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A NOTE ON THURSTON-WINKELNKEMPER'S CONSTRUCTION OF CONTACT FORMS ON 3-MANIFOLDS

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1. Introduction

A *contact structure* on a closed oriented 3-manifold M^3 is a completely non-integrable plane field on it. The existence of a contact structure on M^3 was first proved by Martinet [14] and then Lutz [12], [13] for each homotopy class of plane fields. A co-orientable contact structure is given as the kernel of a 1-form α with $\alpha \wedge d\alpha > 0$ or < 0, which is called a positive or negative *contact form* respectively. Thurston and Winkelnkemper [17] deduced the existence of a contact form in an elegant way from Alexander's theorem [1] on open-book decompositions.

Eliashberg [2] showed that the most of these contact structures are, however, too flexible for geometrical interest (see §2). On the other hand, he completely characterized more historic examples related closely to the complex analysis in several variables, namely, the strictly pseudo-convex boundary of compact Stein surfaces ([4]). The symplectic fillability is one of its significant generalizations. Recently Loi and Piergallini [11] translated Eliashberg's characterization of compact Stein surfaces into the language of Lefschetz fibration by using Gompf's method [8]. They also showed that M^3 is realizable as the boundary of a Stein surface if and only if it admits an open-book decomposition whose monodromy map is a composition of right-handed Dehn-twists. In this note, *we improve Thurston-Winkelnkemper's construction so that we obtain a symplectically fillable contact structure in the case where the monodromy map of a given open-book decomposition is a composition of right-handed Dehn-twists along mutually disjoint curves* (Theorem 2). An interesting by-product of our construction is Theorem 3 in §4 which gives a deformation of symplectically fillable contact structures into a foliation with a Reeb component. Note that a foliation admitting a Reeb component itself is not symplectically fillable.

I would like to thank the referee for noticing the importance of Theorem 3.

2. Preliminaries

Let Σ be an embedded surface in M^3 equipped with a contact structure ξ. Then $T\Sigma \cap \xi$ defines a singular foliation on Σ, which we call the characteristic foliation of ξ on Σ. If there is a contractible closed leaf l of the characteristic foliation on Σ, then the disk on Σ bounded by l is called an over-twisted disk for ξ. Any contact structure ξ on M^3 is classified into *over-twisted* one or *tight* one depending on whether there exists an embedded surface containing an over-twisted disk or not. As Eliashberg showed in [2], the isotopy classification of over-twisted contact structures reduces to the homotopy classification of 2-plane distributions, so they are very flexible.

We say that a co-oriented plane field ξ on M^3 is *symplectically fillable* if there is a compact symplectic manifold (W^4, ω) satisfying the following conditions.
1) M^3 is the coherently oriented boundary of W^4 whose orientation is determined by the volume form $\omega \wedge \omega$.
2) The restriction $\omega|_\xi$ on $M^3 = \partial W^4$ never vanish.
Then we call (W^4, ω) a symplectic filling of the pair (M^3, ξ). For a negative contact structure, we can also define the symplectic fillability by changing the orientation of M^3 or alternatively by replacing $\omega\wedge\omega$ by $-\omega\wedge\omega$ in the above condition 1). As shown in Gromov [10] and Eliashberg [3], symplectically fillable contact structures are tight. The symplectic fillability can be considered as a generalization of the following Stein fillability. A *compact Stein surface* X is a compact complex surface admitting a strictly plurisubharmonic function $\phi : X \to \mathbf{R}$ with $\phi^{-1}(\max\phi) = \partial X$. Then, as ∂X is strictly pseudo-convex, the complex tangency defines a positive contact structure ξ on ∂X. We say that ξ is *Stein fillable*. Eliashberg [4] showed how to build compact Stein surfaces. One can find many such examples in Gompf [8]. A knot on M^3 is called a Legendrian knot of the contact structure ξ if it is everywhere tangent to ξ. A closed leaf of a characteristic foliation is an example of Legendrian knot. The canonical framing on a Legendrian knot is determined by contact planes if ξ is co-orientable. Then a theorem of Eliashberg [4] can be stated as follows (see also Eliashberg [5] for more general statement on symplectic fillings).

