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Osaka University
AUTOMORPHISMS OF POLYNOMIAL ALGEBRAS AND HOMOTOPY COMMUTATIVITY IN H-SPACES

JOHN R. HUBBUCK

(Received December 27, 1968)

1. In an earlier paper [16], the author proved that a connected, finite complex, which is a homotopy commutative $H$-space, has the homotopy type of a torus. The main purpose of this note is to give a proof of the weaker result, see Theorem B, that a homotopy associative, homotopy commutative $H$-space structure on a connected finite complex, implies that the complex is homotopically equivalent to a torus. As a torus is an Eilenberg-MacLane space, there is just one $H$-space multiplication up to homotopy. The reasons for presenting a second proof are twofold. First the details of this proof are much simpler and second the techniques of the proof can be generalized in a straightforward manner to give other results. As in the earlier proof, considerable use is made of results of W. Browder.

First we recall some definitions. $X$ is an $H$-space if there exists a continuous multiplication $m:X \times X \to X$, with unit. $X$ is a homotopy associative $H$-space if, in addition, $m(m \times 1) = m(1 \times m): X \times X \times X \to X$, and $X$ is a homotopy commutative $H$-space if it is an $H$-space and $m = m T: X \times X \to X$, where $T$ is the switching map for cartesian product. We shall also require a condition that an $H$-space be strictly associative and strongly homotopy commutative, in the sense of Sugawara; Theorem 4.3 of [23] implies that a suitable definition is that the classifying space of $X$, as constructed in [14], $B_X$ say, is itself an $H$-space.

In this paper $X$ will always be a connected, non-contractible, finite complex. When $X$ is an $H$-space, the rational cohomology ring $H^*(X, \mathbb{Q})$ is an exterior algebra on odd dimensional generators, and more, it is a connected, graded Hopf algebra with comultiplication induced by $m$. $H^*(X, \mathbb{Q})$ is said to be primitive if there exist generators $x_i$ for the exterior algebra such that

$$m^*(x_i) = x_i \oplus 1 + 1 \otimes x_i \in H^*(X \times X, \mathbb{Q}) , \quad \text{for each } i.$$ 

The results on homotopy commutative $H$-spaces are Theorems A, B and C.

**Theorem A**. Let $X$ be a homotopy commutative $H$-space with $H^*(X, \mathbb{Q})$
primitive, then $X$ has the homotopy type of a torus.

Theorem B. Let $X$ be a homotopy associative, homotopy commutative $H$-space, then $X$ has the homotopy type of a torus.

Theorem C. Let $X$ be a strictly associative, strongly homotopy commutative $H$-space, then $X$ has the homotopy type of a torus.

Theorem C was first proved by W. Browder by other methods. Clearly Theorem A implies Theorem B, and it can be shown that Theorem B implies Theorem C, see [23], but we shall give three slightly different proofs.

Let $Q_p$ be the ring of rational numbers whose denominators are not divisible by the prime number $p$. Let $Y$ be a complex with finite skeletons, whose cohomology ring with $Q_p$-coefficients is a finitely generated polynomial algebra, possibly truncated at height greater than $p$, with even dimensional generators, $y_i, 1 \leq i \leq n$.

A map $f: Y \to Y$ is a $p$-map if $f^*(y_i) = ky_i$ modulo decomposable elements of the polynomial algebra.

The two results which lead to the proofs of Theorems A, B and C are as follows,

Theorem D. Suppose that $Y$ supports a $p$-map, then each $y_i$ has dimension 2.

Theorem E. $(p = 2)$ Suppose that $Y$ supports a $(-1)$-map, then each $y_i$ has dimension 2.

The proofs of Theorems D and E are in essence extremely simple. We just write down the condition that a $k$-map induces a homomorphism in the complex $K$-theory which commutes with the Adams operators of [3]. The technical result used is a suitable interpretation of the "Integrality Theorem on the Chern Character" of [2]. Theorems D and E will be proved in the framework of a simple axiomatic theory, as in [17], which is developed in §2. In §3 we prove the analogues of Theorems D and E in this framework, and in §4 Theorems A, B and C are proved.

I am grateful to two people in particular in connection with this paper. Theorem A arose out of a discussion of Theorem B with Dr. I.M. James and many of the details of the simple axiomatic scheme of §2 were worked out with the assistance of Professor J.F. Adams.

2. Let $M$ be a finitely generated $Q_p$-module, filtered by submodules

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_{g-1} \supset M_g = 0,$$

which possess linear maps $\psi^k$ such that $\psi^k(M_i) \subset M_i$, for each $i$. 

