ 2\ 4\ 8\ 10\ 3\ 11\ 5\ 9\ 12\ 6\ 13\ 14\ 7\ 15\ 16\ 17\ (18) (21\ 22) (20\ (19\ 23)) \]

Then

\[ xy = (16\ 17\ 18\ 21\ 22\ 7\ 15\ 14\ 1\ 4\ 2\ 3\ 11\ 12\ (19\ 23\ 20) (5\ 13\ 8) (6\ 9\ 10) \]
Next assume that \( y=(15) (7\ 14)\ldots \). Since
\[
b^a = (1) (2) (4) (9) (3\ 8) (11) (5\ 10) (13) (6\ 12) (15) (7\ 14) (19\ 21)
(18\ 23) (20\ 22) (16\ 17),
\]
\[
b^a y = (1) (2\ 4) (16\ 17) (15) (7) (14)\ldots .
\]
Thus \( b^a y \) is of order two. Hence from the same reason as above,
\[
y = (1) (2\ 4) (9\ 11) (3\ 10) (5\ 8) (13) (6\ 12) (15) (7\ 14) (16) (17) (18)
(21\ 22) (23) (19\ 20).
\]
Then
\[
xy = (16) (17) (18) (21) (22) (7\ 14\ 15) (1\ 4\ 2) (3\ 10\ 13) (19\ 20\ 23) (5\ 12\ 9)
(6\ 8\ 11).
\]

Now we show that \( \Delta(21, 22) = \{18\} \). Assume that \( y \) is of the first form. We have already proved that for \( I(\langle a, b \rangle) = \{1, 2, 4\} \) and \( I(ab) - I(\langle a, b \rangle) = \{16, 17\} \), \( \Delta(16, 17) = \{1, 2, 4\} \) or \( \Gamma_4 \) and \( \Delta(16, 1) = \{17\} \), \( \Delta(1, 2) \), \( \Delta(1, 4) \), \( \Lambda_1 \) or \( \Lambda_4 \), where \( \Gamma_4 \) is the \( \langle a, b \rangle \)-orbit of length three and different from \( \Delta(1, 2) \), \( \Delta(1, 4) \) and \( \Delta(2, 4) \) and \( \Lambda_1 \) or \( \Lambda_4 \) is the \( \langle a, b \rangle \)-orbit of length six. Since \( I(\langle x, y \rangle) = \{16, 17, 18\} \) and \( I(xy) - I(\langle x, y \rangle) = \{21, 22\} \), by the same argument for \( \langle x, y \rangle \) as is used for \( \langle a, b \rangle \), \( \Delta(21, 22) \) is \( \{16\} \), \( \{17\} \), \( \{18\} \) or the \( \langle x, y \rangle \)-orbit which is of length three and different from \( \Delta(16, 17) \), \( \Delta(16, 18) \) and \( \Delta(17, 18) \), and \( \Delta(21, 18) = \{22\} \), \( \Delta(18, 16) \), \( \Delta(18, 17) \), \( \{5, 13, 6\} \) or \( \{6, 9, 10\} \), where \( \Delta(16, 17) = \{7, 14, 15\} \), \( \Delta(16, 18) = \{1, 2, 4\} \) and \( \{5, 13, 6\} \cup \{6, 9, 10\} \) is the \( \langle x, y \rangle \)-orbit of length six. In the plane \( P(x) \), \( \Delta(16, 17) \) \( \exists \) 7, \( \Delta(16, 18) \) \( \exists \) 4 and \( \Delta(7, 4) \) \( \exists \) 3. Hence \( \Delta(17, 18) \) \( \exists \) 3. Since \( (17, 18)^{xy} = \Delta(17, 18) \), \( \Delta(17, 18) = \{3, 11, 12\} \). Thus \( \Delta(21, 22) = \{16\} \), \( \{17\} \), \( \{18\} \) or \( \{19, 23, 20\} \) and \( \Delta(21, 18) = \{22\} \), \{1, 2, 4\} and \{6, 10\}. On the other hand \( \Delta(21, 22) = \Delta(21, 18) \langle ab \rangle = \{18\} \), \{1, 2, 4\}, \{8, 5, 13\}, \{10, 9, 6\} or \{12, 3, 11\}. Thus \( \Delta(21, 22) = \{18\} \). Next assume that \( y \) is of the second form. Then \( \Delta(16, 17) \) and \( \Delta(16, 18) \) are the same set as above and \( \Delta(17, 18) = \{3, 10, 13\} \). Hence similarly \( \Delta(21, 22) = \{18\} \).

