Endo, S. and Miyata, T.
Osaka J. Math.
13 (1976), 109–122

ON THE PROJECTIVE CLASS GROUP OF FINITE GROUPS

Dedicated to Professor Kiiti Morita on his 60th birthday

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(Received October 17, 1974)
(Revised May 19, 1975)

In this paper we will continue the investigation of integral representations of finite groups done in [3], [4] and [5]. We will here be concerned mainly with the projective class group of nilpotent and symmetric groups.

Let Σ be a (finite dimensional) semi-simple Q-algebra and let Λ be a Z-order in Σ. We will mean by the projective class group of Λ the class group defined by using all locally free, projective Λ-modules and denote it by $C(\Lambda)$.

Let Π be a finite group. A finitely generated Z-free Π-module is briefly called a Π-module. A Π-module is called a permutation Π-module if it can be expressed as a direct sum of $\{Z\Pi/\Pi_i\}$ where each Π_i is a subgroup of Π. Further a Π-module M is called a quasi-permutation Π-module if there exists an exact sequence: $0\to M\to S\to S'\to 0$ where S and S' are permutation Π-modules.

As is well known, the projective class group $C(Z\Pi)$ of the group algebra $Z\Pi$ can be written as follows:

$$C(Z\Pi)=\{[\mathfrak{A}]-[Z\Pi]\mid \mathfrak{A}(\neq 0) \text{ is a projective ideal of } Z\Pi\} .$$

We define the subgroups $\tilde{C}(Z\Pi)$, $C^q(Z\Pi)$ and $\tilde{C}^q(Z\Pi)$ of $C(Z\Pi)$ as follows:

$$\tilde{C}(Z\Pi)=\{[\mathfrak{A}]-[Z\Pi]\in C(Z\Pi)\mid \mathfrak{A}\oplus X\cong Z\Pi\oplus X \text{ for some } \Pi\text{-module } X\},$$

$$C^q(Z\Pi)=\{[\mathfrak{A}]-[Z\Pi]\in C(Z\Pi)\mid \mathfrak{A}\oplus S_1\cong S_2 \text{ for some permutation } \Pi\text{-modules } S_1 \text{ and } S_2\},$$

$$\tilde{C}^q(Z\Pi)=\{[\mathfrak{A}]-[Z\Pi]\in C(Z\Pi)\mid \mathfrak{A}\oplus S\cong Z\Pi\oplus S \text{ for some permutation } \Pi\text{-module } S\}.$$

Let Ω_Π be a maximal Z-order in $Q\Pi$ containing $Z\Pi$ and let $\psi_\Pi\colon C(Z\Pi)\to C(\Omega_\Pi)$ be the epimorphism induced by $\Omega_\Pi\otimes_{Z\Pi}\cdot$. Then the sequence $0\to\tilde{C}(Z\Pi)\to C(Z\Pi)\xrightarrow{\psi_\Pi}C(\Omega_\Pi)\to 0$ is exact.

In [3] and [4] we raised the following problem:

'*For a finite group* Π $\tilde{C}(Z\Pi) = C^q(Z\Pi)(=\tilde{C}^q(Z\Pi))$?'

and showed that the answer to this is affirmative for a fairly extensive class of finite groups but it is negative for the alternating group on 8 symbols.

In §2 we give

[I] *If* Π *is a finite nilpotent group, then* $\tilde{C}^q(Z\Pi)=\tilde{C}(Z\Pi)=C^q(Z\Pi)$.

A finite group Π is said to be of split type over Q if every simple component of $Q\Pi$ is isomorphic to a full matrix algebra over its center. In the previous paper [4] we proved the assertion [I] under the additional assumption that Π is of split type over Q. We will prove [I], using the Mayer-Vietoris sequence in algebraic K-theory ([1]).

Let S_n, A_n denote the symmetric, alternating group on n symbols, respectively. In §3 we give

[II] $\tilde{C}^q(ZS_n) = \tilde{C}(ZS_n) = C^q(ZS_n) = C(ZS_n)$ *for any* $n \geqq 1$.

Let $G(Q\Pi)$ be the Grothendieck group of the category of all finitely generated $Q\Pi$-modules and define $B(Q\Pi)$ to be the subring of $G(Q\Pi)$ generated by all the classes of permutation $Q\Pi$-modules. It is well known that $B(QS_n)=G(QS_n)$ for any $n\geqq 1$. However the following result on the alternating group, which will be proved in §4, seems new.

[III] $B(QA_n) = G(QA_n)$ for any $n \geqq 3$.

We would like to express our gratitude to H. Hijikata for his valuable suggestion, and to I. Reiner and S. Ullom for their helpful comments.

1. Some lemmas on special elementary groups

Let C_{2^l}, $l\geqq 0$, be the cyclic group of order 2^l, i.e., $C_{2^l}=\langle\sigma\,|\,\sigma^{2^l}=\mathrm{I}\rangle$. Let H_{2^l}, $l\geqq 2$, be the (generalized) quaternion group of order 2^{l+1}, i.e., $H_{2^l}=\langle\sigma, \tau\,|\,\sigma^{2^l}=\mathrm{I}, \sigma^{2^{l-1}}=\tau^2, \tau^{-1}\sigma\tau=\sigma^{-1}\rangle$ and let D_{2^l}, $l\geqq 2$, be the dihedral group of order 2^{l+1}, i.e., $D_{2^l}=\langle\sigma, \tau\,|\,\sigma^{2^l}=\tau^2=\mathrm{I}, \tau^{-1}\sigma\tau=\sigma^{-1}\rangle$. Define the groups SD_{2^l} and SC_{2^l}, $l\geqq 3$, of order 2^{l+1} by $SD_{2^l}=\langle\sigma, \tau\,|\,\sigma^{2^l}=\tau^2=\mathrm{I}, \tau^{-1}\sigma\tau=\sigma^{-1+2^{l-1}}\rangle$ and $SC_{2^l}=\langle\sigma, \tau\,|\,\sigma^{2^l}=\tau^2=\mathrm{I}, \tau^{-1}\sigma\tau=\sigma^{1+2^{l-1}}\rangle$.

Let H denote one of the groups C_{2^l}, H_{2^l}, D_{2^l}, SD_{2^l} and SC_{2^l}. Define $\Sigma(H)=QH/(\sigma^{2^{l-1}}+\mathrm{I})$ and $\Lambda(H)=ZH/(\sigma^{2^{l-1}}+\mathrm{I})$ and denote the images of σ and τ in $\Lambda(H)$ by x and y, respectively. Put

$$K(H) \text{ (resp. } R(H)) = \begin{cases} Q(x) & \text{(resp. } Z[x]) \quad \text{when } H=C_{2^l} \\ Q(x+x^{-1}) & \text{(resp. } Z[x+x^{-1}]) \quad \text{when } H=H_{2^l} \text{ or } D_{2^l} \\ Q(x-x^{-1}) & \text{(resp. } Z[x-x^{-1}]) \quad \text{when } H=SD_{2^l} \\ Q(x^2) & \text{(resp. } Z[x^2]) \quad \text{when } H=SC_{2^l}\,. \end{cases}$$

Then $\Sigma(H)$ is a central simple $K(H)$-algebra and is the unique H-faithful simple component of QH, and $\Lambda(H)$ is an $R(H)$-order in $\Sigma(H)$. Further let

$$\alpha_H = \begin{cases} 2 & \text{when } H=C_1=\{I\} \\ x-1 & \text{when } H=C_{2^l},\ l\geqq 1 \\ x+x^{-1}-2 & \text{when } H=H_{2^l} \text{ or } D_{2^l} \\ x-x^{-1} & \text{when } H=SD_{2^l} \\ x^2-1 & \text{when } H=SC_{2^l} \end{cases}$$

and put $\mathfrak{p}(H)=\alpha_H R(H)$. Then $\mathfrak{p}(H)$ is the unique prime ideal of $R(H)$ containing 2 and $R(H)/\mathfrak{p}(H)\simeq Z/2Z$.

