Watanabe, A.
Osaka J. Math.
28 (1991), 85-92

NOTE ON A p-BLOCK OF A FINITE GROUP WITH ABELIAN DEFECT GROUP II

Dedicated to Professor Tuyosi Oyama on his 60th birthday

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(Received March 19, 1990)

1. Introduction

In this paper we study the multiplicities of the lower defect groups of a p-block of a p-solvable group with abelian defect group. Let G be a finite group, $(K, \mathfrak{o}, F)$ be a p-modular system such that K is a splitting field for all subgroups of G, and $B_1, B_2, \cdots, B_\tau \cdots$ be the p-blocks of G. We denote by E_τ the block idempotent of FG corresponding to B_τ and for a conjugacy class C of G we denote also by C the class sums in the group rings KG, $\mathfrak{o}G$ and FG.

(*) *With each B_τ it is possible to associate $k(B_\tau)$ conjugacy classes C_j^τ, $1 \leq j \leq k(B_\tau)$, such that the following conditions are satisfied.*
(i) *Each conjugacy class of G is associated with exactly one p-block.*
(ii) *$\{C_j^\tau E_\tau \mid 1 \leq j \leq k(B_\tau)\}$ is an F-basis of $Z(FG)E_\tau$.*

In the above, $l(B_\tau) = \#\{C_j^\tau \mid C_j^\tau$ is a p-regular class, $1 \leq j \leq k(B_\tau)\}$. We shall call an association of the conjugacy classes of G with the p-blocks which satisfy (i) and (ii) in (*) an association of the conjugacy classes of G with the p-blocks shortly. We put $m(B_\tau, Q) = \#\{C_j^\tau \mid C_j^\tau$ has Q as a defect group, $1 \leq j \leq k(B_\tau)\}$ and $m(B_\tau, Q)' = \#\{C_j^\tau$ is a p-regular class and C_j^τ has Q as a defect group, $1 \leq j \leq k(B_\tau)\}$ for a p-subgroup Q of G. When $m(B_\tau, Q) \neq 0$, Q is called a lower defect group of B_τ and $m(B_\tau, Q)$ is the multiplicity of Q as a lower defect group of B_τ. If Q is a lower defect group of B_τ, then Q is contained in a defect group of B_τ. For further properties of lower defect groups, see Feit [3, chapter V, §10], Nagao-Tsushima [8, chapter 5, §11] and Watanabe [11, §§2 and 3]. The following is our main result.

Theorem. *Let G be a p-solvable group and B be a p-block of G with abelian defect group D. Let b be a p-block of $C_G(D)$ such that $b^G = B$. For a subgroup P of D we have*

$$m(b^{N_G(P)}, P) = m(b^{N_G(D) \cap N_G(P)}, P),$$
$$m(b^{N_G(P)}, P)' = m(b^{N_G(D) \cap N_G(P)}, P)'.$$

In the same situation as in the above theorem Broué conjectured in [1, §6] that (G, B) and $(N_G(D), b^{N_G(D)})$ are of the same type for an arbitrary G. If this is true, then our theorem will follow by [1, Theorem 4.8]. However we shall prove it directly by character theoretic methods.

It is well known that under the assumption of the theorem $k(B)=k(b^{N_G(D)})$ and $l(B)=l(b^{N_G(D)})$ (see Fong [4, Theorem (3C)], Okuyama-Wajima [9, Theorem] and Dade [2]). To prove our theorem we shall use [9, Theorem 2] which was a key result for the proof of [9, Theorem]. When D is cyclic, our theorem holds for arbitrary groups by Broué [1, Theorem 1.5] and Linckelmann [7], since in that case a lower defect group of B is either trivial or conjugate to D.

For the notations, see [3], [8], [11] and Isaacs [6].

2. Preliminary results

Let H be a normal subgroup of G and ζ be a G-invariant (ordinary) irreducible character of H. We set $\bar{G}=G/H$ and $\bar{x}=xH\in\bar{G}$. We say that $\bar{x}$ is ζ-special if ζ is extendible to $\langle H, x, c\rangle$ for all $c\in G$ with $[x, c]\in H$, according to [6, chapter 11, Problems] (Gallagher says that x is good for ζ in [5]). We also say that a conjuacy class $\bar{C}$ of $\bar{G}$ is ζ-special if it consists of ζ-special elements. By [5, Theorem], the number of irreducible constituents of ζ^G is equal to the number of conjugacy classes of $\bar{G}$ which are ζ-special.