Let $N = \sum_{0 \leq i \leq k-1} N_i$ be the associated graded module, that is $N_i \cong M_i/M_{i+1}$.

We assume,

A1. $N$ is a free $Q_p$-module.

A2. $\psi^k \psi^l = \psi^l \psi^k$, for all $l$ and $k$.

A3. If $u \in M_n$, then $(\psi^k(u) - k^n u) \in M_{n+1}$, for each $k$.

A3'. If $u \in M_n$, then there exist $n+1$ elements $v_i \in M_{n+k(p-1)}$, $0 \leq i \leq n$, such that $\psi^k(u) = \sum_{0 \leq i \leq n} p^{n-i} v_i$.

A4. $M$ possesses a multiplication which gives it the structure of a commutative filtered ring, that is $M$ is a commutative ring and $M_i \cdot M_j \subseteq M_{i+j}$.

A5. $\psi^k$ is a ring homomorphism for each $k$.

A6. $\psi^p(u) = u^p \mod p$, for each $u \in M$.

We call such an $M$ a multiplicative $\psi^k$-module over $Q_p$. This is consistent with the usage in [17] where $p=2$. A simple induction argument shows that A3 and A3' are equivalent to A3 of [17] in the presence of A1 and A2.

The ring structure on $M$ induces a graded ring structure on $N$ in a natural way. The same symbols are used for $M$ and $N$ considered as $Q_p$-modules or as rings. The application is to take $M = K(Y, Q_p)$, where $Y$ is a finite complex whose integral homology is free of $p$-torsion. $K(Y, Q_p)$ is unitary $K$-theory with $Q_p$-coefficients, defined by taking the tensor product of the integral $K$-theory of $[8]$ with $Q_p$. The spectral sequence of $[8]$ with $Q_p$-coefficients collapses, since $H^*(Y, Q_p)$ is torsion free. Therefore if $ GK(Y) = \sum_{i \geq 0} K_{2i}(Y, Q_p)/K_{2i+1}(Y, Q_p) $ is the graded ring corresponding to the filtered ring $K(Y, Q_p)$ with the $CW$-filtration, then $H^{even}(Y, Q_p) \cong GK(Y)$. Further this isomorphism preserves the usual ring structure which both sides possess. Now set $M_i = K_{2i}(Y, Q_p)$ and let $\psi^k$ be the Adams operators of $[3]$. A1 is true by hypothesis on $Y$ and A2, A3, A5 and A6, as well as the fact $\psi^k$ is a linear map which preserves the $CW$-filtration, are well-known properties of the Adams operators. Under torsion free hypotheses, A3' is equivalent to the $p$-primary part of Theorem 1 of [2]. It follows almost immediately from Proposition 5.6 of [6], (see Lemma 5.5 of [6]). We have proved Lemma 2.1.

**Lemma 2.1.** $M = K(Y, Q_p)$ is a multiplicative $\psi^k$-module over $Q_p$.

We return to the general case. Let $M = M^1 \oplus M^2$ be a direct sum of modules of the type we are considering, where $M^1$ is an ideal in $M$ and $\psi^k(M^1) \subseteq M^1$ for each $k$. The next result is clear.
Lemma 2.2. The quotient module $M^2$ is a multiplicative $\psi^k$-module over $Q_p$.

Let $I_s: M_s \to N_s$ be the quotient map. Consider $N$ filtered by submodules $\sum_{i \geq s} N_i$. Then $N$ and $M$ are isomorphic as filtered $Q_p$-modules, since $N$ is free (c.f. Lemma 2.5 of [17]), and if $J:N \to M$ realizes this isomorphism, the composition $I_sJ$, restricted to $N_s$, is the identity map. For each $k$ and $J$, define homomorphisms $\Phi^*_j : N \to N$ by requiring that the following diagram is commutative,

\[
\begin{array}{ccc}
N & \xrightarrow{\Phi^*_j} & N \\
J & \downarrow & J \\
M & \xrightarrow{\psi^k} & M
\end{array}
\]

Lemma 2.3. $\Phi^*_l \Phi^*_j = \Phi^*_j \Phi^*_l$, for all $l$ and $k$.

Proof. This follows from A2.

Lemma 2.4. Let $x \in N_n$, then $\Phi^*_j (x) = k^n x \mod \sum_{i \geq 0} N_{n+i}$.

Proof. This follows from A3.