Assume that \( y \) is of the first form. The proof for the second form of \( y \) is similar. Since \( \Delta(21, 22) = \{18\} \), there is an involution \( u \) fixing 21, 22, 18 and commuting with \( x \). Then \( u \) fixes \( I(x) \). If \( u \) fixes 16 or 17, then \( xy \) fixes \( I(u) \), contrary to (3.4) Thus \( u = (16\ 17)\ldots \) or \( (16 \ i) (17 \ j)\ldots , i, j \in \{3, 4, 7, 19\} \).

First assume that \( u = (21) (22) (18) (16\ 17)\ldots \). Then by the incidence structure of \( P(x) \),
\[
u = (21) (22) (18) (16\ 17) (3\ 4) (17) (19)\ldots .
\]
Furthermore \( \Delta(16, 17)^w = \Delta(16, 17) \). Hence \( \{14, 15\}^w = \{14, 15\} \). If \( u \) fixes \{14, 15\} pointwise, then \( I(u) \cap I(u^a) = \{21, 22, 18, 7, 14, 15\} \), contrary to (vi) of (3.1). If \( u \) has a 2-cycle \( (14\ 15) \), then
\[
au = (16) (17) (7) (14) (15) (18\ 19)\ldots .
\]
Thus \( au \) is an involution. Then \( ab \) fixes the subset \{16, 17, 7, 14, 15\} of \( I(au) \). Hence \( ab \) fixes \( I(au) \), contrary to (3.4).

Next suppose that \( u=(21) (22) (18) (16 i) (17 j) \cdots, i, j \in \{3, 4, 7, 19\} \). Then by the incidence structure of \( P(x) \), \( u=(21) (22) (18) (16 3) (17 4) (7) (19) \cdots \) or \( (21) (22) (18) (16 4) (17 3) (7) (19) \cdots \). Then \( \Delta(16, 17)^u = \Delta(3, 4) \). This is a contradiction since \( |\Delta(16, 17)| = 3 \) and \( |\Delta(3, 4)| = 1 \).

(iv) Assume that \( \Delta(16, 17) = \{1\} \). The proof in the case \( \Delta(16, 17) = \{2\} \) or \( \{4\} \) is similar. We have proved that \( \Delta(16, 1) = \{17\}, \Gamma_1, \Gamma_2, \Lambda_1 \) or \( \Lambda_2 \), \( \Delta(16, 2) = \{17\}, \Gamma_1, \Gamma_2, \Lambda_1 \) or \( \Lambda_2 \). Hence by assumption, \( \Delta(16, 1) = \{17\}, \Delta(16, 2) \neq \{17\} \) and \( \Delta(16, 4) \neq \{17\} \). Furthermore since \( \Delta(17, 1) = \Delta(16, 1)^u, \Delta(17, 1) = \{16\} \). Suppose that \( \Delta(16, 2) = \Gamma_3 \). Since \( \Delta(17, 2) = \Delta(16, 2)^u, \Delta(17, 2) = \Gamma_3 \). Thus \( \Delta(2, 6) = \{4, 16, 17\} \). Hence there is an involution \( u \) fixing 2, 6, 16. Then since \( \Delta(16, 2) = \Gamma_3 \),

\[
u = (16) (2) (6) (12 13) (4 17) \cdots .
\]

Then by (3.18), \( (ab)^u = (ab)^{-1} \). Hence \( u \) fixes 1. Thus \( u \) fixes \( \Delta(16, 1) \). This is a contradiction since \( \Delta(16, 1) = \{17\}, \Gamma_1, \Gamma_2, \Lambda_1 \) or \( \Lambda_2 \). In the same way \( \Delta(16, 2) = \Gamma_3 \). Therefore \( \Delta(16, 2) = \Gamma_1, \Lambda_1 \) or \( \Lambda_2 \) and \( \Delta(16, 4) = \Gamma_1, \Lambda_1 \) or \( \Lambda_2 \).

