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THE THURSTON NORM AND THREE-DIMENSIONAL SEIBERG-WITTEN THEORY

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In October 1994 a flurry of activity started around a set of equations discovered by Seiberg and Witten [7]. By November there were three papers published on the Seiberg-Witten equations. One of those papers was Kronheimer and Mrowka's proof of the Thom conjecture [4]. The Thom conjecture was independently proved by Morgan, Szabo, and Taubes, and generalizations and variants of the Thom conjecture will be appearing soon [5], [2], [6]. The result in this paper is a three-dimensional version of the Thom conjecture.

The Thom conjecture is that an algebraic curve in CP^2 realizes the minimal genus in a given homology class. In a general algebraic surface,

$$2g-2\geq|K\cdot F|+F\cdot F \tag{1}$$

when F is an embedded surface of genus g with positive self intersection and K is the canonical class. The same inequality holds when there is an algebraically non-trivial number of solutions to the equations

$$\begin{aligned} F_A^+ &= (\psi,\bar{\psi}) \\ \partial\!\!\!/_A\psi &= 0 \end{aligned} \tag{2}$$

on a line bundle with first Chern class K. The equations in (2) are the Seiberg-Witten equations, which we mentioned previously.

In 1986 Thurston defined a norm on the second homology of irreducible, atoroidal, 3 manifolds [9]. The norm of any non-trivial integral homology class is defined to be the minimum of $2g-2$ over all embedded surfaces representing the class. Our main theorem is exactly a lower bound on this quantity.

Theorem. *If the algebraic number of solutions to the* 3*D Seiberg-Witten equations is non zero, then*

$$2g-2\geq|c_1(L)\cap F|$$

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where F is an embedded surface of genus g.

The arguments in this paper are very similar to the arguments in Kronheimer-Mrowka, so we will give short arguments for the points which are essentially already made there. We begin by defining Spin_c-structures. The group SO_n has a 2-fold cover called Spin(n), which is used to make a group $\mathrm{Spin}(n) \times_{\mathbf{Z}_2} S^1$, called $\mathrm{Spin}_c(n)$. In three dimensions we have

$$\mathrm{Spin}_c(3) = (Sp_1 \times S^1)/\sim \qquad (q,\lambda) \sim (\pm q, \pm\lambda).$$

where Sp_1 is the group of unit quaternions. We are especially interested in the following four representations of $\mathrm{Spin}_c(3)$.

$$\begin{aligned}
&\mu_{Sp_1}: \mathrm{Spin}_c(3) \times sp_1 \to sp_1; (q,\lambda)\alpha = q\alpha q^{-1}\\
&\mu_{\mathbf{H}}: \mathrm{Spin}_c(3) \times \mathbf{H} \to \mathbf{H}; (q,\lambda)x = qx\lambda^{-1}\\
&\mu_{\bar{\mathbf{H}}}: \mathrm{Spin}_c(3) \times \mathbf{H} \to \mathbf{H}; (q,\lambda)x = \lambda x q^{-1}\\
&\mu_L: \mathrm{Spin}_c(3) \times \mathbf{C} \to \mathbf{C}; (q,\lambda)z = z\lambda^2.
\end{aligned} \tag{3}$$

Recall the definition of a twisted product. Namely, $P \tilde{\times} V = P \times V/\sim$ where $(p,v) \sim (pg^{-1}, gv)$, when P is a right G-space and V is a left G-space.

DEFINITION. Let P be a principle $\mathrm{Spin}_c(3)$ bundle over M^3, then a *Spin_c-structure* is an orientation preserving isomorphism

$$\xi: P \tilde{\times} sp_1 \to T^*M$$

where

$W = P \tilde{\times} \mathbf{H}$ is the bundle of spinors,
$\bar{W} = P \tilde{\times} \bar{\mathbf{H}}$ is the bundle of conjugate spinors, and

$L = P \tilde{\times} \mathbf{C}$ is the associated line bundle.

There are three interesting maps between these vector bundles: Clifford multiplication, conjugation, and a spinor pairing.

$$\begin{gathered}
c: T^*M \otimes W \to W;\\
\xi^{-1}([p,\alpha]) \otimes [p,x] \mapsto [p,\bar{\alpha}x]\\
-: W \to \bar{W}\\
[p,x] \mapsto [p,\bar{x}]\\
(\,,): W \otimes \bar{W} \to T^*M\\
[p, x \otimes y] \mapsto \xi(\tfrac{1}{4}\mathrm{Im}(xiy)).
\end{gathered} \tag{4}$$

It is easy to check that the above maps are well defined. It is also important to realize that $\langle\alpha,\beta\rangle=\mathrm{Re}(\alpha\bar{\beta})$ is a natural pairing on $P\tilde{\times}sp_1$, so that a Spin_c structure automatically gives a manifold a Riemannian metric.