For an irreducible character χ of G we denote by e_χ the centrally primitive idempotent of KG corresponding to χ.

Lemma 1. *Let H be a normal subgroup of G, ζ be a G-invariant irreducible character of H and $\chi_1, \chi_2, \cdots, \chi_s$ be the irreducible constituents of ζ^G. Then we have the following.*

(i) (See [6, chapter 11, problem 11.15]). *For $x\in G$, $\bar{x}$ is ζ-special if and only if $\chi_i\neq 0$ on xH for some i $(1\leq i\leq s)$.*

(ii) *Let $C_1, C_2, \cdots, C_s$ be the conjugacy classes of G such that $\{C_je_\zeta \mid j=1, 2, \cdots, s\}$ is a K-basis of $Z(KG)e_\zeta$ and $x_j\in C_j$ $(j=1, 2, \cdots, s)$. Then $\{\bar{x}_j \mid j=1, 2, \cdots, s\}$ forms a complete set of representatives for the ζ-special conjugacy classes of $\bar{G}$.*

Proof. Let ζ_x be an extension of ζ to $\langle H, x\rangle$. Then we can set $\chi_{i\langle H,x\rangle}=\zeta_x(\sum_{\lambda\in\mathrm{Irr}(\langle\bar{x}\rangle)} n_{i,x,\lambda}\lambda)$, where $n_{i,x,\lambda}$ is a non-negative integer and λ is regarded as a character of $\langle H, x\rangle$. It is clear that $\bar{x}$ is ζ-special if and only if $\zeta_x^c=\zeta_x$ for all $c\in G$ with $[x, c]\in H$. If we set $\rho_{i,x}=\sum_{\lambda\in\mathrm{Irr}(\langle\bar{x}\rangle)} n_{i,x,\lambda}\lambda$, then

$$\chi_i(xh)=\zeta_x(xh)\rho_{i,x}(\bar{x}) \qquad (h\in H)\,. \tag{1}$$

(i) Suppose that $\bar{x}$ is not ζ-special, Then for some $c\in G$ with $[x, c]\in H$,

$\zeta_x^c=\zeta_x\mu$, where μ is a non-trivial linear character of $\langle\bar{x}\rangle$. Since $(\chi_{i\langle H,x\rangle})^c=\chi_{i\langle H,x\rangle}$, $n_{i,x,\lambda}=n_{i,x,\mu\lambda}=n_{i,x,\mu^2\lambda}=\cdots$ for any $\lambda\in\mathrm{Irr}(\langle\bar{x}\rangle)$. Hence $\rho_{i,x}(\bar{x})=0$ and $\chi_i(xh)=0$ for any $h\in H$ and χ_i.

As is well known the rank of the matrix $(\chi_i(z))$ $(1\leq i\leq s,\ z\in G)$ is equal to s and hence it is equal to the number of ζ-special conjugacy classes of $\bar{G}$ by Gallagher's theorem. On the other hand for any $x\in G$, vectors $(\chi_1(xh), \chi_2(xh), \cdots, \chi_s(xh))$, $h\in H$ are linearly dependent by (1). Combining these with the if part, if $\bar{x}$ is ζ-special then $\chi_i\neq 0$ on xH for some i.

(ii) We note that $Z(KG)e_\zeta=\sum_{i=1}^{s}Ke_{\chi_i}$, since $e_\zeta=\sum_{i=1}^{s}e_{\chi_i}$. For C_j in (ii) we have

$$C_je_\zeta=\sum_{i=1}^{s}\frac{|G:C_G(x_j)|\chi_i(x_j)}{\chi_i(1)}e_{\chi_i}\neq 0\,.$$

Hence by (i), $\bar{x}_j$ must be ζ-special. Let $h\in H$ and C be a conjugacy class of G containing x_jh. From (1) we obtain

$$\begin{aligned}Ce_\zeta&=\sum_{i=1}^{s}(|G:C_G(x_jh)|(\zeta_{x_j}(x_jh)\rho_{i,x_j}(\bar{x}_j)/\chi_i(1))e_{\chi_i}\\&=((|G:C_G(x_jh)|/|G:C_G(x_j)|)(\zeta_{x_j}(x_jh)/(\zeta_{x_j}(x_j))C_je_\zeta\,.\end{aligned}$$

So by the assumption, if $j\neq k$ then $\bar{x}_j$ and $\bar{x}_k$ are not conjugate. Therefore (ii) holds. This completes the proof of the lemma.