Now let \( x \) be an involution fixing 16, 17 and commuting with \( a \). Since \( \Delta(16, 17) = \{1\}, x \) fixes 1. If \( x \) fixes 2 or 4, then \( ab \) fixes at least four points of \( I(x) \) and so fixes \( I(x) \), contrary to (3.4). Next if \( x \) fixes 3 or 5, then by the incidence structure of \( P(a) \), \( x \) fixes 2 or 4. This is a contradiction. Thus \( x \) fixes three points 1, 6, 7 of \( I(a) \). Hence \( x=(2 4) (3 5), (2 3) (4 5) \) or \( (2 5) (3 4) \) on \( I(a)-\{1, 6, 7\} \). First we determine the form of \( x \) in each cases.

(iv.i) Assume that \( x=(16) (17) (1) (6) (7) (2 4) (3 5) \cdots \). Then \( \Delta(1, 2)^x = \Delta(1, 4) \). Hence \( \{8, 9\}^x = \{10, 11\} \). If \( x=(8 10) (9 11) \cdots \), then

\[
abx = (1) (2 4) (16) (17) (3 11 8 5 9 10) \cdots ,
\]

contrary to (vi) of (3.1). Hence \( x=(8 11) (9 10) \cdots \). Then

\[
abx = (1) (2 4) (16) (17) (3 10) (8 5) (9 11) \cdots .
\]

This shows that \( abx \) is of order two and so \( (ab)^x = (ab)^{-1} \). Hence

\[
x = (1) (2 4) (3 5) (8 11) (9 10) (6) (12 13) (7) (14 15) (16) (17) \cdots .
\]

Since \( |I(x)| = 7 \), \( x \) fixes exactly two points of \( \{18, 19, \ldots , 23\} \). Without loss we may assume that \( x \) fixes 18, 19. Then

\[
x = (1) (2 4) (3 5) (8 11) (9 10) (6) (12 13) (7) (14 15) (16) (17) (18) (19) (21 22) (20 23) ,
\]

which is denoted by \( x_1 \).
(iv.ii) Assume that \(x = (16) (17) (1) (6) (7) (2 3) (4 5) \ldots\). Then \(\{1\}\) and \(\{16, 17\}\) are \(\langle a, b, x \rangle\)-orbits, and the remaining orbits are unions of \(\{2\} \cup \Gamma_x, \{4\} \cup \Gamma_x, \Gamma_x, \Gamma_x, \Gamma_x\) and \(\Lambda_1 \cup \Lambda_2\). If \(\{8, 9\}^x = \{12, 13\}\) or \(\{14, 15\}\) then \(\{2\} \cup \Gamma_x \cup \Gamma_x, \Gamma_x, \Gamma_x\) is a \(\langle a, b, x \rangle\)-orbit of length seven, contrary to (3.11). Thus \(\{8, 9\}^x = \{12, 13\}\), \(\{14, 15\}\). In the same way \(\{10, 11\}^x = \{12, 13\}\), \(\{14, 15\}\).

Next if \(\{8, 9, 10, 11\}^x = \Lambda_1 \cup \Lambda_2\), then \(\{2, 4\} \cup \Gamma_x \cup \Gamma_x \cup \Lambda_1 \cup \Lambda_2\) or \(\{2, 4\} \cup \Gamma_x \cup \Gamma_x \cup \Lambda_1 \cup \Lambda_2\) is a \(\langle a, b, x \rangle\)-orbit. Then \(\{2, 4\} \cup \Gamma_x \cup \Gamma_x \cup \Lambda_1 \cup \Lambda_2\) is a \(\langle a, b, x \rangle\)-orbit of length seven, contrary to (3.11) and the assumption (4) respectively. Thus \(\{8, 9\}^x = \{8, 9\}\), \(\{8, 9\}^x = \{10, 11\}\) or \(\{10, 11\}^x = \{10, 11\}\).

