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Osaka University
We shall extend some theorems of potential theory in space to abstract Riemann surfaces. In the present article we shall be concerned with Evans-Selberg's theorem on Riemann surfaces with null-boundary.

G. C. Evans and H. Selberg\(^2\) proved the following theorem. \textit{Given a closed set }\(F\) \textit{of capacity zero in space, then there exists a positive mass distribution on }\(F\) \textit{whose potential is positively infinite at every point of }\(F\). We shall extend this theorem to abstract Riemann surfaces with null-boundary.

Let \(R^*\) be a Riemann surface with null-boundary and \(\{R_n\} (n=0, 1, 2, \ldots)\) be its exhaustion with compact relative boundaries \(\{\partial R_n\}\). Put \(R=R^*-R_0\). Let \(G_n(z, p)\) be the Green's function of \(R_n-R_0\) with pole at \(p\). Clearly, \(G_n(z, p) \uparrow G(z, p)\) as \(n \to \infty\). Since \(\frac{\partial G_n(z, p)}{\partial n} ds \leq 2\pi\) for every \(n\), \(G(z, p)\) is not constant infinity and harmonic in \(R\) except at \(p\) where \(G(z, p)\) has a logarithmic singularity.

Take \(M\) large so that the set \(V_M(p)=E[z \in R: G(z, p) \geq M]\) is compact in \(R\). Let \(\omega_n(z)\) be a harmonic function in \(R_n-R_0-V_M(p)\) such that \(\omega_n(z)=0\) on \(\partial R_0+\partial V_M(p)\) and \(\omega_n(z)=M\) on \(\partial R_n\). Then since \(R^*\) is a Riemann surface with null-boundary, \(\lim_{n \to \infty} \omega_n(z)=0\). Let \(\bar{G}_n(z, p), G_n(z, p)\) and \(G_0(z, p)\) be harmonic functions in \(R_n-R_0-V_M(p)\) such that \(\bar{G}_n(z, p)=G_n(z, p)=M\) on \(\partial V_M(p)\), \(\bar{G}_n(z, p)=G_n(z, p)=G_n(z, p)=0\) on \(\partial R_0\) and \(\bar{G}_n(z, p)=M, \frac{\partial G_n'(z, p)}{\partial n}=0\) and \(G_n(z, p)=0\) on \(\partial R_n\) respectively. Since \(0<G_n'(z, p)<M\) on \(\partial R_n\), we have by the maximum principle

\[
G_n(z, p) \leq G_n'(z, p) \leq \bar{G}_n(z, p), \quad G_n(z, p) \leq G(z, p) \leq \bar{G}_n(z, p)
\]

and

\[
0 \leq \bar{G}_n(z, p) - G_n(z, p) = M \omega_n(z).
\]

\(^1\) Resumé of this part is reported in Proc. Japan Acad. 32, 1956.

Hence
\[ \lim_{n \to \infty} G(z, p) = \lim_{n \to \infty} G_n^\prime (z, p) = \lim_{n \to \infty} G_n(z, p) = G(z, p). \]

Then by Green's formula and by the compactness of \( V_M(p) \)
\[ \int_{\partial R_0} \frac{\partial G(z, p)}{\partial n} \, ds = \int_{\partial R_0} \lim_{n \to \infty} \frac{\partial G_n'(z, p)}{\partial n} \, ds = - \int_{\partial V_M(p)} \lim_{n \to \infty} \frac{\partial G_n'(z, p)}{\partial n} \, ds = 2\pi. \]

\( G(z, p) \) is called the Green's function of \( R \) with pole at \( p \).

After R. S. Martin\(^3\) we shall define the ideal boundary points as follows: let \( G(z, p) \) be the Green's function of \( R \) with pole at \( p \). Then by definition, the flux of \( G(z, p) \) along \( \partial R_0 \) is \( 2\pi \) and \( G(z, p) \) is positive. Consider now a sequence of points \( \{ p_i \} \) of \( R \) having no point of accumulation in \( R + \partial R_0 \). In any compact part of \( R \), the corresponding functions \( G(z, p_i) \) \( (i = 1, 2, \ldots) \) form, from some \( i \) on, a bounded sequence of harmonic functions—thus a normal family. A sequence of these functions, therefore, is convergent in every compact part of \( R \) to a positive harmonic function. A sequence \( \{ p_i \} \) of \( R \) having no point of accumulation in \( R + \partial R_0 \), for which the corresponding \( G(z, p_i) \)' have the property just mentioned, that is, converges to a harmonic function—will be called fundamental. Two fundamental sequences are called equivalent if their corresponding \( G(z, p_i) \)' have the same limit. The class of all fundamental sequences equivalent to a given one determines an \textit{ideal boundary point} of \( R \). The set of all the ideal boundary points of \( R \) will be denoted by \( B \) and the set \( R + B \), by \( \bar{R} \). The domain of definition of \( G(z, p) \) may now be extended by writing \( G(z, p) = \lim_{i \to \infty} G(z, p_i) (z \in R, p \in B) \), where \( \{ p_i \} \) is any fundamental sequence determining \( p \). For \( p \) in \( B, G(z, p) \) is positive, harmonic and \( \int_{\partial R_0} \frac{\partial G(z, p)}{\partial n} \, ds = 2\pi \) and further \( G(z, p) \) is unbounded in \( R \), because if \( G(z, p) \) is bounded in \( R, G(z, p) \equiv 0 \) by the maximum principle. This contradicts \( \int_{\partial R_0} \frac{\partial G(z, p)}{\partial n} \, ds = 2\pi \). Evidently, the function \( G(z, p) \) is characteristic of the point \( p \) in the sense that the identity of two points of \( \bar{R} \) is equivalent to the equality of their corresponding \( G(z, p) \)' as a function of \( z \). The function \( \delta(p_1, p_2) \) of two points \( p_1 \) and \( p_2 \) in \( \bar{R} \) is defined by
\[
\delta(p_1, p_2) = \sup_{z \in \bar{R} - R_0} \left| \frac{G(z, p_1)}{1 + G(z, p_1)} - \frac{G(z, p_2)}{1 + G(z, p_2)} \right|.
\]

\(^3\) R. S. Martin: Minimal positive harmonic functions. Trans. Amer. Math. Soc. 39, 1941.
Evidently, \( \delta(p_1, p_2) = 0 \) is equivalent to \( G(z, p_1) = G(z, p_2) \) for all points \( z \) in \( R_1 - R_0 \). Therefore we have \( G(z, p_1) = G(z, p_2) \) for all points in \( R \), that is \( \delta(p_1, p_2) = 0 \) implies \( p_1 = p_2 \) and it is clear that \( \delta(p_1, p_2) \) satisfies the axioms of distance. Therefore \( \delta(p_1, p_2) \) can be considered as the distance between two points \( p_1 \) and \( p_2 \) of \( \overline{R} \). The topology induced by this metric is homeomorphic to the original topology when it is restricted in \( R \). Since \( G(z, p_1)(p_1 \in \overline{R}) \) is also a normal family, both \( (R - R^1) + \partial R^1 + B \) and \( B \) are closed and compact. For fixed \( z \), \( G(z, p) \) is continuous with respect to this metric (we denote shortly it by \( \delta \)-continuous) as a function of \( p \) in \( \overline{R} \) except at \( z = p \).

First we shall prove the following

**Lemma 1.** Let \( G_i \) be a compact or non-compact domain with an analytic relative boundary \( \partial G_i \) (\( i = 1, 2, \ldots, k \)). Let \( U_i(z) \) (\( i = 1, 2, \ldots, k \)) be a function which is harmonic in \( R - G_i \) and on \( \partial G_i \), such that the Dirichlet integral of \( U_i(z) \) taken over \( R - G_i \) is finite. Then there exists a sequence of compact curves \( \{\gamma_n\} \) such that \( \gamma_n \) separates \( B \) from \( dR \), \( \{\gamma_n\} \) clusters at \( B \) and that \( \int_{\gamma_n \cap G_i} \frac{\partial U_i(z)}{\partial n} \) ds tends to zero as \( n \to \infty \), for every \( i \).

