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<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 1976, 13(1), p. 109-122</td>
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<td><strong>Version Type</strong></td>
<td>VoR</td>
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<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/5252">https://doi.org/10.18910/5252</a></td>
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Osaka University
ON THE PROJECTIVE CLASS GROUP OF FINITE GROUPS

Dedicated to Professor Kiiti Morita on his 60th birthday

SHIZUO ENDO AND TAKEHIKO MIYATA

(Received October 17, 1974)
(Revised May 19, 1975)

In this paper we will continue the investigation of integral representations of finite groups done in [3], [4] and [5]. We will here be concerned mainly with the projective class group of nilpotent and symmetric groups.

Let $\Sigma$ be a (finite dimensional) semi-simple $Q$-algebra and let $A$ be a $Z$-order in $\Sigma$. We will mean by the projective class group of $A$ the class group defined by using all locally free, projective $A$-modules and denote it by $C(A)$.

Let $\Pi$ be a finite group. A finitely generated $Z$-free $\Pi$-module is briefly called a $\Pi$-module. A $\Pi$-module is called a permutation $\Pi$-module if it can be expressed as a direct sum of $\{\Pi_i/\Pi_i\}$ where each $\Pi_i$ is a subgroup of $\Pi$. Further a $\Pi$-module $M$ is called a quasi-permutation $\Pi$-module if there exists an exact sequence: $0 \to M \to S \to S' \to 0$ where $S$ and $S'$ are permutation $\Pi$-modules.

As is well known, the projective class group $C(Z\Pi)$ of the group algebra $Z\Pi$ can be written as follows:

$C(Z\Pi) = \{[\mathfrak{A}] - [Z\Pi] | \mathfrak{A}(\neq 0) \text{ is a projective ideal of } Z\Pi\}$.

We define the subgroups $C(Z\Pi)_I$, $C^q(Z\Pi)$ and $C^q(Z\Pi)$ of $C(Z\Pi)$ as follows:

$C(Z\Pi)_I = \{[\mathfrak{A}] - [Z\Pi] \subset C(Z\Pi) | \mathfrak{A} \oplus X \cong Z\Pi \oplus X \text{ for some } \Pi\text{-module } X\},$

$C^q(Z\Pi) = \{[\mathfrak{A}] - [Z\Pi] \subset C(Z\Pi) | \mathfrak{A} \oplus S_1 \cong S_2 \text{ for some permutation } \Pi\text{-module } S_1 \text{ and } S_2\},$

$C^q(Z\Pi) = \{[\mathfrak{A}] - [Z\Pi] \subset C(Z\Pi) | \mathfrak{A} \oplus S \cong Z\Pi \oplus S \text{ for some permutation } \Pi\text{-module } S\}.$

Let $\Omega_\Pi$ be a maximal $Z$-order in $Q\Pi$ containing $Z\Pi$ and let $\varphi_\Pi: C(Z\Pi) \to C(\Omega_\Pi)$ be the epimorphism induced by $\Omega_\Pi \otimes^\Pi$. Then the sequence $0 \to C(Z\Pi) \to C(\Omega_\Pi) \to 0$ is exact.

In [3] and [4] we raised the following problem:
For a finite group $\Pi$, $\overline{C}(Z\Pi) = C^*(Z\Pi)(= \overline{C}(Z\Pi))$?

and showed that the answer to this is affirmative for a fairly extensive class of finite groups but it is negative for the alternating group on 8 symbols.

In §2 we give

[I] If $\Pi$ is a finite nilpotent group, then $\overline{C}(Z\Pi) = \overline{C}(Z\Pi) = C^*(Z\Pi)$.

A finite group $\Pi$ is said to be of split type over $\mathbb{Q}$ if every simple component of $Q\Pi$ is isomorphic to a full matrix algebra over its center. In the previous paper [4] we proved the assertion [I] under the additional assumption that $\Pi$ is of split type over $\mathbb{Q}$. We will prove [I], using the Mayer-Vietoris sequence in algebraic $K$-theory ([1]).

Let $S_n, A_n$ denote the symmetric, alternating group on $n$ symbols, respectively. In §3 we give

[II] $\overline{C}(ZS_n) = \overline{C}(ZS_n) = \overline{C}(ZS_n) = C(ZS_n)$ for any $n \geq 1$.

Let $G(Q\Pi)$ be the Grothendieck group of the category of all finitely generated $Q\Pi$-modules and define $B(Q\Pi)$ to be the subring of $G(Q\Pi)$ generated by all the classes of permutation $Q\Pi$-modules. It is well known that $B(QS_n) = G(QS_n)$ for any $n \geq 1$. However the following result on the alternating group, which will be proved in §4, seems new.

[III] $B(QA_n) = G(QA_n)$ for any $n \geq 3$.

We would like to express our gratitude to H. Hijikata for his valuable suggestion, and to I. Reiner and S. Ullom for their helpful comments.

1. Some lemmas on special elementary groups

Let $C_{2^l}, l \geq 0$, be the cyclic group of order $2^l$, i.e., $C_{2^l} = \langle \sigma | \sigma^{2^l} = 1 \rangle$. Let $H_{2^l}, l \geq 2$, be the (generalized) quaternion group of order $2^{l+1}$, i.e., $H_{2^l} = \langle \sigma, \tau | \sigma^{2^l} = 1, \tau^{-1}\sigma\tau = \sigma^{-1}\rangle$ and let $D_{2^l}, l \geq 2$, be the dihedral group of order $2^{l+1}$, i.e., $D_{2^l} = \langle \sigma, \tau | \sigma^{2^l} = 1, \tau^{-1}\sigma\tau = \sigma^{-1}\rangle$. Define the groups $SD_{2^l}$ and $SC_{2^l}, l \geq 3$, of order $2^{l+1}$ by $SD_{2^l} = \langle \sigma, \tau | \sigma^{2^l} = 1, \tau^{-1}\sigma\tau = \sigma^{-1}\rangle$ and $SC_{2^l} = \langle \sigma, \tau | \sigma^{2^l} = 1, \tau^{-1}\sigma\tau = \sigma^{-1}\rangle$.

Let $H$ denote one of the groups $C_{2^l}, H_{2^l}, D_{2^l}, SD_{2^l}$ and $SC_{2^l}$. Define $\Sigma(H) = QH/(\sigma^{2^l} + 1)$ and $\Lambda(H) = ZH/(\sigma^{2^l} + 1)$ and denote the images of $\sigma$ and $\tau$ in $\Lambda(H)$ by $x$ and $y$, respectively. Put

$$K(H) \ (\text{resp.} \ R(H)) = \begin{cases} Q(x) \ (\text{resp.} \ Z[x]) & \text{when} \ H = C_{2^l} \\ Q(x + x^{-1}) \ (\text{resp.} \ Z[x + x^{-1}]) & \text{when} \ H = H_{2^l} \ \text{or} \ D_{2^l} \\ Q(x) \ (\text{resp.} \ Z[x]) & \text{when} \ H = SD_{2^l} \\ Q(x^{-1}) \ (\text{resp.} \ Z[x^{-1}]) & \text{when} \ H = SC_{2^l} \end{cases}$$
Then $\Sigma(H)$ is a central simple $K(H)$-algebra and is the unique $H$-faithful simple component of $QH$, and $A(H)$ is an $R(H)$-order in $\Sigma(H)$. Further let

$$
\alpha_H = \begin{cases} 
2 & \text{when } H=C_2 = \{1\} \\
 2x-1 & \text{when } H=C_d, l \geq 1 \\
 2x-x^{-1} - 2 & \text{when } H=H_d' \text{ or } D_d' \\
 2x-x^{-1} & \text{when } H=SD_d \\
 x^2-1 & \text{when } H=SC_d 
\end{cases}
$$

and put $\psi(H) = \alpha_H R(H)$. Then $\psi(H)$ is the unique prime ideal of $R(H)$ containing 2 and $R(H)/\psi(H) \cong \mathbb{Z}/2\mathbb{Z}$.