Let K be an algebraic number field and let Σ be a central simple K-algebra. We say Σ to be of split type if it is isomorphic to a full matrix algebra over K. For a (finite or infinite) prime $\mathfrak{p}$ of K we denote by $\hat{K}_{\mathfrak{p}}$ the completion of K at $\mathfrak{p}$ and put $\hat{\Sigma}_{\mathfrak{p}}=\hat{K}_{\mathfrak{p}}\otimes_K \Sigma$. We say Σ to be of locally split type if, for every finite prime $\mathfrak{p}$ of K, $\hat{\Sigma}_{\mathfrak{p}}$ is isomorphic to a full matrix algebra over $\hat{K}_{\mathfrak{p}}$.

Lemma 1.1. (1) *If $H=C_{2^l}$, D_{2^l}, SD_{2^l} or SC_{2^l}, $\Sigma(H)$ is of split type.* (2) *$\Sigma(H_{2^l})$ is of locally split type if and only if $l\geqq 3$.*

Proof. The assertion (1) is evident and the assertion (2) may be well known. But for completeness we here give a proof of (2). It is noted that $\Sigma(H_{2^l})$ is the quaternion algebra over the real field $K(H_{2^l})$. Accordingly, for a prime $\mathfrak{p}$ of $K(H_{2^l})$, $\widehat{\Sigma(H_{2^l})}_{\mathfrak{p}}=M_2(\widehat{K(H_{2^l})}_{\mathfrak{p}})$ if and only if the equation $X^2+Y^2+1=0$ has a solution in $\widehat{K(H_{2^l})}_{\mathfrak{p}}$, i.e., if and only if $\left(\frac{-1,\,-1}{\mathfrak{p}}\right)=1$. For every finite prime $\mathfrak{p}$ of $K(H_{2^l})$ with $\mathfrak{p}\neq\mathfrak{p}(H_{2^l})$ we have $\left(\frac{-1,\,-1}{\mathfrak{p}}\right)=1$. On the other hand, for every real prime $\mathfrak{p}$ of $K(H_{2^l})$ we have $\left(\frac{-1,\,-1}{\mathfrak{p}}\right)=-1$. All infinite primes of $K(H_{2^l})$ are real and the number of them is 2^{l-2}. Since $\prod_{\mathfrak{p}}\left(\frac{-1,\,-1}{\mathfrak{p}}\right)=1$ where $\mathfrak{p}$ runs over all primes of $K(H_{2^l})$, we see that $\left(\frac{-1,\,-1}{\mathfrak{p}(H_{2^l})}\right)=1$ if and only if $l\geqq 3$.

For any positive integer n we denote by $\Phi_n(t)$ the n-th cyclotomic polynomial and by ζ_n a primitive n-th root of 1.

From now we assume that $m\geqq 1$ is an odd integer. Let C_m be the cyclic group of order m, i.e., $C_m=\langle\mu\,|\,\mu^m=I\rangle$. Define $K(C_m)=QC_m/(\Phi_m(\mu))=Q(\zeta_m)$ and $R(C_m)=ZC_m/(\Phi_m(\mu))=Z[\zeta_m]$. A finite group E is said to be a special elementary group if $E=C_m\times H$ where $H=C_{2^l}$, H_{2^l}, D_{2^l}, SD_{2^l} or SC_{2^l}. Let $E=C_m\times H$ where $H=C_{2^l}$, H_{2^l}, D_{2^l}, SD_{2^l} or SC_{2^l}. Define $\Sigma(E)=K(C_m)\otimes_Q \Sigma(H)$

$=QE/(\Phi_m(\mu)), \Phi_{2^l}(\sigma))$ and $\Lambda(E)=R(C_m)\underset{Z}{\otimes}\Lambda(H)=ZE/(\Phi_m(\mu), \Phi_{2^l}(\sigma))$, and further put $K(E)=K(C_m)\underset{Q}{\otimes}K(H)$ and $R(E)=R(C_m)\underset{Z}{\otimes}R(H)$. Since m is odd, $K(E)$ is a field and $R(E)$ is the ring of all algebraic integers in $K(E)$. We see that $\Sigma(E)$ is a central simple $K(E)$-algebra and is the unique E-faithful simple component of QE and that $\Lambda(E)$ is an $R(E)$-order in $\Sigma(E)$.

Lemma 1.2. *For any special elementary group E, $\Lambda(E)$ is a quasi-permutation E-module.*

Proof. Let $E=C_m\times H$ where $H=C_{2^l}, H_{2^l}, D_{2^l}, SD_{2^l}$ or SC_{2^l}. Then we have $\Lambda(E)=ZE/(\Phi_{2^l m}(\sigma\mu))$. Hence we can prove the assertion by the argument using a zigzag path as in the proof of [3], (2.3).

Lemma 1.3. *Let $E=C_m\times H$ where $H=H_{2^l}, D_{2^l}$ or SD_{2^l}. Let $\Omega(E)$ be a maximal $R(E)$-order in $\Sigma(E)$ containing $\Lambda(E)$. Then $\alpha_H\Omega(E)\subseteq\Lambda(E)$.*

Proof. For brevity we write $K=K(E)$ and $R=R(E)$. Now we have $\Sigma(E)=K+Kx+Ky+Kxy$ and $\Lambda(E)=R+Rx+Ry+Rxy$. Assume that $H=H_{2^l}$. Let $z=x^{2^{l-2}}$. Then $\Sigma(E)=K+Kz+Ky+Kzy$, and $z^2=y^2=-1$ and $zy+yz=0$. Denote by trd the reduced trace of $\Sigma(E)$. We note that, for any element $v=a_1+a_2z+a_3y+a_4zy$ of $\Sigma(E)$, $a_i\in K$, we have $\mathrm{trd}(v)=2a_1$. Then we can find the K-basis of $\Sigma(E)$ which is dual to $\{1, x, y, xy\}$ with respect to trd as follows: $u_1=\dfrac{x^{-2}-1}{x^2+x^{-2}-2}$, $u_2=\dfrac{x-x^{-1}}{x^2+x^{-2}-2}$, $u_3=\dfrac{-(x^2-1)y}{x^2+x^{-2}-2}$, $u_4=\dfrac{-(x-x^{-1})y}{x^2+x^{-2}-2}$. It is easy to see that $\alpha_H u_i\in\Lambda(H)_2$ for $1\leqq i\leqq 4$. Since $\mathrm{trd}(\Omega(E))\subseteq R$, we have $\Omega(E)\subseteq Ru_1+Ru_2+Ru_3+Ru_4$ and hence $\alpha_H\Omega(E)_2\subseteq\Lambda(E)_2$. It is obvious that $\Omega(E)_p=\Lambda(E)_p$ for any prime $p\neq 2$. Thus we have $\alpha_H\Omega(E)\subseteq\Lambda(E)$.

For the case where $H=D_{2^l}$ or SD_{2^l} we can prove the assertion in a similar manner.

We here consider the case where $E=C_m\times H_4$. Let $u=\dfrac{1}{2}(1+x+y+xy)\in\Sigma(H_4)(\subseteq\Sigma(E))$ and put $\Gamma(E)=\Lambda(E)+R(C_m)u$. Let $\mathfrak{c}(E)=\Gamma(E)(1+x)(=(1+x)\Gamma(E))$.

