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<td>Author(s)</td>
<td>Kôno, Susumu; Tamamura, Akie</td>
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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 24(3) P.481–P.498</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1987</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/5328">https://doi.org/10.18910/5328</a></td>
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<tr>
<td>DOI</td>
<td>10.18910/5328</td>
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Kôno, S. and Tamamura, A.
Osaka J. Math.
24 (1987), 481–498

J-GROUPS OF THE SUSPENSIONS OF THE STUNTED LENS SPACES MOD \( p \)

Dedicated to Professor Masahiro Sugawara on his 60th birthday

Susumu KÔNO and Akie TAMAMURA

(Received April 15, 1986)

1. Introduction

The purpose of this paper is to determine the \( J \)-groups of the suspensions of the stunted lens spaces. In order to state our theorem, we recall some notation in [3] and [7].

Let \( p \) be a prime and \( S^{2t+1} \) be the unit \((2t+1)\)-sphere in the complex \((t+1)\)-space. Then the \((2t+1)\)-dimensional standard lens space mod \( p \) is the orbit space

\[
L'(p) = S^{2t+1}/Z_p, \quad Z_p = \{\exp(2\pi u\sqrt{-1}/p) | u = 0, 1, \ldots, p-1\}
\]

where the action is given by \( z(z_0, \ldots, z_t) = (zz_0, \ldots, zz_t) \). Let \([z_0, \ldots, z_t] \in L'(p)\) denote the class of \((z_0, \ldots, z_t) \in S^{2t+1}\). The space \( L^k(p) (k \leq t) \) is naturally imbedded in \( L'(p) \) by identifying \([z_0, \ldots, z_t] \) with \([z_0, \ldots, z_k, 0, \ldots, 0]\). Denote the subspace

\[
L_0^k(p) = \{[z_0, \ldots, z_t] \in L^k(p) | z_k : \text{real}, z_k \geq 0 \}.
\]

Then \( L'(p) - L_0^k(p) \) and \( L_0^k(p) - L_0^{k-1}(p) (k \leq t) \) are \((2k+1)\)- and \(2k\)-cells respectively, which make \( L'(p) \) a finite \( CW \)-complex.

Let \( \nu_q(s) \) denote the exponent of the prime \( q \) in the prime power decomposition of \( s \) and \( m(s) \) the function defined on positive integers as follows:

\[
\nu_q(m(s)) = \begin{cases} 
0 & \text{if } q \neq 2, s \equiv 0 \pmod{q-1} \\
1 + \nu_q(s) & \text{if } q \neq 2, s \equiv 0 \pmod{q-1} \\
1 & \text{if } q = 2, s \equiv 0 \pmod{2} \\
2 + \nu_q(s) & \text{if } q = 2, s \equiv 0 \pmod{2}.
\end{cases}
\]

For non-negative integers \( r, t \) and \( n \) with \( t > n \), we set

\[
h(r, t, n) = \begin{cases} 
\min \{\nu_q(r)+1, [(t+r)/(p-1)]-[(n+r)/(p-1)]\} & (r > 0) \\
[t/(p-1)]-[n/(p-1)] & (r = 0).
\end{cases}
\]
Main result is the following theorem.

**Theorem.** For an odd prime $p$, we have

(1) $\mathcal{J}(S^{2r+1}(L_0'(p)/L_0^a(p))) \cong 0$.

(2) $\mathcal{J}(S^{2r}(L_0'(p)/L_0^a(p))) \cong Z^h_{k(r,t,n)}$.

(3) $\mathcal{J}(S^{2r+1}(L_0'(p)/L_0^a(p))) \cong \mathcal{J}(S^{2e+2r+1})$.

(4) i) If $n+r+1 \equiv 0 \pmod{(p-1)}$, then

$$\mathcal{J}(S^{2r}(L_0(p)/L_0^a(p))) \cong Z^h_{k(r,t,n+1)} \oplus \mathcal{J}(S^{2e+2r+2})$$.

ii) If $n+r+1 \equiv 0 \pmod{(p-1)}$, then

$$\mathcal{J}(S^{2r}(L_0(p)/L_0^a(p))) \cong Z^i \oplus Z^m_{m(n+r+1)} \oplus Z^h_{k(r,t,n+1)}$$.

where $i = \min\{h(r,t,n+1), \nu_{p}(n+1)\}$.

(5) $\mathcal{J}(S^{2r+1}(L_1'(p)/L_0^a(p))) \cong \mathcal{J}(S^{2t+2r+1})$.

(6) $\mathcal{J}(S^{2r}(L_1'(p)/L_0^a(p))) \cong Z^h_{k(r,t,n)} \oplus \mathcal{J}(S^{2h+2r+1})$.

(7) $\mathcal{J}(S^{2r+1}(L_1'(p)/L_0^a(p))) \cong \mathcal{J}(S^{2t+2r+1}) \oplus \mathcal{J}(S^{2e+2r+3})$.

(8) $\mathcal{J}(S^{2r}(L_1'(p)/L_0^a(p))) \cong \mathcal{J}(S^{2t+2r+1}) \oplus \mathcal{J}(S^{2t+2r+1})$.

**Remark 1.** The $J$-groups of the spheres are well known (cf. [3, Examples (3.5) and (3.6)] and [12]):

$$\mathcal{J}(S^n) \cong \begin{cases} Z^m_{m(n,2)} & (n \equiv 0 \pmod{4}) \\ Z_2 & (n \equiv 1, 2 \pmod{8}) \\ 0 & \text{(otherwise)} \end{cases}.$$

**Remark 2.** The partial results for the case $r=n=0$ in the parts (2) and (6) or the case $n=0$ in the part (2) have been obtained in [7, Theorem 2] and [10, Theorem 3.8]. The corresponding result for the case $p=2$ we have been shown in [11].

The paper is organized as follows. In section 2 we give preliminaries. In section 3 we give proofs of parts (2) and (4) i). In section 4 we prove the part (4) ii). The proofs of the other parts are given in the final section.

**2. Preliminaries**

In this section we prepare some lemmas which are needed to prove the theorem. From now on, $p$ denotes an odd prime.

**Lemma 2.1.** Let $r$ be a positive integer and let $k$ and $j$ be integers with $k \equiv j \pmod{p}$, then
$k' - j' \equiv r(k-j)j'^{-1} \pmod{p^s(r)+2}$.

