<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>Corrections to: &quot;Vanishing theorems for cohomology groups associated to discrete subgroups of semisimple Lie groups&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Raghunathan, M. S.</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 16(1) P.295-P.299</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1979</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/5281">https://doi.org/10.18910/5281</a></td>
</tr>
<tr>
<td><strong>DOI</strong></td>
<td>10.18910/5281</td>
</tr>
<tr>
<td><strong>rights</strong></td>
<td></td>
</tr>
</tbody>
</table>
CORRECTIONS TO
"VANISHING THEOREMS FOR COHOMOLOGY GROUPS
ASSOCIATED TO DISCRETE SUBGROUPS OF
SEMISIMPLE LIE GROUPS"

M.S. RAGHUNATHAN

(Received June 19, 1978)

The paper referred to in the title appeared in this journal in 1966 (vol. 3, pp. 243–256). Unfortunately Theorem 1 of the paper as it stands is incorrect. There is a wrong assertion made on p. 249 in the proof of Assertion III (about representations of $SL(2)$). (I am indebted to Dr. Ruh and Dr. Im Hof for drawing my attention to the error). We give here a modified (weaker) result with the requisite additional arguments for its proof.

The notation is as in the 1966 paper: $G$ will denote a connected real simple Lie group and $\rho$ a representation of $G$ on a complex vector space $F$; $K$ will be a maximal compact subgroup and $\Gamma$ a discrete uniform subgroup; $X = G/K$ and $H^*(\Gamma, X, \rho)$ will be the $p$th cohomology of the complex of $F$-equivariant smooth forms on $X$ (if $\Gamma$ acts fixed point free on $X$, these are simply the Eilenberg-Maclane groups of $\Gamma$ with coefficients in $\rho$). Let $g_0$ (resp. $\mathfrak{t}_0$) denote the Lie algebra (resp. Lie subalgebra) of $G$ (resp. $K$). Let $g$ (resp. $\mathfrak{t}$) denote the complexification of $g_0$ (resp. $k_0$). Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{f}$ in $g$ w.r.t. the Killing form. Let $\mathfrak{h}_0$ be a Cartan-subalgebra of $\mathfrak{k}_0$ and $\mathfrak{h}_0 \supseteq \mathfrak{h}_0$ a Cartan-subalgebra of $g_0$. Let $\mathfrak{h}$ (resp. $\mathfrak{h}_0$) be the $C$-span of $\mathfrak{h}_0$ (resp. $\mathfrak{h}_0$) in $g$. Let $\mathfrak{h}_0 \subseteq \mathfrak{h} \cap \mathfrak{p}$. Let $\varphi$ denote the Killing form on $g$ and $\Delta$ be the root system of $g$ w.r.t. $\mathfrak{h}$. For $\alpha \in \Delta$, let $H_\alpha$ be the unique element of $\mathfrak{h}$ such that $\varphi(H_\alpha, H_\alpha) = \alpha(H)$ for all $H \in \mathfrak{h}$. Let $\mathfrak{h}^* = \sum_{\alpha \in \Delta} \mathbb{R} H_\alpha = i\mathfrak{h} \oplus \mathfrak{h}_0$. Let $\theta$ be the Cartan-involution of $g_0$ determined by $\mathfrak{t}_0$ as well as its extension to $g$. Let

$$A = \{ \alpha \in \Delta | \theta(E_\alpha) = E_\alpha \}$$
$$B = \{ \alpha \in \Delta | \theta(\alpha) = -\alpha \}$$
$$C = \{ \alpha \in \Delta | \theta(E_\alpha) = -E_\alpha \}$$

(Here $E_\alpha$ is a root vector corresponding to $\alpha$); then $\Delta = A \cup B \cup C$. Also $\theta$ stabilises $\mathfrak{h}$ as well as $\Delta$, $A \cup C$ and $B$; moreover $\theta(\alpha) = -\alpha$ if $\alpha \in A \cup C$. We will say in the sequel that an order on the (real) dual of $\mathfrak{h}^*$ is admissible if it is obtained in the following manner: let $H_1, \ldots, H_t$ be an orthonormal basis of $\mathfrak{h}^*$.
so chosen that \( H_1, \cdots, H_s \) constitute a basis of \( \mathfrak{h}_k \); and \( \alpha \in \text{dual of } \mathfrak{h}^* \) is positive if the first nonvanishing \( \alpha(H_i) \) is positive. If we denote by \( O \) one such an order for any subset \( E \) in the dual of \( \mathfrak{h} \), \( E^+(O) \) denotes the positive elements in \( E \).