Lemma 1.4. (1) *$\mathfrak{c}(C_m\times H_4)\subseteq\Lambda(C_m\times H_4)$ and $\Gamma(C_m\times H_4)/\mathfrak{c}(C_m\times H_4)\cong Z/2Z\underset{Z}{\otimes}Z[\zeta_m]\underset{Z}{\otimes}Z[\zeta_3]$.* (2) *$\Gamma(C_m\times H_4)$ is a hereditary $R(C_m)$-order in $\Sigma(C_m\times H_4)$.*

Proof. (1) It is evident that $\mathfrak{c}(C_m\times H_4)\subseteq\Lambda(C_m\times H_4)$. Hence we have only to prove the second assertion. Now it suffices to show that $\Gamma(H_4)/\mathfrak{c}(H_4)\cong(Z/2Z)[X]/(X^2+X+1)$, because $\Gamma(C_m\times H_4)=Z[\zeta_m]\underset{Z}{\otimes}\Gamma(H_4)$ and $\mathfrak{c}(C_m\times H_4)=Z[\zeta_m]\underset{Z}{\otimes}\mathfrak{c}(H_4)$. Define the ring homomorphism $f\colon \Gamma(H_4)\to(Z/2Z)[X]/(X^2+X+1)$ by $f(1)=f(x)=f(y)=\bar{1}$ and $f(u)=\bar{X}$ where $\bar{X}$ denotes the image of

X in $(Z/2Z)[X]/(X^2+X+1)$. It is easy to see that f is an epimorphism and $\mathrm{Ker}\, f=\mathfrak{c}(H_4)$. Therefore f induces an isomorphism $\bar{f}: \Gamma(H_4)/\mathfrak{c}(H_4)\to (Z/2Z)[X]/(X^2+X+1)$. (2) Let $\mathfrak{p}$ be a prime ideal of $R(C_m)$. If $2\in\mathfrak{p}$, it follows from (1) that $\mathfrak{c}(C_m\times H_4)_\mathfrak{p}$ coincides with the Jacobson radical of $\Gamma(C_m\times H_4)_\mathfrak{p}$. Since $\mathfrak{c}(C_m\times H_4)_\mathfrak{p}$ is principal in $\Gamma(C_m\times H_4)_\mathfrak{p}$, $\Gamma(C_m\times H_4)_\mathfrak{p}$ is a hereditary $R(C_m)_\mathfrak{p}$-order in $\Sigma(C_m\times H_4)$. On the other hand, if $2\notin\mathfrak{p}$, then $\mathfrak{p}$ is unramified in $\Gamma(C_m\times H_4)$ and so $\Gamma(C_m\times H_4)_\mathfrak{p}$ is a maximal $R(C_m)_\mathfrak{p}$-order in $\Sigma(C_m\times H_4)$. Consequently $\Gamma(C_m\times H_4)$ is a hereditary $R(C_m)$-order in $\Sigma(C_m\times H_4)$.

2. Nilpotent groups

We state without proof a result due to J. Milnor which will play an essential part in this section.

Proposition 2.1 ([1], X, (1.10)). *Let Σ be a semi-simple Q-algebra and let Λ, Γ be Z-orders in Σ with $\Lambda\subseteq\Gamma$. Let $\mathfrak{c}$ be a two-sided ideal of Γ contained in Λ such that $\mathfrak{c}\Sigma=\Sigma$. Then there exists an exact (Mayer-Vietoris) sequence:*

$$K_1(\Lambda)\to K_1(\Gamma)\oplus K_1(\Lambda/\mathfrak{c})\to K_1(\Gamma/\mathfrak{c})\to K_0(\Lambda)\to K_0(\Gamma)\oplus K_0(\Lambda/\mathfrak{c})\to K_0(\Gamma/\mathfrak{c})\,.$$

Let Σ be a semi-simple Q-algebra and let Λ, Γ be Z-orders in Σ with $\Lambda\subseteq\Gamma$. Let ψ^Γ_Λ: $C(\Lambda)\to C(\Gamma)$ denote the natural epimorphism induced by $\Gamma\underset{\Lambda}{\otimes}\cdot$. For any ring A we denote by $U(A)$ the group of all units of A.

In the following proposition we use the same notation as in §1.

Proposition 2.2. *Let $E=C_m\times H$ be any special elementary group. Let $\Omega(E)$ be a maximal $R(E)$-order in $\Sigma(E)$ containing $\Lambda(E)$. Then the map $\psi^{\Omega(E)}_{\Lambda(E)}$: $C(\Lambda(E))\to C(\Omega(E))$ is an isomorphism.*

Proof. In the case where $H=C_{2^l}$ this is obvious. We first assume that $H\neq H_4$, C_{2^l}, SC_{2^l} or that $H=H_4$ and $Q(\zeta_m)$ is a splitting field for H_4. By (1.3) we have $\alpha_H\Omega(E)\subseteq\Lambda(E)$, and therefore we can apply (2.1) to $\Lambda(E)$, $\Omega(E)$, $\alpha_H\Omega(E)$. Then we get the exact sequence: $K_1(\Omega(E))\oplus K_1(\Lambda(E)/\alpha_H\Omega(E))\xrightarrow{f} K_1(\Omega(E)/\alpha_H\Omega(E))\xrightarrow{g} K_0(\Lambda(E))\xrightarrow{h} K_0(\Omega(E))\oplus K_0(\Lambda(E)/\alpha_H\Omega(E))$. Since, by (1.1), $\Sigma(E)$ is of locally split type, we have $\Omega(E)/\alpha_H\Omega(E)\cong M_2(R(E)/\alpha_H R(E))$ and so $K_1(\Omega(E)/\alpha_H\Omega(E))\cong U(R(E)/\alpha_H R(E))$. The inclusion map $R(E)/\alpha_H R(E)\subseteq\Lambda(E)/\alpha_H\Omega(E)\subseteq\Omega(E)/\alpha_H\Omega(E)$ induces a homomorphism ϕ: $U(R(E)/\alpha_H R(E))\to K_1(\Lambda(E)/\alpha_H\Omega(E))\to K_1(\Omega(E)/\alpha_H\Omega(E))=U(R(E)/\alpha_H R(E))$. Then it is easy to see that $\mathrm{Im}\,\phi=U(R(E)/\alpha_H R(E))^2$. However, since $R(E)/\alpha_H R(E)=Z[\zeta_m]/2Z[\zeta_m]$, the order of $U(R(E)/\alpha_H R(E))$ is odd, hence $U(R(E)/\alpha_H R(E))^2=U(R(E)/\alpha_H R(E))$. Therefore ϕ is an epimorphism and then so is f. Since $\mathrm{Ker}\,\psi^{\Omega(E)}_{\Lambda(E)}=\mathrm{Ker}\,h=\mathrm{Im}\,g$, this implies that $\psi^{\Omega(E)}_{\Lambda(E)}$: $C(\Lambda(E))\to C(\Omega(E))$ is an isomorphism.