We prove Theorem by Fong's reductions. In the following we show how the multipilcities of lower defect groups behave under Fong's second reductions. For a conjugacy class C of G we denote by $D(C)$ a defect gruop of C. For subgroups K and L of G we write $K=_GL$, if K and L are G-conjugate.

Lemma 2. *Let H be a normal p'-subgroup of G, ζ be an irreducible character of H which is extendible to G and $\tilde{\zeta}$ be an extension of ζ to G. Let $\bar{B}$ be a p-block of $\bar{G}$ and $B=\tilde{\zeta}\bar{B}$, i.e., B is the p-block of G which contains $\tilde{\zeta}\chi$, $\chi\in\bar{B}$. Then for a p-subgroup Q of G we have*

$$m(B, Q)=m(\bar{B}, \bar{Q})\quad\textit{and}\quad m(B, Q)'=m(\bar{B}, \bar{Q})'\,,$$

where $\bar{Q}=QH/H$.

Proof. Let $\bar{B}_1, \bar{B}_2, \cdots, \bar{B}_r$ be the p-blocks of $\bar{G}$ and $B_\tau=\tilde{\zeta}\bar{B}_\tau$. Then it is well known that $B_1, B_2, \cdots, B_r$ are the p-blocks of G which covers the p-block of H containing ζ and that $k(B_\tau)=k(\bar{B}_\tau)$ and $l(B_\tau)=l(\bar{B}_\tau)$. Let C_j^τ, $1\leq j\leq k(B_\tau)$, be the conjugacy classes of G associated with B_τ in an association of the conjugacy classes of G with the p-blocks. Since $\bar{e}_\zeta=\sum_{\tau=1}^{r}E_\tau$, $\{C_j^\tau E_\tau\,|\,1\leq j\leq k(B_\tau),\ 1\leq\tau\leq r\}$ forms an F-basis of $Z(FG)\bar{e}_\zeta$, where $\bar{e}_\zeta$ is the block idempotent of FH corresponding to ζ. Hence $\{C_j^\tau e_\zeta \mid 1 \leq j \leq k(B_\tau),\ 1 \leq \tau \leq r\}$ forms a K-basis of $Z(KG)e_\zeta$. Let $x_j^\tau \in C_j^\tau$. By Lemma 1, $\{\bar{x}_j^\tau \mid 1 \leq j \leq k(B_\tau),\ 1 \leq \tau \leq r\}$ forms a set of representatives for the conjugacy classes of $\bar{G}$, because ζ is extendible to G and hence any conjugacy class of $\bar{G}$ is ζ-special.

Let f be the isomorphism from $Z(KG)e_\zeta$ to $Z(K\bar{G})$ defined by $f(e_{\tilde{\zeta}\bar{\chi}}) = e_{\bar{\chi}}$ for $\bar{\chi} \in \mathrm{Irr}(\bar{G})$. Let C be an arbitrary conjugacy class of G, x an element of C and $\bar{C}$ be the conjugacy class of $\bar{G}$ containing $\bar{x}$. Since $Ce_\zeta = \sum_{\bar{\chi} \in \mathrm{Irr}(\bar{G})} (|G : C_G(x)| \tilde{\zeta}(x)\bar{\chi}(\bar{x})/(\zeta(1)\bar{\chi}(\bar{1})))e_{\tilde{\zeta}\bar{\chi}}$ and $\bar{C} = \sum_{\bar{\chi} \in \mathrm{Irr}(\bar{G})} (|\bar{G} : C_{\bar{G}}(\bar{x})| \bar{\chi}(\bar{x})/\bar{\chi}(\bar{1}))e_{\bar{\chi}}$, by the second orthogonality relation we have

