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Osaka University
1. Introduction

Let $X$ be a finite dimensional complex. We consider the general problem of classifying real $n$-plane bundles over $X$, which are in a natural one-to-one correspondence with $[X; BO(n)]$, the set of homotopy classes of maps from $X$ to $BO(n)$. Alternatively, if $\xi$ is a real stable bundle over $X$, we consider the problem of classifying $l$-plane bundles stably equivalent to $\xi$, which are in one-to-one correspondence with $[X; BO(n);l] = \{i\}^{-1}[f] \subseteq [X; BO(n)]$, where $f: X \to BO$ is any map which classifies $\xi$.

Line bundles over $X$ are classified by $\omega_1 \in H^1(X; \mathbb{Z}_2)$, while 2-plane bundles are classified by $\omega_1$ and $W_2 \in H^2(X; \mathbb{Z}[\omega_1])$, a (twisted if $\mathbb{R}^2 \neq O$) integer class which reduces to $\omega_2$. Oriented 3-plane bundles over a 4-complex were classified by Dold and Whitney [3], while James and Thomas enumerated $n$-plane bundles over an $n$-complex for $n$ odd, $n$-plane bundles over an $(n+1)$-complex for $n = 3(4)$, and oriented $n$-plane bundles over an $n$-complex for $n$ even. [4] (Note that "oriented" and "orientable" are equivalent concepts for bundles of odd, but not even, dimensions.) In [9] this result was extended to the case of $n$-plane bundles over an $n$-complex for all $n$ while in [6] the James and Thomas result was restated for a few low-dimensional cases in a somewhat more explicit form.

In [7], a spectral sequence approach was used, which (in theory) completely enumerates $[X; Y]$ in all cases where $X$ is a finite complex and $\pi_*(Y)$ is Abelian. In fact, all real and complex bundles over $P_k$, $k \leq 5$, are tabulated.

Nomura [16] has classified $n$-plane bundles over $P_{n+1}$ for $n \equiv 1(4)$ in most cases, and $n$-plane bundles over $P_{n+2}$ for $n \equiv 3(4)$ is some cases.

In the present paper, we use the approach of affine actions, developed in [9]. For the sake of space, it shall be assumed here that the reader is familiar with the constructions and notations of that paper. A general enumeration result is given for $n$-plane bundles over $X$ of dimension, $m$, provided $m \leq 2n-2$ (the metastable assumption), and provided $m \leq n+2$. We give specific results for $X=P_m$, real projective $m$-space, if $m=n+1$, $n \geq 3$, or if $m \leq n+2$, $n \equiv 3(4)$, $n \geq 7$. 

REAL $n$-PLANE BUNDLES OVER AN $(n+1)$-COMPLEX

LAWRENCE L. LARMORE

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After Nomura, let \( N_n(\xi; X) \) be the number of equivalence classes of \( n \)-plane bundles over \( X \) stably equivalent to \( \xi \). Let \( \eta \) be the Hopf bundle over \( P_m \), if \( m \geq 1 \). Clearly \( N_n(\eta; P_m) = 0 \) if \( m > n \) and \( \left( \begin{array}{c} k \\ n \end{array} \right) \) is odd, since \( w_{n+1}(\eta) \neq 0 \).

In the cases covered by Theorems 1.1 and 1.2, that is the only obstruction to \( n \)-dimensionality, in effect.

**Theorem 1.1.** Let \( n \geq 3, \left( \begin{array}{c} k \\ n+1 \end{array} \right) \) even.

Case I [James and Thomas]: if \( n = 3(4) \)

\[
N_n(\eta; P_{n+1}) = \begin{cases} 
2 \text{ if } \left( \begin{array}{c} k-1 \\ n-1 \end{array} \right) \text{ is even} \\
1 \text{ if } \left( \begin{array}{c} k-1 \\ n-1 \end{array} \right) \text{ is odd}
\end{cases}
\]

Case II [Nomura, except for the last case]: if \( n = 1(4) \)

\[
N_n(\eta; P_{n+1}) = \begin{cases} 
1 \text{ if } \left( \begin{array}{c} k \\ 2 \end{array} \right) \text{ odd, } \left( \begin{array}{c} k-1 \\ n-1 \end{array} \right) \text{ odd} \\
2 \text{ if } \left( \begin{array}{c} k \\ 2 \end{array} \right) \text{ odd, } \left( \begin{array}{c} k-1 \\ n-1 \end{array} \right) \text{ even} \\
2 \text{ if } \left( \begin{array}{c} k-1 \\ 2 \end{array} \right) \text{ even, } \left( \begin{array}{c} k-1 \\ n \end{array} \right) \text{ odd} \\
3 \text{ if } \left( \begin{array}{c} k \\ 2 \end{array} \right) \text{ even, } \left( \begin{array}{c} k-1 \\ n-1 \end{array} \right) \text{ even}
\end{cases}
\]

Case III: if \( n = 0(4) \)

\[
N_n(\eta; P_{n+1}) = \begin{cases} 
2 \text{ if } k \text{ odd} \\
3 \text{ if } k = 0(4), \left( \begin{array}{c} k-1 \\ n-1 \end{array} \right) \text{ odd} \\
2 \text{ if } k = 0(4), \left( \begin{array}{c} k-1 \\ n-1 \end{array} \right) \text{ even, } \left( \begin{array}{c} k \\ n \end{array} \right) \text{ odd} \\
5 \text{ if } k = 0(4), \left( \begin{array}{c} k-1 \\ n-1 \end{array} \right) \text{ even, } \left( \begin{array}{c} k \\ n \end{array} \right) \text{ even} \\
2 \text{ if } k = 2(4), \left( \begin{array}{c} k \\ n \end{array} \right) \text{ odd} \\
5 \text{ if } k = 2(4), \left( \begin{array}{c} k \\ n \end{array} \right) \text{ even}
\end{cases}
\]

Case IV: if \( n = 2(4) \)
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\[ N_n(k; \eta; P_{n+1}) = \begin{cases} 
1 & \text{if } k \text{ odd, } n=6 \\
2 & \text{if } k \text{ even, } n=6, \left(\frac{k}{n-1}\right) \text{ even} \\
2 & \text{if } k=3(4), n \geq 10 \\
4 & \text{if } k=1(4), n \geq 10 \\
6 & \text{if } k \text{ even, } n \geq 10, \left(\frac{k-1}{n-1}\right) \text{ even} \\
4 & \text{if } k \text{ even, } n \geq 10, \left(\frac{k}{n-1}\right) \text{ odd} \\
1 & \text{if } k \text{ even, } n=6, \left(\frac{k}{n-1}\right) \text{ odd} 
\end{cases} \]

**Theorem 1.2.** Let \( n=3(4), n \geq 7, \text{ and } \left(\frac{k}{n+1}\right) \text{ even. Then} \)

\[ N_n(k; \eta; P_{n+1}) = \begin{cases} 
2 & \text{if } \left(\frac{k-1}{n-1}\right) \text{ odd [Nomura]} \\
3 & \text{if } \left(\frac{k-1}{n-1}\right) \text{ even} 
\end{cases} \]

Theorems 1.1 and 1.2 are condensed versions of 3.8 and 3.10, below.

**2. The Main theory**

We shall utilize the techniques of Becker, McClendon, and the author with regards to constructions over, and over-and-under a fixed space. [2, 8, 12]

If \( \pi: E \rightarrow B \) is a fibration and \( f: X \rightarrow B \) is a map, let \([X; E]_f\) be the set of fiber-homotopy classes of liftings of \( f \) to \( E \), and let \([X; E; f]=(\pi_\lambda)^{-1}[f] \subset [X; E]\) be the set of homotopy classes of liftings of \( f \) to \( E \). Recall that \([X; E; f]\) is the set of orbits of a left action [9, 15]:

\[ \mu: \pi_\lambda(B^X, f) \times [X; E]_f \rightarrow [X; E]_f \]