A Spin_c structure is very rich. There are two other principal bundles which may be constructed from a Spin_c structure. The maps

$$S^1 \subsetneq Sp_1\times S^1 \to \mathrm{Spin}_c(3)$$

$$Sp_1 \subsetneq Sp_1\times S^1 \to \mathrm{Spin}_c(3)$$

may be used to define principal bundles,

$$Q_{SO}=P/S^1 \quad \text{and} \quad Q_L=P/Sp_1$$

and a diagram:

$$\begin{array}{ccc} P & \xrightarrow{\pi_{SO}} & Q_{SO} \\ {\scriptstyle \pi_L}\downarrow & & \downarrow \\ Q_L & \to & M \end{array}$$

The bundle of orthonormal frames of the cotangent bundle of M is isomorphic Q_{SO} by the Spin_c structure, so we have a metric connection

$$\omega\in\Gamma(T^*Q_{SO}\otimes sp_1) .$$

Let $A\in\Gamma(T^*Q_L\otimes i\boldsymbol{R})$ be a connection on L, then

$$B=\pi_{SO}^*\omega+\pi_L^*A$$

is a connection on P. This gives us covariant defivatives on all of the associated vector bundles.

DEFINITION. The composition $\not{\partial}_A=c\circ\nabla_A$ is the *twisted Dirac operator* where $\nabla_A:\Gamma(W)\to\Gamma(T^*M\otimes W)$ is the covariant derivative and c is Clifford multiplication.

We can now write down the three-dimensional Seiberg-Witten equations. The equations are for a connection A, a scalar function ϕ, and a spinor field ψ. The equations are:

$$*F_A=id\phi+i\delta+2i(\psi,\bar{\psi}) , \tag{5.1}$$

$$\not{\partial}_A\psi=\psi(\tfrac{1}{2}\phi i) . \tag{5.2}$$

There equations are just the 4-dimensional equations on the manifold $\boldsymbol{R}\times M$ with a connection of the form $A' = A + \phi dt$. It turns out that any solution of these equations has $\phi \equiv 0$. We include ϕ so that we will have an elliptic complex later.

Lemma. *If (A,ϕ,ψ) is a solution to the 3D Seiberg-Witten equations* (5), *and δ is divergence free, then $\phi \equiv 0$ or $\psi \equiv 0$.*

Proof. Clearly (A,ϕ,ψ) is a solution if and only if $I_\delta(A,\phi,\psi) = 0$, where

$$I_\delta(A,\phi,\psi) \equiv \int_M |i * F_A + d\phi + \delta + 2(\psi,\bar{\psi})|^2 + 2|\partial\!\!\!/_A\psi - \psi(\tfrac{1}{2}\phi i)|^2 dvol \ .$$

By expanding the norms, we get

$$\begin{aligned} I_\delta(A,\phi,\psi) = \int_M & [|F_A|^2 + |d\phi|^2 + |\delta|^2 + \tfrac{1}{4}|\psi|^4 \\ & + 2|\partial\!\!\!/_A\psi|^2 + \tfrac{1}{2}\phi^2|\psi|^2 + 4(i * F_A, (\psi,\bar{\psi})) \\ & + 2(i * F_A, \delta) + 4((\psi,\bar{\psi}),\delta)]dvol \\ & + \int_M 4(\partial\!\!\!/_A\psi, \psi(\tfrac{1}{2}\phi i))dvol \\ & + \int_M 2id\phi \wedge F_A + 2d\phi \wedge *\delta + 4d\phi \wedge *(\psi,\bar{\psi}) \ . \end{aligned}$$

An easy computation shows that

$$d * (\psi,\bar{\psi}) = -\frac{1}{2}(\partial\!\!\!/_A\psi, \psi i)dvol \ .$$

Integration by parts will thus cancel out the last term in the last integral and the second-to-last integral. Integration by parts and the Bianchi identity kill the first term in the last integral. Using integration by parts and the fact that δ is divergence free will then show:

$$I_\delta(A, -\phi, \psi) = I_\delta(A, \phi, \psi).$$

We conclude that if (A,ϕ,ψ) is a solution to the δ equations, then $(A, -\phi, \psi)$ is a solution to the δ equations. Looking at the equations we see that $\phi\psi \equiv 0$. To finish the proof we need to know that $\psi(U) = \{0\}$ for an open set U implies that $\psi \equiv 0$, so that either $\phi \equiv 0$ or $\psi \equiv 0$. This isolated singularities result follows from a theorem of Agmon and Nirenberg [1]. The relevant theorem states that if P_t is a family of self-adjoint operators satisfying