Theorem ([4], [8]). *A compact oriented* 4*-manifold X admits a complex structure as a Stein surface if and only if it has a handle decomposition described below.*

Let X_1 be some handle body with 0*- and* 1*-handles. Then ∂X_1 admits the unique Stein fillable positive contact structure ξ_0, i.e., the standard one. X is obtained from X_1 by attaching* 2*-handles H_i along Legendrian knots K_i of ξ_0* $(i = 1, \ldots, m)$*. The framing for attaching each H_i is obtained from the canonical framing on K_i by adding a negative twist.*

We recall here Alexander's theorem [1] on open-book decompositions.

Theorem ([1]). *Every closed oriented* 3*-manifold M^3 has a link L, called a fibred link, whose exterior is a total space of a Seifert surface bundle over the circle. That is, M^3 admits the following decomposition $\mathcal{O}$ for some compact oriented surface F in M^3 whose boundary is parallel to the link L.*

$$M^3 \cong (F \times [0, 2\pi]/f) \cup_{\mathrm{id}} (L \times D^2) \qquad (\partial F \cong L, \quad \mathbf{R}/2\pi\mathbf{Z} \cong \partial D^2).$$

Here $f : F \times \{2\pi\} \to F \times \{0\}$ *is the attaching diffeomorphism relative to* ∂F.

We call f the *monodromy map* of $\mathcal{O}$, which is determined by $\mathcal{O}$ up to isotopy and conjugation in the group of orientation preserving diffeomorphisms relative to ∂F. This decomposition $\mathcal{O}$ is called an *open-book decomposition* of M^3.

Let C be a simple closed curve on F. Let $f : F \to F$ be a diffeomorphism supported in a tubular neighbourhood $N(C)$ of C in F. Then we say that f is a *right-handed* (resp. *left-handed*) *Dehn-twist* along C if f carries a short arc γ crossing $N(C)$ to a long arc winding once around C toward the right (resp. the left). Note that the notion of right or left is well-defined under the presence of the orientation of F and is independent of the choice of the orientation of γ.

3. Construction of filling

Thurston and Winkelnkemper [17] constructed positive and negative contact forms on an arbitrary closed oriented 3-manifold M^3 equipped with an open-book decomposition $M^3 \cong (F \times [0, 2\pi]/f) \cup_{\mathrm{id}} (L \times D^2)$ in the following way:

Construction 1 ([17]). Put $T = F \times [0, 2\pi]/f$ and $N(1) = L \times D^2$ with $\varphi \in \mathbf{R}/2\pi\mathbf{Z}$, $\theta \in L$ and $(r, \varphi) \in D^2$ (polar coordinate with radius $r \in [0, 1]$) as their coordinates. We may extend $N(1)$ to a larger tubular neighbourhood $N(3/2) = \{(\theta, r, \varphi) \mid 0 \leq r \leq 3/2\}$ in which the fiber is given by $\varphi = \text{const.}$ $(1 \leq r \leq 3/2)$. For any volume form Ω on F, we can modify f among its isotopy class into a diffeomorphism preserving Ω. Thus we may regard Ω as defined on every fibre F_φ in T. We choose Ω with $\Omega = d\alpha_0$ for some 1-form α_0 satisfying $\alpha_0|_{T \cap N(3/2)} = (2 - r)d\theta$.

Next, we take smooth functions λ and μ on $N(1)$ depending only on r which satisfy the following conditions.

$$\lambda(r) = 2 - r^2 \ \left(0 \leq r \leq \frac{1}{2}\right), \quad \lambda'(r) < 0 \ (0 < r < 1), \quad \lambda(1) = 1, \quad \lambda'(1) = -1,$$

$$\mu(r) = r^2 \ \left(0 \leq r \leq \frac{1}{2}\right), \quad \mu'(r) > 0 \ (0 < r < 1), \quad \mu(1) = 1 \quad \text{and} \quad \mu'(1) = 0.$$

We put

$$\alpha = \begin{cases} \alpha_0 & (\text{on } T) \\ \lambda d\theta & (\text{on } N(1)) \end{cases} \quad \text{and} \quad \beta = \begin{cases} d\varphi & (\text{on } T) \\ \mu d\varphi & (\text{on } N(1)) \end{cases}$$

where $d\varphi$ also denotes its pull-back by the projection $F \times [0, 2\pi]/f \to \mathbf{R}/2\pi\mathbf{Z}$. Taking a constant K which is larger than the maximal absolute value of the function h on T determined by $\alpha_0 \wedge d\alpha_0 = h\Omega \wedge d\varphi$, we see that $\pm\alpha + K\beta$ are the required positive and negative contact forms. This completes the construction.

