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<th>On a distance function on the set of differentiable structures</th>
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<tr>
<td>Author(s)</td>
<td>Shikata, Yoshihiro</td>
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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 3(1) P.65–P.79</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1966</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/5262">https://doi.org/10.18910/5262</a></td>
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<td>DOI</td>
<td>10.18910/5262</td>
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1. Introduction

Let $X$ be a given compact closed orientable topological manifold and let $\Sigma = \{\sigma_i\}$ denote the set of differentiable structures on $X$. In this paper one defines a pseudo distance $\rho$ on $\Sigma$ which is allowed to take $\infty$ as one of its values. It is proved that $\rho$ is actually a distance function, namely that $\rho(\sigma_1, \sigma_2) = 0$ implies $\sigma_1 = \sigma_2$. More strongly, one proves the following

**Theorem 1.** There exists a positive $\varepsilon_1$, depending on dimension of $X$, such that if $\rho(\sigma_1, \sigma_2) \leq \varepsilon_1$ then $\sigma_1$ and $\sigma_2$ are differentiably equivalent.

The proof is given in Part II. In Part I one investigates relations between the distance and the combinatorial equivalence.

It is seen that $\rho(\sigma_1, \sigma_2) < \infty$ if $\sigma_1$ and $\sigma_2$ are combinatorially equivalent, and the following theorem is proved:

**Theorem 2.** There exists a positive $\varepsilon_2$, depending only on dimension of $X$, such that if $\rho(\sigma_1, \sigma_2) \leq \varepsilon_2$ then $\sigma_1$ and $\sigma_2$ are combinatorially equivalent.

In order to assure non triviality of the distance function the following remark might be sufficient. By J. Milnor and I. Tamura there is found a compact combinatorial manifold which admits two smoothings having different integral Pontrjagin classes, therefore using the result of [S] which asserts that any two differentiable structures of distance less than $1/2 \log 3/2$ have the same integral Pontrjagin classes, one sees that the distance between these structures is finite but not less than $1/2 \log 3/2$.

Although the distance has been allowed to take $\infty$ as its value, it is possible to restrict $\Sigma$ so that the distance always gives finite value, by introducing a notion of Lipschitz manifold and compatible smoothing, as appears in a sequel.
PART I

2. Definition of a pseudo distance.

Let $X_i$ ($i=1, 2$) be metric spaces with metrics $d_i$ ($i=1, 2$). Then the size $I(f)$ (rel. $d_1, d_2$) of a homeomorphism $f$ of $X_i$ onto $X_2$ is defined to be

\[ I(f) = \begin{cases} \inf \{ k \geq 1 | \forall x, y \in X_i, d_i(x, y)/k \leq d_2(f(x), f(y)) \leq kd_i(x, y) \} , \\ \infty, \text{ if the above set of } k \text{ is empty.} \]

A homeomorphism $f$ is said to be (regular) Lipschitz relative to $d_1, d_2$, if the size $I(f)$ (rel. $d_1, d_2$) is bounded, and the size $I(f)$ (rel. $d_1, d_2$) is sometimes called the Lipschitz constant of $f$.

The following properties 1)–3) of the size are elementary.

1) $I(id.)$ (rel. $d_1, d_2$) = 1,
2) $I(f)$ (rel. $d_1, d_2$) = $I(f^{-1})$ (rel. $d_2, d_1$),
3) Let $X_3, d_3$ be the third metric space and its metric, and let $f: X_1 \rightarrow X_2, g: X_2 \rightarrow X_3$ be homeomorphisms, then

\[ I(gf) \leq \{ I(g) \ (rel. d_2, d_3) \} \{ I(f) \ (rel. d_1, d_2) \}. \]

Now let $M_i$ ($i=1, 2$) be compact differentiable manifolds and let $f$ be a homeomorphism of $M_i$ onto $M_2$, then since $M_i$ admit Riemannian metrics $\rho_i$ ($i=1, 2$), one can define the size $I(f)$ (rel. $\rho_1, \rho_2$) of $f$. If $h_i$ are diffeomorphisms on $M_i$ ($i=1, 2$), $h_1^*\rho_1$ also are Riemannian metrics on $M_i$, and obviously

\[ I(h_2^*f_1)(rel. \rho_1, \rho_2) = I(f)(rel. h_1^*f_1, h_2^*f_2). \]

Thus denoting by $I(f)$ the infimum of the sizes $I(f)$ (rel. $\rho_1, \rho_2$) taken on all Riemannian metrics $\rho_i$ on $M_i$ ($i=1, 2$) one sees that, for any diffeomorphisms $h_i$ on $M_i$,

\[ I(h_i f h_i) = I(f). \]

A differentiable manifold $M$ which is homeomorphic to a topological manifold $X$ is said to be a smoothing of $X$ and two smoothings $M_i, M_2$ are said to be equivalent if there is a diffeomorphism between them. The equivalence classes are called differentiable structures on $X$.

Define a real valued function $\rho(M_i, M_2)$ of smoothings $M_i$ ($i=1, 2$) by

\[ \rho(M_i, M_2) = \inf \{ \log I(h_2^{-1} h_1) | h_i: M_i \rightarrow X \text{ is homeomorphism} \}, \]

where the infimum is taken over all the homeomorphisms $h_i$ of $M_i$ onto $X$ ($i=1, 2$).

Then obviously,
4) \( \rho(M_1, M_2) \) depends only on the equivalence classes \( \sigma_i \) \((i=1, 2)\) of smoothings \( M_i \) and therefore can be denoted by \( \rho(\sigma_1, \sigma_2) \).