For an irreducible representation \( \rho \) of \( G \), let \( \Lambda_{\rho}(O) \) be the highest weight of \( \rho \) w.r.t. \( \mathfrak{h} \) and the order \( O \). Let \( \Sigma_+(O) = C^+(O) \cup \{ \alpha \in B^+(O) | \alpha > \theta(\alpha) \} \) and \( \Sigma_{\geq}(O) = \{ \alpha \in \Sigma_+(O) | \varphi(\Lambda_{\rho}(O), \alpha \neq 0) \} \). With this notation we have

**Theorem** Let \( \rho \) be a finite dimensional irreducible representation of \( G \). Then if \( \Sigma_{\geq}(O) \) contains (strictly) more than \( q \) elements for every admissible \( O \), then \( H^p(\Gamma, X, \rho) = 0 \) for \( 0 < p < q \).

As is deduced in [1] from the work of Matsushima and Murakami, the vanishing of \( H^p(\Gamma, X, \rho) = 0 \), \( 0 < p < q \), will follow from the following:

Let \( X_1, \cdots, X_n \) be a basis of \( \mathfrak{g} \) with \( \varphi(X_i, X_j) = \pm \delta_{ij} \) and \( X_i \in k \) for \( 1 \leq i \leq N \) forming a basis of \( K \). Let \( c' = -\sum_{1 \leq i \leq N} X_i^2 \) and \( c = -\sum_{1 \leq i \leq N} X_i^2 + \sum_{N < i \leq N} X_i^2 \) in the universal enveloping algebra \( U(\mathfrak{g}) \) of \( \mathfrak{g} \). We denote by \( \rho \) the extension of \( \rho \) to \( U(\mathfrak{g}) \) as well. Let \( E \) be the \( \rho^k \)th exterior power of \( \rho \) and \( T^k_\rho : F \otimes E \rightarrow F \otimes E \) be the endomorphism

\[
T^k_\rho = 2(\rho \otimes 1)(c) + (1 \otimes \sigma)(c') - (\rho \otimes 1)(c') - (\rho \otimes \sigma)(c')
\]

Let \( \langle , \rangle \) denote the hermitian scalar product on \( F \otimes E \) deduced from the involution \( \theta \) and \( \varphi \) on \( \mathfrak{g}_0 \). Then the hermitian quadratic form,

\[
\eta \mapsto \langle T^k_\rho \eta, \eta \rangle
\]

is positive definite for \( p \leq q \) provided the hypothesis of the theorem holds.

As in [1], let \( E = \sum_{\nu \in \Lambda(\mathfrak{m})} E_\nu \) be a decomposition of \( E \) into its \( \mathfrak{m} \)-irreducible components and similarly \( F = \sum_{\lambda \in \Lambda(\mathfrak{m})} F_\lambda \) a decomposition of \( F \) into irreducible \( \mathfrak{m} \)-modules. The indexing sets are the dominant weights w.r.t. an admissible order \( O \) on the dual of \( \mathfrak{h}^* \). We denote by \( V_{\lambda, \mu} \) the \( \mathfrak{m} \)-irreducible components of \( V_{\lambda, \mu} = F_\lambda \otimes E_\mu \); then each \( V_{\lambda, \mu} \) is contained in an eigen-space of \( T^k_\rho \) and let \( a(\lambda, \mu, \nu) \) be the corresponding eigen-value. Let \( a(\lambda, \mu) = a(\lambda, \mu, \lambda + \mu) \).

We assume in the sequel that \( G \) is simple (over \( \mathbb{R} \))

**Assertion I.** \( a(\lambda, \mu, \nu) \geq a(\lambda, \mu) \) for all \( \lambda \in \Lambda(\mathfrak{m}), \mu \in \Lambda(\mathfrak{m}) \); equality occurs only if \( \nu = \lambda + \mu \).

This is proved in [1].

**Assertion II.** Let \( f_\lambda \) be a dominant weight vector of \( F_\lambda, \lambda \in \Lambda(\mathfrak{m}) \). Suppose that there exists \( \alpha \in B^+(\mathfrak{m}) \) such that \( E_{\alpha} f_\lambda \neq 0 \in F_\lambda, \lambda_1 \in \Lambda(\mathfrak{m}) \); then \( a(\lambda, \mu) > a(\lambda_1, \mu) \).

For a proof see [1].
Assertion III. Suppose that $f_\lambda$ is a (non-zero) dominant weight vector in $F_\lambda$ and that $E_\alpha f_\lambda = 0$ for all $\alpha \in A^+(O) \cup B^+(O)$. Then if $\alpha(\lambda, \mu) = 0$ for some $\mu \in M(O)$, there exists an admissible order $O'$ on the dual of $\mathfrak{h}^*$ such that $f_\lambda$ is dominant for $O'$ (i.e. $E_\alpha f = 0$ for all $\alpha \in \Delta^+(O')$).