First assume that \(\{8, 9\}^x = \{8, 9\}\). If \(x\) fixes \(\{8, 9\}\) pointwise, then
\[
bx = (1) (8) (2 3 9) (16 17) \ldots,
\]
contrary to (vi) of (3.1). Hence \(x\) has a 2-cycle \((8 9)\). Then
\[
a = (1) (2 3) (4 5) (6) (7) (8) (9) (16 17) \ldots,
\]
\[
bax = (1) (2 3 9) (8) (16 17) (11 4 5) (13 6) (15 7) \ldots.
\]
Thus \(bax\) is of order three and so \(ax = (11) (10) (12 13) (14 15) \ldots\) or \((11) (10) (12 15) (13 14) \ldots\). If \(ax\) is of the first form, then \(x\) fixes \(\{1, 6, 7, 16, 17, 12, 13, 14, 15\}\) pointwise, contrary to (i) of (3.1). Thus \(ax\) is of the second form. Hence
\[
x = (1)(2 3) (4 5) (6) (7) (8 9) (10 11) (12 14) (13 15) (16) (17) \ldots.
\]
Then
\[
ax = (1) (2) (8) (9) (10 11) (12 14) (13 15) (16) (17) \ldots.
\]
This shows that \(bx\) is of order four. Furthermore since \(x\) fixes exactly two points of \(\{18, 19, \ldots, 23\}\), we may assume that \(x\) is of the form
\[
\]
which is denoted by \(x_2\).

Next assume that \(\{8, 9\}^x = \{10, 11\}\). Then \(x = (8 11) (9 10) \ldots\) or \((8 10) (9 11) \ldots\). If \(x\) is of the first form, then
\[
abx = (1) (2 3 4 5 8) (16) (17) \ldots,
\]
contrary to (iv) of (3.1). Thus \(x\) is of the second form. Then
\[
a = (1) (2 3) (4 5) (6) (7) (16 17) (8 11) (9 10) \ldots,
\]
\[
bax = (1) (2) (3 10 9) (4 5 8 11) (16) (17) \ldots,
\]
\[
(bax)^x = (1) (2 10) (3 9) (4 8) (5 11) (16) (17) \ldots.
\]
Thus \(bax\) is of order four and \((bax)^x\) is an involution commuting with \(b\) and \(ax\).
Since $I(b) = \{1, 2, 4, 8, 10, 12, 14\}$, $(bax)^2$ fixes $\{12, 14\}$ pointwise. Since $|I(bax)| = 3$, $bax = (1 2 14)\cdots$ and so $ax = (12 14)\cdots$. Hence


This implies that $x$ and $ax$ fix exactly two and four points of $\{18, 19, \ldots, 23\}$ respectively. Hence we may assume that $x$ is of the form


which is denoted by $x_3$.

Finally assume that $\{10, 11\} = \{10, 11\}$. If $x$ fixes $\{10, 11\}$ pointwise, then

$$bx = (1)(4 5 11)(10)(16 17)\cdots,$$

contrary to (v) of (3.1). Thus $x$ has a 2-cycle $(10 11)$. Then

$$ax = (1)(2 3)(4 5)(6)(7)(16 17)(10)(11)\cdots,$$


This shows that $bax$ is of order three. Hence $ax$ fixes 9 and so 8. Thus $x = x_2$.

(iv.iii) Assume that $x = (16)(17)(1)(6)(7)(2 5)(3 4)\cdots$. Then $\{1\}$ and $\{16, 17\}$ are $\langle a, b, x \rangle$-orbits and the remaining orbits are unions of $\{2\} \cup \Gamma_2$, $\{4\} \cup \Gamma_3$, $\Gamma_4$ and $\Lambda_1 \cup \Lambda_2$. Hence by the same argument as in the case (iv.ii), $x$ is one of the following elements:


We have proved that $\Delta(1, 2) = \Gamma_1$, $\Lambda_1$ or $\Lambda_2$ and $\Delta(16, 4) = \Gamma_2$, $\Lambda_1$ or $\Lambda_2$. In the following we treat these cases separately and show that there is no group which satisfies the assumption of Lemma 1 and contains $\langle a, b, x \rangle$, $1 < i < 5$.

(iv.iv) Assume that $\Delta(16, 2) = \Lambda_1$. The proof in the case $\Delta(16, 2) = \Lambda_2$, $\Delta(16, 4) = \Lambda_1$ or $\Delta(16, 4) = \Lambda_2$ is similar.

