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## A NOTE ON SOME PERIODICITY OF Ad-COHOMOLOGY GROUPS OF LENS SPACES

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### 1. Introduction

Let  $p$  be an odd prime and  $q=p^r$ . We choose a positive integer  $k$  such that the class of  $k$  in  $\mathbf{Z}/p^2$  generates the group of units  $(\mathbf{Z}/p^2)^\times$ .

Let  $K^*$  be the  $K$ -cohomology theory and  $K_{(p)}^*$  its  $p$ -localized theory. The Adams operation  $\psi^k$  on  $K$  induces a stable operation  $\psi^k$  on  $K_{(p)}^*$ . We denote by  $K_{(p)}$  the spectrum which represents the  $K_{(p)}^*$ -cohomology theory. Since stable operations induce maps of spectra, we have the following cofibration of spectra

$$K_{(p)} \xrightarrow{1-\psi^k} K_{(p)} \longrightarrow C_{1-\psi^k}.$$

We define a spectrum  $Ad$  as  $\Sigma^{-1}C_{1-\psi^k}$  and its associated cohomology theory  $Ad^*$ . When  $k$  is a prime power, the associated connective theory of  $Ad^*$  coincides with the cohomology theory defined by Seymour [9] and Quillen [8].

Let  $m$  and  $n$  be positive integers. We identify  $\mathbf{Z}/m$  with the set of  $m$ -th root of 1 in  $C$ , and  $S^{2n+1}$  with the unit sphere in  $C^{n+1}$ . The complex vector space structure on  $C^{n+1}$  induces a  $\mathbf{Z}/m$ -action on  $S^{2n+1}$  and we define the standard Lens space mod  $m$  as  $S^{2n+1}(\mathbf{Z}/m)$ . As is well known, the standard Lens space  $L^n(m)$  has a  $CW$ -complex structure

$$L^n(m) = \bigcup_{i=1}^{2n+1} e^i$$

and we denote its  $2n$ -skeleton by  $L_0^n(m)$ . Since the canonical inclusion  $C^{n+1} \subset C^{n+2}$  induces a cellular inclusion  $L^n(m) \subset L^{n+1}(m)$ , we have a  $CW$ -complex  $L^\infty(m) = \text{colim } L^n(m)$ . This space  $L^\infty(m)$  is a classifying space  $B\mathbf{Z}/m$  of  $\mathbf{Z}/m$ . We consider the case  $m=p^r$ . Main results are the following.

**Theorem 1.1.** *Let  $M(n)=r+[(n-1)/(p-1)]$ . For any integers  $i, j$  satisfying  $i-j \equiv 0 \pmod{(p-1)p^{M(n)-1}}$ , there holds the following isomorphisms.*

$$\begin{aligned} \widetilde{Ad}^{2i}(L_0^n(p^r)) &\cong \widetilde{Ad}^{2j}(L_0^n(p^r)) \\ \widetilde{Ad}^{2i+1}(L_0^n(p^r)) &\cong \widetilde{Ad}^{2j+1}(L_0^n(p^r)). \end{aligned}$$

**Theorem 1.2.** *We put  $N=r+v_p(i)$ . Let  $j$  be an integer which satisfies  $v_p(i)=v_p(j)$  and  $i-j\equiv 0 \pmod{(p-1)p^{N-1}}$ . Then, there holds the following isomorphism:*

$$\widetilde{Ad}^{2i}(L_0^n(p^r)) \cong \widetilde{Ad}^{2j}(L_0^n(p^r)).$$

We used the computer of Osaka City University computer center to calculate the samples of the group structure of  $Ad^*(L_0^n(p^r))$ .

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**2. Preliminaries**

Let  $G$  be a compact Lie group and  $R(G)$  be its complex representation ring. The augmentation ideal  $I$  is the kernel of the induced ring homomorphism  $R(G)\rightarrow R(\{1\})$ .