Let $K$ be an algebraic number field and let $\Sigma$ be a central simple $K$-algebra. We say $\Sigma$ to be of split type if it is isomorphic to a full matrix algebra over $K$. For a (finite or infinite) prime $\mathfrak{p}$ of $K$ we denote by $\hat{K}_\mathfrak{p}$ the completion of $K$ at $\mathfrak{p}$ and put $\hat{\Sigma}_\mathfrak{p} = \hat{K}_\mathfrak{p} \otimes \Sigma$. We say $\Sigma$ to be of locally split type if, for every finite prime $\mathfrak{p}$ of $K$, $\hat{\Sigma}_\mathfrak{p}$ is isomorphic to a full matrix algebra over $\hat{K}_\mathfrak{p}$.

**Lemma 1.1.**

1. If $H=C_d', D_d', SD_d'$ or $SC_d'$, $\Sigma(H)$ is of split type.
2. $\Sigma(H_d')$ is of locally split type if and only if $l \geq 3$.

Proof. The assertion (1) is evident and the assertion (2) may be well known. But for completeness we here give a proof of (2). It is noted that $\Sigma(H_d')$ is the quaternion algebra over the real field $K(H_d')$. Accordingly, for a prime $\mathfrak{p}$ of $K(H_d')$, $\hat{\Sigma}_\mathfrak{p}(H_d') \cong M_2(K(H_d'))$ if and only if the equation $X^2 + Y^2 + 1 = 0$ has a solution in $K(H_d')$, i.e., if and only if $\left( \frac{-1, -1}{\mathfrak{p}} \right) = 1$. For every finite prime $\mathfrak{p}$ of $K(H_d')$ with $\mathfrak{p} \neq \mathfrak{p}(H_d')$ we have $\left( \frac{-1, -1}{\mathfrak{p}} \right) = 1$. On the other hand, for every real prime $\mathfrak{p}$ of $K(H_d')$ we have $\left( \frac{-1, -1}{\mathfrak{p}} \right) = -1$. All infinite primes of $K(H_d')$ are real and the number of them is $2^{l-2}$. Since $\prod_{\mathfrak{p}} \left( \frac{-1, -1}{\mathfrak{p}} \right) = 1$ where $\mathfrak{p}$ runs over all primes of $K(H_d')$, we see that $\left( \frac{-1, -1}{\mathfrak{p}(H_d')} \right) = 1$ if and only if $l \geq 3$.

For any positive integer $n$ we denote by $\Phi_n(t)$ the $n$-th cyclotomic polynomial and by $\zeta_n$ a primitive $n$-th root of 1.

From now we assume that $m \geq 1$ is an odd integer. Let $C_m$ be the cyclic group of order $m$, i.e., $C_m = \langle \mu | \mu^m = 1 \rangle$. Define $K(C_m) = Q(C_m/\Phi_m(\mu)) = Q(\zeta_m)$ and $R(C_m) = Z[C_m/\Phi_m(\mu)] = Z[\zeta_m]$. A finite group $E$ is said to be a special elementary group if $E = C_m \times H$ where $H=C_d', H_d', D_d', SD_d'$ or $SC_d'$. Let $E = C_m \times H$ where $H=C_d', H_d', D_d', SD_d'$ or $SC_d'$. Define $\Sigma(E) = \Sigma(C_m) \otimes \Sigma(H)$.
\[ \Lambda(E) = R(C_m) \otimes_R H = \mathbb{Z}E(\Phi_m(\sigma \mu)) \]
and further put \( K(E) = K(C_m) \otimes K(H) \) and \( R(E) = R(C_m) \otimes R(H) \). Since \( m \) is odd, \( K(E) \) is a field and \( R(E) \) is the ring of all algebraic integers in \( K(E) \). We see that \( \Sigma(E) \) is a central simple \( K(E) \)-algebra and is the unique \( E \)-faithful simple component of \( QE \) and that \( \Lambda(E) \) is an \( R(E) \)-order in \( \Sigma(E) \).

**Lemma 1.2.** For any special elementary group \( E \), \( \Lambda(E) \) is a quasi-permutation \( E \)-module.

**Proof.** Let \( E = C_m \times H \) where \( H = D_\nu \), \( D_\nu^* \), \( S D_\nu \) or \( SC_\nu \). Then we have \( \Lambda(E) = \mathbb{Z}E(\Phi_m(\sigma \mu)) \). Hence we can prove the assertion by the argument using a zigzag path as in the proof of [3], (2.3).

**Lemma 1.3.** Let \( E = C_m \times H \) where \( H = H_2^\nu, D_2^\nu \) or \( S D_2^\nu \). Let \( \Omega(E) \) be a maximal \( R(E) \)-order in \( \Sigma(E) \) containing \( \Lambda(E) \). Then \( \alpha_H \Omega(E) \subseteq \Lambda(E) \).

**Proof.** For brevity we write \( K = K(E) \) and \( R = R(E) \). Now we have \( \Sigma(E) = K + Kx + Ky + Kxy \) and \( \Lambda(E) = R + Rx + Ry + Rxy \). Assume that \( H = H_2^\nu \). Let \( z = x^{\nu^2-2} \). Then \( \Sigma(E) = K + Kz + Ky + Kzy \), and \( z^2 = y^2 = -1 \) and \( zy + yz = 0 \). Denote by \( \text{trd} \) the reduced trace of \( \Sigma(E) \). We note that, for any element \( v = a_1 + a_2 z + a_3 y + a_4 zy \) of \( \Sigma(E) \), \( a_i \in K \), we have \( \text{trd}(v) = 2a_1 \). Then we can find the \( K \)-basis of \( \Sigma(E) \) which is dual to \( \{1, x, y, xy\} \) with respect to \( \text{trd} \) as follows:

\[
\begin{align*}
  u_1 &= \frac{x^2 - 1}{x^2 + x^2 - 2}, \\
  u_2 &= \frac{x - x^{-1}}{x^2 + x^2 - 2}, \\
  u_3 &= -\frac{(x^2 - 1)y}{x^2 + x^2 - 2}, \\
  u_4 &= -\frac{(x - x^{-1})y}{x^2 + x^2 - 2}.
\end{align*}
\]

It is easy to see that \( \alpha_H u_i \in \Lambda(H) \) for \( 1 \leq i \leq 4 \). Since \( \text{trd}(\Omega(E)) \subseteq R \), we have \( \Omega(E) \subseteq Ru_i + Ru_i + Ru_i + Ru_i \) and hence \( \alpha_H \Omega(E) \subseteq \Lambda(E) \). It is obvious that \( \Omega(E)_p = \Lambda(E)_p \) for any prime \( p \neq 2 \). Thus we have \( \alpha_H \Omega(E) \subseteq \Lambda(E) \).

