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ON THE STRUCTURE OF THE AUGMENTATION QUOTIENTS RELATIVE TO AN N_p-SERIES

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1. Introduction

Let G be a group with lower central series $G=G_1\supseteqq G_2\supseteqq G_3\supseteqq\cdots\supseteqq G_n\supseteqq G_{n+1}\supseteqq\cdots$, and define

$$W_n(G)=\sum\bigotimes_{i=1}^{n} Sp^{a_i}(G_i/G_{i+1}),$$

where $\sum$ runs over all non-negative integers $a_1, a_2, \cdots, a_n$ such that $\sum ia_i=n$, and $Sp^{a_i}(G_i/G_{i+1})$ is the a_i-th symmetric power of the abelian group G_i/G_{i+1}. Let $I(G)$ be the augmentation ideal of G in $\boldsymbol{Z}G$. We denote by $Q_n(G)$ the additive groups $I^n(G)/I^{n+1}(G)$ for $n\geqq 1$. Some results are known about the structure of $Q_n(G)$.

It is well known that $Q_1(G)\simeq W_1(G)$ for any group G. G. Losey [3] proved that $Q_2(G)\simeq W_2(G)$ for any finitely generated group G. Tahara [6], [7] proved that $Q_3(G)\simeq W_3(G)/R_4^*$ and $Q_4(G)\simeq W_4(G)/R_5^*$ hold for any finite group G, where R_4^* and R_5^* are precisely determined subgroups of $W_3(G)$ and $W_4(G)$. Furthermore Sandling and Tahara [5] proved that $Q_n(G)\simeq W_n(G)$ $(n\geqq 1)$ if G_i/G_{i+1} is free abelian for any $i\geqq 1$.

Let p be a prime number. In the first half of this paper we restrict our attention to groups of exponent p, and prove that

$$Q_n(G)\simeq W_n(G)/R_{n+1}\qquad (n\geqq 1),$$

where R_{n+1} is a precisely determined subgroup of $W_n(G)$ (Theorem 8). As its corollaries we have a well known result 1), and a new result 2) as follows:

1) $D_n(G)=G_n$ for any such group G, where $D_n(G)$ is the n-th dimension subgroup of G (Corollary 9).

2) Let G be a finite group with lower central series

$$G=G_1\supseteqq G_2\supseteqq\cdots\supseteqq G_c\supseteqq G_{c+1}=1.$$

If this series is an N_p-series then $Q_n(G)\simeq W_n(G)$ for $n<p$ (Remark 12).

In the latter half we prove that $Q_p(G)\simeq W_p(G)$ if the lower central series of G is an N_p-series (Theorem 13). Furthermore we construct a subgroup

R_{p+2} of $W_{p+1}(G)$ for which $Q_{p+1}(G) \simeq W_{p+1}(G)/R_{p+2}$ holds if the lower central series of G is an N_p-series (Theorem 14). As for dimension subgroup problem, we will show that $D_n(G) = G_n$ for all $n \geqq 1$, if the lower central series of G is an N_p-series (Theorem 15).

2. Notations and definitions

Let G be a finite p-group of order p^m, and let $\mathfrak{H}$ be a fixed finite N_p-series

$$G = H_1 \supseteqq H_2 \supseteqq \cdots \supseteqq H_c \supseteqq H_{c+1} = 1 ,$$

that is $[H_i, H_j] \leqq H_{i+j}$ for all $i, j \geqq 1$, and $H_i^p \leqq H_{ip}$ for all $i \geqq 1$. The series $\mathfrak{H}$ defines a weight function ω of G in the usual way; $\omega(g) = i$ if $g \in H_i - H_{i+1}$, $\omega(g) = \infty$ if $g = 1$. Conditions of N_p-series imply that $\omega([g, h]) \geqq \omega(g) + \omega(h)$ for all $g, h \in G$, and $\omega(g^p) \geqq p\omega(g)$ for all $g \in G$. Since each factor H_i/H_{i+1} is an elementary abelian p-group, we can put

$$t_i = \operatorname{rank}(H_i/H_{i+1}) , \qquad i = 1, 2, \cdots, c .$$

We fix an ordered uniqueness basis Φ for G;

$$\Phi = \{x_1, x_2, \cdots, x_m\} , \quad \omega(x_1) \leqq \omega(x_2) \leqq \cdots \leqq \omega(x_m) .$$

Let Λ_n be the $\boldsymbol{Z}$-linear span in $\boldsymbol{Z}G$ of all the elements

$$(g_1 - 1)(g_2 - 1) \cdots (g_k - 1) , \quad \sum \omega(g_i) \geqq n .$$

Then

$$I(G) = \Lambda_1 \supseteqq \Lambda_2 \supseteqq \cdots \supseteqq \Lambda_n \supseteqq \cdots$$

is a series of ideals of $\boldsymbol{Z}G$ with the property that $\Lambda_i \Lambda_j \subseteqq \Lambda_{i+j}$ for all $i, j \geqq 1$. This filtration determines a family of $\boldsymbol{Z}G$-modules $Q_n(\mathfrak{H}) = \Lambda_n/\Lambda_{n+1}$ for all $n \geqq 1$. These modules are called the *augmentation quotients of G relative to* $\mathfrak{H}$.

A *proper sequence* $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ is an ordered m-tuple of non-negative integers α_i; α is *basic* if $0 \leqq \alpha_i < p$ for all i. The weight of $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ is $W(\alpha) = \sum \omega(x_i)\alpha_i$. Let A_n be the set of all proper sequences of weight n. Corresponding to each proper (basic) sequence $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m)$, we have the *proper* (*basic*) *product*

$$P(\alpha) = (x_1 - 1)^{\alpha_1}(x_2 - 1)^{\alpha_2} \cdots (x_m - 1)^{\alpha_m} .$$

We define $i_\alpha = \max\{i \colon \alpha_i \neq 0\}$ if $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m) \neq \boldsymbol{0} = (0, 0, \cdots, 0)$ and $i_{\boldsymbol{0}} = 1$. We set $W_n(\mathfrak{H}) = \sum \bigotimes_{i=1}^{n} Sp^{a_i}(H_i/H_{i+1})$, where $\sum$ runs over all non-negative integers $a_1, a_2, \cdots, a_n$ such that $\sum i a_i = n$, and $Sp^{a_i}(H_i/H_{i+1})$ is the a_i-th symmetric power of the abelian group H_i/H_{i+1}. Define $m_\alpha(n)$ to be the least non-negative integer such that $W(\alpha) + m_\alpha(n)(p-1)\omega(x_{i_\alpha}) \geqq n$.

G. Losey and N. Losey [3] proved the following:

Lemma 1. *For any $n \geqq 1$, Λ_n has a free $\boldsymbol{Z}$-basis*

$$B_n = \{p^{m_\alpha(n)}P(\alpha)\colon \alpha \neq 0 \textit{ basic}\}\ .$$

3. The structure of $Q_n(\mathfrak{H})$ and its applications

In this section we deal only with groups of exponent p. Let G be a finite p-group of order p^m with exponent p. Then any N-series $\mathfrak{H}$: $G=H_1 \supseteqq H_2 \supseteqq \cdots \supseteqq H_c \supseteqq H_{c+1}=1$ is an N_p-series.

DEFINITION 2.