The function \( \rho(\sigma_1, \sigma_2) \) defined on the set \( \Sigma \) of the differentiable structures on \( X \) has the following properties 5–7), which are easily deduced from 1–3):

5) \( \rho(\sigma_1, \sigma_1) = 0 \).
6) \( \rho(\sigma_1, \sigma_2) = \rho(\sigma_2, \sigma_1) \).
7) \( \rho(\sigma_1, \sigma_3) \leq \rho(\sigma_1, \sigma_2) + \rho(\sigma_2, \sigma_3) \), where \( \sigma_3 \) also is a differentiable structure on \( X \).

In the other words,

**Proposition 1.** \( \rho \) gives a pseudo distance on the set \( \Sigma \) of the equivalence classes of differentiable structures on a compact topological manifold \( X \).

The following non symmetric versions of the usual Lipschitz conditions are sometimes useful and referred simply as Lipschitz conditions.

A map \( f \) of a metric space \( X_1 \) into a metric space \( X_2 \) is said to satisfy Lipschitz condition with a positive \( \lambda \), if

\[
|d_{X_1}(f(x), f(y)) - d_{X_2}(x, y)| \leq \lambda d_{X_1}(x, y)
\]

for all \( x, y \in X_1 \). Also \( f \) is said to satisfy local Lipschitz condition at \( p \in X_1 \), if there exists a neighbourhood \( U(p) \) of \( p \) such that

\[
|d_{X_1}(f(x), f(y)) - d_{X_2}(x, y)| \leq \lambda d_{X_1}(x, y)
\]

for all \( x, y \in U(p) \).

Obviously a (regular) Lipschitz map with Lipschitz constant \( I(f) \) satisfies both global and local Lipschitz conditions with \( \lambda = \sqrt{I(f)} - 1 \).

Conversely, a map satisfying the (global) Lipschitz condition with \( \lambda < 1 \) is (regular) Lipschitz and has the Lipschitz constant less than \( 1/\sqrt{1-\lambda} \).

**Proposition 2.** For any positive \( \lambda < 1 \), there is a positive \( \rho \) such that \( \log I(f) < \rho \) implies the Lipschitz condition with \( \lambda \) for \( f \). Conversely, for a given \( \rho > 0 \), there is a positive \( \lambda \) such that the Lipschitz condition with \( \lambda \) for \( f \) yields \( \log I(f) < \rho \).

3. Local properties of Lipschitz maps.

Throughout the section, \( R^n \) denotes the Euclidean \( n \)-space with the usual norm \( \| \cdot \| \) and, unless otherwise stated, \( f \) is a map of \( R^n \) into \( R^N \) sending \( 0 \in R^n \) to \( 0 \in R^N \).

**Lemma 1.** Assume that \( f \) satisfies the local Lipschitz condition with \( \lambda \) in an open neighbourhood \( U(0) \) of \( 0 \), then
\begin{align*}
|\langle f(x), f(y) \rangle - \langle x, y \rangle | & \leq 2\lambda(|x|^2 + |y|^2) \\
for all \ x, y \in U(0).
\end{align*}

Let \( s \) be an \( n \)-simplex in \( \mathbb{R}^n \) having 0 as one of its vertices, then, the \( s \)-approximation \( f_s \) of a map \( f \) is the linear map which agrees to \( f \) on each vertices \( v_i \) of \( s \), that is, \( f_s \) is the linear map characterized by \( f_s(v_i) = f(v_i) \) for all vertices \( v_i \) of \( s \).

Unless otherwise stated, a simplex \( s \) in \( U(0) \) is understood to have 0 as one of its vertices.

Denote by \( \delta(s) \) the diameter of a simplex \( s \) in \( \mathbb{R}^n \), and denote by \( \theta(s) \) the fullness \( \text{vol}(s)/\langle \delta(s) \rangle^n \) of \( s \).

**Lemma 2.** For a positive \( \theta \), there is a positive \( \lambda(\theta, n) \) depending only on \( \theta, n \), such that, if \( f \) satisfies the local Lipschitz condition with \( \lambda(\theta, n) \) in \( U(0) \), then for any \( n \)-simplex \( s \) in \( U(0) \) of fullness \( \theta(s) \geq \theta \), it holds that

\[ \theta(f_s(s)) \geq \theta/2. \]

**Proof.** Choose a sufficiently large \( A(n) \) such that, if \( \max |x_{ij} - a_{ij}| \leq \max |a_{ij}| \), then

\[ |\det x_{ij} - \det a_{ij}| \leq A(n) \max |x_{ij} - a_{ij}| \max |a_{ij}|^{n-1}. \]

Substitute \( x_{ij} \) by \( \langle f(v_i), f(v_j) \rangle \) and \( a_{ij} \) by \( \langle v_i, v_j \rangle \) where \( v_k \) \((k=1, \ldots, n)\) are the vertices of \( s \) except the origin 0. Then it follows that

\[ (n!)^2 |\text{vol}^2 f_s(s) - \text{vol}^2(s)| \leq 4A(n)\lambda \delta^n(s). \]

Hence,

\[ \theta^2(f_s(s))/\langle \sqrt{1 - \lambda} \rangle^{2n} \geq \theta^2(s) - 4\lambda A(n)/n^2 \]

and the conclusion follows easily.

