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## THE AUSLANDER-REITEN QUIVER OF A RING OF RIGHT LOCAL TYPE

Dedicated to Professor Manabu Harada on his 60<sup>th</sup> birthday

HIDETO ASASHIBA

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Throughout this paper,  $A$  is a right and left artinian ring and  $J$  is the Jacobson radical of  $A$ . All modules are assumed to be finitely generated over  $A$ . As well known Auslander-Reiten sequences (abbreviated AR-sequences) exist over arbitrary algebras and over representation-finite rings [6], [7], [27]. An AR-sequence of modules over a ring  $A$  and the indecomposable decomposition of the middle term of this sequence define a mesh in the Auslander-Reiten quiver (AR-quiver for short) of  $A$ , and meshes determine the AR-quiver of  $A$ . For the representation-finite rings the computation of AR-quiver gives all indecomposable modules and all non-isomorphisms between them up to isomorphisms. In the algebra case we have a general way to compute AR-sequences stopping at any non-projective modules [11], [14] although the computation of the decomposition of the middle terms is not clear. However this computation is heavily depends on the existence of the selfduality of algebras. Hence for an arbitrary artinian ring even for a representation-finite ring, this computation is not available. Over such a ring we therefore have to compute AR-sequences individually.

In this paper we will compute (1) all the AR-sequences including the indecomposable decompositions of the middle terms (Theorem I), (2) full subquivers of the AR-quiver which give the whole of the AR-quiver by gluing together and (3) all the meshes (including values) in the AR-quiver (Theorem II), over a special type of representation-finite rings, namely over a ring of right local type. As a corollary we will obtain that the AR-quiver of a ring  $A$  of right local type is a *well valued* translation quiver (i.e. the value  $(a, b)$  of any arrow of the AR-quiver maps to  $(b, a)$  by the translation between arrows associated to the AR-translation; see section 3 for detail) iff  $A$  is, in addition, a ring of left colocal type. Further for a ring  $A$  defined by a bimodule  $M$  over division rings, we will give a necessary and sufficient condition for  $A$  to be of right local type in terms of the dimension sequence of  $M$ .

In section 1 we first quote Sumioka's result on a ring of right local type,

and investigate further properties of this kind of rings for the later use. Section 2 is devoted to the computation of AR-sequences over a ring of right local type. Finally in section 3 we draw the form of the AR-quiver of a ring of right local type and compute its value to obtain Theorem II. Further we give an example of a ring of right local type but not of left colocal type, which we presented in a local conference [3], and an example of a ring of left colocal type but not of right local type.

For a module  $M$ , we denote by  $|M|$  and by  $h(M)$  the composition length and the *height* (=the Lowey length) of  $M$ , respectively. And  $\#I$  denotes the cardinality of a set  $I$ .

I would like to thank Dr. M. Hoshino for his useful advice, which enabled me to remove the assumption in the original version that our ring in the title is also of left colocal type, by showing me Lemmas 2.2, 2.4 and to add the section 3 in the revised version, and Professor Sumioka for his short proof of Lemma 1.7.

## 1. Rings of right local type

Recall that  $A$  is said to be a ring of *right local type* in case every indecomposable right  $A$ -module is local (i.e. its top is simple), and dually  $A$  is called a ring of *left colocal type* in case every indecomposable left  $A$ -module is colocal (i.e. its socle is simple). Sumioka [23], [24] had made a precise investigation of rings of right local type and of rings of left colocal type (see also Tachikawa [25], [26] and Yoshii [28]). His result is summarized as follows:

**Theorem 1.0** (Sumioka [24]). *Consider the following conditions:*

- (LR) (a)  $A$  is left serial;
- (b)  $J_A$  is a direct sum of uniserial modules; and
- (c)  $|eJ/eJ^2| \leq 2$  for any primitive idempotent  $e$  of  $A$  with  $eJ/eJ^2$  not homogeneous.

(L)  $[D_1(U): D_2(U)]_r \leq 2$  for any uniserial left  $A$ -module  $U$  with  $|U| \geq 2$  (see [23] or [24] for the definition of  $D_i(U)$ ).

(R)  $[D_1(U): D_2(U)]_l \leq 2$  for any uniserial left  $A$ -module  $U$  with  $|U| \geq 2$ .

Then

(1) If  $A$  is a ring of left colocal type, then  $A$  satisfies the conditions (LR) and (L).

(2) If  $A$  is a ring of right local type, then  $A$  satisfies the conditions (LR) and (R).

(3) If  $A$  satisfies the conditions (LR), (L) and (R), then  $A$  is a ring of right local type and of left colocal type.

REMARK 1.1. Let  $A$  be a left serial ring. Then the following are equi-

valent (see [24, Remark 5] for detail):

- (1)  $|eJ/eJ^2| \leq 2$  for every primitive idempotent  $e$  of  $A$ .
- (2)  $A$  satisfies the conditions (LR) (c) and (L).

With this remark in mind we have the following corollary.

**Corollary 1.2.** *Assume that  $A$  is a ring of right local type. Then  $A$  is a ring of left colocal type iff  $|eJ/eJ^2| \leq 2$  for every primitive idempotent  $e$  of  $A$ .*

**Proposition 1.3.** *If  $A$  is a ring of right local type and  $e$  is a primitive idempotent of  $A$  with  $eJ^2 \neq 0$ , then  $|eJ/eJ^2| \leq 2$ .*

Proof. See [2]. //

By Theorem 1.0 and Proposition 1.3, we can determine the form of every projective indecomposable right module over a ring of right local type as follows.

**Proposition 1.4.** *Let  $A$  be a ring of right local type and  $e$  a primitive idempotent of  $A$ . Then  $eA$  has one of the following forms:*

- (a)  $eA$  is uniserial.
- (b)  $eJ = K \oplus L$  for some non-zero uniserial modules  $K, L$  which are not isomorphic to each other.
- (c)  $eJ = K \oplus L$  for some non-zero uniserial modules  $K, L$  which are isomorphic to each other.
- (d)  $eJ$  is a homogeneous semisimple module of length  $> 2$ .

**DEFINITION 1.5.** Let  $M$  be a subset of  $A$ . Then  $\mathbf{d}(M) := \sup \{n \geq 0 \mid M \subseteq J^n\}$  is called the *depth* of  $M$ . When  $M = \{x\}$  for some  $x \in A$ , we simply denote  $\mathbf{d}(\{x\})$  by  $\mathbf{d}(x)$ . By definition  $\mathbf{d}(0) = \infty$ .

**NOTATION 1.6.** Let  $a \in A$ . Then we denote by  $a \cdot$  the left multiplication by  $a$ .

The following determines the form of submodules of projective indecomposable modules  $P$  with  $|PJ/PJ^2| \leq 2$  over a ring of right local type. This is taken from Harada [15]. For the benefit of the reader we give a proof which is due to Sumioka.

**Lemma 1.7.** *Let  $A$  be a left serial ring and  $e$  a primitive idempotent of  $A$ . If  $eJ = U_1 \oplus U_2$  with each  $U_i$  uniserial and  $M \leq eJ$ , then there is an automorphism  $\alpha$  of  $eA$  such that  $\alpha(M) = U_1 J^s \oplus U_2 J^t$  for some  $s, t \geq 0$ .*

Proof. Put  $d := \mathbf{d}(M) \geq 1$ . Then  $M \leq eJ^d = U_1 J^{d-1} \oplus U_2 J^{d-1}$ , and there exists some  $u \in M$  such that  $\mathbf{d}(u) = d$  with  $u = uf$  for some primitive idempotent  $f$  of  $A$ . Write  $u = u_1 + u_2$  for some  $u_i \in U_i$ . Then  $\mathbf{d}(u_i) = d$  for some  $i$ , say  $i = 1$ . This implies that  $u_1 A = U_1 J^{d-1}$ . Since  $Af$  is uniserial, we have  $au = u_1$  for some

$a \in eAe \setminus eJe$ . Noting that  $aM \geq u_1A = U_1J^{d-1}$ , we have  $aM = aM \cap eJ^d = U_1J^{d-1} \oplus (aM \cap U_2J^{d-1})$ . Hence  $a \cdot$  gives a desired map. //

**Proposition 1.8.** *Assume that  $A$  is a ring of right local type. Then every indecomposable right  $A$ -module has one of the following forms:*

(a)  $eA/(KJ^s \oplus LJ^t)$  for some primitive idempotent  $e$  of  $A$  with  $eJ = K \oplus L$  where  $K$  and  $L$  are uniserial modules and  $s, t \geq 0$ .