Proof. Since $k \equiv j \pmod{p}$, we have

$$k' - j' = (k-j)\sum_{i=0}^{s-1} k^i j'^{-i} \equiv (k-j)\sum_{i=1}^{s-1} j'^{-i} \pmod{p^2}$$

$$= r(k-j)j'^{-1}.$$ 

This proves the lemma for the case $v_p(r) = 0$. Moreover,

$$\sum_{i=0}^{s-1} (k^i)(j')^{s-i-1} \equiv \sum_{i=1}^{s-1} (r(k-j)j'^{-1} + j')^{s-i-1} \pmod{p^2}$$

$$= \sum_{i=1}^{s-1} (p(p-1)/2)r(k-j)j'^{-1} + (j')^{s-i-1} \pmod{p^2}$$

$$= p(j')^{s-1} \pmod{p^2}.$$ 

Assume that

$$k' - j' \equiv r(k-j)j'^{-1} \pmod{p^s(r)+2}.$$ 

Then we have

$$k^{br} - j^{br} = (k' - j')\sum_{i=0}^{s-1} (k^i)(j')^{s-i-1}$$

$$\equiv r(k-j)j'^{-1}p(j')^{s-1} \pmod{p^s(r)+2}$$

$$= pr(k-j)j'^{br-1}.$$ 

Thus the lemma is proved by the induction on $v_p(r)$. q.e.d.

Lemma 2.2. Let $r$ be a positive integer with $r \equiv 0 \pmod{(p-1)}$. Then, for each $k$ prime to $p$, we have

$$k' - 1 \equiv r(1-k^{br-1}) \pmod{p^s(r)+2}.$$ 

Proof. Since $k^{br-1} \equiv 1 \pmod{p}$ for each $k$ prime to $p$, we have

$$k' - 1 = (k^{br-1})^{(p-1)} - 1^{(p-1)}$$

$$\equiv (r/(p-1))(k^{br-1} - 1) \pmod{p^s(r)+2}$$

$$\equiv (1-p)(r/(p-1))(k^{br-1} - 1) \pmod{p^s(r)+2}$$

$$= r(1-k^{br-1})$$

by Lemma 2.1 and the equality $v_p(r/(p-1)) = v_p(r)$. q.e.d.

The following equalities in the polynomial ring $\mathbb{Z}[x]$ are obtained by making use of the binomial theorem.

$$\sum_{i=1}^{s-1} (j') \sum_{k=1}^{i} (-1)^{i-k} x^k = x^j.$$ (2.3)

The following equalities in the polynomial ring $\mathbb{Z}[x]$ are obtained by making use of the binomial theorem.

$$\sum_{i=1}^{s-1} (j') \sum_{k=1}^{i} (-1)^{i-k} x^k = x^j.$$ (2.3)
In the rest of this paper we fix a positive integer \( t \). Set \( C^{2k+1} = L^k(p) - L^0(p) \) and \( C^{2k} = L^k(p) - L^{k-1}(p) \) for \( 0 \leq k \leq t \). Then the lens space \( L^t(p) \) has the cell decomposition

\[
L^t(p) = C^0 \cup C^1 \cup C^2 \cup \cdots \cup C^{2t+1},
\]

\[
\partial(C^{2k+1}) = 0, \quad \partial(C^{2k}) = pC^{2k-1}.
\]

Denote by \( c^* \) the dual cochain of \( C^* \). Then we have the following lemma.

**Lemma 2.4.** For each integer \( n \) with \( 0 \leq n < t \), we have

1. \( H^*(L^0(p), L^0(p)) \approx \sum_{i=n+1}^{t} Z_p \{c^{2i}\} \)

where \( Z_p \{c^{2i}\} \) means the cyclic group of order \( p \) generated by \( c^{2i} \).

2. \( H^*(L^0(p), L^0(p); Z_2) = 0 \).

The following lemma can be obtained by making use of Lemma 2.4 and the Atiyah-Hirzebruch spectral sequence for \( K \)-theory and \( KO \)-theory (cf. [8]).

**Lemma 2.5.** The orders of \( \tilde{K}^{-r}(L^0(p)/L^0(p)) \) and \( \tilde{KO}^{-r}(L^0(p)/L^0(p)) \) are divisors of \( p^{t-n} \). Precisely,

1. \( \text{ord } \tilde{K}^{-r}(L^0(p)/L^0(p)) = \begin{cases} 1 & (r: \text{odd}) \\ p^{t-n} & (r: \text{even}) \end{cases} \)

2. \( \text{ord } \tilde{KO}^{-r}(L^0(p)/L^0(p)) = \begin{cases} 1 & (r: \text{odd}) \\ p^{(2t+r)/4-[2n+r)/4] & (r: \text{even}) \end{cases} \)

where \( \text{ord } G \) means the order of a finite group \( G \).

Considering the \( Z_p \)-action on \( S^{2t+1} \times \mathcal{C} \) given by

\[
\exp(2\pi \sqrt{-1}/p) (z, u) = (z \cdot \exp(2\pi \sqrt{-1}/p), u \cdot \exp(2\pi \sqrt{-1}/p))
\]

for \( (z, u) \in S^{2t+1} \times \mathcal{C} \), we have a complex line bundle

\[
\eta: (S^{2t+1} \times \mathcal{C})/Z_p \to L^t(p).
\]

Set

\[
\sigma = \eta^{-1} \in \tilde{K}(L^t(p)).
\]

We also denote by \( \sigma \) the restriction of \( \sigma \) to \( L^0(p) \). Then the following proposition is well known.

**Proposition 2.6** (Kambe [6, Theorem 1 and Lemma 2.5]).

1. \( K(L^0(p)) \approx Z[\sigma]/(\sigma^{t+1}, (\sigma + 1)^p - 1) \).
2. \( \tilde{K}(L^0(p)) \) is the direct sum of cyclic groups generated by \( \sigma, \sigma^2, \cdots, \sigma^{p-1} \). The order of \( \sigma^t \) is \( p^{(t-1)/(p-1)+1} \).
From this we obtain the following result.

