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# ON A DISTANCE FUNCTION ON THE SET OF DIFFERENTIABLE STRUCTURES

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## 1. Introduction

Let $X$ be a given compact closed orientable topological manifold and let $\Sigma=\{\sigma_i\}$ denote the set of differentiable structures on $X$. In this paper one defines a pseudo distance $\rho$ on $\Sigma$ which is allowed to take $\infty$ as one of its values. It is proved that $\rho$ is actually a distance function, namely that $\rho(\sigma_1, \sigma_2)=0$ implies $\sigma_1=\sigma_2$. More strongly, one proves the following

**Theorem 1.** *There exists a positive $\varepsilon_1$ depending on dimension of $X$, such that if $\rho(\sigma_1, \sigma_2)\leqq\varepsilon_1$ then $\sigma_1$ and $\sigma_2$ are differentiably equivalent.*

The proof is given in Part II. In Part I one investiates relations between the distance and the combinatorial equivalence.

It is seen that $\rho(\sigma_1, \sigma_2)<\infty$ if $\sigma_1$ and $\sigma_2$ are combinatorially equivalent, and the following theorem is proved:

**Theorem 2.** *There exists a positive $\varepsilon_2$, depending only on dimension of $X$, such that if $\rho(\sigma_1, \sigma_2)\leqq\varepsilon_2$ then $\sigma_1$ and $\sigma_2$ are combinatorially equivalent.*

In order to assure non triviality of the distance function the following remark might be sufficient. By J. Milnor and I. Tamura there is found a compact comibinatorial manifold which admits two smoothings having different integral Pontrjagin classes, therefore using the result of [S] which asserts that any two differentiable structures of distance less than $1/2\log 3/2$ have the same integral Pontrjagin classes, one sees that the distance between these structures is finite but not less than $1/2\log 3/2$.

Although the distance has been allowed to take $\infty$ as its value, it is possible to restrict $\Sigma$ so that the distaece always gives finite value, by introducing a notion of Lipschitz manifold and compatible smoothing, as appears in a sequel.

# PART I

## 2. Definition of a pseudo distance.

Let $X_i$ $(i=1, 2)$ be metric spaces with metrics $d_i$ $(i=1, 2)$. Then the *size* $\mathfrak{I}(f)$ (rel. $d_1, d_2$) of a homeomorphism $f$ of $X_1$ onto $X_2$ is defined to be

$$\mathfrak{I}(f) = \begin{cases} \inf \{k \geqq 1 \mid \forall x, y \in X_1,\ d_1(x, y)/k \leqq d_2(f(x), f(y)) \leqq k d_1(x, y)\}\ , \\ \infty,\ \textit{if the above set of } k \textit{ is empty.} \end{cases}$$

A homeomorphism $f$ is said to be (regular) *Lipschitz* relative to $d_1$, $d_2$, if the size $\mathfrak{I}(f)$ (rel. $d_1, d_2$) is bounded, and the size $\mathfrak{I}(f)$ (rel. $d_1, d_2$) is sometimes called the *Lipschitz constant* of $f$.

The following properties 1)–3) of the size are elementary.

1) $\mathfrak{I}(id.)$ (rel. $d_1, d_1) = 1$,

2) $\mathfrak{I}(f)$ (rel. $d_1, d_2) = \mathfrak{I}(f^{-1})$ (rel. $d_2, d_1$),

3) Let $X_3$, $d_3$ be the third metric space and its metric, and let $f: X_1 \to X_2$, $g: X_2 \to X_3$ be homeomorphisms, then

$$\mathfrak{I}(gf)\ (\text{rel.}\, d_1, d_3) \leqq \{\mathfrak{I}(g)\ (\text{rel.}\, d_2, d_3)\}\, \{\mathfrak{I}(f)\ (\text{rel.}\, d_1, d_2)\}\ .$$

Now let $M_i$ $(i=1, 2)$ be compact differentiable manifolds and let $f$ be a homeomorphism of $M_1$ onto $M_2$, then since $M_i$ admit Riemannian metrics $\rho_i$ $(i=1, 2)$, one can define the size $\mathfrak{I}(f)$ (rel. $\rho_1, \rho_2$) of $f$. If $h_i$ are diffeomorphisms on $M_i$ $(i=1, 2)$, $h_i^{\#}\rho_i$ also are Riemannian metrics on $M_i$, and obviously

$$\mathfrak{I}(h_2 f h_1)(\text{rel.}\, \rho_1, \rho_2) = \mathfrak{I}(f)(\text{rel.}\, h_1^{\#} f_1, h_2^{\#} f_2)\,.$$

Thus denoting by $\mathfrak{I}(f)$ the infimum of the sizes $\mathfrak{I}(f)$ (rel. $\rho_1, \rho_2$) taken on all Riemannian metrics $\rho_i$ on $M_i$ $(i=1, 2)$ one sees that, for any diffeomorphisms $h_i$ on $M_i$,

$$\mathfrak{I}(h_2 f h_1) = \mathfrak{I}(f)\,.$$

A differentiable manifold $M$ which is homeomorphic to a topological manifold $X$ is said to be a *smoothing* of $X$ and two smoothings $M_1$, $M_2$ are said to be *equivalent* if there is a diffeomorphism between them. The equivalence classes are called *differentiable structures* on $X$.