(b)  $eA/(\bigoplus_{i=1}^m S_i)$  for some  $0 \leq m \leq n$ , where  $eJ = \bigoplus_{i=1}^n S_i$ ,  $S_1 \cong \cdots \cong S_n$  are simple and  $n \geq 3$ .

Proof. (a) follows by Lemma 1.7. (b) follows by Lemma 1.13 below. //

**DEFINITION 1.9.** Let  $\alpha = (\alpha_i)^T: X \rightarrow \bigoplus_{i=1}^n Y_i$  be a homomorphism of right modules and let  $1 \leq j \leq n$ . Then we say that  $\alpha$  is  $j$ -fusible in case  $\alpha_j = \sum_{i \neq j} \beta_i \alpha_i$  for some  $(\beta_i)_{i \neq j}: \bigoplus_{i \neq j} Y_i \rightarrow Y_j$ . Further  $\alpha$  is said to be *fusible* in case  $\alpha$  is  $j$ -fusible for some  $1 \leq j \leq n$ , and  $\alpha$  is said to be *infusible* if  $\alpha$  is not fusible.

**REMARK 1.10.** (1) In this paper we use this definition for the case  $n=2$  as follows:

Let  $X_i < Y_i$  for each  $i=1, 2$  and let  $\alpha: X_1 \rightarrow X_2$  be an isomorphism. Then  $\alpha$  is extendable to  $Y_1 \rightarrow Y_2$  or  $\alpha^{-1}$  is extendable to  $Y_2 \rightarrow Y_1$  iff  $(\sigma_1, \sigma_2 \alpha)^T: X_1 \rightarrow Y_1 \oplus Y_2$  is fusible, where  $\sigma_i$  are the inclusion maps  $X_i \rightarrow Y_i$ .

(2) In the above definition, if all  $Y_i$  are indecomposable, then  $\alpha$  is infusible iff it is *left minimal* in the sense of Auslander-Reiten [8].

**Lemma 1.11.** Let  $X \xrightarrow{(\alpha_i)^T} \bigoplus_{i=1}^n Y_i \xrightarrow{(\beta_i)} Z \rightarrow 0$  be an exact sequence of right modules. Then

(1) For every  $1 \leq j \leq n$ ,  $(\alpha_i)^T$  is  $j$ -fusible iff  $\beta_j$  is a section.

(2) Assume that  $X$  is simple and all  $Y_i$  are local. Then  $(\alpha_i)^T$  is infusible iff  $Z$  is indecomposable.

Proof. See [1]. //

The above lemma gives a characterization of a ring of right local type, which had been obtained with the following proof before [24, Proposition 1.3].

**Proposition 1.12.** *The following are equivalent:*

(1)  $A$  is of right local type.

(2) If  $S$  is a simple right  $A$ -module, and  $L_1, L_2$  are local right  $A$ -modules, then any homomorphism  $(\alpha_1, \alpha_2)^T: S \rightarrow L_1 \oplus L_2$  is fusible.

Proof. (1)  $\Rightarrow$  (2). Clear from Lemma 1.11 (2).

(2)  $\Rightarrow$  (1). We show that any  $M_A$  is expressed as a direct sum of local modules

by induction on  $|M|$ . It suffices to show that  $M$  has a local direct summand. Let  $S$  be a simple submodule of  $M$ . Then by induction hypothesis  $M/S = \bigoplus_{i=1}^n L_i/S$  for some  $S \leq L_i \leq M$  with all  $L_i/S$  local. If  $S \not\leq L_i J$  for some  $i$ , then  $L_i = S \oplus L'_i$  for some  $L'_i$ , which must be a local direct summand of  $M$ . Hence we may suppose that every  $L_i$  is local, and  $n > 1$ . Let  $\sum_{i=1}^n L_i = \bigoplus_{i=2}^n M$  with every  $M_i$  local. Then we have an exact sequence

$$0 \rightarrow S \xrightarrow{(\alpha_i)^T} L_1 \oplus M_2 \oplus \cdots \oplus M_n \xrightarrow{(\beta_i)} M \rightarrow 0$$

in an obvious way. By (2),  $(\alpha_1, \alpha_2)^T$  is fusible, whence so is  $(\alpha_i)^T$ . Thus  $\beta_i$  is a section for some  $i$  by Lemma 1.11 (1). //

The following is an immediate consequence of the above proposition.

**Lemma 1.13.** *Let  $A$  be a ring of right local type and  $e$  a primitive idempotent of  $A$  with  $eJ$  a homogeneous semisimple module of length  $> 2$ . Then  $eA/I \cong eA/I'$  iff  $|I| = |I'|$  for any  $I, I' \leq eJ$ .*

**Lemma 1.14.** *Let  $X_i \leq Y_i$ , for each  $i = 1, 2$ , and  $\alpha: X_1 \rightarrow X_2$  an isomorphism. Put  $M$  to be the cokernel of the map  $(\sigma_1, \sigma_2 \alpha)^T: X_1 \rightarrow Y_1 \oplus Y_2$ , where  $\sigma_i$  are the inclusion maps. Then  $\text{soc } M \cong \text{soc } Y_2$  iff  $\alpha$  is maximal, i.e., for any  $H > X_1$  any homomorphism from  $H$  to  $Y_2$  is not an extension of  $\alpha$ .*

Proof. See [23]. //

Next we state the dual notions and the dual statements of the above. See [24] for the proof of Lemma 1.18. The remaining proofs are left to the reader.

**DEFINITION 1.15.** Let  $\beta = (\beta_i): \bigoplus_{i=1}^n Y_i \rightarrow Z$  be a homomorphism of right modules and let  $1 \leq j \leq n$ . Then we say that  $\beta$  is *j-cofusible* in case  $\beta_j = \sum_{i \neq j} \beta_i \alpha_i$  for some  $(\alpha_i)_{i \neq j}^T: Y_j \rightarrow \bigoplus_{i \neq j} Y_i$ . Further  $\beta$  is said to be *cofusible* in case  $\beta$  is *j-cofusible* for some  $1 \leq j \leq n$ , and  $\alpha$  is said to be *coinfusible* if  $\alpha$  is not cofusible.

**REMARK 1.16.** (1) In this paper we use this definition for the case  $n=2$  as follows:

Let  $X_i < Y_i$  for each  $i = 1, 2$  and let  $\beta: Y_1/X_1 \rightarrow Y_2/X_2$  be an isomorphism. Then  $\beta$  is liftable to  $Y_1 \rightarrow Y_2$  or  $\beta^{-1}$  is liftable to  $Y_2 \rightarrow Y_1$  iff  $(\beta \pi_1, \pi_2): Y_1 \oplus Y_2 \rightarrow Y_2/X_2$  is cofusible, where  $\pi_i$  are the canonical epimorphisms  $Y_i \rightarrow Y_i/X_i$ .

(2) If in the above definition all  $Y_i$  are indecomposable, then  $\beta$  is coin fusible iff it is *right minimal* in the sense of [8].

**Lemma 1.17.** *Let  $0 \rightarrow X \xrightarrow{(\alpha_i)^T} \bigoplus_{i=1}^n Y_i \xrightarrow{(\beta_i)} Z$  be an exact sequence of right*

modules. Then

- (1) For every  $1 \leq j \leq n$ ,  $(\beta_i)$  is  $j$ -cofusible iff  $\alpha_j$  is a retraction.
- (2) Assume that  $Z$  is simple and all  $Y_i$  are colocal. Then  $(\beta_i)$  is coinfusible iff  $X$  is indecomposable.