Furthermore, if \( \dim X \leq 2n \), where each fiber of \( \pi \) is \( n \)-connected (the metastable assumption), \([X; E]_f\) is an Abelian affine group and \( \mu \) is an affine action, i.e., \( \mu(\alpha, \cdot) \) is an affine automorphism for each \( \alpha \in \pi_\lambda(B^X, f) \). In that case, we also have a left action \( \gamma \) of \( \pi_\lambda(B^X, f) \) on \([X; E]_f, \) the difference group of \([X; E]_f\). Writing \( ax \) for \( \gamma(\alpha, x) \) for all \( x \in [X; E]_f \), we have \( \alpha(a+x)=a+ax \) for all \( a \in [X; E]_f, x \in [X; E]_f \). More generally, if \( h^* \) is any \( B \)-twisted cohomology theory, a left action

\[ \gamma: \pi_\lambda(B^X, f) \times h^*(X, A, f) \rightarrow h^*(X, A, f) \]

can always be defined, for a subcomplex \( A \); and \( 7 \) is functorial in all the obvious ways.
Let $G = O$, $U$, or $Sp$, and let $r = 1, 2,$ or $4$, respectively. Let $\pi : BG(n) \to BG$ be a fibration replacing the usual inclusion, and let $S_n = S_nG$ be the $\Omega BG$-spectrum associated with $\pi$ (see [8]). Let $\Gamma^{n+1} \in H^1(BG, S_n)$ be the single obstruction to section of $\pi$. If $\xi$ is a stable vector bundle over a complex $X$ of dimension $m$, classified by $f : X \to BG$, we can define a characteristic class $\Gamma^{n+1} \xi \in H^1(X; f^{-1}S_n)$ the single (metastable) obstruction to $n$-dimensionality of $\xi$. We have [2, 8]:

**REMARK 2.1.** If $\text{g.d. } \xi \leq n$ (g.d. = geometric dimension), $\Gamma^{n+1} \xi = 0$. This condition is sufficient if $m \leq 2r + 2r - 3$.

**Proof.** The fiber of $BG(n) \to BG$ is $r(n+1) - 2$ connected.

Now let $A_n(\xi; X) = [X \times BG(n)]$, the set of $n$-plane bundles stabilized to $\xi$, an Abelian affine group in the metastable range, i.e., $m \leq 2r + 2r - 4$.

**REMARK 2.2.** (I) If $a, b \in A_n(\xi; X)$, a unique difference class $\Delta(a, b) \in H^0(X; f^{-1}S_n)$ is defined, such that $\Delta(a, c) = \Delta(a, b) + \Delta(b, c)$ for all $a, b, c \in A_n(\xi; X)$. (II) If $m \leq 2r + 2r - 4$, $\Delta(a, b) = 0$ if and only if $a = b$; while for any $a \in A_n(\xi; X)$ and $x \in H^0(X; f^{-1}S_n)$ there exists $b = a + x \in A_n(\xi; X)$ such that $\Delta(a, b) = x$. (III) If $m \leq 2r + 2r - 4$, $H^0(X; f^{-1}S_n)$ corresponds in a natural way to $A_n(\xi; X)$, the difference group of $A_n(\xi; X)$; provided the latter is non-empty.

Recall that $\Lambda_\xi : KG^{-1}(X) \simeq \pi_1(BG, \xi)$ in a natural way [4]. We thus have a left action:

$$\mu : KG^{-1}(X) \times A_n(\xi; X) \to A_n(\xi; X)$$

and $V_n(\xi; X)$, the set of equivalence classes of $n$-plane bundles stably equivalent to $\xi$, corresponds in a natural way to the set of orbits of $\mu$. Write $aa$ for $\mu(a, a)$ for any $a \in KG^{-1}(X) \mu \in A_n(\xi; X)$. Thus $N_n(\xi; X)$ is simply the cardinality of $V_n(\xi; X)$

We summarize our general results for classification of real bundles in low codimension cases. Throughout, let $\xi$ be a real stable bundle over a complex $X$ of dimension $m$, classified by $f : X \to BO$; let $Y = K(Z_2, 1) \times K(Z_2, 2)$, and let $\beta : BO \to Y$ be a map such that $\beta*(t_1 \otimes 1) = w_1$ and $\beta*(1 \otimes t_2) = w_2$. Without loss of generality, $X$ is connected, thus we may identify $H^0(X; Z_2)$ with $Z_2$.

**Theorem 2.3.** Let $m \leq 2n - 2, m \leq n + 2$. Then (I) There is an $\Omega$-$\text{spectra } \mathcal{L}_n$ such that $H^i(X; (\beta f)^{-1}\mathcal{L}_n) \cong H^i(X; f^{-1}S_n)$ (canonical isomorphism) (II) There is a universal characteristic class $\phi^{n+1} \in H^1(BO; \beta^{-1}\mathcal{L}_n)$ such that $\phi^{n+1} = f^*\phi^{n+1} = f^*\Gamma^{n+1} \xi$. (III) The constructions of $\mathcal{L}_n, \phi^{n+1}$ are independent of $X$ and $\xi$.

**Proof.** Let $S_n^{(2)}$ be the second stage of the Postnikov tower for $S_nO, S_n^{(3)}$ has homotopy width three or less. By McClendon [14], $S_n^{(2)} = \beta^{-1}\mathcal{L}_n$ for some
\[ \Omega Y \text{-spectrum } \mathcal{L}_m \text{, since } \beta \text{ is a 3-equivalence. Let } \phi^m = P_{21}^m, \text{ where } P: S_n \rightarrow S_n^{(2)} \text{ is the projection. The remaining details are trivial, and we are done.} \]

Let \( h_i = \sigma w_{i+1} \in H^i(\Omega Z_2) \) for all \( i \geq 0 \), where \( \sigma \) is the looping suspension. We have a short exact sequence:

\[ 0 \rightarrow [X; Spin] \rightarrow KO^{1-1}(X) \xrightarrow{h_{m+1}h_1} H^i(X; \mathbb{Z}_2) + H^i(X; \mathbb{Z}_2) \approx \pi_1(Y^X, \beta f) \rightarrow 0 \]

From 2.3 and [9, diagram (3-1)], we have

**REMARK 2.4.** Let \( m \leq 2n-2, m \leq n+2 \). Then there is a homomorphism \( \nu_\xi: [X; Spin] \rightarrow A_n(\xi; X) \) such that \( \mu(\alpha, \alpha) = a + \nu_\xi \alpha \) for all \( \alpha \in [X; Spin] \), \( a \in A_n(\xi; X) \).

We may express \( \nu_\xi \) in a very specific way. For any \( \alpha \in KO^{1-1}(X) \beta \), \( F_{\xi, \alpha} = \beta \circ f \circ p_X \), where \( F_{\xi, \alpha}: X \times S \rightarrow BO \) is a homotopy of \( f \) representing \( \Lambda_\xi \alpha \), and \( p_X: X \times S \rightarrow \text{Is the projection.} \)

Thus

\[ H^i(X \times S; (\beta \circ F_{\xi, \alpha})^{-1} \mathcal{L}_n) = H^i(X; (\beta \circ f)^{-1} \mathcal{L}_n) + H^i(X \times S, X(\beta \circ f \circ p_X)^{-1} \mathcal{L}_n) \]

and \( \nu_\xi \alpha \) can be uniquely defined by the equation

\[ F_{\xi, \alpha} \ast \phi^m = \phi^m \ast 1 + \nu_\xi \alpha \ast \sigma \]

where \( \sigma \in \pi_3(S) \) is the fundamental class of \( S \) in stable cohomotopy.

We now consider the action

\[ \gamma: KO^{1-1}(X) \times A_n(\xi; X) \rightarrow A_n(\xi; X) \]

where we write \( ax \) for \( \gamma(\alpha, x) \). 2.5 can be restated as follows:

**REMARK 2.5.** For \( m \leq 2n-2, m \leq n+2 \), and fixed \( x \in A_n(\xi; X) \), \( \alpha x \) depends only on \( h_0 \alpha \) and \( h_1 \alpha \). Equivalently, if \( h_0 \alpha = h_1 \alpha = 0 \), \( \alpha x = x \) for all \( x \in A_n(\xi; X) \).

We now give very specific expressions for \( \gamma \) in the case \( m = n+1 \), and the case \( m = n+2 \) for \( m \equiv 3(4) \). We shall assume that the reader is familiar with the procedure for construction of the first two stages of the relative modified Postnikov tower for \( BO(n) \rightarrow BO \) in all cases.

Generally, for any \( u \in H^i(X; Z_2) \), let \( Z[u] \) be the sheaf of integers over \( X \) twisted by \( u \); thus \( Z[0] = Z \), and \( Z[u+v] = Z[u] \otimes Z[v] \). Let \( \rho: H^i(X; Z[u]) \rightarrow H^i(X; Z_2) \) be reduction modulo 2, and let \( \delta[u]: H^i(X; Z_2) \rightarrow H^{i+1}(X; Z[u]) \) be the Bokstein homomorphism associated with the exact sequence of sheaves \( Z[u] \rightarrow Z[u] \rightarrow Z_2 \). Let \( \omega_i = \omega_i \xi \), the \( i^{th} \) Stiefel Whitney class of \( \xi \), for all \( i \geq 0 \).