1) Define the p-sequences of numbers $\{a_k^0\}_{k=0}^{\infty}$, $\{a_k^1\}_{k=0}^{\infty}$, $\cdots$, $\{a_k^{p-1}\}_{k=0}^{\infty}$ as follows:

$$\begin{pmatrix} a_0^0 \\ a_0^1 \\ a_0^2 \\ \vdots \\ a_0^{p-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{pmatrix} a_1^0 \\ a_1^1 \\ a_1^2 \\ \vdots \\ a_1^{p-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} a_{k+1}^0 \\ a_{k+1}^1 \\ a_{k+1}^2 \\ \vdots \\ a_{k+1}^{p-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -\binom{p}{1} \\ 0 & 1 & & 0 & -\binom{p}{2} \\ & & \ddots & & \vdots \\ 0 & & 0 & 1 & -\binom{p}{p-1} \end{pmatrix} \begin{pmatrix} a_k^0 \\ a_k^1 \\ a_k^2 \\ \vdots \\ a_k^{p-1} \end{pmatrix}$$

for $k \geqq 1$.

Note that the next identity holds for any $x \in G$ of order p and for any non-negative integer n:

$$(x-1)^n = a_n^0 \cdot 1 + a_n^1(x-1) + \cdots + a_n^{p-1}(x-1)^{p-1}$$

2) Let $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_m)$ be a proper sequence and $\beta=(\beta_1, \beta_2, \cdots, \beta_m)$ be a basic sequence. We define the integer C_α^β as $C_\alpha^\beta = a_{\alpha_1}^{\beta_1} a_{\alpha_2}^{\beta_2} \cdots a_{\alpha_m}^{\beta_m}$.

We can express $P(\alpha)$ as a $\boldsymbol{Z}$-linear combination of basic products by the following:

Lemma 3. *Let $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_m)$ be a proper sequence with $W(\alpha)=n$, then*

1) $P(\alpha) = \sum_{\beta\,:\,\text{basic}} C_\alpha^\beta P(\beta)$,

2) $p^{m_\beta(n)} | C^\beta_\alpha$ *for any basic sequence* β,

3) *if* α *is basic then* $C^\beta_\alpha \neq 0$ *if and only if* $\beta = \alpha$.

Proof. Expand each $(x_i - 1)^{\alpha_i}$ as in Definition 2. Then we have

$$\begin{aligned} P(\alpha) &= (x_1-1)^{\alpha_1}(x_2-1)^{\alpha_2}\cdots(x_m-1)^{\alpha_m} \\ &= \{\sum_{\beta_1=0}^{p-1} a^{\beta_1}_{\alpha_1}(x_1-1)^{\beta_1}\}\{\sum_{\beta_2=0}^{p-1} a^{\beta_2}_{\alpha_2}(x_2-1)^{\beta_2}\}\cdots\{\sum_{\beta_m=0}^{p-1} a^{\beta_m}_{\alpha_m}(x_m-1)^{\beta_m}\} \\ &= \sum_{\beta_1,\beta_2,\cdots,\beta_m=0}^{p-1} a^{\beta_1}_{\alpha_1} a^{\beta_2}_{\alpha_2}\cdots a^{\beta_m}_{\alpha_m}(x_1-1)^{\beta_1}(x_2-1)^{\beta_2}\cdots(x_m-1)^{\beta_m} \\ &= \sum_{\beta:\text{basic}} C^\beta_\alpha P(\beta)\,. \end{aligned}$$

Thus 1) is obtained. Since $\{p^{m_\beta(n)}P(\beta) | \beta \neq 0\text{: basic}\}$ is a basis system of Λ_n, $P(\alpha)$ is uniquely expressed as a $\boldsymbol{Z}$-linear combination of $p^{m_\beta(n)}P(\beta)$ with $\beta \neq 0$ basic. On the other hand $\{P(\beta) | \beta\text{: basic}\}$ is a basis system of $\boldsymbol{Z}G$. So $P(\alpha)$ is uniquely expressed as a $\boldsymbol{Z}$-linear combination of $P(\beta)$, β basic. Then uniqueness of coefficients implies that $p^{m_\beta(n)} | C^\beta_\alpha$ for all basic sequence β. 3) is trivial from 1).

Definition 4. Let α be a proper sequence with $W(\alpha) = n$. For any basic sequence β, we put $D^\beta_\alpha = C^\beta_\alpha / p^{m_\beta(n)} \in \boldsymbol{Z}$. Therefore

$$P(\alpha) = \sum_{\beta:\text{basic}} D^\beta_\alpha p^{m_\beta(n)} P(\beta)\,.$$

Note that $D^\beta_\beta = 1$ if β is a basic sequence with $W(\beta) \geqq 1$.

Lemma 5 (Passi and Vermani [4]). *Let* p *be a prime number and* $H = \langle a \rangle$ *be a cyclic group of order* p^m. *Then*

$$p^{m-1}(a-1)^{(r+1)(p-1)+1} \equiv (-1)^{(r+1)} p^{m+r}(a-1) \mod I^{(r+1)(p-1)+2}(H)$$

for all $r \geqq 0$.

Corollary 6. *Let* $x \in \Phi$, *then*

$$(x-1)^{r(p-1)+1} \equiv (-1)^r p^r (x-1) \mod \Lambda_{\{r(p-1)+2\}\omega(x)}$$

for all $r \geqq 0$.

Proof. We set $m=1$ in Lemma 5, then we have

$$(x-1)^{(r+1)(p-1)+1} \equiv (-1)^{(r+1)} p^{(r+1)}(x-1) \mod I^{(r+1)(p-1)+2}(\langle x \rangle)$$

for all $r \geqq 0$. This trivially holds for $r = -1$. Then we have

$$(x-1)^{r(p-1)+1} \equiv (-1)^r p^r (x-1) \mod I^{r(p-1)+2}(\langle x \rangle) \qquad \text{for } r \geqq 0\,.$$

Since $I^{r(p-1)+2}(\langle x\rangle)=(x-1)^{r(p-1)+2}\boldsymbol{Z}\langle x\rangle$, we have $I^{r(p-1)+2}(\langle x\rangle)\subseteqq\Lambda_{\{r(p-1)+2\}\omega(x)}$. So the result follows.

Lemma 7.

1) *$W_n(\mathfrak{H})$ is an elementary abelian p-group of order p^r, where $r=\sum\prod_{i=1}^{n}\times\binom{a_i+t_i-1}{a_i}$, and $\sum$ runs over all non-negative integers $a_1, a_2,\cdots, a_n$ such that $\sum_{i=1}^{n} ia_i=n$.*

2) *Regard $W_n(\mathfrak{H})$ as vector space over $\boldsymbol{Z}/p\boldsymbol{Z}$, then $\{\overset{\alpha_1}{\bigotimes}\bar{x}_1\overset{\alpha_2}{\bigotimes}\bar{x}_2\cdots\overset{\alpha_m}{\bigotimes}\bar{x}_m: \alpha\in A_n\}$ is a basis system of $W_n(\mathfrak{H})$, where*

$$\overset{\alpha_1}{\bigotimes}\bar{x}_1\overset{\alpha_2}{\bigotimes}\bar{x}_2\cdots\overset{\alpha_m}{\bigotimes}\bar{x}_m=\begin{cases}\cdots\bigotimes\overbrace{\bar{x}_i\vee\cdots\vee\bar{x}_i}^{\alpha_i}\vee\overbrace{\bar{x}_{i+1}\vee\cdots\vee\bar{x}_{i+1}}^{\alpha_{i+1}}\bigotimes\cdots \\ \quad \text{if } \omega(x_i)=\omega(x_{i+1}), \\ \cdots\bigotimes\overbrace{\bar{x}_i\vee\cdots\vee\bar{x}_i}^{\alpha_i}\otimes\overbrace{\bar{x}_{i+1}\vee\cdots\vee\bar{x}_{i+1}}^{\alpha_{i+1}}\bigotimes\cdots \\ \quad \text{if } \omega(x_i)<\omega(x_{i+1})\end{cases}$$

and $\bar{x}_i=x_iH_{\omega(x_i)+1}$

Proof. Easy to prove.