**Proof.** Let \( \omega_n'(z) \) be a harmonic function in \( R_n - R_0 \) such that \( \omega_n'(z) = 1 \) on \( \partial R_n \) and \( \omega_n'(z) = 0 \) on \( \partial R_0 \). Then \( \lim_{n \to \infty} \omega_n'(z) = 0 \), since \( R^* \) is a Riemann surface with null-boundary. Hence, for any given number \( n' \) there exists a number \( n_0 \) such that \( \omega_n'(z) < \frac{1}{2} \) in \( R_{n'} - R_0 \), for any \( n \geq n_0 \). We denote by \( \omega_n(z) \) a harmonic function in \( R_n - R_0 \) which vanishes on \( \partial R_0 \) and assumes a constant value \( M_n \) on \( \partial R_n \) and whose flux along \( \partial R_0 \) is \( 2\pi \). It is evident that \( \omega_n(z) = M_n \omega_n'(z) \) and \( \lim_{n \to \infty} M_n = \infty \). Then for a number \( n' \) chosen in the manner above stated, the niveau curve with height \( \geq \frac{M_n}{2} \) is contained in \( R_{n'} - R_0 \).

Put

\[
\rho_n(z) = \rho e^{i\theta},
\]

where \( \rho_n(z) \) is the conjugate harmonic function of \( \omega_n(z) \).

Let \( U(z) \) be one of \( U_i(z) \) and put

\[
L(r) = \int_{C_r} \left| \frac{\partial U(z)}{\partial r} \right| rd\theta = \int_{C_r} \left| \frac{\partial U(z)}{\partial n} \right| ds,
\]

where \( C_r \) is the part of the niveau curve \( C_r \) of \( \omega_n(z) \) with height \( r \) contained in \( R - G_i \).

Suppose that there exist two positive constants \( \eta \) and \( \delta \) and infinitely many numbers \( n \) with the property as follows: there exists a
closed set $F_n$ in the interval $(e^{M_n}, e^{M_n/2})$ such that \( \lim_{n \to \infty} \frac{\text{mes } F_n}{(e^{M_n} - e^{M_n/2})} = \eta \) and that \( L(r) \geq \delta \) for any \( r \in F_n \). Since \( \int_{C_r} d\theta = 2\pi, \int_{C_r} d\theta \leq 2\pi \). Then by Schwarz's inequality, we have

\[
D_{R-G}(U(z)) = \int \int \left( \frac{1}{r} \left( \frac{\partial^2 U(z)}{\partial r^2} \right) + \frac{1}{r^2} \left( \frac{\partial U(z)}{\partial \theta} \right) \right) r dr d\theta \geq \frac{1}{2\pi} \int_1^{e^{M_n}} \frac{L^2(r)}{r} dr,
\]

\[
> \frac{1}{2\pi} \frac{e^{M_n}}{r} \frac{L^2(r)}{r} dr \geq \frac{1}{2\pi} \frac{e^{M_n}}{e^{M_n/2}} \frac{\delta^2}{r^2} dr = \frac{M_n}{4\pi} \eta \delta^2.
\]

Let \( n \to \infty \). Then the right hand side diverges. This contradicts the finiteness of \( D(U(z)) \). Hence there exists a sequence of exceptional sets \( \{E_n\} \) in the intervals \((e^{M_n}, e^{M_n/2})\) such that \( \lim_{n \to \infty} \frac{\text{mes } E_n}{(e^{M_n} - e^{M_n/2})} = 0 \) and that \( r \notin E_n \) implies \( L(r) < \delta_n \), where \( \lim \delta_n = 0 \).

Returning to case of \( U_i(z) \), let \( \{E_{i,n}\} \) be a sequence of exceptional sets corresponding to \( U_i(z) \) and \( \{\delta_{i,n}\} \) be the corresponding quantities of \( \{E_{i,n}\} \). Then we see that \( \sum_{i=1}^k \text{mes } E_{i,n} \) and \( \max \delta_{i,n} \) tend to zero as \( n \to \infty \). On the other hand, the niveau curves with height \( \geq \frac{M_n}{2} \) are are contained in \( R-R_\gamma \), since \( \omega_\gamma(z) < \frac{M_n}{2} \) in \( R_n-R_\gamma \). It follows that every \( C_r \) with \( r \in (e^{M_n}, e^{M_n/2}) \) clusters at \( B \) as \( n \to \infty \) and that \( \sum_{i=1}^k E_{i,n} \) clusters at \( B \) as \( n \to \infty \). Consider a niveau curve \( C_r \) above mentioned as \( \gamma_n \). Then we have the lemma.

Next, we shall consider the behaviour of \( G(z, p) \) \( \langle p \in R \rangle \).

**Lemma 2.** Put \( V_m(p) = \{z \in R : G(z, p) \geq m\} \). Then \( \int_{V_m(p)} \frac{\partial G(z, p)}{\partial n} d\sigma < 2\pi m \) and the Dirichlet integral \( D_{R-V_m(p)}(G(z, p)) \leq 2\pi m \), where \( p \in R \) and \( m \geq 0 \).

Proof. We shall prove the lemma in three cases:

Case 1. \( p \in R \) and \( V_m(p) \) is compact.

Case 2. \( p \in R \) and \( V_m(p) \) is non-compact.

Case 3. \( p \in B \).

Case 1. \( p \in R \) and \( V_m(p) \) is compact. Let \( \omega_n(z) \) be a harmonic function in \( R_n-R_\gamma-V_m(p) \) such that \( \omega_n(z) = 1 \) on \( \partial R_n \) and \( \omega_n(z) = 0 \) on

4) In the sequel, \( \frac{\partial}{\partial n} \) means derivative with respect to inner normal with the exception that \( \frac{\partial G(z, p)}{\partial n} \) on the niveau curves of \( G(z, p) \) means derivative with respect to inner or outer normal so that \( \frac{\partial G(z, p)}{\partial n} \geq 0 \).
\[ \partial R_0 + \partial V_m(p) \]. Since \( R^* \) is a Riemann surface with null-boundary, \( \lim_{n \to \infty} \omega_n(z) = 0 \). Let \( \bar{G}_n(z, p) \) and \( G_n(z, p) \) be harmonic functions in \( R_n - R_0 - V_m(p) \) such that \( \bar{G}_n(z, p) = G_n(z, p) = m \) on \( \partial V_m(p) \), \( \bar{G}(z, p) = G_n(z, p) = 0 \) on \( \partial R_0 \) and \( \bar{G}_n(z, p) = m \) on \( \partial R_n \) and \( G_n(z, p) = 0 \) on \( \partial R_n \) respectively.

Then

\[ \bar{G}_n(z, p) > G(z, p) > G_n(z, p) \] and \( 0 < \bar{G}_n(z, p) - G_n(z, p) = m \omega_n(z) \).

Hence \( \lim_{n \to \infty} \bar{G}_n(z, p) = G(z, p) = \lim_{n \to \infty} G_n(z, p) \).

The Dirichlet integral of \( G_n(z, p) \) taken over \( R_n - R_0 - V_m(p) \) is

\[ m \int_{\partial V_m(p)} \frac{\partial G_n(z, p)}{\partial n} ds \].

Therefore, we have by Fatou's lemma
\[ D_{R-V_m(p)}(G(z, p)) \leq \lim_{\nu \to \infty} D_{R_n-R_0-V'_m(p)}(G(z, p)) = \lim_{n \to \infty} m \int_{V'_m(p)} \frac{\partial G(z, p)}{\partial n} \, ds \]

\[ = m \int_{\partial V'_m(p)} \frac{\partial G(z, p)}{\partial n} \, ds = 2\pi m, \]

because \( \int_{\partial V'_m(p)} \frac{\partial G(z, p)}{\partial n} \, ds = 2\pi \) is clear by the compactness of \( V_m(p) \).