$$\left|\left\langle \left(\frac{dP_t}{dt}\right) v, v \right\rangle\right| \leq K_1 \|P_t v\| \, \|v\|,$$

and if

$$\left\|\frac{dw}{dt} - P_t w\right\| \leq K_2 \|w\|$$

with w vanishing for some initial interval, then w is zero for all time. When A is in radial gauge $\not{\partial}_A = \frac{\partial}{\partial r} + L_r$ where L_r is self-adjoint with respect to the metric on the sphere of radius r. By conjugating by appropriate metric factors we can get a family of operators that is self-adjoint with respect to a specific metric. The spinor field will no longer be in the kernel of the operator, but it will satisfy the right inequalities. □

We will now discuss the symmetries of these equations. The gauge group is the group of sections of the adjoint bundle. In our case the adjoint bundle to Q_L is trivial since S^1 is abelian. The gauge group is therefore $\mathscr{G} = \mathrm{Maps}(M, S^1)$. The gauge group acts on the triples (A, ϕ, ψ) by

$$(A, \phi, \psi) \cdot g = (A - 2g^{-1} dg, \phi, \psi g^{-1}) \tag{6}$$

One can check that $(A, \phi, \psi) \cdot g$ is a solution to the 3D equations (5) whenever (A, ϕ, ψ) is a solution. In the next few parts of this paper we will show that there are a finite number of gauge equivalence classes of solutions to the 3D equations; that the classes come with signs; and that the algebraic number of solutions only depends on the first Chern class of the line bundle L.

The first step in defining the algebraic number of solutions is to show that the moduli space of 3D solutions is compact. One might think that the compactness follows from the 4D compachness result, because the 3D equations are just a dimensional reduction of the original equations, but there really is something to check, because there are examples of dimensional reduction that do not preserve compactness. See Hass [H]. The heart of the compactness result is the following bound on the size of ψ.

Lemma. *If (A, ϕ, ψ) is a solution to the 3D equations (5), then*

$$|\psi|^2 \leq \max\{0, 2|\delta| - s\}. \tag{7}$$

Proof. We will use the Bochner-Lichnerowicz-Weitzenböck formula:

$$\not{\partial}_A{}^* \not{\partial}_A \psi = \nabla_A{}^* \nabla_A \psi + \frac{s}{4}\psi - \frac{1}{2} * F_A \cdot \psi.$$

In this formula s is the scalar curvature, and we are using a pairing,

$\Gamma((T^*M\otimes \boldsymbol{C})\otimes\Gamma(W)\to\Gamma(W)$, given by $(\alpha\otimes z)\otimes\psi\mapsto\alpha\psi z$ in the last term.

When $|\psi|^2$ is at its maximum $0\leq\langle\nabla_A{}^*\nabla_A\psi,\psi\rangle$ by the second derivative test. ($\nabla_A{}^*\nabla_A$ is the negative sum of the second derivatives.) Now use the Weitzenbock formula and the equations (5) to get the result. □

Compactness will now follow by an argument similar to the 4-dimensional case.

Lemma. *If there are no solutions to the* 3D *equations with* $\psi\equiv 0$, *then the space of gauge equivalence classes of solutions is compact.*

Proof. Given a sequence of solutions (A_n, ϕ_n, ψ_n), we will find a sequence of gauge transformations, g_n, so that $(A_n, \phi_n, \psi_n)\cdot g_n$ has a convergent subsequence. We have truly great bounds on ϕ_n, and the previous lemma gives us a good bound on ψ_n. Pick a fixed smooth connection, B, on L. Hodge theory allows us to write

$$\frac{1}{i}(A_n - B) = da_n + \delta b_n + \omega_n$$

where ω_n is harmonic. By picking a gauge transformation in the right component of the gauge group we may arrange for all of the ω_n to be in the same fundamental domain of the action of $H^1(M;\boldsymbol{Z})$ on $H^1(M;\boldsymbol{R})$. We can even pick a gauge transformation g_n so that

$$\frac{1}{i}(A_n\cdot g_n - B) = \delta b_n + \omega_n.$$

The first 3D Seiberg-Witten equation will then give a bound on δb_n. The equations will then give bounds on all of the derivatives of $(A_n, \phi_n, \psi_n)\cdot g_n$, so that Rellich's lemma may be used to find a convergent subsequence. □

The next step is to show that the space of solutions mod gauge equivalence is a collection of signed points. The following definition just establishes some useful notation.