For the case where \( H = D_\nu^* \) or \( S D_\nu \) we can prove the assertion in a similar manner.

We here consider the case where \( E = C_m \times H_4 \). Let \( u = \frac{1}{2}(1 + x + y + xy) \in \Sigma(H_4) \subseteq \Sigma(E) \) and put \( \Gamma(E) = \Lambda(E) + R(C_m)u \). Let \( \epsilon(E) = \Gamma(E)(1 + x)(1 + x \Gamma(E)) \).

**Lemma 1.4.** (1) \( \iota(C_m \times H_4) \subseteq \Lambda(C_m \times H_4) \) and \( \Gamma(C_m \times H_4)/\iota(C_m \times H_4) \simeq \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}[\zeta_m] \otimes \mathbb{Z}[\zeta_3] \). (2) \( \Gamma(C_m \times H_4) \) is a hereditary \( R(C_m) \)-order in \( \Sigma(C_m \times H_4) \).

**Proof.** (1) It is evident that \( \iota(C_m \times H_4) \subseteq \Lambda(C_m \times H_4) \). Hence we have only to prove the second assertion. Now it suffices to show that \( \Gamma(H_4)/\iota(H_4) \simeq \mathbb{Z}/2\mathbb{Z}[X]/(X^3 + X + 1) \), because \( \Gamma(C_m \times H_4) = \mathbb{Z}[\zeta_m] \otimes \Gamma(H_4) \) and \( \iota(C_m \times H_4) = \mathbb{Z}[\zeta_m] \otimes \iota(H_4) \). Define the ring homomorphism \( f: \Gamma(H_4) \to \mathbb{Z}/2\mathbb{Z}[X]/(X^3 + X + 1) \) by \( f(1) = f(x) = f(y) = 1 \) and \( f(u) = \bar{X} \) where \( \bar{X} \) denotes the image of
It is easy to see that \( f \) is an epimorphism and \( \text{Ker } f = c(H_i) \). Therefore \( f \) induces an isomorphism \( \hat{f}: \Gamma(H_i)c(H_i) \to (Z/2Z)[X]/(X^2 + X + 1) \). (2) Let \( p \) be a prime ideal of \( R(C_m) \). If \( 2 \in p \), it follows from (1) that \( c(C_m \times H_i)_p \) coincides with the Jacobson radical of \( \Gamma(C_m \times H_i)_p \). Since \( c(C_m \times H_i)_p \) is principal in \( \Gamma(C_m \times H_i)_p \), \( \Gamma(C_m \times H_i)_p \) is a hereditary \( R(C_m) \)-order in \( \Sigma(C_m \times H_i) \). On the other hand, if \( 2 \notin p \), then \( p \) is unramified in \( \Gamma(C_m \times H_i) \) and so \( \Gamma(C_m \times H_i)_p \) is a maximal \( R(C_m)_p \)-order in \( \Sigma(C_m \times H_i) \). Consequently \( \Gamma(C_m \times H_i) \) is a hereditary \( R(C_m) \)-order in \( \Sigma(C_m \times H_i) \).

2. Nilpotent groups

We state without proof a result due to J. Milnor which will play an essential part in this section.

**Proposition 2.1** ([1], X, (1.10)). Let \( \Sigma \) be a semi-simple \( Q \)-algebra and let \( \Lambda, \Gamma \) be \( Z \)-orders in \( \Sigma \) with \( \Lambda \subseteq \Gamma \). Let \( c \) be a two-sided ideal of \( \Gamma \) contained in \( \Lambda \) such that \( c \Sigma = \Sigma \). Then there exists an exact (Mayer-Vietoris) sequence:

\[
K_1(\Lambda) \to K_1(\Gamma) \oplus K_1(\Lambda/c) \to K_0(\Gamma/c) \to K_0(\Lambda) \to K_0(\Gamma/c).
\]

Let \( \Sigma \) be a semi-simple \( Q \)-algebra and let \( \Lambda, \Gamma \) be \( Z \)-orders in \( \Sigma \) with \( \Lambda \subseteq \Gamma \).

Let \( \psi: C(\Lambda) \to C(\Gamma) \) denote the natural epimorphism induced by \( \Gamma \). For any ring \( A \) we denote by \( U(A) \) the group of all units of \( A \).

In the following proposition we use the same notation as in §1.

**Proposition 2.2.** Let \( E = C_m \times H \) be any special elementary group. Let \( \Omega(E) \) be a maximal \( R(E) \)-order in \( \Sigma(E) \) containing \( \Lambda(E) \). Then the map \( \psi_{\Lambda(E)}^\Omega(E): C(\Lambda(E)) \to C(\Omega(E)) \) is an isomorphism.

Proof. In the case where \( H = C_2 \) this is obvious. We first assume that \( H \neq H_4, C_2, SC_2 \), or that \( H = H_4 \) and \( Q(\zeta_m) \) is a splitting field for \( H_4 \). By \( (1.3) \) we have \( \alpha_H \Omega(E) \subseteq \Lambda(E) \), and therefore we can apply (2.1) to \( \Lambda(E), \Omega(E), \alpha_H \Omega(E) \). Then we get the exact sequence:

\[
K_1(\Omega(E)) \oplus K_1(\Lambda(E)/\alpha_H \Omega(E)) \to K_1(\Omega(E)/\alpha_H \Omega(E)) \to K_1(\Omega(E)/\alpha_H \Omega(E)) \to K_0(\Lambda(E)/\alpha_H \Omega(E)).
\]

