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AUTOMORPHISMS OF POLYNOMIAL ALGEBRAS AND HOMOTOPY COMMUTATIVITY IN H-SPACES

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1. In an earlier paper [16], the author proved that a connected, finite complex, which is a homotopy commutative H-space, has the homotopy type of a torus. The main purpose of this note is to give a proof of the weaker result, see Theorem B, that a homotopy associative, homotopy commutative H-space structure on a connected finite complex, implies that the complex is homotopically equivalent to a torus. As a torus is an Eilenberg-MacLane space, there is just one H-space multiplication up to homotopy. The reasons for presenting a second proof are twofold. First the details of this proof are much simpler and second the techniques of the proof can be generalized in a straightforward manner to give other results. As in the earlier proof, considerable use is made of results of W. Browder.

First we recall some definitions. X is an H-space if there exists a continuous multiplication $m: X\times X\to X$, with unit. X is a homotopy associative H-space if, in addition, $m(m\times 1)\simeq m(1\times m)$: $X\times X\times X\to X$, and X is a homotopy commutative H-space if it is an H-space and $m\simeq m\ T: X\times X\to X$, where T is the switching map for cartesian product. We shall also require a condition that an H-space be strictly associative and strongly homotopy commutative, in the sense of Sugawara; Theorem 4.3 of [23] implies that a suitable definition is that the classifying space of X, as constructed in [14], B_X say, is itself an H-space.

In this paper X will always be a connected, non-contractible, finite complex. When X is an H-space, the rational cohomology ring $H^*(X, Q)$ is an exterior algebra on odd dimensional generators, and more, it is a connected, graded Hopf algebra with comultiplication induced by m. $H^*(X, Q)$ is said to be primitive if there exist generators x_i for the exterior algebra such that

$$m^*(x_i) = x_i\otimes 1+1\otimes x_i \in H^*(X\times X, Q)\ , \qquad \text{for each } i.$$

The results on homotopy commutative H-spaces are Theorems A, B and C.

Theorem A[1]. *Let X be a homotopy commutative H-space with $H^*(X, Q)$*

1. Theorems A,B,C,D and a weaker version of Theorem E were contained in the author's doctoral thesis submitted to the University of Oxford.

primitive, then X has the homotopy type of a torus.

Theorem B[1]**.** *Let X be a homotopy associative, homotopy commutative H-space, then X has the homotopy type of a torus.*

Theorem C[1]**.** *Let X be a strictly associative, strongly homotopy commutative H-space, then X has the homotopy type of a torus.*

Theorem C was first proved by W. Browder by other methods. Clearly Theorem A implies Theorem B, and it can be shown that Theorem B implies Theorem C, see [23], but we shall give three slightly different proofs.

Let Q_p be the ring of rational numbers whose denominators are not divisible by the prime number p. Let Y be a complex with finite skeletons, whose cohomology ring with Q_p-coefficients is a finitely generated polynomial algebra, possibly truncated at height greater than p, with even dimensional generators, y_i, $1 \leq i \leq n$.

A map $f: Y \to Y$ is a k-map if $f^*(y_i) = ky_i$ modulo decomposable elements of the polynomial algebra.

The two results which lead to the proofs of Theorems A, B and C are as follows,

Theorem D[1]**.** *Suppose that Y supports a p-map, then each y_i has dimension* 2.

Theorem E[1]**.** $(p=2)$ *Suppose that Y supports a* (-1)*-map, then each y_i has dimension* 2.

The proofs of Theorems D and E are in essence extremely simple. We just write down the condition that a k-map induces a homomorphism in the complex K-theory which commutes with the Adams operators of [3]. The technical result used is a suitable interpretation of the "Integrality Theorem on the Chern Character" of [2]. Theorems D and E will be proved in the framework of a simple axiomatic theory, as in [17], which is developed in §2. In §3 we prove the analogues of Theorems D and E in this framework, and in §4 Theorems A, B and C are proved.

I am grateful to two people in particular in connection with this paper. Theorem A arose out of a discussion of Theorem B with Dr. I.M. James and many of the details of the simple axiomatic scheme of §2 were worked out with the assistance of Professor J.F. Adams.