$$f(Ce_\zeta) = ((|G: C_G(x)|/|\bar{G}: C_{\bar{G}}(\bar{x})|)(\tilde{\zeta}(x)/\zeta(1)))\bar{C} \quad \text{and} \tag{2}$$
$$f^{-1}(\bar{C}) = \sum_y (\zeta(1)\tilde{\zeta}(y^{-1})/|H|)y\,,$$

where y ranges over the set of elements of G such that $\bar{y}$ and $\bar{x}$ are $\bar{G}$-conjugate. In particular $f(Ce_\zeta) \in Z(\mathfrak{o}\bar{G})$ and $f^{-1}(\bar{C}) \in Z(\mathfrak{o}G)e_\zeta$. Therefore f induces an isomorphism from $Z(\mathfrak{o}G)e_\zeta$ to $Z(\mathfrak{o}\bar{G})$, and hence an isomorphism from $Z(FG)E_\tau$ to $Z(F\bar{G})E_{\bar{B}_\tau}$ for any τ, where $E_{\bar{B}_\tau}$ is the block idempotent of $Z(F\bar{G})$ corresponding to $\bar{B}_\tau$.

Let $\bar{C}_j^\tau$ be the conjugacy class of $\bar{G}$ which contains $\bar{x}_j^\tau$. By the above argument and (2), $\{\bar{C}_j^\tau E_{\bar{B}_\tau} \mid 1 \leq j \leq k(\bar{B}_\tau)\}$ forms an F-basis of $Z(F\bar{G})E_{\bar{B}_\tau}$ and $D(\bar{C}_j^\tau) =_{\bar{G}} \overline{D(C_j^\tau)}$. On the other hand, if Q and Q' are p-subgroups of G and if they are not G-conjugate, then $\bar{Q}$ and $\bar{Q}'$ are not $\bar{G}$-conjugate, because H is a p'-subgroup. Further C_j^τ is a p-regular class if and only if $\bar{C}_j^\tau$ is a p-regular class. From these and the definitions of $m(B, Q)$ and $m(B, Q)'$, we have $m(B_\tau, Q) = m(\bar{B}_\tau, \bar{Q})$ and $m(B_\tau, Q)' = m(\bar{B}_\tau, \bar{Q})'$ for any τ. Since B is one of $B_1, B_2, \cdots, B_r$ we obtain the equalities in the lemma.

Lemma 3. *Let Z be a central p'-subgroup of G, λ be a linear character of Z and $\hat{C}_1, \hat{C}_2, \cdots, \hat{C}_s$ be the λ-special conjugacy classes of G/Z. If B is a unique p-block of G that covers the p-block $\{\lambda\}$ of Z, then for a p-subgroup Q we have*

$$m(B, Q) = \sharp\{\hat{C}_j \mid D(\hat{C}_j) =_{G/Z} QZ/Z\} \quad \text{and}$$
$$m(B, Q)' = \sharp\{\hat{C}_j \mid \hat{C}_j \text{ is } p\text{-regular with } D(\hat{C}_j) =_{G/Z} QZ/Z\}\ .$$

Proof. First we note that, by the assumption and Gallagher's theorem, the irreducible characters in B are the irreducible constituents of λ^G. In particular $s = k(B)$. We put $\hat{G} = G/Z$, $\hat{Q} = QZ/Z$ and $\hat{x} = xZ$ $(x \in G)$. Let $\{C_j \mid 1 \leq j \leq k(B)\}$ be the conjugacy classes associated with B in an association of the conjugacy classes of G with the p-blocks, and let $y_j \in C_j$. Then $\{C_j e_\lambda \mid 1 \leq j \leq k(B)\}$ is a K-basis of $Z(KG)e_\lambda$. Hence by Lemma 1, $\{\hat{y}_j \mid 1 \leq j \leq k(B)\}$ forms a set of representatives for the λ-special conjugacy classes of $\hat{G}$.

So we may assume that $\hat{\mathfrak{y}}_j$ belongs to $\hat{C}_j$ $(j=1, 2, \cdots, k(B))$ by a suitable renumbering indices. Since Z is a central p'-subgroup of G, $D(\hat{C}_j)=_{\hat{G}}\widehat{D(\bar{C}_j)}$. Therefore we obtain the equalities in the lemma by the same way as in the proof of Lemma 2.