**Theorem 2.6.** Let \( n \equiv 1(4) \), \( n \geq 5 \), \( \dim X = m \leq n+1 \), and assume \( \omega_{n+1} = 0 \).
Let \( \theta = Sq^2 + w_2 \). Then (1) We have an exact sequence:

\[ H^{n-1}(X; Z_2) \xrightarrow{\theta} H^{n+1}(X; Z_2) \xrightarrow{\lambda} A_n(\xi; X) \xrightarrow{p} H^n(X; Z_2) \rightarrow 0 \]

(II) If \( x \in H^n(X; Z_2) \) and \( z \in p^{-1}x \), then \( 2z = \lambda Sq^1x \).

(III) If \( x \in H^n(X; Z_2) \) and \( z \in p^{-1}x \) and if \( \alpha \in KO^{-1}(X) \),
\[
\alpha z = z + \lambda((h_1\alpha + w_2 U h_2\alpha) U x)
\]

**Theorem 2.7.** Let \( \rho = 3(4), n \geq 7, m \leq n+2 \); and assume \( w_{n+1} = 0 \). Let \( \theta = (Sq^2 + w_2 U + w_1 U) Sq^4 \).

- (I) We have an exact sequence:
\[
H^{n-1}(X; Z_2) \to Sq^{n+2}(X; Z_2) \to A_n^0(\xi; X) \xrightarrow{\delta} H^n(X; Z_2) \to 0
\]

- (II) If \( x \in H^n(X; Z_2) \) and \( z \in p^{-1}x \), then \( 2z = \lambda (Sq^2 + w_1 U + w_1 U)x \).

- (III) If \( x \in H^n(X; Z_2) \) and \( z \in p^{-1}x \), and \( \alpha \in KO^{-1}(X) \), then \( \alpha z = z + \lambda((h_1\alpha + w_2 U h_2\alpha) U x) \).

**Theorem 2.8.** Let \( n = 0(4), n \geq 4, m \leq n+1 \); and assume \( \delta[w_i]w = 0 \).

- (I) We have an exact sequence:
\[
H^{n-1}(X; Z[w_i]) + H^n(X; Z_2) \to H^{n+1}(X; Z_2) \to 0
\]

- (II) If \( x \in H^n(X; Z[w_i]), y \in H^{n+1}(X; Z_2), 2x = 0 \), and \( z \in p^{-1}(x, y) \), pick \( w \in H^{n+1}(X; Z_2) \) such that \( \delta[w_i]w = x \). Then \( 2z = \lambda (Sq^1 + (Sq^2 + w_1 U) w) \).

- (III) If \( x \in H^n(X; Z[w_i]), y \in H^{n+1}(X; Z_2) \) and \( z \in p^{-1}(x, y) \), and if \( \alpha \in KO^{-1}(X) \), then \( \alpha z = (-1)^{h_1\alpha + \lambda((h_1\alpha + w_1 U h_2\alpha) U x + h_3\alpha)) \).

**Theorem 2.9.** Let \( n = 2(4), n \geq 6, m \leq n+1 \); and assume \( \delta[w_i]w = 0 \).

- (I) We have an exact sequence:
\[
H^{n-1}(X; Z[w_i]) + H^n(X; Z_2) \to H^{n+1}(X; Z_2) \to 0
\]

- (II) If \( x \in H^n(X; Z[w_i]), y \in H^{n+1}(X; Z_2), 2x = 0 \), and \( z \in p^{-1}(x, y) \), pick \( w \in H^{n+1}(X; Z_2) \) such that \( \delta[w_i]w = x \). Then
\[
2z = \lambda (Sq^1 + (Sq^2 + w_1 U) y). \]

- (III) If \( x \in H^n(X; Z[w_i]), y \in H^{n+1}(X; Z_2), \) and \( z \in p^{-1}(x, y) \), then for any \( \alpha \in KO^{-1}(X) \), \( \alpha z = (-1)^{h_1\alpha + \lambda((h_1\alpha + w_1 U h_2\alpha) U x + h_3\alpha)) \).

Proof of 2.6-2.9. (I) is obtained from the McClendon spectral sequence [12], while (II) is computed using the extension results of [11]. (III) follows from 4.2, 4.6, and 4.7. (Note that \( w_i = h_2\alpha \), \( w_2 = h_2\alpha + w_1 U h_3\alpha \), in the notation of §4.)
In the range \( m^2n - 2 \), \( m^n - 2 \), knowledge of the three things suffices to enumerate \( V_n(\xi; X) \): namely, \( \nu_k \), \( \beta \), and a function \( s_a : H^q(X; Z_2) + H^r(XZ_2) \to A_n^q(\xi; X) \), for any fixed \( a \in A_n(\xi; X) \). [9, Thm 3.1] We shall see that \( s_a \) is determined by its values on generators, although it is not a homomorphism.

Without loss of generality, \( X \) is connected. Let \( \rho \in KO^{-1}(X) \) be classified by a map which takes \( X \) to a single point of \( O \) which does not lie in the identity component. For any \( u \in H^1(X; Z_2) \), let \( \psi[u] \in KO^{-1}(X) \) be classified by the composition \( X \to P_\infty \to SO \) where \( W \) is the Whitehead map; recall that \( W^*h_i = u^i \) for all \( i \). We define (for fixed \( a \in A_n(\xi; X) \)), for \( (x, y) \in H^q(XZ_2) + H^r(XZ_2) \):

\[
s_a(x, y) = \alpha x + a - a \quad \text{where} \quad \alpha = \begin{cases} \psi[v] & \text{if } x = 0 \\ \nu \psi[y] & \text{if } x = 1 \end{cases}
\]

For any \( (x, y) \in H^q(XZ_2) + H^r(XZ_2) \), and any \( z \in A_n(\xi; X) \), let \( \gamma'(x, y, z) = (x, y)z = \rho x + z \in A_n^q(\xi; X) \). It is clear that \( \gamma' \) is an action if \( m \leq 2n - 2 \) and \( m \leq n + 2 \).

From 2.4 and Theorem 3.1 of [9], we immediately have:

REMARK 2.10. Let \( X \) be a connected complex of dimension \( m \), where \( m \leq 2n - 2 \), \( m \leq n + 2 \); and let \( \xi \) be a real stable vector bundle over \( X \) which has an \( n \)-dimensional stabilization \( a \). Let \( \Gamma \subset H^1(XZ_2) \) be any generating set. Then, in order to enumerate \( V_n(\xi; X) \), it is sufficient to compute

(i) the Abelian group \( A_n(\xi; X) \)
(ii) the homomorphism \( \nu : [X; Spin] \to A_n(\xi; X) \)
(iii) the action \( \gamma' : H^q(X; Z_2) + H^r(X; Z_2) \times A_n(\xi; X) \to A_n^q(\xi; X) \)
(iv) \( s_a(1, 0) = s_a a \in A_n^q(\xi; X) \), and \( s_a(0, u) = s_a(\psi[u]) \in A_n^q(\xi; X) \) for all \( u \in \Gamma \).

The following lemma, which follows from James and Thomas [4, 1.4] will be a useful aid in computing \( s_a \). Let \( \rho^\wedge : A_n^q(\xi; X) \to H^*(X; Z_2) \) be the reduction defined in the obvious way, to wit, if \( x = a - b \), where \( a \) and \( b \) are stabilized vector bundles, classified by liftings \( g_a, g_b : X \to BO \) of \( f : X \to BO \), \( \rho^\wedge x \) is the difference class of \( g_a \) and \( g_b \) defined by \( w_{n+1} \) in \( BO \).

**Lemma 2.11.** Let \( \xi \) be a real stable bundle over a connected complex \( X \) of dimension \( m \), \( m \leq 2n - 2 \), and let \( a \in A_n(\xi; X) \). (I) If \( \alpha \in KO^{-1}(X) \), \( \rho^\wedge(\alpha a - a) = \sum_{i=0}^n h_i a \cup w_{n-i} \xi \). (II) If \( m \leq n + 2p \), \( a = w_n \xi \), and \( \rho^\wedge s_a(\psi[u]) = \sum_{i=0}^n u^i \cup w_{n-i} \xi \) for all \( u \in H^1(X; Z_2) \).