(iv.iv.i) Assume that there is an involution $x$. Then $\Delta(16, 4) = \Delta(16, 2)^x = \Lambda_1$. Hence $\Delta(16, 18) \ni \{2, 4\}$. Moreover in the plane $P(x)$ $\Delta(1, 6) \ni 7$ and $\Delta(1, 16) \ni 17$. Hence $(16, 18) \ni 6$ or 7. Thus $\Delta(16, 18) = \{2, 4, 6\}$ or $\{2, 4, 7\}$. If $\Delta(16, 18) = \{2, 4, 6\}$, then there is an involution $u$ fixing 16, 18, 2. Since $\Delta(16, 2) = \Lambda_1$,


Then $\Delta(16, 13) = \Delta(16, 4)^{uab} = \Lambda_{uab} = \Lambda_1$. Hence $\Delta(16, 18) \ni 13$, which is a contradiction. Similarly $\Delta(16, 18) \ni \{2, 4, 7\}$. Thus the proof in the case (iv.iv.i) is complete.
(iv.iv.ii) First assume that there is an involution $x_2$. Then $\Delta(16, 3) = \Delta(16, 2)^{x_2} = \Delta^{x_2} = \{18, 20, 23\}$. Hence $\Delta(16, 18) \ni \{2, 3\}$. Moreover in the plane $P(x_2)$ $\Delta(16, 18) \ni \{6\}$ or $\{7\}$. If $\Delta(16, 18) = \{2, 3, 6\}$, then there is an involution $u$ fixing 16, 18, 2. Since $\Delta(16, 2) = \Lambda$, 

$$u = (16) (18) (2) (21 22) (3 6) \cdots.$$ 

On the other hand $\Delta(16, 22) = \Delta(16, 18)^{x_2} = \{2, 9, 13\}$ and $\Delta(16, 21) = \Delta(16, 22)^{x_2} = \{2, 8, 12\}$. Then since $\Delta(16, 22)^{x_2} = \Delta(16, 21)$, $\{9, 13\}^{x_2} = \{8, 12\}$. If $u = (8 9) (12 13) \cdots$, then

$$abu = (16) (2) (22) (9) (6 12 3 8) \cdots,$$

contrary to (iii) of (3.1). Next if $u = (8 13) (9 12) \cdots$, then

$$au = (2) (3 6) (8 12) (9 13) \cdots.$$

Thus $au$ is of order two. Hence $I(u) \ni \{2, 18, 16\}$. Thus $I(u) \cap I(x_2) \ni \{16, 17, 18\}$ and $I(u) \neq I(x_2)$, contrary to (vi) of (3.1). Similarly $\Delta(16, 18) \ni \{2, 3, 7\}$. Thus there is no group containing $x_2$.

Next assume that there is the involution $x_3$. Since $x_3$ is the same permutation as $x_2$ with 3 and 5 interchanged, by the same argument as is used for $x_2$ there is no group containing $x_3$.

(iv.iv.iii) First assume that there is the involution $x_3$. Since

$$abx_3 = (1) (2 3 11 8) (4 5 9 10) (6 14 7 12) (13 15) (16) (17) (18 23 19 21) (20 22),$$

$\Delta(16, 11) = \Delta(16, 2)^{abx_3} = \{19, 22, 23\}$ and $\Delta(16, 8) = \Delta(16, 11)^{abx_3} = \{21, 10, 19\}$. Hence $\Delta(16, 19) \ni \{11, 18\}$. Since $x_3$ fixes $\Delta(16, 19)$, $\Delta(16, 19) \ni \{11, 8\}^{x_3} = \{11, 9, 8, 10\}$, which is a contradiction.

Next assume that there is the involution $x_4$. Since $x_3$ is the same permutation as $x_4$ with 3 and 5 interchanged, by the same argument as is used for $x_3$ there is no group containing $x_4$.