Since, by \( (1.1) \), \( \Sigma(E) \) is of locally split type, we have \( \Omega(E)/\alpha_H \Omega(E) \cong M_2(R(E)/\alpha_H R(E)) \) and so \( K_1(\Omega(E)/\alpha_H \Omega(E)) \cong U(R(E)/\alpha_H R(E)) \). The inclusion map \( R(E)/\alpha_H R(E) \subseteq \Lambda(E)/\alpha_H \Omega(E) \subseteq \Omega(E)/\alpha_H \Omega(E) \) induces a homomorphism \( \phi: U(R(E)/\alpha_H R(E)) \to K_1(\Lambda(E)/\alpha_H \Omega(E)) \to K_1(\Omega(E)/\alpha_H \Omega(E)) \). Then it is easy to see that \( \text{Im } \phi = U(R(E)/\alpha_H R(E))^2 \). However, since \( R(E)/\alpha_H R(E) = \mathbb{Z}[\zeta_m]/2\mathbb{Z}[\zeta_m], \) the order of \( U(R(E)/\alpha_H R(E))^2 \) is odd, hence \( U(R(E)/\alpha_H R(E))^2 = U(R(E)/\alpha_H R(E)). \) Therefore \( \phi \) is an isomorphism and then so is \( f \). Since \( \text{Ker } \psi_{\Lambda(E)}^\Omega(E) = \text{Ker } h = \text{Im } g \), this implies that \( \psi_{\Lambda(E)}^\Omega(E): C(\Lambda(E)) \to C(\Omega(E)) \) is an isomorphism.
Next assume that \( H = SC \). In this case we have \( \Sigma(E) \cong M_s(K(E)) \). Define \( \Omega'(E) = \text{End}_{R(E)}(\Sigma(E)(y+1)) \cong M_s(R(E)) \). Then we can regard \( \Omega'(E) \) as a maximal \( R(E) \)-order in \( \Sigma(E) \) containing \( \Lambda(E) \). Because \( C(\Omega(E)) \cong C(\Omega'(E)) \), we may assume that \( \Omega(E) = \Omega'(E) \). Now we have \( \Lambda(E) = \left\{ \left[ \begin{array}{cc} a+b & c+d \\ c-a & b-a \end{array} \right] \right\} \subseteq \Omega(E) = M_s(R(E)) \). Hence \( 2\Omega(E) \subseteq \Lambda(E) \) and \( \Lambda(E)/2\Omega(E) = \left\{ \left[ \begin{array}{c} a \\ b \end{array} \right] \right\} \). Applying (2.1) to \( \Lambda(E), \Omega(E), 2\Omega(E) \), we get the exact sequence: \( \mathbb{K}_1(\Omega(E)) \xrightarrow{f} \mathbb{K}_1(\Lambda(E)/2\Omega(E)) \xrightarrow{g} \mathbb{K}_1(\Lambda(E)/2\Omega(E)) \). Since \( \mathbb{K}_1(\Lambda(E)/2\Omega(E)) \cong U(R(E)/2R(E)) \), we see that the composed map \( \psi_j : \mathbb{K}_1(\Omega(E)) \xrightarrow{f} \mathbb{K}_1(\Lambda(E)/2\Omega(E)) \xrightarrow{g} \mathbb{K}_1(\Omega(E)/2\Omega(E)) \) coincides with the determinant map \( \det : U(A(E)/2\Omega(E)) \to U(R(E)/2R(E)) \) as in the preceding case, in order to show that \( \psi_j : \Lambda(E)/2\Omega(E) \to \Omega(E) \) is an epimorphism.

Finally we will treat the case where \( H = H_4 \). We have \( C(\Omega(E)) \cong C(\Omega'(E)) \) for any other maximal order \( \Omega(E) \) in \( \Sigma(E) \) containing \( \Lambda(E) \). Hence we may assume that \( \Gamma(E) \subseteq \Omega(E) \). By (1.4) \( \Gamma(E) \) is a hereditary order in \( \Sigma(E) \) and so, according to [4], (2.4), \( \psi_j : \mathbb{K}_1(\Gamma(E)) \to \mathbb{K}_1(\Omega(E)) \) is an isomorphism. Because \( \psi_j : \mathbb{K}_1(\Gamma(E)) \cong \mathbb{K}_1(\Omega(E)) \) is an isomorphism if and only if \( \psi_j : \mathbb{K}_1(\Omega(E)) \to \mathbb{K}_1(\Omega(E)) \) is an isomorphism. If \( \Omega(E) \) is a splitting field for \( H_4 \), it has already been shown that \( \psi_j : \mathbb{K}_1(\Omega(E)) \to \mathbb{K}_1(\Omega(E)) \) is an isomorphism, and hence \( \psi_j : \mathbb{K}_1(\Omega(E)) \to \mathbb{K}_1(\Omega(E)) \) is also an isomorphism. Assume that \( \Omega(E) \) is not a splitting field for \( H_4 \). Now it suffices to show that \( \psi_j : \mathbb{K}_1(\Omega(E)) \to \mathbb{K}_1(\Omega(E)) \) is an isomorphism. Applying (2.1) to \( \Lambda(E), \Gamma(E), \psi_j \), we get the exact sequence: \( \mathbb{K}_1(\Gamma(E)) \oplus \mathbb{K}_1(\Lambda(E)/\psi_j(\Gamma(E))) \xrightarrow{f} \mathbb{K}_1(\Gamma(E)/\psi_j(\Gamma(E))) \xrightarrow{g} \mathbb{K}_1(\Lambda(E)/\psi_j(\Gamma(E))) \). Since \( \mathbb{K}_1(\Gamma(E)/\psi_j(\Gamma(E))) \cong \mathbb{K}_1(\Lambda(E)/\psi_j(\Gamma(E))) \cong \mathbb{K}_1(\Omega(E)) \), the order of \( \mathbb{K}_1(\Gamma(E)/\psi_j(\Gamma(E))) \) is odd and \( \mathbb{K}_1(\Lambda(E)/\psi_j(\Gamma(E))) \) is odd. Therefore the order of \( \mathbb{K}_1(\Gamma(E)/\psi_j(\Gamma(E))) \) is odd. Because \( \mathbb{K}_1(\Lambda(E)/\psi_j(\Gamma(E))) \) is odd, it follows that the order of \( \mathbb{K}_1(\Lambda(E)/\psi_j(\Gamma(E))) \) is odd. It is well known that \( \Omega(E) \) is a
splitting field for \( H \), and so we have \( 3^m \). Let \( \tilde{E} = C_3 \times E \). Then we have \( \mathcal{A}(\tilde{E}) = \mathbb{Z}[\zeta_3] \otimes \mathcal{A}(E) \) and \( \Gamma(\tilde{E}) = \mathbb{Z}[\zeta_3] \otimes \Gamma(E) \). Therefore we can construct the commutative diagram:

\[
\begin{array}{ccc}
C(\mathcal{A}(E)) & \xrightarrow{\varphi_{\mathcal{A}(E)}} & C(\Gamma(E)) \\
\phi_A & & \phi_T \\
C(\mathcal{A}(\tilde{E})) & \xrightarrow{\varphi_{\mathcal{A}(\tilde{E})}} & C(\Gamma(\tilde{E}))
\end{array}
\]

where \( \phi_A \) and \( \phi_T \) denote the homomorphisms induced by \( \mathbb{Z}[\zeta_3] \otimes \cdot \). Since \( Q(\zeta_{3m}) \) is a splitting field for \( H \), \( \varphi_{\mathcal{A}(\tilde{E})} \) is an isomorphism as shown above and hence \( \text{Ker } \varphi_{\mathcal{A}(\tilde{E})} \subseteq \text{Ker } \phi_A \). Composing \( \phi_A \) with the restriction map \( C(\mathcal{A}(\tilde{E})) \) to \( C(\mathcal{A}(E)) \), we see that the exponent of \( \text{Ker } \phi_A \) is at most 2 and therefore the order of \( \text{Ker } \varphi_{\mathcal{A}(E)} \) is a power of 2. However the order of \( \text{Ker } \varphi_{\mathcal{A}(\tilde{E})} \) is odd. Thus we must have \( \text{Ker } \varphi_{\mathcal{A}(\tilde{E})} = 0 \). This shows that \( \varphi_{\mathcal{A}(\tilde{E})}: C(\mathcal{A}(E)) \to C(\Gamma(\tilde{E})) \) is an isomorphism, which completes the proof of the proposition.

Lemma 2.3. Let \( \Pi \) be a finite group. Let \( \mathcal{A}_\Pi \) be a \( Z \)-order in \( Q\Pi \) containing \( Z\Pi \) which is a quasi-permutation \( \Pi \)-module and let \( \mathcal{O}_\Pi \) be a maximal \( Z \)-order in \( Q\Pi \) containing \( \mathcal{A}_\Pi \). Assume that \( \psi_{\mathcal{O}_\Pi}^\mathcal{A}_\Pi: C(\mathcal{A}_\Pi) \to C(\mathcal{O}_\Pi) \) is an isomorphism. Then \( C^*(Z\Pi) = C(Z\Pi) \).