Define a real valued function $\rho(M_1, M_2)$ of smoothings $M_i$ $(i=1, 2)$ by

$$\rho(M_1, M_2) = \inf \{\log \mathfrak{I}(h_2^{-1} h_1) \mid h_i : M_i \to X \text{ is homeomorphism}\}\ ,$$

where the infimum is taken over all the homeomorphisms $h_i$ of $M_i$ onto $X$ $(i=1, 2)$.

Then obviously,

4) $\rho(M_1, M_2)$ depends only on the equivalence classes $\sigma_i$ $(i=1, 2)$ of smoothings $M_i$ and therefore can be denoted by $\rho(\sigma_1, \sigma_2)$.

The function $\rho(\sigma_1, \sigma_2)$ defined on the set $\Sigma$ of the differentiable structures on $X$ has the following properties 5–7), which are easily deduced from 1–3):

5) $\rho(\sigma_1, \sigma_1)=0$.

6) $\rho(\sigma_1, \sigma_2)=\rho(\sigma_2, \sigma_1)$.

7) $\rho(\sigma_1, \sigma_3)\leqq\rho(\sigma_1, \sigma_2)+\rho(\sigma_2, \sigma_3)$, where $\sigma_3$ also is a differentiable structure on $X$.

In the other words,

**Proposition 1.** *$\rho$ gives a pseudo distance on the set $\Sigma$ of the equivalence classes of differentiable structures on a compact topological manifold $X$.*

The following non symmetric versions of the usual Lipschitz conditions are sometimes useful and referred simply as Lipschitz conditions.

A map $f$ of a metric space $X_1$ into a metric space $X_2$ is said to satisfy *Lipschitz condition* with a positive $\lambda$, if

$$|d_2^2(f(x), f(y))-d_1^2(x, y)|\leqq\lambda d_1^2(x, y)$$

for all $x, y\in X_1$. Also $f$ is said to satisfy *local Lipschitz condition* at $p\in X_1$, if there exists a neighbourhood $U(p)$ of $p$ such that

$$|d_2^2(f(x), f(y))-d_1^2(x, y)|\leqq\lambda d_1^2(x, y)$$

for all $x, y\in U(p)$.

Obviously a (regular) Lipschitz map with Lipschitz constant $\mathrm{l}(f)$ satisfies both global and local Lipschitz conditions with $\lambda=\sqrt{\mathrm{l}^2(f)-1}$.

Conversely, a map satisfying the (global) Lipschitz condition with $\lambda<1$ is (regular) Lipschitz and has the Lipschitz constant less than $1/\sqrt{1-\lambda}$.

**Proposition 2.** *For any positive $\lambda<1$, there is a positive $\rho$ such that $\log \mathrm{l}(f)<\rho$ implies the Lipschitz condition with $\lambda$ for $f$. Conversely, for a given $\rho>0$, there is a positive $\lambda$ such that the Lipschitz condition with $\lambda$ for $f$ yields $\log \mathrm{l}(f)<\rho$.*

### 3. Local properties of Lipschitz maps.

Throughout the section, $R^n$ denotes the Euclidean $n$-space with the usual norm $|\quad|$ and, unless otherwise stated, $f$ is a map of $R^n$ into $R^N$ sending $0\in R^n$ to $0\in R^N$.

**Lemma 1.** *Assume that $f$ satisfies the local Lipschitz condition with $\lambda$ in an open neighbourhood $U(0)$ of $0$, then*

$$|\langle f(x), f(y)\rangle - \langle x, y\rangle| \leqq 2\lambda(|x|^2 + |y|^2)$$

*for all* $x, y \in U(0)$.

Let $s$ be an $n$-simplex in $R^n$ having 0 as one of its vertices, then, the *s-approximation* $f_s$ of a map $f$ is the linear map which agrees to $f$ on each vertices $v_i$ of $s$, that is, $f_s$ is the linear map characterized by $f_s(v_i) = f(v_i)$ for all verteces $v_i$ of $s$.

Unless otherwise stated, a simplex $s$ in $U(0)$ is understood to have 0 as one of its vertices.

Denote by $\delta(s)$ the *diameter* of a simplex $s$ in $R^n$, and denote by $\theta(s)$ the *fullness* $\mathrm{vol}(s)/(\delta(s))^n$ of $s$.

**Lemma 2.** *For a positive* $\theta$, *there is a positive* $\lambda(\theta, n)$ *depending only on* $\theta$, $n$, *such that, if* $f$ *satisfies the local Lipschitz condition with* $\lambda(\theta, n)$ *in* $U(0)$, *then for any n-simplex* $s$ *in* $U(0)$ *of fullness* $\theta(s) \geqq \theta$, *it holds that*

$$\theta(f_s(s)) \geqq \theta/2\,.$$

Proof. Choose a sufficiently large $A(n)$ such that, if $\max|x_{ij} - a_{ij}| \leqq \max|a_{ij}|$, then

$$|\det x_{ij} - \det a_{ij}| \leqq A(n) \max|x_{ij} - a_{ij}| \max|a_{ij}|^{n-1}\,.$$

Substitute $x_{ij}$ by $\langle f(v_i), f(v_j)\rangle$ and $a_{ij}$ by $\langle v_i, v_j\rangle$ where $v_k$ $(k=1, \cdots, n)$ are the vertices of $s$ except the origin 0. Then it follows that

$$(n!)^2|\mathrm{vol}^2 f_s(s) - \mathrm{vol}^2(s)| \leqq 4A(n)\lambda\delta^{2n}(s)\,.$$

Hence,

$$\theta^2(f_s(s))/(\sqrt{1-\lambda})^{2n} \geqq \theta^2(s) - 4\lambda A(n)/n!^2$$

and the conclusion follows easily.