2. Let M be a finitely generated Q_p-module, filtered by submodules

$$M = M_0 \supset M_1 \supset M_2 \supset, \cdots, \supset M_{g-1} \supset M_g = 0\,,$$

which possess linear maps ψ^k such that $\psi^k(M_i) \subset M_i$, for each i.

Let $N = \sum_{0 \leq i \leq g-1} N_i$ be the associated graded module, that is $N_i \cong M_i / M_{i+1}$.

We assume,

A1. N is a free Q_p-module.

A2. $\psi^k \psi^l = \psi^l \psi^k$, for all l and k.

A′3. If $u \in M_n$, then $(\psi^k(u) - k^n u) \in M_{n+1}$, for each k.

A″3. If $u \in M_n$, then there exist $n+1$ elements $v_i \in M_{n+i(p-1)}$, $0 \leq i \leq n$, such that $\psi^p(u) = \sum_{0 \leq i \leq n} p^{n-i} v_i$.

A4. M possesses a multiplication which gives it the structure of a commutative filtered ring, that is M is a commutative ring and $M_i \cdot M_j \subset M_{i+j}$.

A5. ψ^k is a ring homomorphism for each k.

A6. $\psi^p(u) = u^p$ mod. p, for each $u \in M$.

We call such an M a multiplicative ψ^k-module over Q_p. This is consistent with the usage in [17] where $p=2$. A simple induction argument shows that A′3 and A″3 are equivalent to A3 of [17] in the presence of A1 and A2.

The ring structure on M induces a graded ring structure on N in a natural way. The same symbols are used for M and N considered as Q_p-modules or as rings. The application is to take $M = K(Y, Q_p)$, where Y is a finite complex whose integral homology is free of p-torsion. $K(Y, Q_p)$ is unitary K-theory with Q_p-coefficients, defined by taking the tensor product of the integral K-theory of [8] with Q_p. The spectral sequence of [8] with Q_p-coefficients collapses, since $H^*(Y, Q_p)$ is torsion free. Therefore if $GK(Y) = \sum_{i \geq 0} K_{2i}(Y, Q_p)/K_{2i+1}(Y, Q_p)$ is the graded ring corresponding to the filtered ring $K(Y, Q_p)$ with the CW-filtration, then $H^{\text{even}}(Y, Q_p) \cong GK(Y)$. Further this isomorphism preserves the usual ring structure which both sides possess. Now set $M_i = K_{2i}(Y, Q_p)$ and let ψ^k be the Adams operators of [3]. A1 is true by hypothesis on Y and A2, A′3, A5 and A6, as well as the fact ψ^k is a linear map which preserves the CW-filtration, are well-known properties of the Adams operators. Under torsion free hypotheses, A″3 is equivalent to the p-primary part of Theorem 1 of [2]. It follows almost immediately from Proposition 5.6 of [6], (see Lemma 5.5 of [6]). We have proved Lemma 2.1.

Lemma 2.1. *$M = K(Y, Q_p)$ is a multiplicative ψ^k-module over Q_p.*

We return to the general case. Let $M = M^1 \oplus M^2$ be a direct sum of modules of the type we are considering, where M^1 is an ideal in M and $\psi^k(M^1) \subset M^1$ for each k. The next result is clear.

Lemma 2.2. *The quotient module M^2 is a multiplicative ψ^k-module over Q_p.*

Let $I_s\colon M_s \to N_s$ be the quotient map. Consider N filtered by submodules $\sum_{i\geq s} N_i$. Then N and M are isomorphic as filtered Q_p-modules, since N is free (c.f. Lemma 2.5 of [17]), and if $J\colon N \to M$ realizes this isomorphism, the composition $I_s J$, restricted to N_s, is the identity map. For each k and J, define homomorphisms $\Phi^k_J\colon N \to N$ by requiring that the following diagram is commutative,

$$\begin{array}{ccc} N & \xrightarrow{\Phi^k_J} & N \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J} \\ M & \xrightarrow[\psi^k]{} & M \end{array}$$

Lemma 2.3. $\Phi^k_J \Phi^l_J = \Phi^l_J \Phi^k_J$, *for all l and k.*

Proof. This follows from A2.

Lemma 2.4. *Let $x \in N_n$, then $\Phi^k_J(x) = k^n x \bmod. \sum_{i>0} N_{n+i}$.*

Proof. This follows from A3.