Proof. Let \( [\mathfrak{A}] - [Z\Pi] \) be an element of \( C(Z\Pi) \). Since \( \psi_{\mathcal{O}_\Pi}^\mathcal{A}_\Pi \) is an isomorphism, we have \( \mathfrak{A} \oplus \mathcal{A}_\Pi \oplus \mathcal{O}_\Pi \cong Z\Pi \oplus \mathcal{A}_\Pi \oplus \mathcal{O}_\Pi \). There exists an exact sequence:

\[ 0 \to A_\Pi \oplus A_\Pi \oplus S \to S' \to 0 \]

where \( S \) and \( S' \) are permutation \( \Pi \)-modules. Then we easily see that \( \mathfrak{A} \oplus S \oplus S' \cong Z\Pi \oplus S \oplus S' \). This shows that \( [\mathfrak{A}] - [Z\Pi] \subseteq C^*(Z\Pi) \).

We are now ready to prove our main theorem.

Theorem 2.4. Let \( \Pi \) be any finite nilpotent group. Then \( C^*(Z\Pi) = C(Z\Pi) = C^*(Z\Pi) \).

Proof. It has been proved in [4], (3.2) that \( C^*(Z\Pi) = C(Z\Pi) \). Hence we only need to show that \( C^*(Z\Pi) = C(Z\Pi) \). Let \( Q\Pi = \sum_{i=1}^{r} \Sigma_i \) be the decomposition of \( Q\Pi \) into simple algebras. Applying [6], (14.3) or (14.5) to every \( \Sigma_i \) we can find a subgroup \( \Pi_i \) of \( \Pi \) and a simple component \( \Sigma'_i \) of \( Q\Pi_i \) such that \( \text{End}_{\Sigma_i}(Q\Pi_i \otimes \Sigma'_i) \cong \Sigma_i \) and \( \Pi_i/\text{Ker}(\Pi_i \to \Sigma'_i) \) is a special elementary group. Let \( E_i = \Pi_i/\text{Ker}(\Pi_i \to \Sigma'_i) \). Then \( \Sigma'_i \) can be identified with \( \Sigma(E_i) \). By (1.2) \( \Lambda(E_i) \) is a quasi-permutation \( \Pi_i \)-module, and therefore, if we put \( L_i = Z\Pi_i \otimes \Lambda(E_i) \), then \( L_i \) is a quasi-permutation \( \Pi \)-module. Let \( \Omega_3(E_i) \) be a maximal \( R(E_i) \)-order in \( \Sigma(E_i) \) containing \( \Lambda(E_i) \). Define \( L_i = \text{End}_{R(E_i)}(L_i) \) and \( \Omega_i = \text{End}_{R(E_i)}(L_i, \Omega(E_i)) \).
Then $A_i$ and $Q_i$ are $R(E_i)$-orders in $\Sigma_i$ with $A_i \subseteq Q_i$. Since $L_i$ is a free $A(E_i)$-module, $A_i$(resp. $Q_i$) is Morita equivalent to $A(E_i)$ (resp. $Q(E_i)$), and hence $Q_i$ is a maximal $R(E_i)$-order in $\Sigma_i$. Furthermore we see that $A_i$ is a quasi-permutation $\Pi_i$-module. By the Morita theorem we have $C(A_i) \cong C(A(E_i))$ and $C(Q_i) \cong C(Q(E_i))$. However, according to (2.2), $\psi_{\Pi_i}^E_i: C(A(E_i)) \to C(Q(E_i))$ is an isomorphism. Therefore $\psi_{\Pi_i}^E_i: C(A_i) \to C(Q_i)$ is also an isomorphism. We further put $A_{ii} = \bigoplus_{i=1}^{n} A_i$ and $Q_{ii} = \bigoplus_{i=1}^{n} Q_i$. Then $A_{ii}$ and $Q_{ii}$ are $Z$-orders in $QII$ with $ZII \subseteq A_{ii} \subseteq Q_{ii}$ and $Q_{ii}$ is a maximal order in $QII$. Here $A_{ii}$ is a quasi-permutation $\Pi$-module and $Q_{ii}$ is a maximal $\Pi$-order in $QII$. Thus we conclude by (2.3) that $C^q(ZII) = C(ZII)$, which completes the proof of the theorem.

3. Symmetric groups

Let $\Pi$ be a finite group and let $Q_{ii}$ denote a maximal order in $QII$ containing $ZII$. For a $\Pi$-module $M$ we denote by $\gamma_M$ the number of all isomorphism types of $\Pi$-modules, $L$, such that, for each prime $p \mid \Pi$, $L_p \cong M_p$ and $\Omega_{ii}L \oplus \Omega_{ii} \cong \Omega_{ii}M \oplus \Omega_{ii}$. For each prime $p \mid \Pi$ we denote by $\Pi^{(p)}$ a $p$-Sylow subgroup of $\Pi$.

We here prove the following proposition which will play a central part in §3 and §4.

**Proposition 3.1.** Let $\Pi$ be a finite group which is a direct product of a subgroup $\Pi'$ and a $p$-subgroup $P'$. Assume that $\Pi'$ is a semidirect product of a cyclic group $C$ of order prime to $p$ by an abelian $p$-group $P$ such that the action of $P$ on $C$ induces an isomorphism of $P$ onto $(\text{Aut } C)^{(p)}$. In the case where $p=2$, assume further that $P'$ is of split type over $Q$. Then $C^q(ZII) = C(ZII)$ and $B(QII) = G(QII)$.

**Proof.** In order to show that $C^q(ZII) = C(ZII)$ it suffices by [4], (2.2) to show that there exists a $ZII$-faithful, quasi-permutation $\Pi$-module $N$ with $|\gamma_N| = 1$. We will construct such $\Pi$-module $N$. Let $C = \langle \sigma \rangle$ and $n = \lvert C \rvert$. Let $n = q_1^{r_1}q_2^{r_2} \cdots q_t^{r_t}$ be the decomposition of $n$ into primes where $q_i$, $q_2$, ..., $q_t$ are distinct primes. Then $|\text{Aut } C| = \prod_{i=1}^{t} q_i^{r_i-1}(q_i - 1)$. Here we may assume that $p \nmid q_i - 1$ for $1 \leq i \leq s$ but $p \nmid q_i - 1$ for $s + 1 \leq i \leq t$. For every $1 \leq i \leq s$ let $c_i$ be a positive integer such that $p^{c_i} \mid q_i - 1$ but $p^{c_i+1} \nmid q_i - 1$. Since $QC/(\Phi_m(\sigma)) = Q(\xi_{n}) = Q(\xi_{q_1^{r_1}})Q(\xi_{q_2^{r_2}}) \cdots Q(\xi_{q_t^{r_t}})$, we have $P \cong (\text{Aut } C)^{(p)} = (\text{Aut } Q(\xi_{q_1^{r_1}})Q(\xi_{q_2^{r_2}}) \cdots Q(\xi_{q_t^{r_t}}))^{(p)}$, and therefore $P$ can be expressed as the direct product of the cyclic groups $\langle \tau_i \rangle$ of order $p^{c_i}$, $1 \leq i \leq s$, such that $\langle \tau_i \rangle \cap Q(\xi_{q_{j}}^{r_j}) = (\text{Aut } Q(\xi_{q_{j}}^{r_j}))^{(p)}$ but $\langle \tau_j \rangle \cap Q(\xi_{q_{j}}^{r_j}) = 1$ for $j \neq i$, $1 \leq j \leq t$.