For convenience we write x_i instead of $\bar{x}_i$. Let $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_m)\neq 0$ be a basic sequence. Then we call α to be *regular for n* if $W(\alpha)+m_\alpha(n)(p-1)\times\omega(x_{i_\alpha})=n$.

Theorem 8. *Let R_{n+1} be the submodule of $W_n(\mathfrak{H})$ generated by the elements of the form*

$$\overset{\alpha_1}{\bigotimes}x_1\overset{\alpha_2}{\bigotimes}x_2\cdots\overset{\alpha_m}{\bigotimes}x_m-\sum_{\beta:\,\text{regular for } n}D_\alpha^\beta(-1)^{m_\beta(n)}\overset{\beta_1}{\bigotimes}x_1\overset{\beta_2}{\bigotimes}x_2\cdots\overset{\beta_{i_\beta-1}}{\bigotimes}x_{i_\beta-1}\overset{\beta_{i_\beta}+m_\beta(n)(p-1)}{\bigotimes}x_{i_\beta},$$

where $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_m)$ runs over all elements of A_n. Then Λ_n/Λ_{n+1} is isomorphic to $W_n(\mathfrak{H})/R_{n+1}$ for all $n\geqq 1$.

Proof. We shall divide the proof in the following four steps.

Step 1. We define a homomorphism ψ_n from Λ_n to $W_n(\mathfrak{H})/R_{n+1}$ which is defined on the basis of Λ_n. Let $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_m)$ be a basic sequence with $W(\alpha)\geqq 1$. Then

$$P(\alpha)=(x_1-1)^{\alpha_1}(x_2-1)^{\alpha_2}\cdots(x_{i_\alpha-1}-1)^{\alpha_{i_\alpha-1}}(x_{i_\alpha}-1)^{\alpha_{i_\alpha}}.$$

Define the image of $p^{m_\alpha(n)}P(\alpha)$ under ψ_n as follows:

1) If α is regular for n then

$$\psi_n(p^{m_\alpha(n)}P(\alpha)) = (-1)^{m_\alpha(n)}\overset{\alpha_1}{\bigtriangledown}x_1\overset{\alpha_2}{\bigtriangledown}x_2\cdots\overset{\alpha_{i_\alpha-1}}{\bigtriangledown}x_{i_\alpha-1}\overset{\alpha_{i_\alpha}+m_\alpha(n)(p-1)}{\bigtriangledown}x_{i_\alpha}+R_{n+1}\,.$$

2) If α is not regular for n then

$$\psi_n(p^{m_\alpha(n)}P(\alpha)) = R_{n+1}\,.$$

Then we shall show that $\psi_n(\Lambda_{n+1})=R_{n+1}$ and hence ψ_n induces a homomorphism ψ_n^* from Λ_n/Λ_{n+1} to $W_n(\mathfrak{H})/R_{n+1}$.

It suffices to prove it on the $\boldsymbol{Z}$-basis of Λ_{n+1}. Let $p^{m_\alpha(n+1)}P(\alpha)\in B_{n+1}$. By the definition of $m_\alpha(n)$ we have $m_\alpha(n)\leqq m_\alpha(n+1)\leqq m_\alpha(n)+1$. If $m_\alpha(n+1)=m_\alpha(n)$ then α is not regular for n since $W(\alpha)+m_\alpha(n)(p-1)\omega(x_{i_\alpha})=W(\alpha)+m_\alpha(n+1)(p-1)\omega(x_{i_\alpha})\geqq n+1$. Therefore by the definition of ψ_n we have

$$\psi_n(p^{m_\alpha(n+1)}P(\alpha)) = \psi_n(p^{m_\alpha(n)}P(\alpha)) = R_{n+1}\,.$$

If $m_\alpha(n+1)=m_\alpha(n)+1$ then

$$\psi_n(p^{m_\alpha(n+1)}P(\alpha)) = p\psi_n(p^{m_\alpha(n)}P(\alpha)) = R_{n+1}\,,$$

since $W_n(\mathfrak{H})$ is an elementary abelian p-group. So the result follows.

Step 2. We define a linear transformation ϕ_n from $W_n(\mathfrak{H})$ to Λ_n/Λ_{n+1} as follows: By Lemma 7 $\{\overset{\alpha_1}{\bigtriangledown}x_1\overset{\alpha_2}{\bigtriangledown}x_2\cdots\overset{\alpha_m}{\bigtriangledown}x_m;\ \alpha\in A_n\}$ is a basis system of $W_n(\mathfrak{H})$. Note that G. Losey and N. Losey proved that Λ_n/Λ_{n+1} is an elementary abelian p-group. Define the image of $\overset{\alpha_1}{\bigtriangledown}x_1\overset{\alpha_2}{\bigtriangledown}x_2\cdots\overset{\alpha_m}{\bigtriangledown}x_m$ under ϕ_n as the element

$$(x_1-1)^{\alpha_1}(x_2-1)^{\alpha_2}\cdots(x_m-1)^{\alpha_m}+\Lambda_{n+1}\,,$$

and extend it $\boldsymbol{Z}/p\boldsymbol{Z}$-linearly.

Then we shall show that $\phi_n(R_{n+1})=\Lambda_{n+1}$, so ϕ_n induces a homomorphism ϕ_n^* from $W_n(\mathfrak{H})/R_{n+1}$ to Λ_n/Λ_{n+1}. Let $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_m)$ be a proper sequence with $W(\alpha)=n$. Then

$$\begin{aligned}
&\phi_n\Big(\overset{\alpha_1}{\bigtriangledown}x_1\overset{\alpha_2}{\bigtriangledown}x_2\cdots\overset{\alpha_m}{\bigtriangledown}x_m-\sum_{\beta:\text{ regular for }n}D_\alpha^\beta(-1)^{m_\beta(n)}\overset{\beta_1}{\bigtriangledown}x_1\overset{\beta_2}{\bigtriangledown}x_2\\
&\qquad\cdots\overset{\beta_{i_\beta-1}}{\bigtriangledown}x_{i_\beta-1}\overset{\beta_{i_\beta}+m_\beta(n)(p-1)}{\bigtriangledown}x_{i_\beta}\Big)\\
&=(x_1-1)^{\alpha_1}(x_2-1)^{\alpha_2}\cdots(x_m-1)^{\alpha_m}-\sum_{\beta:\text{ regular for }n}D_\alpha^\beta(-1)^{m_\beta(n)}(x_1-1)^{\beta_1}(x_2-1)^{\beta_2}\\
&\qquad\cdots(x_{i_\beta-1}-1)^{\beta_{i_\beta-1}}(x_{i_\beta}-1)^{\beta_{i_\beta}+m_\beta(n)(p-1)}+\Lambda_{n+1}\\
&=\sum_{\gamma:\text{ basic}}D_\alpha^\gamma p^{m_\gamma(n)}(x_1-1)^{\gamma_1}(x_2-1)^{\gamma_2}\cdots(x_{i_\gamma-1}-1)^{\gamma_{i_\gamma-1}}(x_{i_\gamma}-1)^{\gamma_{i_\gamma}}\\
&\quad-\sum_{\beta:\text{ regular for }n}D_\alpha^\beta(-1)^{m_\beta(n)}(x_1-1)^{\beta_1}(x_2-1)^{\beta_2}\cdots\\
&\quad(x_{i_\beta-1}-1)^{\beta_{i_\beta-1}}(x_{i_\beta}-1)^{\beta_{i_\beta}+m_\beta(n)(p-1)}+\Lambda_{n+1}
\end{aligned}$$

$$= \sum_{\beta:\ \text{regular for } n} D_\alpha^\beta (x_1-1)^{\beta_1}(x_2-1)^{\beta_2}\cdots(x_{i_\beta-1}-1)^{\beta_{i_\beta-1}}\{p^{m_\beta(n)}(x_{i_\beta}-1)^{\beta_{i_\beta}}$$
$$-(-1)^{m_\beta(n)}(x_{i_\beta}-1)^{\beta_{i_\beta}+m_\beta(n)(p-1)}\}+\Lambda_{n+1}\,.$$