**Lemma 3.** *Let* $s$, $t$ *be n-simplexes in* $U(0)$ *such that*

$$\beta\delta \leq \delta(s),\ \delta(t) \leq \delta, \quad \theta \leq \theta(s),\ \theta(t)$$

*then, if* $f$ *satisfies the local Lipschitz condition with* $\lambda$ *in* $U(0)$,

$$|f_s(x) - f_t(x)|^2 \leq 8|x|^2\lambda n^2(1 + 1/\beta)/n!^4\theta^4\,.$$

*for any* $x \in U(0)$

Proof. Making use of the fullness $\theta(s)$, one gets [Wy, p. 126] that

(3.1) $\qquad n!\theta(s)\delta(s) \leq |v_i| \leq \delta(s)$, *for all vertex* $v_i$ *of* $s$,

and that,

(3.2) $\quad |x_i| \leq |\sum x_i v_i|/n!\theta(s)|v_i|$ *for any linear combination* $\sum x_i v_i$.

Therefore, one easily sees that

$$|f_s(x)-f_t(x)|^2 \leq 2\lambda n^2 |x|^2 \{2/(n!\theta(s))^4+2/(n!\theta(t))^4 \\ +2(\delta^2(s)+\delta^2(t))/(n!\theta(s))^2\delta(s)(n!\theta(t))^2\delta(t)\} .$$

Hence the evaluation in Lemma follows easily.

**Proposition 3.** *In case of $n=N$, there exists a positive $\mu(n, \beta, \theta)$ depending only upon $n, \beta, \theta$, such that, if $f$ of $R^n$ into $R^n$ satisfies the local Lipschitz condition with $\mu(n, \beta, \theta)$ in $U(0)$ and if $s=(0, v_1 \cdots v_{n-1}, p)$, $t=(0, v_1 \cdots v_{n-1}, q)$ are properly joined $n$-simplexes at the $(n-1)$ face $F=(0, v_1 \cdots v_{n-1})$ which satisfy*

$$s, t \subset U(0) \qquad \beta\delta \leqq \delta(s),\ \delta(t) \leqq \delta,\ \theta \leqq \theta(s),\ \theta(t) ,$$

*then the simplexes $f_s(s)$ and $f_t(t)$ are non degenerate and properly joined, in the other words, non degenerate simplexes $f_s(s)$ and $f_t(t)$ are separated each other by the $(n-1)$-simplex $f_s(F)=f_t(F)$.*

Proof. Assume that $f$ satisfies the Lipschitz condition with the constant $\lambda_0=\lambda(n, \theta)$ in Lemma 1, then (see (3.1))

$$\text{any height of } f_s(s) \geqq n!\theta(f_s(s))\delta(f_s(s)) \\ \geqq n!\sqrt{1-\delta_0}\beta\delta\theta/2 ,$$

in particular

$$\text{dist}\,(f_s(p), \text{ plane of } f_s(F)) \geqq n!\sqrt{1-\lambda_0}\beta\delta\theta/2 ,$$

therefore if $\lambda$ is closen so small that

$$4\sqrt{\lambda(1+1/\beta)}n/n!^2\theta^2 < n!\sqrt{1-\lambda_0}\beta\theta/2$$

then by Lemma 3, both $f_s(p)$ and $f_t(p)$ lie in the same side of the plane of $f_s(F)=f_t(F)$.

**Lemma 4.** *If $f$ of $R^n$ into $R^N$ satisfies the Lipschitz condition with $\lambda$ in $U(0)$, then for an $n$-simplex $s$ in $U(0)$ and for $x \in U(0)$ such that*

$$\alpha\delta(s) \leqq |x| \leqq \delta(s) ,$$

*it holds that*

$$|f(x)-f_s(x)|^2 \leqq 2\lambda |x|^2 \{1+2n^2/(n!\theta(s))^4+2n/(n!\theta(s))^2+2n/\alpha n!\theta(s)\}$$

Proof. A calculation using (3, 1), (3, 2) shows that

$$|f(x)-f_s(x)|^2 \leqq 2\lambda |x|^2(1+2n^2/(n!\theta(s))^4) \\ +4\lambda |x| (\textstyle\sum(|x|^2+|v_i|^2)/n!\theta(s)|v_i|) .$$

Since $\alpha\delta(s) \leqq |x| \leqq \delta(s)$,

$$\sum(|x|^2 + |v_i|^2/|v_i|) = |x|\sum(|x|/|v_i| + |v_i|/|x|)$$
$$\leqq |x|(n/n!\theta(s) + n/\alpha).$$

Then one gets the evaluation.

Consider now the case $f(0)$ is not necessarily 0, and let $f(p)=q$, then a parallel translation $g(x)=f(x+p)-q$ maps 0 into 0 and the local Lipschitz condition for $f$ at $p$ yields that for $g$ at 0. Therefore defining $s$-approximation $f_s$ of $f$ by the parallel translation of $g_s$, one gets properties of $f_s$ similar to those in Lemmas 1–4 and in Proportion 3.

In particular, since, for any $x \in s$, there found a vertex $v$ of $s$ such that

$$|x-v| \geqq n!\theta(s)\delta(s)/2$$

(see (3, 1)). Lemma 4 applied to a parallel translation of $f$ implies

**Corollary 1.** *Under the same condition as in Lemma* 4,

$$|f(x)-f_s(x)|^2 \leqq 2\lambda\delta^2(s)\{1+2n^2/(n!\theta(s))^4+6n/(n!\theta(s))^2\}$$

*for all* $x \in s$.