**Corollary 2.7.** Let \( u \) be a positive integer with \( u = s(p-1) + j \) for \( 1 \leq j \leq p-1 \). Then, in \( K(L_0(p)) \),

\[
\sigma^u \equiv (-p)^s \sigma^j
\]

modulo the subgroup generated by

\[
\{p^{s+1} \sigma^1, \ldots, p^{s+1} \sigma^j, p^s \sigma^{j+1}, \ldots, p^s \sigma^{p-1}\}.
\]

**Proof.** By making use of the relation \((\sigma + 1)^p = 1\), we obtain inductively

\[
\sigma^{j+p-1} = \sum_{i=1}^{p-1} \{(-1)^{j-i} (p^i + j - i)(i-k+1) \} \sigma^i
\]

\[
+ \sum_{i=1}^{p-1} (-1)^{j-i+1} (p^i + j - i)(i-k+1) \sigma^i,
\]

for \( 1 \leq j \leq p-1 \). Set integers \( B_{i,j} \) \((1 \leq i < p-1, 1 \leq j \leq p-1)\) by

\[
B_{i,j} = \begin{cases} 
\sum_{i=1}^{j-1} (-1)^{j-i-1} (p^i + j - i)(i-k+1) & (1 \leq i < j) \\
(-1)^{j-i} (p^i + j - i)(i-k+1) & (j \leq i \leq p-1).
\end{cases}
\]

Then we have

\[
\sigma^{j+p-1} = \sum_{i=1}^{p-1} B_{i,j} \sigma^i
\]

and

\[
B_{i,j} \equiv \begin{cases} 
0 \pmod{p^2} & (1 \leq i < j) \\
-p \pmod{p^2} & (i=j) \\
0 \pmod{p} & (j < i \leq p-1).
\end{cases}
\]

This proves the case \( s = 1 \). Now suppose the result true for some value of \( s \), that is

\[
\sigma^{j+(s+1)(p-1)} = \sum_{i=1}^{p-1} A_i \sigma^i
\]

with

\[
A_i \equiv \begin{cases} 
0 \pmod{p^{s+1}} & (1 \leq i < j) \\
(-p)^s \pmod{p^{s+1}} & (i=j) \\
0 \pmod{p^s} & (j < i \leq p-1).
\end{cases}
\]

Then we have

\[
\sigma^{j+(s+1)(p-1)} = \sum_{i=1}^{p-1} A_i \sigma^{i+p-1}
\]

\[
= \sum_{i=1}^{p-1} A_i (\sum_{k=1}^{s+1} B_{k,i} \sigma^k)
\]

\[
= \sum_{i=1}^{p-1} (\sum_{k=1}^{s+1} A_i B_{k,i}) \sigma^k.
\]
It follows from the inductive hypothesis that

\[ A_i B_{k,i} \equiv \begin{cases} 
0 & \text{(mod } p^{s+2} \text{) (} k < i \text{ or } i < j \text{)} \\
(-p)^{s+1} & \text{(mod } p^{s+2} \text{) (} i = k = j \text{)} \\
0 & \text{(mod } p^{s+1} \text{) (otherwise)}. 
\end{cases} \]

Hence

\[ \sum_{k=1}^{i-1} A_i B_{k,i} \equiv \begin{cases} 
0 & \text{(mod } p^{s+2} \text{) (} 1 \leq k < j \text{)} \\
(-p)^{s+1} & \text{(mod } p^{s+2} \text{) (} k = j \text{)} \\
0 & \text{(mod } p^{s+1} \text{) (} j < k \leq p - 1 \text{)}. 
\end{cases} \]

Thus the proof is completed by the induction with respect to \( s \).

We define the function

\[ (2.8) \quad \mu: \mathbb{Z} \to \mathbb{Z} \]

by setting \( \mu(k) \) to be the remainder of \( k \) divided by \( p \) for every \( k \in \mathbb{Z} \). Set

\[ (2.9) \]

\[
\begin{align*}
(1) \quad x_i &= I'(\eta' - 1) \in K(S^{pL_0})(p) \quad \text{for each integer } i, \\
(2) \quad y_i &= I'(\eta - 1)' \in K(S^{pL_0})(p) \quad \text{for each positive integer } i,
\end{align*}
\]

where \( I \) denotes the isomorphism defined by the Bott periodicity. Then, following properties are obtained by the proof of \([10, \text{Theorem 3.8}]\) and the equalities of \((2.3)\).

\[ (2.10) \]

\[
\begin{align*}
(1) \quad x_i &= x_{\mu(i)} , \\
(2) \quad y_i &= \sum_{j=1}^{i-1} \binom{i}{j} (-1)^{i-j} x_j \quad (i > 0) , \\
(3) \quad x_i &= \sum_{j=1}^{i-1} \binom{i}{j} y_j \quad (i > 0) , \\
(4) \quad \text{For Adams operation } \psi^k, \text{ we have } \\
\psi^k(x_i) &= k^ix_{hi} .
\end{align*}
\]

For each \( i \) prime to \( p \), \( N(i) \) denote the integer chosen to satisfy the property

\[ (2.11) \quad iN(i) \equiv 1 \quad (\text{mod } p^i) \]

Let \( w \) be the remainder of \( r \) divided by \( p - 1 \) and set \( \nu = p - 1 - w \). Then \( 1 \leq \nu \leq p - 1 \), and

\[
\begin{align*}
\sum_{i=1}^{\nu} \binom{i}{j} (-1)^{r-i} N(i^r) &\equiv \sum_{i=1}^{\nu} \binom{i}{j} (-1)^{r-i} r^i \\ &\equiv \nu! \\ &\equiv 0 \quad (\text{mod } p)
\end{align*}
\]
J-GROUPS OF THE SUSPENSIONS

by [10, Lemma 3.7]. For \( 1 \leq j \leq p-1 \), we put

\[
Y_j = y_j - \sum_{i=1}^{j} (-1)^{i-1} N(i^{\text{pr}}) N_s y_s
\]

where \( N_s = N(\sum_{i=1}^{j} \psi_i (i^{\text{pr}})) \). Then we have the following.