**Proposition 2.1** (Atiyah, Atiyah-Segal). *There holds the following natural isomorphism:*

$$\alpha: R(G)_I^\wedge \rightarrow K(BG).$$

In case  $G=S^1$ ,  $R(S^1)=\mathbf{Z}[\mathbf{H}, \mathbf{H}^{-1}]$  where  $\mathbf{H}$  is the canonical representation of  $S^1$ . Let  $x$  be  $e(\mathbf{H})=\mathbf{H}-1$ , the euler class of  $\mathbf{H}$ . Since  $I=(x)$ ,  $K(BS^1)=\mathbf{Z}[x]_x^\wedge=\mathbf{Z}[[x]]$  and  $K^1(BS^1)=0$ . We consider the case  $G=\mathbf{Z}/q$ . There is the canonical inclusion  $i: \mathbf{Z}/q\subset S^1$ , and we write  $i^*(\mathbf{H})=\mathbf{H}$  and  $i^*(x)=x$ . Then  $R(\mathbf{Z}/q)=\mathbf{Z}[\mathbf{H}]/(\mathbf{H}^q-1)$  and  $I=(x)$ . Thus  $K(B\mathbf{Z}/q)=(\mathbf{Z}[x]/((x+1)^q-1))_x^\wedge$  and  $K^1(B\mathbf{Z}/q)=0$ . We denote  $R=K_{(p)}(B\mathbf{Z}/q)=(\mathbf{Z}_{(p)}[x]/((x+1)^q-1))_x^\wedge$ . The definition of the completion induces

**Lemma 2.2.** *There exists a continuous surjection*

$$\mathbf{Z}_{(p)}[x]_x^\wedge/((x+1)^q-1) \rightarrow (\mathbf{Z}_{(p)}[x]/((x+1)^q-1))_x^\wedge.$$

The  $K$ -ring of finite Lens spaces is given by Mahammed [7]. That is

$$(2.3) \quad \begin{aligned} K(L_0^n(q)) &= \mathbf{Z}[x]/((x+1)^q-1, x^{n+1}) \\ K^1(L_0^n(q)) &= 0, \end{aligned}$$

where  $x$  is the restriction of  $x$ , the euler class of the canonical line bundle.

**Lemma 2.4.** *The ring  $K_{(p)}(L_0^n(q))$  is isomorphic to  $R/(x^{n+1})$  and the inclusion  $L_0^n(q)\subset L^\infty(q)$  induces the projection  $R\rightarrow R/(x^{n+1})$ .*

We denote  $L_0^{\infty,n}(q)=L^\infty(q)/L_0^n(q)$  and  $L_{0,0}^{m,n}(q)=L_0^m(q)/L_0^n(q)$ . Then we have the following exact sequence.

$$(2.5) \quad 0 \rightarrow \tilde{K}(L_0^{\infty,n}(q)) \rightarrow K(L^\infty(q)) \rightarrow K(L_0^n(q)) \rightarrow 0.$$

Thus we have

**Lemma 2.6.**  $\tilde{K}(L_0^{\infty,n}(q)) = x^{n+1} R$   
 $\tilde{K}^1(L_0^{\infty,n}(q)) = 0.$

It is easy to see that  $L_{0,0}^{n+1,n}(q) = S^{2n+1} \cup_q e^{2n+2}.$  Thus we have

**Lemma 2.7.**  $\tilde{K}(L_{0,0}^{n+1,n}(q)) \cong \mathbf{Z}/q$   
 $\tilde{K}^1(L_{0,0}^{n+1,n}(q)) = 0.$

On the other hand, the exact sequence

$$0 \rightarrow \tilde{K}(L_0^{\infty,n+1}(q)) \rightarrow \tilde{K}(L_0^{\infty,n}(q)) \rightarrow \tilde{K}(L_{0,0}^{n+1,n}(q)) \rightarrow 0$$

shows that  $\tilde{K}(L_{0,0}^{n+1,n}(q)) = x^n R/x^{n+1},$  which is a cyclic group generated by  $x^n.$  So we have

**Proposition 2.8.** *When  $m - n \equiv 0 \pmod{p - 1},$  we put  $t = \min(r, \nu_p(m) + 1).$  Then*

$$\tilde{Ad}^{2m}(L_{0,0}^{n+1,n}(q)) \cong \tilde{Ad}^{2m+1}(L_{0,0}^{n+1,n}(q)) \cong \begin{cases} \mathbf{Z}/p^t & \text{if } n - m \equiv 0 \pmod{p - 1} \\ 0 & \text{if } n - m \not\equiv 0 \pmod{p - 1}. \end{cases}$$

### 3. Proof of Theorems

We identify  $K_{(p)}^{2m}(X)$  with  $K_{(p)}(X)$  using the Bott periodicity. We denote the stable operation  $\psi^k$  on  $K_{(p)}^{2m}(X)$  by  $\psi^{k,m}.$  Then  $\psi^{k,m} = k^{-m} \psi^k$  under this identification.