(iv.v) Assume that $\Delta(16, 2) = \Gamma_1$ and $(16, 4) = \Gamma_2$. Then $\Delta(17, 2) = \Delta(16, 2)^{x_4} = \Gamma_1$. Hence $\Delta(2, 3) = \{1, 16, 17\}$ and consequently there is an involution $u$ fixing 2, 3, 16. Since $\Delta(16, 2) = \Gamma_1$,

$$u = (16) (2) (3) (8 9) (1 17) \cdots.$$

Then since $\Delta(16, 2) = \Gamma_1$, $(ab)^u = (ab)^{-1}$ by (3.18). Hence $u$ fixes 4. Then

$$au = (2) (4) (3) (9) (1 17 16) \cdots.$$

Thus $au$ is of order three. In the cases (i), (ii) and (iii) we have proved that for $I(<a, b>) = \{1, 2, 4\}$ and $I(ab) = I(<a, b>) = \{16, 17\}$, $\Delta(16, 17) = \{1\}$, $\{2\}$ or $\{4\}$. Since $I(<a, u>) = \{2, 3, 4\}$ and $I(ab) = I(<a, b>) = \{8, 9\}$, by the same
argument for \( \langle a, u \rangle \) as is used for \( \langle a, b \rangle \) \( \Delta(8, 9) = \{2\}, \{3\} \) or \( \{4\} \). If \( \Delta(8, 9) = \{2\} \), then \( \Delta(2, 8) \supseteq 9 \). Since \( b \) fixes \( \Delta(2, 8) \), \( \Delta(2, 8) \supseteq 9 \). Thus \( \Delta(2, 8) \supseteq \{3, 9\} \) and so \( |\Delta(2, 8)| = 3 \). However since \( au \) fixes \( \Delta(2, 8) \) and \( \Delta(2, 8) \cap I(au) = \{3, 9\}, |\Delta(2, 8)| = 3 \). Thus we have a contradiction. In the same way \( \Delta(8, 9) = \{3\} \). Thus \( \Delta(8, 9) = \{3\} \).

Now there is an involution \( v \) fixing 8, 9 and commuting with \( a \). Then \( v \) fixes 3 and \( I(a) \). If \( v \) fixes 2 or 4, then \( au \) fixes at least four points of \( I(v) \). Hence \( au \) fixes \( I(v) \), contrary to (3.4). Hence \( v = (8) (9) (3) (5) (6) \). Since \( av = va \) and \( I(\langle a, v \rangle) = \{3, 5, 6\} \), by (3.19) \( \Delta(3, 5) = \{6\} \). If there is the involution \( x = (2 \leq i \leq 5) \), then \( \Delta(3, 5) = \{\Delta(2, 4) \}. This is a contradiction since \( |\Delta(2, 4)| = 3 \). Next suppose that there is the involution \( x \). Since \( u \) fixes \( \{16, 4\} \) pointwise and \( \Delta(16, 4) = \Gamma \), \( u = (5) (10) (11) \cdots, (10) (5) (11) \cdots \) or \( (11) (5) (10) \cdots \). If \( u = (5) (10) \cdots \), then
\[
au = (2) (4) (3) (8) (9) (5) (10) (11) (17) (16) \cdots ,
\]
contrary to (ii) of (3.1). Next suppose that \( u = (10) (5) (11) \cdots \). Then
\[
bu = (2) (4) (17) (16) (3) (8) (9) (5) (10) (11) \cdots .
\]
Thus \( bu \) is of order three. Then since \( \Delta(2, 4) = \Delta(2, 4) \) and \( |I(au)| = |I(bu)| = 5 \),
\[
\]
Thus \( \Delta(5, 8) = \Delta(3, 5) = \{6\} \). Since \( v \) fixes 5, 8, \( v \) fixes 13 and so 12. Thus \( I(v) \cap I(v^5) = \{3, 5, 6, 12, 13\} \), contrary to (vi) of (3.1). Finally suppose that \( u = (11) (5) (10) \cdots \). Then
\[
b^5u = (2) (4) (17) (16) (3) (9) (8) (5) (10) (11) \cdots .
\]
Hence by the same argument as above we have a contradiction.

Thus the proof of Lemma 1 is complete.

**Remark.** Let \( G \) be one of the groups of Lemma 1. Then it is not difficult to prove that \( G \) satisfies the assumptions of Lemma 1.

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**References**