It is clear that in general there will not exist $J:N \to M$ which is a ring isomorphism. However, it is true that when $N$ is a truncated polynomial algebra there exists a ring isomorphism $J:N \to M$. There are two approaches. The better is to use a general $K$-theory splitting theorem of J.F. Adams from which the two key lemmas that follow, Lemma 2.5 and Lemma 2.9, are easy consequences. However, this work has not yet been published and so we state Lemma 2.5 and Lemma 2.9 without proof and in an appendix give direct proofs, which were the author's original proofs.

Lemma 2.5. Suppose that there exists $J:N \to M$ which is a ring isomorphism, then there exists a ring isomorphism $K:N \to M$ such that

\[\Phi^*_k (N_n) \subseteq \sum_{i \geq 0} N_{n+i(p-1)},\]

for all $k$ and $n$.

Lemma 2.6. For $K$ as in Lemma 2.5, if $x \in N_n$, then $\Phi^*_k (x) = \sum_{i \leq t \leq s} p^n - t x_i$ for some $t$, where $x_i \in N_{n+i(p-1)}$ and $x_0 = x$.

Proof. This follows from A'3 and A"3.

The next two lemmas also follow easily from the axioms.
Lemma 2.7. If \( J : N \to M \) is a ring isomorphism, \( \Phi^J \) is a ring homomorphism.

Lemma 2.8. If \( J : N \to M \) is a ring isomorphism, \( \Phi^J(x) = x^p \mod p \), for all \( x \in N \).

Now let \( N \) be a truncated polynomial algebra of height \( k \) over \( \mathbb{Q}_p \), that is, \( N \) has a homogeneous multiplicative basis \( 1, x_1, x_2, \ldots, x_m \) of elements of dimensions \( 0 < n_1 \leq n_2 \leq \cdots \leq n_m \), where \( x_1^i x_2^j \cdots x_m^r = 0 \) if and only if \( \sum_{1 \leq i \leq m} i \geq k \), and there are no other relations.

Let \( N \) be such a truncated polynomial algebra.

Lemma 2.9. There exists a ring isomorphism \( J : N \to M \).

A ring homomorphism \( f : M \to M \) is natural if it commutes with the \( \psi^k \). It induces a linear map \( f_J : N \to N \), defined by requiring that the following diagram is commutative,

\[
\begin{array}{ccc}
N & \xrightarrow{f} & N \\
\downarrow{J} & & \downarrow{J} \\
M & \xrightarrow{f} & M
\end{array}
\]

There are four properties of such homomorphisms with which we are concerned, the proofs of Lemmas 2.10, 2.11 and 2.13 are straightforward.

First suppose that we have a direct sum decomposition of \( M \) as in Lemma 2.2, and a natural ring homomorphism \( f : M \to M \) such that \( f(M^i) \subseteq M^i \), then

Lemma 2.10. The natural map \( f : M \to M \) induces a natural map of the quotient module \( f_2^2 : M^2 \to M^2 \).

Lemma 2.11. \( f_J \Phi^J = \Phi^J f_J \), for each \( k \).

Lemma 2.12. Suppose that \( \Phi^k_{\mathcal{K}}(N_n) \subseteq \sum_{i \geq 0} N_{n+i}^{p^{-1}} \), for all \( n \) and \( k \), then \( f_{\mathcal{K}}(N_n) \subseteq \sum_{i \geq 0} N_{n+i}^{p^{-1}} \).

Proof. First we show that \( f_{\mathcal{K}}(N_n) \subseteq \sum_{i \geq 0} N_{n+i} \) arguing by induction in \( N_n \) for decreasing \( n \). If \( N_g = 0 \) and \( x \in N_{g-1} \), let \( f_{\mathcal{K}}(x) = \sum_{0 \leq i \leq g-1} y_{i} \), where \( y_0 \neq 0 \) has dimension \( t \) and the dimension of each \( y_i \) is greater than \( t \) for \( i > 0 \). Consider \( f_{\mathcal{K}}(\Phi_{\mathcal{K}}^k(x)) = \Phi_{\mathcal{K}}^k(f_{\mathcal{K}}(x)) \), by Lemma 2.11, in dimension \( t \). Lemma 2.4 implies that \( k^{g-1}y_0 = k^t y_0 \), which implies that \( t = g - 1 \) as required. Therefore assume the result in \( N_i \) for \( i > n \) and let \( x \in N_n \). As before let \( f_{\mathcal{K}}(x) = \sum_{0 \leq i \leq g} y_i \), where \( y_0 \neq 0 \) has dimension \( t \) and \( y_i, i > 0 \), has dimension greater than \( t \). Therefore once more applying Lemma 2.11 and Lemma 2.4, in dimension \( t \) we have \( k^t y_0 = k^t y_0 \), which
implies \( n = t \) and completes the proof that \( f_\mathcal{K} \) preserves the filtration on \( N \).