Next assume that $H=SC_{2^l}$. In this case we have $\Sigma(E)\cong M_2(K(E))$. Define $\Omega'(E)=\mathrm{End}_{R(E)}(\Lambda(E)(y+1))\cong M_2(R(E))$. Then we can regard $\Omega'(E)$ as a maximal $R(E)$-order in $\Sigma(E)$ containing $\Lambda(E)$. Because $C(\Omega(E))\cong C(\Omega'(E))$ we may assume that $\Omega(E)=\Omega'(E)$. Now we have $\Lambda(E)=\left\{\begin{bmatrix} a+b & c+d \\ (c-d)x^2 & a-b\end{bmatrix}\middle| a, b, c, d\in R(E)\right\}\subseteq\Omega(E)=M_2(R(E))$. Hence $2\Omega(E)\subseteq\Lambda(E)$ and $\Lambda(E)/2\Omega(E)$ $=\left\{\begin{bmatrix} \bar{a} & \bar{b} \\ \bar{b}\bar{x}^2 & \bar{a}\end{bmatrix}\middle| \bar{a}, \bar{b}\in R(E)/2R(E)\right\}$. Applying (2.1) to $\Lambda(E)$, $\Omega(E)$, $2\Omega(E)$, we get the exact sequence: $K_1(\Omega(E))\oplus K_1(\Lambda(E)/2\Omega(E))\xrightarrow{f}K_1(\Omega(E)/2\Omega(E))\xrightarrow{g}K_0(\Lambda(E))$ $\xrightarrow{h}K_0(\Omega(E))\oplus K_0(\Lambda(E)/2\Omega(E))$. Since $\Omega(E)/2\Omega(E)=M_2(R(E)/2R(E))$, we have $K_1(\Omega(E)/2\Omega(E))\cong U(R(E)/2R(E))$. We see that the composed map $U(\Lambda(E)/2\Omega(E))\to K_1(\Lambda(E)/2\Omega(E))\to K_1(\Omega(E)/2\Omega(E))\cong U(R(E)/2R(E))$ coincides with the determinant map det: $U(\Lambda(E)/2\Omega(E))$ $(\subseteq M_2(R(E)/2R(E))\to U(R(E)/2R(E))$. As in the preceding case, in order to show that $\psi^{\Omega(E)}_{\Lambda(E)}\colon C(\Lambda(E))\to C(\Omega(E))$ is an isomorphism, it suffices to show that det: $U(\Lambda(E)/2\Omega(E))\to U(R(E)/2R(E))$ is an epimorphism. Let be $\bar{\alpha}$ the image of $\alpha_H=x^2-1$ in $R(E)/2R(E)$ and let $t=2^{l-3}$. Let $\bar{u}$ be any element of $U(R(E)/2R(E))$. Then we can write $\bar{u}=\bar{a}_0+\bar{a}_1\bar{\alpha}+\bar{a}_2\bar{\alpha}^2+\cdots+\bar{a}_{2t-1}\bar{\alpha}^{2t-1}$, $\bar{a}_i\in Z[\zeta_m]/2Z[\zeta_m]$. Since m is odd, there exist $\bar{b}_i, \bar{c}_i\in Z[\zeta_m]/2Z[\zeta_m]$ such that $(\bar{b}_0+\bar{b}_1\bar{\alpha}+\cdots+\bar{b}_{t-1}\bar{\alpha}^{t-1})^2=\bar{a}_0+\bar{a}_2\bar{\alpha}^2+\cdots+\bar{a}_{2t-2}\bar{\alpha}^{2t-2}$ and $(\bar{c}_0+\bar{c}_1\bar{\alpha}+\cdots+\bar{c}_{t-1}\bar{\alpha}^{t-1})^2=\bar{a}_1+\bar{a}_3\bar{\alpha}^2+\cdots+\bar{a}_{2t-1}\bar{\alpha}^{2t-2}$. Let $\bar{a}=(\bar{b}_0+\bar{c}_0)+(\bar{b}_1+\bar{c}_1)\bar{\alpha}+\cdots+(\bar{b}_{t-1}+\bar{c}_{t-1})\bar{\alpha}^{t-1}$ and $\bar{b}=\bar{c}_0+\bar{c}_1\bar{\alpha}+\cdots+\bar{c}_{t-1}\bar{\alpha}^{t-1}$. Then we have $\bar{u}=\bar{a}^2+\bar{b}^2\bar{x}^2=\det\begin{bmatrix} \bar{a} & \bar{b} \\ \bar{b}\bar{x}^2 & \bar{a}\end{bmatrix}$. This proves that det: $U(\Lambda(E)/2\Omega(E))$ $\to U(R(E)/2R(E))$ is an epimorphism.

Finally we will treat the case where $H=H_4$. We have $C(\Omega(E))\cong C(\Omega'(E))$ for any other maximal order $\Omega'(E)$ in $\Sigma(E)$ containing $\Lambda(E)$. Hence we may assume that $\Gamma(E)\subseteq\Omega(E)$. By (1.4) $\Gamma(E)$ is a hereditary order in $\Sigma(E)$ and so, according to [4], (2.4), $\psi^{\Omega(E)}_{\Gamma(E)}\colon C(\Gamma(E))\to C(\Omega(E))$ is an isomorphism. Because $\psi^{\Omega(E)}_{\Lambda(E)}=\psi^{\Omega(E)}_{\Gamma(E)}\cdot\psi^{\Gamma(E)}_{\Lambda(E)}$, $\psi^{\Omega(E)}_{\Lambda(E)}$ is an isomorphism if and only if $\psi^{\Gamma(E)}_{\Lambda(E)}$ is an isomorphism. If $Q(\zeta_m)$ is a splitting field for H_4, it has already been shown that $\psi^{\Omega(E)}_{\Lambda(E)}$ is an isomorphism, and hence $\psi^{\Gamma(E)}_{\Lambda(E)}$ is also an isomorphism. Assume that $Q(\zeta_m)$ is not a splitting field for H_4. Now it suffices to show that $\psi^{\Gamma(E)}_{\Lambda(E)}\colon C(\Lambda(E))\to C(\Gamma(E))$ is an isomorphism. Applying (2.1) to $\Lambda(E)$, $\Gamma(E)$, $\mathfrak{c}(E)$, we get the exact sequence: $K_1(\Gamma(E))\oplus K_1(\Lambda(E)/\mathfrak{c}(E))\xrightarrow{f}K_1(\Gamma(E)/\mathfrak{c}(E))\xrightarrow{g}$ $K_0(\Lambda(E))\xrightarrow{h}K_0(\Gamma(E))\oplus K_0(\Lambda(E)/\mathfrak{c}(E))$. Since by (1.4) $\Gamma(E)/\mathfrak{c}(E)\cong Z/2Z\otimes_Z Z[\zeta_m]\otimes_Z Z[\zeta_3]$, the order of $U(\Gamma(E)/\mathfrak{c}(E))$ is odd and $K_1(\Gamma(E)/\mathfrak{c}(E))\cong U(\Gamma(E)/\mathfrak{c}(E))$. Therefore the order of $\mathrm{Ker}\, h=\mathrm{Im}\, g$ is odd. Because $\mathrm{Ker}\,\psi^{\Gamma(E)}_{\Lambda(E)}=\mathrm{Ker}\, h$, it follows that the order of $\mathrm{Ker}\,\psi^{\Gamma(E)}_{\Lambda(E)}$ is odd. It is well known that $Q(\zeta_3)$ is a splitting field for H_4, and so we have $3\not| m$. Let $\tilde{E}=C_3\times E$. Then we have $\Lambda(\tilde{E})=Z[\zeta_3]\otimes_Z \Lambda(E)$ and $\Gamma(\tilde{E})=Z[\zeta_3]\otimes_Z \Gamma(E)$. Therefore we can construct the commutative diagram:

$$\begin{array}{ccc} C(\Lambda(E)) & \xrightarrow{\psi^{\Gamma(E)}_{\Lambda(E)}} & C(\Gamma(E)) \\ \downarrow \phi_\Lambda & & \downarrow \phi_\Gamma \\ C(\Lambda(\tilde{E})) & \xrightarrow{\psi^{\Gamma(\tilde{E})}_{\Lambda(\tilde{E})}} & C(\Gamma(\tilde{E})) \end{array}$$

where ϕ_Λ and ϕ_Γ denote the homomorphisms induced by $Z[\zeta_3]\otimes_Z \cdot$. Since $Q(\zeta_{3m})$ is a splitting field for H_4, $\psi^{\Gamma(\tilde{E})}_{\Lambda(\tilde{E})}$ is an isomorphism as shown above and hence $\operatorname{Ker}\psi^{\Gamma(E)}_{\Lambda(E)}\subseteq\operatorname{Ker}\phi_\Lambda$. Composing ϕ_Λ with the restriction map $C(\Lambda(\tilde{E}))\to C(\Lambda(E))$, we see that the exponent of $\operatorname{Ker}\phi_\Lambda$ is at most 2 and therefore the order of $\operatorname{Ker}\psi^{\Gamma(E)}_{\Lambda(E)}$ is a power of 2. However the order of $\operatorname{Ker}\psi^{\Gamma(E)}_{\Lambda(E)}$ is odd. Thus we must have $\operatorname{Ker}\psi^{\Gamma(E)}_{\Lambda(E)}=0$. This shows that $\psi^{\Gamma(E)}_{\Lambda(E)}$: $C(\Lambda(E))\to C(\Gamma(E))$ is an isomorphism, which completes the proof of the proposition.