It is clear that in general there will not exist $J\colon N \to M$ which is a ring isomorphism. However, it is true that when N is a truncated polynomial algebra there exists a ring isomorphism $J\colon N \to M$. There are now two approaches. The better is to use a general K-theory splitting theorem of J.F. Adams from which the two key lemmas that follow, Lemma 2.5 and Lemma 2.9, are easy consequences. However, this work has not yet been published and so we state Lemma 2.5 and Lemma 2.9 without proof and in an appendix give direct proofs, which were the author's original proofs.

Lemma 2.5. *Suppose that there exists $J\colon N \to M$ which is a ring isomorphism, then there exists a ring isomorphism $K\colon N \to M$ such that*

$$\Phi^k_K(N_n) \subset \sum_{i\geq 0} N_{n+i(p-1)},$$

for all k and n.

Lemma 2.6. *For K as in Lemma 2.5, if $x \in N_n$, then $\Phi^n_K(x) = \sum_{0\leq i\leq t} p^{n-i} x_i$ for some t, where $x_i \in N_{n+i(p-1)}$ and $x_0 = x$.*

Proof. This follows from A′3 and A″3.

The next two lemmas also follow easily from the axioms.

Lemma 2.7. *If $J:N\to M$ is a ring isomorphism, Φ_J^k is a ring homomorphism.*

Lemma 2.8. *If $J:N\to M$ is a ring isomorphism, $\Phi_J^p(x)=x^p$ mod p, for all $x\in N$.*

Now let N be a truncated polynomial algebra of height k over $\underset{\sim}{Q}_p$, that is, N has a homogeneous multiplicative basis $1, x_1, x_2, \cdots, x_m$ of elements of dimensions $0<n_1\leq n_2\leq\cdots\leq n_m$, where $x_1^{\nu_1}x_2^{\nu_2}\cdots x_m^{\nu_m}=0$ if and only if $\sum_{1\leq i\leq m}\nu_i\geq k$, and there are no other relations.

Let N be such a truncated polynomial algebra.

Lemma 2.9. *There exists a ring isomorphism $J:N\to M$.*

A ring homomorphism $f:M\to M$ is natural if it commutes with the ψ^k. It induces a linear map $f_J:N\to N$, defined by requiring that the following diagram is commutative,

$$\begin{array}{ccc} N & \xrightarrow{f_J} & N \\ J\downarrow & & \downarrow J \\ M & \xrightarrow[f]{} & M \end{array} .$$

There are four properties of such homomorphisms with which we are concerned, the proofs of Lemmas 2.10, 2.11 and 2.13 are straightforward.

First suppose that we have a direct sum decomposition of M as in Lemma 2.2, and a natural ring homomorphism $f:M\to M$ such that $f(M^1)\subset M^1$, then

Lemma 2.10. *The natural map $f:M\to M$ induces a natural map of the quotient module f^2; $M^2\to M^2$.*

Lemma 2.11. *$f_J\Phi_J^k=\Phi_J^k f_J$, for each k.*

Lemma 2.12. *Suppose that $\Phi_K^k(N_n)\subset\sum_{i\geq 0}N_{n+i(p-1)}$, for all n and k, then $f_K(N_n)\subset\sum_{i\geq 0}N_{n+i(p-1)}$.*

Proof. First we show that $f_K(N_n)\subset\sum_{i\geq 0}N_{n+i}$ arguing by induction in N_n for decreasing n. If $N_g=0$ and $x\in N_{g-1}$, let $f_K(x)=\sum_{0\leq i\leq s}y_i$, where $y_0\neq 0$ has dimension t and the dimension of each y_i is greater than t for $i>0$. Consider $f_K\Phi_K^k(x)=\Phi_K^k(f_K(x))$, by Lemma 2.11, in dimension t. Lemma 2.4 implies that $k^{g-1}y_0=k^t y_0$, which implies that $t=g-1$ as required. Therefore assume the result in N_i for $i>n$ and let $x\in N_n$. As before let $f_K(x)=\sum_{0\leq i\leq s}y_i$ where $y_0\neq 0$ has dimension t and y_i, $i>0$, has dimension greater than t. Therefore once more applying Lemma 2.11 and Lemma 2.4, in dimension t we have $k^n y_0=k^t y_0$, which implies $n=t$ and completes the proof that f_K preserves the filtration on N.