Then our result can be stated as follows.

Theorem 2. *In the above* Construction 1, *assume that there exists a finite* (*possibly empty*) *union* C *of mutually disjoint oriented simple closed curves* C_i ($i \in I$) *on* F. *Take their mutually disjoint closed tubular neighbourhoods* $N(C_i) = C_i \times [-1, 1]$ *with coordinates* (θ_i, ρ) *and put* $N(C) = \bigcup_i N(C_i)$. *Suppose that the monodromy* f *is the composition of right-handed* (*resp. left-handed*) *Dehn-twists along* C, *that is,* f *satisfies the following two conditions for* $\tau = 1$ (*resp.* -1).

1) $\quad f|_{F-N(C)}$ *is the identity map.*

2) $\quad f|_{C_i \times [-1,1]}(\theta_i, \rho) = (\theta_i + 2\pi\tau g_0(\rho), \rho)$ *for each* $i \in I$.

Here the function $g_0 : [-1, 1] \to [0, 1]$ *satisfies* $g_0 \equiv 0$ *on* $[-1, -3/4]$ *and* $g_0 \equiv 1$ *on* $[3/4, 1]$ *and increases on* $(-3/4, 3/4)$. *Then we can choose* Ω *and* α_0 *such that* $\ker(\alpha+\beta)$ (*resp.* $\ker(-\alpha+\beta)$) *is a symplectically fillable positive* (*resp. negative*) *contact structure. Moreover, if* C *is empty, then* M^3 *is a connected sum of some copies of* $S^2 \times S^1$ *and we can take a pair of fillable contact structures* $\ker(\pm\alpha + \beta)$.

Proof. We prove the theorem only in the case where $\tau = 1$ since in the other case it can be proved by changing the orientation of M^3. For sufficiently small $\epsilon > 0$, put $r = \epsilon\rho$ and $g(r) = g_0(\rho)$. We regard $N(C)$ as $C \times [-\epsilon, \epsilon]$ with coordinates (θ_i, r) ($i \in I$) throughout the proof. First we show that we can choose Ω and α_0 satisfying

$$\Omega|_{C_i \times [-\epsilon,\epsilon]} = d\theta_i \wedge dr$$

and

$$\alpha_0|_{C_i \times [-\epsilon,\epsilon] \times [0,2\pi]/f} = -(r + \epsilon k_i)d\theta_i - (r + \epsilon k_i)\varphi g'(r)dr$$

for some constant k_i for each $i \in I$. Put

$$\alpha_0' = \begin{cases} -(r + \epsilon k_i)d\theta_i - (r + \epsilon k_i)\varphi g'(r)dr & \begin{pmatrix} \text{on } C_i \times [-\epsilon, \epsilon] \times [0, 2\pi]/\sim, \\ \text{where } (\theta_i, r, 2\pi) \sim (f(\theta_i, r), 0) \end{pmatrix} \\ (2 - r)d\theta & (\text{ on } T \cap N(3/2) \text{ }) \end{cases}$$

and let V_j ($j \in J$) denote the connected components of $F - \mathrm{int}(N(C))$. Then we define the "distance" $d(j)$ from V_j to ∂F by

$$d(j) = \min\{\#(\gamma \cap C) \mid \gamma \text{ is a path joining a point on } V_j \text{ with a point on } \partial F\}.$$

Take an integer l which is larger than $d(j)$ for any $j \in J$. Let p_i and q_i be two distinct elements of J satisfying $V_{p_i} \cap N(C_i) \neq \emptyset$, $V_{q_i} \cap N(C_i) \neq \emptyset$ and $d(p_i) \leq d(q_i)$.