**Lemma 1.18.** Let  $X_i \leq Y_i$ , for each  $i=1, 2$ , and  $\beta: Y_1/X_1 \rightarrow Y_2/X_2$  an isomorphism. Put  $M$  to be the kernel of the map  $(\beta\pi_1, \pi_2): Y_1 \oplus Y_2 \rightarrow Y_2/X_2$ , where  $\pi_i$  are the canonical epimorphisms  $Y_i \rightarrow Y_i/X_i$ . Then  $\text{top } M \cong \text{top } Y_1$  iff  $\beta$  is comaximal, i.e., for any  $H < X_2$  any homomorphism from  $Y_1$  to  $Y_2/H$  is not a lift of  $\beta$ .

By Lemma 1.17 (2) and Lemma 1.18, we get the following, which states that a ring of right local type has a property dual to the property [25, Theorem 5.3 II] of a ring of left colocal type.

**Lemma 1.19.** Let  $A$  be a ring of right local type, and  $Y_1, Y_2$  uniserial right  $A$ -modules such that  $|Y_1| \geq |Y_2| > 2$ . Then any isomorphism  $\text{top } Y_1 \rightarrow \text{top } Y_2$  is comaximal or is liftable to an epimorphism  $Y_1 \rightarrow Y_2$ .

*Proof.* We keep the notation of Lemma 1.18. Let  $\beta: Y_1/X_1 \rightarrow Y_2/X_2$  be an isomorphism, where  $X_i := Y_i J$ . If  $\beta$  is not comaximal, then  $M$  is not local by Lemma 1.18. Thus  $M$  is decomposable. Hence the assertion follows by Lemma 1.17(2). //

The following is a key for the computation of values of AR-quiver of a ring of right local type.

**Proposition 1.20.** Let  $A$  be a ring of right local type and  $e$  a primitive idempotent of  $A$  such that  $eJ = K \oplus L$ ;  $K, L$  are non-zero and uniserial. Then the following are equivalent:

- (1)  $eA/(KJ^s \oplus LJ^{s+1}) \cong eA/(KJ^{s+1} \oplus LJ^s)$  for some  $s \geq 0$ .
- (2)  $eA/(KJ^s \oplus LJ^{s+1}) \cong eA/(KJ^{s+1} \oplus LJ^s)$  for all  $s \geq 0$ .
- (3)  $eA/(K \oplus LJ) \cong eA/(KJ \oplus L)$ .
- (4)  $eA/K \cong eA/L$ .
- (5)  $K \cong L$ .
- (6)  $\text{top } K \cong \text{top } L$ .

*Proof.* The proof proceeds as follows: (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (1), (3)  $\Rightarrow$  (4)  $\Rightarrow$  (5)  $\Rightarrow$  (6)  $\Rightarrow$  (3).

(1)  $\Rightarrow$  (2). An isomorphism of (1) is liftable to some automorphism  $\alpha$  of  $eA$  which gives rise to an isomorphism  $KJ^s \oplus LJ^{s+1} \rightarrow KJ^{s+1} \oplus LJ^s$ . This also induces an isomorphism  $KJ^{s+1} \oplus LJ^{s+2} \rightarrow KJ^{s+2} \oplus LJ^{s+1}$  since the radical maps to the radical. Thus  $\alpha$  gives  $eA/(KJ^{s+1} \oplus LJ^{s+2}) \cong eA/(KJ^{s+2} \oplus LJ^{s+1})$ . Conversely, taking ‘‘over socles’’ the last isomorphism implies the first one. Hence (1) implies (2).

(2) $\Rightarrow$ (3) $\Rightarrow$ (1). Trivial.

(3) $\Rightarrow$ (4). We may assume that  $|eA/K| \leq |eA/L|$ . Let  $\varphi: eA/(KJ \oplus L) \rightarrow eA/(K \oplus LJ)$  be an isomorphism. This induces an automorphism  $\beta$  of  $eA/eJ$ . Since  $\beta$  is liftable to  $\varphi$ , Lemma 1.19 says that  $\beta$  is liftable to an epimorphism  $eA/L \rightarrow eA/K$ , which induces an isomorphism  $\psi: eA/M \rightarrow eA/K$  for some  $L \leq M \leq eJ$ . Further  $\psi$  is liftable to an automorphism of  $eA$ , which gives rise to an isomorphism  $M \rightarrow K$ . The since  $M = L \oplus (M \cap K) \cong K$  is uniserial, we have  $M = L$ .

(4) $\Rightarrow$ (5) $\Rightarrow$ (6). Trivial.

(6) $\Rightarrow$ (3). Assume (6). Then there is an isomorphism between the radicals of the left and right hand sides of (3), which is extendable to a desired isomorphism by Proposition 1.12. //

**Lemma 1.21.** *Let  $A$  be a ring of right local type. Then  $A$  is right coserial, i.e., every injective indecomposable right  $A$ -module is uniserial.*

Proof. This is clear because  $\text{soc}^2 E$  is local for any injective indecomposable right  $A$ -module  $E$ . //

Let  $A$  be a ring of right local type. Then by Lemma 1.21, any uniserial right  $A$ -module is quasi-injective. Now let  $U$  be a uniserial right  $A$ -module of length  $n \geq 2$ . For each  $1 \leq i \leq n$ , we set  $D^i(U) := \text{End}_A(UJ^{i-1}/UJ^i)$ , which clearly is a division ring. As in Sumioka [23], we can define a ring monomorphism  $\mu_{ij}: D^j(U) \rightarrow D^i(U)$  for all  $1 \leq i < j \leq n$  as follows: For every  $x \in D^j(U)$ ,  $x$  is extendable to some  $y \in \text{End}_A(UJ^{i-1}/UJ^i)$ , which induces an element  $\bar{y} \in D^i(U)$ . Then the correspondence  $\mu_{ij}: x \mapsto \bar{y}$  is easily seen to be well-defined and a monomorphism. Further we have commutativity relations  $\mu_{ij} = \mu_{it} \mu_{tj}$  for all  $1 \leq i < t < j \leq n$ . Thus we can regard  $D^1(U) \supseteq D^2(U) \supseteq \dots \supseteq D^n(U)$  by  $\mu_{ij}$ 's. Under this notation we obtain the following by Lemma 1.19.

**Proposition 1.22.** *Let  $A$  be a ring of right local type, and  $U$  a uniserial right  $A$ -module of length  $n > 2$ . Then  $D^2(U) = D^3(U) = \dots = D^n(U)$ .*

Proof. We have only to show that  $\mu_{2n}$  is surjective. Let  $x \in D^2(U)$ . Then  $x$  is extendable to some  $y \in \text{End}_A(U/UJ^2)$ , which induces an element  $\mu_{12}(x) \in D^1(U)$ . Since  $\mu_{12}(x)$  is liftable to  $y$ , Lemma 1.19 insists that  $\mu_{12}(x)$  is liftable to some  $z \in \text{End}_A(U)$ . Put  $t := z|UJ^{n-1}$ . Then  $t \in D^n(U)$  and  $\mu_{12}(x) = \mu_{1n}(t) = \mu_{12} \mu_{2n}(t)$ . Thus  $x = \mu_{2n}(t)$ . //

**DEFINITION 1.23.** Let  $A$  be a ring of right local type, and  $S$  a simple right  $A$ -module with the projective cover  $P$ . Then analogous to [25], we say that  $S$  is of *1st kind* if  $PJ/PJ^2$  is square-free, and that  $S$  is of *2nd kind* if it is not of 1st kind, equivalently if  $PJ/PJ^2$  is a homogeneous semisimple module of length  $\geq 2$ . Note that by Proposition 1.20,  $S$  is of 1st (resp. 2nd) kind iff  $P$  is of the from (a) or (b) (resp. (c) or (d)) in Proposition 1.4.



For division rings  $D^1(U)$  and  $D^2(U)$  we have the following, which gives another key for the computation of values of  $AR$ -quiver of a ring of right local type.