We need the following Lemma.

Lemma. Let $E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be the standard basis of $SL(2)$. Let $\tau$ be a finite dimensional irreducible representation of $SL(2)$ and $v$ a weight vector for $H$ of weight $\lambda$. Then

$$\tau(E_+ E_- + E_- E_+)v = \lambda' \cdot v$$

with $\lambda' \geq |\lambda|$; also equality occurs if and only if $\lambda$ is dominant (positive or negative).

Proof. Let $b = E_+ E_- + E_- E_+ + H^2/2$ be the Casimir element. Then $\tau(b)$ is a scalar equal to $\lambda_0(H)^2/2 + \lambda_0(H)$ where $\lambda_0$ is the positive dominant weight. Thus

$$\tau(E_+ E_- + E_- E_+)v = (\tau(b) - \tau(H^2/2))v = \lambda_0(H)^2/2 + \lambda_0(H) - \lambda(H)^2/2.$$ 

Now for any weight $\nu$ of $\tau$, $\nu(H)^2 \leq \lambda_0(H)^2$, equality occuring if and only if $\nu$ is extremal. The lemma is now immediate.

Let $f_\lambda$ be a unit dominant weight vector in $F_\lambda$ ($\lambda \in L(O)$). Then we have

$$\rho(c)f_\lambda = \sum_{\alpha \in A^+ \cap \{0\}} \rho(E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha) + \sum_{1 \leq i \leq 1} \rho(H_i)^2 \cdot f_\lambda$$

$$= \left( \sum_{\alpha \in A^+ \cap \{0\}} \sum_{\alpha \in B^+ \cap \{0\}} \lambda(H_a + H_{\theta(a)})/2 \right) \cdot f_\lambda$$

$$+ \left( \sum_{1 \leq i \leq r} \lambda_{\alpha}(H_i)^2 \right) f_\lambda + \sum_{\alpha \in C^+ \cap \{0\}} \rho(E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha) f_\lambda$$

We have used here the fact $E_\alpha f_\lambda = 0$ for $\alpha \in A^+(O) \cup B^+(O)$. Also, we have

$$\rho(c)f_\lambda = \sum_{1 \leq i < j} \lambda(H_i)^2 + \sum_{\alpha \in \Sigma_1 \cap \{0\}} \lambda(H_a + H_\theta(a))/2 \cdot f_\lambda$$

where $\Sigma_1 \cap \{0\} = A^+(O) \cup \{\alpha \in B^+(O) | \alpha > \theta(\alpha)\}$. Also for dominant weight vector $e_\mu \in E_\mu (\mu \in M(O))$ of unit length

$$\sigma(c')e_\mu = \sum_{1 \leq i < j} \mu(H_i)^2 + \sum_{\alpha \in \Sigma_1 \cap \{0\}} \mu(H_a + H_\theta(a))/2 \cdot e_\mu$$

$$(\rho \otimes \sigma)(c')(f \otimes e_\mu) = \left( \sum_{1 \leq i \leq r} \lambda + \mu \right) (\mu (H_i)^2$$

$$+ \sum_{\alpha \in \Sigma_1 \cap \{0\}} (\lambda + \mu)(H_a + H_\theta(a))/2 ) f_\lambda \otimes e_\mu$$

leading to:

$$\langle T_{c'}f_\lambda \otimes e_\mu, f_\lambda \otimes e_\mu \rangle = 2 \sum_{\alpha \in \Sigma_1 \cap \{0\}} \langle \rho(E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha) f_\lambda \otimes e_\mu, f_\lambda \otimes e_\mu \rangle$$

$$+ 2 \sum_{1 \leq i \leq 1} \langle \rho(H_i)^2 f_\lambda \otimes e_\mu, f_\lambda \otimes e_\mu \rangle$$

$$+ \sum_{\alpha \in B^+ \cap \{0\}} \lambda(H_a + H_\theta(a))$$

$$- 2 \sum_{1 \leq i < j} \lambda(H_i) \mu(H_j) \mu(H_i).$$
Consider now \( \langle \rho(E\alpha E\alpha + E_{-\alpha}E\alpha) f_\lambda \otimes e_\mu, f_\lambda \otimes e_\mu \rangle \) for \( \alpha \in C^+(O) \). We claim that this can be zero if and only if \( f_\lambda \) is an extremal vector for \( g(\alpha) \). To see this we decompose \( F_\lambda \) into \( g(\alpha) \)-irreducible components

\[
F_\lambda = \sum_{\ell} V(\ell)
\]