Now assume the result of the lemma in \( N_i \), for \( i > n \), since it is certainly true in \( N_{n-1} \). Let \( x \in N_n \) and \( f_\mathcal{K}(x) = \sum_{0 \leq u \leq s} y_u \), where \( y_u \neq 0 \) say, for some \( u \) in \( 0 \leq u \leq s \), is the element of least dimension, \( t \) say, not of the form \( n+i(p-1) \) for some \( i \). Again Lemma 2.11 and Lemma 2.4 imply \( k^sy_s = k^ty_t \) which is impossible, and the lemma is proved.

**Lemma 2.13.** Let \( f: N \rightarrow M \) be a ring isomorphism, then \( f_J \) is a ring homomorphism.

In all the notations introduced above, except \( f_J \), when it is clear which isomorphism \( J: N \rightarrow M \) we are using, the suffix \( J \) will be omitted.

3. Let \( N \) be a truncated polynomial algebra of height greater than \( p \) over \( \mathbb{Q}_p \), with generators \( 1, x_1, x_2, \ldots, x_m \). By setting \( M' = (M)^{p+1} \) in Lemma 2.2, it can be assumed that the height of \( N \) is precisely \( p+1 \). We assume that a choice of generators has been made and so there is a natural base for \( N \) over \( \mathbb{Q}_p \). It is convenient to have an alternative notation for the generators; we write \( y_q \) for a generator where it is assumed that the dimension is \( q \).

A linear map \( f:M \rightarrow M \), which preserves the filtration, induces a map of the associated graded module, \( f_*: N \rightarrow N \). Clearly \( f_* \) and \( f_J \) restricted to \( N_n \) have images which coincide in \( N_n \). We define a natural ring homomorphism to be a \( k \)-map if \( f_\mathcal{K}(x_i) = kx_i \) modulo decomposable elements of \( N \).

Using Lemma 2.9, it is assumed that an isomorphism \( K: N \rightarrow M \) has been chosen as in Lemma 2.5. Note that if \( M \), a truncated polynomial algebra of height greater than \( p+1 \) has a \( k \)-map, then \( M^2 \) as constructed above, a truncated polynomial algebra of height \( p+1 \), has a \( k \)-map, by Lemma 2.10 and Lemma 2.13.

**Theorem 3.1.** Suppose that \( f:M \rightarrow M \) is a \( p \)-map, then each generator of \( N \) has dimension 1.

Proof\(^2\). Let \( y_q \) be a generator of highest possible dimension. We suppose that \( q > 1 \) and obtain a contradiction.

Lemma 2.11 implies that \( f_\mathcal{K}\Phi'(y_q) = \Phi'(f_\mathcal{K}(y_q)) \). It is sufficient to consider this equality modulo \( p^2 \), and in particular the component in dimension \( qp \), the highest non-trivial dimension.

First consider the left-hand side. Lemma 2.6 implies that

\[
\Phi'(y_q) = p^2q_{p-1} + p^2 \mod p^2 \tag{3.1}
\]

2. In fact, it is sufficient that \( f:M \rightarrow M \) is natural and a \( p \)-map modulo \( p^2 \).

3. The proof can be expressed in terms of the homomorphisms \( S_j \) introduced in [17].
where $z_i \in N_i$. Now if $y_i$ is any generator of dimension $t$

$$f_K(y_i) = p y_i + w_i + w_i + \text{higher dimensional elements},$$

where $w_i \in N_i$ and $w_i$ is decomposable.

Now for dimensional reasons $z_{qp}$ is a sum of scalar multiples of products of $p$ generators, and since $f_K$ is a ring homomorphism, Lemma 2.13, $f_K(z_{qp}) = p^q z_{qp}$. Also since $q > 1, pq - (p - 1) > q$, and so $z_{qp - (p - 1)}$ is decomposable. In the expression (3.2), again for dimensional reasons, $w_t$ does not involve $y_q$ (or any other generator of dimension $q$). Thus if $f_K(z_{q - (p - 1)})$ is not independent of $y_q^n$, when we express $z_{q - (p - 1)}$ in terms of the given base, it must involve $y_q^n y_i$, for some generator $y_i$, with $r > 0$. The coefficient of $y_q^n$ in $f_K(y_q^n y_i)$ is zero mod $p'$, since $f_K$ is a ring homomorphism. Thus the coefficient of $y_q^n$ in $f_K \Phi^p(y_q)$ mod $p'$ is zero.