**Lemma 3.1.** *Under the above identification, when  $m - m' \equiv 0 \pmod{p - 1},$*

$$\text{Im}((1 - \psi^{k,m}) - (1 - \psi^{k,m'})) \subset p^{1 + \nu_p(m - m')} \text{Im}(\psi^k).$$

Kobayashi, Murakami and Sugawara [6] have computed the explicit abelian group structure of  $K(L_0^n(q)).$  One corollary of their results is

**Proposition 3.2.**

$$p^{r + [(n-1)/(p-1)]} \tilde{K}_{(p)}(L_0^n(q)) = 0.$$

Let  $M(n) = r + [(n-1)/(p-1)]$  and  $M' = (p-1)p^{M(n)-1}.$  Then

**Corollary 3.3.** *Under the above identification*

$$1 - \psi^{k,m} = 1 - \psi^{k,m+M'} : K(L_0^n(q)) \rightarrow K(L_0^n(q)).$$

When  $i - j \equiv 0 \pmod{M'},$  we consider the following commutative diagram where horizontal lines are exact:

$$\begin{array}{ccccccccc}
 0 & \rightarrow & Ad^{2i}(L_0^n(q)) & \rightarrow & K^{2i}(L_0^n(q)) & \rightarrow & K^{2i}(L_0^n(q)) & \rightarrow & Ad^{2i+1}(L_0^n(q)) & \rightarrow & 0 \\
 & & & & \downarrow \cong & & \downarrow \cong & & & & \\
 0 & \rightarrow & Ad^{2j}(L_0^n(q)) & \rightarrow & K^{2j}(L_0^n(q)) & \rightarrow & K^{2j}(L_0^n(q)) & \rightarrow & Ad^{2j+1}(L_0^n(q)) & \rightarrow & 0,
 \end{array}$$

where the vertical isomorphism are induced from the Bott periodicity. This completes the proof of Theorem 1.1.

We took  $k$  as a generator of  $(\mathbf{Z}/p^2)^\times$ . It is easy to see the following lemmata.

**Lemma 3.4.** *The class of  $k$  in  $\mathbf{Z}/p^r$  is a generator of  $(\mathbf{Z}/p^r)^\times$  for every  $r$ .*

**Lemma 3.5.** *Let  $W=(p-1)p^{r-1}/2$ . Then  $k^W \equiv -1 \pmod{p^r}$ .*

Let  $U=(p-1)/2$ . Lemma 3.5 induces

**Corollary 3.6.**

$$k^{(p-1)p^{r-2}U} \times k^{(p-1)p^{r-2}/2} \equiv -1 \pmod{p^r}.$$

DEFINITION 3.7.

$$A = \prod_{i=0}^{p-1} (\mathbf{H}^{k^i(p-1)p^{r-2}} - 1) = \prod_{i=0}^{p-1} (\psi^{k^{(p-1)p^{r-2}}})^i (\mathbf{H} - 1)$$

We compute the action of the Adams operation on the element  $A$ , and using Corollary 3.6 we have

**Lemma 3.8.** *There exists a natural number  $t$  such that*

$$(\psi^k)^{(p-1)^{r-2}/2} A = -\mathbf{H}^t A.$$

The proof is only a computation. The natural number  $t = -\sum_{j=1}^{p-1} k^{j(p-1)p^{r-2}}$ , but we don't need it. Let  $Q = (p-1)p^{r-2}/2$ .

**Lemma 3.9.** *There exists a natural number  $s$  such that*

$$(\psi^k)^{(p-1)p^{r-2}/2} (\mathbf{H}^s A) = -\mathbf{H}^s A.$$

Proof. Using Lemma 3.8, we need to solve the following equation in  $\mathbf{Z}/q$ .

$$k^{(p-1)^{r-2}/2} s + t = s.$$

Since  $Q$  is not a multiple of  $p-1$ ,  $v_p(1-k^Q) = 0$ . This implies that  $1-k^Q$  is invertible in  $\mathbf{Z}/p^r$ . Thus the above equation has a solution.