We may assume that the orientation of C_i coincides with that of ∂V_{p_i}. Put $k_i = x^{l-d(p_i)}$ for a positive constant x. Then we have

$$\int_{\partial V_j \times \{\varphi\}} \alpha_0' = 2\pi\epsilon \left(a_j x^{l-d(j)+1} + b_j x^{l-d(j)} + c_j x^{l-d(j)-1} + d_j\right)$$

for each j with $d(j) \neq 0$. Here $a_j(> 0), b_j, c_j(\leq 0)$ and d_j are constants independent of x. Thus we can choose x such that $\int_{\partial V_j \times \{\varphi\}} \alpha_0' > 0$ holds for any j with $d(j) \neq 0$. On the other hand,

$$\int_{\partial V_j \times \{\varphi\}} \alpha_0' = 2\pi\#(\partial V_j \cap \partial F) + \epsilon \cdot \text{const.} > \int_{(V_k \times \{\varphi\}) \cap N(3/2)} d\alpha_0'$$

holds for sufficiently small $\epsilon > 0$ for each j with $d(j) = 0$. Then we can extend α_0' to be a 1-form α_0 on T satisfying the following conditions.
1) $\alpha_0|_{(F-N(C))\times\{\varphi\}}$ is the pull-back of a 1-form on $F - N(C)$ by the natural projection $p : (F - N(C)) \times \mathbf{R}/2\pi\mathbf{Z} \to F - N(C)$.
2) $d\alpha_0|_{(F-N(C))\times\{\varphi\}} = p^*\Omega$ is the pull-back of a volume form Ω on $F - N(C)$.
Note that we can take K $(> \max\{(r + \epsilon k_i)^2 g'(r) \mid i \in I, r \in [-\epsilon, \epsilon]\})$ in Construction 1 as small as we need. Assume that we can take $K = 1$.

Now we can regard $\left(F - C \times (-\epsilon, \epsilon)\right) \times S^1$ as $\left(F - C \times (-\epsilon, \epsilon)\right) \times \partial D^2$, where D^2 is the disk with radial coordinate $s \in [0, 1]$. Put $\sigma = +$ or $-$ and $\omega^\sigma = d(\sigma\alpha + s^2 d\varphi)$. Then ω^σ is a symplectic form on $\left(F - C \times (-\epsilon, \epsilon)\right) \times D^2$. Moreover, the volume form $\sigma\omega^\sigma \wedge \omega^\sigma$ determines the orientation to which the orientation of $\left(F - C \times (-\epsilon, \epsilon)\right) \times \partial D^2$ $(\subset M^3)$ is coherent. We will extend ω^+ (resp. ω^σ in the case when $C = \emptyset$) to be a symplectic filling of $(M^3, \ker(\alpha + \beta))$ (resp. of $(M^3, \ker(\sigma\alpha + \beta))$) by the following two steps.

STEP 1 (filling near L). We embed $\{\theta\} \times D^2 (\subset L \times D^2)$ into $\mathbf{R}^3$ with coordinate (x, y, z) by putting

$$w = x + yi, \quad z = h_0(r), \quad |w| = h_1(r) \quad \text{and} \quad \arg w = \varphi.$$

Here $h_0(r)$ and $h_1(r)$ are smooth increasing functions defined on $[0, 1]$ satisfying

i) $$h_0(r) = r - \frac{1}{2} \quad \text{and} \quad h_1(r) = 1 \quad \text{near} \quad r = 1$$

and

ii) $$h_0(r) = \frac{r^2}{4} \quad \text{and} \quad h_1(r) = r \quad \text{near} \quad r = 0.$$

Then any point on the region $R_0 = \left\{(w, z) \mid h_0 \circ h_1^{-1}(|w|) \leq z \leq 1/2\right\}$ can be presented by $z = h_0(r)$, $|w| = sh_1(r)$ and $\arg w = \varphi$, where $s \in [0, 1]$ is determined by each point but $(0, 0)$ (see Fig. 1).

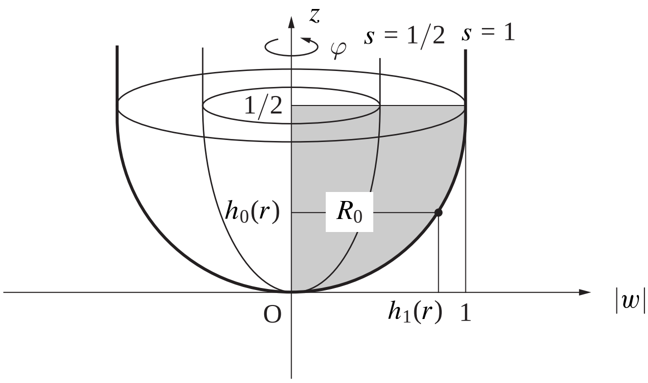

Fig. 1.