DEFINITION.

$$S(M,L) = \{(A,\phi,\psi)\mid *F_A = id\phi + i\delta + 2i(\psi,\bar{\psi}),\ \not{\partial}\psi = \psi(\tfrac{1}{2}\phi i)\}$$

is the *space of* 3D *Seiberg-Witten* solutions, and

$$R(M,L) = S(M,L)/\,\mathrm{Gauge}$$

is the *Seiberg-Witten character variety.*

We split the problem of showing that $R(M,L)$ is a collection of signed points into three parts: show that $R(M,L)$ is a manifold, show that it zero-dimensional, and show that it is oriented. To show that $R(M,L)$ is a manifold, we will use the implicit function theorem to see that $S(M,L)$ is a manifold, and we will show that the gauge group acts freely on $S(M,L)$. The slice theorem will then imply that $R(M,L)$ is a manifold.

If (A,ϕ,ψ) has non-trivial stabilizer, then

$$\psi\cdot g^{-1}=\psi$$

with g non-trivial (see Line (6)). This implies that $\psi=0$ on the open set where $g\neq 1$. The argument at the end of the $\phi\equiv 0$ lemma now says that $\psi\equiv 0$. If we plug this into the 3D equations and use $c_1(L)=\frac{1}{2\pi i}F_A$, we get:

$$2\pi * c_1(L)=d\phi+\delta.$$

Projecting onto the harmonic subspace of $\Gamma(\wedge^1 M)$ gives:

$$2\pi \mathrm{pr}_{\mathscr{H}^1(g)}(*c_1(L))=\mathrm{pr}_{\mathscr{H}^1(g)}(\delta).$$

This motivates the following definition.

DEFINITION. A pair (ξ,δ) is *bad* if $2\pi\mathrm{pr}_{\mathscr{H}^1(g)}(*c_1(L))=\mathrm{pr}_{\mathscr{H}^1(g)}(\delta)$. The pair is *good* othervise. Here ξ is a spin$_c$-structure with associated line bundle, L, inducing a metric, g.

The above discussion proves the following lemma.

Lemma. *If (ξ,δ) is good, then the gauge group acts freely on $S(M,L)$.*

It is really easy to show that good pairs and families of good pairs exist. Good pairs and families of good pairs may even be constructed in such a way that δ is divergence free.

Lemma. *If $\dim H^1(M;\boldsymbol{R})\geq 1$ and ξ_0 is a given spin$_c$ structure, then there is a δ_0 and an $\varepsilon>0$ so that (ξ_0,δ) is good whenever $\|\delta-\delta_0\|_{L^2(g_0)}<\varepsilon$. Furthermore, if $\dim H^1(M;\boldsymbol{R})\geq 2$, and (ξ_0,δ_0), (ξ_1,δ_1) are good pairs with ξ_0 and ξ_1 in the same path component of the space of spin$_c$ structures then there are paths $\xi_t,\delta_t,\varepsilon_t>0$ so that (ξ_t,δ) is good whenever $\|\delta-\delta_t\|_{L^2(g_t)}<\varepsilon_t$.*

Proof. By Hodge theory, $H^1(M;\boldsymbol{R})\cong\mathscr{H}^1(g_0)$, so just pick δ_0 to be any element of $\mathscr{H}^1(g_0)$ except $2\pi\mathrm{pr}_{\mathscr{H}^1(g_0)}(*c_1(L))$ and pick $\varepsilon=\|2\pi\mathrm{pr}_{\mathscr{H}^1(g_0)}(*c_0(L))-\delta_0\|_{L^2(g_0)}$. We will construct a path of good pairs in four segments. In the first segment leave ξ_t fixed at ξ_0 and define to be the straight path from δ_0 to $\mathrm{pr}_{\mathscr{H}^1(g_0)}(\delta_0)$. To construct the next segment, start by using Hodge theory to show that $\mathscr{H}^1 \to (\text{spin}_c\text{-structures}) \times \Gamma(\wedge^1 M)$ is a vector bundle where

$$\underline{\mathscr{H}}^1 = \{(\xi, \delta, \alpha) \in \text{spin}_c\text{-structures} \times \Gamma(\wedge^1 M) \times \Gamma(\wedge^1 M) \mid \alpha \in \mathscr{H}^1(g(\xi))\}.$$

The first Chern class of L defines a section of $\underline{\mathscr{H}}^1$. As long as the fiber of $\underline{\mathscr{H}}^1$ is non-trivial, $\underline{\mathscr{H}}^1_* = \underline{\mathscr{H}}^1 - (\text{image of the } c_1\text{-section})$ will be a fibration.