Now assume the result of the lemma in N_i, for $i>n$, since it is certainly true in N_{g-1}. Let $x\in N_n$ and $f_K(x)=\sum_{0\leq i\leq s} y_i$, where $y_u\neq 0$ say, for some u in $0\leq u\leq s$, is the element of least dimension, t say, not of the form $n+i(p-1)$ for some i. Again Lemma 2.11 and Lemma 2.4 imply $k^n y_s=k^t y_s$ which is impossible, and the lemma is proved.

Lemma 2.13. *Let $J:N\to M$ be a ring isomorphism, then f_J is a ring homomorphism.*

In all the notations introduced above, except f_J, when it is clear which isomorphism $J:N\to M$ we are using, the suffix J will be omitted.

3. Let N be a truncated polynomial algebra of height greater than p over Q_p, with generators $1, x_1, x_2, \cdots, x_m$. By setting $M^1=(M_1)^{p+1}$ in Lemma 2.2, it can be assumed that the height of N is precisely $p+1$. We assume that a choice of generators has been made and so there is a natural base for N over Q_p. It is convenient to have an alternative notation for the generators; we write y_q for a generator where it is assumed that the dimension is q.

A linear map $f:M\to M$, which preserves the filtration, induces a map of the associated graded module, $f_*:N\to N$. Clearly f_* and f_J restricted to N_n have images which coincide in N_n. We define a natural ring homomorphism to be a k-map if $f_*(x_i)=kx_i$ modulo decomposable elements of N.

Using Lemma 2.9, it is assumed that an isomorphism $K:N\to M$ has been chosen as in Lemma 2.5. Note that if M, a truncated polynomial algebra of height greater than $p+1$ has a k-map, then M^2 as constructed above, a truncated polynomial algebra of height $p+1$, has a k-map, by Lemma 2.10 and Lemma 2.13.

Theorem 3.1. *Suppose that $f:M\to M$ is a p-map*[2]*, then each generator of N has dimension* 1.

Proof[3]. Let y_q be a generator of highest possible dimension. We suppose that $q>1$ and obtain a contradiction.

Lemma 2.11 implies that $f_K\Phi^p(y_q)=\Phi^p(f_K(y_q))$. It is sufficient to consider this equality modulo p^2, and in particular the component in dimension qp, the highest non-trivial dimension.

First consider the left-hand side. Lemma 2.6 implies that

$$\Phi^p(y_q)=pz_{qp-(p-1)}+z_{qp} \bmod p^2 \tag{3.1}$$

2. In fact, it is sufficient that $f:M\to M$ is natural and a p-map modulo p^2.
3. The proof can be expressed in terms of the homomorphisms S^i_J introduced in [17].

where $z_i \in N_i$. Now if y_t is any generator of dimension t

(3.2) $$f_K(y_t) = py_t + w_t + w_{t+(p-1)} + \text{higher dimensional elements},$$

where $w_i \in N_i$ and w_t is decomposable.

Now for dimensional reasons z_{qp} is a sum of scalar multiples of products of p generators, and since f_K is a ring homomorphism, Lemma 2.13, $f_K(z_{qp}) = p^p z_{qp}$. Also since $q>1, pq-(p-1)>q$, and so $z_{qp-(p-1)}$ is decomposable. In the expression (3.2), again for dimensional reasons, w_t does not involve y_q (or any other generator of dimension q). Thus if $f_K(z_{qp-(p-1)})$ is not independent of y_q^p, when we express $z_{qp-(p-1)}$ in terms of the given base, it must involve $y_p^r y_s$ for some generator y_s, with $r>0$. The coefficient of y_q^p in $f_K(y_q^r y_s)$ is zero mod p^r, since f_K is a ring homomorphism. Thus the coefficient of y_q^p in $f_K \Phi^p(y_q)$ mod p^2 is zero.

Now consider $\Phi^p(f_K(y_q))$. Let $f_K(y_q) = py_q + w_q +$ higher dimensional elements, where w_q is decomposable and of dimension q. But if w has dimension greater than q, $\Phi^p(w)=0$ mod p^2, by Lemma 2.6. Also since Φ^p is a ring homomorphism, Lemma 2.7, and w_q is decomposable, Lemma 2.8 implies that $\Phi^p(w_q)=0$ mod p^2. Thus again using Lemma 2.8, the coefficient of y_q^p in $\Phi^p(f_K(y_q))$ mod p^2 is p, which gives the required contradiction.