**Lemma 2.13.** Let \( j \) be an integer with \( 1 \leq j \leq p-1 \) and \( k \) an integer prime to \( p \), then we have

\begin{enumerate}
  \item \( Y_j \equiv y_j \pmod{p y_s} \) (\( j > v \)).
  \item \( Y_s = 0 \).
  \item \( Y_1 = -N_s \sum_{i=1}^{\psi_i} (-1)^{r-i} N(i^{\text{pr}}) (\psi^{i^{\text{pr}}} - 1) y_1 \).
  \item \( Y_j = \sum_{i=1}^{j} (-1)^{i-1} N(i^{\text{pr}}) (\psi^{i^{\text{pr}}} - 1)y_1 + Y_1 \).
  \item \( (\psi^{k-1}) (y_j) = k \sum_{i=1}^{\psi_i} (-1)^{i-1} \sum_{u=1}^{\psi(u)} (\mu (1/2)) y_s + Y_j \)
    \begin{align*}
    &
    \begin{cases}
    0 & (\text{mod } p y_s^{(r+1)} y_s) \quad (j < v) \\
    (1-k^{r-1}) y_s & (\text{mod } p y_s^{(r+2)} y_s) \quad (j = v) \\
    0 & (\text{mod } p y_s^{(r+2)} y_s) \quad (j > v).
    \end{cases}
    \end{align*}
\end{enumerate}

**Proof.**

\(1\) Since

\[
\sum_{i=1}^{j} (-1)^{i-1} N(i^{\text{pr}}) \equiv \sum_{i=1}^{j} (-1)^{i-1} i^{\text{pr}} = 0 \pmod{p}
\]

by [10, Lemma 3.7], (1) is obtained by the definition (2.12).

\(2\) From the definition of \( N_s \), we have

\[
Y_s = y_s - (\sum_{i=1}^{\psi_i} (-1)^{r-i} N(i^{\text{pr}})) N_s y_s = 0.
\]

\(3\) By making use of the properties (2.10) and the definition of \( N(i) \), we have

\[
Y_1 = y_1 - N_s y_s = -N_s (y_s - \sum_{i=1}^{\psi_i} (-1)^{r-i} N(i^{\text{pr}}) y_i)
\]

\[
= -N_s (\sum_{i=1}^{\psi_i} (-1)^{r-i} (i^{\text{pr}} - N(i^{\text{pr}}) y_i)
\]

\[
= -N_s (\sum_{i=1}^{\psi_i} (-1)^{r-i} N(i^{\text{pr}}) (i^{\text{pr}} y_i - y_i)
\]

\[
= -N_s (\sum_{i=1}^{\psi_i} (-1)^{r-i} N(i^{\text{pr}}) (\psi^{i^{\text{pr}}} - 1) y_i)
\]

\(4\) Similarly, we have

\[
Y_j = y_j - \sum_{i=1}^{j} (-1)^{i-1} N(i^{\text{pr}}) (y_1 - Y_1)
\]
by the properties (2.10).

(5) It follows from the definition (2.12), by making use of the properties (2.10) and the equalities (2.3), that

\[
\sum_{u=1}^{l-1} \left( \frac{j}{u} \right) Y_u = x_j - \sum_{u=1}^{l-1} \left( \frac{j}{u} \right) \sum_{u=1}^{l-1} \left( \frac{j}{u} \right) (-1)^{l-j-i} N(i^{pr}) N_{y_u}
\]

\[
= x_j - N(i^{pr}) N_{y_u}.
\]

Hence we have

\[
(\psi - 1)y_j = \left( \sum_{u=1}^{l-1} \left( \frac{j}{u} \right) \right) (-1)^{l-j-i}\psi x_i - y_j
\]

\[
= \left( \sum_{u=1}^{l-1} \left( \frac{j}{u} \right) \right) (-1)^{l-j-i}k^s x_u - y_j
\]

\[
= \left( \sum_{u=1}^{l-1} \left( \frac{j}{u} \right) \right) (-1)^{l-j-i}k^s \left( \sum_{u=1}^{l-1} \left( \frac{\mu(ki)}{u} \right) Y_u + N(\mu(ki)^{pr}) N_{y_u} \right) - y_j
\]

\[
= k^r \sum_{i=1}^{l-1} \left( \frac{j}{i} \right) (-1)^{l-j-i} \sum_{u=1}^{l-1} \left( \frac{\mu(ki)}{u} \right) Y_u
\]

\[
+ k^r \sum_{i=1}^{l-1} \left( \frac{j}{i} \right) (-1)^{l-j-i} N(\mu(ki)^{pr}) N_{y_u}
\]

\[- Y_j - \sum_{i=1}^{l-1} \left( \frac{j}{i} \right) (-1)^{l-j-i} N(i^{pr}) N_{y_u}
\]

\[
= k^r \sum_{i=1}^{l-1} \left( \frac{j}{i} \right) (-1)^{l-j-i} \sum_{u=1}^{l-1} \left( \frac{\mu(ki)}{u} \right) Y_u - Y_j
\]

\[
+ \sum_{i=1}^{l-1} \left( \frac{j}{i} \right) (-1)^{l-j-i} (k^r N(\mu(ki)^{pr}) - N(i^{pr})) N_{y_u}
\]

by the properties (2.10). Lemma 2.1 shows

\[
(\psi - 1)y_j - k^r \sum_{i=1}^{l-1} \left( \frac{j}{i} \right) (-1)^{l-j-i} \sum_{u=1}^{l-1} \left( \frac{\mu(ki)}{u} \right) Y_u + Y_j
\]

\[
= \sum_{i=1}^{l-1} \left( \frac{j}{i} \right) (-1)^{l-j-i} N(\mu(ki)^{pr}) N(i^{pr}) (k^r i^{pr} - \mu(ki)^{pr}) N_{y_u}
\]

\[
\equiv \sum_{i=1}^{l-1} \left( \frac{j}{i} \right) (-1)^{l-j-i} N(\mu(ki)^{pr}) N(i^{pr}) (k^r i^{pr} - (ki)^{pr}) N_{y_u} \pmod{p^s r^{(s+2)} y_u}
\]

\[
\equiv \sum_{i=1}^{l-1} \left( \frac{j}{i} \right) (-1)^{l-j-i} N(\mu(ki)^{pr}) r(k-k^s) k^{r-1} N_{y_u} \pmod{p^s r^{(s+2)} y_u}
\]

\[
\equiv \sum_{i=1}^{l-1} \left( \frac{j}{i} \right) (-1)^{l-j-i} (1-k^{r-1}) r N_{y_u} \pmod{p^s r^{(s+2)} y_u}.
\]

Since \(1-k^{r-1}) r \equiv 0 \pmod{p^s r^{(s+2)}} \), (5) is obtained by [10, Lemma 3.7].
Lemma 2.14. Let $u$ and $j$ be integers with $1 \leq u \leq j \leq p-1$. Then, for each $k$ prime to $p$, we have
\[
\sum_{i=1}^{j-1}(-1)^{j-i} \binom{k}{i}^{u+j} \equiv \begin{cases} 0 & (\text{mod } p) \ (u<j) \\ k^{u} & (\text{mod } p) \ (u=j). \end{cases}
\]
Proof. From [10, Lemma 3.7], we have
\[
\sum_{i=1}^{j-1}(-1)^{j-i} \binom{k}{i}^{u+j} \equiv \sum_{i=1}^{j-1}(-1)^{j-i} \mu(k)^{u+j} \equiv k^{u} \sum_{i=1}^{j-1}(-1)^{j-i} \mu(k)^{u+j} \ (\text{mod } p)
\]
\[
= \begin{cases} 0 & (u<j) \\ k^{u}(u!) & (u=j). \end{cases}
\]
q.e.d.