**4. Simplexwise positive maps.**

Let $K$ be a pseudo $n$-manifold which may have boundary $\partial K$.

A map $f$ of $K$ into $R^n$ is said to be *simplexwise positive* if for each $n$-simplex $s \in K$, $f$ is smooth and one to one in $s$ and Jacobian $Jf$ of $f$ in $s$ is positive there.

**Lemma 5.** *If $f$ is simplexwise positive in $K$, then for any interior point $p$ of $K$ there exists a neighbourhood $U(p)$ of $p$ in $K$ such that $f$ restricted on $U(p)$ is one to one.*

Proof. Let $p$ be an interior point of a simplex $\sigma$ in $K$ (dimension unspecified), then, since $f$ is one to one in all simplexes in $K$, $f(p)$ is covered only once by $f(\overline{\mathrm{St}}(\sigma))$. Take a sufficiently fine subdivision $K'$ of $K$ for which the closed star $\overline{\mathrm{St}}'(\sigma')$ in $K'$ of a simplex $\sigma'$ having $p$ in its intorior, is contained in $\mathrm{St}(\sigma)$. An application of LEMMA 15a of [Wy p. 369] to a simplexwise positive map $f$ in $K$ (therefore, so is in $K'$ and in $\overline{\mathrm{St}}'(\sigma')$) and a combinatorial $n$-manifold $\overline{\mathrm{St}}'(\sigma')$, shows that $f$ is one to one, when considered only in an inverse image $f^{-1}(R)$ of an open set $R$ in $R^n - f(\partial(\overline{\mathrm{St}}(\sigma')))$ containing $f(p)$.

Let $f$, $g$ be simplexwise differentiable maps of $K$ into $R^n$. A homotopy $h_t$ between $f$ and $g$ is called a *non degenerate homotopy*, if, for each

$t$, $h_t$ is simplexwise differentiable and the non degenerate Jacobian $Jh_t$ of $h_t$ gives a homotopy between $Jf$ and $Jg$. Obviously one gets

**Lemma 6.** *If a simplexwise one to one map $f$ of $K$ into $R^n$ is non degenerately homotopic to a simplexwise positive map $g$ of $K$ into $R^n$ then $f$ itself is simplexwise positive.*

When one specialize the manifold $K$ to that imbedded in $R^n$ and satisfying

$$\theta \leq \theta(s) \quad \text{and} \quad \beta\delta \leq \delta(s) \leq \delta\,.$$

for each $n$-simplex $s$ of $K$, one can apply Proposition 3 to get

**Proposition 4.** *Let $K$ be a pseudo manifold as above and let $\mu(n, \beta, \theta)$ be the constant in Proposition 3. Then if a map $g$ of $K$ into $R^n$ satisfies the Lipschitz condition with $\mu(n, \beta, \theta)$, the simplicial approximation $g_K$ of $g$ on $K$ is simplexwise positive and, therefore, is locally homeomorphic at any interior point $p$ of $K$.*

**Corollary 2.** *Using the same notations as in Proposition 4, if a simplexwise one to one map $f$ of $K$ into $R^n$ is non degenerately homotopic to $g_K$, then $f$ also is simplexwise positive, and therefore, is locally homeomorphic at any interior point $p$ of $K$.*

## 5. A proof of the combinatorial equivalence.

First we fix notations; $M$ is an orientable compact connected Rimannian $n$-manifold, isometrically imbedded in an Euclidean $N$-space $R^N$, and $V(M)$ is the tubular neighbourhood of $M$ with the projection $\pi$ along the normal plane field $\eta$.

The tangent $n$-plane at $p \in M$ is denoted by $T_p(M)$ or simply by $T_p$, then the local projection $\pi_p$ along $\eta$ of a part of $T_p$ into $M$ is defined in some neighbourhood of 0 in $T_p$, and so is the local orthogonal projection $\Pi_p$ along the fixed plane $\eta(p)$ of a part of $T_p$ into $M$.

If $\pi_p^*$, $\Pi_p^*$ are defined to be local projections of a part of $V(M)$ into $T_p$ along $\eta$ and along $\eta(p)$, respectively, then obviously $\pi = \pi_p \cdot \pi_p^*$.

The following lemmas are elementary and their proofs are found, for instance, essentially in [Wy p. 117–174].

**Lemma 7.** *Given $\varepsilon > 0$, for each $p \in M$, there exists a neighbourhood $U_1(p)$ of $p$ in $M$ such that if a simplex $\sigma$ of fullness $> \varepsilon$ in $R^N$ has its all vertices in $U_1(p)$, then both $\pi_p^*$, $\Pi_p^*$ are non degenerate and one to one on $\sigma$ and, moreover, $\pi_p^*$, $\Pi_p^*$ are non degenerately homotopic on $\sigma$.*

**Lemma 8.** *For any positive $\kappa$ and for each $p \in M$, there exists a* *neighbourhood* $U_2(p)$ *of* $p$ *in* $M$ *such that both* $\pi_p^*$ *and* $\Pi_p^*$ *restricted on* $U_2(p)$, *satisfy the Lipschitz condition with* $\kappa$ *relative to the Riemannian metric on* $M$ *and the usual norm on* $T_p$.

The existence of a certain triangulation of $M$, which is very useful for our purpose, is proved in [Wy. p. 124–135].