**Proof.** (I) follows immediately from the James and Thomas formula, and (II) is an immediate corollary of (I).

3. **Applications to projective spaces**

**Tensor products.** Let \( \xi \) be a stable real, or complex vector bundle over a
complex $X$, and let $L$ be a line bundle over $X$. Let $\xi^{(\infty)}$ be the virtual ra-plane bundle representing $\xi$; we define $L \otimes \xi$ to be the stable bundle represented by the virtual $n$-plane bundle $L \otimes \xi^{(\infty)}$. Let $t$ and $t_n$, respectively, classify $\otimes$ and $\otimes_n$ such that we have a commutative diagram:

\[
\begin{array}{ccc}
BG(1) \times BG(n) & \xrightarrow{t} & BG(n) \\
\downarrow 1 \times \pi & & \downarrow \pi \\
BG(1) \times BG & \xrightarrow{t_n} & EG
\end{array}
\]

If $a \in A_n(\xi; X)$, we can thus define $L \otimes a \in A_n(L \otimes \xi; X)$ as follows: if $g: X \to BG(n)$ classifies $a$, $t(l, g)$ classifies $L \otimes a$; where $l: X \to BG(1)$ classifies $L$.

REMARK 3.1. (I) $L \otimes: A_n(\xi; X) \to A_n(L \otimes \xi; X)$ is one-to-one and onto: in fact its inverse is $L' \otimes$, where $L'$ is the line bundle conjugate to $L$ (note that $L' = L$ in the real case). (II) In the metastable range, i.e., $\dim X \leq 2rn + 2r - 4$, $L \otimes$ is an affine isomorphism.

Proof. (I) is obvious; (II) requires some manipulation of the base spaces; we leave the details to the reader.

Let $l \otimes_n: BG \to BG$ be given by $l \otimes_n f' = t_n(l, f')$ for any $f': X \to BG$. If $f: X \to BG$ classifies $\xi$, and $\Lambda_\xi: \pi_i(BG^x, f) \approx KG^{-1}(X)$ is the James-Thomas isomorphism, let $\delta_\xi, L, n: KG^{-1}(X) \to KG^{-1}(X)$, the composition

\[
KG^{-1}(X) \xrightarrow{(\Lambda_\xi)^{-1}} \pi_i(BG^x, f) \xrightarrow{(l \otimes_n)^\xi} \pi_i(BG^x, l \otimes_n f', L \otimes_n \xi) \approx KG^{-1}(X)
\]

In the metastable range, define $L \otimes x \in A_n(L \otimes \xi; X)$ for all $x \in A_n(\xi; X)$ by $L \otimes (a - b) = L \otimes a - L \otimes b$, for all $a, b \in A_n(\xi; X)$. By simply chaising the definitions, we can easily check that:

REMARK 3.2. (I) For all $a \in A_n(\xi; X)$ and all $\alpha \in KG^{-1}(X)$, $L \otimes_\alpha a = \delta_\xi, L, n, a(L \otimes a)$. (II) If $m \leq 2nr + 2r - 4$, $L \otimes \alpha x = \delta_\xi, L, n, \alpha(L \otimes x)$ for all $x \in A_n(\xi; X)$ and $\alpha \in KG^{-1}(X)$.

And from the splitting theorem, we can compute

Lemma 3.3. In the real case, i.e., $G = O$, $t_n \ast w_h = \sum_{i=\frac{n}{2}}^{n-k+1} \binom{i}{k} \ast w_i$ $\otimes w_{h-i}$. (Where it is understood that if $a < 0$, $b \geq 0$, $\binom{a+b}{a}$ is defined (modulo 2) to be $\binom{a+b}{a+b-a}$ sufficiently large $N$.)

The null-element. Let $\xi$ be real, henceforth. We define $v_{\xi, n} \in KO^{-1}(X)$ to be the element corresponding to $N_{\xi, n} \in \pi_1(BO^x, f)$ where $N_{\xi, n}$ is the composition

\[
\begin{array}{ccc}
X \times S^{-} & \xrightarrow{j} & BO(1) \times BO \\
\downarrow t_n & & \downarrow t \times t_n \\
& & BO
\end{array}
\]
where $\mathcal{E}_i$ represents the generator of $\pi_i BO(1)$, and $T$ reverses coordinates. We call the $v_{\xi,n}$ null-element of $\xi$ in dimension $n$.

**Lemma 3.4.** Let $\xi$ be real. Then (I) $v_{\xi,n}$ is functorial in $X$, i.e., if $g: X' \to X$ is a map and $\xi' = g^{-1} \xi$, then $v_{\xi',n} = g^* v_{\xi,n}$. (II) For all $a \in A_n(\xi, X)$, $v_{\xi,n} a = a$. (III) For all $k \geq 0$, $h_k v_{\xi,n} = \binom{n-k}{1} v_{\xi,n}$. 

Proof. (I) is obvious. (II) holds since, if $g: X \to BO(n)$ is a lifting of $f$ which represents $a$, $N_{\xi,n}$ lifts to $t_0(\mathcal{E}_i \times g) \circ TX \times S \to BO(n)$, a self-homotopy of $g$. (III) follows immediately from 3.3.

**Projective spaces.** Henceforth, let $P_r$ be real projective $r$-space for any integer $r > 0$, and let $u \in H^1(P_r; \mathbb{Z})$ be the generator, if $r > 1$. Let $\psi = \psi[u] \in KO^{-1}(P_r)$. It is well-known and easily computable that

**Lemma 3.5.** Let $r \geq 1$. Then (I) if $r \neq 3(4)$, $KO^{-1}(P_r) \cong Z_s + Z_2$ with generators $p$ and $\psi$. (II) If $r = 3(4)$, $KO^{-1}(P_r) \cong Z_s + Z_2 + Z$ with generators $p$, $\psi$, and $\tau$, where $\tau \in [P_r; \text{Spin}]$ is classified by $P_r - \text{Spin} \to Spin$, and $\mathcal{E}_r$ represents the generator of $\pi_r(\text{Spin}) \cong Z$. (III) $KO^0(P_r) \cong [P_r; BO] \cong Z_s$, generated by $\eta$, the canonical line bundle over $P_r$ (i.e., $w_1(\eta) = u$), where $s$ is the number of positive integers less than or equal to $r$ which are equivalent to 0, 1, 2, or 4 modulo 8. (IV) For all $i \geq 1$, $h_i p = 0$, and $h_i \psi = u^i$; while $h_0 p = 1$ and $h_0 \psi = 0$. If $r = 2(4)$, $h_r \tau = u^r$ for $r = 2$ or 6; while $h_r \tau = 0$ if $i \neq r$ or $r > 6$. (V) For any $0 \leq k < 2^s$, $w_i(\eta^k) = \binom{k}{i} u^i$ for all $i \geq 0$.

**Lemma 3.6.** For $r \geq 1$ and $0 \leq k < 2^s$ (where $s$ is computed as in 3.5 (III)), (I) $\delta_{k+1,n} \psi = \psi$. (II) $\delta_{k+1,n} \rho = p + \psi$. (III) If $r = 2(4)$, $\delta_{k+1,n} \tau = \tau$.

Proof. Clearly $\delta_{k+1,n} \psi$ and $\delta_{k+1,n} \rho$ must be 2-torsion elements; the stated results for (I) and (II) are the only answers which agree with 3.3. To prove (III), consider the covering map $c: S^r \to P_r$. Now $c^{-1} \eta = 1$, the trivial line bundle, and $\text{Ker } c'$ is generated by $p$ and $\psi$. Thus $\delta_{k+1,n} \tau = \tau + \lambda_1 \rho + \lambda_2 \psi$ for $\lambda_1, \lambda_2 \in Z_s$. Again by 3.3, $\lambda_1 = \lambda_2 = 0$.

**Lemma 3.7.** Let $r \geq 1$, $0 \leq k < 2^s$. Then

\[
\nu_{k+1,n} = \begin{cases} 
0 & \text{if } n \text{ is even} \\
P & \text{if } n \text{ odd, } k \text{ even} \\
\rho + \psi & \text{if } n \text{ odd, } k \text{ odd}
\end{cases}
\]

Proof. For $r \neq 2(4)$, the stated result is the only possible which agrees with 3.3. If $r = 2(4)$, the result still holds, by 3.4(I), since $k_7$ lives in $P_{r+2}$.
Notation. We give standard names, \(x_{00}\), \(x_{01}\), and \(x_{10}\), for generators of the Abelian group \(A_n^\omega(k\eta; P)\) (which can be computed by one of the theorems 2.6-2.9), for \(r = n+1\), \(n \geq 3\). If \(n \equiv 3(4)\), let \(x_{00} = \lambda u^{n+1}\), while if \(n = 3(4)\), let \(x_{00} = 0\). Let \(x_{00}\) be defined by the equation \(px_{00} = u^n\) if \(n\) is odd, \(px_{00} = (\delta u^{n-1}, 0)\) if \(n\) and \(k\) are both even. If \(n\) is even and \(k\) is odd, \(x_{00}\) shall not be defined. If \(n\) is even, \(x_{10}\) shall be defined by the equation \(px_{10} = (\delta u^{n+1})\). For odd \(n\), \(x_{10}\) is not defined.

