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ON THE DISTRIBUTION OF *k*-TH POWER FREE INTEGERS

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Abstract

Let $X^{(k)}(n)$ be the indicator function of the set of k-th power free integers. In this paper, we study refinements of the density theorem $S_N^{(k)}(m) := (1/N)\sum_{n=1}^{N} X^{(k)}(m+n) \to 1/\zeta(k)$, ζ being the Riemann zeta function. The following is one of our results;

$$\lim_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M}\left(N\left(S_N^{(k)}(m)-\frac{1}{\zeta(k)}\right)\right)^2 \asymp N^{1/k}.$$

The method we take here is a compactification of $\mathbb{Z}$; we extend $S_N^{(k)}$ to a random variable on a probability space $(\hat{\mathbb{Z}}, \lambda)$ in a natural way, where $\hat{\mathbb{Z}}$ is the ring of finite integral adeles and λ is the shift invariant normalized Haar measure. Then we investigate the rate of L^2-convergence of $S_N^{(k)}$, from which the above asymptotic result is derived.

1. Introduction

For $k \in \{2, 3, \dots\}$, let $X^{(k)}(n)$, $n \in \mathbb{Z}$, be the indicator function of the set of k-th power free integers, i.e.,

$$X^{(k)}(n) := \begin{cases} 1, & (\forall p\colon \text{prime},\ p^k \nmid n), \\ 0, & (\exists p\colon \text{prime},\ p^k \mid n), \end{cases}$$

and let $S_N^{(k)}(m)$, $m \in \mathbb{Z}$, denote the frequency of k-th power free integers between $m+1$ and $m+N$, i.e.,

$$S_N^{(k)}(m) := \frac{1}{N}\sum_{n=1}^{N} X^{(k)}(m+n).$$

Then it is well known that for each $m \in \mathbb{Z}$,

$$\lim_{N\to\infty} S_N^{(k)}(m) = \frac{1}{\zeta(k)}, \tag{1}$$

where ζ is the Riemann zeta function (cf. [4]).

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Many researchers have been interested in estimating the error $S_N^{(k)}(m) - 1/\zeta(k)$. Under the Riemann hypothesis, there is a conjecture about this;

$$\forall \varepsilon > 0, \quad N\left(S_N^{(k)}(m) - \frac{1}{\zeta(k)}\right) = O(N^{1/2k+\varepsilon}), \quad N \to \infty. \tag{2}$$

As is mentioned in [8], this conjecture should hold, but it is quite unlikely that it will be proved in near future, because it is related to the Riemann hypothesis so closely. In particular, in the case of $k = 2$, there have been many challenges to this conjecture, assuming the Riemann hypothesis, such as [1, 2, 3, 5, 7]. Refer to [8] for an overview of this topic.

In this paper, we study the probabilistic aspects of this problem. We take here a compactification method which has been developed by [9, 10]. Let us give an overview of this paper.

In Section 2, the ring of finite integral adeles $\hat{\mathbb{Z}}$, which is a well-known compactification of $\mathbb{Z}$ in number theory, as well as some related basic notions, is introduced. Since $\hat{\mathbb{Z}}$ is a compact metric group with respect to addition, there exists a unique normalized Haar measure λ defined on the Borel field $\mathcal{B}$ of $\hat{\mathbb{Z}}$. In Section 3, it is noted that the mapping $x \mapsto x + 1$ is a λ-preserving ergodic shift on the probability space $(\hat{\mathbb{Z}}, \mathcal{B}, \lambda)$. By this fact, since we can extend the functions $X^{(k)}(n)$ and $S_N^{(k)}(n)$ on $\mathbb{Z}$ to $L^1(\hat{\mathbb{Z}}, \mathcal{B}, \lambda)$-functions in a natural way, we get the following law of large numbers

$$\lim_{N\to\infty} S_N^{(k)}(x) = \mathbf{E}[X^{(k)}] = \frac{1}{\zeta(k)}, \quad \lambda\text{-a.e.} \quad x \in \hat{\mathbb{Z}}, \tag{3}$$

which is the adelic version of (1).

The main aim of this paper is to study the convergence rate of the law of large numbers (3). With the help of the explicit formula for the random variable $S_N^{(k)}$ given in Section 4, we can estimate the rate of convergence in Section 5 as follows;

$$\mathbf{E}\left[\left(N\left(S_N^{(k)} - \frac{1}{\zeta(k)}\right)\right)^2\right] \asymp N^{1/k}.$$

Finally, in Section 6, the last estimate is translated into the language of $\mathbb{Z}$ as

$$\lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} \left(N\left(S_N^{(k)}(m) - \frac{1}{\zeta(k)}\right)\right)^2 \asymp N^{1/k} \quad \text{(Corollary 6.3)}.$$

This may be called as a mean square version of the conjecture (2). It should be noted that we do not need the Riemann hypothesis to prove this and nevertheless get the same exponent as in the conjecture.

2. Basic notions

This section deals with some basic notions and some known results needed for this paper. For proof of lemmas, see [10].

For a prime p, the p-adic metric d_p is defined by

$$d_p(x, y) := \inf\{p^{-l};\ p^l \mid (x - y)\}, \quad x, y \in \mathbb{Z}.$$

The completion of $\mathbb{Z}$ by d_p is denoted by $\mathbb{Z}_p$. By extending the algebraic operations '$+$' and '$\times$' in $\mathbb{Z}$ continuously to those in $\mathbb{Z}_p$, the compact metric space $(\mathbb{Z}_p, d_p)$ becomes a ring, called the ring of p-adic integers. In particular, $(\mathbb{Z}_p, d_p)$ is a compact abelian group with respect to '$+$'. According to the general theory of compact groups, there is a unique normalized Haar measure λ_p with respect to '$+$' on the measurable space $(\mathbb{Z}_p, \mathcal{B}(\mathbb{Z}_p))$, where $\mathcal{B}(\mathbb{Z}_p)$ denotes the Borel field of $\mathbb{Z}_p$.