**Case 2.** \( p \in R \) and \( V_m(p) \) is non-compact. Take \( M \) large enough so that \( V_M(p) \) is compact. Then by the results of the case 1, \( 2\pi M \geq D_{R-V_M(p)}(G(z, p)) > D_{R-V_m(p)}(G(z, p)) \). Consider \( G(z, p) \) as \( U(z) \) in lemma 1. Then there exists a sequence of compact curves \( \{\gamma_n\} \) clustering at \( B \) such that \( \gamma_n \) separates \( B \) from \( \partial R_0 \) and \( \lim_{n \to \infty} \int_{\gamma_n-V'_m(p)} \frac{\partial G(z, p)}{\partial n} \, ds = 0. \)

Denote by \( R_n' \) the compact component of \( R \) bounded by \( \gamma_n \) and \( \partial R_0 \). On the other hand, it is obvious that

\[ \int_{\gamma_n-V'_m(p)} \frac{\partial G(z, p)}{\partial n} \, ds = \int_{\partial V'_m(p)} \frac{\partial G(z, p)}{\partial n} \, ds + \int_{\gamma_n-V'_m(p)} \frac{\partial G(z, p)}{\partial n} \, ds = 0. \]

Since \( \{\gamma_n\} \) clusters at \( B \) and \( \frac{\partial G(z, p)}{\partial n} \geq 0 \) on \( \partial V_m(p) \), by mentioning to the above equality, we have

\[ \int_{\partial V'_m(p)} \frac{\partial G(z, p)}{\partial n} \, ds = 2\pi. \]

The Dirichlet integral of \( G(z, p) \) is

\[ D_{R_n-V'_m(p)}(G(z, p)) = \int_{V'_m(p)} G(z, p) \frac{\partial G(z, p)}{\partial n} \, ds + \int_{\gamma_n-V'_m(p)} G(z, p) \frac{\partial G(z, p)}{\partial n} \, ds. \]

Since \( \{\gamma_n\} \) clusters at \( B \) and the second term on the right hand side tends to zero as \( n \to \infty \), we have

\[ D_{R-V_m(p)}(G(z, p)) = 2\pi m. \]

**Case 3.** \( p \in B \). Let \( \{p_i\} \) be a fundamental sequence determining \( p \). Consider the Dirichlet integral \( D_{R_n-R_0-V_m(p)}(G(z, p)) \). For any given positive number \( \varepsilon \), we can find a narrow strip \( S \) such that the interior of \( S \) contains \( \partial V_m(p) \cap (R_n-R_0) \), \( D_{R_n-R_0-S-V_m(p)}(G(z, p)) \geq D_{R_n-R_0-V'_m(p)}(G(z, p)) - \varepsilon \) and that \( R-V_m(p) \supset R_0 - R_0 - S - V_m(p) \) for any \( i \geq i_0(S, \varepsilon) \), where \( i(S, \varepsilon) \) is a suitable number depending on \( S \) and \( \varepsilon \), because \( G(z, p_i) \) converges uniformly to \( G(z, p) \) and hence the niveau curves \( \partial V_m(p) \) tend to \( \partial V_m(p) \) as \( i \to \infty \), (Fig. 1). Since the derivatives of \( G(z, p_i) \) converge uniformly to those of \( G(z, p) \) as \( i \to \infty \), we have
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\[ D_{R_n - R_0 - \mathcal{V}_m(p)}(G(z, p)) \leq \lim_{i \to \infty} D_{R - \mathcal{V}_m(p)}(G(z, p)) \leq 2\pi m. \]

By letting \( \varepsilon \to 0 \) and then \( n \to \infty \),

\[ D_{R - \mathcal{V}_m(p)}(G(z, p)) \leq 2\pi m. \]

Hence, by lemma 1, we can prove the existence of a sequence of compact curves \( \{\gamma_n\} \) such that \( \gamma_n \) separates \( B \) from \( \partial R_0 \) and \( \{\gamma_n\} \) clusters at \( B \) and that \( \lim_{n \to \infty} \int_{\gamma_n - \mathcal{V}_m(p)} \frac{\partial G(z, p)}{\partial n} \, ds = 0 \). Therefore we have

\[ \int_{\gamma_n - \mathcal{V}_m(p)} \frac{\partial G(z, p)}{\partial n} \, ds = \int_{\gamma_n - \partial R_0} \frac{\partial G(z, p)}{\partial n} \, ds = 2\pi. \]

Thus we have the lemma.

**Lemma 3.** (Extension of Green's formula). Let \( q \) be a point in \( R - \mathcal{V}_m(p) \). Then for every point \( p \in R \),

\[ \frac{1}{2\pi} \int_{\partial \mathcal{V}_m(p)} G(z, q) \frac{\partial G(z, p)}{\partial n} \, ds = G(q, p). \] (1)

Proof. Since \( q \in R \), there exists a number \( r' \) such that \( R - R_0 \ni q \), whence there exists a constant \( L \) such that \( G(z, q) \leq L \) in \( R - R_0 \). Hence by lemma 2, \( D_{R - R_0}(G(z, q)) \leq D_{R - \mathcal{V}_m(p)}(G(z, q)) \leq 2\pi L \) and \( D_{R - \mathcal{V}_m(p)}(G(z, p)) \leq 2\pi m \). Therefore by lemma 1, there exists a sequence of compact curves \( \{\gamma_n\} \) such that \( \gamma_n \) separates \( B \) from \( \partial R_0 \), \( \{\gamma_n\} \) clusters at \( B \) and that both \( \int_{\gamma_n - \mathcal{V}_m(p)} \frac{\partial G(z, q)}{\partial n} \, ds \) and \( \int_{\gamma_n - \partial R_0} \frac{\partial G(z, p)}{\partial n} \, ds \) tend to zero as \( n \to \infty \).

Denote by \( R_n \) the component bounded by \( \gamma_n \) and \( \partial R_0 \). Suppose \( R_n \subset R_0 \). Apply the Green's formula to \( G(z, p) \) and \( G(z, q) \) in \( R_n - \mathcal{V}_m(p) \). Then

\[ \int_{\partial \mathcal{V}_m(p) \cap R_n} G(z, q) \frac{\partial G(z, p)}{\partial n} \, ds = 2\pi G(q, p) + \int_{\partial \mathcal{V}_m(p) \cap R_n} G(z, p) \frac{\partial G(z, q)}{\partial n} \, ds \]

\[ + \int_{\gamma_n - \mathcal{V}_m(p)} G(z, p) \frac{\partial G(z, q)}{\partial n} \, ds - \int_{\gamma_n - \mathcal{V}_m(p)} G(z, q) \frac{\partial G(z, p)}{\partial n} \, ds. \]

We shall see that every term, except the first, on the right hand side tends to zero as \( n \to \infty \). In fact,

\[ | \int_{\partial \mathcal{V}_m(p) \cap R_n} G(z, p) \frac{\partial G(z, q)}{\partial n} \, ds | \leq | G(z, p) | \int_{\partial \mathcal{V}_m(p) \cap R_n} G(z, q) \, ds | \leq m \int_{\gamma_n - \mathcal{V}_m(p)} \frac{\partial G(z, q)}{\partial n} \, ds \leq m \int_{\gamma_n - \partial R_0} \frac{\partial G(z, q)}{\partial n} \, ds, \]

\[ | \int_{\gamma_n - \mathcal{V}_m(p)} G(z, p) \frac{\partial G(z, q)}{\partial n} \, ds | \leq m \int_{\gamma_n - \mathcal{V}_m(p)} \frac{\partial G(z, q)}{\partial n} \, ds \text{ and } | \int_{\gamma_n - \partial R_0} G(z, q) \frac{\partial G(z, p)}{\partial n} \, ds | \leq L \int_{\gamma_n - \mathcal{V}_m(p)} \frac{\partial G(z, p)}{\partial n} \, ds. \]

On the other hand, \( G(z, q) \frac{\partial G(z, p)}{\partial n} \geq 0 \) on
\[ \partial V_m(p). \] Therefore we have the lemma.