We now have $QII \cong \bigoplus_{m=1}^{n} QII/(\Phi_m(\sigma))$. We easily see that $QII/(\Phi_m(\sigma))$
Define $\Sigma_n = Q \Pi_1^l(\Phi_n(\sigma))$ and $\Lambda_n = Z \Pi_1^l(\Phi_n(\sigma))$. Then $\Lambda_n$ is a quasi-permutation $\Pi'$-module (cf. [4]). Because $P = (\text{Aut} C)^{(p)}$, $\Sigma_n$ (resp. $\Lambda_n$) is isomorphic to the trivial crossed product of $Q(\zeta_n)$ (resp. $Z[\zeta_n]$) and $P$. Further define $L_n = \Lambda_n/\Lambda_n(\tau_1-1, \tau_2-1, \ldots, \tau_s-1) = Z(\zeta_n)$. Then $L_n$ is also a quasi-permutation $\Pi'$-module and $\text{End}_{Z}(L_n) \cong Z[\zeta_n]^p$. Let $Q \Pi' = \bigoplus_{k=1}^d \Sigma'_k$ be the decomposition of $Q \Pi'$ into simple algebras. For each $1 \leq k \leq d$ we denote by $F'_k$ the center of $\Sigma'_k$ and by $R'_k$ the ring of all algebraic integers in $Q(\zeta_n) \otimes F'_k$. In the proof of (2.5) we have shown that there exists a quasi-permutation $P'$-module $L'_k$ such that $\text{End}_{Z}(L'_k) \cong R'_k$. Since $p \mid n$, $Q(\zeta_n)^p \otimes F'_k$ is a field and $Z[\zeta_n]^p \otimes R'_k$ is the ring of all algebraic integers in $Q(\zeta_n)^p \otimes F'_k$. We have $\text{End}_{Q}(Q \Pi \otimes Q \Pi) \cong Q \Pi_k^p \otimes F'_k$ and $\text{End}_{Z}(L_k \otimes L'_k) \cong Z[\zeta_n]^p \otimes R'_k$, and therefore, by [3], $\langle \gamma \rangle \neq 1$. Let $N_k = \bigoplus_{k=1}^d (L_n \otimes L'_k)$. Then $N_k$ is a $Z \Pi(\Phi_n(\sigma))$-faithful, quasi-permutation $\Pi'$-module with $|\gamma_{N_k}| = 1$.

Let $m \mid n$, $m < n$ and let $m = q_1^{\alpha_1} \cdots q_s^{\alpha_s} \cdots q_t^{\alpha_t}$ be the decomposition of $m$ into primes where $1 \leq i_1 < \cdots < i_s \leq t$ and $s+1 \leq j_1 < \cdots < j_t \leq s$. We define $\Pi_m = \Pi(\sigma_m)$, $C_m = C(\sigma_m) = \langle \sigma_m \rangle$, $P_m = \prod_{i=1}^s \tau_{i_k}$ and $P'_m = (\prod_{i=1}^s \tau_{j_k}) \times P'$. Further let $\Pi'_m$ be the semidirect product of $C_m$ by $P_m$ with the action of $P_m$ on $C_m$ induced by that of $P$ on $C$. Then $\Pi'_m$ can be identified with the direct product of $\Pi'_m$ and $P'_m$, and the action of $P_m$ on $C_m$ induces an isomorphism of $P_m$ onto $(\text{Aut} C)^{(p)}$. We have $Z \Pi(\Phi_n(\sigma)) \cong Z \Pi_m(\Phi_m(\sigma_m))$. Therefore, applying the preceding method to $\Pi'_m$, we can construct a $Z \Pi(\Phi_n(\sigma))$-faithful, quasi-permutation $\Pi'$-module $N_m$ with $|\gamma_{N_m}| = 1$. If we put $N = \bigoplus_{m \mid n} N_m$, then $N$ is a $Z \Pi(\Phi_n(\sigma))$-faithful, quasi-permutation $\Pi'$-module with $|\gamma_N| = 1$ as required. This proves that $\mathcal{C}(Z \Pi) = \mathcal{C}(Z \Pi)$.

Let $V$ be any simple $Z \Pi$-module. In the above proof we see that there exists a quasi-permutation $\Pi'$-module $L$ such that $Q \Pi \cong V$. Hence the class of $V$ in $G(Z \Pi)$ is contained in $B(Z \Pi)$. This shows that $B(Q \Pi) = G(Z \Pi)$, which completes the proof.

Lemma 3.2. Let $S_n$ be the symmetric group on $n$ symbols. Let $E$ be a maximal hyperelementary subgroup of $S_n$ at a prime $p$. Then:

$$E \cong H \times S^{(p)}_{l_i} \times S^{(p)}_{l_2} \times \cdots \times S^{(p)}_{l_s},$$

where $H$ is a semidirect product of a cyclic group $C$ of order prime to $p$ by an abelian $p$-group $P$ such that the action of $P$ on $C$ induces an isomorphism of $P$ onto $(\text{Aut} C)^{(p)}$, and every $S_{l_i}$ denotes the symmetric group on $l_i$ symbols.

Proof. This lemma may be well known. However, for completeness, we
will give a proof of it. Since $E$ is hyperelementary at $p$, there exists a cyclic normal subgroup $C = \langle \sigma \rangle$ of $E$ of order prime to $p$ such that $E/C$ is a $p$-group. We have $E \subseteq N_{S_n}(C)$ and therefore $E$ is conjugate to $C \cdot N_{S_n}(C)^{\langle p \rangle}$ in $N_{S_n}(C)$ because $E$ is maximal hyperelementary. Let $\sigma = \sigma_1^{(r_1)} \cdots \sigma_t^{(r_t)} \cdots \sigma_1^{(r_1)} \cdots \sigma_t^{(r_t)}$ be the decomposition of $\sigma$ into cycles which do not contain common symbols where $n \geq r_1 > r_2 > \cdots > r_t \geq 1$ and every $\sigma_j^{(r_j)}$ is an $r_j$-cycle. We denote the Euler function by $\varphi(\cdot)$. Let $m = |C|$ and let $\{k_1, k_2, \ldots, k_{\varphi(m)}\}$ be the set of all integers $k$ such that $(k, m) = 1$ and $1 \leq k < m$. Then, for every $k_h$, $1 \leq h \leq \varphi(m)$, there exists $\tau_{k_h} \in S_n$ such that $\tau_{k_h}^{-1} \sigma_j^{(r_j)} \tau_{k_h} = (\sigma_j^{(r_j)})^{k_h}$ for all $1 \leq i \leq t$ and $1 \leq j \leq l_i$. Put $K = \langle \sigma_1^{(r_1)}, \ldots, \sigma_i^{(r_i)}, \ldots, \sigma_t^{(r_t)}, \tau_{k_1}, \ldots, \tau_{k_{\varphi(m)}} \rangle \subseteq N_{S_n}(C)$ and $P = K^{\langle p \rangle}$. Then the action of $P$ on $C$ induces an isomorphism of $P$ onto $(\text{Aut } C)^{\langle p \rangle}$. Further, for each $1 \leq i \leq t$, let $S_i$ denote the symmetric group on $l_i$ symbols $\{\sigma_1^{(r_1)}, \sigma_2^{(r_2)}, \ldots, \sigma_t^{(r_t)}\}$. Each $S_i$ can be regarded as a subgroup of $N_{S_n}(C)$, and we have $N_{S_n}(C) = K \times S_1 \times S_2 \times \cdots \times S_t$. Hence $N_{S_n}(C)^{\langle p \rangle} = P \times S_1^{(p)} \times S_2^{(p)} \times \cdots \times S_t^{(p)}$, and so $E \cong C \cdot N_{S_n}(C)^{\langle p \rangle} \cong CP \times S_1^{(p)} \times S_2^{(p)} \times \cdots \times S_t^{(p)}$. This concludes the proof of the lemma.