**Lemma 3.** Let \( s, t \) be \( n \)-simplexes in \( U(0) \) such that

\[ \beta \delta \leq \delta(s), \delta(t) \leq \delta, \quad \theta(s), \theta(t) \]

then, if \( f \) satisfies the local Lipschitz condition with \( \lambda \) in \( U(0) \),

\[ |f_s(x) - f_t(x)|^2 \leq 8|x|^2 \lambda n^2(1 + 1/\beta)/n! \theta^4. \]

for any \( x \in U(0) \)

**Proof.** Making use of the fullness \( \theta(s) \), one gets [Wy, p. 126] that

\[ n! \theta(s) \delta(s) \leq |v_i| \leq \delta(s), \quad \text{for all vertex} \ v_i \ of \ s, \]

and that,

\[ |x_i| \leq |\sum x_i v_i|/n! \theta(s)|v_i| \quad \text{for any linear combination} \ \sum x_i v_i. \]
Therefore, one easily sees that

\[ |f_s(x) - f_t(x)|^2 \leq 2\lambda n^2 |x|^2 \left\{ 2/(n!\theta(s)) + 2/(n!\theta(t)) \right\}^2 + 2(\delta(s) + \delta(t))/(n!\theta(s))^{3/2}(\delta(s)n!\theta(t))^{3/2}. \]

Hence the evaluation in Lemma follows easily.

**Proposition 3.** In case of \( n = N \), there exists a positive \( \mu(n, \beta, \theta) \) depending only upon \( n, \beta, \theta \), such that, if \( f \) of \( R^n \) into \( R^n \) satisfies the local Lipschitz condition with \( \mu(n, \beta, \theta) \) in \( U(0) \) and if \( s = (0, v_1, \ldots, v_{n-1}, p) \), \( t = (0, v_1, \ldots, v_{n-1}, q) \) are properly joined \( n \)-simplexes at the \( (n-1) \) face \( F = (0, v_1, \ldots, v_{n-1}) \) which satisfy

\[ s, t \subset U(0) \quad \beta \delta \leq \delta(s), \delta(t) \leq \delta, \theta \leq \theta(s), \theta(t), \]

then the simplexes \( f_s(s) \) and \( f_t(t) \) are non degenerate and properly joined, in the other words, non degenerate simplexes \( f_s(s) \) and \( f_t(t) \) are separated each other by the \( (n-1) \)-simplex \( f_s(F) = f_t(F) \).

**Proof.** Assume that \( f \) satisfies the Lipschitz condition with the constant \( \lambda_s = \lambda(n, \theta) \) in Lemma 1, then (see (3.1))

\[ \text{any height of } |f_s(s)| \geq n!\theta(f_s(s)) \delta(f_s(s)) \geq n! \sqrt{1 - \delta \theta / (2 \lambda)}, \]

in particular

\[ \text{dist } (f_s(p), \text{plane of } f_s(F)) \geq n! \sqrt{1 - \delta \theta / (2 \lambda)}, \]

therefore if \( \lambda \) is close enough small that

\[ 4\sqrt{\lambda(1 + 1/\beta)} n! \theta < n! \sqrt{1 - \delta \theta / (2 \lambda)} \]

then by Lemma 3, both \( f_s(p) \) and \( f_t(p) \) lie in the same side of the plane of \( f_s(F) = f_t(F) \).

**Lemma 4.** If \( f \) of \( R^n \) into \( R^N \) satisfies the Lipschitz condition with \( \lambda \) in \( U(0) \), then for an \( n \)-simplex \( s \) in \( U(0) \) and for \( x \in U(0) \) such that

\[ a\delta(s) \leq |x| \leq \delta(s), \]

it holds that

\[ |f(x) - f_s(x)|^2 \leq 2\lambda |x|^2 \left\{ 1 + 2n^2/(n!\theta(s))^2 + 2n/(n!\theta(s))^2 \right\} + 4\lambda |x| (\sum |x_i|^2/n!\theta(s)|v_i|). \]

**Proof.** A calculation using (3.1), (3.2) shows that

\[ |f(x) - f_s(x)|^2 \leq 2\lambda |x|^2 \left( 1 + 2n^2/(n!\theta(s))^2 \right) + 4\lambda |x| (\sum |x_i|^2/n!\theta(s)|v_i|). \]
Since \( \alpha \delta(s) \leq |x| \leq \delta(s) \),

\[
\sum(|x|^2 + |v_i|^2/|v_i|) = |x| \sum(|x|/|v_i| + |v_i|/|x|)
\leq |x|(n/n!\theta(s) + n/\alpha).
\]

Then one gets the evaluation.

Consider now the case \( f(0) \) is not necessarily 0, and let \( f(p) = q \),
then a parallel translation \( g(x) = f(x + p) - q \) maps 0 into 0 and the local Lipschitz condition for \( f \) at \( p \) yields that for \( g \) at 0. Therefore defining \( s \)-approximation \( f_s \) of \( f \) by the parallel translation of \( g_s \), one gets properties of \( f_s \) similar to those in Lemmas 1-4 and in Proposition 3.

In particular, since, for any \( x \in s \), there found a vertex \( v \) of \( s \) such that

\[
|x - v| \geq n!\theta(s)\delta(s)/2
\]

(see (3,1)). Lemma 4 applied to a parallel translation of \( f \) implies

**Corollary 1.** Under the same condition as in Lemma 4,

\[
|f(x) - f_s(x)|^2 \leq 2\lambda\delta^2(s)\{1 + 2n^2/(n!\theta(s))^2 + 6n/(n!\theta(s))^3\}
\]

for all \( x \in s \).