We give, as a slight generalization of [4], (2.5),

Lemma 2.3. *Let Π be a finite group. Let Λ_Π be a Z-order in $Q\Pi$ containing $Z\Pi$ which is a quasi-permutation Π-module and let Ω_Π be a maximal Z-order in $Q\Pi$ containing Λ_Π. Assume that $\psi^{\Omega_\Pi}_{\Lambda_\Pi}$: $C(\Lambda_\Pi)\to C(\Omega_\Pi)$ is an isomorphism. Then $\tilde{C}^q(Z\Pi)=\tilde{C}(Z\Pi)$.*

Proof. Let $[\mathfrak{A}]-[Z\Pi]$ be an element of $\tilde{C}(Z\Pi)$. Since $\psi^{\Omega_\Pi}_{\Lambda_\Pi}$ is an isomorphism, we have $\mathfrak{A}\oplus\Lambda_\Pi\oplus\Lambda_\Pi\cong Z\Pi\oplus\Lambda_\Pi\oplus\Lambda_\Pi$. There exists an exact sequence: $0\to\Lambda_\Pi\oplus\Lambda_\Pi\to S\to S'\to 0$ where S and S' are permutation Π-modules. Then we easily see that $\mathfrak{A}\oplus S\oplus S'\cong Z\Pi\oplus S\oplus S'$. This shows that $[\mathfrak{A}]-[Z\Pi]\in\tilde{C}^q(Z\Pi)$.

We are now ready to prove our main theorem.

Theorem 2.4. *Let Π be any finite nilpotent group. Then $\tilde{C}^q(Z\Pi)=\tilde{C}(Z\Pi)$ $=C^q(Z\Pi)$.*

Proof. It has been proved in [4], (3.2) that $\tilde{C}^q(Z\Pi)=C^q(Z\Pi)$. Hence we only need to show that $\tilde{C}^q(Z\Pi)=\tilde{C}(Z\Pi)$. Let $Q\Pi=\bigoplus_{i=1}^{s}\Sigma_i$ be the decomposition of $Q\Pi$ into imple algebras. Applying [6], (14.3) or (14.5) to every Σ_i we can find a subgroup Π_i of Π and a simple component Σ_i' of $Q\Pi_i$ such that $\operatorname{End}_{\Sigma_i'}(Q\Pi\otimes_{Q\Pi_i}\Sigma_i')\cong\Sigma_i$ and $\Pi_i/\operatorname{Ker}(\Pi_i\to\Sigma_i')$ is a special elementary group. Let $E_i=\Pi_i/\operatorname{Ker}(\Pi_i\to\Sigma_i')$. Then Σ_i' can be identified with $\Sigma(E_i)$. By (1.2) $\Lambda(E_i)$ is a quasi-permutation Π_i-module, and therefore, if we put $L_i=Z\Pi\otimes_{Z\Pi_i}\Lambda(E_i)$, then L_i is a quasi-permutation Π-module. Let $\Omega(E_i)$ be a maximal $R(E_i)$-order in $\Sigma(E_i)$ containing $\Lambda(E_i)$. Define $\Lambda_i=\operatorname{End}_{\Lambda(E_i)}(L_i)$ and $\Omega_i=\operatorname{End}_{\Omega(E_i)}(L_i\Omega(E_i))$.

Then Λ_i and Ω_i are $R(E_i)$-orders in Σ_i with $\Lambda_i \subseteq \Omega_i$. Since L_i is a free $\Lambda(E_i)$-module, Λ_i(resp. Ω_i) is Morita equivalent to $\Lambda(E_i)$ (resp. $\Omega(E_i)$), and hence Ω_i is a maximal $R(E_i)$-order in Σ_i. Furthermore we see that Λ_i is a quasi-permutation Π_i-module. By the Morita theorem we have $C(\Lambda_i) \cong C(\Lambda(E_i))$ and $C(\Omega_i) \cong C(\Omega(E_i))$. However, according to (2.2), $\psi^{\Omega(E_i)}_{\Lambda(E_i)}$: $C(\Lambda(E_i)) \rightarrow C(\Omega(E_i))$ is an isomorphism. Therefore $\psi^{\Omega_i}_{\Lambda_i}$: $C(\Lambda_i) \rightarrow C(\Omega_i)$ is also an isomorphism. We further put $\Lambda_\Pi = \bigoplus_{i=1}^{s} \Lambda_i$ and $\Omega_\Pi = \bigoplus_{i=1}^{s} \Omega_i$. Then Λ_Π and Ω_Π are Z-orders in $Q\Pi$ with $Z\Pi \subseteq \Lambda_\Pi \subseteq \Omega_\Pi$ and Ω_Π is a maximal order in $Q\Pi$. Here Λ_Π is a quasi-permutation Π-module and $\psi^{\Omega_\Pi}_{\Lambda_\Pi}$: $C(\Lambda_\Pi) \rightarrow C(\Omega_\Pi)$ is an isomorphism. Thus we conclude by (2.3) that $\tilde{C}^q(Z\Pi) = \tilde{C}(Z\Pi)$, which completes the proof of the theorem.

3. Symmetric groups

Let Π be a finite group and let Ω_Π denote a maximal order in $Q\Pi$ containing $Z\Pi$. For a Π-module M we denote by $|\gamma_M|$ the number of all isomorphism types of Π-modules, L, such that, for each prime $p \mid |\Pi|$, $L_p \cong M_p$ and $\Omega_\Pi L \oplus \Omega_\Pi \cong \Omega_\Pi M \oplus \Omega_\Pi$. For each prime $p \mid |\Pi|$ we denote by $\Pi^{(p)}$ a p-Sylow subgroup of Π.

We here prove the following proposition which will play a central part in §3 and §4.

Proposition 3.1. *Let Π be a finite group which is a direct product of a subgroup Π' and a p-subgroup P'. Assume that Π' is a semidirect product of a cyclic group C of order prime to p by an abelian p-group P such that the action of P on C induces an isomorphism of P onto* (Aut $C)^{(p)}$ *In the case where $p=2$, assume further that P' is of split type over Q. Then $\tilde{C}^q(Z\Pi)=\tilde{C}(Z\Pi)$ and $B(Q\Pi)=G(Q\Pi)$.*

Proof. In order to show that $\tilde{C}^q(Z\Pi)=\tilde{C}(Z\Pi)$ it suffices by [4], (2.2) to show that there exists a $Z\Pi$-faithful, quasi-permutation Π-module N with $|\gamma_N|=1$. We will construct such Π-module N. Let $C=\langle\sigma\rangle$ and $n=|C|$. Let $n=q_1^{l_1}q_2^{l_2}\cdots q_t^{l_t}$ be the decomposition of n into primes where $q_1, q_2, \cdots, q_t$ are distinct primes. Then $|\text{Aut}\, C| = \prod_{i=1}^{t} q_i^{l_i-1}(q_i-1)$. Here we may assume that $p \mid q_i-1$ for $1 \leqq i \leqq s$ but $p \nmid q_i-1$ for $s+1 \leqq i \leqq t$. For every $1 \leqq i \leqq s$ let c_i be a positive integer such that $p^{c_i} \mid q_i-1$ but $p^{c_i+1} \nmid q_i-1$. Since $QC/(\Phi_n(\sigma)) \cong Q(\zeta_n)$ $= Q(\zeta_{q_1^{l_1}})Q(\zeta_{q_2^{l_2}})\cdots Q(\zeta_{q_t^{l_t}})$, we have $P \cong (\text{Aut}\, C)^{(p)} = (\text{Aut}_Q\, Q(\zeta_{q_1^{l_1}})Q(\zeta_{q_2^{l_2}})\cdots$ $Q(\zeta_{q_t^{l_t}}))^{(p)}$, and therefore P can be expressed as the direct product of the cyclic groups $\langle\tau_i\rangle$ of order p^{c_i}, $1 \leqq i \leqq s$, such that $\langle\tau_i\rangle / Q(\zeta_{q_i^{l_i}}) = (\text{Aut}_Q\, Q(\zeta_{q_i^{l_i}}))^{(p)}$ but $\langle\tau_i\rangle / Q(\zeta_{q_j^{l_j}}) = \{\mathrm{I}\}$ for $j \neq i$, $1 \leqq j \leqq t$.