We shall consider the behaviour of the topology induced by \( \delta \)-metric.

**Corollary.** Let \( v_n(p) \) be a \( \delta \)-neighbourhood of \( p \in \bar{R} \), that is \( v_n(p) = \{ z \in R : \delta(z, p) < \frac{1}{n} \} \). Then for any given \( V_m(p) \), there exists a neighbourhood \( v_n(p) \) such that

\[ V_m(p) \supset (v_n(p) \cap R). \]

**Proof.** The assertion is evident for \( p \in R \), because our topology is homeomorphic to the original one in \( R \). Hence it is sufficient to prove the corollary for \( p \in B \). Suppose that the assertion is false. Then there exists a number \( m_0 \) such that \( V_m(p) \supset (v_n(p) \cap R) \) for infinitely many numbers \( n \). Hence we can find a sequence of points \( \{ q_i \} \) in \( R - V_m(p) \), tending to \( p \) with respect to \( \delta \)-metric. Let \( m \geq 3m_0 \). Then we can find a number \( n_0 \) by lemma 2, such that

\[ \int \limits_{\partial V_m(p) \cap (R_n - R_0)} \frac{\partial G(z, p)}{\partial n} ds \geq \pi. \]

Since \( q_i \in R - V_m(p) \), we have by (1),

\[ \int \limits_{\partial V_m(p) \cap (R_n - R_0)} G(z, q_i) \frac{\partial G(z, p)}{\partial n} ds < \int \limits_{\partial V_m(p)} G(z, q_i) \frac{\partial G(z, p)}{\partial n} ds = 2\pi G(q_i, p) \leq 2\pi m_0. \]

Since \( \frac{\partial G(z, p)}{\partial n} \geq 0 \) on \( \partial V_m(p) \), there exists one point \( z_i \) on \( \partial V_m(p) \cap (R_n - R_0) \) such that \( G(z_i, q_i) \leq 2m_0 \). Let \( i \) tend to \( \infty \). They by the compactness of \( \partial V_m(p) \cap (R_n - R_0) \), we have \( G(z_0, p) \leq 2m_0 \), where \( z_0 \) is one of limiting points of \( \{ z_i \} \). This contradicts \( G(z_0, p) = m \geq 3m_0 \). Therefore we have the corollary.

If two points \( p \) and \( q \) are contained in \( R \), we have, by definition \( G_n(p, q) = G_n(q, p) \), where \( G_n(z, p) \) and \( G_n(z, q) \) are Green's functions of \( R_n - R_0 \) with pole \( p \) and \( q \) respectively. Hence, by letting \( n \to \infty \), we have \( G(p, q) = G(q, p) \). Next, suppose \( p \in B \) and \( q \in R \). Let \( \{ p_i \} \) be one of fundamental sequences determining \( p \). Then, since \( G(p_i, q) = G(q, p_i) \) and since \( G(z, p_i) \) converges to \( G(z, p) \) uniformly in every compact set of \( R \), \( G(p_i, q) \) has a limit denoted by \( G(p, q) \) as \( p_i \to p \). More generally, suppose that a sequence \( \{ p_i \} \) of \( \bar{R} \) tends to \( p \) with respect to \( \delta \)-metric and that \( q \) belongs to \( R \). Then we have

\[ G(q, p) = \lim_{i \to \infty} G(q, p_i) = \lim_{i \to \infty} G(p_i, q). \]

Hence \( G(z, q)(q \in R) \) has a limit when \( z \) tends to \( p \in \bar{R} \) with respect to
δ-metric. In this case we define the value of \( G(z, q) \) at \( p \) as this limit denoted by \( G(p, q) \). Thus we have the following

**Lemma 4.** If at least one of two points \( p \) and \( q \) is contained in \( R \), then
\[
G(p, q) = G(q, p). \tag{2}
\]

\( G(z, q) \) is defined in \( \overline{R} \) for \( q \in R \) but \( G(z, q) \) has been defined only in \( R \) for \( q \in B \). In what follows, we shall define \( G(z, q) \) in \( \overline{R} \), even in case \( q \in B \). For this purpose, we shall prove the following

**Lemma 5.** Suppose that \( p \) and \( q \) are contained in \( \overline{R} \). Let \( V_m(p) = E[z \in R : G(z, p) \geq m] \) and \( V'_m(p) = E[z \in R : G(z, p) \geq m'] \), where \( m < m' \), i.e. \( V_m(p) \supset V'_m(p) \). Then
\[
2\pi G_{V'_m(p)}(p, q) = \int_{\partial V'_m(p)} G(z, q) \frac{\partial G(z, p)}{\partial n} ds \geq \int_{\partial V_m(p)} G(z, q) \frac{\partial G(z, p)}{\partial n} ds = 2\pi G_{V_m(p)}(p, q).
\]

Proof. At first, if \( p \in R \), since \( G(z, q) \) is harmonic in \( \overline{R} \) for \( q \in R \), \( 2\pi G(p, q) = \int_{\partial V_m(p)} G(z, q) \frac{\partial G(z, p)}{\partial n} ds \) for every \( V_m(p) \) such that \( V_m(p) \nsubseteq q \). Next, if \( p \in B \) and \( q \in R \), we have also by (1), \( 2\pi G(p, q) = \int_{\partial V_m(p)} G(z, q) \frac{\partial G(z, p)}{\partial n} ds \) for \( V_m(p) \nsubseteq q \). Hence our assertion

![Fig. 2.](image-url)
is clear if either $p$ or $q$, at least belongs to $R$. Therefore it is sufficient to prove the lemma when both $p$ and $q$ belong to $B$. Let $\{q_i\}$ be a fundamental sequence determining $q$. $V_m(p)$ may consist of at most a enumerably infinite number of domains $D_l$ $(l = 1, 2, \cdots)$, (Fig. 2).

Let $D$ be one of them. Let $G_{D,n}(z, q_j)$ be a harmonic function in $D \cap (R_n-R_0)$ such that $G_{D,n}(z, q_j)=G(z, q_j)$ on $\partial D \cap (R_n-R_0)$ and $G_{D,n}(z, q_j) = 0$ on $\partial R_n \cap D$. Then we have by Green’s formula

$$G(z, q_j) > G_{D,n}(z, q_j) = \frac{1}{2\pi} \int_{\partial D \cap (R_n-R_0)} G(\xi, q_j) \frac{\partial G^D(\xi, z)}{\partial n} \, ds,$$

where $G^D(\xi, z)$ is the Green’s function of $D \cap (R_n-R_0)$ with pole at $z$.

Since $G^D(\xi, z)$ is increasing with respect to $n$, $\frac{\partial G^D(\xi, z)}{\partial n} > \frac{\partial G^D(\xi, z)}{\partial n}$ at every point $\xi$ on $\partial D$, where $G^D(\xi, z)$ is the Green’s function of $D$. Hence

$$G(z, q_j) \geq G_D(z, q_j) = \lim_{n \to \infty} G_{D,n}(z, q_j) = \frac{1}{2\pi} \int_{\partial D} G(\xi, q_j) \frac{\partial G^D(\xi, z)}{\partial n} \, ds.$$

We call $G_D(z, q_j)$ the solution of Dirichlet problem in $D$ with boundary value $G(z, q_j)$ on $\partial D$. Let $q_j$ tend to $q$. Then, since $G(\xi, q_j)$ tends to $G(\xi, q)$ at every point $\xi$ on $\partial D$, we have by Fatou’s lemma

$$G(z, q) = \lim_{j \to \infty} G(z, q_j) \geq \lim_{j \to \infty} G_D(z, q_j) \geq \frac{1}{2\pi} \int_{\partial D} \lim_{j \to \infty} G(\xi, q_j) \frac{\partial G^D(\xi, z)}{\partial n} \, ds = G_D(z, q), \quad (3)$$

where $G_D(z, q)$ is the solution of Dirichlet problem in $D$ with the boundary value $G(z, q)$.