Thus $m=1$ and the theorem is proved.

We now restrict attention to the case $p=2$ and consider a k-map with $k=-1$.

Lemma 3.2. *Let f: $M \to M$ be a (-1)-map, then if $u_i = K(x_i)$, $1 \leq i \leq m$, $f(u_i) = -u_i$ modulo decomposable elements of M, for all i.*

Proof. It is necessary to show that $f_K(x_i) = -x_i$ modulo decomposable elements of N. Neglect all decomposable elements, that is consider M a truncated polynomial algebra of height 2. Suppose that the result is false and let y_q be a generator of highest possible dimension for which it is false and y_t a generator of least possible dimension which occurs in $f_K(y_q)+y_q$. Thus $f_K(y_q) = -y_q + ay_t +$ elements of equal or higher dimension, where $a \neq 0$ and is in Q_2. Consider the coefficient of y_t in $\Phi^k(f_K(y_q)) = f_K \Phi^k(y_q)$. If the coefficient of y_t in $\Phi^k(y_q)$ is b, on the left-hand side we obtain $-b+ak^t$ and on the right-hand side $k^q a - b$, which implies that $q=t$. This is impossible by the definition of f.

Theorem 3.3. *Suppose that f: $M \to M$ is a (-1)-map, then each generator of N has dimension 1.*

Proof[3]. As in the proof of Theorem 3.1, let y_q be a generator of highest possible dimension q. We suppose that $q>1$ and obtain a contradiction.

Consider the equality

$$f_K \Phi^2(y_q) - \Phi^2(y_q) = \Phi^2(f_K(y_q)) - \Phi^2(y_q)\,.$$

Again it is sufficient to consider the components in dimension $2q$ taking the value modulo 4. By Lemma 2.6

$$\Phi^2(y_q) = 2z_{2q-1} + z_{2q} \bmod 4 \tag{3.3}$$

where $z_i \in N_i$, and if y_t is any generator of dimension t,

$$f_K(y_t) = -y_t + \text{decomposable elements} \tag{3.4}$$

by Lemma 3.2. But $f_K(z_{2q}) = (-1)^2 z_{2q} = z_{2q}$ and $f_K(z_{2q-1}) = (-1)^2 z_{2q-1} = z_{2q-1}$ by (3.4) since the product of three generators vanishes and z_{2q-1} and z_{2q} are decomposable. Thus

$$f_K \Phi^2(y_q) - \Phi^2(y_q) = 2\{f_K(z_{2q-1}) - z_{2q-1}\} + \{f_K(z_{2q}) - z_{2q}\} \bmod 4$$

is zero, modulo 4.

Now consider $\Phi^2(f_K(y_q)) - \Phi^2(y_q)$. If w is any decomposable element $\Phi^2(z)=0 \bmod 4$, using Lemma 2.7 and Lemma 2.8. Therefore by (3.4),

$$\Phi^2(f_K(y_q)) - \Phi^2(y_q) = 2\Phi^2(y_q) \bmod 4$$

which equals $2y_q^2$, by Lemma 2.8.

This is the required contradiction, and the theorem is proved.

Finally we remark that Theorem 3.1 and Theorem 3.3 are just two of a number of results which can be proved by these methods. The author hoqes to return to these and similar questions at a later date.

4. Let $H^*(Y, Q_p)$ be free of p-torsion. Then $H^*(Y, Z)$ is free of p-torsion by the universal coefficient theorem and therefore so is $H_*(Y, Z)$. In particular when Y is a finite complex, we may apply the results of the previous section with $M=K(Y, Q_p)$, by Lemma 2.1. Thus Theorem D and Theorem E follow from Theorem 3.1 and Theorem 3.3 respectively, at least for finite complexes. If Y is not finite but has finite skeletons, consider a skeleton Y^t of Y of sufficiently high dimension to perform all arguments with $H^*(Y^t, Z)$ free of p-torsion and $M=K(Y^t, Q_p)$. Combining Lemmas 2.2, 2.9, 2.10 and 2.12, it may be assumed that M is a truncated polynomial algebra of height $p+1$ over Q_p, and again Theorems D and E follow from Theorem 3.1 and Theorem 3.3.

Next consider Theorems A and B. The next two theorems are quite basic to the proofs. They arise from the work of W. Browder.