The proof of the following proposition will be reduced to the preceding lemma.

Proposition 1. *Let H be a normal p'-subgroup of G, ζ be a G-invariant irreducible character of H and $\bar{C}_1, \bar{C}_2, \cdots, \bar{C}_s$ be the ζ-special conjugacy classes of $\bar{G}$. If B is a unique p-block of G that covers the p-block $\{\zeta\}$ of H, then for a p-subgroup Q of G we have*

$$m(B, Q) = \#\{\bar{C}_j \mid D(\bar{C}_j) =_{\bar{G}} \bar{Q}\} \quad \textit{and}$$
$$m(B, Q)' = \#\{\bar{C}_j \mid \bar{C}_j \textit{ is } p\textit{-regular with } D(\bar{C}_j) =_{\bar{G}} \bar{Q}\}\ .$$

Proof. By [4] or [3, chapter X], there exists a central extension

$$1 \to \tilde{Z} \to \tilde{G} \xrightarrow{f} G \to 1$$

of G by a p'-group $\tilde{Z}$ such that there is a normal subgroup $\tilde{H}$ of $\tilde{G}$ isomorphic to H by f and that the irreducible character $\zeta\circ f$ of $\tilde{H}$ is extended to $\tilde{G}$. Let $\tilde{\zeta}$ be an extension of $\zeta\circ f$ to $\tilde{G}$ and $\tilde{B}$ be the inflation of B, *i.e.*, $\tilde{B}$ is the p-block of $\tilde{G}$ containing $\chi\circ f$, $\chi\in B$. Since $\tilde{Z}$ is a p'-group, there exists a p-subgroup $\tilde{Q}$ of $\tilde{G}$ such that $f(\tilde{Q})=Q$. By Lemma 2, we have

$$m(B, Q) = m(\tilde{B}, \tilde{Q}) \quad \text{and} \quad m(B, Q)' = m(\tilde{B}, \tilde{Q})'\ . \tag{3}$$

We can set $\tilde{B}=\tilde{\zeta}\tilde{B}'$ with $\tilde{B}'$ a p-block of $\tilde{G}/\tilde{H}$. Then by Lemma 2 again,

$$m(\tilde{B}, \tilde{Q}) = m(\tilde{B}', \tilde{Q}\tilde{H}/\tilde{H}) \quad \text{and} \quad m(\tilde{B}, \tilde{Q})' = m(\tilde{B}', \tilde{Q}\tilde{H}/\tilde{H})'\ . \tag{4}$$

Let $\{\lambda\}$ be the p-block of $\tilde{Z}\tilde{H}/\tilde{H}$ covered by $\tilde{B}'$. Then λ^{-1} (as a character of $\tilde{Z}$) is a unique constituent of $\zeta_{\tilde{Z}}$. By the assumption, $\tilde{B}'$ is a unique p-block of $\tilde{G}/\tilde{H}$ covering $\{\lambda\}$. We put $\hat{G}=(\tilde{G}/\tilde{H})/(\tilde{Z}\tilde{H}/\tilde{H})$ and $\hat{Q}=((\tilde{Q}\tilde{H}/\tilde{H})(\tilde{Z}\tilde{H}/\tilde{H}))/(\tilde{Z}\tilde{H}/\tilde{H})$. Then f induces an isomorphism $\hat{f}$ from $\hat{G}$ to $\bar{G}$. For $x\in G$, let $\hat{x}$ be the element of $\hat{G}$ with $\hat{f}(\hat{x})=\bar{x}$. We see that $\bar{x}$ is ζ-special if and only if $\hat{x}$ is λ-special. Therefore if $\hat{C}_j=\hat{f}^{-1}(\bar{C}_j)$, then $\hat{C}_1, \hat{C}_2, \cdots, \hat{C}_s$ are the λ-special conjugacy classes of $\hat{G}$. Hence by Lemma 3 we have

$$m(\tilde{B}', \tilde{Q}\tilde{H}/\tilde{H}) = \#\{\hat{C}_j \mid D(\hat{C}_j) =_{\hat{G}} \hat{Q}\} = \#\{\bar{C}_j \mid D(\bar{C}_j) =_{\bar{G}} \bar{Q}\} \quad \text{and}$$
$$m(\tilde{B}', \tilde{Q}\tilde{H}/\tilde{H})' = \#\{\hat{C}_j \mid \hat{C}_j \ \text{ is } p\text{-regular with } D(\hat{C}_j) =_{\hat{G}} \hat{Q}\} = \#\{\bar{C}_j \mid \bar{C}_j$$
$$\text{is } p\text{-regular with } D(\bar{C}_j) =_{\bar{G}} \bar{Q}\}\ .$$

Combining these with (3) and (4), we get the proposition.