4. Simplexwise positive maps.

Let \( K \) be a pseudo \( n \)-manifold which may have boundary \( \partial K \).

A map \( f \) of \( K \) into \( \mathbb{R}^n \) is said to be **simplexwise positive** if for each \( n \)-simplex \( s \subset K \), \( f \) is smooth and one to one in \( s \) and Jacobian \( Jf \) of \( f \) in \( s \) is positive there.

**Lemma 5.** If \( f \) is simplexwise positive in \( K \), then for any interior point \( p \) of \( K \) there exists a neighbourhood \( U(p) \) of \( p \) in \( K \) such that \( f \) restricted on \( U(p) \) is one to one.

**Proof.** Let \( p \) be an interior point of a simplex \( \sigma \) in \( K \) (dimension unspecified), then, since \( f \) is one to one in all simplexes in \( K \), \( f(p) \) is covered only once by \( f(\text{St}(\sigma)) \). Take a sufficiently fine subdivision \( K' \) of \( K \) for which the closed star \( \overline{\text{St}'(\sigma')} \) in \( K' \) of a simplex \( \sigma' \) having \( p \) in its interior, is contained in \( \text{St}(\sigma) \). An application of LEMMA 15a of [Wy p. 369] to a simplexwise positive map \( f \) in \( K \) (therefore, so is in \( K' \) and in \( \overline{\text{St}'(\sigma')} \)) and a combinatorial \( n \)-manifold \( \overline{\text{St}'(\sigma')} \), shows that \( f \) is one to one, when considered only in an inverse image \( f^{-1}(R) \) of an open set \( R \) in \( \mathbb{R}^n - f(\partial(\overline{\text{St}'(\sigma')})) \) containing \( f(p) \).

Let \( f, g \) be simplexwise differentiable maps of \( K \) into \( \mathbb{R}^n \). A homotopy \( h_t \) between \( f \) and \( g \) is called a **non degenerate homotopy**, if, for each
t, h, is simplexwise differentiable and the non degenerate Jacobian Jhₜ of hₜ gives a homotopy between Jf and Jg. Obviously one gets

Lemma 6. If a simplexwise one to one map f of K into Rⁿ is non degenerately homotopic to a simplexwise positive map g of K into Rⁿ then f itself is simplexwise positive.

When one specialize the manifold K to that imbedded in Rⁿ and satisfying

\[ \theta \leq \theta(s) \quad \text{and} \quad \beta \delta \leq \delta(s) \leq \delta. \]

for each n-simplex s of K, one can apply Proposition 3 to get

Proposition 4. Let K be a pseudo manifold as above and let \( \mu(n, \beta, \theta) \) be the constant in Proposition 3. Then if a map g of K into Rⁿ satisfies the Lipschitz condition with \( \mu(n, \beta, \theta) \), the simplicial approximation \( g_κ \) of g on K is simplexwise positive and, therefore, is locally homeomorphic at any interior point p of K.

Corollary 2. Using the same notations as in Proposition 4, if a simplexwise one to one map f of K into Rⁿ is non degenerately homotopic to \( g_κ \), then f also is simplexwise positive, and therefore, is locally homeomorphic at any interior point p of K.

5. A proof of the combinatorial equivalence.

First we fix notations; M is an orientable compact connected Rimanian n-manifold, isometrically imbedded in an Euclidean N-space \( R^N \), and \( V(M) \) is the tubular neighbourhood of M with the projection \( \pi \) along the normal plane field \( η \).

The tangent n-plane at \( p ∈ M \) is denoted by \( T_p(M) \) or simply by \( T_p \), then the local projection \( \pi_p \) along \( η \) of a part of \( T_p \) into \( M \) is defined in some neighbourhood of 0 in \( T_p \), and so is the local orthogonal projection \( \Pi_p \) along the fixed plane \( v(p) \) of a part of \( T_p \) into \( M \).

If \( \pi_p^*, \Pi_p^* \) are defined to be local projections of a part of \( V(M) \) into \( T_p \) along \( η \) and along \( η(p) \), respectively, then obviously \( \pi = \pi_p \cdot \pi_p^* \).

The following lemmas are elementary and their proofs are found, for instance, essentially in [Wy p. 117-174].

Lemma 7. Given \( \varepsilon > 0 \), for each \( p ∈ M \), there exists a neighbourhood \( U_p(p) \) of \( p \) in \( M \) such that if a simplex \( σ \) of fullness \( > \varepsilon \) in \( R^N \) has its all vertices in \( U_p(p) \), then both \( \pi_p^*, \Pi_p^* \) are non degenerate and one to one on \( σ \) and, moreover, \( \pi_p^*, \Pi_p^* \) are non degenerately homotopic on \( σ \).

Lemma 8. For any positive \( κ \) and for each \( p ∈ M \), there exists a
neighbourhood $U_\varepsilon(p)$ of $p$ in $M$ such that both $\pi^*_p$ and $\Pi^*_p$ restricted on $U_\varepsilon(p)$, satisfy the Lipschitz condition with $\kappa$ relative to the Riemannian metric on $M$ and the usual norm on $T_p$.

The existence of a certain triangulation of $M$, which is very useful for our purpose, is proved in [Wy. p. 124–135].