Theorem 4.1. *Let X be a finite connected complex and a homotopy commutative H-space, then $H_*(X, Z)$ is free of 2-torsion.*

Proof. This follows from Theorem 8.5 of [11].

Theorem 4.2. *Let X be a finite connected complex and an H-space and suppose that the exterior algebra H^* (X, Q) is generated by one dimensional generators, then X has the homotopy type of a torus.*

Proof. First we show that X is a cohomology torus.

Suppose that $H_*(X, Z)$ has p-torsion, then $H^*(X, Z_p)$ is not an exterior algebra on odd dimensional generators, by Theorem 4.9 of [9], the converse of a classical theorem of Borel. But in the notation of [9], the p-dimension and the rational dimension of X coincide, by Theorem 7.1 of [9] and hence if $H^m(X, Q)=0$ for $m>n$, then $H^m(X, Z_p)=0$ for $m>n$. This contradicts the fact that $H^*(X, Q)$ is an exterior algebra on generators of dimension one, but $H^*(X, Z_p)$ is a Hopf algebra which is not. We deduce that $H^*(X, Z)$ is a cohomology torus.

Therefore there is a map f: $X \rightarrow S^1 \times S^1 \times \cdots \times S^1$ inducing an isomorphism of cohomology, since $S^1=K(Z, 1)$, and the conclusion of the theorem follows from a simple application of Lemma 6.2 of [10].

The hypotheses of both Theorem A and Theorem B imply that X is a finite connected complex which is a homotopy commutative H-space with $H^*(X, Q)$ primitive. We consider the cohomology and the K-theory with Q_2-coefficients of the projective plane of such an X. For basic K-theoretic properties, we refer to [5], [8] and [7]. The details of the cohomology of the projective plane which are required follow from the results of [12]. They follow by straightforward calculation and most are given in [17] and so we just summarize them here.

Since $H_*(X, Z)$ is free of 2-torsion, Theorem 3.1, $H^*(X, Q_2)$ and $K^*(X, Q_2)$ are exterior algebras on odd dimensional generators, that is in $K^*(X, Q_2)$ the generators may be chosen to lie in $K^1(X, Q_2)$. Write $\tilde{h}^q(\ )$ for either $\tilde{H}^q(\ , Q_2)$ or $\tilde{K}^q(\ , Q_2)$, the index in $\tilde{K}^q(\ , Q_2)$ being taken mod 2. We have the exact sequence.

$$(4.1) \qquad \longrightarrow \tilde{h}^q(X) \xrightarrow{\ \phi\ } \tilde{h}^q(X \mathbin{⋇} X) \xrightarrow{\ \lambda\ } \tilde{h}^{q+2}(P_2X) \xrightarrow{\ i\ } \tilde{h}^{q+1}(X) \longrightarrow$$

where $\phi=m^*-\pi_1^*-\pi_2^*$, π_i: $X \times X \rightarrow X$, $i=1$ or 2, being projection onto the i-th factor; which essentially (3.1) of [12].

Then $h^*(P_2X)=A \oplus B$, where A is a truncated polynomial algebra of height 3 on generators $u_1, u_2, \cdots, u_m$ such that $v_j=i(u_j)$, $1 \leq j \leq m$, form a multiplicative basis for the exterior algebra $h^*(X)$ and $\lambda(v_i \otimes v_j)=u_iu_j$, and B is an ideal, free of 2-torsion, generated over Q_2 by all elements of the form $\lambda(v_1^{\alpha_1} v_2^{\alpha_2} \cdots v_m^{\alpha_m} \otimes v_1^{\beta_1} v_2^{\beta_2} \cdots v_m^{\beta_m})$ where each α_i and β_i is zero or one, $\Sigma\alpha_i>0$, $\Sigma\beta_i>0$ and $\Sigma(\alpha_i+\beta_i)>2$.

Let $H^*(P_2X, Q_2)=A' \oplus B'$ and $K^*(P_2X, Q_2)=A'' \oplus B''$, then set $M=$

$K(P_2X, Q_2)$. Then by Lemma 2.1 we may apply the results of the last two sections to M.

Lemma 4.3. *$\psi^k(B'') \subset B''$ for each k.*

Proof. See Lemma 6.4 of [17].