3. Proof of Theorem

We may assume that P is a normal subgroup of G. Putting $B_0=b^{N_G(D)}$, it suffices to show that

$$(5)\qquad m(B, P)=m(B_0, P) \quad\text{and}\quad m(B, P)'=m(B_0, P)'.$$

We prove (5) by induction on $|G|$. Let $H=O_{p'}(G)$ and suppose that B covers a p-block $\{\zeta\}$ of H containing an irreducible character ζ. As is well known, there exists a p-block B_1 of $T(\zeta)$ such that B_1 covers $\{\zeta\}$ and $B_1^G=B$, where $T(\zeta)$ is the inertial group of ζ in G. Replacing ζ by an G-conjugate of ζ if necessary, we may assume D is a defect group of B_1. In particular ζ is D-invariant. Let θ be the irreducible character of $C_H(D)$ which is the Glauberman correspondent of ζ. By the uniqueness of θ, $T(\theta)=T(\zeta)\cap N_G(D)$, where $T(\theta)$ is the inertial group of θ in $N_G(D)$. Let B_2 be the p-block of $T(\zeta)\cap N_G(D)$ such that $B_1=B_2^{T(\zeta)}$. Since θ is a unique irreducible character of $C_H(D)$ such that $p\nmid(\zeta_{C_H(D)}, \theta)$, it is easy to see that B_2 covers $\{\theta\}$. Hence $B_2^{N_G(D)}$ is defined and $B_0=B_2^{N_G(D)}$ by the first main theorem on p-blocks as $B=B_2^G$. By [11, Lemma 2] and by the same way as in the proof of [11, Theorem 1], we have

$$m(B, P)=m(B_1, P), \quad m(B, P)'=m(B_1, P)',$$
$$m(B_0, P)=m(B_2, P) \quad\text{and}\quad m(B_0, P)'=m(B_2, P)'.$$

So if $T(\zeta)<G$, then by induction (5) holds.

Suppose that $T(\zeta)=G$. Then by [4, §3], B is the unique p-block of G covering $\{\zeta\}$ and D is a Sylow p-subgroup of G. Since G is p-solvable and D is abelian, $G=HN_G(D)$ and $O_{p'}(N_G(D))=C_H(D)$. Hence B_0 is also the unique p-block of $N_G(D)$ covering $\{\theta\}$.

We put $\bar{G}=G/H$, $\bar{P}=PH/H$ and $\bar{x}=xH(x\in G)$. Suppose that x is an element of $N_G(D)$ such that $\bar{x}$ is ζ-special. Let $y\in N_G(D)$ with $[x, y]\in C_H(D)$ (and hence $[x, y]\in H$). Then ζ is extendible to $\langle H, x, y\rangle$. By [6, (8.15)], ζ is extendible to HD. Hence by [6, (11.31)], ζ is extendible to $\langle HD, x, y\rangle$. Therefore by [9, Theorem 2, (3)], θ is extendible to $\langle C_H(D), x, y\rangle$. So $xC_H(D)$ is θ-special. Similarly, we can show that if x is an element of $N_G(D)$ such that $xC_H(D)$ is θ-special then $\bar{x}$ is ζ-special.