**Triangulation Theorem.** There are defined positive functions $\beta(n, N) < 1$ and $\theta(n, N)$ of $n, N$ such that for any positive $\varepsilon$, there are a positive $\delta < \varepsilon$ and an oriented finite combinatorial $n$–manifold $K(\varepsilon) \subset V(M)$ which satisfies the following properties 1), 2).

1) The restriction of $\pi$ to $K(\varepsilon)$ is a homeomorphism onto $M$ and for each $n$–simplex $s$ in $K(\varepsilon)$, $\pi$ is differentiable and non degenerate on $s$.

2) For each point $p \in M$, there is a combinatorial $n$–submanifold $K(p)$ of $K(\varepsilon)$ satisfying the following:

\begin{enumerate}
  \item[(2.1)] The simplicial approximation $(\pi^*_p)_K$ of $\pi^*_p$ on $K(p)$ is isomorphic and the image contains a neighbourhood of $0$ of $T_p$.
  \item[(2.2)] The isomorphic image $L(p)$ of $K(p)$ by $(\pi^*_p)_K$ satisfies that 
    \[
    \text{diam}(L(p)) < 10\delta
    \]
    and for all $n$–simplex $s$ in $L(p)$,
    \[
    \theta(n, N) \leq \theta(s), \beta(n, N)\delta \leq \delta(s) \leq \delta.
    \]
\end{enumerate}

Now let $M'$ be a second compact connected Riemannian $n$–manifold, isometrically imbedded in $R^N$, and let $V(M') \pi'$ etc. denote the corresponding notions defined for $M' \subset R^N$ such as tubular neighbourhood, projection along the normal plane field etc.

Assume that a map $f$ of $M$ onto $M'$ satisfies the Lipschitz condition with $\mu(n, \beta(n, N), \theta(n, N))/2$, (See Prop 2). Then by Lemmas 2, 7, 8 and by Triangulation Theorem, using the compactness of $M, M'$, one easily verifies:

**Assertion.** For some $\varepsilon > 0$, $K(\varepsilon)$ satisfies the following properties 1)–3):

1) For all $p \in M$, $\pi_p$ is defined on $L(p)$ and both $\pi^*_p, \Pi^*_p$ are defined on a neighbourhood of $q = f(p)$ in $R^N$ containing both $f\pi_p(L(p))$ and $(f\pi_p)L(L(p))$, moreover, $\pi_p$ is one to one on $\pi^*_p(f\pi_p)L(L(p))$.

2) For each simplex $\sigma$ in $L(p)$, both $\pi^*_q, \Pi^*_q$ are one to one on the simplex $(f\pi_p)L(\sigma)$ in $R^N$ and are non degenerately homotopic to each other on it.

3) The map $\Pi^*_q f\pi_p$ satisfies the Lipschitz condition with $\mu(n, \beta(n, N), \theta(n, N))$.

Hence, from Proposition 4 and the property 3) above, $(\Pi^*_q f\pi_p)L$ is
simplexwise positive and, since $\Pi^*_{\Phi}$ is linear, simplexwise one to one map $\pi^*_{\Phi}(f_{\Phi})_L$ is non degenerately homotopic to $\Pi^*_{\Phi}(f_{\Phi})_L=(\Pi^*_{\Phi}f_{\Phi})_L$, (see 2) above). Therefore $\pi^*_{\Phi}(f_{\Phi})_L$ is locally homeomorphic at $0 \in L(p)$ and so is $\pi^*_{\Phi}(f_{\Phi})_L\cdot(\pi^*_{\Phi}K)\cdot=\pi^*_{\Phi}(f_{\Phi})_L$ at $p^*\in K(\varepsilon)$ which is mapped to $p$ by $\pi$ (see (2.1) of Triangulation Theorem).

Thus the composition $\pi'_{\Phi}\circ\pi^*_{\Phi}(f_{\Phi})_L=\pi'(f_{\Phi})_K$ is locally homeomorphic at $p^*\in K(\varepsilon)$.

These facts can be unified as follows:

**Proposition 4.** Let $M, M'$ be oriented compact connected Riemannian $n$-manifolds, isometrically imbedded in $R^N$ and let $f$ be a map of $M$ onto $M'$ which satisfies the Lipschitz condition with $\mu(n, \beta(n, N), \theta(n, N))/2$ relative to the Riemannian metrics, then there exists a triangulation $K(\varepsilon)$ of $M$ such that the simplexwise differentiable and simplexwise non degenerate map $F=\pi(f_{\Phi})_L$ is a homeomorphism of $K(\varepsilon)$ onto $M'$, that is, there is a combinatorial equivalence between $K(\varepsilon)$ and $M'$.

**Proof.** Since $F$ is locally homeomorphic, $F: K(\varepsilon)\to M'$ is a covering of $M'$, (see, for instance, [Hu, p. 105]). And since $F$ is homotopic to a homeomorphic map $f_{\Phi}$, every point in $M'$ is covered only once by $F(K(\varepsilon))$, indicating that $F$ itself is a homeomorphism.

According to J. Nash and N. Kuiper [N], every Riemannian manifold $M$ is isometrically imbedded in $R^N$, and an upper bound of $N$ can be given as a function $N(n)$ of the dimension $n$ of $M$. Thus letting $\lambda(n)=\mu(n, \beta(n, N(n)))\theta(n, N(n))/4$, for instance, one gets

**Theorem 1.** Let $M, M'$ be oriented compact connected Riemannian $n$-manifolds. Then if there exists a map $f$ on $M$ onto $M'$ of which Lipschitz constant $I(f)$ is less than a certain positive $\lambda(n)$, which is a function only in $n$, the differentiable manifolds $M, M'$ admit a common triangulation.