For the second segment let ξ_t be any path from ξ_0 to ξ_1, and Δ_t be any path from δ_0 to δ_1. By the path-lifting property of fibrations there is a path in $\underline{\mathscr{H}}^1_*$ which covers the path (ξ_t, Δ_t) starting at the point $(\xi_t, \delta_0, \text{pr}_{\mathscr{H}^1(g_0)}(\delta_0))$ [S]. Let δ_t be the α component of the covering path in the second segment.

In the third segment let ξ_t be fixed and pick δ_t to be a path from the end point of the second segment to $\text{pr}_{\mathscr{H}^1(g_1)}(\delta_1)$ which does not go through $2\pi \text{pr}_{\mathscr{H}^1(g_1)}(c_1(L))$. We can do this because $\dim \mathscr{H}^1(g_1) \geq 2$. Finish with the straight path from $(\xi_1, \text{pr}_{\mathscr{H}^1(g_1)}(\delta_1))$ to (ξ_1, δ_1).

The set $S(M,L)$ may be written as the roots of a function. By putting the function into general position (choose a generic δ) we can force zero to be a regular value, thereby forcing $S(M,L)$ to be a manifold. To be exact, define a function:

$$G: \mathscr{A} \times \Gamma(\wedge^0 M \oplus \wedge^1 M) \times (\Gamma(W) - \{0\}) \to \Gamma(\wedge^1 M \oplus W);$$

$$G(A, \phi, \delta, \psi) = \begin{bmatrix} \frac{1}{i} * F_A - d\phi - \delta - 2(\psi, \bar{\psi}) \\ \not{\partial}_A \psi - \psi(\frac{1}{2}\phi i) \end{bmatrix}.$$

Lemma. $G \pitchfork \{0\}$.

Proof. Compute the derivative of G, $T_{(A,\phi,\delta,\psi)}G$:

$$TG: \Gamma(\wedge^1 M \oplus \wedge^0 M \oplus \wedge^1 M \oplus W) \to \Gamma(\wedge^1 M \oplus W);$$

$$TG(a, f, \gamma, s) = \begin{bmatrix} *da - df - \text{Im}(\psi i \bar{s}) - \gamma \\ \not{\partial}_A s - \frac{1}{2}c(ia \otimes \psi) - \frac{1}{2}\phi s i^{-}\frac{1}{2}\psi f i \end{bmatrix}.$$

We need to show that TG is surjective. Let $(\omega, \tau) \in TG^{\perp}$. Picking a, f and s all to be zero, we see that $\omega = 0$. Once ω is known to be zero the right choice of a will show that τ must be zero. (Remember that ψ cannot be zero on an open set.) Thus, $TG^{\perp} = \{0\}$ and TG is onto.

The above lemma and the following corollary even work if we restrict to the class of divergence free δ's.

Corollary. *If the pair* (ξ_0, δ) *is good for all* δ *within* ε *of* δ_0, *then there is a*

δ_1 so that $R_{(\xi_0,\delta_1)}(M,L)$ is a smooth manifold.

Proof. Let $\pi : G^{-1}(0) \to \Gamma(\wedge^1 M)$ be the projection onto the δ component. By the Sard-Smale theorem, there is a regular value of π within ε of δ_0. If δ_1 is that regular value, then $S_{(\xi_0,\delta_1)}(M,G) = \pi^{-1}(\delta_1)$ is a smooth manifold freely acted on by the gauge group. The slice theorem will then show that $R(M,L)$ is a smooth manifold. □

We will call (ξ_0,δ_1) a very good pair. As a small aside, notice that by repeating the arguments from the previous lemma and corollary, we can prove the following fact.

Fact. If ξ_1 and ξ_2 are two spin$_c$ structures in the same path component, and (ξ_1,δ_1), (ξ_2,δ_2) are very good pairs, then there is a cobordism between $R_{(\xi_1,\delta_1)}(M,L)$ and $R_{(\xi_2,\delta_2)}(M,L)$.

The above fact and the following lemma are the main ingredients in the proof that the number of points in $R(M,L)$ only depends on the line bundle, L.