We now come to the main theorem of this section.

**Theorem 3.3.** Let $S_n$, $n \geq 1$, be the symmetric group on $n$ symbols. Then $C^q(ZS_n) = C(ZS_n) = C(ZS_n) = C(ZS_n)$.

**Proof.** Since $Q$ is a splitting field for $S_n$, we have $C(Q_{S_n}) = 0$, hence $C(ZS_n) = C(ZS_n)$. Therefore we only need to show that $C^q(ZS_n) = C(ZS_n)$. According to the induction theorem ([4], §1), it suffices to prove that, for every maximal hyperelementary subgroup $E$ of $S_n$, $C^q(ZE) = C(ZE)$. However $Q$ is also a splitting field for $S_n$, $l \geq 1$ (e.g. [8], (5.9)). Therefore this follows immediately from (3.1) and (3.2).

**Remark 3.4.** (1) $C(ZS_n) = 0$ for $n \leq 4$. (2) For every odd prime $p$ $|C(ZS_p)| \geq \frac{1}{2} (p-1)$ and $|C(ZS_{p+1})| \geq \frac{1}{2} (p-1)$.

**Proof.** The assertion (1) is well known. We will prove the assertion (2) only on $S_p$ because we can prove the one on $S_{p+1}$ in the same way. Let $\sigma$ be a $p$-cycle in $S_p$ and define $K = N_{S_p}(\langle \sigma \rangle)$. Then $K$ can be expressed as a semidirect product of the cyclic subgroup $\langle \sigma \rangle$ by a cyclic subgroup $H$ of order $p-1$ such that the action of $H$ on $\langle \sigma \rangle$ induces an isomorphism of $H$ onto $\text{Aut } \langle \sigma \rangle$. We have $N_{S_p}(K) = K$, and if $K \neq \rho^{-1} K \rho$, $\rho \in S_p$, then $K \cap \rho^{-1} K \rho \subseteq \mu^{-1} H \mu$ for some $\mu \in K$. Let $f : C(ZH) \rightarrow C(ZK)$ and $g : C(ZK) \rightarrow C(ZS_p)$ be the natural homomorphism induced by $ZK \otimes_{\mathbb{Z}_p} \cdot$ and $ZS_p \otimes_{\mathbb{Z}_p} \cdot$, respectively.

Using the Mackey's subgroup theorem we see that $\text{Ker } g \subseteq \text{Im } f$, and so $|C(ZK)|/|C(ZH)| \leq |C(ZS_p)|$. By virtue of [7], (1.2) we have $|C(ZK)| = \ldots$
\[ \frac{1}{2} (p-1) | C(ZH) |. \] This shows that \( |C(ZS_p)| \geq \frac{1}{2} (p-1). \)

4. Alternating groups

In this section we will be concerned with alternating groups.

**Proposition 4.1.** Let \( A_n, n \geq 3, \) be the alternating group on \( n \) symbols. Then both \( C(ZA_n)/C^a(ZA_n) \) and \( G(QA_n)/B(QA_n) \) are 2-groups.

**Proof.** Let \( E' \) be a maximal hyperelementary subgroup of \( A_n \) at a prime \( p \). Then there exists a maximal hyperelementary subgroup \( E \) of \( S_n \) at \( p \) such that \( E' = E \cap A_n \). We can write \( E = CP \) where \( C \) is a cyclic normal subgroup of \( E \) with \( p \nmid |C| \) and \( P \) is a \( p \)-subgroup of \( E \). Assume that \( p \) is odd. If \( |C| \) is odd, then \( E' = E \) and therefore, by (3.1) and (3.2), \( C(ZE') = C(ZE') \) and \( B(QE') = G(QE') \). If \( |C| \) is even, then \( C \) is expressible as a direct product of a subgroup \( C' \) with \( 2 \nmid |C'| \) and a 2-subgroup \( C_2 \). Let \( C' = C_2 \cap E' \). Then \( E' = C_2 \times C_2' \) and so, again by (3.1), \( C^a(ZE') = C(ZE') \) and \( B(QE') = G(QE') \). Next assume that \( p = 2 \). Then the Artin exponent of \( E' \) is a power of 2 ([9], §7), and therefore, by the Artin induction theorem ([4], §1), both \( C(ZE')/C^a(ZE') \) and \( G(QE')/B(QE') \) are 2-groups. Applying the Witt-Berman induction theorem ([4], §1), we can conclude that both \( C(ZA_n)/C^a(ZA_n) \) and \( G(QA_n)/B(QA_n) \) are 2-groups.

**Remark 4.2.** (1) \( C(ZA_n) = 0 \) for \( n \leq 5 \). (2) For every prime \( p \) with \( p \equiv 3 \) mod 4 \( |C(ZA_p)| \geq \frac{1}{2} (p-1) \) and \( |C(ZA_{p+1})| \geq \frac{1}{2} (p-1) \) and for every prime \( p \) with \( p \equiv 1 \) mod 4 \( |C(ZA_p)| \geq \frac{1}{4} (p-1) \) and \( |C(ZA_{p+1})| \geq \frac{1}{4} (p-1) \). (3) \( C^a(ZA_7) = C^a(ZA_7) = C(ZA_7) = 0 \) and \( C^a(ZA_8) = C^a(ZA_8) = C(ZA_8) \).

**Proof.** The assertion (1) can easily be shown, using the induction theorem, and the assertion (2) can be shown in the same way as in (3.4), (2). The assertion (3) follows from the induction theorem and [4], (5.5).

**Lemma 4.3.** Let \( \Pi = \langle \sigma \rangle \) be a cyclic group of order \( n \) and let \( n = p_1 p_2 \cdots p_r \) be the decomposition of \( n \) into primes where \( p_1, p_2, \ldots, p_r \) are distinct primes. Let \( \chi_1 \) and \( \chi_2 \) be rational characters of \( \Pi \). Assume that, for every proper subgroup \( \Pi' \) of \( \Pi \), \( \chi_1|\Pi' = \chi_2|\Pi' \). Then:

\[ (\chi_1, 1_{\Pi}) - (\chi_2, 1_{\Pi}) = (\chi_1(\sigma) - \chi_2(\sigma)) \prod_{i=1}^r \left( 1 - \frac{1}{p_i} \right), \]

where \( 1_{\Pi} \) denotes the one dimensional trivial character of \( \Pi \).