**Proposition 1.24.** *Let  $A$  be a ring of right local type, and  $U$  a uniserial right  $A$ -module of length  $\geq 2$  with the projective cover  $P$ . Then*

- (1) *top  $U$  is of 1st kind iff  $D^1(U) = D^2(U)$ ; and*
- (2) *If top  $U$  is of 2nd kind, then*
  - (a)  *$U$  is injective, and*
  - (b)  *$[D^1(U) : D^2(U)]_r = 2$ ,  $[D^1(U) : D^2(U)]_l = |PJ/PJ^2|$ .*

**Proof.** (1) ( $\Rightarrow$ ). Assume that  $D^1(U) \neq D^2(U)$ , say  $x \in D^1(U) \setminus D^2(U)$ . Construct the kernel  $M$  of the map  $(x\pi, \pi) : U/UJ^2 \oplus U/UJ^2 \rightarrow \text{top } U$ , where  $\pi : U/UJ^2 \rightarrow \text{top } U$  is the canonical epimorphism. Then by Lemma 1.18 we have  $\text{top } M \cong \text{top } U$ , and  $MJ/MJ^2 \cong UJ/UJ^2 \oplus UJ/UJ^2$  is a homogeneous semisimple module of length 2. Thus top  $U$  is of 2nd kind.

(1) ( $\Leftarrow$ ). Assume that top  $U$  is of 2nd kind. Choose some  $PJ^2 \leq I \leq PJ$  so that  $M := P/I$  has the radical of the form  $S \oplus T$  with  $S \cong T$  simple modules. Then by Lemma 1.13 and Proposition 1.20 we have  $M/S \cong M/T \cong U/UJ^2$ . Hence we get an exact sequence of the form

$$0 \rightarrow M \rightarrow U/UJ^2 \oplus U/UJ^2 \xrightarrow{(\alpha, \beta)} \text{top } U \rightarrow 0,$$

where  $\alpha, \beta$  are epimorphisms. Since  $M$  is indecomposable, we have  $\bar{\alpha}\bar{\beta}^{-1} \in D^1(U) \setminus D^2(U)$  by Lemma 1.17 (2), where  $\bar{\alpha}, \bar{\beta}$  are the maps induced from  $\alpha, \beta$ , respectively. Thus  $D^1(U) \neq D^2(U)$ .

(2) (a). It is enough to show that  $N := U/UJ^2$  is injective. Let  $E$  be the injective hull of  $N$ , and  $F := \text{soc}^3 E$ . Suppose that  $N$  is not injective. Then  $F$  is a uniserial module of length 3, and  $N = FJ$ . Note that  $D^1(N) = D^2(F)$  and  $D^2(N) = D^3(F)$ . Then since top  $N$  is of 2nd kind, the above shows that  $D^1(N) \neq D^2(N)$ , which contradicts Lemma 1.22.

(2) (b). The following is easily verified:

**Claim.** *Let  $x_1, \dots, x_n \in D^1(U)$ . Then  $\{x_1, \dots, x_n\}$  is linearly dependent in  $D^1(U)_{D^2(U)}$  iff the homomorphism  $(x_i \pi) : (U/UJ^2)^{(n)} \rightarrow \text{top } U$  is cofusible, where  $\pi : U/UJ^2 \rightarrow \text{top } U$  is the canonical epimorphism.*

If  $n \geq 3$ , then  $\text{Ker}(x_i \pi)$  is not local, whence  $(x_i \pi)$  is cofusible by Lemma 1.17(2). Thus  $[D^1(U) : D^2(U)]_r = 2$  because  $D^1(U) \neq D^2(U)$  by (1) above. For the remaining part we apply [4, Theorem and Lemma 2]. We may assume that  $|U| = 2$ , whence that  $U = P/M$  for some maximal submodule  $M$  of  $PJ$ . Note that in this case  $D^1(U) = D$  and  $D^2(U) = D(M)$  under the notation of [4]. Put  $X^* := \bigcap_{\alpha} \alpha^{-1}(X)$  for any  $X \leq PJ$ , where  $\alpha$  runs through  $\text{End}_A P$ . Then the above mentioned statements guarantee that  $[D^1(U) : D^2(U)]_l = |PJ/M^*|$  because  $PJ^2 \leq M \leq$

$PJ$  and  $P/M$  is quasi-injective. Since  $\text{top } U = \text{top } P$  is of 2nd kind, Proposition 1.20 and Lemma 1.13 claim that for any maximal submodule  $X$  of  $PJ$ ,  $P/M \cong P/X$ , whence  $M^* = X^* \leq X$ . Thus  $PJ^2 \leq M^* \leq \text{rad}(PJ) = PJ^2$ , i.e.,  $M^* = PJ^2$ . //

**REMARK 1.25.** In the setting of the proof of (2) (b), let  $v \in PJ \setminus PJ^2$ . Then by the above lemma, we have  $[D^1(U): D^2(U)]_v = [D_1(v): D_2(v)]_v$ , and  $[D^1(U): D^2(U)]_v = [D_1(v): D_2(v)]_v$ , where  $D_i(v)$  are division rings defined in [24]. We expect that these together with Proposition 1.4, Lemma 1.21 and Theorem 1.0 will give a complete characterization of a ring of right local type.

## 2. AR-sequences over a ring of right local type

Recall that for indecomposable right  $A$ -modules  $X, Y$ , the bimodule  ${}_{F(Y)}\text{Irr}(X, Y)_{F(X)}$  of irreducible maps is, by definition, the factor module  $\text{rad}(X, Y)/\text{rad}^2(X, Y)$  where  $\text{rad}$  is the radical of the category of finitely generated right  $A$ -modules, and  $F(X) := \text{End}_A(X)/\text{rad}(X, X)$ . The following holds for arbitrary artinian rings; see, for instance, [14] or [19] for the proof in the algebra case.

**Lemma 2.1.** *Let  $A$  be an arbitrary artinian ring, and  $X_1, \dots, X_n, Y$  indecomposable right  $A$ -modules with  $X_i \cong X_j$  for any  $i \neq j$ , and  $\alpha_{ij}: X_i \rightarrow Y$  a homomorphism for every  $1 \leq i \leq n, 1 \leq j \leq d_i$ . Then the homomorphism  $(\alpha_{ij})_{i,j}: \bigoplus_{i=1}^n X_i^{(d_i)} \rightarrow Y$  is irreducible iff every  $\alpha_{ij}$  is irreducible and  $\{\bar{\alpha}_{i1}, \dots, \bar{\alpha}_{id_i}\}$  is a linearly independent set in  $\text{Irr}(X_i, Y)_{F(X_i)}$  for every  $i$ , where  $\bar{\alpha}_{ij}$  is the image of  $\alpha_{ij}$  in  $\text{Irr}(X_i, Y)$ .*

In the rest of this section, we assume that  $A$  is a ring of right local type. We now give all the irreducible maps between indecomposable right  $A$ -modules.

**Lemma 2.2.** *Let  $\alpha: X \rightarrow Y$  be an epimorphism between local right  $A$ -modules. Then  $\alpha$  is irreducible iff  $\text{Ker } \alpha$  is simple.*

**Proof.** ( $\Rightarrow$ ). This part holds for an arbitrary ring by [16, Lemma 1.3], and this is trivial in our case.

( $\Leftarrow$ ). By assumption  $Y$  is not projective, so we have an AR-sequence (\*) stopping at  $Y$ , which gives rise to a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Ker } \alpha & \xrightarrow{\beta} & X & \xrightarrow{\alpha} & Y \longrightarrow 0 \\
 & & \gamma \downarrow & & \delta \downarrow & & \downarrow = \\
 (*) & & 0 & \longrightarrow & T & \longrightarrow & E \longrightarrow Y \longrightarrow 0,
 \end{array}$$

Consider the left square (S), which is a pushout and pullback diagram. Then

since  $(\beta, \gamma)^T$  is fusible by Proposition 1.12,  $\delta$  is a section by Lemma 1.11 (1). Thus  $\alpha$  is irreducible. //

**REMARK 2.3.** If  $A$  is not of right local type, then the implication  $(\Leftarrow)$  is not true in general. For instance, let  $A^{op}$  be the algebra defined by the quiver  $1 \rightarrow 2 \leftarrow 3$ . Then the canonical epimorphism  $(k \rightarrow k \leftarrow 0) \rightarrow (k \rightarrow 0 \leftarrow 0)$  gives a counterexample.