Note that \(x_{01}\) is always uniquely defined, but may be zero, while \(x_{00}\) and \(x_{10}\) are not always defined, have indeterminacy \(x_{01}\), and are never zero. Intuitively, if \(e^{\eta}\) and \(e^{\eta+1}\) are the bottom two cells of the Stiefel manifold \(V_n\) and \(\nu, \eta\) are the generators of the stable 0- and 1-stems in the homotopy of spheres, we may write \(x_{00} = e^{\eta} \otimes u^n\), \(x_{01} = e^{\eta} \otimes u^{n+1}\), and \(x_{10} = e^{\eta+1} \otimes \eta \otimes u^{n+1}\).

Define endomorphisms \(\gamma\) and \(\delta\) on \(A_n^\omega(k\eta; P_{n+1})\) for \(r = n+1\), \(n > 3\), as follows: \(\gamma(x_{00}) = x_{10}\) and \(\delta(x_{00}) = 0\) if \(x_{00}\) exists; and \(\gamma(x_{10}) = x_{10}\) if \(x_{10}\) exists.

Theorem 3.8. Let \(n \geq 3\), and let \(k\) be any integer such that \(\nu_{k\eta}\) is even. (I) If \(\nu_{k\eta}\) is zero, while if \(n = 6\), \(\nu_{k\eta} = \pm x_{10}\). (II) The group \(A_n^\omega(k\eta; P_{n+1})\), the automorphisms \(\gamma(p, )\) and \(\gamma(\nu, )\) on \(A_n^\omega(k\eta; P_{n+1})\), the elements \(s_a p, s_a \nu \in A_n^\omega(k\eta; P_{n+1})\) (for some choice of \(a \in A_n^\omega(k\eta; P_{n+1})\)), and the resulting value of \(N_n(k\eta; P_{n+1})\) are as in Table A.

Proof. The groups \(A_n^\omega(k\eta; P_{n+1})\), with their generators and relations, come from theorems 2.5-2.8, (I) and (II). The actions \(\gamma(p, )\) and \(\gamma(\nu, )\) are then from 2.5-2.8 (III). If \(n \equiv 2(4)\), \(\nu_{k\eta} = 0\) since \(P_{n+1}; \text{Spin}\) is even. If \(n = 2(4)\), \(P_{n+1}; \text{Spin}\) is generated by \(r\) (cf. 3.5) and \(\nu_{k\eta} = \pm x_{10}\) by 2.11 if \(n = 6\). If \(n > 14\), \(P_{n+1} \rightarrow \text{Spin}(n)\) can be lifted to \(\text{Spin}(n)\) (see Barratt and Mahowald [1]). If \(n = 10\) and \(k = 3(4)\), \(\nu_{k\eta} = 0\) since \(\rho^\omega: A_{10}^\omega(k\eta; P_{n+1}) \rightarrow H^{10}(P_{11}; Z_2)\) is mono. Let \(n = 10\), \(e_1 = 4 + \Delta, 0 \leq j \leq 2\). Since \(\pi_{10}(\text{BSpin})\) is onto, \(\pi_{10}(\text{BSpin})\) classifies a quaternionic bundle. Let \((A_{k\eta})^{-1}: KO^{-1}(P_{11}) \approx \pi_{10}(\text{BOP}_{11}, f)\) the James-Thomas isomorphism. Then \((A_{k\eta})^{-1}\tau\) is represented by \(F_{\tau, k\eta}: P_{11} \times S \rightarrow BO\) which classifies \(Q \otimes j\eta\), where \(Q\) is a bundle with a quaternionic structure. One may easily verify that \(e_0 Q = 0\), hence \(g.d.Q \leq 8\). Thus \(F_{\tau, k\eta}\) lifts to \(BO(10)\), whence \(\nu_{k\eta} = 0\).

The values of \(s_a p\) and \(s_a \nu\) can now be computed up to the natural indeterminacy caused by the choice of \(a\), using 2.11, 3.1, 3.2, 3.3, 3.6, and 3.7; and in the case \(n = 2(4)\), by lemma 3.9 below. This completes the proof of 3.8.

Lemma 3.9. (I) Let \(n = 2(4)\), \(\begin{pmatrix} k \\ 2 \end{pmatrix}\) even, \(\begin{pmatrix} k \end{pmatrix}\) even. Then \(\rho a = a\) for some \(a \in A_n^\omega(k\eta; P_{n+1})\). (II) Let \(n = 2(4), k = 1(4)\), \(\begin{pmatrix} k \\ n+1 \end{pmatrix}\) even. Then \((\rho + \psi)a = a\) for some \(a \in A_n^\omega(k\eta, P_{n+1})\).
### Table A: n-plane bundles over $P_{n+1}$ of stable type $k\eta$, for $n \geq 3$, $\binom{k}{n+1}$ even.

<table>
<thead>
<tr>
<th>$n \equiv 3(4)$</th>
<th>$Z_2$</th>
<th>generators &amp; relations</th>
<th>$\tau(\rho, \cdot)$</th>
<th>$\tau(\psi, \cdot)$</th>
<th>$s_\rho\delta$</th>
<th>$s_\psi\delta$</th>
<th>$N_s(k\eta; P_{n+1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n=1(4)$, $k=0(4)$</td>
<td>$Z_4$</td>
<td>$x_{00} = 0$, $x_{01} = 1$</td>
<td>1 + $\chi_0$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2 if $\binom{k-1}{n-1}$ even, 1 if $\binom{k-1}{n-1}$ odd</td>
</tr>
<tr>
<td>$n=1(4)$, $k=1(4)$</td>
<td>$Z_4$</td>
<td>$x_{00} = 0$, $x_{01} = 1$</td>
<td>1 + $\chi_0$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2 if $\binom{k-1}{n-1}$ even, 1 if $\binom{k-1}{n-1}$ odd</td>
</tr>
<tr>
<td>$n=2(4)$, $k=0(4)$</td>
<td>$Z_4$</td>
<td>$x_{00} = 0$, $x_{01} = 1$</td>
<td>1 + $\chi_0$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2 if $\binom{k-1}{n-1}$ even, 1 if $\binom{k-1}{n-1}$ odd</td>
</tr>
<tr>
<td>$n=2(4)$, $k=1(4)$</td>
<td>$Z_4$</td>
<td>$x_{00} = 0$, $x_{01} = 1$</td>
<td>1 + $\chi_0$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2 if $\binom{k-1}{n-1}$ even, 1 if $\binom{k-1}{n-1}$ odd</td>
</tr>
<tr>
<td>$n=2(4)$, $k=2(4)$</td>
<td>$Z_4$</td>
<td>$x_{00} = 0$, $x_{01} = 1$</td>
<td>1 + $\chi_0$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2 if $\binom{k-1}{n-1}$ even, 1 if $\binom{k-1}{n-1}$ odd</td>
</tr>
<tr>
<td>$n=2(4)$, $k=3(4)$</td>
<td>$Z_4$</td>
<td>$x_{00} = 0$, $x_{01} = 1$</td>
<td>1 + $\chi_0$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2 if $\binom{k-1}{n-1}$ even, 1 if $\binom{k-1}{n-1}$ odd</td>
</tr>
<tr>
<td>$n=2(4)$, $k=4(4)$</td>
<td>$Z_4$</td>
<td>$x_{00} = 0$, $x_{01} = 1$</td>
<td>1 + $\chi_0$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2 if $\binom{k-1}{n-1}$ even, 1 if $\binom{k-1}{n-1}$ odd</td>
</tr>
<tr>
<td>$n=2(4)$, $k=5(4)$</td>
<td>$Z_4$</td>
<td>$x_{00} = 0$, $x_{01} = 1$</td>
<td>1 + $\chi_0$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2 if $\binom{k-1}{n-1}$ even, 1 if $\binom{k-1}{n-1}$ odd</td>
</tr>
</tbody>
</table>

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Proof. (I) Let Σ be the line bundle classified by \( P_{n+1} \times S^2 \to BO(1) \). If \( k=4i+j = 0 \) or 1, \( 4i \eta \) has a quaternionic structure, and \( e_{n+2}(4i \eta) = 0 \), thus \( g.d.(k\eta) = n-1 \). Thus \( P_{n+1} \times S \to BO \) classifies \( k\eta \otimes \Sigma \), hence can be lifted to \( BO(n) \). Thus \( pa = a \) for some \( a \).