**Triangulation Theorem.** *There are defined positive functions* $\beta(n, N) < 1$ *and* $\theta(n, N)$ *of* $n$, $N$ *such that for any positive* $\varepsilon$, *there are a positive* $\delta < \varepsilon$ *and an oriented finite combinatorial n-manifold* $K(\varepsilon) \subset V(M)$ *which satisfies the following properties* 1), 2).

1) *The restriction of* $\pi$ *to* $K(\varepsilon)$ *is a homeomorphism onto* $M$ *and for each n–simplex s in* $K(\varepsilon)$, $\pi$ *is differentiable and non degenerate on s.*

2) *For each point* $p \in M$, *there is a combinatorial n-submanifold* $K(p)$ *of* $K(\varepsilon)$ *satisfying the following*:

(2. 1) *The simplicial approximation* $(\pi_p^*)_K$ *of* $\pi_p^*$ *on* $K(p)$ *is isomorphic and the image contains a neighbourhood of* 0 *of* $T_p$.

(2. 2) *The isomorphic image* $L(p)$ *of* $K(p)$ *by* $(\pi_p^*)_K$ *satisfies that*

$$\operatorname{diam}(L(p)) < 10\delta$$

*and for all n–simplex s in* $L(p)$,

$$\theta(n, N) \leq \theta(s),\ \beta(n, N)\delta \leq \delta(s) \leq \delta\,.$$

Now let $M'$ be a second compact connected Riemannian $n$-manifold, isometrically imbedded in $R^N$, and let $V'(M')$ $\pi'\cdots$etc. denote the corresponding notions defined for $M' \subset R^N$ such as tubular neighbourhood, projection along the normal plane field etc.

Assume that a map $f$ of $M$ onto $M'$ satisfies the Lipschitz condition with $\mu(n, \beta(n, N), \theta(n, N))/2$, (See Prop 2). Then by Lemmas 2, 7, 8 and by Triangulation Theorem, using the compactness of $M$, $M'$, one easily verifies:

*Assertion.* For some $\varepsilon > 0$, $K(\varepsilon)$ satisfies the following properties 1)–3):

1) For all $p \in M$, $\pi_p$ is defined on $L(p)$ and both $\pi_p^{*\prime}$, $\Pi_p^{*\prime}$ are defined on a neighbourhood of $q = f(p)$ in $R^N$ containing both $f\pi_p(L(p))$ and $(f\pi_p)_L(L(p))$, moreover, $\pi_p'$ is one to one on $\pi_q^{*\prime}(f\pi_p)_L(L(p))$.

2) For each simplex $\sigma$ in $L(p)$, both $\pi_q^{*\prime}$, $\Pi_q^{*\prime}$ are one to one on the simplex $(f\pi_p)_L(\sigma)$ in $R^N$ and are non degenerately homotopic to each other on it.

3) The map $\Pi_q^{*\prime} f\pi_p$ satisfies the Lipschitz condition with $\mu(n, \beta(n, N), \theta(n, N))$.

Hence, from Proposition 4 and the property 3) above, $(\Pi_q^{*\prime} f\pi_p)_L$ is simplexwise positive and, since $\Pi_q^{*\prime}$ is linear, simplexwise one to one map $\pi_q^{*\prime}$ $(f\pi_p)_L$ is non degenerately homotopic to $\Pi_q^{*\prime}(f\pi_p)_L=(\Pi_q^{*\prime}f\pi_p)_L$, (see 2) above). Therefore $\pi_q^{*\prime}(f\pi_p)_L$ is locally homeomorphic at $0\in L(p)$ and so is $\pi_q^{*\prime}(f\pi_p)_L\cdot(\pi_p^*)_K=\pi_q^{*\prime}(f\pi)_K$ at $p^*$ in $K(\varepsilon)$ which is mapped to $p$ by $\pi$ (see (2.1) of Triangulation Theorem).

Thus the composition $\pi_q'\circ\pi_q^{*\prime}(f\pi)_K=\pi'(f\pi)_K$ is locally homeomorphic at $p^*\in K(\varepsilon)$.

These facts can be unified as follows:

**Proposition 4.** *Let $M$, $M'$ be oriented compact connected Riemannian $n$-manifolds, isometrically imbedded in $R^N$ and let $f$ be a map of $M$ onto $M'$ which satisfies the Lipschitz condition with $\mu(n, \beta(n, N), \theta(n, N))/2$ relative to the Riemannian metrics, then there exists a triangulation $K(\varepsilon)$ of $M$ such that the simplexwise differentiable and simplexwise non degenerate map $F=\pi'(f\pi)_K$ is a homeomorphism of $K(\varepsilon)$ onto $M'$, that is, there is a combinatorial equivalence between $K(\varepsilon)$ aud $M'$.*

Proof. Since $F$ is locally homeomorphic, $F: K(\varepsilon)\to M'$ is a covering of $M'$, (see, for instance, [Hu, p. 105]). And since $F$ is homotopic to a homeomorphic map $f\pi$, every point in $M'$ is covered only once by $F(K(\varepsilon))$, indicating that $F$ itself is a homeomorphism.