**Lemma 2.4.** *Let  $0 \rightarrow X \xrightarrow{(\alpha_i)^T} \bigoplus_{i=1}^n Y_i \xrightarrow{(\beta_i)} Z \rightarrow 0$  be an AR-sequence with each  $Y_i$  indecomposable. Then  $\beta_i$  is an epimorphism for some  $i$ , and  $n \leq 2$ .*

**Proof.** We may suppose that  $n \geq 2$ . Since  $Z$  is local,  $\beta_i$  is an epimorphism for some  $i$ , say  $i=1$ . Then this sequence yields a pushout and pullback diagram:

$$\begin{array}{ccc} X & \xrightarrow{\alpha_1} & Y_1 \\ \alpha' \downarrow & & \downarrow \beta_1 \\ Y' & \xrightarrow{\beta'} & Z \end{array}$$

where  $Y' := \bigoplus_{i \neq 1} Y_i$ ,  $\alpha' := (\alpha_i)_{i \neq 1}^T$  and  $\beta' := -(\beta_i)_{i \neq 1}$ . Since  $\alpha'$  is also an epimorphism,  $Y'$  is local, whence  $n \leq 2$ . //

The next gives all the irreducible monomorphisms between indecomposables, which follows by the above lemma considering “left boundaries” of Figures 2 and 3 below. Since we do not use this later and also follows by Theorem I, the precise proof is left to the reader.

**Lemma 2.5.** *Let  $\alpha: X \rightarrow Y$  be a monomorphism between indecomposable right  $A$ -modules. Then  $\alpha$  is irreducible iff we have a commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & U & \longrightarrow & X \longrightarrow 0 \\ & & \downarrow = & & \downarrow \beta & & \downarrow \alpha \\ 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & Y \longrightarrow 0 \end{array}$$

*with exact rows such that  $P$  is indecomposable projective and  $\beta$  is an irreducible map, i.e., an inclusion map of some direct summand of  $PJ$ . (Hence  $P$  is necessarily the projective cover of  $Y$ .) If this is the case,  $X$  is uniserial since so is  $U$ .*

The following gives all the AR-sequences over a ring of right local type.

**Theorem I.** *Let  $A$  be a ring of right local type. Then an exact sequence*

of right  $A$ -modules is an  $AR$ -sequence iff it is isomorphic to one of the following sequences:

$$(I) \quad 0 \rightarrow eJ/eJ^{i+1} \xrightarrow{(\mu_1, -\varepsilon_1)^T} eA/eJ^{i+1} \oplus eJ/eJ^i \xrightarrow{(\varepsilon_2, \mu_2)} eA/eJ^i \rightarrow 0$$

where  $eA$  is uniserial and  $i \geq 1$ .

$$(II) \quad 0 \rightarrow K/KJ^{s+1} \xrightarrow{(\mu_1, -\varepsilon_1)^T} eA/KJ^{s+1} \oplus K/KJ^s \xrightarrow{(\varepsilon_2, \mu_2)} eA/KJ^s \rightarrow 0$$

where  $eJ = K \oplus L$  and  $K, L (\neq 0)$  are uniserial,  $s \geq 0$ ,  $KJ^s \neq 0$ .

$$(III) \quad 0 \rightarrow eA/(KJ^{s+1} \oplus LJ^{t+1}) \xrightarrow{(\varepsilon'_1, -\varepsilon'_2)^T} eA/(KJ^{s+1} \oplus LJ^t) \oplus eA/(KJ^s \oplus LJ^{t+1}) \\ \xrightarrow{(\varepsilon_1, \varepsilon_2)} eA/(KJ^s \oplus LJ^t) \rightarrow 0$$

where  $eJ = K \oplus L$ , and  $K, L$  are uniserial,  $s, t \geq 0$  and  $KJ^s, LJ^t \neq 0$ .

$$(IV) \quad 0 \rightarrow S \xrightarrow{\mu} eA \xrightarrow{\varepsilon} eA/S \rightarrow 0$$

where  $eJ = S_1 \oplus \cdots \oplus S_n$ ,  $S := S_1 \cong \cdots \cong S_n$  are simple and  $n > 2$ .

$$(V) \quad 0 \rightarrow eA/(S_2 \oplus \cdots \oplus S_{i-1}) \xrightarrow{(\varepsilon'_1, -\varepsilon'_2)^T} eA/(S_1 \oplus \cdots \oplus S_{i-1}) \oplus eA/(S_2 \oplus \cdots \oplus S_i) \\ \xrightarrow{(\varepsilon_1, \varepsilon_2)} eA/(S_1 \oplus \cdots \oplus S_i) \rightarrow 0$$

where  $eJ = S_1 \oplus \cdots \oplus S_n$ ,  $S := S_1 \cong \cdots \cong S_n$  are simple and  $n > 2$ ,  $2 \leq i \leq n$  (for convenience we set  $S_2 \oplus \cdots \oplus S_1 = 0$ ).

In the above,  $\varepsilon_*$ ,  $\varepsilon'_*$  are the canonical epimorphisms and  $\mu_*$  are the inclusion maps, and  $e$  is a primitive idempotent of  $A$ .

REMARK 2.6. In the above list of sequences, the middle term is indecomposable in the case (I),  $i=1$ , and in the case (II),  $s=0$ .

**Proof of Theorem I.** Let  $(e): 0 \rightarrow K \xrightarrow{\sigma} E \xrightarrow{\tau} C \rightarrow 0$  be an exact sequence. Then by definition  $(e)$  is an  $AR$ -sequence if

- (1)  $(e)$  is non-split;
- (2)  $K$  and  $C$  are indecomposable; and
- (3)  $\tau$  is right almost split.

Let  $(e)$  be any one of the sequences in the theorem. First it is easy to see that  $(e)$  is exact. The statements (1) and (2) are trivial because each component map of  $\sigma$  is non-isomorphism and  $K, C$  are local. Hence noting that all the non-projective indecomposables appear as the module  $C$ , it only remains to prove the statment (3). By definition the statement (3) is trivially equivalent to

the following:

(3') For any non-isomorphism  $\alpha: X \rightarrow C$  from any indecomposable module  $X$ , there exists a homomorphism  $\beta: X \rightarrow E$  with  $\alpha = \tau\beta$ .

In the cases (I), (II), (IV) we directly construct the desired homomorphism  $\beta$ , and at the same time we construct the  $\beta$  for any *non-epimorphism*  $\alpha$  in the cases (III) and (V) for the later use. For the cases (III) and (V) we show the statement (3) as follows. Let  $E_1$  and  $E_2$  be indecomposable modules in the middle term  $E$  so that  $(e)$  has the form  $0 \rightarrow K \xrightarrow{(\varepsilon'_1, -\varepsilon'_2)^T} E_1 \oplus E_2 \xrightarrow{(\varepsilon_1, \varepsilon_2)} C \rightarrow 0$ . By Lemma 2.4 it suffices to show that  $(\varepsilon_1, \varepsilon_2)$  is irreducible. To this end we can apply Lemma 2.1 since by Lemma 2.2,  $\varepsilon_1$  and  $\varepsilon_2$  are irreducible. If  $E_1 \cong E_2$ , then the assertion is trivial. Hence we have only to show that the set  $\{\bar{\varepsilon}_1, \bar{\varepsilon}_2\varphi\}$  is linearly independent in  $\text{Irr}(E_1, C)_{F(E_1)}$  under the assumption that there is an isomorphism  $\varphi: E_1 \rightarrow E_2$ .

**(i) Proof of (3') for the cases (I), ..., (V) when  $\alpha$  is not an epimorphism**

We may assume that  $X = fA/I$  for some primitive idempotent  $f$  of  $A$  and for some  $I \leq fJ$  and that  $\alpha = a \cdot$  for some  $a \in eJf$ .