(II) If \( \Lambda = (4) \), \( (k-1)0^2 \otimes \Sigma \) has a quaternionic structure, hence \( F_{p+\psi, \Lambda} \), which classifies \( k(\eta \otimes \Sigma) \), lifts to \( BO(n) \) thus \( pa = \psi a \) for some \( a \).

**Theorem 3.10**. Let \( n = 3(4) \), \( n \geq 7 \), and let \( k \) be any integer such that \( \binom{k}{n+1} \) is even. Let \( x_{00} \) be an element of \( A_0^*(k\eta; P_{n+2}) \) with \( p_{00} = u \) and let \( x_{11} = \lambda u^{n+2} \). Then

(i) \( A_n^0(k\eta; P_{n+2}) \approx \mathbb{Z}_4, 2x_{00} = x_{11} \) for \( k = 0 \), \( 3(4) \),
\( A_n^0(k\eta; P_{n+2}) \approx \mathbb{Z}_4, 2x_{00} = 2x_{11} = 0 \) for \( k = 1, 2(4) \);

Furthermore \( A_n^0(k\eta; P_{n+2}) \to A_n^0(k\eta; P_{n+1}) \) is onto.

(ii) \( \gamma(\psi, x_{00}) = x_{00} + x_{11} \), and \( \gamma(p, x_{00}) = x_{00} + \binom{k}{1} x_{11} \).

(iii) For any \( a \in A_n^0(k\eta; P_{n+2}) \), \( s_a \psi = \frac{k-1}{n-1} x_{00} \), with indeterminacy \( x_{11} \) (based on the choice of \( a \)). Furthermore, \( s_a \rho = 0 \) for \( k \) even, \( s_a \rho = s_a \psi \) for \( k \) odd, both with no indeterminacy.

Proof. Similar to 3.8.

For the sake of uniformity, the above information is displayed in table B.

**Table B**: \( n \)-plane bundles over \( P_{n+2} \) of stable type \( k \) for \( n = 3(4), n \geq 7, \binom{k}{n+1} \) even.

| \( A_n^0(k\eta; P_{n+2}) \) \( \tau(\rho, \psi) \) \( s_a \rho \) \( s_a \psi \) \( N_a(k\eta; P_{n+2}) \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( k \equiv 0(4) \) \( \mathbb{Z}_4 \) \( x_{00}, x_{11} \) \( 2x_{00} = x_{11} \) \( 2x_{11} = 0 \) \( 1 \) \( 1 + x \) \( 0 \) \( \binom{k-1}{n-1} x_{00} \) \( 3 \) if \( \binom{k-1}{n-1} \) even \( \) if \( \binom{k-1}{n-1} \) odd |
| \( k \equiv 1(4) \) \( \mathbb{Z}_2 + \mathbb{Z}_2 \) \( x_{00}, x_{11} \) \( 2x_{00} = 2x_{11} = 0 \) \( 1 + Z \) \( 1 + Z \) \( \frac{k}{n-1} x_{00} \) |
| \( k \equiv 2(4) \) \( \mathbb{Z}_2 + \mathbb{Z}_2 \) \( x_{00}, x_{11} \) \( 2x_{00} = 2x_{11} = 0 \) \( 1 \) \( 1 + x \) \( 0 \) |
| \( k \equiv 3(4) \) \( \mathbb{Z}_4 \) \( x_{00}, x_{11} \) \( 2x_{00} = x_{11} \) \( 2x_{11} = 0 \) \( 1 + x \) \( 1 + x \) \( \frac{k}{n} x_{00} \) |

1), 2) The corrected statement of Theorem 3.10, and a corresponding correction of Table B, were supplied by the referee.
4. Actions

Let $A^*$ be any $Y$-twisted cohomology theory satisfying the axioms given in [2]. Fix, for the moment, a C.W.-pair $(X, A)$ and a map $f: X \to Y$. We have a natural left action

$$\gamma: \mathcal{H} \times h^*(X, A, f) \to h^*(X, A, f)$$

where $\mathcal{H} = \pi_1(Y; f)$, defined as follows: if $\alpha \in \mathcal{H}$ is represented by $F_\alpha: X \times S \to Y$, where $F_\alpha(x, *)=f(x)$ for all $x \in X$, let $F_\alpha$ also denote the composition $X \times I \to X \times S \to Y$. We have isomorphisms, where $i_0$ and $i_1: X \to X \times I$ are the inclusions along 0 and 1, respectively:

$$h^k(X \times f, A \times f, F_\alpha) \cong h^k(X, A, f)$$

Let $\gamma(\alpha, x) = \alpha x = i_0(i_1)^{-1}x$ for all $x \in h^*(X, A, f)$.

Let $\varepsilon \in \mathbb{Z}$; we shall write $(-1)^\varepsilon = \pm 1 \in \mathbb{Z}$. For $\alpha \in \mathcal{H}$, we shall define a long exact action sequence for the pair $(\varepsilon, a)$ and the theory $h^*$:

\[(4-1) \quad \ldots \to v \to h^{k-1}(X, A, f) \xrightarrow{\phi_0} h^{k-1}(X, A, f) \xrightarrow{\gamma} h^{k+1}(X \times T_\varepsilon, X \times jS \cup A \times T_\varepsilon, F_\alpha) \xrightarrow{v} h^k(X, A, f) \xrightarrow{\phi_0} h^{k+1}(X \times T_\varepsilon, X \times jS \cup A \times T_\varepsilon, F_\alpha) \to \ldots \]

where $T_\varepsilon$ is the torus if $\varepsilon=0$, the Klein bottle if $\varepsilon=1$, $S \to T_\varepsilon \xrightarrow{\pi} S$ is the fibration, and $j: S \to T$ is the section. By a slight abuse of notation, write $F_\alpha: X \times T \xrightarrow{j} X \times S \xrightarrow{\pi} Y$. For $\alpha \in h^*(X, A, f)$, $\phi_\alpha x$ is given to be $\alpha x - (-1)^\varepsilon x$. We let $v$ be the composition

$$h^{k+1}(X \times T_\varepsilon, X \times jS \cup A \times T_\varepsilon, F_\alpha) \xrightarrow{(1 \times j)^*} h^{k+1}(X \times iS, A \times iS \cup X, f \circ p_X) \xrightarrow{\gamma} h^k(X, A, f)$$

where $p_X$ is projection to $X$ and $\gamma$ is suspension. Let $X$ be defined by commutativity of the diagram:

\[
\begin{array}{ccc}
  h^k(X, A, f) & \xrightarrow{\chi} & h^{k+1}(X \times T_\varepsilon, X \times jS \cup A \times T_\varepsilon, F_\alpha) \\
  h^k(X \times iS, X \cup A \times iS, f \circ p_X) & \cong & h^{k+1}(X \times (iS \cup S) \cup A \times T_\varepsilon, F_\alpha) \\
  h^{k+1}(X \times iS \times I, X \times L \cup A \times iS \times I, F_\alpha) & \xrightarrow{\{F\}^*} & h^{k+1}(X \times iS \times I, X \times L \cup A \times iS \times I, F_\alpha) \\
\end{array}
\]
where \( q: X \times iS \times I \to X \times T \) is the obvious quotient map, \( L=iS \times \{0, 1\} \cup \{\ast\} \times I \subseteq iS \times I \), and \( F_s(x, y, u) = (F_a \circ q)(y, 1-t(1-u)) \) for all \( t, u \in I, x \in X, y \in iS \).

It is fairly straightforward to prove that the action sequence is exact and natural with respect to stable \( Y \)-twisted cohomology operations, i.e., \( \phi_a^* \psi = \nu \psi = \nu \psi \) and \( X \psi = \psi X \) for any such operation \( \psi \). We leave the details to the reader.

Now suppose that \( 'h^* \), \( 'A^* \) are \( Y \)-twisted cohomology theories classified by \( \Omega_Y \)-spectra \( 'E \), \( 'E \), respectively. Let \( \psi: 'h^* \to 'h^* \) be a stable \( Y \)-twisted cohomology operation of degree, say, \( rf \), classified by an \( \Omega_Y \)-spectrum map \( \psi: 'E \to 'E \). If we let \( A^* \) be the \( Y \)-twisted cohomology theory classified by \( E \), the fiber of \( \psi \), we obtain the long exact sequence associated to \( \psi \):

\[
\begin{align*}
(4-2) \quad '&h^{k-1}(X, A, f) &\xrightarrow{\psi} ''h^{k+d-1}(X, A, f) &\xrightarrow{\lambda} &h^k(X, A, f) \\
&\xrightarrow{p} &'h^k(X, A, f) &\xrightarrow{\psi} &''h^{k+d}(X, A, f) \to \ldots
\end{align*}
\]

where \( \lambda, p \) are now stable \( Y \)-twisted cohomology operations of degrees \( d-1 \) and 0, respectively.