3. Proofs of parts (2) and (4) i) of Theorem
We begin with the method which is used in the proof of [7, Theorem 2].

Lemma 3.1. Let $X$ be a finite CW-complex and assume that $\widetilde{KO}(X)$ has an odd order. Then the real restriction
\[
\rho: \mathcal{K}(X) \to \widetilde{KO}(X)
\]
is an epimorphism. In particular, if $\mathcal{K}(X)$ also has an odd order, then
\[
\ker \rho = (1-\tau)\mathcal{K}(X),
\]
and
\[
\ker J \circ \rho = \sum_{k}(\psi^k-1)\mathcal{K}(X))
\]
where $\tau: \mathcal{K}(X) \to \mathcal{K}(X)$ is the conjugation and $J: \widetilde{KO}(X) \to J(X)$ is the natural projection.

Proof. Let $c: \widetilde{KO}(X) \to \mathcal{K}(X)$ be the complexification. Since $\rho \circ c = 2: \widetilde{KO}(X) \to \mathcal{K}(X)$ is an isomorphism, $c$ is a monomorphism and $\rho$ is an epimorphism.

We now turn to the case in which $\mathcal{K}(X)$ also has an odd order. Since $\rho = \rho \circ \tau$, $\rho((1-\tau)\mathcal{K}(X)) = 0$. Conversely, assume $\rho(y) = 0$ for some $y \in \mathcal{K}(X)$, then $y + \tau(y) = c \circ \rho(y) = 0$. Since $\mathcal{K}(X)$ has an odd order, $y = 2x$ for some $x \in \mathcal{K}(X)$, and the equality $2y = y - \tau(y) = 2(1-\tau)x$ implies $y = (1-\tau)x$. Therefore
\[
\ker \rho = (1-\tau)\mathcal{K}(X).
\]
Since $\widetilde{KO}(X)$ has a finite order,
\[
\ker J = \sum_{k}(\psi^k-1)\widetilde{KO}(X))
\]
It follows from the compatibility of the Adams operations with the real restriction
(cf. [4, Lemma A 2]), that \( \ker J \circ \rho \) coincides with the subgroup generated by the elements of \( \ker \rho \) and \( \sum \kappa \left( \psi^k \right) \). Since \( \tau = \psi^{-1} : \bar{K}(X) \to \bar{K}(X) \), we have
\[
\ker \rho = (1-\tau)\bar{K}(X) \subset \sum \kappa \left( \psi^k \right) \bar{K}(X).
\]
Therefore,
\[
\ker J \circ \rho = \sum \kappa \left( \psi^k \right) \bar{K}(X).
\]
This completes the proof.

From Lemma 2.5, we have
\[
\bar{K}'(L_0(p)/L_0(p)) \cong \bar{K}'(L_0(p)) \cong 0
\]
for each odd integer \( i \). Therefore we have a short exact sequence
\[
0 \to \bar{K}(S^{2r}(L_0(p)/L_0(p))) \overset{i_{2r}}{\to} \bar{K}(S^{2r}(L_0(p))) \overset{i_{2r}}{\to} \bar{K}(S^{2r}(L_0(p))) \to 0.
\]
We now put
\[
V_n = \ker i_n,
\]
\[
U_n = \sum_{k \equiv 0(\mod p)} (\psi^k - 1)V_n.
\]
Then we have the following property.

(3.4) The group \( V_n \) is the direct sum of cyclic groups generated by
\[
p^{(s-1)/(g-1)} \psi^i, \quad (i = 1, \ldots, p-1).
\]
The order of \( p^{(s-1)/(g-1)} \psi^i \) is \( p^{(s-1)/(g-1)} - [(s-1)/(g-1)] \).

Moreover we have the following.

**Lemma 3.5.** Assume \( r > 0 \). Then \( U_n \) is the subgroup of \( V_n \) generated by
\[
p^{(s-1)/(g-1)+1} Y_i, \quad (i = 1, \ldots, p-1)
\]
and \( p^{(s-1)/(g-1)+2} Y_s \).

In the case \( r = 0 \), \( U_n \) is the subgroup of \( V_n \) generated by
\[
p^{(s-1)/(g-1)+1} Y_i, \quad (i = 1, \ldots, p-1).
\]

**Proof.** Put \( s = [n/(p-1)] \) and \( j = n - s(p-1) \), and consider the case \( r > 0 \). The lemma is true for \( U_i \) by (2.12), Lemma 2.13 and Proposition 2.6. Assume that the lemma is true for \( U_{n+1} \), that is
\[
U_{n+1} = \langle p^{j+1} Y_i | 1 \leq i \leq j+1 \rangle \cup \{ p^s Y_i | j+2 \leq i \leq p-1 \}
\]
\[
\cup \{ p^{s+p(r)+[(s+1)-r]/(g-1)+2} Y_s \}.
\]
Then, by Lemmas 2.13 and 2.14, we have
\[(\psi^k - 1) (p^j y_{j+1}) \equiv \begin{cases} (k^{t+j+1} - 1) p^i Y_{j+1} \pmod{U_n} & (j+1 \equiv v) \\ -(k^{t-1} - 1) p^i y_s \pmod{U_n} & (j+1 = v) \end{cases}.\]

Since \(U_n = \langle U_{n+1} \cup \{ (\psi^k - 1) (p^j y_{j+1}) | k \equiv 0 \pmod{p} \} \rangle\), the lemma is true for \(U_n\). The proof of the case \(r = 0\) is similar to the above proof. q.e.d.