Put $G^M(z, q) = \min \{M, G(z, q)\}$. Then $G^M(z, q)$ is superharmonic in $R$. Let $\bar{G}^M_n(z, q)$, $G^M_n(z, q)$ and $\bar{G}^M(z, q)$ be harmonic functions in $D \cap (R_n-R_0)$ such that $\bar{G}^M_n(z, q) = G^M_n(z, q) = G^M(z, q) = G^M(z, q)$ on $\partial D \cap (R_n-R_0)$ and $\bar{G}^M_n(z, q) = M$, $G^M_n(z, q) = G^M(z, q)$ and $G^M(z, q) = 0$ on $\partial R_n \cap D$ respectively. Then $\bar{G}^M_n(z, q) > G^M_n(z, q) > G^M_n(z, q)$ and $G^M(z, q) \leq M \omega_n(z)$, where $\omega_n(z)$ is a harmonic function in $R_n-R_0$ such that $\omega_n(z) = 0$ on $\partial R_0$ and $\omega_n(z) = 1$ on $\partial R_n$, whence

$$G^M_D(z, q) = \lim_{n \to \infty} \bar{G}^M_n(z, q) = \lim_{n \to \infty} G^M_n(z, q) = \lim_{n \to \infty} G^M(z, q).$$

Evidently, $G^M_D(z, q)$ is the solution of Dirichlet problem in $D$ with the boundary value $G^M(z, q)$ on $\partial D$ and $G^M_D(z, q) = \frac{1}{2\pi} \int_{\partial D} G^M(\xi, q_j) \frac{\partial G^M(\xi, z)}{\partial n} \, ds$. 

Evidently, $G^M_D(z, q)$ is the solution of Dirichlet problem in $D$ with the boundary value $G^M(z, q)$ on $\partial D$ and $G^M_D(z, q) = \frac{1}{2\pi} \int_{\partial D} G^M(\xi, q_j) \frac{\partial G^M(\xi, z)}{\partial n} \, ds$. 

Evidently, $G^M_D(z, q)$ is the solution of Dirichlet problem in $D$ with the boundary value $G^M(z, q)$ on $\partial D$ and $G^M_D(z, q) = \frac{1}{2\pi} \int_{\partial D} G^M(\xi, q_j) \frac{\partial G^M(\xi, z)}{\partial n} \, ds$. 

Evidently, $G^M_D(z, q)$ is the solution of Dirichlet problem in $D$ with the boundary value $G^M(z, q)$ on $\partial D$ and $G^M_D(z, q) = \frac{1}{2\pi} \int_{\partial D} G^M(\xi, q_j) \frac{\partial G^M(\xi, z)}{\partial n} \, ds$. 

Evidently, $G^M_D(z, q)$ is the solution of Dirichlet problem in $D$ with the boundary value $G^M(z, q)$ on $\partial D$ and $G^M_D(z, q) = \frac{1}{2\pi} \int_{\partial D} G^M(\xi, q_j) \frac{\partial G^M(\xi, z)}{\partial n} \, ds$.
Therefore
\[
\lim_{k \to \infty} G_D^M(z, q) = G_D(z, q).
\]

In the sequel, we denote briefly by \( G_{V_{m'} \cdot p}(z, q) \) the function which is equal to \( G_{D_{l}}(z, q) \) which is the solution of Dirichlet problem in \( D_l \) with boundary value \( G(z, q) \), in every domain \( D_l \) \((l=1, 2, \ldots)\).

Consider the Dirichlet integral of \( G_{V_{m'} \cdot p}^M(z, q) \) which is equal to the solution of Dirichlet problem \( G_{D_l}^M(z, q) \) with the boundary value \( G^M(z, q) \), in every domain \( D_l \). Then by Dirichlet principle
\[
\sum_l D_{D_l \cap \left(R_n - R_0\right)}(G_{n}^M(z, q)) \leq \sum_l D_{D_l \cap R}(G_{n}^M(z, q)) = D_{V_{m'} \cdot q}(G_{m'}^M(z, q)) \leq 2\pi M,
\]
because the Dirichlet integral of \( G^M(z, q) \) over \( R \) equals \( D_{R - V_{m} \cdot q}(G(z, q)) \leq 2\pi M \). Let \( n \to \infty \). Then
\[
D_{V_{m'} \cdot p}(G_{V_{m'} \cdot p}^M(z, q)) \leq \lim_{n \to \infty} \sum_l D_{D_l}(G_{D_l, n}^M(z, q)) \leq 2\pi M.
\]
Since \( D_{V_{m'} \cdot p}(G_{V_{m'} \cdot p}(z, q)) \) and \( D_{R - V_{m'} \cdot p}(G(z, p)) (\leq 2\pi M') \) are bounded, there exists, by lemma 1, a sequence of compact curves \( \{\gamma_n\} \) separating \( B \) from \( 0 \) such that \( \{\gamma_n\} \) clusters at \( B \) and that both \( L_1(\gamma_n) = \int_{\gamma_n \cap V_{m'}(p)} \frac{\partial G(z, p)}{\partial n} \ ds \)
and \( L_2(\gamma_n) = \int_{\gamma_n \cap V_{m'}(p)} \frac{\partial G_M(z, p)}{\partial n} \ ds \) tend to zero as \( n \to \infty \). Denoting by \( R_n \) the compact component of \( R \) bounded by \( \gamma_n \) and \( \partial R_0 \), apply the Green's formula to \( G_{V_{m'} \cdot p}^M(z, q) \) and \( G(z, p) \) in \( (V_{m}(p) - V_{m'}(p)) \cap R_n \). Then
\[
\int_{\gamma_n \cap V_{m'}(p)} G_{M}^M(z, q) \frac{\partial G(z, p)}{\partial n} \ ds - \int_{\gamma_n \cap V_{m'}(p)} G_{m'}^M(z, q) \frac{\partial G(z, p)}{\partial n} \ ds
\]
\[
= \int_{\gamma_n \cap V_{m'}(p)} G(z, p) \frac{\partial G_{m'}^M(z, q)}{\partial n} \ ds + \int_{\gamma_n \cap V_{m'}(p)} G(z, p) \frac{\partial G_{m'}^M(z, q)}{\partial n} \ ds.
\]
It can be proved, as in lemma 3, that every term on the right hand side tends to zero as \( n \to \infty \), by the fact that \( L_i(\gamma_n) \) \((i=1, 2)\) tends to zero. Now \( G_{V_{m'} \cdot p}(z, q) \) \( \frac{\partial G(z, p)}{\partial n} \geq 0 \) on \( \partial V_{m}(p) + \partial V_{m'}(p) \). Hence
\[
\int_{\gamma_n \cap V_{m'}(p)} G_{V_{m'} \cdot p}(z, q) \frac{\partial G(z, p)}{\partial n} \ ds = \int_{\gamma_n \cap V_{m'}(p)} G_{V_{m'} \cdot p}(z, q) \frac{\partial G(z, p)}{\partial n} \ ds.
\]
By letting \( M \to \infty \) and by (3)
Thus we have the lemma.

**Definition of \( G(z, q) \) for \( z \) and \( q \) belonging to \( \bar{R} \).**

Since \( G_{v_{m,p}}(p, q) = \frac{1}{2\pi} \int_{\partial V_{m,p}(q)} G(z, q) \frac{\partial G(z, p)}{\partial n} \, ds \) is increasing with respect to \( m \), \( G_{v_{m,p}}(p, q) \) has a limit as \( m \to \infty \) which we denote by \( G(p, q) \). We define the value of \( G(z, q)(q \in R) \) at \( p \in \bar{R} \) by this limit. It is easily seen that this definition of \( G(p, q) \) coincides with what was given previously in case either \( p \) or \( q \) is contained in \( R \). In fact, it is evident that \( G_{v_{m,p}}(p, q) = \frac{1}{2\pi} \int_{\partial V_{m,p}(q)} G(z, q) \frac{\partial G(z, p)}{\partial n} \, ds = G(p, q) \) for \( p \in R \) and \( V_{m}(p) \notin q \) and that, by (1) \( G_{v_{m,p}}(p, q) = \frac{1}{2\pi} \int_{\partial V_{m,p}(q)} G(z, q) \frac{\partial G(z, p)}{\partial n} \, ds = G(q, p) = \lim_{m \to \infty} G(q, p) = \lim_{m \to \infty} G(p, q) = G(p, q) \) for \( p \in B \) and \( q \in R \), where \( \{p_i\} \) is a fundamental sequence determining \( p \).