Proof of Theorem B. Let $m(2)\colon X \to X$ be the H-space squaring map. If X is a homotopy associative, homotopy commutative H-space, the following diagram is homotopy commutative,

$$\begin{array}{ccc} X \times X & \xrightarrow{m} & X \\ {\scriptstyle m(2)\times m(2)}\downarrow & & \downarrow{\scriptstyle m(2)} \\ X \times X & \xrightarrow[m]{} & X \end{array}$$

and hence

$$\begin{array}{ccc} X * X & \xrightarrow{m'} & SX \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ X * X & \xrightarrow[m']{} & SX \end{array}$$

is homotopy commutative, where $X*X$ is the join of X with itself and SX is the suspension of X. This induces a map of the cofibre sequence of P_2X into itself, where again the squares are homotopy commutative,

$$\begin{array}{ccccccccc} X * X & \xrightarrow{m'} & SX & \xrightarrow{i} & P_2X & \xrightarrow{j} & S(X*X) & \longrightarrow \\ {\scriptstyle h}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle Sh}\downarrow & \\ X * X & \xrightarrow[m']{} & SX & \xrightarrow[i]{} & P_2X & \xrightarrow[j]{} & S(X*X) & \longrightarrow \end{array} \tag{4.2}$$

Lemma 4.4. *$f^!(B'') \subset B''$.*

Proof. This follows from the definition of B'' and the homotopy commutativity of the third square in (4.2).

Lemma 4.5. *A'' is a multiplicative ψ^k-module over Q_2 and a truncated polynomial algebra of height 3 which supports a 2-map.*

Proof. We apply Lemma 2.10 with $M^1 = B''$, $M^2 = A''$ and $f\colon M \to M$ the map $f^!\colon K(P_2X, Q_2) \to K(P_2X, Q_2)$ of (4.2). The conditions of Lemma 2.10 are ensured by Lemma 4.3 and Lemma 4.4.

Therefore by Theorem 3.1 with $p=2$, each generator of A'' has dimension 1 corresponding to cohomological dimension 2 and so each generator of $H^*(X, Q)$ has dimension 1. Theorem B now follows from Theorem 4.2.

Proof of Theorem A. The proof is very similar to that of Theorem B, replacing Theorem 3.1 with Theorem 3.3. As in [16], we use a geometric construction of James. More precisely in §3 of [19], using only homotopy commutativity, James constructs a diagram

$$\begin{array}{ccccc} SX & \longrightarrow & P_2X & \longrightarrow & SX \# SX \\ \downarrow l & & \downarrow f & & \downarrow T \\ SX & \longrightarrow & P_2X & \longrightarrow & SX \# SX \end{array} \tag{4.3}$$

in which the squares are homotopy commutative, where l turns the suspension coordinate upside down and T is the switching map for smash product.

The proof of Theorem A now follows, using Theorem 3.3, in a very similar manner to that of Theorem B.

Proof of Theorem C.

Choose a prime p such that $H^*(X, Z)$ has no p-torsion, (this is true for all p, a result of Browder, but we don't need this fact). Then $H^*(X, Q_p)$ is an exterior algebra on generators of dimensions $2n_1-1$, $2n_2-1$, $\cdots$, $2n_m-1$ and $H^*(B_X, Q_p)$ is a polynomial algebra on generators of dimensions $2n_1, 2n_2, \cdots, 2n_m$. Now if B_X is an H-space, a p-map is induced by the H-space p-th power map on B_X (with respect to some fixed order of multiplication) and it follows from Theorem D, since B_X can be taken with finite skeletons, that $n_i=1$, $1\leq i\leq m$. The proof of Theorem C is completed using Theorem 4.2.

Appendix

The purpose of this appendix is to give direct proofs of Lemma 2.5 and Lemma 2.9. We shall need the following result.