According to J. Nash and N. Kuiper [N], every Riemannian manifold $M$ is isometrically imbedded in $R^N$, and an upper bound of $N$ can be given as a function $N(n)$ of the dimension $n$ of $M$. Thus letting $\lambda(n)=\mu(n, \beta(n, N(n))\theta(n, N(n))/4$, for instance, one gets

**Theorem 1.** *Let $M$, $M'$ be oriented compact connected Riemannian $n$-manifolds. Then if there exists a map $f$ on $M$ onto $M'$ of which Lipschitz constant $\mathfrak{l}(f)$ is less than a certain positive $\lambda(n)$, which is a function only in $n$, the differentiable manifolds $M$, $M'$ admit a common triangulation.*

In particular Theorem 1 implies (see Prop 2)

**Theorem 2.** *Let $\sigma_1$, $\sigma_2$ be differentiable structures on an orientable compact connected topological $n$-manifold. Then there exists a positive $\rho(n)$ depending only on $n$ such that $\rho(\sigma_1, \sigma_2)<\rho(n)$ implies the combinatorial equivalence of $\sigma_1$ and $\sigma_2$.*

## PART II

### 1. $\phi$ approximation of a Lipschitz map.

Let $\phi$ be a non negative smooth function on $R$ satisfying (1.1)–(1.3):

(1.1) $\quad \mathrm{Car}\,\phi\subset[-1, 1], \quad \max\phi = 1.$

(1.2) $\quad \mathrm{Car}\,\phi'\subset[-1, -1/2]\cup[1/2, 1], \quad \max|\phi'|\leq 4.$

(1.3) $\quad \max|\phi''|\leq 32.$

For a positive $\delta$, define $\kappa(n)$ and $\phi_\delta(x)$ by

$$\kappa(n) = \int\phi(|x|)\,dv,$$

$$\phi_\delta(x) = \phi(|x|/\delta)/\kappa(n)\delta^n, \tag{1.4}$$

where $\int dv$ denotes the integration over $R^n$ by the standard volume element $dv$. Then obviously

$$\int\phi_\delta(x)dv = 1, \tag{1.5}$$

$$\mathrm{Car}\,\phi_\delta\subset U_\delta(0), \ \max\phi_\delta = 1/\kappa(n)\delta^n. \tag{1.1$'$}$$

Let $\partial_\xi f$ denote the differential in $\xi\in T_x(R^n)$:

$$\partial_\xi f = \lim_{t\to 0} f(x+\xi t)-f(x)/t.$$

Then an easy calculation shows that, for $\xi, \eta\in T_0(R^n)$,

$$\mathrm{Car}\,\partial_\xi\phi_\delta\subset U_{\delta/2}(0)'\cap U_\delta(0), \tag{1.2$'$}$$

$$\max|\partial_\xi\phi_\delta(x)|\leqq 4|\xi|/\kappa(n)\delta^{n+1},$$

$$\max|\partial_\eta\partial_\xi\phi_\delta(x)|\leqq 64|\xi|\,|\eta|/\kappa(n)\delta^{n+2}. \tag{1.3$'$}$$

Define $\phi_\delta(x, p)$ (or simply $\phi(x, p)$) for $x, p\in R^n$ by

$$\phi_\delta(x, p) = \phi_\delta(x-p).$$

Denote simply by $\int f\,dv$ the componentwise integration over $R^n$ of an $R^N$ valued function $f$ on $R^n$:

$$\int f\,dv = \left(\int f_1(x)dv, \cdots, \int f_N(x)\,dv\right).$$

And define the $\phi_\delta(x, p)$ approximation (or simply $\phi$ approximation) $\phi(f)$ of a map $f$ of $R^n$ into $R^N$ by

$$(\phi(f))(p) = \int\phi_\delta(x, p)f(x)dv.$$

Then, if $f$ satisfies the local Lipschitz condition with $\lambda^2$ on $U_\delta(0)$, that is, if

$$||f(x)-f(y)|^2-|x-y|^2|\leq\lambda^2|x-y|^2$$

for all $x, y\in U_\delta(0)$, one gets the following evaluations (1.6) (1.7):
(1.6) *For some positive* $\mu_0=\mu_0$ $(n, N)$,

$$|\phi(f)(p)-f(p)|\leqq\mu_0(1+\lambda)\delta\,.$$

(1.7) *If* $\sigma$ *is an n-simplex having* $p$ *as one of its vertices and such that*

$$\rho(\sigma)\geqq\eta,\ \delta(\sigma)=\delta\,,$$

*then there is a positive* $\mu_1=\mu_1$ $(n, N, \eta)$, *such that*

$$|\partial_\xi\phi(f)-f_\sigma(\xi)|\leqq\mu_1\lambda|\xi|$$

*for* $\xi\in T_0(R^n)$ *and for the* $\sigma$ *approximation* $f_\sigma$ *of* $f$.

Proof. (1.6) is easily deduced from (1.1)′, and $\mu_0$ is given by

$$\mu_0=\sqrt{N}\gamma(n)/\kappa(n)\,,$$

where $\gamma(n)$ is the ratio of the volume of the $n$-ball $U_\delta(0)$ to $\delta^n$. (1.7) is obtained with the aid of Lemma 4 of part I which shows that, if $\delta(\sigma)/2\leq|x-p|\leq\delta(\sigma)$, then

$$|f(x)-f_\sigma(x)|^2\leqq2\lambda^2\beta(n, \eta)|x|^2$$

for some $\beta=\beta(n, \eta)$. And one deduces (1.7) from (1.2)′ as follows:

$$\begin{aligned}|\partial_\xi\phi(f)-f_\sigma(\xi)| &= \left|\int\partial_\xi\phi(x, p)\{f(x)-f_\sigma(x)\}dv\right|\\ &\leqq\lambda\sqrt{2N\beta}\,4\gamma(n)|\xi|/\kappa(n)\\ &\leqq\lambda\mu_1(n, N, \eta)|\xi|\,,\end{aligned}$$

where $\mu_1(n, N, \eta)=4\sqrt{2N\beta}\,\gamma(n)/\kappa(n)$.