**Definition of Superharmonicity at a point \( p \in \bar{R} \).**

Suppose a function \( U(z) \) in \( \bar{R} \). If \( U(p) > \frac{1}{2\pi} \int_{\partial V_{m,p}(q)} U(z) \frac{\partial G(z, p)}{\partial n} \, ds \) holds for the niveau curves of \( G(z, p) \), we say that \( U(z) \) is superharmonic in the weak sense at a point \( p \).

In what follows, we shall show that \( G(z, q) \) (\( z \) and \( q \in \bar{R} \)) defined as above, has the essential properties of the logarithmic potential in the plane. Now we have the following

**Theorem 1.** The Green's function in \( \bar{R} \) has the following properties:

1) \( G(p, p) = \infty \).
2) \( G(z, q) \) is lower semicontinuous in \( \bar{R} \) with respect to \( \delta \)-metric.
3) \( G(z, q) \) is superharmonic in the weak sense at every point of \( \bar{R} \).
4) \( G(p, q) = G(q, p) \).

Proof. 1) and 3) are clear by the definition of \( G(z, q) \).

Proof of 2). Suppose that \( \{p_i\} \) tends to \( p \) with respect to \( \delta \)-metric. Since \( G_{v_{m,p}}(p, q) = \frac{1}{2\pi} \int_{\partial V_{m,p}(q)} G(z, q) \frac{\partial G(z, p)}{\partial n} \, ds \), there exists a number \( n_0 \) for any given positive number \( \varepsilon \) such that

\[
G_{v_{m,p}}(p, q) \leq \frac{1}{2\pi} \int_{\partial V_{m,p}(q) \cap (R_N - R_0)} G(z, q) \frac{\partial G(z, p)}{\partial n} \, ds + \varepsilon, \text{ for } n \geq n_0.
\]

Here \( (R_{n_0} - R_0) \cap \partial V_{m}(p) \) is composed of at most a finite number of
analytic curves. We make a narrow strip $S$ in $R_{n_0+1} - R_0$ such that the interior of $S$ contains $\partial V_m(p) \cap (R_{n_0} - R_0)$ and $\partial S$ cuts $\partial V_m(p)$ orthogonally at the end points of $\partial V_m(p) \cap (R_{n_0} - R_0)$. We divide $S$ into a finite number of narrow strips $S_l$ ($l = 1, 2, \cdots, k$) so that $\partial S_l$ intersects $\partial V_m(p)$ with angles being not equal to $0$ or $\pi$ and map $S_l$ onto a rectangle: $0 \leq \text{Im} \xi \leq \delta$ ($\delta$ is sufficiently small), $-1 \leq \text{Re} \xi \leq 1$, on the $\xi$-plane so that every vertical line: $\text{Re} \xi = s$ ($-1 \leq s \leq 1$) intersects only once $\partial V_m(p_l)$ for $j = J_0$, where $J_0$ is a suitable number. This is possible, because $G(z, p_l)$ tend to $G(z, p)$ that is, $\partial V_m(p_l)$ tends to $\partial V_m(p)$ and the derivatives of $G(z, p_l)$ tend to those of $G(z, p)$ on $R_{n_0} - R_0$. We make a point $\alpha_j$ of $\partial V_m(p_l)$ correspond to a point $\alpha$ of $\partial V_m(p)$ so that $\text{Re} \alpha_j = \text{Re} \alpha$. Then we have

$$\lim_{j \to \infty} \left( \int_{\partial V_m(p)} G(z, q) \frac{\partial G(z, p)}{\partial n} \right) \to \partial G(\alpha, p) \frac{\partial n}{\partial n}$$

because $\frac{\partial G(\alpha_j, p_l)}{\partial n} ds \geq 0$ and uniformly bounded in $S$, $\frac{\partial G(\alpha_j, p_l)}{\partial n} ds$ tends to $\partial G(\alpha, p) \frac{\partial n}{\partial n} ds$ and $G(\alpha_j, p_l) \to G(\alpha, p)$. Hence

$$\lim_{j \to \infty} 2\pi G_{V_m(p_l)}(p_l, q) = \lim_{j \to \infty} \left( \int_{\partial V_m(p_l)} G(z, q) \frac{\partial G(z, p)}{\partial n} \right) \frac{\partial n}{\partial n}$$

$$\geq \lim_{j \to \infty} \left( \int_{\partial V_m(p_l)} G(z, q) \frac{\partial G(z, p)}{\partial n} \right) \frac{\partial n}{\partial n} \geq \int_{\partial V_m(p)} G(z, q) \frac{\partial G(z, p)}{\partial n} \frac{\partial n}{\partial n}$$

$$-\varepsilon = 2\pi G_{V_m(p)}(p, q) - \varepsilon,$$
whence by letting $\varepsilon \to 0$,
\[
\lim_{\varepsilon \to 0} G_{V_m(p)}(p, q, \varepsilon) \geq G_{V_m(p)}(p, q).
\]
Hence $G_{V_m(p)}(p, q)$ is lower semicontinuous at $p$ for fixed $m$. Since $G_{V_m(p)}(p, q) \uparrow G(p, q)$, $G(p, q)$ is also lower semicontinuous at $p$. Therefore $G(z, q)$ is lower semicontinuous in $\bar{R}$.

Proof of 4). If $p$ or $q$ belongs to $R$, 4) is clear by (2). We suppose that both $p$ and $q$ belong to $B$. Let $\xi$ and $\eta$ be points in $R$. Then by (1) and (2) we have the following

\[
G(p, \eta) = G(\eta, p) = \frac{1}{2\pi} \int_{\partial V_m(p)} G(z, \eta) \frac{\partial G(z, \eta)}{\partial n} ds \quad \text{for} \quad \eta \notin V_m(p), \quad (4)
\]

\[
G(p, \eta) = G(\eta, p) \geq \frac{1}{2\pi} \int_{\partial V_m(p)} G(z, \eta) \frac{\partial G(z, \eta)}{\partial n} ds \quad \text{for} \quad \eta \in V_m(p). \quad (5)
\]

Since $G_{V_m(p)}(p, q) = \frac{1}{2\pi} \int_{\partial V_m(p)} G(\xi, q) \frac{\partial G(\xi, q)}{\partial n} ds$ and since $\{V_m(q)\}$ clusters at $B$ as $n \to \infty$, there exists a number $n$ for any given positive number $\varepsilon$, such that

\[
G_{V_m(p)}(p, q) - \varepsilon \leq \frac{1}{2\pi} \int_{\partial V_m(p)} G(\xi, q) \frac{\partial G(\xi, q)}{\partial n} ds,
\]

where $\partial V_m(p)$ is the part of $\partial V_m(p)$ outside of $V_n(q)$.