DEFINITION 2.1. (i) Let $\{p_i\}_{i=1}^{\infty}$, $2 = p_1 < p_2 < \cdots$, be the sequence of all primes.
(ii) Put

$$\hat{\mathbb{Z}} := \prod_{i=1}^{\infty} \mathbb{Z}_{p_i}, \quad \lambda := \prod_{i=1}^{\infty} \lambda_{p_i}.$$

For $x = (x_i)$, $y = (y_i) \in \hat{\mathbb{Z}}$, we define

$$d(x, y) := \sum_{i=1}^{\infty} \frac{1}{2^i} d_{p_i}(x_i, y_i), \quad x + y := (x_i + y_i), \quad xy := (x_i y_i).$$

By these definitions, $\hat{\mathbb{Z}}$ becomes a ring, called *the ring of finite integral adeles*. $(\hat{\mathbb{Z}}, d)$ is again a compact metric space, and both '$+$' and '$\times$' are continuous. In particular, $(\hat{\mathbb{Z}}, d)$ is a compact abelian group with respect to '$+$' and its normalized Haar measure on the Borel field $\mathcal{B}$ is nothing but λ.

DEFINITION 2.2. (i) We identify $\mathbb{Z}$ with the diagonal set $\{(n, n, \dots) \in \mathbb{Z} \times \mathbb{Z} \times \cdots\} \subset \hat{\mathbb{Z}}$.
(ii) For $\mathbb{N} \ni m \geq 2$ and $l \in \{0, 1, \dots, m-1\}$, we define $m\hat{\mathbb{Z}} + l := \{mx + l;\ x \in \hat{\mathbb{Z}}\}$. Then we have $\hat{\mathbb{Z}} = \bigcup_{l=0}^{m-1}(m\hat{\mathbb{Z}} + l)$, which is a disjoint union (Lemma 2.5 (iii)). So, for $x \in \hat{\mathbb{Z}}$ and $\mathbb{N} \ni m \geq 2$, there exists a unique $l \in \{0, 1, \dots, m-1\}$ such that $x - l \in m\hat{\mathbb{Z}}$. This l is denoted by $x \bmod m$. For $m = 1$, we always set $x \bmod m := 0$. Obviously, if $x \in \mathbb{Z}$, this definition coincides with the usual modulo operation.
(iii) For $x, y \in \hat{\mathbb{Z}}$, we define the greatest common divisor of x and y by

$$\gcd(x, y) := \sup\{m \in \mathbb{N};\ (x \bmod m) = (y \bmod m) = 0\}.$$

Obviously, for $x, y \in \mathbb{Z}$, this definition coincides with the usual gcd.

Lemma 2.3. $\mathbb{N}' := \{(n, n, \dots) \in \hat{\mathbb{Z}}; n \in \mathbb{N}\}$ *is dense in* $\hat{\mathbb{Z}}$.

Lemma 2.4. (i) *Let* p *be a prime and* $j \in \mathbb{N}$. *Then* $p^j \mathbb{Z}_p$ *is closed and open.*
(ii) *Let* p, q *be distinct primes and* $j \in \mathbb{N}$. *Then we have* $p^j \mathbb{Z}_q = \mathbb{Z}_q$.

Lemma 2.5. *Let* $m \in \mathbb{N}$ *and* $l \in \{0, 1, \dots, m-1\}$.
(i) *The set* $(m\hat{\mathbb{Z}} + l)$ *is closed and open.*
(ii) $\rho_m : \hat{\mathbb{Z}} \to \{0, 1\}$ *is continuous, where* $\rho_m(x) = \begin{cases} 1 & \text{if} \quad x \bmod m = 0, \\ 0 & \text{otherwise.} \end{cases}$
(iii) $\hat{\mathbb{Z}} = \bigcup_{l=0}^{m-1} (m\hat{\mathbb{Z}} + l)$, *which is a disjoint union.*

Corollary 2.6. *For any* $l \in \mathbb{Z}$, *the mapping*

$$\hat{\mathbb{Z}} \ni x \mapsto \frac{(l + x) \bmod m}{m} \in [0, 1)$$

is continuous.

Lemma 2.7. *For any* $l \in \mathbb{Z} \setminus \{0\}$ *and any* $A \in \mathcal{B}$, *we have* $lA \in \mathcal{B}$ *and*

$$\lambda(lA) = \frac{1}{|l|} \lambda(A).$$

Lemma 2.8. *If* $f : \hat{\mathbb{Z}} \to \mathbb{C}$ *is continuous, then*

$$\int_{\hat{\mathbb{Z}}} f(x) \lambda(dx) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} f(n), \quad \forall n_0 \in \mathbb{Z}.$$

The convergence is uniform in $n_0 \in \mathbb{Z}$.

3. The law of large numbers

In what follows, we fix an integer $k \geq 2$. Let $X^{(k)} : (\hat{\mathbb{Z}}, \mathcal{B}, \lambda) \to \{0, 1\}$ be a natural extension of $X^{(k)}(n)$ defined by

$$X^{(k)}(x) = \prod_p (1 - \rho_{p^k}(x)).$$

If we put

$$B^{(k)} := \bigcap_p (\hat{\mathbb{Z}} \setminus p^k \hat{\mathbb{Z}}) \subset \hat{\mathbb{Z}},$$

then it is clear that $X^{(k)} = \mathbf{1}_{B^{(k)}}$, and thus,

$$\mathbf{E}[X^{(k)}] = \lambda(B^{(k)}) = \prod_p \left(1 - \frac{1}{p^k}\right) = \frac{1}{\zeta(k)}.$$

Next, we consider a shift θ

$$\theta : (\hat{\mathbb{Z}}, \mathcal{B}, \lambda) \to (\hat{\mathbb{Z}}, \mathcal{B}, \lambda),$$
$$x \mapsto x + 1.$$

Recall that $\{\theta^n(1)\}_{n=0,1,2,\ldots}$ $(= \mathbb{N}')$ is dense in $\hat{\mathbb{Z}}$ (Lemma 2.3). Therefore, according to [11, Theorem 1.9], the shift θ is ergodic. Now applying the ergodic theorem for $X^{(k)}$ yields

$$S_N^{(k)}(x) := \frac{1}{N}\sum_{n=1}^{N} X^{(k)}(x+n) = \frac{1}{N}\sum_{n=1}^{N} X^{(k)}(\theta^n x) \xrightarrow{N\to\infty} \mathbf{E}[X^{(k)}] = \frac{1}{\zeta(k)} \quad (\lambda\text{-a.e. } x \in \hat{\mathbb{Z}}),$$

which is the adelic version of (1).