We now examine the following question. Suppose we know the action of \( H \) on \( 'h^*(X, A, f) \) and \( ''h^*(X, A, f) \). How can we determine the action of \( H \) on \( h^*(X, A, f) \)? We give a partial answer, which suffices for our applications to vector bundles.

Let \( \alpha \in \text{Ker} \nu / r \) in the sequence above, and suppose \( \alpha \in \mathcal{H} \) for some \( \alpha \in \mathcal{H}, \epsilon \in \mathbb{Z} \). If \( z \in h^k(X, A, f) \) and \( p_z = z \), it is clear that \( (-1)^k z \) differs from \( \alpha z \) by \( \lambda w \) for some \( w \in h^{k+d-1}(X, A, f) \). This element \( w = \Phi_a \psi \) clearly has indeterminacy. In fact (as is trivial to check), we have a homomorphism

\[
\Phi_a = \lambda^{-1} \phi_a \nu^{-1}: \text{Ker} \psi \cap \text{Ker} \phi_a \to ''h^*(X, A, f) / \text{Im} \psi + \text{Im} \phi_a
\]

Analogous to diagram (3-2) of [11] and Fig. 2 of [10], we have a commutative diagram with exact rows and columns:

\[
\begin{array}{ccc}
'h^{k-1}(X, A, f) & \xrightarrow{\psi} & ''h^{k+d-1}(X, A, f) \\
\downarrow{\lambda} & & \downarrow{\lambda} \\
h^k(X, A, f) & \xrightarrow{\phi_a^* \psi} & h^k(X, A, f)
\end{array}
\]

\[
\begin{array}{ccc}
'h^{k-1}(X, A, f) & \xrightarrow{\psi} & ''h^{k+d-1}(X, A, f) \\
\downarrow{\lambda} & & \downarrow{\lambda} \\
h^k(X, A, f) & \xrightarrow{\phi_a^* \psi} & h^k(X, A, f)
\end{array}
\]
We define a homomorphism:

\[ \Phi_a = X^{-1} \psi^{-1} : \text{Ker } \psi \cap \text{Ker } \phi_a \to h^*(X, A, f) / \text{Im } \psi + \text{Im } \phi_a \]

and, analogous to Theorem 2.5 of [10] and Theorem 3.2 of [11], we have

**Theorem 4.1.** \( \Phi_a = - \Phi_a \)

**Proof.** Analogous to that of Theorem 2.5, [10].

The usefulness of \( \Phi_a \) as a computational tool is illustrated by the following remark:

**REMARK 4.2.** Let \( \mathcal{E} : \mathcal{A} \to \mathbb{Z}_2 \) be a homomorphism such that (in sequence (4.22)) \( \alpha = (-1)^{t(x)} \) and \( \gamma = (-1)^{t(y)} \) for all \( \alpha \in \mathcal{A} \) and all \( x \in h^*(X, A, f) \), \( y \in h^{*+d}(X, A, f) \). If \( \Gamma \subset \mathcal{A} \) is a generating set, knowledge of \( \Phi_a \) for all \( x \in \text{Ker } \psi \subset h^*(X, A, f) \) and all \( \alpha \in \Gamma \) suffices to determine the action of \( \mathcal{A} \) on \( h^*(X, A, f) \).

**Proof.** We leave to the reader.

We proceed to compute \( \Phi_a \) in certain cases which are applicable to vector bundles. Henceforth, assume that \( h^* \) is an ordinary twisted cohomology theory of type \( Z \) or \( \mathbb{Z}_2 \), i.e., with coefficients in a sheaf \( G \) over \( Y \), where each stalk of \( G \) is isomorphic to either \( Z \) or \( \mathbb{Z}_2 \). Thus \( G = \mathbb{Z}_2 \) or \( G = \mathbb{Z}[u] \) for some \( u \in H^1(Y; \mathbb{Z}) \). The same assumption shall be made concerning \( h^* \), namely that it is also an ordinary twisted theory of type \( Z \) or \( \mathbb{Z}_2 \), and is determined by a sheaf \( H \) over \( Y \).

We shall make specific computations of the homomorphisms

\[
\begin{align*}
\nu : & h^*(X \times T, X \times jS \cup A \times T, F_a) \to h^*(X, A, f) \\
\psi : & h^*(X \times T, X \times jS \cup A \times T, F_a) \to h^*(X \times T, X \times jS \cup A \times T, F_a) \\
\chi : & h^*(X, A, f) \to h^*(X \times T, X \times jS \cup A \times T, F_a) \\
\phi_a : & h^*(X, A, f) \to h^*(X, A, f) \\
\phi_a : & h^*(X, A, f) \to h^*(X, A, f)
\end{align*}
\]

in each possible case. If \( h^* \), \( h^* \) are in fact direct sums of ordinary theories, no additional complication ensues, since \( \psi \) will be a matrix of ordinary operations, and \( \nu, \chi, \) and \( \phi_a \) shall each be a vector consisting of the corresponding homomorphisms in the ordinary component cases.

Recall that \( F_a : X \times S \to Y \) classifies \( \alpha \in \mathcal{A} \); as before, we let \( F_a = F_a \circ (1X \pi) : X \times T \to Y \). If \( y \in H(Y; \mathbb{Z}_2) \), let \( \overline{y} \in H^{*+1}(X \times Z) \) be defined by the equation \( F_a \overline{y} = f^*y \otimes 1 + a \otimes y \otimes \sigma \) where \( \sigma \in H^1(S; \mathbb{Z}) \) is the fundamental class of \( S \). We shall assume \( X \) is connected, thus \( \overline{u} \in H^0(X; \mathbb{Z}_2) \) if \( u \in H^1(Y; \mathbb{Z}_2) \). If \( \overline{u} = \overline{\epsilon} \), and \( y \in H^k(Y; \mathbb{Z}[u]) \), let \( \overline{y} \in H^{*+1}(X; \mathbb{Z}[f^*u]) \) be defined by \( F_a \overline{y} = f^*y \otimes 1 + a \otimes y \otimes \sigma \).
The cohomology of $T_e$ is well-known and easily computed. Let $a = \rho \pi^* \sigma \in H^i(T_e; \mathbb{Z}_2)$, and let $B \in H^j(T_e; \mathbb{Z}[\mathfrak{a}])$ be uniquely defined by the equation $j^* B = 0$, $i^* B = \sigma$; and let $b = \rho B$. We have:

REMARK 4.3. $H^k(T_e, \mathbb{Z}_2)$ is generated by $a$ and $b$, subject only to the relations $a^2 = \eta$ and $b^2 = \delta b a$.

Lemma 4.4. We have isomorphisms (for any $u \in H^i(Y; \mathbb{Z}_2)$):

(I) $\iota: H^k(X, A, \mathbb{Z}_2) \rightarrow H^k(X, A, \mathbb{Z}_2)^H$, where $\iota(x, x') = x \otimes b + x' \otimes ab$.

(II) $L: H^k(X, A; \mathbb{Z}[\mathfrak{u}]) \rightarrow H^k(X, A; \mathbb{Z}[\mathfrak{u}])^H$, where $\epsilon(x, x') = x \otimes B + x' \otimes \pi^* \sigma \cup B$ if $\mathfrak{u} = \epsilon$.

(III) $\iota: H^k(X, A; \mathbb{Z}_2) \approx H^k(X \times TX \times jS \cup A \times T_e; Z[f^* u])$, where $\epsilon(x) = \delta(F \cup [x])$; if $\mathfrak{u} = \delta$.

Lemma 4.5. Consider the action sequence for $h^*(X, A, f) = H^*(X, A; f^{-1} G)$, where $G = \mathbb{Z}_2$ or $\mathbb{Z}[\mathfrak{u}]$, $u \in H^i(Y; \mathbb{Z}_2)$. Then

(I) $\psi: H^k(X, A; \mathbb{Z}_2) \rightarrow H^k(X, A; \mathbb{Z}_2)^H$.

(II) $\psi: H^k(X, A; \mathbb{Z}[\mathfrak{u}]) \rightarrow H^k(X, A; \mathbb{Z}[\mathfrak{u}])^H$.

(III) $\psi: H^k(X, A; \mathbb{Z}_2) \rightarrow H^k(X, A; \mathbb{Z}_2)^H$, where $\epsilon(x) = \delta(F \cup [x])$; if $\mathfrak{u} = \delta$.