Lemma. *Let ξ_1 and ξ_2 be two* spin$_c$*-structures with isomorphic line bundles. If ξ_1 and ξ_2 induce the same metric on M, then there is a bijection from $R_{(\xi_1,\delta)}(M,L)$ to $R_{(\xi_2,\delta)}(M,L)$.*

Proof. Let $\xi_i : P_i \tilde{\times} sp_1 \to T^*M$, $i = 1,2$ be two spin$_c$ strucures. We will first show that $P_1 \cong P_2$. Let $\Delta_G^* P_1 \times P_2 = \{(x,y) \in P_1 \times P_2 \mid \pi(x) = \pi(y)\} / \sim$ where $(x,y) \sim (xg,yg)$. There is an isomorphism from P_1 to P_2 if and only if $\Delta_G^* P_1 \times P_2$ has a section. The relation between a section and an isomorphism is,

$$\sigma[x] = [x, \zeta(y)]$$

where $\sigma : M \cong P_1 / G \to \Delta_G^* P_1 \times P_2$ and $\zeta : P_1 \to P_2$. Now $\Delta_G^* P_1 \times P_2$ is a principal Spin$_c$(3) bundle with the action $[x,y] \cdot (q,\lambda) = [(x(q,\lambda), y)]$. There is no obstruction to extending a section from the 0-skeleton to the 1-skeleton since $\pi_0(\mathrm{Spin}_c(3)) = 1$, and no obstruction to extending a section from the 2-skeleton to the 3-skeleton since $\pi_2(\mathrm{Spin}_c(3)) = 1$. The only relevant obstruction is therefore the first Chern class of the line bundle $(\Delta_G^* P_1 \times P_2) \tilde{\times} \mathbf{C}$.
But we have

$$\begin{aligned} L_1 \otimes L_2^{-1} \equiv (\Delta_G^* P_1 \times P_2) \tilde{\times}_{\mu L_1 \otimes \mu L_2^{-1}} (\mathbf{C} \times \mathbf{C}) &\cong (\Delta_G^* P_1 \times P_2) \tilde{\times} \mathbf{C} \\ [(x,y), z_1 \otimes z_2] &\mapsto [[x,y], z_1 z_2] \end{aligned}$$

where L_i is the line bundle associated to ξ_i. Thus

$$c_1((\Delta_G^* P_1 \times P_2) \tilde{\times} C) = c_1(L_1 \otimes L_2^{-1})$$
$$= c_1(L_1) - c_1(L_2) = 0.$$

If $\zeta : P_1 \to P_2$ is an isomorphism, then there is a unique section, $h \in \Gamma(P_1 \times_{\mathrm{Ad}} GL_3^+(\boldsymbol{R}))$ so that

$$\xi_1[p, \alpha] = \xi_2[\zeta(p), h(p)\alpha].$$

Varying the isomorphism gives us an identification:

$$\begin{matrix}\text{spin}_c\text{-structures} \\ \text{with line bundle } L_1\end{matrix} \cong \Gamma(P_1 \times_{\mathrm{Ad}} GL_3^+(\boldsymbol{R})) / \Gamma(\mathrm{Ad}\, P_1)$$

The assumption that ξ_1 and ξ_2 induce the same metric means that we can take $h \in \Gamma(P_1 \times_{\mathrm{Ad}} SO_3)$. There is a $\tau_1(p) \in Sp_1$ defined up to sign so that $h(p)(\alpha) = \tau_1(p)\alpha\overline{\tau_1(p)}$. This gives us a well defined section $\tau \in \Gamma(P_1 \tilde{\times} SO_4)$, by $\tau(p) = [\tau_1(p), \tau_1(p)] \in Sp_1 \times_{\boldsymbol{Z}_2} Sp_1 \cong SO_4$. This map τ induces a bijection from the space of solutions with spin$_c$-structure ξ_1 to the space of solutions with spin$_c$-structure ξ_2 given by $A, \psi \mapsto A', \psi'$ where $\psi' = \zeta(\tau \cdot \psi)$, and A' is the unique connection so that $\not{\partial}_{A'}\psi' = 0$. □

We have already seen that $R(M,L)$ is a manifold, so we need to show that it is zero-dimensional and oriented. To show this we will use the general principal that locally, a space looks just like its linearization. The implicit function is one example of this principal, the slice theorem is another. If a group G acts on a manifold, N then there is an evaluation map $\mathrm{ev}_{x_0} : G \to N$. This linerizes to $T_e \mathrm{ev}_{x_0} : T_e G \to T_{x_0} N$. The slice theorem says that N/G is locally homeomorphic to $\mathrm{coker}(T_e \mathrm{ev}_{x_0}) / G_{x_0}$, at $[x_0]$, where $G_{x_0} = \{g \in G \mid x_0 \cdot g = x_0\}$ is the stabilizer of x_0. In our case we have

$$\mathscr{G}_{\mathrm{ev}} \underset{(A_0, \phi_0, \psi_0)}{\to} \mathscr{A} \times \Gamma(\wedge^0 M \oplus W) \underset{H}{\to} \Gamma(\wedge^1 M \oplus W)$$

where

$$H(A, \phi, \psi) = \begin{bmatrix} \frac{1}{i} * F_A - d\phi - \delta - 2(\psi, \bar{\psi}) \\ \not{\partial}_A \psi - \psi(\frac{1}{2}\phi i) \end{bmatrix}.$$