We now turn to the proof of the part (2) of Theorem. From (3.4), Lemmas 2.13, 3.1 and 3.5 we obtain

\[
J(S^r(L_0(p)/L_0^n(p))) \cong V_n | U_n
\]

\[
\mathbb{R} = \begin{cases} \langle p^{(n-v)/(p-1)+1} y_r \rangle & (r = 0) \\ \langle p^{(n-v)/(p-1)+1} y_r \rangle \langle p^{(n+r)/(p-1)+2} y_n \rangle & (r > 0) \end{cases}
\]

Then, the equality

\[
\frac{((t-v)/(p-1)) + 1 - ((n-v)/(p-1)) + 1}{((t+r)/(p-1)) - ((n+r)/(p-1))}
\]

establishes the part (2) of Theorem.

We turn now to the proof of the part (4) i) of Theorem. By the above proof, we have the following lemma.

**Lemma 3.6.** If \(n + r + 1 \equiv 0 \pmod{p-1}\), then the quotient map

\[
q_n: S^r(L_0(p)/L_0^{n+1}(p)) \to S^{r+1}(L_0(p)/L_0^{n+1}(p))
\]

induces the isomorphism

\[
J(q_n^*): J(S^r(L_0(p)/L_0^{n+1}(p))) \to J(S^{r+1}(L_0(p)/L_0^{n+1}(p))).
\]

In the exact sequence of triple \((L_0(p), L_0^{n+1}(p), L^n(p))\), we have

\[
\tilde{KO} - r + 1(L_0(p)/L_0^{n+1}(p)) \cong 0
\]

by Lemma 2.5. Hence, we have an exact sequence,

\[
\tilde{KO}(S^r(L_0(p)/L_0^{n+1}(p))) \xrightarrow{f} \tilde{KO}(S^{r+1}(L_0(p)/L_0^n(p))) \xrightarrow{g} \tilde{KO}(S^{2r+2}) \to 0.
\]

Therefore, the row of the commutative diagram

\[
\begin{array}{ccccc}
J(q_n^*) & & J(f) & & J(g) \\
J(S^r(L_0(p)/L_0^{n+1}(p))) & \to & J(S^{r+1}(L_0(p)/L_0^n(p))) & \to & J(S^{2r+2}) \to 0
\end{array}
\]

is exact by [3, Theorem (3.12)]. Since the map \(J(q_n^*)\) is the isomorphism by Lemma 3.6, we have a split short exact sequence

\[
0 \to J(S^r(L_0(p)/L_0^{n+1}(p))) \to J(S^{r+1}(L_0(p)/L_0^n(p))) \to J(S^{2r+2}) \to 0.
\]
This completes the proof.

4. Proof of the part (4) ii) of Theorem

In this section we assume that \(n+r+1 \equiv 0 \pmod{(p-1)}\). We set \((n+1-v)/(p-1) = s\) where \(v\) denotes the integer defined in section 2. By making use of the isomorphisms

\[
\tilde{K}(S^{2r}(L_5(p)/L_6^0(p))) \simeq V_n
\]

and

\[
\tilde{K}(S^{2r}(L_5(p)/L_6^{s+1}(p))) \simeq V_{n+1},
\]

we obtain the following commutative diagram, in which the row is exact:

\[
\begin{array}{cccc}
0 & \rightarrow & V_{n+1} & \rightarrow & \tilde{K}(S^{2r}(L_5(p)/L_6^s(p))) & \rightarrow & \tilde{K}(S^{2n+2r+2}) & \rightarrow & 0 \\
& & \downarrow f_1 & & \downarrow \phi & & \downarrow f_3 & & \\
& & V_{n+1} & \rightarrow & V_n.
\end{array}
\]  

(4.1)

It follows from Corollary 2.7 that we have \(x \in \tilde{K}(S^{2r}(L_5(p)/L_6^s(p)))\) such that \(f_3(x) = p^s y_s^n\) and \(f_6(x)\) is a generator of \(\tilde{K}(S^{2n+2r+2})\). Since \(\tilde{K}(S^{2n+2r+2})\) is isomorphic to \(Z\), we have a direct sum decomposition

\[
\tilde{K}(S^{2r}(L_5(p)/L_6^s(p))) \simeq f_1(V_{n+1}) \oplus Z \{x\}
\]

where \(Z \{x\}\) means the infinite cyclic group generated by \(x\).

For the Adams operation, we have the following lemma.

Lemma 4.2. (1) For each integer \(k\) prime to \(p\), we have

\[
\psi^h(x) \equiv k^{s+r+1} x - (k^{s+r+1}-1+1(p) f_1(p^{s+1} y_s)) \pmod{f_1(U_{n+1})}.
\]

(2) If \(k \equiv 0 \pmod{p}\), then we have

\[
\psi^h(x) \equiv k^{s+r+1} x - (k^{s+r+1} \pmod{p}) f_1(p^{s+1} y_s) \pmod{f_1(U_{n+1})}.
\]

Proof. (1) We necessarily have

\[
\psi^h(x) \equiv \alpha f_1(p^{s+1} y_s) + \beta x \pmod{f_1(U_{n+1})}
\]

for some integers \(\alpha\) and \(\beta\) by (2.12), (3.4), Lemmas 2.13 and 3.5. By using the \(\psi\)-map \(f_3\), we see that \(\beta = k^{s+r+1}\). Now project into \(V_n; f_1(p^{s+1} y_s)\) maps into \(p^{s+1} y_s\) and \(x\) into \(p^s y_s^n\), and we see that

\[
(k^{s+r+1} + p \alpha) (p^s y_s^n) \equiv \psi^h(p^s y_s^n) \pmod{U_{n+1}}.
\]
It follows from Lemma 2.13 that
\[(k^{*+r+1} + p \alpha) (p^s y_s) \equiv ((1 - k^{s+1}) r + 1) p^s y_s \pmod{U_n+1}.\]

This implies that
\[\alpha f_i(p^{s+1} y_s) \equiv -((k^{*+r+1} - 1) + r(k^{s+1} - 1)) / p f_i(p^{s+1} y_s) \pmod{f_i(U_{n+1})}\]
and
\[\psi^h(x) \equiv \alpha f_i(p^{s+1} y_s) + k^{s+1} x \pmod{f_i(U_{n+1})}\]
\[\equiv k^{s+1} x - ((k^{*+r+1} - 1) + r(k^{s+1} - 1)) / p f_i(p^{s+1} y_s) \pmod{f_i(U_{n+1})}.
\]

(2) If \( k \equiv 0 \pmod{p} \), then
\[\psi^h(\sigma) = \psi^h(\eta - 1) = (\psi^h(\eta) - 1) = (\eta^h - 1) = (1 - 1)^i = 0,\]
and
\[\psi^h(y_i) = 0.\]

Hence, the desired result is obtained by using the similar method used in the proof of (1).