## 2. $\alpha$-neighbourhood of $\phi$.

If no confusion occurs, the notations Car $g$, $\int g\,dv$, and $\partial_\xi g$ of a function $g$ of two variables $x$, $p$ denote the carrier, integration in $x$ of $g(x)=g(x, p_0)$, and the differential in $p$ of $g(p)=g(x_0, p)$, respectively.

A smooth function $g$ on $R^n\times R^n$ is said to be in $\alpha$-neighbourhood of $\phi_\delta(x, p)$, if

(2. 1) $$|g(x, p)-\phi_\delta(x, p)|<1/\delta^n,\ \text{Car}\ g(x, p)\subset U_1(p)\,,$$

(2. 2) $$|\partial_\xi g(x, p)-\partial_\xi\phi_\delta(x, p)|<\alpha|\xi|/\delta^{n+1}\,,$$

(2. 3) $$\int g(x, p)dv=1\,,$$

where $U_1(p)$ denotes the open ball

$$U_1(p) = \{y \in R^n / |y-p| < 2\delta\}.$$

Let $g(f)$ be the $g$-average of a map $f$ of $R^n$ into $R^n$:

$$g(f)(p) = \int g(x, p) f(x) dv.$$

Then if $f$ satisfies the $\lambda^2$-Lipschitz condition on $U_1(p)$ and if $g$ is in $\alpha$-neighbourhood of $\phi_\delta$, one easily gets the following:

$$|g(f)(p) - \phi_\delta(f)(p)| < \mu_2(1+\lambda)\delta, \tag{2.4}$$

$$|\partial_\xi g(f) - \partial_\xi \phi_\delta(f)| < \mu_2(1+\lambda)\alpha|\xi|, \tag{2.5}$$

where $\mu_2$ is given by

$$\mu_2 = 2^{n+1}\sqrt{N}\gamma(n).$$

Combining (2,4) (2,5) with (1,6) (1,7), one gets the following

**Proposition 1.** *Let $\phi_\delta(x, p)$ be the function defined in* 1) *and assume that a map $f$ satisfies the $\lambda^2$-Lipschitz condition on $U_{2\delta}(p)$, then there exist positive numbers $\mu_0$, $\mu_1$,$\cdot$ $\mu_2$ such that, for any function $g$ in $\alpha$-neighbourhood of $\phi_\delta$,*

$$|g(f)(p) - f(p)| < (\mu_0 + \mu_2)(1+\lambda)\delta, \tag{2.6}$$

$$|\partial_\xi g(f) - f_\sigma(\xi)| < (\mu_1\lambda + \mu_2(1+\lambda)\alpha\} |\xi|. \tag{2.7}$$

**Corollary 1.** *There exists $\lambda_0 = \lambda_0 (n, N, \eta) > 0$, such that, if $f$ satisfies the Lipschitz condition with $\lambda_0^2$ and if $g$ is in $\alpha$-neighbourhood of $\phi_\delta(x, p)$ for some small $\alpha$, then the $g$-average $g(f)$ is non degenerate at $p$.*

### 3. Proof of main theorem.

Let $M$ be a compact manifold with Riemannian metric $\rho$. Denote by $dV$ the volume element on $M$ and define a function $G_\delta(X, P)$ on $M \times M$ by

$$G_\delta(X, P) = \phi(\rho(X, P)/\delta) / \int_M \phi(\rho(X, P)/\delta) dV,$$

where $\phi$ is the function defined in 1).

The $G_\delta(X, P)$-average $G_\delta(F)$ of a map $F$ of $M$ into $R^N$ is defined to be

$$G_\delta(F)(P) = \int_M G_\delta(X, P) F(X) dV.$$

Obviously $G_\delta(F)$ is a smooth map of $M$ into $R^N$. If no confusion occurs, denote simply by corresponding small letters $x, p, \cdots$ the points in $T_Q(M)$ which are mapped by the exponential map $E_Q$ to $X, P, \cdots$ in $M$, and let $f_Q(x)$, $\rho_Q(x, p) \cdots$ denote the composition maps $F(E_Q(x))$, $\rho(E_Q(p))$. Then if $\delta$ is sufficiently small and if $Q$ is near to $P$, one gets

$$G_\delta(X, P) = \phi(\rho_Q(x, p)/\delta) \Big/ \int \phi(\rho_Q(x, p)/\delta) e_Q(x) dv\,, \tag{3. 1}$$

$$G_\delta(F)(P) = \int g_\delta(x, p) f_Q(x) e_Q(x) dv\,, \tag{3. 2}$$

where $dv$ is the volume element in $T_p(M)$ and $e_p(x)$ is given by

$$e_Q(x) = \det dE_Q(x)\,.$$

*Assertion.* For any $\alpha > 0$, there exists $\delta > 0$, such that the function $g(x, p)$ of two variables defined by

$$g(x, p) = g_\delta(x, p) e_Q(x)$$

is in $\alpha$-neighbourhood of $\phi_\delta(x, p)$, provided $p$ is sufficiently near to 0.