Now consider $\Phi^p(f_K(y_q))$. Let $f_K(y_q) = p y_q + w_q + \text{higher dimensional elements}$, where $w_q$ is decomposable and of dimension $q$. But if $w$ has dimension greater than $q$, $\Phi^p(w) = 0$ mod $p'$, by Lemma 2.6. Also since $\Phi$ is a ring homomorphism, Lemma 2.7, and $w_q$ is decomposable, Lemma 2.8 implies that $\Phi^p(w_q) = 0$ mod $p^2$. Thus again using Lemma 2.8, the coefficient of $y_q^n$ in $\Phi^p(f_K(y_q))$ mod $p^2$ is $p$, which gives the required contradiction.

Thus $m = 1$ and the theorem is proved.

We now restrict attention to the case $p = 2$ and consider a $k$-map with $k = -1$.

**Lemma 3.2.** Let $f: M \to M$ be a $(-1)$-map, then if $u_i = K(x_i)$, $1 \leq i \leq m$, $f(u_i) = -u_i$ modulo decomposable elements of $M$, for all $i$.

**Proof.** It is necessary to show that $f_K(x_i) = -x_i$ modulo decomposable elements of $N$. Neglect all decomposable elements, that is consider $M$ a truncated polynomial algebra of height 2. Suppose that the result is false and let $y_q$ be a generator of highest possible dimension for which it is false and $y_i$ a generator of least possible dimension which occurs in $f_K(y_q) + y_q$. Thus $f_K(y_q) = -y_q + a y_i + \text{elements of equal or higher dimension}$, where $a \neq 0$ and $a = b$. Consider the coefficient of $y_i$ in $\Phi^k(f_K(y_q)) = f_K \Phi^k(y_q)$. If the coefficient of $y_i$ in $\Phi^k(y_q)$ is $b$, on the left-hand side we obtain $-b + ak$ and on the right-hand side $ka - b$, which implies that $q = t$. This is impossible by the definition of $f$.

**Theorem 3.3.** Suppose that $f: M \to M$ is a $(-1)$-map, then each generator of $N$ has dimension 1.

**Proof.** As in the proof of Theorem 3.1, let $y_q$ be a generator of highest possible dimension $q$. We suppose that $q > 1$ and obtain a contradiction.

Consider the equality
Again it is sufficient to consider the components in dimension $2q$ taking the value modulo 4. By Lemma 2.6

\[(3.3) \quad \Phi^2(y_q) = 2z_{2q-1} + z_{2q} \mod 4\]

where $z_t \in N_t$, and if $y_t$ is any generator of dimension $t$,

\[(3.4) \quad f_K(y_t) = -y_t + \text{decomposable elements}\]

by Lemma 3.2. But $f_K(z_{2q}) = (-1)^q z_{2q} = z_{2q}$ and $f_K(z_{2q-1}) = (-1)^q z_{2q-1} = z_{2q-1}$ by (3.4) since the product of three generators vanishes and $z_{2q-1}$ and $z_{2q}$ are decomposable. Thus

\[
f_K\Phi^2(y_q) - \Phi^2(y_q) = 2(f_K(z_{2q-1}) - z_{2q-1}) + f_K(z_{2q}) - z_{2q} \mod 4
\]

is zero, modulo 4.

Now consider $\Phi^2(f_K(y_q)) - \Phi^2(y_q)$. If $w$ is any decomposable element $\Phi^2(w) = 0 \mod 4$, using Lemma 2.7 and Lemma 2.8. Therefore by (3.4),

\[
\Phi^2(f_K(y_q)) - \Phi^2(y_q) = 2\Phi^2(y_q) \mod 4
\]

which equals $2y_q^2$, by Lemma 2.8.

This is the required contradiction, and the theorem is proved.

Finally we remark that Theorem 3.1 and Theorem 3.3 are just two of a number of results which can be proved by these methods. The author hopes to return to these and similar questions at a later date.

4. Let $H^*(Y, \mathbb{Q}_p)$ be free of $p$-torsion. Then $H^*(Y, \mathbb{Z})$ is free of $p$-torsion by the universal coefficient theorem and therefore so is $H_*(Y, \mathbb{Z})$. In particular when $Y$ is a finite complex, we may apply the results of the previous section with $M = K(Y, \mathbb{Q}_p)$, by Lemma 2.1. Thus Theorem D and Theorem E follow from Theorem 3.1 and Theorem 3.3 respectively, at least for finite complexes. If $Y$ is not finite but has finite skeletons, consider a skeleton $Y^t$ of $Y$ of sufficiently high dimension to perform all arguments with $H^*(Y^t, \mathbb{Z})$ free of $p$-torsion and $M = K(Y^t, \mathbb{Q}_p)$. Combining Lemmas 2.2, 2.9, 2.10 and 2.12, it may be assumed that $M$ is a truncated polynomial algebra of height $p+1$ over $\mathbb{Q}_p$, and again Theorems D and E follow from Theorem 3.1 and Theorem 3.3.