We now recall some definition in [3]. Set \( Y = K(S^2(L_0(p)) \cap \mathbb{L}(p)) \) and let \( f \) be a function which assigns to each integer \( k \) a non-negative integer \( f(k) \).

Given such a function \( f \), we define \( Y_f \) to be the subgroup of \( Y \) generated by \( \{k^{f(k)}(\psi^h(y-1)(y) | k \in \mathbb{Z}, y \in Y \} \), that is
\[Y_f = \langle \{k^{f(k)}(\psi^h(y-1)(y) | k \in \mathbb{Z}, y \in Y \} \rangle.
\]
Then the kernel of the homomorphism \( J'': Y \rightarrow J''(Y) \) coincides with \( \bigcap_f Y_f \) where the intersection runs over all functions \( f \).

Suppose that \( f \) satisfies
\[(4.3) \ f(k) \geq t + \max \{\nu_q(m(n+r+1)) | q \) is a prime divisor of \( k \}\]
for every \( k \in \mathbb{Z} \). In the following calculation we put \( n+r+1 = u \) for the sake of simplicity. From Lemmas 4.2 and 2.2, we have
\[k^{f(k)}(\psi^h(y-1)(x)) = k^{f(k)}(k^n - 1)x - k^{f(k)}((k^n - 1) + r(k^{s+1} - 1)) / p f_i(p^{s+1} y_s) \pmod{f_i(U_{n+1})}\]
\[= k^{f(k)}(k^n - 1)x - k^{f(k)} N(u/p^{s+1}(u)) ((u(k^n - 1) + ur(k^{s+1} - 1)) / p^{s+1} y_s) f_i(p^{s+1} y_s) \pmod{f_i(U_{n+1})}\]
\[= (k^{f(k)}(k^n - 1) + p^{s+1} y_s) ((u(k^n - 1) - r(k^{s+1} - 1)) / p^{s+1} y_s) f_i(p^{s+1} y_s) \pmod{f_i(U_{n+1})}\]
\[= (k^{f(k)}(k^n - 1) + p^{s+1} y_s) x - N(u/p^{s+1}(u)) (n+1) f_i(p^{s+1} y_s)).\]

By virtue of [3, Theorem (2.7) and Lemma (2.12)],
\begin{align*}
    &\langle f_1(U_{n+1}) \cup \{k^{(b)}(\psi^k-1)(x) \mid k \in \mathbb{Z}\} \rangle \\
    &= \langle f_1(U_{n+1}) \cup \{m(u)q^{s(k)+1}(p^s(k)+1)x - N(u/p^s(k)) (n+1)f_1(p^{s+1}y_0)\} \rangle \\
    &= \langle f_1(U_{n+1}) \cup \{m(u)x - (m(u)/p^s(k)+1)N(u/p^s(k)) (n+1)f_1(p^{s+1}y_0)\} \rangle.
\end{align*}

Therefore,

\[ Y_f = \langle f_1(U_{n+1}) \cup \{n+r+1)x - Mf_1(p^{s+1}y_0)\} \rangle \]

where \( M=(m(n+r+1)/p^s(n+r+1))N((n+r+1)/p^s(n+r+1)) (n+1) \). Since this is true for every function \( f \) which satisfies (4.3), we have

\[ (4.4) \quad f''(Y) = Y/\langle f_1(U_{n+1}) \cup \{n+r+1)x - Mf_1(p^{s+1}y_0)\} \rangle. \]

We now recall the notation of \( h(r, t, n+1) \):

\[ h(r, t, n+1) = \begin{cases} 
    \min \{\nu_p(r)+1, [(t+r)/(t-1)]-(n+r+1)/(t-1)\} & (r>0) \\
    [t/(t-1)]-(n+1)/(t-1) & (r=0). 
\end{cases} \]

Then we have the following lemma.

**Lemma 4.5.**

\[ J''(K(S^2(L^0(p)/L^1(p)))) = \mathbb{Z}_p \otimes \mathbb{Z}[m(n+r+1)p^{k(r,t,n+1)}]^{-i} \]

where \( i=\min \{\nu_p(n+1), h(r, t, n+1)\} \).

**Proof.** By (4.4), we have

\[ J''(K(S^2(L^0(p)/L^1(p)))) = \langle \{x, f_1(p^{s+1}y_0)\} \rangle/\langle \{m(n+r+1)x - Mf_1(p^{s+1}y_0), p^{k(r,t,n+1)}f_1(p^{s+1}y_0)\} \rangle. \]

Since \( \nu_p(M)=\nu_p(n+1) \), the greatest common divisor of \( M \) and \( p^{k(r,t,n+1)} \) equals to \( p^i \). Choose integers \( a \) and \( b \) with

\[ aM + bp^{k(r,t,n+1)} = p^i. \]

Then, it is easily seen that

\[ \langle \{m(n+r+1)x - Mf_1(p^{s+1}y_0), p^{k(r,t,n+1)}f_1(p^{s+1}y_0)\} \rangle = \langle \{m(n+r+1)p^{k(r,t,n+1)}-i)x, p^{i}(a(m(n+r+1)/p)x-f_1(p^{s+1}y_0)) \rangle \]

and

\[ \langle \{x, f_1(p^{s+1}y_0)\} \rangle = \langle \{x, a(m(n+r+1)/p)x-f_1(p^{s+1}y_0)\} \rangle. \]

This proves the lemma.

Now, by virtue of the above lemma, the proof of Theorem (4) ii) is completed by the following lemma.
Lemma 4.6. $J(S^2(L_0(p)/L^s(p))) \cong J''(\tilde{K}(S^2(L_0(p)/L^s(p))))$.

Proof. In this proof we put $A = S^2(L_0(p)/L^s(p))$, $B = S^2(L_0(p)/L_0^{+1}(p))$ and $C = S^{2s+2r+2}$ for the sake of simplicity.