We now have $Q\Pi \cong \bigoplus_{m \mid n} Q\Pi/(\Phi_m(\sigma))$. We easily see that $Q\Pi/(\Phi_n(\sigma))=$

$Q\Pi'/(\Phi_n(\sigma))\otimes_Q QP'$ and $Z\Pi/(\Phi_n(\sigma))\otimes_Z ZP'$. Define $\Sigma_n = Q\Pi'/(\Phi_n(\sigma))$ and $\Lambda_n = Z\Pi'/(\Phi_n(\sigma))$. Then Λ_n is a quasi-permutation Π'-module (cf. [4]). Because $P=(\mathrm{Aut}\ C)^{(p)}$, Σ_n (resp. Λ_n) is isomorphic to the trivial crossed product of $Q(\zeta_n)$ (resp. $Z[\zeta_n]$) and P. Further define $L_n = \Lambda_n/\Lambda_n(\tau_1 - \mathrm{I}, \tau_2 - \mathrm{I}, \cdots, \tau_s - \mathrm{I}) = Z[\zeta_n]$. Then L_n is also a quasi-permutation Π'-module and $\mathrm{End}_{Z\Pi'}(L_n) \cong Z[\zeta_n]^P$. Let $QP' = \bigoplus_{k=1}^{d} \Sigma'_k$ be the decomposition of QP' into simple algebras. For each $1 \leqq k \leqq d$ we denote by F'_k the center of Σ'_k and by R'_k the ring of all algebraic integers in F'_k. In the proof of (2.5) we have shown that there exists a quasi-permutation P'-module L'_k such that $\mathrm{End}_{ZP'}(L'_k) = R'_k$. Since $p \mid n$, $Q(\zeta_n)^P \otimes_Q F'_k$ is a field and $Z[\zeta_n]^P \otimes_Z R'_k$ is the ring of all algebraic integers in $Q(\zeta_n)^P \otimes_Q F'_k$. We have $\mathrm{End}_{Q\Pi}(QL_n \otimes_Q QL'_k) \cong Q(\zeta_n)^P \otimes_Q F'_k$ and $\mathrm{End}_{Z\Pi}(L_n \otimes_Z L'_k) \cong Z[\zeta_n]^P \otimes_Z R'_k$, and therefore, by [3], §3, (E'), $|\gamma_{L_n \otimes_Z L'_k}| = 1$. Let $N_n = \bigoplus_{k=1}^{d} (L_n \otimes_Z L'_k)$. Then N_n is a $Z\Pi/(\Phi_n(\sigma))$-faithful, quasi-permutation Π-module with $|\gamma_{N_n}| = 1$.

Let $m \mid n$, $m < n$ and let $m = q_{i_1}^{l_1'} \cdots q_{i_r}^{l_r'} q_{j_1}^{l_1''} \cdots q_{j_u}^{l_u''}$ be the decomposition of m into primes where $1 \leqq i_1 < \cdots < i_r \leqq s$ and $s+1 \leqq j_1 < \cdots < j_u \leqq t$. We define $\Pi_m = \Pi/\langle \sigma^m \rangle$, $C_m = C/\langle \sigma^m \rangle = \langle \sigma_m \rangle$, $P_m = \prod_{k=1}^{r} \langle \tau_{i_k} \rangle$ and $P'_m = (\prod_{i \neq i_1, \cdots, i_r} \langle \tau_i \rangle) \times P'$. Further let Π'_m be the semidirect product of C_m by P_m with the action of P_m on C_m induced by that of P on C. Then Π_m can be identified with the direct product of Π'_m and P'_m, and the action of P_m on C_m induces an isomorphism of P_m onto $(\mathrm{Aut}\ C_m)^{(p)}$. We here have $Z\Pi/(\Phi_m(\sigma)) \cong Z\Pi_m/(\Phi_m(\sigma_m))$. Therefore, applying the preceding method to Π_m, we can construct a $Z\Pi/(\Phi_m(\sigma))$-faithful, quasi-permutation Π-module N_m with $|\gamma_{N_m}| = 1$. If we put $N = \bigoplus_{m \mid n} N_m$, then N is a $Z\Pi$-faithful, quasi-permutation Π-module with $|\gamma_N| = 1$ as required. This proves that $\tilde{C}^q(Z\Pi) = \tilde{C}(Z\Pi)$.

Let V be any simple $Q\Pi$-module. In the above proof we see that there exists a quasi-permutation Π-module L such that $QL \cong V$. Hence the class of V in $G(Q\Pi)$ is contained in $B(Q\Pi)$. This shows that $B(Q\Pi) = G(Q\Pi)$, which completes the proof.

Lemma 3.2. *Let S_n be the symmetric group on n symbols. Let E be a maximal hyperelementary subgroup of S_n at a prime p. Then:*

$$E \cong H \times S_{l_1}^{(p)} \times S_{l_2}^{(p)} \times \cdots \times S_{l_t}^{(p)},$$

where H is a semidirect product of a cyclic group C of order prime to p by an abelian p-group P such that the action of P on C induces an isomorphism of P onto $(\mathrm{Aut}\ C)^{(p)}$, *and every S_{l_i} denotes the symmetric group on l_i symbols.*

Proof. This lemma may be well known. However, for completeness, we will give a proof of it. Since E is hyperelementary at p, there exists a cyclic normal subgroup $C=\langle\sigma\rangle$ of E of order prime to p such that E/C is a p-group. We have $E\subseteq N_{S_n}(C)$ and therefore E is conjugate to $C\cdot N_{S_n}(C)^{(p)}$ in $N_{S_n}(C)$ because E is maximal hyperelementary. Let $\sigma=\sigma_1^{(r_1)}\cdots\sigma_{l_1}^{(r_1)}\sigma_1^{(r_2)}\cdots\sigma_{l_2}^{(r_2)}\cdots\sigma_1^{(r_t)}\cdots\sigma_{l_t}^{(r_t)}$ be the decomposition of σ into cycles which do not contain common symbols where $n\geqq r_1>r_2>\cdots>r_t\geqq 1$ and every $\sigma_j^{(r_i)}$ is an r_i-cycle. We denote the Euler function by $\varphi(\cdot)$. Let $m=|C|$ and let $\{k_1=1, k_2, \cdots, k_{\varphi(m)}\}$ be the set of all integers k such that $(k, m)=1$ and $1\leqq k<m$. Then, for every k_h, $1\leqq h\leqq\varphi(m)$, there exists $\tau_{k_h}\in S_n$ such that $\tau_{k_h}^{-1}\sigma_j^{(r_i)}\tau_{k_h}=(\sigma_j^{(r_i)})^{k_h}$ for all $1\leqq i\leqq t$ and $1\leqq j\leqq l_i$. Put $K=\langle\sigma_1^{(r_1)}, \cdots, \sigma_{l_1}^{(r)}, \cdots, \sigma_1^{(r_t)}, \cdots, \sigma_{l_t}^{(r_t)}, \tau_{k_1}, \cdots, \tau_{k_{\varphi(m)}}\rangle\subseteq N_{S_n}(C)$ and $P=K^{(p)}$. Then the action of P on C induces an isomorphism of P onto $(\mathrm{Aut}\, C)^{(p)}$. Further, for each $1\leqq i\leqq t$, let S_{l_i} denote the symmetric group on l_i symbols $\{\sigma_1^{(r_i)}, \sigma_2^{(r_i)}, \cdots, \sigma_{l_i}^{(r_i)}\}$. Each S_{l_i} can be regarded as a subgroup of $N_{S_n}(C)$, and we have $N_{S_n}(C)=K\times S_{l_1}\times S_{l_2}\times\cdots\times S_{l_t}$. Hence $N_{S_n}(C)^{(p)}=P\times S_{l_1}^{(p)}\times S_{l_2}^{(p)}\times\cdots\times S_{l_t}^{(p)}$, and so $E\cong C\cdot N_{S_n}(C)^{(p)}\cong CP\times S_{l_1}^{(p)}\times S_{l_2}^{(p)}\times\cdots\times S_{l_t}^{(p)}$. This concludes the proof of the lemma.