In particular Theorem 1 implies (see Prop 2)

**Theorem 2.** Let $\sigma_1, \sigma_2$ be differentiable structures on an orientable compact connected topological $n$-manifold. Then there exists a positive $\rho(n)$ depending only on $n$ such that $\rho(\sigma_1, \sigma_2)\leq\rho(n)$ implies the combinatorial equivalence of $\sigma_1$ and $\sigma_2$.

**PART II**

1. $\phi$ approximation of a Lipschitz map.

Let $\phi$ be a non negative smooth function on $R$ satisfying (1.1)-(1.3):
(1.1) \[ \text{Car } \phi \subset [-1, 1], \quad \max \phi = 1. \]
(1.2) \[ \text{Car } \phi' \subset [-1, -1/2] \cup [1/2, 1], \quad \max |\phi'| \leq 4. \]
(1.3) \[ \max |\phi''| \leq 32. \]

For a positive \( \delta \), define \( \kappa(n) \) and \( \phi_{\delta}(x) \) by

\[
\kappa(n) = \int \phi(|x|) \, dv,
\]
\[
\phi_{\delta}(x) = \phi(|x|/\delta)/\kappa(n)\delta^n,
\]

where \( \int dv \) denotes the integration over \( \mathbb{R}^n \) by the standard volume element \( dv \). Then obviously

\[
\int \phi_{\delta}(x) dv = 1,
\]

(1.1)' \[ \text{Car } \phi_{\delta} \subset U_{\delta}(0), \quad \max \phi_{\delta} = 1/\kappa(n)\delta^n. \]

Let \( \partial_t f \) denote the differential in \( \xi \in T_x(\mathbb{R}^n) \):

\[
\partial_t f = \lim_{t \to 0} f(x + \xi t) - f(x)/t.
\]

Then an easy calculation shows that, for \( \xi, \eta \in T_{\delta}(\mathbb{R}^n) \),

(1.2)' \[ \text{Car } \partial_{\xi} \phi_{\delta} \subset U_{\delta/\eta}(0) \cap U_{\delta}(0), \]
\[ \max |\partial_{\xi} \phi_{\delta}(x)| \leq 4 |\xi|/\kappa(n)\delta^{n+1}, \]

(1.3)' \[ \max |\partial_{\xi} \partial_{\eta} \phi_{\delta}(x)| \leq 64 |\xi| |\eta|/\kappa(n)\delta^{n+2}. \]

Define \( \phi_{\delta}(x, \rho) \) (or simply \( \phi(x, \rho) \)) for \( x, \rho \in \mathbb{R}^n \) by

\[ \phi_{\delta}(x, \rho) = \phi_{\delta}(x - \rho). \]

Denote simply by \( \int f \, dv \) the componentwise integration over \( \mathbb{R}^n \) of an \( \mathbb{R}^N \) valued function \( f \) on \( \mathbb{R}^n \):

\[ \int f \, dv = \left( \int f_1(x) dv, \ldots, \int f_N(x) dv \right). \]

And define the \( \phi_{\delta}(x, \rho) \) approximation (or simply \( \phi \) approximation) \( \phi(f) \) of a map \( f \) of \( \mathbb{R}^n \) into \( \mathbb{R}^N \) by

\[ (\phi(f))(\rho) = \int \phi_{\delta}(x, \rho) f(x) dv. \]

Then, if \( f \) satisfies the local Lipschitz condition with \( \lambda^2 \) on \( U_{\delta}(0) \), that is, if
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\[ \| f(x) - f(y) \|^2 - \| x - y \|^2 \leq \lambda^2 \| x - y \|^2 \]

for all \( x, y \in U_\delta(0) \), one gets the following evaluations (1.6) (1.7):

(1.6) For some positive \( \mu_0 = \mu_0(n, N) \),

\[ |\phi(f)(x) - f(y) - f(z)| \leq \mu_0(1 + \lambda) \delta . \]

(1.7) If \( \sigma \) is an \( n \)-simplex having \( p \) as one of its vertices and such that

\[ \rho(\sigma) \geq \eta, \delta(\sigma) = \delta, \]

then there is a positive \( \mu_1 = \mu_1(n, N, \eta) \), such that

\[ |\partial_f \phi(f) - f_{\sigma}(\xi)| \leq \mu_1 \lambda |\xi| \]

for \( \xi \in T_0(R^n) \) and for the \( \sigma \) approximation \( f_{\sigma} \) of \( f \).

Proof. (1.6) is easily deduced from (1.1)', and \( \mu_0 \) is given by

\[ \mu_0 = \sqrt{N\gamma(n)} / \kappa(n), \]

where \( \gamma(n) \) is the ratio of the volume of the \( n \)-ball \( U_\delta(0) \) to \( \delta^n \). (1.7) is obtained with the aid of Lemma 4 of part I which shows that, if \( \delta(\sigma)/2 \leq |x - p| \leq \delta(\sigma) \), then

\[ |f(x) - f_{\sigma}(x)|^2 \leq 2\lambda^2 \beta(n, \eta) |x|^2 \]

for some \( \beta = \beta(n, \eta) \). And one deduces (1.7) from (1.2)' as follows:

\[ \left| \partial_f \phi(f) - f_{\sigma}(\xi) \right| = \left| \int \partial_f \phi(x, p) \{ f(x) - f_{\sigma}(x) \} dv \right| \]

\[ \leq \lambda \sqrt{2N\beta} 4 \gamma(n) |\xi| / \kappa(n) \]

\[ \leq \lambda \mu_1(n, N, \eta) |\xi| , \]

where \( \mu_1(n, N, \eta) = 4\sqrt{2N\beta} \gamma(n) / \kappa(n) \).