Consider the homomorphisms

$$J''(\tilde{K}(B)) \xrightarrow{J''(\rho_1)} J''(\tilde{K}(A)) \xrightarrow{J''(c_1)}$$

where $\rho_1$ and $c_1$ are the real restriction and the complexification. Then, by making use of Lemma 2.5, Lemma 3.1 and [3, Lemma (3.8) and Theorem (3.12)], we see that $J''(\ker \rho_1) \cong 0$ and so $J''(\rho_1)$ and $J''(c_1)$ are the isomorphisms. Since $p$ is an odd prime and $n+r+1 \equiv 0 \pmod{p-1}$, there are following cases.

i) If $n+r+1 \equiv 0 \pmod{4}$, then we have the commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & J''(\tilde{K}(B)) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \tilde{K}(B) \\
\end{array}
\]

Then we have the following commutative and exact diagram:

\[
\begin{array}{ccc}
0 & \longrightarrow & \tilde{K}(B) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \tilde{K}(A) \\
\end{array}
\]

of exact sequences. By making use of [3, Lemma (3.8) and Theorem (3.12)], we have the following commutative and exact diagram:

\[
\begin{array}{ccc}
0 & \longrightarrow & \tilde{K}(B) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \tilde{K}(A) \\
\end{array}
\]

Since $J''(f)$ is a monomorphism by (4.4), $J''(c_2)$ is also an isomorphism. Therefore we obtain

$$J(A) \cong J''(\tilde{K}(A)) \cong J''(\tilde{K}(A)) .$$

ii) If $n+r+1 \equiv 2 \pmod{4}$, then we have a commutative diagram
of exact sequences. Inspecting the diagram, we see that $p_2$ is an epimorphism and

$$f|_{\ker p_1}: \ker p_1 \to \ker p_2$$

is an isomorphism. By making use of [3, Lemma (3.8) and Theorem (3.12)] we have the following commutative and exact diagram:

$$
\begin{array}{cccccc}
0 & \to & J''(\ker p_1) & \to & J''(\ker p_2) & \to & 0 \\
\vert & & \vert & & \vert & & \\
J''(\mathcal{K}(B)) & \to & J''(\mathcal{K}(A)) & \to & J''(\mathcal{K}(C)) & \to & 0 \\
\vert & & \vert & & \vert & & \\
J''(\mathcal{K}O(B)) & \to & J''(\mathcal{K}O(A)) & \to & J''(\mathcal{K}O(C)) & \to & 0 \\
\vert & & \vert & & \vert & & \\
0 & & 0 & & 0 & & \\
\end{array}
$$

It follows from the first part of the proof that $J''(\ker p_1)^0$. Thus we have

$$J'\mathcal{K}(A)\equiv J''(\mathcal{K}O(A))\equiv J''(\mathcal{K}(A)).$$

This completes the proof of the lemma.

5. **Proofs of the other parts of Theorem**

In this section we complete the proof of Theorem. We begin with the part (1). Since $\mathcal{K}O(S^{2r+1}(L_0(p)/L_0^+(p)))^0$, we have

$$J(S^{2r+1}(L_0(p)/L_0^+(p)))^0.$$
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\[ L^\nu(p) \), we have an isomorphism

\[ \tilde{KO}(S^{2r+1}(L^0(p)/L^\nu(p))) \cong \tilde{KO}(S^{2n+2r+3}). \]  

Hence we obtain

\[ \tilde{J}(S^{2r+1}(L^0(p)/L^\nu(p))) \cong \tilde{J}(S^{2n+2r+3}). \]

This proves the part (3).

Similarly, by making use of the exact sequences of the triple \((L^1(p), L^0(p), L^\nu(p))\), we have an isomorphism

\[ \tilde{J}(S^{2r+1}(L^1(p)/L^\nu(p))) \cong \tilde{J}(S^{2n+2r+1}). \]

This proves the part (5).

We now turn to the part (7). Put \( A=S^{2r+1}(L^\nu(p)/L^\nu(p)) \), \( B=S^{2r+1}(L^0(p)/L^\nu(p)) \), \( C=S^{2r+2+2} \), and \( D=S^{2r+1}(L^1(p)/L^\nu(p)) \). Then, we have the commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \tilde{KO}(A) & \longrightarrow & \tilde{KO}(B) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{KO}(C) & \longrightarrow & \tilde{KO}(D) & \longrightarrow & 0 \\
\end{array}
\]

where the row is exact. It follows from (5.1) that \( i \) is an isomorphism and \( \tilde{KO}(B) \cong \tilde{KO}(S^{2n+2r+3}) \). Hence the row sequence splits as an exact sequence of \( \psi \)-groups. Thus we have

\[ \tilde{J}(S^{2r+1}(L^\nu(p)/L^\nu(p))) \cong \tilde{J}(C) \oplus \tilde{J}(B) \cong \tilde{J}(S^{2n+2r+3}) \]

This proves the part (7).

We now turn to the part (8). Put \( A=S^{2r}(L^\nu(p)/L^\nu(p)), B=S^{2r}(L^0(p)/L^\nu(p)), C=S^{2r+2+1}, D=S^{2r}(L^1(p)/L^\nu(p)), \) and \( E=S^{2r}(L^{1+1}(p)/L^{1+1}(p)). \) Then we have the commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \tilde{KO}(A) & \longrightarrow & \tilde{KO}(B) & \longrightarrow & 0 \\
\downarrow j & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \tilde{KO}(C) & \longrightarrow & \tilde{KO}(D) & \longrightarrow & \tilde{KO}(E) \\
\downarrow \delta_1 & & \downarrow \delta_2 & & \downarrow \delta_2 & & \\
\tilde{KO}(C) & = & \tilde{KO}(C) & & \tilde{KO}(C) & & \\
\end{array}
\]

of exact sequences. Since \( \tilde{KO}(E) \) has an odd order and \( 2\tilde{KO}(C) \cong 0 \), we have
This implies that \( \delta_i \) is an isomorphism. Hence the middle row sequence of the diagram splits as an exact sequence of \( \psi \)-groups. Therefore we have

\[
\tilde{J}(S^n(L^t(\mathcal{p}))/L^s(\mathcal{p}))) \cong \tilde{J}(S^{2n}(L^0(\mathcal{p}))/L^s(\mathcal{p}))) \oplus \tilde{J}(S^{2r+2r+1})
\]

This proves the part (8).

Finally we note that the proof of part (6) is similar to that of part (8). q.e.d.

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