By Corollary 6 we have

$$p^{m_\beta(n)}(x_{i_\beta}-1)^{\beta_{i_\beta}}-(-1)^{m_\beta(n)}(x_{i_\beta}-1)^{\beta_{i_\beta}+m_\beta(n)(p-1)}\in\Lambda_{\{\beta_{i_\beta}+m_\beta(n)(p-1)+1\}\omega(x_{i_\beta})}\,.$$

Therefore

$$(x_1-1)^{\beta_1}(x_2-1)^{\beta_2}\cdots(x_{i_\beta-1}-1)^{\beta_{i_\beta-1}}\{p^{m_\beta(n)}(x_{i_\beta}-1)^{\beta_{i_\beta}}-(-1)^{m_\beta(n)}(x_{i_\beta}-1)^{\beta_{i_\beta}+m_\beta(n)(p-1)}\}$$

belongs to Λ_r, where $r=W(\beta)+m_\beta(n)(p-1)\omega(x_{i_\beta})+\omega(x_{i_\beta})\geqq n+1$. Thus we have

$$\phi_n(\overset{\alpha_1}{\ovee}x_1\overset{\alpha_2}{\ovee}x_2\cdots\overset{\alpha_m}{\ovee}x_m-\sum_{\beta:\ \text{regular for } n} D_\alpha^\beta(-1)^{m_\beta(n)}\overset{\beta_1}{\ovee}x_1\overset{\beta_2}{\ovee}x_2$$
$$\cdots\overset{\beta_{i_\beta-1}}{\ovee}x_{i_\beta-1}\overset{\beta_{i_\beta}+m_\beta(n)(p-1)}{\ovee}x_{i_\beta})=\Lambda_{n+1}\,.$$

Consequently we have $\phi_n(R_{n+1})=\Lambda_{n+1}$, and so ϕ_n induces a homomorphism ϕ_n^* from $W_n(\mathfrak{H})/R_{n+1}$ to Λ_n/Λ_{n+1}.

Step 3. We shall prove that $\psi_n^*\circ\phi_n^*$ is the identity map on $W_n(\mathfrak{H})/R_{n+1}$. Since $W_n(\mathfrak{H})/R_{n+1}$ is generated by $\{\overset{\alpha_1}{\ovee}x_1\overset{\alpha_2}{\ovee}x_2\cdots\overset{\alpha_m}{\ovee}x_m+R_{n+1}\colon \alpha\in A_n\}$, it suffices to prove

$$\psi_n^*\circ\phi_n^*(\overset{\alpha_1}{\ovee}x_1\overset{\alpha_2}{\ovee}x_2\cdots\overset{\alpha_m}{\ovee}x_m+R_{n+1})=\overset{\alpha_1}{\ovee}x_1\overset{\alpha_2}{\ovee}x_2\cdots\overset{\alpha_m}{\ovee}x_m+R_{n+1}$$

for any $\alpha\in A_n$. Let $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_m)$ be a proper sequence with $W(\alpha)=n$, namely $\alpha\in A_n$. Then we have

$$\begin{aligned}
&\psi_n^*\circ\phi_n^*(\overset{\alpha_1}{\ovee}x_1\overset{\alpha_2}{\ovee}x_2\cdots\overset{\alpha_m}{\ovee}x_m+R_{n+1})\\
&=\psi_n^*((x_1-1)^{\alpha_1}(x_2-1)^{\alpha_2}\cdots(x_m-1)^{\alpha_m}+\Lambda_{n+1})\\
&=\psi_n^*(\sum_{\beta:\ \text{regular for } n} D_\alpha^\beta p^{m_\beta(n)}(x_1-1)^{\beta_1}(x_2-1)^{\beta_2}\cdots\\
&\quad(x_{i_\beta-1}-1)^{\beta_{i_\beta-1}}(x_{i_\beta}-1)^{\beta_{i_\beta}}+\Lambda_{n+1})\\
&=\sum_{\beta:\ \text{regular for } n} D_\alpha^\beta(-1)^{m_\beta(n)}\overset{\beta_1}{\ovee}x_1\overset{\beta_2}{\ovee}x_2\cdots\\
&\quad\overset{\beta_{i_\beta-1}}{\ovee}x_{i_\beta-1}\overset{\beta_{i_\beta}+m_\beta(n)(p-1)}{\ovee}x_{i_\beta}+R_{n+1}\\
&=\overset{\alpha_1}{\ovee}x_1\overset{\alpha_2}{\ovee}x_2\cdots\overset{\alpha_m}{\ovee}x_m+R_{n+1}\,.
\end{aligned}$$

Step 4. Finally we shall prove that $\phi_n^*\circ\psi_n^*$ is the identity map on Λ_n/Λ_{n+1}. Since Λ_n/Λ_{n+1} is generated by $\{p^{m_\alpha(n)}P(\alpha)+\Lambda_{n+1}\,|\,\alpha\colon \text{regular foı } n\}$, it suffices to prove

$$\phi_n^* \circ \psi_n^*(p^{m_\alpha(n)}P(\alpha)+\Lambda_{n+1}) = p^{m_\alpha(n)}P(\alpha)+\Lambda_{n+1}$$

for such an α. Let $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_m)$ be a sequence regular for n. Then we have

$$\begin{aligned}
&\phi_n^* \circ \psi_n^*(p^{m_\alpha(n)}(x_1-1)^{\alpha_1}(x_2-1)^{\alpha_2}\cdots(x_{i_\alpha-1}-1)^{\alpha_{i_\alpha-1}}(x_{i_\alpha}-1)^{\alpha_{i_\alpha}}+\Lambda_{n+1})\\
&= \phi_n^*((-1)^{m_\alpha(n)} \overset{\alpha_1}{\circledvee} x_1 \overset{\alpha_2}{\circledvee} x_2 \cdots \overset{\alpha_{i_\alpha-1}}{\circledvee} x_{i_\alpha-1} \overset{\alpha_{i_\alpha}+m_\alpha(n)(p-1)}{\circledvee} x_{i_\alpha}+R_{n+1})\\
&= (-1)^{m_\alpha(n)}(x_1-1)^{\alpha_1}(x_2-1)^{\alpha_2}\cdots(x_{i_\alpha-1}-1)^{\alpha_{i_\alpha-1}}(x_{i_\alpha}-1)^{\alpha_{i_\alpha}+m_\alpha(n)(p-1)}+\Lambda_{n+1}\\
&= p^{m_\alpha(n)}(x_1-1)^{\alpha_1}(x_2-1)^{\alpha_2}\cdots(x_{i_\alpha-1}-1)^{\alpha_{i_\alpha-1}}(x_{i_\alpha}-1)^{\alpha_{i_\alpha}}+\Lambda_{n+1}
\end{aligned}$$

by using Corollary 6.

Step 1~Step 4 imply that $\Lambda_n/\Lambda_{n+1} \simeq W_n(\mathfrak{H})/R_{n+1}$ for all $n \geqq 1$.