The above sequence linearizes to

$$0 \to \Gamma(\wedge^0 M) \xrightarrow{L_0} \Gamma(\wedge^0 M \oplus \wedge^1 M \oplus W) \xrightarrow{L_1} \Gamma(\wedge^1 M \oplus W) \to 0$$

where

$$L_0(u)=\begin{bmatrix}0\\ -2du\\ -\psi_0(iu)\end{bmatrix}$$

and

$$L_1(f,a,s)=\begin{bmatrix} *da-df-\mathrm{Im}(si\bar{\psi}_0)\\ \partial\!\!\!/_{A_0}s-\frac{1}{2}c(ia\otimes\psi_0)-\frac{1}{2}\phi_0 si^{-}\frac{1}{2}f\psi_0 i\end{bmatrix}.$$

The slice theorem implies that $R(M,L)$ is locally homeomorohic to $H^1(L_*)/\mathscr{G}_{(A_0,\phi_0,\psi_0)} \cong H^1(L_*)$ since the stablizer is trivial at a good pair.

At this point we can explain why we included the function ϕ in the equations even though any solution has $\phi\equiv 0$. The reason is that with ϕ, L_* is an elliptic complex and without ϕ it is not. To briefly review, if $D:\Gamma(E)\to\Gamma(F)$ is an n-th order differential operator, and $\pi:T^*M\to M$ then $\sigma(D):\pi^*E\to\pi^*F$; $\sigma(D)(\alpha,s)=(\alpha,D(\frac{1}{n!}f^n\cdot s))$ is the symbol of D, where f is a function so that $d_{\pi(\alpha)}f=\alpha$ and $f(\pi(\alpha))=0$. A complex of differential operators is called an elliptic complex if the symbol sequence is exact off of the zero section.

The symbol sequence of L_* is

$$0\to\pi^*\wedge^0M\xrightarrow{\sigma(L_0)}\pi^*(\wedge^0M\oplus\wedge^1M\oplus W)\xrightarrow{\sigma(L_1)}\pi^*(\wedge^1M\oplus W)\to 0,$$

where

$$\sigma(L_0)(\alpha,u)=(0,-2u\alpha,0)\ ,$$

$$\sigma(L_1)(\alpha,f,a,s)=(*(\alpha\wedge a)-fa,c(\alpha\otimes s))\ .$$

We are now in a position to show that $R(M,L)$ is zero-dimensional.

Lemma. $\dim R(M,L)=0$.

Proof. It is an easy exercise to show that $H^0(L_*)\cong T_e\mathscr{G}_{(A_0,\phi_0,\psi_0)}=0$ and $H^2(L_*)=0$ because δ is a regular value. Thus,

$$\begin{aligned}\dim R(M,L)&=\dim H^1(L_*)\\ &=-(\Sigma_k(-1)^k\dim H^k(L_*))\\ &=-\mathrm{Index}\,L_*\\ &=\mathrm{Index}(L_0^*\oplus L_1)\ .\end{aligned}$$

But the index of a differential operator only depends on the highest order part of the operator, and the highest order part of $L_0^*\oplus L_1$ is self-adjoint (with the right metric). So $\dim R(M,L)=\mathrm{Index}(L_0^*\oplus L_1)=0$. □

The same ideas may be used to show that $R(M,L)$ is oriented.

Lemma. *$R(M,L)$ is oriented.*

Proof. An orientation is a section of the unit sphere bundle of the top exterior power of the tangent bundle. Now,

$$\begin{aligned}\Lambda^{\mathrm{Top}}TR(M,L) &\cong \Lambda^{\mathrm{Top}}H^1(L_*)\\ &\cong \Lambda^{\mathrm{Top}}H^0(L_*)^*\otimes\Lambda^{\mathrm{Top}}H^1(L_*)\otimes\Lambda^{\mathrm{Top}}H^2(L_*)^*\\ &\cong(\det(L_*))^*.\end{aligned}$$