Then we can extend ω^σ to $L \times R_0$ by putting $\omega^\sigma = d(\sigma\lambda(r)d\theta + s^2\mu(r)d\varphi)$. Note that we have

$$dx \wedge dy \wedge dz = \frac{sr^3}{2} dr \wedge ds \wedge d\varphi$$

and

$$\sigma\omega^\sigma \wedge \omega^\sigma = \{-4s\lambda'(r)\mu(r)\} d\theta \wedge dr \wedge ds \wedge d\varphi.$$

Thus $\sigma\omega^\sigma \wedge \omega^\sigma > 0$ holds even at $(w, z) = (0, 0)$. Evidently, $d\omega^\sigma = 0$ and $\omega^\sigma|_{\ker(\sigma\alpha+\beta)} \neq 0$ hold. It is easy to see that $\partial(F \times D^2)$ is homeomorphic to the connected sum of some copies of $S^2 \times S^1$. This proves the second half of the theorem.

STEP 2 (filling near $C \times [0, 2\pi]/f$). We embed $C_i \times [-\epsilon, \epsilon] \times [0, 2\pi]/f$ into $\mathbf{C}^2$ with coordinate (z_1, z_2) by putting

$$\begin{aligned} |z_1|^2 &= 1 + rg(r), & \arg z_2 &= (1 - h_2(r))\theta_i + g(r)\varphi, \\ |z_2|^2 &= |z_1|^2 - r, \ \text{ and } & -\arg z_1 &= (1 - h_3(r))\theta_i + (g(r) - 1)\varphi. \end{aligned}$$

Here $h_2(r)$ and $h_3(r)$ are smooth functions on $[-\epsilon, \epsilon]$ satisfying

$$\operatorname{supp}(h_2) = \left[\frac{7}{8}\epsilon, \epsilon\right], \quad \operatorname{supp}(h_3) = \left[-\epsilon, -\frac{7}{8}\epsilon\right],$$
$$h_2(r) = 1 \text{ near } r = \epsilon, \quad \text{and} \quad h_3(r) = 1 \text{ near } r = -\epsilon.$$

Take the region R_i containing $(0, 0)$ and bounded by the image Σ_i of the above embedding and the two hypersurfaces given by $|z_1|^2 = 1 + \epsilon$ and $|z_2|^2 = 1 + \epsilon$ respectively (see Fig. 2).

Then we can extend ω^+ to R_i by setting

$$\omega^+ = d(|z_1|^2) \wedge d(\arg z_1) + d(|z_2|^2) \wedge d(\arg z_2),$$

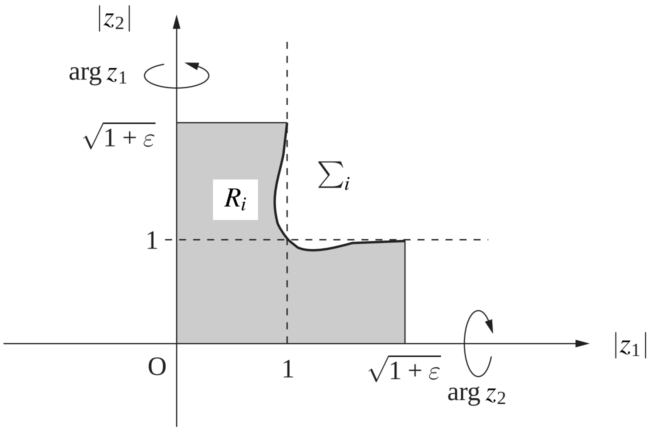

Fig. 2.

which is equal to $d\theta_i \wedge dr + 2sds \wedge d\varphi$ near $r = \epsilon$ (resp. $-\epsilon$) by putting $s = |z_2|$ (resp. $|z_1|$). This is a symplectic form on R_i. Then

$$\begin{aligned}(\alpha+\beta)\wedge\omega^+|_{\Sigma_i} &= (\alpha+\beta)\wedge\{d\theta_i\wedge dr + rg'(r)dr\wedge d\varphi\}\\ &= \{1 - r(r+\epsilon k_i)g'(r)\}d\theta_i\wedge dr\wedge d\varphi \quad \neq 0\end{aligned}$$

holds for sufficiently small $\epsilon > 0$.