Case (I). Since  $C = eA/eJ^i$  is local,  $\text{Im } \alpha \leq eJ/eJ^i$ . Hence  $a \cdot: X \rightarrow eJ/eJ^i$  is defined and is a desired map.

Case (II). Since  $\text{Im } \alpha \leq eJ/KJ^s$ , we can define the maps  $\alpha'$  and  $\sigma$  by the following commutative diagram

$$\begin{array}{ccccc}
 X & & & & \\
 \downarrow \alpha' & \searrow \alpha & & & \\
 L \oplus K/KJ^s & \xrightarrow{\quad} & eJ/KJ^s & & \\
 \downarrow \sigma & \nearrow \gamma & \nwarrow & & \\
 eA/KJ^s & \xrightarrow{(\varepsilon, \mu)} & eA/KJ^{s+1} \oplus K/KJ^s & & 
 \end{array}$$

which is completed by the  $\gamma := \begin{bmatrix} \beta & 0 \\ 0 & id \end{bmatrix}$  above, where  $id$  is the identity map of  $K/KJ^s$  and  $\beta$  is the composite  $L \rightarrow K \oplus L \rightarrow (K \oplus L)/KJ^{s+1} = eJ/KJ^{s+1} \rightarrow eA/KJ^{s+1}$  of the canonical maps. Hence  $\gamma\alpha'$  is a desired map.

Case (III).  $\text{Im } \alpha \leq eJ/(KJ^s \oplus LJ^t) \cong L/LJ^t \oplus K/KJ^s$ . Hence we can define the maps  $\alpha'$  and  $\sigma$  by the following commutative diagram:

$$\begin{array}{ccccc}
 & & X & & \\
 & & \downarrow \alpha' & \searrow \alpha & \\
 & & L/LJ^t \oplus K/KJ^s & \approx & eJ/(LJ^t \oplus KJ^s) \\
 & \swarrow \delta & \downarrow \sigma & & \swarrow \\
 eA/(KJ^{s+1} \oplus LJ^t) \oplus eA/(KJ^s \oplus LJ^{t+1}) & \longrightarrow & eA/(KJ^s \oplus LJ^t) & & 
 \end{array}$$

Define the maps  $\beta: L/LJ^t \rightarrow eA/(KJ^{s+1} \oplus LJ^t)$  and  $\gamma: K/KJ^s \rightarrow eA/(KJ^s \oplus LJ^{t+1})$  as the composite maps of the canonical maps:  $L/LJ^t \rightarrow (K \oplus L)/LJ^t = eJ/LJ^t \rightarrow eJ/(KJ^{s+1} \oplus LJ^t) \rightarrow eA/(KJ^{s+1} \oplus LJ^t)$ ; and  $K/KJ^s \rightarrow (K \oplus L)/KJ^s = eJ/KJ^s \rightarrow eJ/(KJ^s \oplus LJ^{t+1}) \rightarrow eA/(KJ^s \oplus LJ^{t+1})$ , respectively. (Note that  $\beta, \gamma$  are non-isomorphisms.)

Then as easily seen the homomorphism  $\delta := \begin{bmatrix} \beta & 0 \\ 0 & \gamma \end{bmatrix}$  completes the above commutative diagram, whence the homomorphism  $\delta\alpha'$  is a desired map.

Case (IV). Since  $eJ/S$  is semisimple,  $0 = a(fJ) \geq aI$ . Hence  $a \cdot$  gives a desired map  $X \rightarrow eA$ .

Case (V). Again since  $eJ/(S_1 \oplus \cdots \oplus S_i)$  is semisimple, both  $(a \cdot, 0)^T$  and  $(0, a \cdot)^T$  gives a desired map, whose components are non-isomorphisms.

### (ii) Proof of (3') for the cases (I), (II), (IV) when $\alpha$ is an epimorphism

Since the projective cover of  $X$  is isomorphic to  $eA$ , we may assume that  $X = eA/I$  for some  $I \leq eJ$ , and that  $\alpha = a \cdot$  for some  $a \in eAe \setminus eJe$ .

Case (I).  $I = eJ^j$ , for some  $j \geq i+1$ . Since  $a(eJ^j) \leq eJ^j \leq eJ^{i+1}$ ,  $a \cdot: X \rightarrow eA/eJ^{i+1}$  is defined and is a desired map.

Case (II).  $aI \leq KJ^s$ . If  $aI = KJ^s$ , then since  $a$  is a unit,  $a^{-1}KJ^s = I$ , i.e.,  $\text{Ker } \alpha = 0$ , a contradiction. Hence  $aI \leq KJ^{s+1}$ . Thus  $(a \cdot, 0)^T: X \rightarrow eA/KJ^{s+1} \oplus K/KJ^s$  is a desired map.

Case (IV). In this case  $X = eA$  is projective, whence there is nothing to show.

### (iii) Proof of (3) for the cases (III), (V)

Suppose that our assertion is not true. Then  $\varepsilon_1 = \varepsilon_2 \eta + \theta$  for some isomorphism  $\eta: E_1 \rightarrow E_2$  and for some  $\theta \in \text{rad}^2(E_1, C)$ . Let  $\theta = \gamma\beta$  with  $\beta \in \text{rad}(E_1, L)$ ,  $\gamma \in \text{rad}(L, C)$ , where  $L = \bigoplus_{i=1}^n L_i$  with all  $L_i$  local and  $\beta = (\beta_i)^T$ ,  $\gamma = (\gamma_i)$ . Then since  $|C| - |E_1| = 1$  and  $L_i$  are local, every  $\gamma_i \beta_i$  is not an epimorphism,

whence we have  $\gamma_i \beta_i = \varepsilon_1 \delta_i + \varepsilon_2 \zeta_i$  for some  $\delta_i \in \text{rad}(E_1, E_1)$  and  $\zeta_i \in \text{rad}(E_1, E_2)$  by (i) above. Thus  $\varepsilon_1 - \varepsilon_2 \eta = \varepsilon_1 \delta + \varepsilon_2 \zeta$ , where  $\delta := \sum \delta_i$  and  $\zeta := \sum \zeta_i$ . Consequently  $\varepsilon_1 = \varepsilon_2 \lambda$  for some isomorphism  $\lambda$  because  $1_{E_1} - \delta$  and  $\eta + \zeta$  are isomorphisms. This shows that  $(\varepsilon_1, \varepsilon_2)$  is cofusible, a contradiction to the fact that  $\text{Ker}(\varepsilon_1, \varepsilon_2) = K$  is indecomposable. //

### 3. The AR-quiver of a ring of right local type

**DEFINITION 3.1.** A *translation quiver* is a pair  $(Q, \tau)$  of a quiver  $Q$  and a bijection  $\tau: R \rightarrow L$  of subsets of the set  $Q_0$  of vertices of  $Q$  satisfying the conditions

(a)  $Q$  has no loop  $\cdot \rhd$  and no multiple arrow  $\cdot \rightrightarrows \cdot$ ; and

(b) For any  $x \in R$ ,  $(\tau x)^+ = x^-$ ,

where we denote by  $y^+$  and by  $y^-$  the set  $\{z \in Q_0 \mid \text{there exists an arrow } y \rightarrow z \text{ in } Q\}$  and  $\{z \in Q_0 \mid \text{there exists an arrow } z \rightarrow y \text{ in } Q\}$ , respectively for every  $y \in Q_0$ . The full subquiver formed by  $\{x, \tau x\} \cup x^-$  for any  $x \in R$  is called the *mesh* starting at  $\tau x$  and stopping at  $x$ . For each arrow  $\alpha: y \rightarrow x$  in  $Q$  with  $x \in R$  we denote by  $\sigma \alpha$  the unique arrow  $\tau x \rightarrow y$  in  $Q$ .

A *valued translation quiver* is a translation quiver  $(Q, \tau)$  together with a pair  $(d, d')$  (called a *value* of  $(Q, \tau)$ ) of maps from  $Q_1$  ( $:=$  the set of arrows of  $Q$ ) to the set of natural numbers satisfying the condition

(c)  $d(\sigma \alpha) = d'(\alpha)$  for any  $\alpha: y \rightarrow x$  in  $Q_1$  with  $x \in R$ . If further the following is satisfied, then it is called a *well valued translation quiver*:

(d)  $d'(\sigma \alpha) = d(\alpha)$  for any  $\alpha: y \rightarrow x$  in  $Q_1$  with  $x \in R$ .