Corollary 9 (P.M. Cohn [1]). *Let G be a group of prime exponent p. Let $\{H_j\}$ be an N-series for G and $\{\Lambda_j\}$ the canonical filtration of $I(G)$ relative to $\{H_j\}$. Then $D(\Lambda_n)=H_n$ for all $n \geqq 1$.*

Proof. We prove it by induction on n. By standard reduction arguments we may assume that $H_{n+1}=1$, $D(\Lambda_n)=H_n$ and G is finite. Define the homomorphism f from H_n to Λ_n/Λ_{n+1} by $f(x)=(x-1)+\Lambda_{n+1}$. Then $D(\Lambda_{n+1})=\ker f$. Let $x \in H_n$ be an element of $D(\Lambda_{n+1})$. Write x as $x=\prod x_j^{c_j}$ $(0 \leqq c_j < p)$ using elements of uniqueness basis of weight n. Then $f(x)=\sum c_j(x_j-1)+\Lambda_{n+1}$ and $\psi_n^*(f(x))=\sum c_j x_j+R_{n+1}$. Since $f(x)=\Lambda_{n+1}$, $\sum c_j x_j$ can be expressed as a $\boldsymbol{Z}$-linear combination of generators of R_{n+1}. But the elements of uniqueness basis of weight n do not appear in the generators of R_{n+1}. We shall prove it. If an element of uniqueness basis of weight n is in the generators of R_{n+1}, there must exist some proper sequence $\alpha=(0, \cdots, 0, \underset{k}{1}, 0, \cdots, 0)$ of weight n such that

$$x_k - \sum_{\beta:\text{ regular for } n} D_\alpha^\beta(-1)^{m_\beta(n)} \overset{\beta_1}{\circledvee} x_1 \overset{\beta_2}{\circledvee} x_2 \cdots \overset{\beta_{i_\beta-1}}{\circledvee} x_{i_\beta-1} \overset{\beta_{i_\beta}+m_\beta(n)(p-1)}{\circledvee} x_{i_\beta} \neq 0.$$

Now α is a basic sequence, so by Lemma 3 $D_\alpha^\beta \neq 0$ if and only if $\beta=\alpha$. Trivially $m_\alpha(n)=0$ and $D_\alpha^\alpha=1$, so

$$x_k - \sum_{\beta:\text{ regular for } n} D_\alpha^\beta(-1)^{m_\beta(n)} \overset{\beta_1}{\circledvee} x_1 \overset{\beta_2}{\circledvee} x_2 \cdots \overset{\beta_{i_\beta-1}}{\circledvee} x_{i_\beta-1} \overset{\beta_{i_\beta}+m_\beta(n)(p-1)}{\circledvee} x_{i_\beta} = 0.$$

Thus any element of uniqueness basis of weight n does not appear in the generators of R_{n+1}. If some $c_j \neq 0$, $\sum c_j x_j$ is not able to be expressed as a $\boldsymbol{Z}$-linear combination of generators of R_{n+1}. This implies $c_j=0$ for all j, and $x=\prod_j x_j^{c_j}=1$. Therefore the result follows.

Passi and Vermani [4] proved the following

Theorem 10. *Let* $M=Z_{p^{m_1}}\oplus Z_{p^{m_2}}\oplus\cdots\oplus Z_{p^{m_k}}$ $r=\min_{1\leq i<j\leq k}|m_i-m_j|$ *and* $k>1$. *Then* $I^n(G)/I^{n+1}(G)\simeq Sp^n(G)$ *if and only if* $n\leqq p+r(p-1)$.

As a special case of this result we have that if G is an elementary abelian p-group of order$\geqq p^2$ then $I^n(G)/I^{n+1}(G)\simeq Sp^n(G)$ if and only if $n\leqq p$. Our method is available for non-abelian p-group of exponent p and we have a similar result as follows.

Corollary 11. *Let G be a finite p-group of exponent p with N-series* $\mathfrak{H}$: $G=H_1\supseteqq H_2\supseteqq\cdots\supseteqq H_c\supseteqq H_{c+1}=1$ *with* $|H_1/H_2|\geqq p^2$. *Then* $\Lambda_n/\Lambda_{n+1}\simeq W_n(\mathfrak{H})$ *if and only if* $n\leqq p$.

Proof. Let $\Phi=\{x_1, x_2, \cdots, x_m\}$ be the uniqueness basis for G relative to $\mathfrak{H}$. By Theorem 8 $\Lambda_n/\Lambda_{n+1}\simeq W_n(\mathfrak{H})/R_{n+1}$. We shall prove that $R_{n+1}=0$ for $n\leqq p$ and $R_{n+1}\neq 0$ for $n>p$.

Case 1. $n<p$.

Let $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_m)$ be a proper sequence of weight n. Then α is a basic sequence. By Lemma 3, $D_\alpha^\beta\neq 0$ if and only if $\beta=\alpha$. Trivially $m_\alpha(n)=0$ and $D_\alpha^\alpha=1$. These conditions imply that

$$\overset{\alpha_1}{\bigcirc\!\!\!\!\vee}x_1\overset{\alpha_2}{\bigcirc\!\!\!\!\vee}x_2\cdots\overset{\alpha_m}{\bigcirc\!\!\!\!\vee}x_m-\sum_{\beta:\text{ regular for }n}D_\alpha^\beta(-1)^{m_\beta(n)}\overset{\beta_1}{\bigcirc\!\!\!\!\vee}x_1\overset{\beta_2}{\bigcirc\!\!\!\!\vee}x_2\cdots\overset{\beta_{i_\beta-1}}{\bigcirc\!\!\!\!\vee}x_{i_\beta-1}\overset{\beta_{i_\beta}+m_\beta(n)(p-1)}{\bigcirc\!\!\!\!\vee}x_{i_\beta}=0\,.$$

Therefore $R_{n+1}=0$ for $n<p$.

Case 2. $n=p$.

Let $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_m)$ be a proper sequence of weight p. If α is a basic sequence it follows as above that

$$\overset{\alpha_1}{\bigcirc\!\!\!\!\vee}x_1\overset{\alpha_2}{\bigcirc\!\!\!\!\vee}x_2\cdots\overset{\alpha_m}{\bigcirc\!\!\!\!\vee}x_m-\sum_{\beta:\text{ regular for }p}D_\alpha^\beta(-1)^{m_\beta(p)}\overset{\beta_1}{\bigcirc\!\!\!\!\vee}x_1\overset{\beta_2}{\bigcirc\!\!\!\!\vee}x_2\cdots\overset{\beta_{i_\beta-1}}{\bigcirc\!\!\!\!\vee}x_{i_\beta-1}\overset{\beta_{i_\beta}+m_\beta(p)(p-1)}{\bigcirc\!\!\!\!\vee}x_{i_\beta}=0\,.$$

If α is not a basic sequence then α has the form $\alpha=(0, \cdots, 0, \underset{j}{p}, 0, \cdots, 0)$ for some j and $\omega(x_j)=1$. $C_\alpha^\beta=a_0^{\beta_1}\cdots a_0^{\beta_{j-1}}a_p^{\beta_j}a_0^{\beta_{j+1}}\cdots a_0^{\beta_m}\neq 0$ implies $\beta_j\neq 0$ and $\beta_k=0$ $(k\neq j)$. Let β_0 be a basic sequence of the form $\beta_0=(0, \cdots, 0, \underset{j}{1}, 0, \cdots, 0)$. If β is any basic sequence different from β_0, then $C_\alpha^\beta=0$ or β is not regular for p. Clearly $m_{\beta_0}(p)=1$ and $D_\alpha^{\beta_0}=a_p^1/p=-1$. Therefore

$$\overset{p}{\bigcirc\!\!\!\!\vee}x_j-\sum_{\beta:\text{ regular for }p}D_\alpha^\beta(-1)^{m_\beta(p)}\overset{\beta_1}{\bigcirc\!\!\!\!\vee}x_1\overset{\beta_2}{\bigcirc\!\!\!\!\vee}x_2\cdots\overset{\beta_{i_\beta-1}}{\bigcirc\!\!\!\!\vee}x_{i_\beta-1}\overset{\beta_{i_\beta}+m_\beta(p)(p-1)}{\bigcirc\!\!\!\!\vee}x_{i_\beta}=0\,.$$

Thus we have $R_{p+1}=0$.