4. Explicit formula for $S_N^{(k)}$

For each $L \in \mathbb{N}$, let

$$X_L^{(k)}(x) := \prod_{p \le p_L} (1 - \rho_{p^k}(x)),$$
$$S_{N,L}^{(k)}(x) := \frac{1}{N}\sum_{n=1}^{N} X_L^{(k)}(x+n),$$
$$\mathbb{M}_L := \{u = p_1^{\alpha_1}\cdots p_L^{\alpha_L} \in \mathbb{N}\,;\, 0 \le \alpha_1, \ldots, \alpha_L \le L\}.$$

REMARK 1. From now on, if there is no confusion, we will omit $^{(k)}$ in formulas. For example, X will be considered as $X^{(k)}$ and so on.

Lemma 4.1. *For each $N \in \mathbb{N}$,*

(4) $$S_{N,L}(x) \xrightarrow{L\to\infty} S_N(x) \quad (\textit{pointwise convergence}),$$

(5) $$S_{N,L}(x) = \sum_{u \in \mathbb{M}_L} \mu(u)\left(\frac{1}{u^k} - \frac{1}{N}\left(\frac{(N+x) \bmod u^k}{u^k} - \frac{x \bmod u^k}{u^k}\right)\right),$$

where μ denotes the Möbius function.

Proof. The convergence (4) is obvious. We now prove (5). The definition of $S_{N,L}(x)$ gives

$$\begin{aligned} S_{N,L}(x) &= \frac{1}{N}\sum_{n=1}^{N}\prod_{p\le p_L}(1-\rho_{p^k}(x+n)) \\ &= \frac{1}{N}\sum_{n=1}^{N}\left(1+\sum_{r=1}^{L}\sum_{1\le i_1<\cdots<i_r\le L}(-1)^r\rho_{p_{i_1}^k}(x+n)\cdots\rho_{p_{i_r}^k}(x+n)\right) \\ &= \frac{1}{N}\sum_{n=1}^{N}\left(1+\sum_{r=1}^{L}\sum_{1\le i_1<\cdots<i_r\le L}(-1)^r\rho_{p_{i_1}^k\cdots p_{i_r}^k}(x+n)\right) \qquad (6) \\ &= \frac{1}{N}\sum_{n=1}^{N}\sum_{u\mid p_1\cdots p_L}\mu(u)\rho_{u^k}(x+n) \\ &= \sum_{u\in\mathbb{M}_L}\mu(u)\left(\frac{1}{N}\sum_{n=1}^{N}\rho_{u^k}(x+n)\right). \end{aligned}$$

Here we have

$$\begin{aligned} \frac{1}{N}\sum_{n=1}^{N}\rho_{u^k}(x+n) &= \frac{1}{N}\left\lfloor\frac{N+x \bmod u^k}{u^k}\right\rfloor \\ &= \frac{1}{N}\left(\frac{N+x \bmod u^k}{u^k}-\frac{(N+x) \bmod u^k}{u^k}\right) \qquad (7) \\ &= \frac{1}{u^k}-\frac{1}{N}\left(\frac{(N+x) \bmod u^k}{u^k}-\frac{x \bmod u^k}{u^k}\right). \end{aligned}$$

Therefore, substituting (7) into (6), we obtain (5). The lemma is proved. □

The following lemma is a key in this paper.

Lemma 4.2 (cf. [10, Lemma 8]). *For $u, v \in \mathbb{N}$ and $y, z \in \hat{\mathbb{Z}}$, we have*

$$\begin{aligned} &\mathbf{E}\left[\left(\frac{(y+x) \bmod u}{u}-\frac{x \bmod u}{u}\right)\left(\frac{(z+x) \bmod v}{v}-\frac{x \bmod v}{v}\right)\right] \\ &= \frac{(y \bmod (u,v))\wedge(z \bmod (u,v))}{\{u,v\}}\left(1-\frac{(y \bmod (u,v))\vee(z \bmod (u,v))}{(u,v)}\right), \end{aligned}$$

where the expectation $\mathbf{E}$ *works on* x, *and*

$$\begin{aligned} &(u,v) = \gcd(u,v), \\ &\{u,v\} = \operatorname{lcm}(u,v) = \textit{the least common multiple of } u \textit{ and } v. \end{aligned}$$

Proof. We divide the proof into four steps.

STEP 1. For $a, b, c \in \mathbb{N}$ with $(b, c) = 1$ and for $x \in \mathbb{Z}$, it holds that

$$\frac{1}{b}\sum_{s=0}^{b-1}\frac{(x + sac) \bmod ab}{ab} = \frac{x \bmod a}{ab} + \frac{b-1}{2b}. \tag{8}$$

This is shown in the following way. Since $(b, c) = 1$, by a similar argument of [4, Theorem 56], we have

$$\begin{aligned}&\{(x + sac) \bmod ab;\, s = 0, 1, \dots, b-1\}\\ &= \{(x + sa) \bmod ab;\, s = 0, 1, \dots, b-1\}.\end{aligned}$$

Thus, it is enough to prove (8) only for $c = 1$. Moreover, we have

$$\begin{aligned}&\{(x + sa) \bmod ab;\, s = 0, 1, \dots, b-1\}\\ &= \{(x + a + sa) \bmod ab;\, s = 0, 1, \dots, b-1\},\end{aligned}$$

so that we have only to prove (8) for $x = 0, 1, \dots, a-1$. But then, for $s = 0, 1, \dots, b-1$, we have $(x + sa) \bmod ab = x + sa$, consequently,

$$\frac{1}{b}\sum_{s=0}^{b-1}\frac{(x + sa) \bmod ab}{ab} = \frac{1}{b}\sum_{s=0}^{b-1}\frac{x + sa}{ab} = \frac{x}{ab} + \frac{b-1}{2b}.$$

Thus (8) is valid.