For each  $\alpha \in Q_1$ , the diagram  $\tau x \xleftarrow{\sigma \alpha} y \xrightarrow{\alpha} x$ , denoted by  $T(\sigma \alpha, \alpha)$ , is

$$\begin{array}{ccccc} & & y & & \\ \sigma \alpha & \nearrow & & \searrow & \alpha \\ \tau x & & & & x \end{array}$$

called a *triangle* stopping at  $\alpha$ , and the diagram  $x \xleftarrow{\alpha} y \xrightarrow{\sigma^{-1} \alpha} \tau^{-1} x$ , denoted by  $T(\alpha, \sigma^{-1} \alpha)$ , is called a *triangle* starting at  $\alpha$ , where we set  $\tau x$ ,  $\tau^{-1} x$ ,  $\sigma \alpha$  or  $\sigma^{-1} \alpha$  to be zero if it is not defined. A triangle  $T(\alpha, \beta)$  is said to be *projective*, *injective* or *proper* if  $\alpha = 0$ ,  $\beta = 0$  or  $\alpha, \beta \neq 0$ , respectively. Proper triangles are identified with their corresponding full subquiver of  $Q$ . We define  $\sigma T(\alpha, \beta) := T(\sigma \alpha, \sigma \beta)$  if  $\alpha \neq 0$ ,  $\sigma^{-1} T(\alpha, \beta) := T(\sigma^{-1} \alpha, \sigma^{-1} \beta)$  if  $\beta \neq 0$  and  $d(T(\alpha, \beta)) := \begin{cases} d(\alpha) & \text{if } \alpha \neq 0 \\ d'(\beta) & \text{if } \beta \neq 0 \end{cases}$ , which is called a *value* of  $T(\alpha, \beta)$  and is well-defined by the condition (c). The map  $d: T(\alpha, \beta) \mapsto d(T(\alpha, \beta))$  is called a *value* of triangles of  $(Q, \tau)$ . Note that giving a value  $(d, d')$  to a translation quiver  $(Q, \tau)$  is equivalent to giving a value of triangles of  $(Q, \tau)$ . The condition (d) is equivalent to the condition

(d')  $d(\sigma T) = d(\sigma^{-1} T)$  for all proper triangles  $T$ .

**DEFINITION 3.2.** Let  $A$  be a ring with *AR*-sequences. Then the *AR*-

quiver  $\Gamma$  of  $A$  is a valued translation quiver defined as follows:

(a) The set  $\Gamma_0$  of vertices is the set of isomorphism classes of indecomposable right  $A$ -modules. (We identify indecomposables with their isomorphism classes.)

(b) For any  $X, Y$  in  $\Gamma_0$ , the number of arrows from  $X$  to  $Y$  is at most one, and there exists an arrow  $X \rightarrow Y$  iff  $\text{Irr}(X, Y) \neq 0$ .

(c) If  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  is an  $AR$ -sequence, then translation is defined as  $\tau Z := X$ .

(d) The value is defined as  $d(\alpha) := [\text{Irr}(X, Y): F(Y)]_l$  and  $d'(\alpha) := [\text{Irr}(X, Y): F(X)]_r$  for each arrow  $\alpha: X \rightarrow Y$ .

REMARK 3.3. (1) In the above definition, the condition (c) in Definition 3.1 is verified by Lemma 3.4 below.

(2) If  $A$  is an artin algebra, then the  $AR$ -equiver is a well valued translation quiver.

The following two lemmas are well known in the algebra case, which remain valid also in our artinian case. See for instance [20].

**Lemma 3.4.** *Let  $0 \rightarrow X \rightarrow \bigoplus_{i=1}^n Y_i \rightarrow Z \rightarrow 0$  be an  $AR$ -sequence of right  $A$ -modules, where all  $Y_i$  are indecomposable. Then for any indecomposable module  $Y_A$ , we have*

$$[\text{Irr}(X, Y): F(Y)]_l = \# \{i \mid Y \cong Y_i\} = [\text{Irr}(Y, Z): F(Y)]_r.$$

**Lemma 3.5.** (1) *Let  $Z_A$  be an indecomposable projective module and  $ZJ = \bigoplus_{i=1}^n Y_i$  with each  $Y_i$  indecomposable. Then for any indecomposable module  $Y_A$ , we have*

$$[\text{Irr}(Y, Z): F(Y)]_r = \# \{i \mid Y \cong Y_i\}.$$

(2) *Let  $X_A$  be an indecomposable projective module and  $X/\text{soc } X = \bigoplus_{i=1}^n Y_i$  with each  $Y_i$  indecomposable. Then for any indecomposable module  $Y_A$ , we have*

$$[\text{Irr}(X, Y): F(Y)]_l = \# \{i \mid Y \cong Y_i\}.$$

**Theorem II.** *Let  $A$  be a ring of right local type and  $e$  a primitive idempotent of  $A$ . Then*

(1) *The  $AR$ -quiver of  $A$  with the precise form of the full subquiver consisting of the meshes stopping at all the local modules that are factor modules of  $eA$  has the following form: (In the figures we write a value of a triangle inside the triangle if the value is greater than 1, and triangles without indications of values have the value*



1. Further the symbols  $[*$  and  $*$ ] indicate that  $*$  is projective and that  $*$  is injective, respectively.)

(a) In the case that  $eA$  is uniserial. Put  $h := h(eA)$ . See Fig. 1.

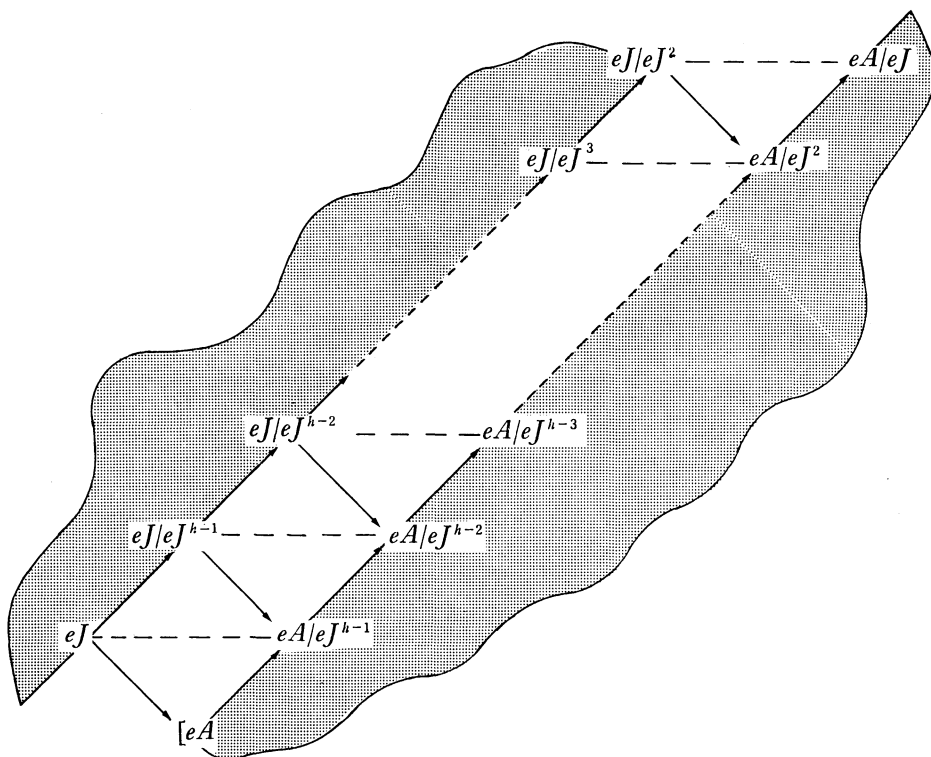


Fig. 1.