Consequently we obtain a symplectic filling

$$\left(\left(\bigsqcup_{j\in J} V_j \times D^2\right) \cup (L \times R_0) \cup \left(\bigsqcup_{i=1}^{m} R_i\right),\ \omega^+\right)$$

of the pair $\left(M^3, \ker(\alpha+\beta)\right)$. This completes the proof. □

4. Further discussions

We obtain the following theorem from our concrete construction in the above proof. Let us compare this result with the theory of confoliations (see Eliashberg and Thurston [6] for details).

Theorem 3. *Suppose that a closed oriented* 3*-manifold* M^3 *admits an open-book decomposition* $\mathcal{O}$ *satisfying the assumption of* Theorem 2. *Then there exist foliations* $\mathcal{F}^{\pm}$ *with Reeb components on* M^3 *which satisfy the following conditions.*

1) $\mathcal{F}^{\sigma}$ *is associated with* $\mathcal{O}$ *by inserting Reeb components along the fibred link of* $\mathcal{O}$ *under the following choice of the co-orientations: The co-orientation of* $\mathcal{F}^{\sigma}$ *determined by* $d\varphi$ *outside the Reeb components agrees with one determined by* σdr *on the toral leaves and one determined by* $\tau d\theta$ *inside them* ($\sigma = +, -$).

2) *$\mathcal{F}^\sigma$ is the limit of smooth deformation of a symplectically fillable positive* (*resp. negative*) *contact structure if* $\tau = 1$ (*resp.* $\tau = -1$) ($\sigma = +, -$). *As a consequence,* $\mathcal{F}^\sigma(\sigma = +, -)$ *satisfy Thurston's inequality even though they admit Reeb components.*

Since symplectially (semi-)fillable foliations are taut, i.e., without dead end components, the above $\mathcal{F}^\pm$ themselves are not fillable. On the other hand, a foliation without Reeb components satisfies Thurston's inequality, however it is still unknown which other foliations satisfy the inequality in general. Our $\mathcal{F}^\pm$ satisfy the inequality while they have "essential" Reeb components, since they are the limits of tight contact structures. Here we say that a Reeb component is essential if we cannot eliminate it by the inverse process of turbularization. Such a phenomenon was pointed out by Mitsumatsu [15] in the case of S^3. The convergence into a non-taut foliation is one of the typical case appearing in Noda's classification [16] of regular projectively Anosov flows on T^2-bundles, but this non-taut foliation has no Reeb components. Recently Giroux announced that all co-orientable contact structures can be constructed by using Thurston-Winkelnkemper's construction. Then, together with the following proof of Theorem 3, it would imply that any co-orientable tight contact structure should be deforemed into a foliation with essential Reeb components which satisfies Thurston's inequality.

Proof of Theorem 3. We use the same notations as in the proof of Theorem 2. Take a function $\phi : [0, 1] \to [0, 1]$ satisfying $\phi(x) = 1$ for $x \in [0, 1/4]$, $\phi(x) = 0$ for $x \in [1/2, 1]$ and $\phi'(x) < 0$ for $x \in (1/4, 1/2)$. Put

$$\eta^\sigma = \begin{cases} \sigma(1 - \phi(r) - \phi(1-r))dr + \tau\phi(r)d\theta + \phi(1-r)d\varphi & (\text{on } N(1)) \\ d\varphi & (\text{on } T) \end{cases}$$

and

$$\eta_t^\sigma = t\eta^\sigma + (1-t)(\tau\alpha + \beta) \ (t \in [0, 1]).$$

Then we have

$$\begin{aligned} \eta_t^\sigma \wedge d\eta_t^\sigma = \tau\Big[&\{-t\phi'(1-r) + (1-t)\mu'(r)\}\{t\phi(r) + (1-t)\lambda(r)\} \\ &- \{t\phi(1-r) + (1-t)\mu(r)\}\{t\phi'(r) + (1-t)\lambda'(r)\}\Big] d\theta \wedge dr \wedge d\varphi \end{aligned}$$

on $N(1)$. Thus we see that $\{\eta_t^\sigma\}_{t\in[0,1)}$ is a family of contact forms and $\mathcal{F}^\pm = \ker(\eta_1^\pm)$ is the required foliations with Reeb components, whose boundary toral leaves are given by $r = 1/2$. Note that $\ker(\eta_t^\sigma)$ is symplectically fillable for any $t \in [0, 1)$ since it is isotopic to $\ker(\tau\alpha + \beta)$ by Gray's stability theorem [9]. This completes the proof. $\square$