STEP 2. By the fact that for $z \in \hat{\mathbb{Z}}$,

$$\begin{aligned}&(z + sac) \bmod ab = (z \bmod ab + sac) \bmod ab,\\ &(z \bmod ab) \bmod a = z \bmod a,\end{aligned}$$

and by Step 1, it is easy to see that for $a, b, c \in \mathbb{N}$ with $(b, c) = 1$ and $x, y \in \hat{\mathbb{Z}}$,

$$\begin{aligned}&\frac{1}{b}\sum_{s=0}^{b-1}\left(\frac{(y + x + sac) \bmod ab}{ab} - \frac{(x + sac) \bmod ab}{ab}\right)\\ &= \frac{1}{b}\left(\frac{(y + x) \bmod a}{a} - \frac{x \bmod a}{a}\right).\end{aligned}$$

Therefore, for any periodic function $f : \hat{\mathbb{Z}} \to \mathbb{R}$ with period ac, we have

$$\begin{aligned}&\mathbf{E}\left[\left(\frac{(y + x) \bmod ab}{ab} - \frac{x \bmod ab}{ab}\right)f(x)\right] \quad \text{(by the shift invariance of } \lambda)\\ &= \frac{1}{b}\sum_{s=0}^{b-1}\mathbf{E}\left[\left(\frac{(y + x + sac) \bmod ab}{ab} - \frac{(x + sac) \bmod ab}{ab}\right)f(x + sac)\right]\end{aligned}$$

$$= \mathbf{E}\left[\frac{1}{b}\sum_{s=0}^{b-1}\left(\frac{(y+x+sac) \bmod ab}{ab} - \frac{(x+sac) \bmod ab}{ab}\right)f(x)\right]$$
$$= \frac{1}{b}\mathbf{E}\left[\left(\frac{(y+x) \bmod a}{a} - \frac{x \bmod a}{a}\right)f(x)\right].$$

STEP 3. Set $a := (u, v)$, $b := u/a$, $c := v/a$ and f to be

$$f(x) := \frac{(z+x) \bmod v}{v} - \frac{x \bmod v}{v}.$$

Then Step 2 implies that

$$\mathbf{E}\left[\left(\frac{(y+x) \bmod u}{u} - \frac{x \bmod u}{u}\right)\left(\frac{(z+x) \bmod v}{v} - \frac{x \bmod v}{v}\right)\right]$$
$$= \mathbf{E}\left[\left(\frac{(y+x) \bmod ab}{ab} - \frac{x \bmod ab}{ab}\right)\left(\frac{(z+x) \bmod ac}{ac} - \frac{x \bmod ac}{ac}\right)\right]$$
$$= \frac{1}{b}\mathbf{E}\left[\left(\frac{(y+x) \bmod a}{a} - \frac{x \bmod a}{a}\right)\left(\frac{(z+x) \bmod ac}{ac} - \frac{x \bmod ac}{ac}\right)\right].$$

By letting y, b, c and $f(x)$ in Step 2 be $z, c, 1$ and

$$\frac{(y+x) \bmod a}{a} - \frac{x \bmod a}{a},$$

respectively, we see that the last line above is equal to

$$\frac{1}{bc}\mathbf{E}\left[\left(\frac{(y+x) \bmod a}{a} - \frac{x \bmod a}{a}\right)\left(\frac{(z+x) \bmod a}{a} - \frac{x \bmod a}{a}\right)\right]. \tag{9}$$

STEP 4. Without loss of generality, we assume that $y \bmod a \le z \bmod a$. By Corollary 2.6, the integrand of (9) is continuous, and it is periodic with period a. Therefore, Lemma 2.8 implies that

$$(9) = \frac{1}{bc}\frac{1}{a}\sum_{s=0}^{a-1}\left(\frac{(y+s) \bmod a}{a} - \frac{s \bmod a}{a}\right)\left(\frac{(z+s) \bmod a}{a} - \frac{s \bmod a}{a}\right). \tag{10}$$

Moreover, it is clear that

$$\frac{(y+s) \bmod a}{a} - \frac{s \bmod a}{a} = \begin{cases} \dfrac{y \bmod a}{a} & \text{if} \quad 0 \le s < a - y \bmod a, \\ \dfrac{y \bmod a}{a} - 1 & \text{if} \quad a - y \bmod a \le s < a, \end{cases}$$

and that

$$\frac{(z+s) \bmod a}{a} - \frac{s \bmod a}{a} = \begin{cases} \dfrac{z \bmod a}{a} & \text{if} \quad 0 \le s < a - z \bmod a, \\ \dfrac{z \bmod a}{a} - 1 & \text{if} \quad a - z \bmod a \le s < a. \end{cases}$$

Finally, dividing the sum (10) into three parts and using the above expressions, we arrive at

$$\begin{aligned}
(10) &= \frac{1}{bc}\frac{1}{a}\Bigg(\sum_{0 \le s < a - z \bmod a} \frac{y \bmod a}{a}\frac{z \bmod a}{a} \\
&\qquad + \sum_{a - z \bmod a \le s < a - y \bmod a} \frac{y \bmod a}{a}\left(\frac{z \bmod a}{a} - 1\right) \\
&\qquad + \sum_{a - y \bmod a \le s < a} \left(\frac{y \bmod a}{a} - 1\right)\left(\frac{z \bmod a}{a} - 1\right)\Bigg) \\
&= \frac{1}{bc}\frac{1}{a}\Bigg(\frac{y \bmod a}{a}\frac{z \bmod a}{a}(a - z \bmod a) \\
&\qquad + \frac{y \bmod a}{a}\left(\frac{z \bmod a}{a} - 1\right)(z \bmod a - y \bmod a) \\
&\qquad + \left(\frac{y \bmod a}{a} - 1\right)\left(\frac{z \bmod a}{a} - 1\right)(y \bmod a)\Bigg) \\
&= \frac{1}{bc}\frac{1}{a}(y \bmod a)\left(1 - \frac{z \bmod a}{a}\right) \\
&= \frac{y \bmod (u, v)}{\{u, v\}}\left(1 - \frac{z \bmod (u, v)}{(u, v)}\right) \\
&= \frac{(y \bmod (u, v)) \wedge (z \bmod (u, v))}{\{u, v\}}\left(1 - \frac{(y \bmod (u, v)) \vee (z \bmod (u, v))}{(u, v)}\right).
\end{aligned}$$

The lemma is proved. □

A small modification of [10, Lemma 9] gives the following.