Case 3. $n>p$.

Since $|H_1/H_2|\geqq p^2$, there exists a proper sequence $\alpha=(n-1, 1, 0, \cdots, 0)$ in A_n. If $C_\alpha^\beta=a_{n-1}^{\beta_1}a_1^{\beta_2}a_0^{\beta_3}\cdots a_0^{\beta_m}\neq 0$ for a basic sequence $\beta=(\beta_1, \beta_2, \cdots, \beta_m)$, then $\beta_2=1$ and $\beta_3=\beta_4=\cdots=\beta_m=0$. Moreover if $\beta=(\beta_1, 1, 0, \cdots, 0)$ is regular for n, then

$$\overset{\beta_1}{\bigcirc\!\!\!\!\vee}\, x_1 \overset{\beta_2+m_\beta(n)(p-1)}{\bigcirc\!\!\!\!\vee}\, x_2=\overset{\beta_1}{\bigcirc\!\!\!\!\vee}\, x_1 \overset{n-\beta_1}{\bigcirc\!\!\!\!\vee}\, x_2\neq \overset{n-1}{\bigcirc\!\!\!\!\vee}\, x_1\vee x_2\,,$$

since $\beta_1<p\leqq n-1$. Thus

$$\overset{n-1}{\bigcirc\!\!\!\!\vee}\, x_1\vee x_2-\sum_{\beta:\ \text{regular for } n} D_\alpha^\beta(-1)^{m_\beta(n)}\overset{\beta_1}{\bigcirc\!\!\!\!\vee}\, x_1\overset{\beta_2}{\bigcirc\!\!\!\!\vee}\, x_2\cdots \overset{\beta_{i_\beta-1}}{\bigcirc\!\!\!\!\vee}\, x_{i_\beta-1} \overset{\beta_{i_\beta}+m_\beta(n)(p-1)}{\bigcirc\!\!\!\!\vee}\, x_{i_\beta}$$
$$=\overset{n-1}{\bigcirc\!\!\!\!\vee}\, x_1\vee x_2-\sum_{\beta:\ \text{regular for } n} D_\alpha^\beta(-1)^{m_\beta(n)}\overset{\beta_1}{\bigcirc\!\!\!\!\vee}\, x_1 \overset{n-\beta_1}{\bigcirc\!\!\!\!\vee}\, x_2\neq 0\,,$$

and hence $R_{n+1}\neq 0$ for all $n>p$. Therefore the result follows.

REMARK 12. When we determine the structure of $Q_n(\mathfrak{H})$ $(n<p)$ for an N_p-series $\mathfrak{H}$: $G=H_1\supseteqq H_2\supseteqq\cdots\supseteqq H_c\supseteqq H_{c+1}=1$, we may assume that $H_p=1$. So we may assume that G has exponent p. Then by Corollary 11 we have $Q_n(\mathfrak{H})\simeq W_n(\mathfrak{H})$ $(n<p)$ for any N_p-series of a finite p-group G. (It is easy to see $R_{n+1}=0$ for $n<p$ if H_1/H_2 is a cyclic group of order p.)

4. The structure of Λ_p/Λ_{p+1} and $\Lambda_{p+1}/\Lambda_{p+2}$

In the previous section we proved that $\Lambda_n/\Lambda_{n+1}\simeq W_n(\mathfrak{H})$ holds for $n<p$ and for any N_p-series $\mathfrak{H}$ of the finite p-group G. In this section we determine the structure of Λ_p/Λ_{p+1} and $\Lambda_{p+1}/\Lambda_{p+2}$.

Theorem 13. *Let G be a finite p-group with N_p-series $\mathfrak{H}$, and $\{\Lambda_j\}$ its canonical filtration of $I(G)$ with respect to $\mathfrak{H}$. Then Λ_p/Λ_{p+1} is isomorphic to $W_p(\mathfrak{H})$.*

Proof. The proof is similar to that of Theorem 8. Since $\{p^{m_\alpha(n)}P(\alpha)+\Lambda_{p+1}|\alpha$: regular for $p\}$ is a basis system of the vector space Λ_p/Λ_{p+1}, we can define a linear transformation ψ: $\Lambda_p/\Lambda_{p+1}\rightarrow W_p(\mathfrak{H})$ as follows: Let α be a regular sequence for p. Then $p^{m_\alpha(n)}P(\alpha)$ is either $p(x_i-1)$ with $\omega(x_i)=1$, or $P(\alpha)$ with $W(\alpha)=p$. We define to be $\psi(p(x_i-1)+\Lambda_{p+1})=-\overset{p}{\bigcirc\!\!\!\!\vee}\, x_i+x_i^p$ and $\psi(P(\alpha)+\Lambda_{p+1})=\psi((x_1-1)^{\alpha_1}(x_2-1)^{\alpha_2}\cdots(x_m-1)^{\alpha_m}+\Lambda_{p+1})=\overset{\alpha_1}{\bigcirc\!\!\!\!\vee}\, x_1\overset{\alpha_2}{\bigcirc\!\!\!\!\vee}\, x_2\cdots\overset{\alpha_m}{\bigcirc\!\!\!\!\vee}\, x_m$. Next we define a linear transformation ϕ: $W_p(\mathfrak{H})\rightarrow\Lambda_p/\Lambda_{p+1}$ by just the same way as ϕ_p which we defined in Step 2 of the proof of Theorem 8. Then we can easily show that $\psi_0\phi$ and $\phi_0\psi$ are the identity maps on $W_p(\mathfrak{H})$ and Λ_p/Λ_{p+1} respectively, and hence $\Lambda_p/\Lambda_{p+1}\simeq W_p(\mathfrak{H})$.

Theorem 14. *Let G be a finite p-group with N_p-series $\mathfrak{H}=\{H_j\}$, $\{\Lambda_j\}$ its canonical filtration of $I(G)$ with respect to $\mathfrak{H}$ and $\Phi=\{x_1, x_2, \cdots, x_m\}$ its uniqueness basis. Then $\Lambda_{p+1}/\Lambda_{p+2}$ is isomorphic to $W_{p+1}(\mathfrak{H})/R_{p+2}$ where R_{p+2} is generated by the elements*

$$x_i\overset{p}{\bigtriangledown}x_j-\overset{p}{\bigtriangledown}x_i\vee x_j-x_i\otimes x_j^p+x_j\otimes x_i^p+[x_i^p, x_j],\ i<j \text{ and } \omega(x_i)=\omega(x_j)=1\,.$$

Proof. Tahara [6] proved that $\Lambda_3/\Lambda_4\simeq W_3(\mathfrak{H})/R_4^*$ holds for any N-series $\mathfrak{H}$ of the finite group G, where R_4^* is the submodule of $W_3(\mathfrak{H})$ generated by the elements

$$\frac{d(j)}{d(i)}\binom{d(i)}{2}\overset{2}{\bigtriangledown}\bar{x}_{1i}\vee\bar{x}_{1j}-\binom{d(j)}{2}\bar{x}_{1i}\overset{2}{\bigtriangledown}\bar{x}_{1j}+\bar{x}_{1i}\otimes\bar{x}_{1j}^{d(j)}$$
$$-\frac{d(j)}{d(i)}\{\bar{x}_{1j}\otimes\bar{x}_{1i}^{d(i)}\}-\frac{d(j)}{d(i)}\overline{[x_{1i}^{d(i)}, x_{1j}]},\ i<j\,.$$

The case $p=2$ of Theorem 14 is directly obtained by this Theorem. Let p be an odd prime. We shall divide the proof in the following 4 steps.