We now come to the main theorem of this section.

Theorem 3.3. *Let S_n, $n\geqq 1$, be the symmetric group on n symbols. Then $\tilde{C}^q(ZS_n)=\tilde{C}(ZS_n)=C^q(ZS_n)=C(ZS_n)$.*

Proof. Since Q is a splitting field for S_n, we have $C(\Omega_{S_n})=0$, hence $\tilde{C}(ZS_n)=C(ZS_n)$. Therefore we only need to show that $\tilde{C}^q(ZS_n)=\tilde{C}(ZS_n)$. According to the induction theorem ([4], §1), it suffices to prove that, for every maximal hyperelementary subgroup E of S_n, $\tilde{C}^q(ZE)=\tilde{C}(ZE)$. However Q is also a splitting field for $S_l^{(2)}$, $l\geqq 1$ (e.g. [8], (5.9)). Therefore this follows immediately from (3.1) and (3.2).

REMARK 3.4. (1) *$C(ZS_n)=0$ for $n\leqq 4$.* (2) *For every odd prime p $|C(ZS_p)|\geqq\frac{1}{2}(p-1)$ and $|C(ZS_{p+1})|\geqq\frac{1}{2}(p-1)$.*

Proof. The assertion (1) is well known. We will prove the assertion (2) only on S_p because we can prove the one on S_{p+1} in the same way. Let σ be a p-cycle in S_p and define $K=N_{S_p}(\langle\sigma\rangle)$. Then K can be expressed as a semidirect product of the cyclic subgroup $\langle\sigma\rangle$ by a cyclic subgroup H of order $p-1$ such that the action of H on $\langle\sigma\rangle$ induces an isomorphism of H onto $\mathrm{Aut}\langle\sigma\rangle$. We have $N_{S_p}(K)=K$, and, if $K\neq\rho^{-1}K\rho$, $\rho\in S_p$, then $K\cap\rho^{-1}K\rho\subseteq\mu^{-1}H\mu$ for some $\mu\in K$. Let $f\colon C(ZH)\rightarrow C(ZK)$ and $g\colon C(ZK)\rightarrow C(ZS_p)$ be the natural homomorphism induced by $ZK\underset{ZH}{\otimes}\cdot$ and $ZS_p\underset{ZK}{\otimes}\cdot$, respectively. Using the Mackey's subgroup theorem we see that $\mathrm{Ker}\, g\subseteq\mathrm{Im}\, f$, and so $|C(ZK)|/|C(ZH)|\leqq|C(ZS_p)|$. By virtue of [7], (1.2) we have $|C(ZK)|=$

$\frac{1}{2}(p-1)|C(ZH)|$. This shows that $|C(ZS_p)| \geqq \frac{1}{2}(p-1)$.

4. Alternating groups

In this section we will be concerned with alternating groups.

Proposition 4.1. *Let A_n, $n \geqq 3$, be the alternating group on n symbols. Then both $\tilde{C}(ZA_n)/\tilde{C}^q(ZA_n)$ and $G(QA_n)/B(QA_n)$ are 2-groups.*

Proof. Let E' be a maximal hyperelementary subgroup of A_n at a prime p. Then there exists a maximal hyperelementary subgroup E of S_n at p such that $E'=E \cap A_n$. We can write $E=CP$ where C is a cyclic normal subgroup of E with $p \nmid |C|$ and P is a p-subgroup of E. Assume that p is odd. If $|C|$ is odd, then $E'=E$ and therefore, by (3.1) and (3.2), $\tilde{C}^q(ZE')=\tilde{C}(ZE')$ and $B(QE')=G(QE')$. If $|C|$ is even, then C is expressible as a direct product of a subgroup C_1 with $2 \nmid |C_1|$ and a 2-subgroup C_2. Let $C_2'=C_2 \cap E'$. Then $E' \simeq C_1P \times C_2'$ and so, again by (3.1), $\tilde{C}^q(ZE')=\tilde{C}(ZE')$ and $B(QE')=G(QE')$. Next assume that $p=2$. Then the Artin exponent of E' is a power of 2 ([9], §7), and therefore, by the Artin induction theorem ([4], §1), both $\tilde{C}(ZE')/\tilde{C}^q(ZE')$ and $G(QE')/B(QE')$ are 2-groups. Applying the Witt-Berman induction theorem ([4], §1), we can conclude that both $\tilde{C}(ZA_n)/\tilde{C}^q(ZA_n)$ and $G(QA_n)/B(QA_n)$ are 2-groups.

REMARK 4.2. (1) $C(ZA_n)=0$ *for* $n \leqq 5$. (2) *For every prime p with $p \equiv 3$* mod 4 $|C(ZA_p)| \geqq \frac{1}{2}(p-1)$ *and* $|C(ZA_{p+1})| \geqq \frac{1}{2}(p-1)$ *and for every prime p with $p \equiv 1$* mod 4 $|C(ZA_p)| \geqq \frac{1}{4}(p-1)$ *and* $|C(ZA_{p+1})| \geqq \frac{1}{4}(p-1)$. (3) $\tilde{C}^q(ZA_7)=\tilde{C}(ZA_7)=C^q(ZA_7)=C(ZA_7) \neq 0$ *and* $\tilde{C}^q(ZA_8)=\tilde{C}(ZA_8) \subsetneqq C^q(ZA_8)=C(ZA_8)$.

Proof. The assertion (1) can easily be shown, using the induction theorem, and the assertion (2) can be shown in the same way as in (3.4), (2). The assertion (3) follows from the induction theorem and [4], (5.5).

Lemma 4.3. *Let $\Pi=\langle\sigma\rangle$ be a cyclic group of order n and let $n=p_1^{l_1}p_2^{l_2}\cdots p_t^{l_t}$ be the decomposition of n into primes where $p_1, p_2, \cdots, p_t$ are distinct primes. Let χ_1 and χ_2 be rational characters of Π. Assume that, for every proper subgroup Π' of Π, $\chi_1/\Pi'=\chi_2/\Pi'$. Then:*

$$(\chi_1, 1_\Pi)-(\chi_2, 1_\Pi) = (\chi_1(\sigma)-\chi_2(\sigma)) \prod_{i=1}^{t}\left(1-\frac{1}{p_i}\right),$$

where 1_Π denotes the one dimensional trivial character of Π.