2. \( \alpha \)-neighbourhood of \( \phi \).

If no confusion occurs, the notations \( \text{Car} \ g, \int g \ dv \), and \( \partial_\xi g \) of a function \( g \) of two variables \( x, p \) denote the carrier, integration in \( x \) of \( g(x) = g(x, p) \), and the differential in \( p \) of \( g(p) = g(x, p) \), respectively.

A smooth function \( g \) on \( R^n \times R^n \) is said to be in \( \alpha \)-neighbourhood of \( \phi_\delta(x, p) \), if

(2.1) \[ |g(x, p) - \phi_\delta(x, p)| < 1/\delta^n, \text{Car} \ g(x, p) \subset U_\delta(p), \]

(2.2) \[ |\partial_\xi g(x, p) - \partial_\xi \phi_\delta(x, p)| < \alpha |\xi| / \delta^{n+1}, \]

(2.3) \[ \int g(x, p) dv = 1 \]
where $U_i(p)$ denotes the open ball

$$U_i(p) = \{y \in \mathbb{R}^n / |y - p| < 2\delta\}.$$ 

Let $g(f)$ be the $g$-average of a map $f$ of $\mathbb{R}^n$ into $\mathbb{R}^n$:

$$g(f)(p) = \int g(x, p) f(x) dv.$$ 

Then if $f$ satisfies the $\lambda^2$-Lipschitz condition on $U_i(p)$ and if $g$ is in $\alpha$-neighbourhood of $\phi_3$, one easily gets the following:

$$|g(f)(p) - \phi_3(f)(p)| < \mu_2(1 + \lambda) \delta,$$

$$|\partial g(f) - \partial \phi_3(f)| < \mu_2(1 + \lambda) \alpha |\xi|,$$

where $\mu_2$ is given by

$$\mu_2 = 2^{n+1} \sqrt{N} \gamma(n).$$

Combining (2.4) (2.5) with (1.6) (1.7), one gets the following

**Proposition 1.** Let $\phi_3(x, p)$ be the function defined in 1) and assume that a map $f$ satisfies the $\lambda^2$-Lipschitz condition on $U_{\phi_3}(p)$, then there exist positive numbers $\mu_0, \mu_1, \mu_2$ such that, for any function $g$ in $\alpha$-neighbourhood of $\phi_3$,

$$|g(f)(p) - f(p)| < (\mu_0 + \mu_2)(1 + \lambda) \delta,$$

$$|\partial g(f) - \partial f(\xi)| < (\mu_1 \lambda + \mu_2(1 + \lambda) \alpha |\xi|).$$

**Corollary 1.** There exists $\lambda_0 = \lambda_0(n, N, \eta) > 0$, such that, if $f$ satisfies the Lipschitz condition with $\lambda_0^2$ and if $g$ is in $\alpha$-neighbourhood of $\phi_3(x, p)$ for some small $\alpha$, then the $g$-average $g(f)$ is non degenerate at $p$.

3. **Proof of main theorem.**

Let $M$ be a compact manifold with Riemannian metric $\rho$. Denote by $dV$ the volume element on $M$ and define a function $G_\delta(X, P)$ on $M \times M$ by

$$G_\delta(X, P) = \phi(\rho(X, P)/\delta)/\int_M \phi(\rho(X, P)/\delta) dV,$$

where $\phi$ is the function defined in 1).

The $G_\delta(X, P)$-average $G_\delta(F)$ of a map $F$ of $M$ into $\mathbb{R}^N$ is defined to be

$$G_\delta(F)(P) = \int_M G_\delta(X, P) F(X) dV.$$

Obviously $G_\delta(F)$ is a smooth map of $M$ into $\mathbb{R}^N$. If no confusion occurs, denote simply by corresponding small letters $x, p, \ldots$ the points
in $T_Q(M)$ which are mapped by the exponential map $E_Q$ to $X, P, \cdots$ in $M$, and let $f_Q(x), \rho_Q(x, \rho) \cdots$ denote the composition maps $F(E_Q(x)), \rho(E_Q(\rho))$. Then if $\delta$ is sufficiently small and if $Q$ is near to $P$, one gets

\[ G_\delta(X, P) = \phi(\rho_Q(x, \rho)/\delta)\int \phi(\rho_Q(x, \rho)/\delta)e_Q(x)dv, \]

\[ G_\delta(F)(P) = \int g_\delta(x, \rho)f_Q(x)e_Q(x)dv, \]

where $dv$ is the volume element in $T_p(M)$ and $e_Q(x)$ is given by

\[ e_Q(x) = \det dE_Q(x). \]

**Assertion.** For any $\alpha>0$, there exists $\delta>0$, such that the function $g(x, \rho)$ of two variables defined by

\[ g(x, \rho) = g_\delta(x, \rho)e_Q(x) \]

is in $\alpha$–neighbourhood of $\phi_\delta(x, \rho)$, provided $\rho$ is sufficiently near to 0.