(b) In the case that  $eJ = K \oplus L$  for some non-zero uniserial modules  $K, L$  which are not isomorphic to each other. Put  $s := h(K)$ ,  $t := h(L)$ . See Fig. 2.

(c) In the case that  $eJ = K \oplus L$  for some non-zero uniserial modules  $K, L$  which are isomorphic to each other. Put  $h := h(K)$ . See Fig. 3.

(d) In the case that  $eJ = S_1 \oplus \cdots \oplus S_n$ ,  $S_1 \cong \cdots \cong S_n$  are simple and  $n > 2$ . See Fig. 4, where  $\begin{pmatrix} 0 \\ 1 \end{pmatrix} := S_1$ ,  $\begin{pmatrix} 1 \\ i \end{pmatrix} := eA / (S_{i+1} \oplus \cdots \oplus S_n)$  ( $S_{n+1} \oplus \cdots \oplus S_n := 0$ ), for all  $0 \leq i \leq n$ .





**Theorem 3.6.** *The AR-quiver of a ring  $A$  of right local type is a well valued translation quiver iff  $A$  is also a ring of left colocal type.*

This suggests us the following.

**Conjecture.** *If  $A$  is both of right local type and of left colocal type, then  $A$  has a selfduality.*

**Corollary 3.7.** *If  $A$  is given by the form  $\begin{pmatrix} F & M \\ 0 & G \end{pmatrix}$  for some division rings  $F, G$  and some  $F$ - $G$ -bimodule  $M$  with  $[M:G]_r = n \geq 1$ , then  $A$  is of right local type iff the dimension sequence of  $M$  is  $(n, 1, 2, 2, \dots, 2, 1)$  ( $n+2$  terms) (if  $n=1$ , it is  $(1, 1, 1)$ ). Further in this case  $A$  is of left colocal type iff  $n \leq 2$ .*

Proof. ( $\Rightarrow$ ). The AR-quiver of  $A$  has just the form in Fig. 4 without the shaded parts, and  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  is projective,  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  is injective, where  $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ . Let  $T_0, T_1, \dots, T_{n+1}$  be the triangles in Fig. 4 in the same order. (Notice that  $T_0$  is projective and  $T_{n+1}$  is injective.) Then by [12], the desired dimension sequence is given by  $(d(T_0), \dots, d(T_{n+1}))$ .

( $\Leftarrow$ ). By [12],  $A$  has exactly  $n+2$  non-isomorphic indecomposable modules. Since  $|eJ| = n$ , there exists  $n+2$  non-isomorphic local right modules, which shows that  $A$  is of right local type.

The rest follows by Corollary 1.2. //

**EXAMPLE 3.8.** In the algebra case, it is known that  $A$  is of right local type iff  $A$  is left serial and  $eJ$  is a direct sum of at most two uniserial modules for every primitive idempotent  $e$  of  $A$  (note that the latter is equivalent to saying that the conditions (LR) and (L) in Theorem 1.0 are satisfied). However this does not hold in the artinian case. We give counterexamples of both implications, which at the same time are an example of a ring of right local type but not of left colocal type and an example of a ring of left colocal type but not of right local type. Let  $d$  be any dimension sequence of length 5. Then [21, section 13] and [12, Proposition 1] guarantee that there exists an  $F$ - $G$ -bimodule  $M$  over division rings  $F, G$  such that the dimension sequence of  $M$  is  $d$ . Let  $A$  be a ring defined by  $M$  as in Corollary 3.7.

(1) (This example had been reported in Japanese [3]). If  $d = (3, 1, 2, 2, 1)$ , then  $A$  is of right local type but not of left colocal type by Corollary 3.7. Further  $A$  gives a counterexample of the implication ( $\Rightarrow$ ).

(2) If  $d = (2, 1, 3, 1, 2)$ , then  $A$  is of left colocal type but not of right local type. In fact  $A^{op}$  is a ring defined by a bimodule over division rings having the dimension sequence  $(1, 2, 2, 1, 3)$  and it has an indecomposable injective module  $I$  with  $|I/\text{soc } I| = 3$ , which gives 5 colocal modules. Thus  $A$  is of left colocal type, and  $A$  is not of right local type by Corollary 3.7. Further  $A$  gives

a counterexample of the implication ( $\Leftarrow$ ).

REMARK 3.9. (1) Rings defined as in Corollary 3.7 by a bimodule over division rings having dimension sequence  $d$  of length 5 is called *rings of type  $I_2(5)$*  having dimension sequence  $d$ . As easily seen these rings are characterized as hereditary rings with just 2 non-isomorphic simples and just 5 non-isomorphic indecomposables. Thus if  $A$  is of type  $I_2(5)$ , then so are  $A^{op}$  and  $D_r(A) := \text{End}_A(E_1 \oplus E_2)$ , where  $E_1$  and  $E_2$  are the injective indecomposable right  $A$ -modules. More precisely, if  $A$  has dimension sequence  $(a_1, \dots, a_5)$ , then  $A^{op}$  has  $(a_2, a_1, \dots)$ , and  $D_r(A)$  has  $(a_5, \dots, a_1)$ . It should be noted that a dimension sequence of length 5 is completely determined by the first two terms.

(2) The  $AR$ -quiver of a ring of type  $I_2(5)$  is as follows:

$$\begin{array}{ccc}
 \begin{array}{c} \left[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{-2} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \xrightarrow{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \\ \searrow \quad \nearrow \\ 1 \quad \left[ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{-2} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right] \quad 3 \end{array} & & \begin{array}{c} \left[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \\ \searrow \quad \nearrow \\ 2 \quad \left[ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \xrightarrow{-3} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right] \quad 2 \end{array} \\
 \\
 \begin{array}{c} \left[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{-3} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \xrightarrow{-2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \\ \searrow \quad \nearrow \\ 1 \quad \left[ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right] \quad 2 \end{array} & & \begin{array}{c} \left[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \xrightarrow{-2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \\ \searrow \quad \nearrow \\ 3 \quad \left[ \begin{pmatrix} 1 \\ 3 \end{pmatrix} \xrightarrow{-2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] \quad 1 \end{array} \\
 \\
 \begin{array}{c} \left[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{-2} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \xrightarrow{-3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \\ \searrow \quad \nearrow \\ 2 \quad \left[ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \xrightarrow{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] \quad 1 \end{array} & & \begin{array}{c} \left[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{-2} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \xrightarrow{-3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \\ \searrow \quad \nearrow \\ 2 \quad \left[ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \xrightarrow{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] \quad 1 \end{array}
 \end{array}$$

In the above, modules  $X$  are presented by the dimension type  $([Xe_1: F]_r, [Xe_2: G]_r)^T$  where  $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ . (Notice that for non-simple indecomposables  $X$ ,  $Xe_1 = \text{top } X$  and  $Xe_2 = XJ = \text{soc } X$ .) Further the dimension sequences are just the sequence of values of triangles, and dimension sequences  $(a, b, \dots)$  and  $(b, a, \dots)$  are placed horizontally to express left modules and right modules. The last two quivers are the same. The  $AR$ -quiver of a ring  $A$  and that of  $D_r(A)$  are placed diagonally (left-up to right-down).

(3) As well known a ring is serial iff it is of (left and right) local type. By Lemma 1.21, rings of local type are coserial. Hence serial rings are coserial. But the converse is not true as the above  $AR$ -sequence of a ring of type  $I_2(5)$  having dimension sequence  $(2, 2, 1, 3, 1)$  shows.

(4) A ring is called a right Tachikawa ring if every indecomposable right module is local or colocal, and is called a Tachikawa ring if both  $A$  and  $A^{op}$  are

right Tachikawa rings. A ring of right local and left colocal type is an example of Tachikawa rings. However the  $AR$ -sequence of a ring of type  $I_2(5)$  having dimension sequence  $(3, 1, 2, 2, 1)$  and that of its opposite ring tell us that a ring of right local type does not need to be a Tachikawa ring.

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Department of Mathematics  
Osaka City University  
Sugimoto, Sumiyoshi-ku  
Osaka 558, Japan