DEFINITION 3.10. We put  $B = \mathbf{H}^s A$  and  $C = \sum_{j=0}^{Q-1} (\psi^k)^j B$ . This element  $C$  is a polynomial of  $\mathbf{H}$ , so we write  $C = C(\mathbf{H})$ . It is easy to see that  $\psi^k B = -B$ ,

$\nu^k C = -C$  and  $B$  and  $C$  are in the ideal  $(H-1)^p$ . So we define  $D(H) = (H-1)^{-p} \times C(H)$ .

**Lemma 3.11.** *The integer  $D(1)$  is prime to  $p$ .*

Proof. Let  $f(X) = (X^n - 1)/(X - 1)$ , then  $f(1) = n$ . Thus

$$D(1) = \sum_{j=0}^{q-1} k^j \times k^{j+2q} \times \dots \times k^{j+2q(p-1)} \\ = k^v (1 - k^q)/(1 - k^p).$$

Since  $\nu_p(1 - k^q) = 0$ , the proof is completed.

**Lemma 3.12.** *The ideal generated by  $C$  coincides with the ideal  $(H-1)^p$  in  $R$ .*

Proof. We write  $D$  as a polynomial of  $x = H - 1$ . Then  $D = D(1) + \text{higher}$ . So  $D$  is an invertible element in  $\mathbb{Z}_{(p)}[x]_x$ . By Lemma 2.2  $D$  is invertible in  $R$ . Since  $C = (H - 1)^p D$ , the Lemma is proved.

**Proposition 3.13.** *When  $n \equiv 0, -1 \pmod{p}$ , then*

$$\widetilde{Ad}^{2m+1}(L_0^{\infty, n}(q)) = \bigoplus_{j=1}^r \mathbb{Z}/p^{j+\nu_p(m)}.$$

Proof. By Lemma 2.6  $\widetilde{K}(L_0^{\infty, n}(q)) = x^{n+1}R$ . The assumption implies  $n = tp - \varepsilon$ , where  $t$  is a positive integer and  $\varepsilon$  is 0 or  $-1$ . We choose an additive basis of  $x^{n+1}R$  as  $\{g_n(i, j); 1 \leq j \leq r, 1 \leq i \leq (p-1)p^{j-1} - 1\}$  where  $g_n(i, j) = (-C)^t \times (H^{p^r - jk^i} + \varepsilon - 1)$ . As same as in computation in [4], we have the required result.

Consider the following exact sequence

$$0 \rightarrow \widetilde{Ad}^{2m}(L_{0,0}^{n, n-1}(q)) \rightarrow \widetilde{Ad}^{2m+1}(L_0^{\infty, n}(q)) \\ \rightarrow \widetilde{Ad}^{2m+1}(L_0^{\infty, n-1}(q)) \rightarrow \widetilde{Ad}^{2m+1}(L_{0,0}^{n, n-1}(q)) \rightarrow 0.$$

By Proposition 2.8, we have

**Corollary 3.14.** *If  $n - m \equiv 0 \pmod{p-1}$ , we have an isomorphism:*

$$\widetilde{Ad}^{2m+1}(L_0^{\infty, n}(q)) \cong \widetilde{Ad}^{2m+1}(L_0^{\infty, n-1}(q)).$$

Proposition 3.13 and Corollary 3.14 imply

**Lemma 3.15.** *Let  $N(m) = r + \nu_p(m)$ . Then*

$$p^{N(m)} \cdot \widetilde{Ad}^{2m+1}(L_0^{\infty, n}(q)) = 0.$$

The exact sequence

$$0 \rightarrow \tilde{A}d^{2m}(L_0^n(q)) \rightarrow \tilde{A}d^{2m+1}(L_0^{\infty,n}(q)) \rightarrow \tilde{A}d^{2m+1}(L^\infty(q))$$

induces

**Corollary 3.16.**  $p^{N(m)} \cdot \tilde{A}d^{2m+1}(L_0^n(q)) = 0$  for any integer  $n$ .

Lemma 3.1 and Corollary 3.16 imply Theorem 1.2. This completes the proof.

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