Lemma 4.3. *For any bounded function $H\colon \mathbb{N} \to \mathbb{R}$, it holds that*

$$\sum_{u,v=1}^{\infty} \frac{|\mu(u)\mu(v)|}{\{u, v\}^k} |H((u, v))| = \sum_{n=1}^{\infty} \frac{|\mu(n)||H(n)|}{n^k} \prod_{p \nmid n}\left(1 + \frac{2}{p^k}\right) < \infty,$$

$$\sum_{u,v=1}^{\infty} \frac{\mu(u)\mu(v)}{\{u, v\}^k} H((u, v)) = \sum_{n=1}^{\infty} \frac{|\mu(n)|H(n)}{n^k} \prod_{p \nmid n}\left(1 - \frac{2}{p^k}\right).$$

Lemma 4.4. *For each* $N \in \mathbb{N}$,

$$\sum_{u=1}^{\infty} \mu(u)\Big(\frac{(N+x) \bmod u^k}{u^k} - \frac{x \bmod u^k}{u^k}\Big) =: T(x, N) \tag{11}$$

is convergent in $L^2(\hat{\mathbb{Z}}, \mathcal{B}, \lambda)$.

Proof. Fix an $N \in \mathbb{N}$. For finite sets $\mathbb{L}$ and $\mathbb{M}$ such that $\mathbb{L} \subset \mathbb{M} \subset \mathbb{N}$, Lemma 4.2 and Lemma 4.3 imply that

$$\begin{aligned}
&\mathbf{E}\Big[\Big(\sum_{u\in\mathbb{M}} \mu(u)\Big(\frac{(N+x) \bmod u^k}{u^k} - \frac{x \bmod u^k}{u^k}\Big) \\
&\qquad - \sum_{u\in\mathbb{L}} \mu(u)\Big(\frac{(N+x) \bmod u^k}{u^k} - \frac{x \bmod u^k}{u^k}\Big)\Big)^2\Big] \\
&= \sum_{u,v\in\mathbb{M}\setminus\mathbb{L}} \mu(u)\mu(v)\mathbf{E}\Big[\Big(\frac{(N+x) \bmod u^k}{u^k} - \frac{x \bmod u^k}{u^k}\Big) \\
&\qquad\qquad \times \Big(\frac{(N+x) \bmod v^k}{v^k} - \frac{x \bmod v^k}{v^k}\Big)\Big] \\
&= \sum_{u,v\in\mathbb{M}\setminus\mathbb{L}} \mu(u)\mu(v)\frac{N \bmod (u,v)^k}{\{u,v\}^k}\Big(1 - \frac{N \bmod (u,v)^k}{(u,v)^k}\Big) \\
&\le N \sum_{u,v\in\mathbb{M}\setminus\mathbb{L}} \frac{|\mu(u)\mu(v)|}{\{u,v\}^k} \\
&\le N \sum_{u,v\in\mathbb{N}\setminus\mathbb{L}} \frac{|\mu(u)\mu(v)|}{\{u,v\}^k} \to 0 \quad \text{as} \quad \mathbb{L} \nearrow \mathbb{N}.
\end{aligned}$$

The lemma is proved. □

By letting $\mathbb{M} \nearrow \mathbb{N}$ in the proof of Lemma 4.4, and then $\mathbb{L} \nearrow \mathbb{N}$, it follows that

$$\sum_{u\in\mathbb{L}} \mu(u)\Big(\frac{(N+x) \bmod u^k}{u^k} - \frac{x \bmod u^k}{u^k}\Big) \xrightarrow{L^2} T(x, N) \quad \text{as} \quad \mathbb{L} \nearrow \mathbb{N}. \tag{12}$$

On the other hand,

$$\sum_{u=1}^{\infty} \frac{\mu(u)}{u^k} = \frac{1}{\zeta(k)} \quad \text{(absolute convergence)}, \tag{13}$$

and $S_{N,L} \xrightarrow{L^2} S_N$ by the bounded convergence theorem. Therefore, using these convergences in the formula (5), we have an explicit formula for $S_N^{(k)}$ as in the following theorem.

Theorem 4.5. *For each $N \in \mathbb{N}$, as an equality in $L^2(\hat{\mathbb{Z}},\mathcal{B},\lambda)$, the following holds;*

$$S_N^{(k)}(x) = \frac{1}{\zeta(k)} - \frac{1}{N}\sum_{u=1}^{\infty}\mu(u)\left(\frac{(N+x) \bmod u^k}{u^k} - \frac{x \bmod u^k}{u^k}\right). \tag{14}$$

5. Estimate of the L^2-norm and limit points in L^2

In Section 4, we proved that $N(S_N(x) - 1/\zeta(k)) = -T(x, N)$ and

$$T(x, N) \overset{L^2}{=} \sum_{u=1}^{\infty}\mu(u)\left(\frac{(N+x) \bmod u^k}{u^k} - \frac{x \bmod u^k}{u^k}\right).$$

We are now in position to give the explicit formula for the L^2-norm of $T(x, N)$. By using Lemma 4.2 and Lemma 4.3, we have

$$\begin{aligned}
\mathbf{E}[|T(x, N)|^2] &= \lim_{U\to\infty}\mathbf{E}\left[\left(\sum_{u\le U}\mu(u)\left(\frac{(N+x) \bmod u^k}{u^k} - \frac{x \bmod u^k}{u^k}\right)\right)^2\right] \\
&= \lim_{U\to\infty}\mathbf{E}\left[\sum_{u,v\le U}\mu(u)\mu(v)\left(\frac{(N+x) \bmod u^k}{u^k} - \frac{x \bmod u^k}{u^k}\right)\right. \\
&\qquad \left.\times\left(\frac{(N+x) \bmod v^k}{v^k} - \frac{x \bmod v^k}{v^k}\right)\right] \\
&= \lim_{U\to\infty}\sum_{u,v\le U}\frac{\mu(u)\mu(v)}{\{u, v\}^k}(N \bmod (u, v)^k)\left(1 - \frac{N \bmod (u, v)^k}{(u, v)^k}\right) \\
&= \sum_{u,v\in\mathbb{N}}\frac{\mu(u)\mu(v)}{\{u, v\}^k}(N \bmod (u, v)^k)\left(1 - \frac{N \bmod (u, v)^k}{(u, v)^k}\right) \\
&= \sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^k}(N \bmod n^k)\left(1 - \frac{N \bmod n^k}{n^k}\right)\prod_{p\nmid n}\left(1 - \frac{2}{p^k}\right),
\end{aligned} \tag{15}$$

where in the last line, we have applied Lemma 4.3 for

$$H(n) = H_N(n) := (N \bmod n^k)\left(1 - \frac{N \bmod n^k}{n^k}\right).$$

The following estimate gives us the upper bound of $\mathbf{E}[|T(x, N)|^2]$.