Let $\bar{C}_1, \bar{C}_2, \cdots, \bar{C}_s$ be the ζ-special conjugacy classes of $\bar{G}$ and C_j' be the conjugacy class of $N_G(D)/C_H(D)$ corresponding to $\bar{C}_j$ by the natural isomorphism from $\bar{G}$ to $N_G(D)/C_H(D)$. Then $C_1', C_2', \cdots, C_s'$ are the θ-special conjugacy classes of $N_G(D)/C_H(D)$ by the above. Putting $G'=N_G(D)/C_H(D)$ and $P'=PC_H(D)/C_H(D)$, we have

$$\#\{\bar{C}_j \mid D(\bar{C}_j)=\bar{P}\}=\#\{C_j' \mid D(C_j')=P'\} \quad\text{and}$$
$$\#\{\bar{C}_j \mid \bar{C}_j \text{ is } p\text{-regular with } D(\bar{C}_j)=\bar{P}\}=\#\{C_j' \mid C_j' \text{ is } p\text{-regular with } D(C_j')=P'\}.$$

So by Proposition 1, the theorem follows.

4. Some consequences

As an immediate consequence of the theorem and [11, §§2 and 3], we get the following.

Corollary. *With the same assumption and notation as in Theorem, for a p-subgroup Q of G we have*

$$m(B, Q) = \sum_R m(b^{N_G(D)}, R) \quad \textit{and} \quad m(B, Q') = \sum_R (b^{N_G(D)}, R)' ,$$

where R ranges over a set of representatives for the $N_G(D)$-conjugacy classes of p-subgroups of $N_G(D)$ which are G-conjugate to Q.

In the same situation as in the above corollary, if R is normal in $N_G(D)$, then $m(b^{N_G(D)}, R) = m(b^{T(b)}, R)$, $m(b^{N_G(D)}, R)' = m(b^{T(b)}, R)'$ and $m(b^{T(b)}, R) = \sum_\pi m(b^{C_G(\pi) \cap T(b)}, R)'$, where $T(b)$ is the inertial group of b in $N_G(D)$ and π ranges over a set of representatives for the $T(b)$-conjugacy classes of D. For $m(b^{T(b)}, R)'$ we have the following

Proposition 2. *Let D be a normal abelian p-subgroup of G, B be a p-block of G with defect group D and b be a p-block of $C_G(D)$ covered by B. Suppose that b is G-invariant and let ζ be the irreducible character in b which is trivial on D. If $\{x_j C_G(D) \mid 1 \leq j \leq r\}$ is a set of representatives for the ζ-special conjugacy classes of $G/C_G(D)$, then for a p-subgroup Q of G we have*

$$m(B, Q)' = \#\{x_j C_G(D) \mid C_D(x_j) =_G Q\} .$$

Proof. Let $\chi_1, \chi_2, \cdots, \chi_r$ be the irreducible constituents of ζ^G. Note that χ_i is trivial on D and is of defect 0 as a character of G/D, since D is a defect group of B. In particular χ_i is irreducible as a modular character and $\chi_{1G_{p'}}$, $\chi_{2G_{p'}}, \cdots, \chi_{rG_{p'}}$ are different from each other, where $G_{p'}$ is the set of all p'-elements of G. Since $\zeta_{(C_G(D))_{p'}}$ is a unique modular irreducible character in b, every modular irreducible character $\phi \in B$ is a constituent of $(\zeta_{(C_G(D))_{p'}})^G$. Hence $\phi = \chi_{iG_{p'}}$ for some i. In particular $r = l(B)$ and the $\chi_{iG_{p'}} (1 \leq i \leq l(B))$ are the modular irreducible characters in B. Let $C_1, C_2, \cdots, C_{l(B)}$ be the p-regular classes associated with B in an association of conjugacy classes of G with the p-blocks, and let $y_j \in C_j$. Then by Osima [10, §4] or [8, chapter 5, §11], the $l(B) \times l(B)$ matrix $(\chi_i(y_j))$ $(1 \leq i, j \leq l(B))$ is non-singular. From the proof of Lemma 1, $\{y_j C_G(D) \mid 1 \leq j \leq l(B)\}$ is a set of representatives for the ζ-special conjugacy classes of $G/C_G(D)$. So we may set $y_j = c_j x_j$ $(c_j \in C_G(D),\ 1 \leq j \leq l(B))$. If Q is a Sylow p-subgroup of $C_G(y_j)$, then Q is a lower defect group of B and hence $Q \leq D$. So we have $D(C_j) =_G Q = C_D(y_j) = C_D(x_j)$. From this and the definition of $m(B, Q)'$ we get the proposition.

Acknowledgement. The author thanks Professor Yukio Tsushima for his kind advice.

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