**Proof.** For every \( m | n \) let \( \alpha_m \) denote the character of \( \Pi \) afforded by the
irreducible $QII$-module $Q(\zeta m)$. Then we can write
\[ \chi_1 - \chi_2 = \sum_{m} c_m \chi_m, \quad c_m \in \mathbb{Z}. \]

Restricting both sides to the subgroup $<\sigma^h_i>$, we see that $c_m = 0$ when $p_i | m$. Hence $\chi_1 - \chi_2 = \sum c_m \alpha_m$. Furthermore, restricting both sides to the subgroup $<\sigma^h_i>$, we see that $c_1 + (p_i - 1)c_{P_j} = 0$ and $c_{P_j} + (p_i - 1)c_{P_j} = 0$ whenever $p_i \in \{p_j, p_{j'}, \ldots, p_j\}$. Clearly $\alpha_m(\sigma)$ coincides with the Möbius function $\mu(m)$. Therefore we have
\[ \chi_1(\sigma) - \chi_2(\sigma) = \sum_{m} c_m \mu(m) \]
\[ = c_1 \left(1 + \sum_{i=1}^{l} \frac{1}{p_i} + \frac{1}{(p_i - 1)(p_i - 1)} + \cdots \right) \]
\[ = c_1 \prod_{i=1}^{l} \frac{p_i}{p_i - 1}. \]

However $(\chi_1, 1n) - (\chi_2, 1n) = (\sum c_m \alpha_m, \alpha_1) = c_1$. Thus we get $(\chi_1, 1n) - (\chi_2, 1n) = (\chi_1(\sigma) - \chi_2(\sigma)) \prod_{i=1}^{l} \left(1 - \frac{1}{p_i}\right)$.

Let $S_n, A_n$ be the symmetric, alternating group on $n$ symbols, respectively. Let $I_n$ denote the image of the restriction map $G(QS_n) \to G(QA_n)$ and let $\tau = (1, 2) \in S_n$. For $\chi \in G(QA_n)$, $\chi \in I_n$ if and only if $\chi' \neq \chi$. Therefore $G(QA_n)/I_n$ is a free abelian group generated by all the classes of irreducible rational characters, $\chi$, of $A_n$ with $\chi' \neq \chi$. Every irreducible rational character $\chi$ of $A_n$ with $\chi' \neq \chi$ is absolutely irreducible. Hence there is a one to one correspondence between pairs, $(\chi, \chi')$, $\chi' \neq \chi$, of irreducible rational characters of $A_n$ and partitions,
\[ n = c_1 + c_2 + \cdots + c_t, \]

such that $c_1 < c_2 < \cdots < c_t$, $2 \nmid c_t$ and $\prod_{i=1}^{t} c_i$ is square. Let $\chi$ be an irreducible rational character of $A_n$ with $\chi' \neq \chi$ and let $n = c_1 + c_2 + \cdots + c_t$ be the partition corresponding to $(\chi, \chi')$. Define $\sigma_{x} = (1, 2, \ldots, c_t) (c_t + 1, \ldots, c_t + c_2) \cdots (c_1 + 1, \ldots, \sum_{i=1}^{t} c_i)$ and let $C_x, C'_{x}$ denote the conjugate classes of $A_n$ containing $\sigma_x, \tau \sigma_x \tau$, respectively. Then we have
\[ \chi(\sigma) = \chi'^{s}(\sigma) = \frac{1}{2} \chi' s_{x}(\sigma) \quad \text{for every } \sigma \in C_x \cup C'_x, \]
\[ \chi(\sigma_x) = \frac{1}{2} (1 + d), \quad \chi'(\sigma_x) = \frac{1}{2} (1 + d) \]
where \( d \) denotes the positive integer with \( d^2 = \prod c_i \). Let \( H_x = \langle \sigma_x \rangle \), let \( K_x^\prime \) be a 2-Sylow subgroup of \( N_{S_n}(H_x) (= N_{A_n}(H_x)) \) and put \( K_x = H_x K_x^\prime \). Let \( n_x \) be the odd part of \( \varphi(d) \). Then \( |H_x| = d^2 \) and \( |K_x^\prime| = \varphi(d)/n_x \). We can here prove

**Lemma 4.4.** Let \( \chi \) be an irreducible rational character of \( A_n \) with \( \chi \in I_n \). Then \( n_x \cdot \chi - 1_{K_x^\prime}^A \subseteq I_n \).

**Proof.** We may assume that \( \chi(\sigma_x) = \frac{1}{2}(1 + d) \) and \( \chi'(\sigma_x) = \frac{1}{2}(1 - d) \). For every proper subgroup \( H' \) of \( H_x \), \( \chi/H' = \chi'/H' \) and \( \chi(\sigma_x) - \chi'(\sigma_x) = d \). Therefore by (4.3) and the Frobenius reciprocity theorem we get \( (\chi, 1_{H_x}^A) - (\chi', 1_{H_x}^A) = (\chi/H_x, 1_{H_x}) - (\chi'/H_x, 1_{H_x}) = \varphi(d) \), and so \( (\chi, \varphi(d) \cdot \chi - 1_{H_x}^A) = (\chi', \varphi(d) \cdot \chi - 1_{H_x}^A) \). Since generators of proper subgroups of \( H_x \) are not of type (*), for any irreducible rational character \( \chi' \) of \( A_n \), with \( \chi' \not\equiv \chi \), \( \chi' \) we have \( (\chi', \varphi(d) \cdot \chi - 1_{H_x}^A) = (\chi', \varphi(d) \cdot \chi - 1_{H_x}^A) \). Therefore \( \varphi(d) \cdot \chi - 1_{H_x}^A \subseteq I_n \). On the other hand, applying the Brauer coefficient theorem ([2], Satz 1) to \( K_x \), we get

\[
|K_x| \cdot 1_{K_x} = |H_x| \cdot 1_{K_x}^A + \sum_{H' \subseteq H_x} b_{H'} \cdot 1_{H'}^A, \quad b_{H'} \in \mathbb{Z}
\]

where \( H' \) runs over all cyclic subgroups \( \not\subseteq H_x \) of \( K_x \). Since generators of subgroups, \( H' \subseteq H_x \), of \( K_x \) are not of type (*), we have \( 1_{H'}^A \subseteq I_n \) and so \( b_{H'} \cdot 1_{H'}^A \subseteq I_n \). Therefore \( |H_x| \cdot 1_{K_x}^A \subseteq |K_x| \cdot 1_{K_x}^A \subseteq I_n \). Thus we have \( |K_x| (n_x \cdot \chi - 1_{K_x}^A) = |H_x| \varphi(d) \cdot \chi - 1_{K_x}^A \subseteq I_n \). Hence \( G(QA_n)/I_n \) is torsion-free, this shows that \( n_x \cdot \chi - 1_{K_x}^A \subseteq I_n \).

We now establish the following:

**Theorem 4.5.** Let \( A_n, n \geq 3 \), be the alternating group on \( n \) symbols. Then \( B(QA_n) = G(QA_n) \).

**Proof.** Since \( B(QS_n) = G(QA_n) \) as is well known, we see that \( I_n \subseteq B(QA_n) \). It follows immediately from (4.4) that \( [G(QA_n): B(QA_n)] \) is odd. However, by (4.1), \( G(QA_n)/B(QA_n) \) is a 2-group. Thus we have \( B(QA_n) = G(QA_n) \).

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**References**