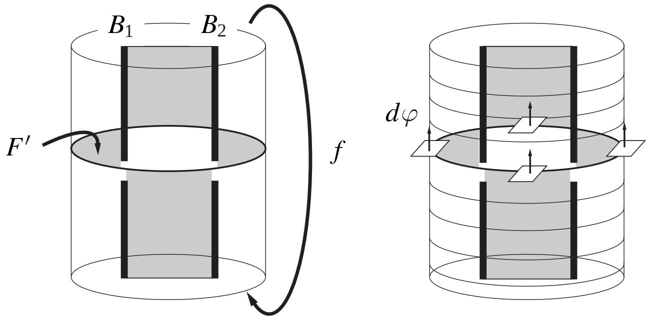

Fig. 3.

REMARK 4. It seems that the right-handedness of Dehn-twists is essential for the symplectic fillability of the positive contact structure constructed above. We cannot expect even the tightness when the monodromy map contains a left-handed twist in general: For example, assume the following four conditions.

1) There is an disk $F'(\subset F)$ with two holes corresponding to $B_1, B_2(\subset \partial F)$.
2) ∂F has another connected component than B_1 and B_2.
3) C is a union of two circles C_1, C_2 parallel to B_1, B_2.
4) f is a composition of a right-handed twist along C_1 and a left-handed twist along C_2 (see Fig. 3).

Then we can take α_0 satisfying $\alpha_0|_{\partial F' - (B_1 \cup B_2)} = 0$. In this case, the positive and negative contact structures determined by $\pm\alpha + K\beta$ on the solid torus $(F' \times [0, 2\pi]/f) \cup_{\mathrm{id}} ((B_1 \cup B_2) \times D^2)$ have over-twisted meridian disks.

REMARK 5. We do not have to assume the disjointness of the twist curves C in many cases. Let f be a right-handed Dehn-twist along a non-separating simple closed curve C_1. Then we can take $k_1 = 0$ in the above proof. In this case, we see that Step 2 is equivalent to attaching a 2-handle along the Legendrian knot C_1 with the same framing as in the theorem of Eliashberg in §2. Thus we can easily generalize Theorem 2 to the case where C have intersections as long as each curve is non-separating on the fibre F. Then we will obtain a Stein fillable contact structure.

In fact, a recent result of Loi and Piergallini [11] says that a closed oriented 3-manifold M^3 can be realized as the oriented boundary of a compact Stein surface if and only if M^3 admits a positive open-book decomposition. Here we say that an open-book decomposition is positive if its monodromy map is a finite composition of finitely many right-handed Dehn-twists. The proof of "if" part relies on the following facts.

1) M^3 admitting a positive open-book also admits another positive open-book for some fibred knot L.
2) The monodromy map of a positive open-book for a fibred knot L can be taken as

the finite composition of right-handed Dehn-twists along non-separating simple closed curves.

We are now interested in Dehn-twists along separating simple closed curves. Let C_1 be a curve which separates the fibre F into two surfaces F_1 and F_2 with $F_2 \cap \partial F = \emptyset$. Suppose that the monodromy map is the right-handed Dehn-twist along C_1. Then we cannot take $k_1 = 0$ in the above proof since the volume of F_2 is positive. We cannot decide directly whether the contact structure obtained in the above proof is Stein fillable or not, in general.

Fukui [7] completely determined the homotopy type of the topological group of diffeomorphisms preserving a given open-book decomposition. The homotopy type is a point, a circle or a 2-torus. The following theorem implies that, in the case stated in the theorem, the generators of the fundamental group are represented by circle actions preserving not only the open-book decomposition but also a suitably constructed contact form. Note that there is another obvious circle action rotating F in the case where F is a disk or an annulus.