Suppose that $\xi$ is on $\partial V_m(p)$, then $\xi \notin V_n(q)$, whence

\[
G(\xi, q) = G(q, \xi) = \frac{1}{2\pi} \int_{\partial V_m(q)} G(\eta, \xi) \frac{\partial G(\eta, q)}{\partial n} ds.
\]

Accordingly we have

\[
G_{V_m(p)}(p, q) - \varepsilon \leq \frac{1}{4\pi^2} \int_{\partial V_m(p)} \int_{\partial V_m(q)} G(\eta, \xi) \frac{\partial G(\eta, q)}{\partial n} ds \frac{\partial G(\xi, p)}{\partial n} ds \leq \frac{1}{4\pi^2} \int_{\partial V_m(p)} \int_{\partial V_m(q)} G(\eta, \xi) \frac{\partial G(\xi, q)}{\partial n} ds \frac{\partial G(\eta, p)}{\partial n} ds.
\]

Now by (4) and (5)

\[
\frac{1}{2\pi} \int_{\partial V_m(p)} G(\xi, \eta) \frac{\partial G(\xi, p)}{\partial n} ds \leq \frac{1}{2\pi} \int_{\partial V_m(p)} G(\xi, \eta) \frac{\partial G(\xi, p)}{\partial n} ds = G(\eta, p) = G(p, \eta) \quad \text{for} \quad \eta \notin V_m(p).
\]

\[
\frac{1}{2\pi} \int_{\partial V_m(p)} G(\xi, \eta) \frac{\partial G(\xi, p)}{\partial n} ds \leq \frac{1}{2\pi} \int_{\partial V_m(p)} G(\xi, \eta) \frac{\partial G(\xi, p)}{\partial n} ds \leq G(\eta, p) = G(p, \eta) \quad \text{for} \quad \eta \in V_m(p).
\]
On the other hand,
\[ G_{V_n}(q, p) = \frac{1}{2\pi} \int_{\partial V_n(q)} G(p, \eta) \frac{\partial G(\eta, q)}{\partial n} \, ds. \]

Hence
\[ G_{V_m}(p, q) \leq \frac{1}{4\pi^2} \int_{\partial V_m(p)} \left( \int_{\partial V_m(q)} G(\xi, \eta) \frac{\partial G(\eta, p)}{\partial n} \, ds \right) \frac{\partial G(\eta, q)}{\partial n} \, ds \leq \frac{1}{2\pi} \int_{\partial V_m(p)} G(\eta, p) \frac{\partial G(\eta, q)}{\partial n} \, ds = G_{V_n}(q, p). \]

Thus by letting \( \varepsilon \to 0 \),
\[ G_{V_m}(p, q) \leq G_{V_n}(q, p). \]

Since the inverse inequality holds for the other pair of \( V_m(p) \) and \( V_n(q) \) and since \( G_{V_m}(p, q) \uparrow G(p, q) \) and \( G_{V_n}(q, p) \uparrow G(q, p) \), we have 4).

**Transfinite Diameter.** Let \( A \) be a \( \delta \)-closed subset of \( B \) (closed with respect to \( \delta \)-metric). We define the transfinite diameter of \( A \) of order \( n \) as follows:
\[ 1/\alpha D_n = \frac{1}{2\pi n C_2} \left( \inf_{p_2, p_1 \in A} \sum_{s \leq t, s = 1}^{n \times n} G(p_s, p_t) \right). \]

Then we have the following:

a) *From the definition, it is clear that \( A_i \supseteq A_j \) implies \( A_i D_n \supseteq A_j D_n. \)

b) *Put \( \bar{\Omega}_m = \bar{R} - R_m + \partial \Omega_m \) and let \( 1/\alpha D_n = \frac{1}{2\pi n C_2} \left( \inf_{p_2} \sum_{p_1 \in \bar{\Omega}_m} G(p_s, p_t) \right). \)

Then every \( p_t \) is situated on \( \partial \Omega_m. \)

In fact,
\[ \sum_{s \leq t} G(p_s, p_t) = \sum_{s \leq t} G(p_s, p_t) + \sum_{t \leq s} G(p_s, p_t). \]

The sum of the first term does not depend on \( p_s \) and by 2) of Theorem 1, \( \sum_{t \leq s} G(p_s, p_t) = U(p_s) \) is superharmonic at every point \( p_s \) of \( \bar{R} \) for fixed \( p_t. \) We make \( V_M(p_i) \) correspond to every point \( p_i \) such that \( U(p_i) \geq M \) in \( V_M(p_i) \), where \( M \geq \min U(p_s) + 1. \) Since \( U(p_i) \) is \( \delta \)-lower semicontinuous, \( U(p_i) \) attains its the minimum \( m_0 \) at \( z_0 \) on a \( \delta \)-closed set \( \bar{\Omega}_m \) (\( \bar{\Omega}_m \) is the closure of \( \Omega_m \)). We show that \( z_0 \in \partial \Omega_m. \) \( U(p_i) \) does not attain its minimum in \( \{ \bar{\Omega}_m - \sum_{t \leq s} V_M(p_t) \} \cap R \) by the minimum principle, because \( U(p_i) \) is harmonic and bounded in \( \{ \bar{\Omega}_m - \sum_{t \leq s} V_M(p_t) \} \cap R \) and \( R^* \) is a Riemann surface with null-boundary. Next, suppose, \( U(z_0) \leq m_0 = \min_{p_s \in \partial \Omega_m} U(p_s) \) \( (z_0 \in B). \) Then by 3) of Theorem 1, \( U(z_0) \geq \frac{1}{2\pi} \int_{\partial V_M(z_0)} U(z) \)
where $M'$ is large so that $V_M'(z) = E[z \in R : G(z, z) \geq M']$ is contained in $\Omega_m$, whence there exists at least one point $z'$ in $\Omega_m \cap R$ such that $U(z') \leq m$. This contradicts the minimum principle. Hence $U(p)$ attains its minimum on $\partial \Omega_m$. Therefore every $p_i$ is on $\partial \Omega_m$.

We can discuss mass distributions on $R$ by $G(z, p)$, that is, the potential of an unit mass at $p$ is given by $G(z, p)$ and we can define also the energy integral of mass distributions as in space. In our case, since $\partial \Omega_m$ is compact, it is easily proved that there exists the unique unit mass distribution $\mu$ on $\Omega_m$ called the equilibrium distribution, whose energy $I(\mu)$ is minimal and that whose potential $U(z) = \int G(z, p) \, d\mu(p)$ is a constant on $\partial \Omega_m$, that is, $U(z) = \omega_m(z)$, where $\omega_m(z)$ is a harmonic function in $R_m - R_0$ such that $\omega_m(z) = 0$ on $\partial R_m$, $\omega_m(z) = M_m$ on $\partial R_m$ and $\int_{\partial R_m} \omega_m(z) \, ds = 2\pi$. Moreover, it is easily proved by (b) as in space that the transfinite diameter $\delta_m = \lim_{m \to \infty} \delta_m D_n$ is equal to $1/I(\mu) = 1/2\pi M_m$.

Given a system of $n$ points $p_1, p_2, \ldots, p_n$ on $A$, we can choose an $(n+1)_s$ point $p (p = p(p_1, p_2, \ldots, p_n))$ on $A$ such that

$$V(p) = (\sum_{i=1}^n G(p, p_i))/2\pi n$$

is minimal, because the above function is $\delta$–lower semicontinuous on $A$. Let $V_n$ be the least upper bound of the minimum above defined as $p_1, p_2, \ldots, p_n$ vary on $A$. Then there exists a system $(p_1^*, p_2^*, \ldots, p_n^*)$ such that

$$V(p, p_1^*, p_2^*, \ldots, p_n^*) \geq V_n - \frac{1}{2\pi n}$$

for $p$ on $A$.

Denote by $V(z)$ the potential

$$V(z) = \frac{1}{2\pi n} \left( \sum_{i=1}^n G(z, p_i^*) \right).$$

This is the potential of a certain distribution of equal point mass on $A$ of total mass unity and it is clear that $V(z) \geq V_n - \frac{1}{2\pi n}$ for all points of $A$ admitting $\infty$ as a possible value of either member. Furthermore, since $V(z)$ is $\delta$–lower semicontinuous, $\lim_{z_j \to q \in A} V(z_j) \geq A V_n - \frac{1}{2\pi n}$ for every sequence $\{z_j\}$ tending to $A$ with respect to $\delta$–metric.

Now, since $G(p_i, p_j) = G(p_j, p_i)$,

$$\left( \frac{n+1}{2} \right) D_{n+1} \leq \frac{1}{2\pi} \min_{p_{i\neq j}} \left( \sum_{i=1}^{n+1} G(p_i, p_j) \right) \leq 2 \cdot \frac{1}{2\pi} \sum_{k=1}^{n+1} \left( \sum_{i=1}^{n+1} G(p_i, p_k) \right).$$
Hence \( A V_n \geq 1/A D_{n+1} \), whence
\[
V(z) \geq 1/A D_{n+1} - \frac{1}{2\pi n}
\]
on \( A \).