Next consider Theorems A and B. The next two theorems are quite basic to the proofs. They arise from the work of W. Browder.

**Theorem 4.1.** Let $X$ be a finite connected complex and a homotopy commutative $H$-space, then $H_*(X, \mathbb{Z})$ is free of $2$-torsion.

**Proof.** This follows from Theorem 8.5 of [11].
Theorem 4.2. Let $X$ be a finite connected complex and an $H$-space and suppose that the exterior algebra $H^*(X, Q)$ is generated by one dimensional generators, then $X$ has the homotopy type of a torus.

Proof. First we show that $X$ is a cohomology torus.

Suppose that $H_*(X, Z)$ has $p$-torsion, then $H^*(X, Z_p)$ is not an exterior algebra on odd dimensional generators, by Theorem 4.9 of [9], the converse of a classical theorem of Borel. But in the notation of [9], the $p$-dimension and the rational dimension of $X$ coincide, by Theorem 7.1 of [9] and hence if $H^m(X, Q)=0$ for $m>n$, then $H^m(X, Z_p)=0$ for $m>n$. This contradicts the fact that $H^*(X, Q)$ is an exterior algebra on generators of dimension one, but $H^*(X, Z_p)$ is a Hopf algebra which is not. We deduce that $H^*(X, Z)$ is a cohomology torus.

Therefore there is a map $f: X \to S^1 \times S^1 \times \cdots \times S^1$ inducing an isomorphism of cohomology, since $S^1 = K(Z, 1)$, and the conclusion of the theorem follows from a simple application of Lemma 6.2 of [10].

The hypotheses of both Theorem A and Theorem B imply that $X$ is a finite connected complex which is a homotopy commutative $H$-space with $H^*(X, Q)$ primitive. We consider the cohomology and the $K$-theory with $Q_2$-coefficients of the projective plane of such an $X$. For basic $K$-theoretic properties, we refer to [5], [8] and [7]. The details of the cohomology of the projective plane which are required follow from the results of [12]. They follow by straightforward calculation and most are given in [17] and so we just summarize them here.

Since $H_*(X, Z)$ is free of $2$–torsion, Theorem 3.1, $H^*(X, Q_2)$ and $K^*(X, Q_2)$ are exterior algebras on odd dimensional generators, that is in $K^*(X, Q_2)$ the generators may be chosen to lie in $K^*(X, Q_2)$ the index in $K^*(X, Q_2)$ being taken mod 2. We have the exact sequence.

$$
\begin{array}{c}
\longrightarrow \tilde{h}^q(X) \xrightarrow{\phi} \tilde{h}^q(X \times X) \xrightarrow{\lambda} \tilde{h}^{q+2}(P_2 X) \xrightarrow{i} \tilde{h}^{q+1}(X) \longrightarrow \\
\end{array}
$$

where $\phi = m^* - \pi^* - \pi^*_2$, $\pi_i: X \times X \to X$, $i=1$ or 2, being projection onto the $i$-th factor; which essentially (3.1) of [12].

Then $h^s(P_2 X) = A \oplus B$, where $A$ is a truncated polynomial algebra of height 3 on generators $u_1, u_2, \cdots, u_m$ such that $v_j = i(u_j)$, $1 \leq j \leq m$, form a multiplicative basis for the exterior algebra $h^*(X)$ and $\lambda(v_i \otimes v_j) = u_i u_j$, and $B$ is an ideal, free of 2–torsion, generated over $Q_2$ by all elements of the form $\lambda(v_1^{\alpha_1} v_2^{\alpha_2} \cdots v_m^{\alpha_m} \otimes v_1^{\beta_1} v_2^{\beta_2} \cdots v_m^{\beta_m})$ where each $\alpha_i$ and $\beta_i$ is zero or one, $\Sigma \alpha_i > 0$, $\Sigma \beta_i > 0$ and $\Sigma (\alpha_i + \beta_i) > 2$.

Let $H^*(P_2 X, Q_2) = A' \oplus B'$ and $K^*(P_2 X, Q_2) = A'' \oplus B''$, then set $M =$
Lemma 4.3. \( \psi^k(B'') \subset B'' \) for each \( k \).

Proof. See Lemma 6.4 of [17].