Step 1. $B_{p+1}=\{p^{m_\alpha(p+1)}P(\alpha)\,|\,\alpha\neq 0\text{: basic}\}$ is classified into following three subsets a)$\sim$c), and we define a homomorphism ψ from Λ_{p+1} to $W_{p+1}(\mathfrak{H})/R_{p+2}$ as follows:

a) $p(x_i-1)(x_j-1)$, $i\leqq j$ and $\omega(x_i)=\omega(x_j)=1$,
$\psi(p(x_i-1)(x_j-1))=-x_i\overset{p}{\bigtriangledown}x_j+x_i\otimes x_j^p+R_{p+2}$,

b) $P(\alpha)=(x_1-1)^{\alpha_1}(x_2-1)^{\alpha_2}\cdots(x_m-1)^{\alpha_m}$, $W(\alpha)=p+1$,
$\psi(P(\alpha))=\overset{\alpha_1}{\bigtriangledown}x_1\overset{\alpha_2}{\bigtriangledown}x_2\cdots\overset{\alpha_m}{\bigtriangledown}x_m+R_{p+2}$,

c) $p^{m_\alpha(p+1)}P(\alpha)$, α not regular for $p+1$, $\psi(p^{m_\alpha(p+1)}P(\alpha))=R_{p+2}$.

Then in the same way as in Step 1 of the proof of Theorem 8, we can easily show $\psi(\Lambda_{p+2})=R_{p+2}$ and hence ψ induces the homomorphism ψ^*; $\Lambda_{p+1}/\Lambda_{p+2}\to W_{p+1}(\mathfrak{H})/R_{p+2}$.

Step 2. We define a linear transformation ϕ from $W_{p+1}(\mathfrak{H})$ to $\Lambda_{p+1}/\Lambda_{p+2}$ by defining it on the basis of $W_{p+1}(\mathfrak{H})$ as follows:

$$\phi(\overset{\alpha_1}{\bigtriangledown}x_1\overset{\alpha_2}{\bigtriangledown}x_2\cdots\overset{\alpha_m}{\bigtriangledown}x_m)=(x_1-1)^{\alpha_1}(x_2-1)^{\alpha_2}\cdots(x_m-1)^{\alpha_m}+\Lambda_{p+2}\,,$$

where $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_m)\in A_{p+1}$. Then we shall prove that $\phi(R_{p+2})=\Lambda_{p+2}$ and ϕ induces the linear transformation ϕ^* from $W_{p+1}(\mathfrak{H})/R_{p+2}$ to $\Lambda_{p+1}/\Lambda_{p+2}$.

Since

$$(x_i-1)^p(x_j-1)=-\sum_{k=1}^{p-1}\binom{p}{k}(x_i-1)^k(x_j-1)+(x_i^p-1)(x_j-1)$$

and

$$(x_i^p-1)(x_j-1)=(x_j-1)(x_i^p-1)+([x_i^p, x_j]-1)+(x_j-1)([x_i^p, x_j]-1)$$
$$+(x_i^p-1)([x_i^p, x_j]-1)+(x_j-1)(x_i^p-1)([x_i^p, x_j]-1)\,,$$

we have

$$\begin{aligned}
&\phi(x_i \overset{p}{\bigcirc\!\!\!\!\vee} x_j - \overset{p}{\bigcirc\!\!\!\!\vee} x_i \vee x_j - x_i \otimes x_j^p + x_j \otimes x_i^p + [x_i^p, x_j]) \\
&= (x_i-1)(x_j-1)^p - (x_i-1)^p(x_j-1) - (x_i-1)(x_j^p-1) \\
&\qquad + (x_j-1)(x_i^p-1) + ([x_i^p, x_j]-1) + \Lambda_{p+2} \\
&= -\sum_{k=2}^{p-1}\binom{p}{k}(x_i-1)(x_j-1)^k + \sum_{k=2}^{p-1}\binom{p}{k}(x_i-1)^k(x_j-1) + \Lambda_{p+2} \\
&= \Lambda_{p+2}\,.
\end{aligned}$$

Thus $\phi(R_{p+2}) = \Lambda_{p+2}$ and ϕ^* is induced.

Step 3. We shall prove that $\phi^* \circ \psi^*$ is the identity map on $\Lambda_{p+1}/\Lambda_{p+2}$. It suffices to prove it on $\{p^{m_\alpha(p+1)}P(\alpha) + \Lambda_{p+2} \mid \alpha\colon \text{regular for } p+1\}$. If $p^{m_\alpha(p+1)}P(\alpha) = p(x_i-1)(x_j-1)$ with $i \leqq j$ and $\omega(x_i) = \omega(x_j) = 1$, then

$$\begin{aligned}
&\phi^* \circ \psi^*(p(x_i-1)(x_j-1) + \Lambda_{p+2}) = \phi^*(-x_i \overset{p}{\bigcirc\!\!\!\!\vee} x_j + x_i \otimes x_j^p + R_{p+2}) \\
&= -(x_i-1)(x_j-1)^p + (x_i-1)(x_j^p-1) + \Lambda_{p+2} \\
&= \sum_{k=1}^{p-1}\binom{p}{k}(x_i-1)(x_j-1)^k + \Lambda_{p+2}\,, \\
&= p(x_i-1)(x_j-1) + \Lambda_{p+2}\,.
\end{aligned}$$

If $p^{m_\alpha(p+1)}P(\alpha) = (x_1-1)^{\alpha_1}(x_2-1)^{\alpha_2}\cdots(x_m-1)^{\alpha_m}$ where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ and $W(\alpha) = p+1$, then

$$\begin{aligned}
&\phi^* \circ \psi^*((x_1-1)^{\alpha_1}(x_2-1)^{\alpha_2}\cdots(x_m-1)^{\alpha_m} + \Lambda_{p+2}) \\
&= \phi^*(\overset{\alpha_1}{\bigcirc\!\!\!\!\vee} x_1 \overset{\alpha_2}{\bigcirc\!\!\!\!\vee} x_2 \cdots \overset{\alpha_m}{\bigcirc\!\!\!\!\vee} x_m + R_{p+2}) \\
&= (x_1-1)^{\alpha_1}(x_2-1)^{\alpha_2}\cdots(x_m-1)^{\alpha_m} + \Lambda_{p+2}\,.
\end{aligned}$$

Now our assertion is proved.

Step 4. Finally we shall show that $\psi^* \circ \phi^*$ is the identity map on $W_{p+1}(\mathfrak{H})/R_{p+2}$. Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m) \in A_{p+1}$. Clearly α has one of the following 4 forms:

a) $\alpha = (0, \cdots, 0, \underset{i}{p+1}, 0, \cdots, 0)$ and $\omega(x_i) = 1$,

b) $\alpha = (0, \cdots, 0, \underset{i}{p}, 0, \cdots, 0, \underset{j}{1}, 0, \cdots, 0)$ and $\omega(x_i) = \omega(x_j) = 1$,

c) $\alpha = (0, \cdots, 0, \underset{i}{1}, 0, \cdots, 0, \underset{j}{p}, 0, \cdots, 0)$ and $\omega(x_i) = \omega(x_j) = 1$,

d) $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m)$, α basic and $W(\alpha) = p+1$.