We give a useful way of expressing $\psi$: $H^k(X \times T_e, X \times jS \cup A \times T_e; F_o \sim G) \rightarrow H^k(X \times T_e, X \times jS \cup A \times T_e; F_o \sim H)$.

Proof of 4.4 and 4.5. Straightforward; by 4.3 and the definitions of $\phi_\alpha$, $\chi$, and $\psi$. We give a useful way of expressing $\psi$: $H^k(X \times T_e, X \times jS \cup A \times T_e; F_o \sim G) \rightarrow H^k(X \times T_e, X \times jS \cup A \times T_e; F_o \sim H)$, for $G$ and $H$ both $\mathbb{Z}_2$ or $\mathbb{Z}[\mathfrak{u}]$.

Let $\mathfrak{S}$ be the mod 2 Steenrod algebra. Let $\kappa: \mathfrak{S} \rightarrow \mathfrak{S}$ be the derivation of degree $-1$ defined by Kristensen [5]; we write $\theta' = \kappa \theta$. The following property also defines $\theta'$: if $x \in H^*(W; \mathbb{Z}_2)$, $y \in H^*(W; \mathbb{Z}_2)$, where $P$ is any space, $\theta(x \cup y) = \theta x \cup y + \theta' x \cup Sq^1 y$.

Lemma 4.6. (I) Let $G = H = \mathbb{Z}_2$, $\psi = y \cup \theta$ for some $y \in H^*(Y; \mathbb{Z}_2)$. Then for $x \in H^k(X; \mathbb{Z}_2)$, $x' \in H^k(X; \mathbb{Z}_2)$, $\psi(x, y) = (f^* x \cup \theta x' + y \cup \theta f^* x \cup \theta y)$. (II) Let $G = Z[u], H = Z[v]$, $\psi = \rho$, reduction modulo 2. // $u = \epsilon, \psi(x, y) = (\rho x, \rho x')$ for $x \in H^k(X, A; \mathbb{Z}[f^* u]), x' \in H^k(X, A; \mathbb{Z}[f^* u])$. If $\mathfrak{u} = \epsilon, \psi(x, y') = (x + (S \mathfrak{u}^* + f^* \mathfrak{u} \cup x')$. (III) Let $G = Z[u], H = Z[v], \psi = \delta [v]$. If $\mathfrak{u} = \epsilon, \psi(x, x') = (x + S \mathfrak{u}^* x' + f^* \mathfrak{u} x)$ for $x \in H^k(X, A; \mathbb{Z}_2), x' \in H^k(X, A; \mathbb{Z}_2)$. If $\mathfrak{u} = \epsilon, \psi(x, x') = (x + (S \mathfrak{u}^* x' + f^* \mathfrak{u} x), x' \in H^k(X, A; \mathbb{Z}_2)$. (IV) Let $G = Z[u], H = Z[v], \psi = D \cup \delta [v]$, where $D \in H^k(Y; \mathbb{Z}_2), x' \in H^k(X, A; \mathbb{Z}[f^* u])$ when $\psi(x, x') = (f^* \cup x, f^* \cup x' \cup \delta [v] \cup x')$ for $x' \in H^k(X, A; \mathbb{Z}_2), x' \in H^k(X, A; \mathbb{Z}[f^* u])$. If $\mathfrak{u} = \epsilon, \psi(x, x') = (f^* \cup x, f^* \cup x' \cup D \cup x')$, where $D \in H^k(Y; \mathbb{Z}_2), x' \in H^k(X, A; \mathbb{Z}[f^* u])$ when $\psi(x, x') = (f^* \cup x, f^* \cup x' \cup D \cup x')$, where $D \in H^k(Y; \mathbb{Z}_2), x' \in H^k(X, A; \mathbb{Z}[f^* u])$.
\[ + f^*(\rho D) \cup x^\ast f^\ast \varepsilon. \quad \text{On the other hand, if } \ast u = \ast \varepsilon, \text{ and } x \in H^k(X, \mathbb{A}; \mathbb{Z}_2), \]
\[ \psi_{\mathbb{A}^*} = \iota(\delta(f^*v) \circ f^*D \cup \delta^2(f^*v)(\rho D) \cup x)) \quad \text{for } \ast v = \ast \varepsilon, \quad \text{while if } \ast v = \ast \varepsilon, \psi_{\mathbb{A}^*} = \iota((\rho f^*D) \cup x + \ast D \cup f^*u \cup x^\ast D \cup \Sigma^q)^a \psi_{\mathbb{A}^*}). \]

Proof. Case (I) is elementary. For cases (II) and (III), the formulas given are the only ones consistent with case (I) in the respective universal examples. Case (IV) is proved somewhat similarly (it is necessary to observe that if \( \ast u = \ast v, (\Sigma^q + f^*D) \cup f^*v) a(\rho D) = f^*(\rho D) \) we leave the details to the reader.

If \( \ast h, \ast h \) can both be expressed as direct sums of ordinary twisted cohomologies of types \( \mathbb{Z}_2 \) and \( \mathbb{Z}, \Phi_{\ast}^\ast \) can be easily computed using 4.1, 4.4, 4.5, and 4.6. We write the specific results in all cases where each has only one summand.

**Corollary 4.7.** Let \( f: X \to Y, (X, \mathbb{A}) \) a C.W.-pair, \( X \) connected, \( \alpha \in \mathbb{H}(X, \mathbb{A}), \) and \( \varepsilon \in \mathbb{Z}_2. \) Let \( \ast h, \ast h \) be twisted cohomology theories determined by sheaves of local coefficients \( G \) and \( H, \) respectively, over \( Y \), and let \( \psi: \ast h^* \to \ast h^* \) be a stable \( Y \)-twisted cohomology operation. Let \( x \in H^k(X, \mathbb{A}, f^{-1}G), \) such that \( \psi_{\mathbb{A}^*} = 0, \phi_{\mathbb{A}^*} \alpha = 0. \) Then (where in each case, \( y_\lambda \in H^k(Y; \mathbb{Z}_2), \lambda \in \Lambda, \) some finite indexing set; \( \theta \in \mathbb{Z}_2 \) for each \( \lambda \in \Lambda, \) and \( u, v \in H^1(Y; \mathbb{Z}_2): \)

(I) \( G = \mathbb{Z}_2, \) \( H = \Sigma_{\lambda \in \Lambda} y_\lambda \cup \theta_\lambda, \) Then \( \Phi_{\ast}^\ast x = \Sigma_{\lambda \in \Lambda} \ast y_\lambda \cup \theta_\lambda x + \varepsilon f^*y \cup \theta_\lambda x. \) (II) \( G = \mathbb{Z}[u], \)

\( H = \mathbb{Z}[v], \) \( \psi = \Sigma_{\lambda \in \Lambda} y_\lambda \cup \theta_\lambda, \) Then \( \Phi_{\ast}^\ast x = \Sigma_{\lambda \in \Lambda} \ast y_\lambda \cup \theta_\lambda x + \varepsilon f^*y \cup \theta_\lambda x + \varepsilon f^*y \cup \theta_\lambda x. \) (III) \( G = \mathbb{Z}_2, \)

\( H = \mathbb{Z}[v], \) \( \psi = \Sigma_{\lambda \in \Lambda} y_\lambda \cup \theta_\lambda, \) Then

where, if \( \varepsilon = \ast u, z \in H^k(X, \mathbb{A}; \mathbb{Z}) \) is chosen such that \( \delta(f^*u)z = x. \) (III) \( G = \mathbb{Z}_2, \)

\( H = \Sigma_{\lambda \in \Lambda} y_\lambda \cup \theta_\lambda, \) Then

where, if \( \varepsilon = \ast v, w \in H^k(X, \mathbb{A}; \mathbb{Z}[f^*v]), \) \( z \in H^k(X, \mathbb{A}; \mathbb{Z}) \) is chosen such that \( \rho w = \Sigma_{\lambda \in \Lambda} f^*y_\lambda \cup \theta_\lambda x. \) (IV) \( G = \mathbb{Z}[u], \)

\( H = \mathbb{Z}[v], \) \( \psi = \Sigma_{\lambda \in \Lambda} y_\lambda \cup \theta_\lambda, \) for some \( D \in H^k(Y; \mathbb{Z}[u] + v), \) then

where (when necessary), \( w \in H^k(X, \mathbb{A}; \mathbb{Z}[f^*v]), z \in H^k(X, \mathbb{A}; \mathbb{Z}_2) \) are chosen by the equations \( \rho w = \ast \rho \) and \( \delta[f^*u]z = x. \)
References