$$\begin{aligned}
\mathbf{E}[|T(x, N)|^2] &= \sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^k}(N \bmod n^k)\left(1 - \frac{N \bmod n^k}{n^k}\right)\prod_{p\nmid n}\left(1 - \frac{2}{p^k}\right) \\
&= \sum_{n^k\le N}(\cdots) + \sum_{n^k>N}(\cdots)
\end{aligned}$$

$$\le \sum_{n^k \le N} \frac{|\mu(n)|}{n^k} \cdot n^k \cdot 1 \cdot 1 + N \sum_{n^k > N} \frac{1}{n^k}$$
$$\le \sum_{n \le N^{1/k}} 1 + N \int_{N^{1/k}}^{\infty} \frac{1}{(t-1)^k}\, dt$$
$$\le N^{1/k} + N \frac{1}{k-1} \frac{1}{(N^{1/k}-1)^{k-1}}$$
$$= \left(1 + \frac{1}{k-1} \frac{1}{(1-N^{-1/k})^{k-1}}\right) N^{1/k}$$
$$\le c_2 N^{1/k} \quad (N \ge 2),$$

where $c_2 = 1 + (1/(k-1))(1-2^{-1/k})^{-k+1}$.

For the lower bound, we need the following lemma.

Lemma 5.1. *Let $\{a_n\}_n$ be a complex sequence. Put $s_n := a_1 + \cdots + a_n$. Assume that there exists a constant $c \in \mathbb{C}$ such that*

$$\frac{s_N}{N} \to c \quad \text{as} \quad N \to \infty. \tag{16}$$

Then, for any $s \in (0, \infty)$,

$$N^s \sum_{n=N}^{\infty} \frac{a_n}{n^{s+1}} \to \frac{1}{s} c \quad \text{as} \quad N \to \infty. \tag{17}$$

Proof. Let $s_x = \sum_{l \le x} a_l$, $(x \in \mathbb{R}^+)$, be an extension of s_n as a function on $\mathbb{R}^+$. Clearly $\lim_{x\to\infty} s_x/x = c$. First, we check the convergence of $\sum_n a_n/n^{s+1}$. For $N, M \in \mathbb{N}$, $N < M$,

$$\sum_{N \le n \le M} \frac{a_n}{n^{s+1}} = \sum_{N \le n \le M} \frac{s_n - s_{n-1}}{n^{s+1}}$$
$$= \sum_{N \le n \le M-1} s_n \left(\frac{1}{n^{s+1}} - \frac{1}{(n+1)^{s+1}}\right) - \frac{s_{N-1}}{N^{s+1}} + \frac{s_M}{M^{s+1}}$$
$$= \sum_{N \le n \le M-1} s_n \int_n^{n+1} \left(-\frac{1}{x^{s+1}}\right)' dx - \frac{s_{N-1}}{N^{s+1}} + \frac{s_M}{M^{s+1}}$$
$$= \sum_{N \le n \le M-1} s_n \int_n^{n+1} \frac{s+1}{x^{s+2}}\, dx - \frac{s_{N-1}}{N^{s+1}} + \frac{s_M}{M^{s+1}}$$
$$= (s+1) \int_N^M \frac{s_x}{x^{s+2}}\, dx - \frac{s_{N-1}}{N^{s+1}} + \frac{s_M}{M^{s+1}}$$
$$= (s+1) \int_N^M \frac{s_x}{x} \frac{dx}{x^{s+1}} - \frac{s_{N-1}}{N-1} \frac{N-1}{N} \frac{1}{N^s} + \frac{s_M}{M} \frac{1}{M^s}.$$

This tells us that $\sum_n a_n/n^{s+1}$ is convergent. Next, letting $M \to \infty$ in the above, and then multiplying this by N^s yield that

$$\begin{aligned} N^s \sum_{n=N}^{\infty} \frac{a_n}{n^{s+1}} &= (s+1)N^s \int_N^{\infty} \frac{s_x}{x} \frac{dx}{x^{s+1}} - \frac{s_{N-1}}{N-1} \frac{N-1}{N} \\ &= (s+1) \int_N^{\infty} \frac{s_x}{x} \frac{1}{(x/N)^{s+1}} \frac{dx}{N} - \frac{s_{N-1}}{N-1} \frac{N-1}{N} \\ &= (s+1) \int_1^{\infty} \frac{s_{Ny}}{Ny} \frac{dy}{y^{s+1}} - \frac{s_{N-1}}{N-1} \frac{N-1}{N}. \end{aligned}$$

From Lebesgue's dominated convergence theorem, the assertion follows immediately. □

It is noted that $\prod_p (1 - 2/p^k) =: c > 0$. In order to find the lower bound, we use the following;

$$\begin{aligned} \mathbf{E}[|T(x,N)|^2] &= \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^k} (N \bmod n^k) \left(1 - \frac{N \bmod n^k}{n^k}\right) \prod_{p \nmid n} \left(1 - \frac{2}{p^k}\right) \\ &\geq \sum_{n^k \geq 2N} (\cdots) \\ &\geq c \sum_{n^k \geq 2N} \frac{|\mu(n)|}{n^k} N \left(1 - \frac{N}{n^k}\right) \\ &\geq \frac{c}{2 \cdot 2^{(k-1)/k}} N^{1/k} \left((2N)^{(k-1)/k} \sum_{n \geq (2N)^{1/k}} \frac{|\mu(n)|}{n^k} \right) \\ &\geq c_1 N^{1/k} \quad (\exists c_1 > 0) \text{ (for } N \text{ being large enough).} \end{aligned}$$

The last inequality holds, because

$$N^{k-1} \sum_{n \geq N} \frac{|\mu(n)|}{n^k} \to \frac{1}{k-1} \frac{6}{\pi^2} \quad \text{as} \quad N \to \infty, \tag{18}$$

by applying Lemma 5.1 for $s = k-1$ and the sequence $\{a_n = |\mu(n)|\}_n$ with $N^{-1}(a_1 + \cdots + a_N) \to 6/\pi^2$ as $N \to \infty$.