Proof. (2, 3) for $g$ is obvious. Also the following (3, 3) (3, 4) are well known

$$dE_Q(0) = id\,, \tag{3. 3}$$

$$\textit{if } x = 0 \textit{ or } p = 0 \textit{ or } x = p, \textit{ then } \rho_Q(x, p) = |x - p|\,. \tag{3. 4}$$

Since $\rho_Q(x, p)$, $|x-p|$ both are smooth, (3, 4) implies the following:

Given $\varepsilon > 0$, there is $\gamma > 0$ such that if $|x| < \gamma$ then

$$|\rho_Q(x, p) - |x - p|| < \varepsilon |x - p|\,,$$
$$|\partial_\xi \rho_Q(x, p) - \partial_\xi(|x - p|)| < \varepsilon |\xi|\,.$$

Therefore one gets the following evaluations:

$$|\phi(\rho_Q(x, p)/\delta) - \phi(|x - p|/\delta)| < 8\varepsilon\,,$$
$$|\partial_\xi \phi(\rho_Q(x, p)/\delta) - \partial_\xi \phi(|x - p|/\delta) < 32\varepsilon |\xi|/\delta\,.$$

Taking (3, 3) into consideration, if $\delta$ and $|p|$ are sufficiently small, one also gets

$$\left| \int \phi(\rho_Q(x, p)/\delta) e_Q(x) dv - \int \phi(|x - p|/\delta) dv \right| < 8\mu_2 \varepsilon' \delta^n\,,$$

$$\left| \partial_\xi \int \phi(\rho_Q(x, p)/\delta) d_Q(x) dv - \partial_\xi \int \phi(|x - p|/\delta) dv \right| < 32\mu_2 \varepsilon' \delta^{n-1} |\xi|\,.$$

From these inequalities, (2, 1) (2, 2) can be deduced easily, and this finishes the proof.

(3. 3) also yields that if $F(X)$ satisfies the Lipschitz condition with $\lambda^2/4$ at $Q\in M$ then $f_Q(x)=F(E_Q(x))$ satisfies that with $\lambda^2$ at $0\in T_Q(M)$. Thus from Proposition 1, Corollary 1 and Assertion, one easily gets

**Proposition 2.** *If $F(X)$ satisfies the Lipschitz condition with $\lambda_0^2/4$ at $Q\in M$, there are $\delta_0$ and a neighbourhood $V(Q)$, in $M$ of $Q$ such that, for any $0<\delta\leqq\delta_0$ and for any $P\in V(Q)$, $G_\delta(X, P)$ is non degenerate and*

$$|G_\delta(F)(P)-F(P)|<(\mu_0+\mu_2)(1+\lambda_0)\delta\,. \tag{3. 4}$$

Now let $F$ be a map of $M$ onto a Riemannian manifold $M'$ isometrically imbedded in $R^N$ and assume that $F$ satisfies the Lipschitz condition with $\lambda_0^2/4$ for each point $Q\in M$. Making $\delta$ small, $G_\delta(F)(M)$ is in the tubular neighbourhood of $M'$ and the composition $\pi'G_\delta(F)$ is defined, where $\pi'$ is the projection of the tubular neighbourhood onto $M'$.

**Proposition 3.** *If $F: M\to M'$ satisfies the Lipschitz condition with $\lambda_1=\lambda_1(n)$ for each $Q\in M$, then $\pi'G_\delta(X, P)$ is non degenerate at any point in $M$.*

Proof. Make $\lambda_1'\leqq\lambda_0^2/4$ so small that $f$ of the $\lambda_1$-Lipschitz condition satisfies

$$\theta(f_\sigma(\sigma))\geqq\varepsilon/2\,,$$

for $\sigma\subset T_Q(M)$ of fullness $\geqq\varepsilon$ and of diameter $\delta$ and for the $\sigma$-approximation $f_\sigma$ of $f_Q$. Then by LEMMA IIa of [Wy p. 123], one concludes that, for some $\delta$, the plane $\Pi(f_\sigma(\sigma))$ is near to $T_{Q'}(M')$ and any vector $\xi\in R^N$ satisfying

$$|\xi-\eta|<\kappa|\eta| \qquad \text{for some} \quad \eta\in\Pi(f_\sigma)\,, \quad \kappa>0\,,$$

is not in $d\pi'$-kernel near $Q'$, therefore, by (2, 7), for a small $\lambda_1\leqq\lambda_1'$ and $\delta$,

$$d\pi' dG_\delta(F)(\xi)\neq 0, \qquad \text{for any} \quad \xi\in T_Q(M)\,.$$

Thus the compactness arguement finishes the proof.

Using the same notations as in Proposition 3, define $H_t(P)$ by

$$H_t(P) = \begin{cases} F(P) & t=0\,, \\ \pi'G_{t\delta}(F)(P) & 0<t\leqq 1\,, \end{cases}$$

then $H_t(P)$ gives a homotopy between $F(P)$ and $\pi'G_\delta(F)(P)$, because the continuity of $H_t(P)$ at $t=0$ is given by (3, 4). Since $F$ is homeomorphic, the fiber of the covering $(M, M', \pi'G_\delta(F))$ (see [Hu p. 105]) consists of a single point, that is, $\pi'G_\delta(F)$ itself is homeomorphic.

**Theorem 1.** *Let $M$, $M'$ be compact connected Riemannian $n$-manifolds. Then, if there exists a map $f$ of $M$ onto $M'$ whose Lipschitz constant* $\mathrm{l}(f)$ *is less than a certain positive, which is a function only in $n$, the differentiable manifolds are diffeomorphic.*

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