We assume as we may that \( V(\ell) \) are mutually orthogonal with respect to the scalar product on \( F \). Let \( f_\lambda = \sum f(\ell) \) with \( f(\ell) \in V(\ell) \). We then have

\[
\langle \rho(E\alpha E\alpha + E_{-\alpha}E\alpha) f(\ell) \otimes e_\mu, f(\ell) \otimes e_\mu \rangle = \sum \langle \rho(\alpha) E\alpha E\alpha f(\ell) \otimes e_\mu, f(\ell) \otimes e_\mu \rangle
\]

Each term on the right hand side is non-negative so that the left hand side is non-negative and equals zero if and only if

\[
\langle \rho(\alpha) E\alpha E\alpha f(\ell) \otimes e_\mu, f(\ell) \otimes e_\mu \rangle = 0
\]

for all \( \ell \). Now by the lemma above, this means that for every \( \ell \), \( f(\ell) \) is an extremal weight vector for \( g(\alpha) \). Also \( E\alpha \) (resp. \( E_{-\alpha} \)) annihilates \( f(\ell) \) if \( \lambda(H_a) > 0 \) (resp. \( \leq 0 \))—\( f(\ell) \) is a weight vector whose weight is \( H_a \rightarrow \lambda(H_a) \) for all \( \ell \). Thus we see that all the \( f(\ell) \) are annihilated by the same root vector: \( E\alpha \) or \( E_{-\alpha} \). Thus \( E\alpha f_\lambda = 0 \) or \( E_{-\alpha} f_\lambda = 0 \) proving our contention. Also since \( E\alpha f_\lambda = 0 \) for all \( \alpha \in A^+(O) \cup B^+(O) \) one sees that \( \lambda(H_a + H_{\theta(a)}) \geq 0 \). Since \( H_i \in \mathfrak{h}_p \) for \( i > p' \), \( \langle \rho(H_i)^2 f_\lambda, f_\lambda \rangle \geq 0 \). Finally

\[
\sum_{1 \leq i \leq p} \lambda(H_i) \mu(H_i) = \langle \lambda, \mu \rangle = \langle \lambda, \sum_{1 \leq i \leq p} \pm (\alpha_i + \theta(\alpha_i))/2 \rangle
\]

for suitable elements \( \alpha_i, 1 \leq i \leq p \), in \( \sum_\tau (O) \). We see then that if \( \langle T^\theta(f \otimes e_\mu), f_\lambda \otimes e_\mu \rangle = 0 \), we must necessarily have the following

(i) \( \rho(H_l) f_\lambda = 0 \), \( p' < i \leq l \)
(ii) \( \lambda(H_a + H_{\theta(a)}) \geq 0 \) for all \( \alpha \in B^+(O) \)
(iii) \( E\alpha f_{-\lambda} = 0 \) or \( E_{-\alpha} f_\lambda = 0 \) for any \( \alpha \in C^+(O) \).

From the fact that \( \rho(H_l) f_\lambda = 0 \) for \( p' < i \leq l \) we see that \( f_\lambda \) is an eigen-vector for all of \( \mathfrak{h} \) with the corresponding weight \( \lambda \) being the unique extension of \( \lambda \) which is zero on \( \mathfrak{h}_p \). Let \( h_\lambda (= h_\lambda) \) be the unique element of \( i\mathfrak{h}_l \) such that \( \phi(h_\lambda, h) = \lambda(h) \) for all \( h \in \mathfrak{h}_l \). Then we take an orthonormal basis \( H_1, \ldots, H_l \) of \( \mathfrak{h}^* \) yielding an order \( O' \) on the dual of \( \mathfrak{h}^* \) with \( H_1 \) being a positive multiple of \( h_\lambda \). Now since \( \lambda(h) \geq 0 \) for all \( \alpha \in A^+(O) \cup B^+(O) \) and \( E\alpha f_\lambda = 0 \) for \( \pm \alpha \in C^+ \) if \( \pm \alpha(h) \geq 0 \), we see that with respect to the order \( O' \), \( f_\lambda \) is an extremal vector. We need only conclude that \( O' \) is admissible and this is obvious. This completes the proof of Assertion III.

**Assertion IV.** Fix an admissible order \( O \) on the dual of \( \mathfrak{h}^* \). Let \( \Lambda \) be the dominant weight of \( \rho \) w.r.t. \( O \) and let \( \lambda_0 \) denote the restriction of \( \lambda \) to \( i\mathfrak{h}_l \); then \( a(\lambda_0, \mu) > 0 \) for all \( \mu \in M(O) \) provided that there are at least \( (g+1) \) roots \( \alpha \in \sum_\tau (O) \) such that \( \Lambda(H_a + H_{\theta(a)}) > 0 \).
The proof is given in [1].

REMARK. We have proved the theorem for the case of simple $g$ but the general case is immediate from this.

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

References