Therefore, for N being large enough, it holds that

$$c_1 N^{1/k} \leq \mathbf{E}\left[\left(N \left(S_N(x) - \frac{1}{\zeta(k)} \right) \right)^2 \right] \leq c_2 N^{1/k}. \tag{19}$$

Consequently, the scaled process

$$Y_N(x) := -\frac{1}{N^{1/2k}} T(x, N) = \frac{1}{N^{1/2k}} \sum_{n=1}^{N} \left(X(x+n) - \frac{1}{\zeta(k)} \right)$$

may be good to consider the limit behavior as $N \to \infty$.

Theorem 5.2. *$\{Y_N\}_{N=1,2,\dots}$ has no limit point in $L^2(\hat{\mathbb{Z}}, \mathcal{B}, \lambda)$.*

Proof. For $0 < N < M$, we consider

$$\mathbf{E}[|Y_M - Y_N|^2] = \mathbf{E}[|Y_M|^2] + \mathbf{E}[|Y_N|^2] - 2\mathbf{E}[Y_M Y_N].$$

Similarly as in showing the equality (15), by using Lemma 4.2 and Lemma 4.3 again, we have

$$\begin{aligned}
\mathbf{E}[T(x, M)T(x, N)] &= \lim_{U\to\infty} \mathbf{E}\Bigg[\left(\sum_{u \le U} \mu(u) \left(\frac{(M+x) \bmod u^k}{u^k} - \frac{x \bmod u^k}{u^k} \right) \right) \\
&\qquad \times \left(\sum_{v \le U} \mu(v) \left(\frac{(N+x) \bmod v^k}{v^k} - \frac{x \bmod v^k}{v^k} \right) \right) \Bigg] \\
&= \lim_{U\to\infty} \sum_{u,v \le U} \mu(u)\mu(v) \mathbf{E}\Bigg[\left(\frac{(M+x) \bmod u^k}{u^k} - \frac{x \bmod u^k}{u^k} \right) \\
&\qquad \times \left(\frac{(N+x) \bmod v^k}{v^k} - \frac{x \bmod v^k}{v^k} \right) \Bigg] \\
&= \lim_{U\to\infty} \sum_{u,v \le U} \frac{\mu(u)\mu(v)}{\{u, v\}^k} H_{M,N}((u, v)) \\
&= \sum_{u,v \in \mathbb{N}} \frac{\mu(u)\mu(v)}{\{u, v\}^k} H_{M,N}((u, v)) \\
&= \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^k} H_{M,N}(n) \prod_{p \nmid n} \left(1 - \frac{2}{p^k} \right),
\end{aligned}$$

where

$$H_{M,N}(n) := ((M \bmod n^k) \wedge (N \bmod n^k)) \left(1 - \frac{(M \bmod n^k) \vee (N \bmod n^k)}{n^k} \right)$$

is a bounded function. Together with the formula (15), we have

$$\mathbf{E}[|Y_M - Y_N|^2] = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^k} \prod_{p \nmid n} \left(1 - \frac{2}{p^k}\right) \left[\frac{1}{M^{1/k}} H_M(n) + \frac{1}{N^{1/k}} H_N(n) - \frac{2}{(MN)^{1/2k}} H_{M,N}(n) \right].$$

Now, it is easy to see that

$$0 \le H_{M,N}(n) \le H_N(n) \wedge H_M(n), \quad \forall n \in \mathbb{N}.$$

Thus, the above series is a positive term one. Moreover, for $n^k \ge 2M$,

$$\frac{M \bmod n^k}{n^k} = \frac{M}{n^k} \le \frac{1}{2}; \quad \frac{N \bmod n^k}{n^k} = \frac{N}{n^k} < \frac{1}{2};$$
$$H_{M,N}(n) = N\left(1 - \frac{M}{n^k}\right) \le N,$$

and hence

$$H_M(n) \ge \frac{1}{2}M; \quad H_N(n) > \frac{1}{2}N; \quad H_{M,N}(n) \le N.$$

Therefore,

$$\begin{aligned} &\mathbf{E}[|Y_M - Y_N|^2] \\ &\ge c \sum_{n^k \ge 2M} \frac{|\mu(n)|}{n^k} \left[\frac{1}{2} M^{(k-1)/k} + \frac{1}{2} N^{(k-1)/k} - \frac{2N}{(MN)^{1/2k}} \right] \\ &= c \left[\frac{1}{2} M^{(k-1)/k} + \frac{1}{2} N^{(k-1)/k} - \frac{2N}{(MN)^{1/2k}} \right] \\ &\quad \times \frac{1}{(2M)^{(k-1)/k}} \left((2M)^{(k-1)/k} \sum_{n \ge (2M)^{1/k}} \frac{|\mu(n)|}{n^k} \right). \end{aligned}$$

Letting $M \to \infty$ and using the convergence (18), we obtain

$$\liminf_{M \to \infty} \mathbf{E}[|Y_M - Y_N|^2] \ge \frac{3c}{(k-1)2^{(k-1)/k}\pi^2} > 0.$$

This implies that $\{Y_N\}_N$ has no limit point in $L^2(\hat{\mathbb{Z}}, \mathcal{B}, \lambda)$. The theorem is proved. □

REMARK 2. (i) Even if each Y_N is normalized, $\{Y_N/\|Y_N\|_2\}_N$ has no limit point in $L^2(\hat{\mathbb{Z}}, \mathcal{B}, \lambda)$, either. Indeed, assume that there is a subsequence $\{Y_{N_j}\}_j$ such that $\{Y_{N_j}/\|Y_{N_j}\|_2\}_j$ converges. Then, taking a subsequence if necessary, we can assume that the subsequence $\{\|Y_{N_j}\|_2\}_j$ also converges. Consequently, $\{Y_{N_j}\}_j = \{\|Y_{N_j}\|_2 \cdot Y_{N_j}/\|Y_{N_j}\|_2\}_j$ converges, which is a contradiction.

(ii) On the other hand, since $\{\|Y_N\|_2\}_N$ is bounded, the sequence of probability measures $\{\lambda \circ Y_N^{-1}\}_N$ on $\mathbb{R}$ is tight. Therefore, for any subsequence $\{N_j\}_j$ there exists a subsubsequence $\{N'_j\}$ such that $\{\lambda \circ Y_{N'_j}^{-1}\}_j$ converges weakly, or $\{Y_{N'_j}\}_j$ converges in distribution.