Proof of Theorem B. Let \( m(2): X \to X \) be the \( H \)-space squaring map. If \( X \) is a homotopy associative, homotopy commutative \( H \)-space, the following diagram is homotopy commutative,

\[
\begin{array}{ccc}
X \times X & \xrightarrow{m} & X \\
\downarrow \quad \downarrow m(2) & & \downarrow m(2) \\
X \times X & \xrightarrow{m(2) \times m(2)} & X
\end{array}
\]

and hence

\[
\begin{array}{ccc}
X \ast X & \xrightarrow{m'} & SX \\
\downarrow h & & \downarrow g \\
X \ast X & \xrightarrow{m'} & SX
\end{array}
\]

is homotopy commutative, where \( X \ast X \) is the join of \( X \) with itself and \( SX \) is the suspension of \( X \). This induces a map of the cofibre sequence of \( P_3X \) into itself, where again the squares are homotopy commutative,

\[
\begin{array}{ccc}
X \ast X & \xrightarrow{m'} & SX \\
\downarrow h & & \downarrow g \\
X \ast X & \xrightarrow{m'} & SX
\end{array}
\]

is homotopy commutative, where \( X \ast X \) is the join of \( X \) with itself and \( SX \) is the suspension of \( X \). This induces a map of the cofibre sequence of \( P_3X \) into itself, where again the squares are homotopy commutative,

\[
\begin{array}{ccc}
X \ast X & \xrightarrow{m'} & SX & \xrightarrow{i} & P_3X & \xrightarrow{j} & S(X \ast X) \\
\downarrow h & & \downarrow g & & \downarrow f & & \downarrow Sh \\
X \ast X & \xrightarrow{m'} & SX & \xrightarrow{i} & P_3X & \xrightarrow{j} & S(X \ast X)
\end{array}
\]

(4.2)

Lemma 4.4. \( f'(B'') \subset B'' \).

Proof. This follows from the definition of \( B'' \) and the homotopy commutativity of the third square in (4.2).

Lemma 4.5. \( A'' \) is a multiplicative \( \psi^* \)-module over \( Q_2 \) and a truncated polynomial algebra of height 3 which supports a 2-map.

Proof. We apply Lemma 2.10 with \( M' = B'' \), \( M'' = A'' \) and \( f: M \to M \) the map \( f': K(P_3X, Q_2) \to K(P_3X, Q_2) \) of (4.2). The conditions of Lemma 2.10 are ensured by Lemma 4.3 and Lemma 4.4.

Therefore by Theorem 3.1 with \( p = 2 \), each generator of \( A'' \) has dimension 1
corresponding to cohomological dimension 2 and so each generator of $H^*(X, \mathbb{Q})$ has dimension 1. Theorem B now follows from Theorem 4.2.

Proof of Theorem A. The proof is very similar to that of Theorem B, replacing Theorem 3.1 with Theorem 3.3. As in [16], we use a geometric construction of James. More precisely in §3 of [19], using only homotopy commutativity, James constructs a diagram

$$
\begin{array}{ccc}
SX & \longrightarrow & P_{\ast}X \\
\downarrow l & & \downarrow f \\
SX & \longrightarrow & P_{\ast}X \\
\end{array}
$$

(4.3)

in which the squares are homotopy commutative, where $l$ turns the suspension coordinate upside down and $T$ is the switching map for smash product.

The proof of Theorem A now follows, using Theorem 3.3, in a very similar manner to that of Theorem B.

Proof of Theorem C.

Choose a prime $p$ such that $H^*(X, \mathbb{Z})$ has no $p$-torsion, (this is true for all $p$, a result of Browder, but we don’t need this fact). Then $H^*(X, \mathbb{Q}_p)$ is an exterior algebra on generators of dimensions $2n_1 - 1, 2n_2 - 1, \ldots, 2n_m - 1$ and $H^*(B_X, \mathbb{Q}_p)$ is a polynomial algebra on generators of dimensions $2n_1, 2n_2, \ldots, 2n_m$. Now if $B_X$ is an $H$-space, a $p$-map is induced by the $H$-space $p$-th power map on $B_X$ (with respect to some fixed order of multiplication) and it follows from Theorem D, since $B_X$ can be taken with finite skeletons, that $n_i = 1, 1 \leq i \leq m$. The proof of Theorem C is completed using Theorem 4.2.

Appendix

The purpose of this appendix is to give direct proofs of Lemma 2.5 and Lemma 2.9. We shall need the following result.

**Lemma.** There exists a homogeneous base for $N\{x_1, x_2, \ldots, x_l, y_1, y_2, \ldots, y_r\}$ such that

(i) $x_1, x_2, \ldots, x_l$ generate the graded ring $N$,

(ii) $y_i, 1 \leq i \leq r$, is a monomial in the $x_i, 1 \leq i \leq l$.