The determinant line bundle extends to a line bundle on $\mathscr{A}\times\Gamma(\wedge^0 M\oplus W)/\mathscr{G}$, which is a simply-connected space, so $(\det(L_*))^*$ is a trivial bundle. It is, therefore, sufficient to pick a non-zero element of $(\det(L_*))^*$. Pick a connection B. Hodge theory then shows that

$$\begin{aligned}(\det(L_*))^*|_{(B,0,0)} \cong{}& \Lambda^{\mathrm{Top}}H^0(M)^*\otimes\Lambda^{\mathrm{Top}}H^0(M)\otimes\Lambda^{\mathrm{Top}}H^1(M)\otimes\Lambda^{\mathrm{Top}}H^1(M)^*\\ &\otimes\Lambda^{\mathrm{Top}}\ker\not\partial_B\otimes\Lambda^{\mathrm{Top}}(\mathrm{coker}\,\not\partial_B)^*.\end{aligned}$$

If V is any non-trivial vector space, then $V^*\otimes V$ has a special element correspoding to the identity map, id: $V\to V$. Furthermore, the twisted Dirac operator $\not\partial_B$ is complex linear, so both $\mathrm{Ker}\,\not\partial_B$ and $\mathrm{coker}\,\not\partial_B$ are naturally oriented. There is therefore a natural element of $(\det(L_*))^*$. □

We have just shown that there is a collection of signed points associated to any very good pair (ξ,δ). In an aside we proved two lemmas which imply that the number of points only depends on the first Chern class of the associated line bundle. In other words, we have shown that quantity defined below is well-defined.

DEFINITON. Let $\lambda(M,L)$ be the number of points in $R(M,L)$ counted with sign.

We may now prove our main theorem. The proof is easier than the analogous proof in four dimensions. In four dimensions it is shown that the Seiberg-Witten equations on $\boldsymbol{R}\times M$ are the gradient flow equations of a certain functional. The flow nature of the equations is then used to show that a family of solutions on a 4-manifold with an ever increasing neck gives rise to an $\boldsymbol{R}$-invariant solution in temporal gauge on the stretched out neck, $\boldsymbol{R}\times M$. The same results are true in the 3-dimensional case, but they are unnecessary. The only result that we need is that $\phi\equiv 0$ for any solution.

Theorem. *If $\lambda(M,L)\neq 0$ then*

$$2g-2\geq |c_1(L)\cap F|$$

where F is an embedded surface of genus g.

Proof. Since $\lambda(M,L)$ is well defined, we may pick a spin$_c$-structure which induces any given metric on M. Pick a metirc with constant sectional curvature equal to 0 or -1 on F and extend it to a metric on M with a metric product neighborhood around F. Finally, squeeze the metric around F. This means replace the metric with

$$dx^2+((x^2+R)/(x^2+1))^2 g_F \ .$$

By direct computation we see that $\max\{0,-s\}\to\infty$ as $R\to 0$. This means that we may assume that the maximum of $-s$ is obtained inside a neighborhood of F. Further computations show that the area of F is $4\pi(g-1)R^2$, and $\max\{0,-s\}\leq\frac{16(1-R)}{R}-\frac{2K}{R^2}$, here $K=0$ or -1, depending on the genus. Pick a solution to the Seiberg-Witten equations on M with a spin$_c$-structure which induces the above metric. Now,

$$\begin{aligned}
|c_1(L)\cap F| &= |-\int_F \frac{1}{2\pi i}F_A| \\
&= |\int_F \frac{1}{2\pi i}(*id\phi+2*(\psi,\bar{\psi})+i*\delta)| \\
&\leq \frac{1}{\pi}\int_F |*(\psi,\bar{\psi})|\mathrm{d}area+\frac{1}{2\pi}\int_F|\delta|\mathrm{d}area \\
&\leq \frac{1}{4\pi}\int_F |\psi|^2\mathrm{d}area+\frac{1}{2\pi}\int_F|\delta|\mathrm{d}area \\
&\leq -\frac{1}{4\pi}\int_F s\,\mathrm{d}area+\frac{1}{\pi}\int_F|\delta|\mathrm{d}area \\
&\leq \frac{1}{4\pi}\int_F \frac{16(1-R)}{R}-\frac{2K}{R^2}\mathrm{d}area+\frac{1}{\pi}\int_F|\delta|\mathrm{d}area \\
&\leq 2g-2+16(g-1)(R-R^2)+\frac{1}{\pi}\int_F|\delta|\mathrm{d}area
\end{aligned}$$

In the above lines we used the Chern-Weil definition of the Chern class, the fact that $\phi\equiv 0$, and the bound on $|\psi|^2$. Since $|c_1(L)\cap F|$ is an integer and δ, and R, may be chosen arbitrarily small, we are done. □

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