**Proof.** (2, 3) for $g$ is obvious. Also the following (3, 3) (3, 4) are well known

\[ dE_Q(0) = id, \]

\[ if \ x=0 \ or \ \rho=0 \ or \ x=\rho, \ then \ \rho_Q(x, \rho) = |x-\rho|. \]

Since $\rho_Q(x, \rho), |x-\rho|$ both are smooth, (3, 4) implies the following:

Given $\varepsilon>0$, there is $\gamma>0$ such that if $|x|<\gamma$ then

\[ |\rho_Q(x, \rho) - |x-\rho|| < \varepsilon |x-\rho|, \]

\[ |\partial_\xi\rho_Q(x, \rho) - \partial_\xi(|x-\rho|| < \varepsilon |\xi|. \]

Therefore one gets the following evaluations:

\[ |\phi(\rho_Q(x, \rho)/\delta) - \phi(\rho_Q(x, \rho)/\delta)| < 8\varepsilon, \]

\[ |\partial_\xi\phi(\rho_Q(x, \rho)/\delta) - \partial_\xi\phi(\rho_Q(x, \rho)/\delta)| < 32\varepsilon |\xi|. \]

Taking (3, 3) into consideration, if $\delta$ and $|\rho|$ are sufficiently small, one also gets

\[ \left| \int \phi(\rho_Q(x, \rho)/\delta)e_Q(x)dv - \int \phi(|x-\rho|/\delta)dv \right| < 8\mu_\varepsilon^\delta n, \]

\[ \left| \partial_\xi \int \phi(\rho_Q(x, \rho)/\delta)d_Q(x)dv - \partial_\xi \int \phi(|x-\rho|/\delta)dv \right| < 32\mu_\varepsilon^\delta n-1 |\xi|. \]

From these inequalities, (2, 1) (2, 2) can be deduced easily, and this finishes the proof.
(3. 3) also yields that if \( F(X) \) satisfies the Lipschitz condition with \( \lambda^2/4 \) at \( Q \in M \) then \( f_\delta(x) = F(E_\delta(x)) \) satisfies that with \( \lambda^2 \) at \( 0 \in T_\delta(M) \). Thus from Proposition 1, Corollary 1 and Assertion, one easily gets

**Proposition 2.** If \( F(X) \) satisfies the Lipschitz condition with \( \lambda^2_0/4 \) at \( Q \in M \), there are \( \delta_0 \) and a neighborhood \( V(Q) \), in \( M \) of \( Q \) such that, for any \( 0 < \delta \leq \delta_0 \) and for any \( P \in V(Q) \), \( G_\delta(X, P) \) is non degenerate and

\[
|G_\delta(F)(P) - F(P)| < (\mu_0 + \mu_0)(1 + \lambda_0)\delta.
\]

Now let \( F \) be a map of \( M \) onto a Riemannian manifold \( M' \) isometrically imbedded in \( R^N \) and assume that \( F \) satisfies the Lipschitz condition with \( \lambda^2_0/4 \) for each point \( Q \in M \). Making \( \delta \) small, \( G_\delta(F)(M) \) is in the tubular neighbourhood of \( M' \) and the composition \( \pi'G_\delta(F) \) is defined, where \( \pi' \) is the projection of the tubular neighbourhood onto \( M' \).

**Proposition 3.** If \( F: M \to M' \) satisfies the Lipschitz condition with \( \lambda_1 = \lambda_1(n) \) for each \( Q \in M \), then \( \pi'G_\delta(X, P) \) is non degenerate at any point in \( M \).

Proof. Make \( \lambda_1' \leq \lambda_0^2/4 \) so small that \( f \) of the \( \lambda_1 \)-Lipschitz condition satisfies

\[
\theta(f_\delta(\sigma)) \geq \varepsilon/2,
\]

for \( \sigma \in T_\delta(M) \) of fullness \( \geq \varepsilon \) and of diameter \( \delta \) and for the \( \sigma \)-approximation \( f_\sigma \) of \( f_\delta \). Then by LEMMA IIa of [Wy p. 123], one concludes that, for some \( \delta \), the plane \( \Pi(f_\delta(\sigma)) \) is near to \( T_\delta(M') \) and any vector \( \xi \in R^N \) satisfying

\[
|\xi - \eta| < \kappa |\eta|
\]

for some \( \eta \in \Pi(f_\delta), \ \kappa > 0 \),

is not in \( d\pi'-\text{kernel near } Q' \), therefore, by (2, 7), for a small \( \lambda_1 \leq \lambda_1' \) and \( \delta \),

\[
d\pi'dG_\delta(F)(\xi) \neq 0,
\]

for any \( \xi \in T_\delta(M) \).

Thus the compactness argument finishes the proof.

Using the same notations as in Proposition 3, define \( H_t(P) \) by

\[
H_t(P) = \begin{cases} 
F(P) & t = 0, \\
\pi'G_{t\delta}(F)(P) & 0 < t \leq 1,
\end{cases}
\]

then \( H_t(P) \) gives a homotopy between \( F(P) \) and \( \pi'G_\delta(F)(P) \), because the continuity of \( H_t(P) \) at \( t = 0 \) is given by (3, 4). Since \( F \) is homeomorphic, the fiber of the covering \( (M, M', \pi'G_\delta(F)) \) (see [Hu p. 105]) consists of a single point, that is, \( \pi'G_\delta(F) \) itself is homeomorphic.
Theorem 1. Let $M, M'$ be compact connected Riemannian $n$-manifolds. Then, if there exists a map $f$ of $M$ onto $M'$ whose Lipschitz constant $l(f)$ is less than a certain positive, which is a function only in $n$, the differentiable manifolds are diffeomorphic.

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References