Proof. Let $x_1, x_2, \ldots, x_l$ be a minimal homogeneous generating set for $N$, then $x_i, 1 \leq i \leq l$, and the other non-zero monomials in the $x_i$ span $N$. For each polynomial relation among the $x_i$ in $N_n$, we can eliminate one monomial from this spanning set, since $N$ is free and $\mathbb{Q}_p$ is a local ring. In fact, if $\sum_{i,j} a_{ij} x_i x_j = 0$ is any polynomial relation, since $N$ is torsion free we may
divide by $p^a$, $a \geq 0$ to ensure that at least one coefficient $a_{ij}$... is a unit in $Q_p$ and hence eliminate the corresponding element $x_i x_j \cdots$. This cannot be an $x_i$, $1 \leq i \leq l$, since these formed a minimal generating set. After a finite number of steps we obtain the required base.

Proof of Lemma 2.5. ($p$ odd)

Choose generators $x_1, x_2, \ldots, x_l$ for the ring $N$ as in the lemma above and let $f: N \to M$ be a ring isomorphism which is also an isomorphism of filtered $Q_p$-modules. Suppose that there exists an isomorphism $K: N \to M$ which is a ring isomorphism and such that, for each $k$, $\Phi_k^i(x_j)$, $1 \leq i \leq l$, is of the required form modulo $\sum N_j$ for all $s \leq t$, where $x_i$ has dimension $q_k$. This is true for $t=1$ which starts the inductive argument. Let $t = a(p-1) + \beta$, $0 \leq \beta < (p-1)$. If $\beta = 0$, $\Phi_k^i(x_j)$ is of the required form modulo $\sum N_j$. If $\beta \neq 0$, let the component of $\Phi_k^i(x_j)$ in dimension $q+t$ be $w_k$. Using Lemma 2.3, consider the component of $\Phi_k^i \Phi_k^j(x_i) = \Phi_k^i \Phi_k^j(x_j)$ in dimension $q+t$ and use the induction hypothesis to obtain

$$k^q (k^t - 1) w_k = p^q (p^t - 1) w_k.$$ 

Now since $t$ is not divisible by $p-1$, we can find $k$ such that $k^t - 1$ is not divisible by $p$, e.g. $k=2$. Therefore $w_k$ is divisible by $p^q$ and we set $w_k = p^q (p^t - 1) w_k$. Define a new ring isomorphism $L: N \to M$ by setting $L(x_i) = K(x_i + w_k)$, $L(x_j) = K(x_j)$ otherwise, and extend $L$ to be a ring isomorphism using the base of the lemma above. It is a straightforward matter to see that $\Phi_L^i(x_j)$ has the required form modulo $\sum N_j$. We perform this operation simultaneously for each $x_i$, $1 \leq i \leq l$ and so complete the induction step and prove the lemma.

Proof of Lemma 2.8. Let $1, x_1, x_2, \ldots, x_m$ be generators for the truncated polynomial algebra of height $k$. Choose $J(x_1), J(x_2), \ldots, J(x_m)$ in $M$ for some isomorphism $f: N \to M$ where if $x_i \in N_{ni}$ then $J(x_i) \in M_{n_i}$. The set of elements $J(x_1)^{\alpha_1} J(x_2)^{\alpha_2} \cdots J(x_m)^{\alpha_m}$ with $\sum \alpha_i \leq k$ form a base for $M$ over $Q_p$ and we use this base. We need to show that $1, J(x_1), J(x_2), \ldots, J(x_m)$ generate a truncated polynomial algebra of height $k$. It is sufficient to show that $J(x_1)^{\alpha_1} J(x_2)^{\alpha_2} \cdots J(x_m)^{\alpha_m} = 0$ if $\sum \alpha_i > k$. Suppose that this is not true, and let $w = J(x_1)^{\beta_1} J(x_2)^{\beta_2} \cdots J(x_m)^{\beta_m}$ be non-zero with $\sum \beta_i > k$ and $q = \sum n_i \beta_i$ chosen as large as is possible. It follows from A3 and A5 that $\Phi^i(w) = k^q w$. But

$$J(x_1)^{\beta_1} J(x_2)^{\beta_2} \cdots J(x_m)^{\beta_m} = J(x_1^{\beta_1} x_2^{\beta_2} \cdots x_m^{\beta_m}) \mod M_{q+1}$$

and the term on the right-hand side is zero. We obtain a contradiction by
once more applying \( A'3 \). This completes the proof of the lemma.

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**References**