Theorem 6. *In the above* Theorem 2, *assume moreover that each connected component of C is parallel to the corresponding component of ∂F. Then we can choose Ω and α_0 so that there is an effective S^1-action preserving not only the open-book decomposition but also the contact form $\tau\alpha + \beta$.*

Proof. We use the same notations as in the proof of Theorem 2. We consider the case where C is connected. We may orient $C = C_1$ as the boundary of the annular component V_1 of $F - \mathrm{int}N(C_1)$. Let L_1 be the corresponding link component. We may assume that an open neighbourhood U_1 of $V_1 \times \mathbf{R}/2\pi\mathbf{Z}$ admits a coordinate (θ, r_1, φ) satisfying

1) $\theta = -\theta_1 + \text{const.}$ and $r_1 = -r + \text{const.}$ on $U_1 \cap N(C_1)$,
2) $r_1 = r + \text{const.}$ on $U_1 \cap N(3/2)$ and
3) θ is an extension of the coordinate function θ defined on $N(3/2)$.

Choose α such that $\alpha|_{V_1} = \tilde{\lambda}(r_1)d\theta$ for some function $\tilde{\lambda}(r_1)$ depending only on r_1. Then the required S^1-action is given by the following transformations for $t \in \mathbf{R}/2\pi\mathbf{Z}$.

1) $\varphi \mapsto \varphi + t$ on $((F - N(C_1) - V_1) \times (\mathbf{R}/2\pi\mathbf{Z})) \cup_{\mathrm{id}} ((L - L_1) \times D^2)$.
2) $\varphi \mapsto \varphi + t$ and $\theta \mapsto \theta + t$ on $(\mathrm{int}V_1 \times (\mathbf{R}/2\pi\mathbf{Z})) \cup (L_1 \times D^2)$.
3) $\varphi \mapsto \varphi + t$ and $\theta_1 \mapsto \theta_1 - tg(r)$ on $N(C_1) \times [0, 2\pi]/f$.

Then we see that this S^1-action preserves the contact form $\tau\alpha + \beta$.

We can prove the theorem in the other case similarly. This ends the proof. □

References

[1] J. Alexander: *A lemma on systems of knotted curves*, Proc. Nat. Acad. Sci. U.S.A. **9** (1923), 93–95.

[2] Y. Eliashberg: *Classification of overtwisted contact structures on* 3*-manifolds*, Invent. Math. **98** (1989), 623–637.

[3] Y. Eliashberg: *Filling by holomorphic discs and its applications*, Lect. Notes L.M.S. 1992.

[4] Y. Eliashberg: *Topological characterization of Stein manifolds of dimension* > 2, Int. J. of Math. **1** (1990), 29–46.

[5] Y. Eliashberg: *Legendrian and transversal knot in tight contact* 3*-manifolds*, Topological Methods in Modern Mathematics, Publish or Perish, (1993), 171–193.

[6] Y. Eliashberg and W. Thurston: *Confoliations*, A.M.S. Univ. Lect. Series **13** (1998).

[7] K. Fukui: *On the homotopy type of some subgroups of* $\mathrm{Diff}(M^3)$, Japan. J. Math. **2** (1976), 249–267.

[8] R. Gompf: *Handlebody construction of Stein surfaces*, Ann. of Math. **148** (1998), 619–693.

[9] J. Gray: *Some global properties of contact structures*, Ann. of Math. **69** (1959), 421–450.

[10] M. Gromov: *Pseudo-holomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), 307–347.

[11] A. Loi and R. Piergallini: *Compact Stein surfaces with boundary as branched covers of* B^4, Invent. Math. **143** (2001), 325–348.

[12] R. Lutz: *Sur quelques propriétés des formes différentielles en dimension trois*, Thèse, Strasbourg, 1971.

[13] R. Lutz: *Structures de contact sur les fibrés principaux en circles de dimension* 3, Ann. Inst. Fourier, **27** (1977), 1–15.

[14] J. Martinet: *Formes de contact sur les variétés de dimension* 3, Proc. Liverpool Singularities Symposium II, Springer Lect. Notes in Math. **209** (1971), 142–163.

[15] Y. Mitsumatsu: *Topology of* 3*-dimensional contact structures*, Memoir of Math. Soc. Japan. **1** (2000) (in Japanese).

[16] T. Noda: *Projectively Anosov flows with differentiable* (*un*)*stable foliations*, Ann. Inst. Fourier Grenoble **50** (2000), 1617–1647.

[17] W. Thurston and H. Winkelnkemper: *On the existence of contact forms*, Proc. A.M.S. **52** (1975), 345–347.

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