Lemma. *There exists a homogeneous base for $N\{x_1, x_2, \cdots, x_l, y_1, y_2, \cdots, y_r\}$ such that*

(i) *$x_1, x_2, \cdots, x_l$ generate the graded ring N,*

(ii) *y_i, $1\leq i\leq r$, is a monomial in the x_i, $1\leq i\leq l$.*

Proof. Let $x_1, x_2, \cdots, x_l$ be a minimal homogeneous generating set for N, then x_i, $1\leq i\leq l$, and the other non-zero monomials in the x_i span N. For each polynomial relation among the x_i in N_n, we can eliminate one monomial from this spanning set, since N is free and Q_p is a local ring. In fact, if $\sum_{i,j\cdots} a_{ij\cdots}x_ix_j\cdots=0$ is any polynomial relation, since N is torsion free we may divide by p^{α}, $\alpha\geq 0$ to ensure that at least one coefficient $a_{ij\dots}$ is a unit in Q_p and hence eliminate the corresponding element $x_i x_j\cdots$. This cannot be an x_i, $1\leq i\leq l$, since these formed a minimal generating set. After a finite number of steps we obtain the required base.

Proof of Lemma 2.5. (p odd)

Choose generators $x_1, x_2, \cdots, x_l$ for the ring N as in the lemma above and let J: $N\to M$ be a ring isomorphism which is also an isomorphism of filtered Q_p-modules. Suppose that there exists an isomorphism K: $N\to M$ which is a ring isomorphism and such that, for each k, $\Phi_K^k(x_i)$, $1\leq i\leq l$, is of the required form modulo $\sum_{j\geq q+s} N_j$ for all $s\leq t$, where x_i has dimension q. This is true for $t=1$ which starts the inductive argument. Let $t=\alpha(p-1)+\beta$, $0\leq\beta<(p-1)$. If $\beta=0$, $\Phi_K^k(x_i)$ is of the required form modulo $\sum_{j>q+s} N_j$. If $\beta\neq 0$, let the component of $\Phi_K^k(x_i)$ in dimension $q+t$ be w_k. Using Lemma 2.3, consider the component of $\Phi^k\Phi^p(x_i)=\Phi^p\Phi^k(x_i)$ in dimension $q+t$ and use the induction hypothesis to obtain

$$k^q(k^t-1)w_p = p^q(p^t-1)w_k\,.$$

Now since t is not divisible by $p-1$, we can find k such that k^t-1 is not divisible by p, e.g. $k=2$. Therefore w_p is divisible by p^q and we set $w_p=p^q(p^t-1)w$. Then $w_k=k^q(k^t-1)w$. Define a new ring isomorphism L: $N\to M$ by setting $L(x_i)=K(x_i+w)$, $L(x_j)=K(x_j)$ otherwise, and extend L to be a ring isomorphism using the base of the lemma above. It is a straightforward matter to see that $\Phi_L^k(x_i)$ has the required form modulo $\sum_{j>q+t} N_j$. We perform this operation simultaneously for each x_i, $1\leq i\leq l$ and so complete the induction step and prove the lemma.

Proof of Lemma 2.8. Let $1, x_1\, x_2, \cdots, x_m$ be generators for the truncated polynomial algebra of height k. Choose $J(x_1), J(x_2), \cdots, J(x_m)$ in M for some isomorphism J: $N\to M$ where if $x_i\in N_{n_i}$ then $J(x_i)\in M_{n_i}$. The set of elements $J(x_1)^{\alpha_1}J(x_2)^{\alpha_2}\cdots J(x_m)^{\alpha_m}$ with $\sum_{1\leq i\leq m}\alpha_i\leq k$ form a base for M over Q_p and we use this base. We need to show that $1, J(x_1), J(x_2), \cdots, J(x_m)$ generate a truncated polynomial algebra of height k. It is sufficient to show that $J(x_1)^{\alpha_1}(J(x_2)^{\alpha_2}\cdots J(x_m)^{\alpha_m}=0$ if $\sum_{1\leq i\leq m}\alpha_i>k$. Suppose that this is not true, and let $w=J(x_1)^{\beta_1}J(x_2)^{\beta_2}\cdots J(x_m)^{\beta_m}$ be non-zero with $\sum_{1\leq i\leq m}\beta_i>k$ and $q=\sum_{1\leq i\leq m}n_i\beta_i$ chosen as large as is possible. It follows from A′3 and A5 that $\Phi^k(w)=k^q w$. But

$$J(x_1)^{\beta_1}J(x_2)^{\beta_2}\cdots J(x_m)^{\beta_m} = J(x_1^{\beta_1}x_2^{\beta_2}\cdots x_m^{\beta_m}) \bmod M_{q+1}$$

and the term on the right-hand side is zero. We obtain a contradiction by once more applying A′3. This completes the proof of the lemma.

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