Since \( A \subset \Omega_m \) for every \( m \) and \( \lim_{m \to \infty} M_m = \infty \).
\[
\infty = 1/A D = \lim_{m \to \infty} 1/A D_n = \lim_{m \to \infty} \left( \sum_{i=1}^{n} G(p_i) / n G_e \right).
\]

Therefore, for any given large number \( M \), we can find a system of \( n(M) \) points \( p_1, p_2, \ldots, p_n \) such that the function
\[
V(z) = \frac{1}{2\pi n} \left( \sum_{i=1}^{n} G(z, p_i) \right) \geq M
\]
on \( A \).

**Theorem 2.** Let \( A \) be a \( \delta \)-closed subset of \( B \). Then there exists a potential \( U(z) \) such that
1. \( U(z) \) is harmonic in \( R \).
2. \( U(z) = 0 \) on \( \partial R_0 \).
3. The flux of \( U(z) \) along \( \partial R_0 \) is \( 2\pi \).
4. \( \lim_{z \to A} U(z) = \infty \).

**Proof.** Let \( N \) be an integer larger than 3. Then since \( \lim_{n \to \infty} A D_n = 0 \), there exists, for any positive integer \( m, n(N, m) \) number of points \( p_1, p_2, \ldots, p_n \) such that
\[
V^m(z) = \frac{1}{2\pi n} \left( \sum_{i=1}^{n} G(z, p_i) \right) \geq M
\]
on \( A \).

Put \( \sum_{n=1}^{\infty} V^m(z)/2^m = U(z) \). Then, clearly \( U(z) \) is the function required.

For an \( F_\sigma \) set of \( R \), the capacity of \( F_\sigma \) is defined usually. Let \( A \) be an \( F_\sigma \) subset of \( R \) of capacity zero. Then both \( A \cap R \) (\( R \) is open) and \( A \cap B \) are \( F_\sigma \) sets. Hence we have at once the following

**Corollary.** Let \( A \) be an \( F_\sigma \) subset of \( R \) of capacity zero. Then there exists a potential \( U(z) \) satisfying the four conditions of Theorem 2.

Let \( \{G_n\} \) be a decreasing sequence of non compact subsurfaces of \( R \) with compact relative boundaries \( \{\partial G_n\} \) such that \( \bigcap_{n \geq 1} G_n = 0 \). Two such sequences \( \{G_n\} \) and \( \{G_n'\} \) are called equivalent if for given \( m \), there exists a number \( n \) such that \( G_m \supset G_n' \) and \( G_m' \supset G_n \). We consider that any equivalent sequences determine an unique ideal boundary component. Denote the set of all the ideal components by \( B \). A topology is introduced on \( R + B + \partial R_0 \) by the usual manner and it is easily seen that \( R + B + \partial R_0 \) and \( B \) are closed and compact. Let \( A \) be a closed subset of \( B \) and let \( A \) be the set of ideal boundary points on \( A \). Then since \( \{G(z, p_i)\} \) for \( p_i \in A \) is a normal family, \( A \) is also a \( \delta \)-closed set. Hence we have

**Theorem 3.** Let \( A \) be the subset of \( B \) on a closed subset \( A \) of \( B \).
Then there exists a harmonic function $U(z)$ satisfying the conditions of Theorem 2 and moreover $5^\circ$. $\lim_{z \to \partial A} U(z) < \infty$.

It is sufficient to prove that the condition $5^\circ$ is satisfied, since the other four conditions are clearly satisfied. Let $q$ be a point of the complementary set of $A$. Then there exists a component $G(q)$ of $R - R_m$ ($m$ is a suitable number with a compact relative boundary $\partial G(q)$ such that $G(q) \ni q$ and $G(q) \cap A = 0$. Then $\max_{z \in \partial G(q)} U(z) \leq M$, which implies $\sup_{z \in \partial G(q)} U(z) \leq M$, by the maximum principle, because $U(z)$ is harmonic and bounded in $G(q)$ and $R^*$ is a Riemann surface with a null-boundary.

**Corollary.** Let $A$ be the subset of $R$ on an $F_\sigma$ subset of $R + B$ of capacity zero. Then there exists a harmonic function $U(z)$ satisfying the conditions of Theorem 3.

R. S. Martin defined the ideal boundary points by the use of the function $K(z, p) = \frac{G(z, p)}{G(0, p)}$, where 0 is a fixed point of $R$. However, in case $R^*$ is a Riemann surface with null-boundary, since $G(z, 0) \geq \delta > 0$ in $R - R_s$, $K(z, p)$ is a multiple of $G(z, p)$, where $R_s \subseteq 0$. $G(z, p)$ plays consequently the same role as $K(z, p)$. Hence Martin's assertions hold even in our case.

Let $U(z)$ be a positive harmonic function in $R$ vanishing on $\partial R_s$. If $U(z) \geq V(z) \geq 0$ implies $V(z) = KU(z)$ for any harmonic function $V(z)$ in $R$, $U(z)$ is called a minimal function. Martin proved that every minimal function is a multiple of some $G(z, p)$ ($p \in B$) and that every positive harmonic function vanishing on $\partial R_s$ is represented uniquely by an integral form of minimal functions.

The condition $5^\circ$ of Theorem 3 is not always satisfied under the assumptions of Theorem 2, that is, a positive harmonic function $U(z)$ such that $U(z) = \infty$ on a $\delta$-closed set $A$ and $U(z) < \infty$ except on $A$ does not always exist.

**Example.** Suppose that there exist $n$ minimal function $G(z, p_i)$ ($i = 1, 2, \ldots, n$) with pole $p_i$ on a boundary component $p$. Then every Green's function $G(z, p^*)$ with pole $p^*$ on $p$, being not minimal, must be a linear from $G(z, p^*) = \sum_{i=1}^{n} c_i G(z, p_i)$ ($c_i \geq 0$, $\sum_{i=1}^{n} c_i = 1$). Put $A = \bigcup_{i=1}^{n} p_i$. Then clearly $A$ is a $\delta$-closed set and $\delta(p^*, A) > 0$. Denote by $U(z)$ a positive harmonic function in Theorem 2, that is, $U(z) = 0$ on $\partial R_s$, $\int_{\partial R_s} \frac{\partial U(z)}{\partial n} ds = 2\pi$ and $U(z) = \infty$ at every point of $A$. Then

5) See 4).

6) Clearly, there exists a fundamental sequence $\{p_i^*\}$ determining $p^*$. 
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\[ U(z) = \int G(z, q_a) \, d\mu(q_a) \quad (q_a \in B). \]

By the symmetry of the Green's function,

\[ U(p^*) = \int G(p^*, q_a) \, d\mu(q_a) = \int \sum_{i=1}^{n} c_i G(q_{a_i}, p_i) \, d\mu(q_a) = \sum_{i=1}^{n} c_i U(p_i). \]

Hence \( U(z) = \infty \) on \( A \) implies \( U(p^*) = \infty \). Therefore any positive harmonic function that is infinite at every point of \( A \) must be infinite at any point of \( B \) lying on \( p \). Thus there exists no positive harmonic function infinite only on \( A \).

As an application to classification of types of Riemann surfaces, we have

**Theorem 4.** \( R^* \) is a Riemann surface with null-boundary, if and only if there exists a harmonic function \( U(z) \) with one negative logarithmic singularity at a point of \( R^* \) such that \( U(z) \) has limit \( \infty \) as \( z \) tends to \( B \).

Proof. If the function above stated exists, \( R^* \) is clearly a Riemann surface with null-boundary and it is easy to construct the function in this theorem from the function in Theorem 3, by putting \( A = B \) and by the alternating process of Schwarz.

Many other applications, for instance, to Nevanlinna's first and second fundamental theorems, will be omitted here.

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