Proof. For every $m|n$ let α_m denote the character of Π afforded by the irreducible $Q\Pi$-module $Q(\zeta_m)$. Then we can write

$$\chi_1-\chi_2=\sum_{m|n} c_m\alpha_m\,, \qquad c_m\in Z\,.$$

Restricting both sides to the subgroup $\langle\sigma^{p_i}\rangle$, we see that $c_m=0$ when $p_i^2|m$. Hence $\chi_1-\chi_2=\sum_{m|p_1p_2\cdots p_t} c_m\alpha_m$. Furthermore, restricting both sides to the subgroup $\langle\sigma^{p_i^{l_i}}\rangle$, we see that $c_1+(p_i-1)c_{p_i}=0$ and $c_{p_{j_1}p_{j_2}\cdots p_{j_s}}+(p_i-1)c_{p_ip_{j_1}\cdots p_{j_s}}=0$ whenever $p_i\notin\{p_{j_1}, p_{j_2}, \cdots, p_{j_s}\}$. Clearly $\alpha_m(\sigma)$ coincides with the Möbius function $\mu(m)$. Therefore we have

$$\begin{aligned}\chi_1(\sigma)-\chi_2(\sigma)&=\sum_{m|p_1p_2\cdots p_t} c_m\mu(m)\\&=c_1\Big(1+\sum_{i=1}^{t}\frac{1}{p_i-1}+\sum_{1\le i<j\le t}\frac{1}{(p_i-1)(p_j-1)}+\cdots\Big)\\&=c_1\prod_{i=1}^{t}\frac{p_i}{p_i-1}\,.\end{aligned}$$

However $(\chi_1, 1_\Pi)-(\chi_2, 1_\Pi)=(\sum_{m|p_1p_2\cdots p_t} c_m\alpha_m, \alpha_1)=c_1$. Thus we get $(\chi_1, 1_\Pi)-(\chi_2, 1_\Pi)=(\chi_1(\sigma)-\chi_2(\sigma))\prod_{i=1}^{t}\Big(1-\frac{1}{p_i}\Big)$.

Let S_n, A_n be the symmetric, alternating group on n symbols, respectively. Let I_n denote the image of the restriction map $G(QS_n)\to G(QA_n)$ and let $\tau=(1, 2)\in S_n$. For $\chi\in G(QA_n)$, $\chi\notin I_n$ if and only if $\chi^\tau\neq\chi$. Therefore $G(QA_n)/I_n$ is a free abelian group generated by all the classes of irreducible rational characters, χ, of A_n with $\chi^\tau\neq\chi$. Every irreducible rational character χ of A_n with $\chi^\tau\neq\chi$ is absolutely irreducible. Hence there is a one to one correspondence between pairs, (χ, χ^τ), $\chi^\tau\neq\chi$, of irreducible rational characters of A_n and partitions,

$$n=c_1+c_2+\cdots+c_t\,, \tag{*}$$

such that $c_1<c_2<\cdots<c_t$, $2\nmid c_i$ and $\prod_{i=1}^{t} c_i$ is square. Let χ be an irreducible rational character of A_n with $\chi^\tau\neq\chi$ and let $n=c_1+c_2+\cdots+c_t$ be the partition corresponding to (χ, χ^τ). Define $\sigma_\chi=(1, 2,\cdots, c_1)\,(c_1+1,\cdots, c_1+c_2)\cdots(\sum_{i=1}^{t-1}c_i+1,\cdots, \sum_{i=1}^{t}c_i)$ and let C_χ, C_χ^τ denote the conjugate classes of A_n containing σ_χ, $\tau\sigma_\chi\tau$, respectively. Then we have

$$\chi(\sigma)=\chi^\tau(\sigma)=\frac{1}{2}\chi^{S_n}(\sigma) \qquad \text{for every } \sigma\notin C_\chi\cup C_\chi^\tau\,,$$

$$\chi(\sigma_\chi)=\frac{1}{2}(1\pm d)\,, \quad \chi^\tau(\sigma_\chi)=\frac{1}{2}(1\mp d)$$

where d denotes the positive integer with $d^2=\prod_{i=1}^{t} c_i$. Let $H_\chi=\langle\sigma_\chi\rangle$, let K'_χ be a 2-Sylow subgroup of $N_{S_n}(H_\chi)$ $(=N_{A_n}(H_\chi))$ and put $K_\chi=H_\chi K'_\chi$. Let n_χ be the odd part of $\varphi(d)$. Then $|H_\chi|=d^2$ and $|K'_\chi|=\varphi(d)/n_\chi$. We can here prove

Lemma 4.4. *Let χ be an irreducible rational character of A_n with $\chi\notin I_n$. Then $n_\chi\cdot\chi-1_{K_\chi}^{A_n}\in I_n$.*

Proof. We may assume that $\chi(\sigma_\chi)=\frac{1}{2}(1+d)$ and $\chi^\tau(\sigma_\chi)=\frac{1}{2}(1-d)$. For every proper subgroup H' of H_χ, $\chi/H'=\chi^\tau/H'$ and $\chi(\sigma_\chi)-\chi^\tau(\sigma_\chi)=d$. Therefore by (4.3) and the Frobenius reciprocity theorem we get $(\chi, 1_{H_\chi}^{A_n})-(\chi^\tau, 1_{H_\chi}^{A_n})=(\chi/H_\chi, 1_{H_\chi})-(\chi^\tau/H_\chi, 1_{H_\chi})=\varphi(d)$, and so $(\chi, \varphi(d)\cdot\chi-1_{H_\chi}^{A_n})=(\chi^\tau, \varphi(d)\cdot\chi-1_{H_\chi}^{A_n})$. Since generators of proper subgroups of H_χ are not of type $(*)$, for any irreducible rational character χ' of A_n with $\chi'\neq\chi, \chi^\tau$ we have $(\chi', \varphi(d)\cdot\chi-1_{H_\chi}^{A_n})=(\chi'^\tau, \varphi(d)\cdot\chi-1_{H_\chi}^{A_n})$. Therefore $\varphi(d)\cdot\chi-1_{H_\chi}^{A_n}\in I_n$. On the other hand, applying the Brauer coefficient theorem ([2], Satz 1) to K_χ, we get

$$|K_\chi|\cdot 1_{K_\chi} = |H_\chi|\cdot 1_{H_\chi}^{K_\chi}+\sum_{H'} b_{H'}\cdot 1_{H'}^{K_\chi}, \qquad b_{H'}\in Z$$

where H' runs over all cyclic subgroups $\neq H_\chi$ of K_χ. Since generators of subgroups, $H'\neq H_\chi$, of K_χ are not of type $(*)$, we have $1_{H'}^{A_n}\in I_n$ and so $\sum_{H'} b_{H'}\cdot 1_{K_\chi}^{A_n}\in I_n$. Therefore $|H_\chi|\cdot 1_{H_\chi}^{A_n}-|K_\chi|\cdot 1_{K_\chi}^{A_n}\in I_n$. Thus we have $|K_\chi|(n_\chi\cdot\chi-1_{K_\chi}^{A_n})=|H_\chi|\varphi(d)\cdot\chi-|K_\chi|\cdot 1_{K_\chi}^{A_n}=|H_\chi|(\varphi(d)\cdot\chi-1_{H_\chi}^{A_n})+(|H_\chi|\cdot 1_{H_\chi}^{A_n}-|K_\chi|\cdot 1_{K_\chi}^{A_n})\in I_n$. Because $G(QA_n)/I_n$ is torsion-free, this shows that $n_\chi\cdot\chi-1_{K_\chi}^{A_n}\in I_n$.

We now establish the following:

Theorem 4.5. *Let A_n, $n\geqq 3$, be the alternating group on n symbols. Then $B(QA_n)=G(QA_n)$.*

Proof. Since $B(QS_n)=G(QA_n)$ as is well known, we see that $I_n\subseteq B(QA_n)$. It follows immediately from (4.4) that $[G(QA_n): B(QA_n)]$ is odd. However, by (4.1), $G(QA_n)/B(QA_n)$ is a 2-group. Thus we have $B(QA_n)=G(QA_n)$.

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