6. Mean square convergence rate

We consider "distribution" of a function $U\colon \mathbb{Z} \to \mathbb{R}$ as follows: If the limit

$$\lim_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M} \exp(\sqrt{-1}tU(m)), \quad t \in \mathbb{R},$$

exists and it coincides with the characteristic function of some probability distribution on $\mathbb{R}$, then we call it the "distribution" of U.

As we expected, the distributions of X, S_N coincide with the "distributions" of the original functions on $\mathbb{Z}$, respectively, namely;

Theorem 6.1. *Let $U = X$ or S_N (in the latter case, $N \in \mathbb{N}$ is fixed). Then for each $t \in \mathbb{R}$, it holds that*

$$\lim_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M} \exp(\sqrt{-1}tU(m)) = \mathbf{E}[\exp(\sqrt{-1}tU(x))]. \tag{20}$$

Proof. It is enough to show (20) for $U = S_N$, since $X \circ \theta = S_1$. The following estimate is obvious.

$$\begin{aligned}
&\left|\frac{1}{M}\sum_{m=1}^{M} \exp(\sqrt{-1}tS_N(m)) - \mathbf{E}[\exp(\sqrt{-1}tS_N(x))]\right| \\
&\le \left|\frac{1}{M}\sum_{m=1}^{M}\{\exp(\sqrt{-1}tS_N(m)) - \exp(\sqrt{-1}tS_{N,L}(m))\}\right| \\
&\quad + \left|\frac{1}{M}\sum_{m=1}^{M} \exp(\sqrt{-1}tS_{N,L}(m)) - \mathbf{E}[\exp(\sqrt{-1}tS_{N,L}(x))]\right| \\
&\quad + \left|\mathbf{E}[\exp(\sqrt{-1}tS_{N,L}(x))] - \mathbf{E}[\exp(\sqrt{-1}tS_N(x))]\right| =: I_1 + I_2 + I_3.
\end{aligned}$$

It is easy to see that

$$|X_L(x) - X(x)| \le \sum_{p>p_L} \rho_{p^k}(x),$$

and hence

$$|S_{N,L}(x) - S_N(x)| \le \frac{1}{N}\sum_{n=1}^{N}\sum_{p>p_L} \rho_{p^k}(x+n).$$

Therefore,

$$\begin{aligned} I_1 &\le \frac{1}{M}\sum_{m=1}^{M}|t|\,|S_{N,L}(m)-S_N(m)| \\ &\le \frac{1}{M}|t|\sum_{m=1}^{M}\frac{1}{N}\sum_{n=1}^{N}\sum_{p>p_L}\rho_{p^k}(m+n) \\ &= \frac{|t|}{N}\sum_{n=1}^{N}\sum_{p>p_L}\frac{1}{M}\sum_{m=1}^{M}\rho_{p^k}(n+m) \\ &\le \frac{|t|}{N}\sum_{n=1}^{N}\sum_{p>p_L}\frac{1}{M}\frac{N+M}{p^k} = |t|\left(1+\frac{N}{M}\right)\sum_{p>p_L}\frac{1}{p^k}, \end{aligned}$$

where in the last inequality, we have used the fact that the number of multiples of p^k in the sequence $\{1, 2, \ldots, N+M\}$ is at most $(N+M)/p^k$.

On the other hand, since $S_{N,L}(x)$ converges to $S_N(x)$ pointwise as $L\to\infty$, it is clear that $I_3\to 0$ as $L\to\infty$ by the bounded convergence theorem. In addition, for fixed L, $I_2\to 0$ as $M\to\infty$ by applying Lemma 2.8 for $f = S_{N,L}$.

Collecting all the above, we see

$$\begin{aligned} &\limsup_{M\to\infty}\left|\frac{1}{M}\sum_{m=1}^{M}\exp(\sqrt{-1}tS_N(m)) - \mathbf{E}[\exp(\sqrt{-1}tS_N(x))]\right| \\ &\le |t|\sum_{p>p_L}\frac{1}{p^k} + \left|\mathbf{E}[\exp(\sqrt{-1}tS_{N,L}(x))] - \mathbf{E}[\exp(\sqrt{-1}tS_N(x))]\right| \xrightarrow{L\to\infty} 0, \end{aligned}$$

which completes the proof. □

Corollary 6.2. *For each $N\in\mathbb{N}$,*

$$\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\left(S_N(m)-\frac{1}{\zeta(k)}\right)^2 = \mathbf{E}\left[\left(S_N-\frac{1}{\zeta(k)}\right)^2\right]. \tag{21}$$

Proof. If, for $M\in\mathbb{N}$, we define a probability measure λ_M on $(\hat{\mathbb{Z}}, \mathcal{B})$ as

$$\lambda_M = \frac{1}{M}\sum_{m=1}^{M}\delta_m,$$

where δ_m is the Dirac measure at $m\in\mathbb{Z}\subset\hat{\mathbb{Z}}$, then Theorem 6.1 for $U = S_N$ asserts that $\lambda_M\circ S_N^{-1}$ converges to $\lambda\circ S_N^{-1}$ weakly as $M\to\infty$. Thus, for any bounded continuous function f on $\mathbb{R}$

$$\begin{array}{ccc}\lim_{M\to\infty}\int_{\mathbb{R}} f(t)\lambda_M \circ S_N^{-1}(dt) & = & \int_{\mathbb{R}} f(t)\lambda \circ S_N^{-1}(dt) \\ \| & & \| \\ \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M} f(S_N(m)) & & \mathbf{E}[f(S_N(x))].\end{array}$$

If we here take

$$f(t) = \left(t - \frac{1}{\zeta(k)}\right)^2 \wedge \left(1 + \frac{1}{\zeta(k)}\right)^2,$$

then $f(S_N(x)) = (S_N(x) - \zeta(k)^{-1})^2$, because $|S_N(x) - \zeta(k)^{-1}| \leq S_N(x) + \zeta(k)^{-1} \leq 1 + \zeta(k)^{-1}$. Thus, the above convergence for this f is just an assertion of Corollary 6.2. □

The convergence (21) together with the estimate (19) gives us the estimate of the mean square convergence rate, namely;

Corollary 6.3.

$$\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\left(N\left(S_N^{(k)}(m) - \frac{1}{\zeta(k)}\right)\right)^2 \asymp N^{1/k}.$$

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