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ be a proper sequence of type a). Since

$$\begin{aligned}
&(x_i-1)^{p+1} + \Lambda_{p+2} = (x_i-1)\Big\{-\sum_{k=1}^{p-1}\binom{p}{k}(x_i-1)^k + (x_i^p-1)\Big\} + \Lambda_{p+2} \\
&= -p(x_i-1)^2 + (x_i-1)(x_i^p-1) + \Lambda_{p+2}\,,
\end{aligned}$$

we have

$$\psi^*\circ\phi^*(\overset{p+1}{\bigcirc\!\!\!\!\vee}\, x_i+R_{p+2}) = \psi^*((x_i-1)^{p+1}+\Lambda_{p+2})$$
$$= \psi^*(-p(x_i-1)^2+(x_i-1)(x_i^p-1)+\Lambda_{p+2})$$
$$= \overset{p+1}{\bigcirc\!\!\!\!\vee}\, x_i - x_i\otimes x_i^p + x_i\otimes x_i^p + R_{p+2}$$
$$= \overset{p+1}{\bigcirc\!\!\!\!\vee}\, x_i + R_{p+2}\,.$$

Let α be a proper sequence of type b). Then

$$\psi^*\circ\phi^*(\overset{p}{\bigcirc\!\!\!\!\vee} x_i\vee x_j+R_{p+2}) = \psi^*((x_i-1)^p(x_j-1)+\Lambda_{p+2})$$
$$= \psi^*(-p(x_i-1)(x_j-1)+(x_i^p-1)(x_j-1)+\Lambda_{p+2})$$
$$= x_i\overset{p}{\bigcirc\!\!\!\!\vee} x_j - x_i\otimes x_j^p + x_j\otimes x_i^p + [x_i^p,\, x_j] + R_{p+2}$$
$$= \overset{p}{\bigcirc\!\!\!\!\vee} x_i\vee x_j + R_{p+2}\,.$$

Let α be a proper sequence of type c). Then

$$\psi^*\circ\phi^*(x_i\overset{p}{\bigcirc\!\!\!\!\vee} x_j+R_{p+2}) = \psi^*((x_i-1)(x_j-1)^p+\Lambda_{p+2})$$
$$= \psi^*(-p(x_i-1)(x_j-1)+(x_i-1)(x_j^p-1)+\Lambda_{p+2})$$
$$= x_i\overset{p}{\bigcirc\!\!\!\!\vee} x_j + R_{p+2}\,.$$

Let α be a basic sequence of type d). Then

$$\psi^*\circ\phi^*(\overset{\alpha_1}{\bigcirc\!\!\!\!\vee} x_1 \overset{\alpha_2}{\bigcirc\!\!\!\!\vee} x_2 \cdots \overset{\alpha_m}{\bigcirc\!\!\!\!\vee} x_m + R_{p+2})$$
$$= \psi^*((x_1-1)^{\alpha_1}(x_2-1)^{\alpha_2}\cdots(x_m-1)^{\alpha_m}+\Lambda_{p+2})$$
$$= \overset{\alpha_1}{\bigcirc\!\!\!\!\vee} x_1 \overset{\alpha_2}{\bigcirc\!\!\!\!\vee} x_2 \cdots \overset{\alpha_m}{\bigcirc\!\!\!\!\vee} x_m + R_{p+2}\,.$$

Step 1$\sim$Step 4 imply that $\Lambda_{p+1}/\Lambda_{p+2}\simeq W_{p+1}(\mathfrak{H})/R_{p+2}$.

Using Theorem 14 we can easily show that $D(\Lambda_{p+2})=H_{p+2}$ for any N_p-series of the finite p-group G. But we can get more powerful result as follows.

Theorem 15. *Let G be a finite p-group with N_p-series $\mathfrak{H}=\{H_i\}$, and $\{\Lambda_i\}$ its canonical filtration of $I(G)$ with respect to $\mathfrak{H}$. Then $D(\Lambda_n)=H_n$ for all $n\geqq 1$.*

Proof. We prove it by induction on n. We may assume $D(\Lambda_n)=H_n$, and $H_{n+1}=1$. We fix an ordered uniqueness basis $\Phi=\{x_1, x_2, \cdots, x_m, y_1, \cdots, y_s\,|\, \omega(x_1)\leqq\omega(x_2)\leqq\cdots\leqq\omega(x_m)<n,\ \omega(y_1)=\omega(y_2)=\cdots=\omega(y_s)=n\}$ for G. Let $x\in H_n$ be an element of $D(\Lambda_{n+1})$. Write x as $x=\prod y_j^{c_j}$ $(0\leqq c_j<p)$. Then

$$
\begin{aligned}
x-1 &= \prod_j y_j^{c_j}-1 = \prod_j \{(y_j-1)+1\}^{c_j}-1 \\
&= \prod_j \left\{\sum_{k_j=0}^{c_j} \binom{c_j}{k_j}(y_j-1)^{k_j}\right\}-1 \\
&= \sum_{k_1=0}^{c_1} \cdots \sum_{k_s=0}^{c_s} \binom{c_1}{k_1}\binom{c_2}{k_2}\cdots\binom{c_s}{k_s}(y_1-1)^{k_1}\cdots(y_s-1)^{k_s}-1 \\
&= c_1(y_1-1)+\cdots+c_s(y_s-1)+\text{higher terms}\,.
\end{aligned}
$$

Note that each $(y_1-1)^{k_1}(y_2-1)^{k_2}\cdots(y_s-1)^{k_s}$ is basic product. As $x-1$ belongs to Λ_{n+1}, $x-1$ is expressed as a $\boldsymbol{Z}$-linear combination of $p^{m_\alpha(n+1)}P(\alpha)$ $\alpha\neq\mathbf{0}$ basic. Write $x-1$ as follows:

$$
x-1 = \sum_{\alpha:\,\text{basic}} a_\alpha p^{m_\alpha(n+1)}P(\alpha) \qquad (a_\alpha\in\boldsymbol{Z})\,.
$$

Let β_j be a basic sequence such that $P(\beta_j)=(y_j-1)$. By uniqueness of coefficients we have $a_{\beta_j}p^{m_{\beta_j}(n+1)}=c_j$ for all j. Since $m_{\beta_j}(n+1)=1$, c_j is a multiple of p. This gives $c_j=0$ for all j, because $0\leqq c_j<p$. So $x=\prod_j y_j^{c_j}=1$. Thus we have $D(\Lambda_{n+1})=H_{n+1}$.

REMARK 16. Corollary 9 is also obtained from this theorem.

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References

[1] P.M. Cohn: *Generalization of a theorem of Magnus*, Proc. London Math. Soc. (3) **2** (1952), 297–310 (see Correction at the end of the same volume).

[2] G. Losey: *On the structure of $Q_2(G)$ for finitely generated groups*, Canad. J. Math. **25** (1973), 353–359.

[3] G. Losey and N. Losey: *Augmentation quotients of some non-abelian finite groups*, Math. Proc. Cambridge Philos. Soc. **85** (1979), 261–270.

[4] I.B.S. Passi and L.R. Vermani: *The associated graded ring of an integral group ring*, Math. Proc. Cambridge Philos. Soc. **82** (1977), 25–33.

[5] R. Sandling and K. Tahara: *Augmentation quotients of group rings and symmetric powers*, Math. Proc. Cambridge Philos. Soc. **85** (1979), 247–252.

[6] K. Tahara: *On the structure of $Q_3(G)$ and the fourth dimension subgroups*, Japan. J. Math. New Ser. **3** (1977), 381–394.

[7] K. Tahara: *The augmentation quotients of group rings and the fifth dimension subgroups*, J. Algebra **71** (1981), 141–173.

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