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# REAL n-PLANE BUNDLES OVER AN (n I,-COMPLEX 

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## 1. Introduction

Let $X$ be a finite dimensional complex. We consider the general problem of classifying real $n$-plane bundles over $X$, which are in a natural one-to-one correspondence with $[X ; B O(n)]$, the set of homotopy classes of maps from $X$ to $B O(n)$. Alternatively, if $\xi$ is a real stable bundle over $X$, we consider the problem of classifying fl-plane bundles stably equivalent to $\xi$, which are in one-to-one correspondence with $[X ; B O(n) ; f]=\left(i_{\sharp}\right)^{-1}[f] \subset[X ; B O(n)]$, where $f: X \rightarrow B O$ is any map which classifies $\xi$.

Line bundles over $X$ are classified by $w_{1} \in H^{1}\left(X ; Z_{2}\right)$, while 2-plane bundles are classified by $w_{1}$ and $W_{2} \in H^{2}\left(X ; Z\left[w_{1}\right]\right)$, a (twisted if $\mathrm{ft}^{\wedge} \varphi \mathrm{O}$ ) integer class which reduces to $w_{2}$. Oriented 3-plane bundles over a 4 -complex were classified by Dold and Whitney [3], while James and Thomas enumerated $n$-plane bundles over an $n$-complex for $n$ odd, $n$-plane bundles over an ( $n+1$ )-complex for $n=3(4)$, and oriented $n$-plane bundles over an $n$-complex for $n$ even. [4] (Note that "oriented" and "orientable" are equivalent concepts for bundles of odd, but not even, dimensions.) In [9] this result was extended to the case of $n$ plane bundles over an $n$-complex for all $n$ while in [6] the James and Thomas result was restated for a few low-dimensional cases in a somewhat more explicit form.

In [7], a spectral sequence approach was used, which (in theory) completely enumerates $\left[X Y\right.$ ] in all cases where $X$ is a finite complex and $\pi_{1}(Y)$ is Abelian. In fact, all real and complex bundles over $P_{k}, k \leqslant 5$, are tabulated.

Nomura [16] has classified $n$-plane bundles over $P_{n+1}$ for $n \equiv 1(4)$ in most cases, and $n$-plane bundles over $P_{n+2}$ for $n=3(4)$ is some cases.

In the present paper, we use the approach of affine actions, developed in [9]. For the sake of space, it shall be assumed here that the reader is familiar with the constructions and notations of that paper. A general enumeration result is given for $n$-plane bundles over $X$ of dimension, $m$, provided $m \leqslant 2 n-2$ (the metastable assumption), and provided $m \leqslant n+2$. We give specific results for $X=P_{m}$, real projective $m$-space, if $m=n+1, n \geqslant 3$, or if $m \leqslant n+2, n \equiv 3$ (4), $n \geqslant 7$.

After Nomura, let $N_{n}(\xi X)$ be the number of equivalence classes of $n$-plane bundles over $X$ stably equivalent to $\xi$. Let $\eta$ be the Hopf bundle over $P_{m}$, if $m \geqslant 1$. Clearly $N_{n}\left(k \eta ; P_{m}\right)=0$ if $m>n$ and $\binom{k}{-}$ is odd, since $w_{n+1}(k \eta) \neq 0$. In the cases covered by Theorems 1.1 and 1.2, that is the only obstruction to $n$-dimensionality, in effect.

Theorem 1.1. Let $n \geqslant 3,\binom{k}{n+1}$ even.
Case I [James and Thomas]: if $n=3(4)$

$$
N_{n}\left(k \eta ; P_{n+1}\right)=\left\{\begin{array}{l}
2 \text { if }\binom{k-1}{n-1} \text { is even } \\
1 \text { if }\binom{k-1}{n-1} \text { is odd }
\end{array}\right.
$$

Case II [Nomura, except for the last case]: if $n \equiv 1(4)$

$$
N_{n}\left(k \eta ; P_{n+1}\right)= \begin{cases}1 \text { if }\binom{k}{2} \text { odd, } & \binom{k-1}{n-1} \text { odd } \\ 2 \text { if }\binom{k}{2} \text { odd, } & \binom{k-1}{n-1} \text { even } \\ 2 \text { if }\binom{(k)}{<} \text { even, } & \left.\mathrm{I}_{\mid n-1}^{k-1} \mid\right) \\ 3 \text { i, odd } \\ 3 \text { if }\binom{k}{2} \text { even, }, & \binom{k-1}{n-1} \text { even }\end{cases}
$$

Case III: if $n \equiv 0(4)$

$$
N_{n}\left(k \eta ; P_{n+1}\right)=\left\{\begin{array}{l}
2 \text { if } k \text { odd } \\
3 \text { if } k \equiv 0(4),\binom{k-1}{n-1} \text { odd } \\
2 \text { if } k \equiv 0(4),\binom{k-1}{n-1} \text { even, }\binom{k}{n} \text { odd } \\
5 \text { if } k \equiv 0(4),\binom{k-1}{n-1} \text { even, }\binom{k}{n} \text { even } \\
2 \text { if } k=2(4),\binom{k}{n} \text { odd } \\
5 \text { if } k=2(4),\binom{k}{n} \text { even }
\end{array}\right.
$$

Case IV: if $n \equiv 2(4)$

$$
N_{n}\left(k \eta ; P_{n+1}\right)=\left\{\begin{array}{l}
1 \text { if } k \text { odd, } n=6 \\
2 \text { if } k \text { even, } n=0,\left(\begin{array}{cc}
n & 1 \\
2 \text { if } \mathrm{ft}=3(4), n \geqslant 10 \\
4 \text { if } k=1(4), n \geqslant 10 \\
6 \text { if } \mathrm{ft} \text { even, } n \geqslant 10, \\
4 \text { if } k \text { even, } n \geqslant 10, & \binom{k-1}{n-1} \text { even } \\
1 \text { if } k \text { even, } n=6, & \binom{k=1}{n=1} \text { odd } \\
1
\end{array}\right) \text { odd }
\end{array}\right.
$$

Theorem 1.2. Let $n=\mathcal{Z}(\mathcal{Y}), \underline{n} \geqslant 7$, and $\binom{k}{n+1}$ even. Then

$$
N_{n}\left(k \eta ; P_{n+2}\right)=\left\{\begin{array}{l}
2 \text { if }\binom{k-1}{n-1} \text { odd [Nomura] } \\
3 \text { if }\binom{k-1}{n-1} \text { even }
\end{array}\right.
$$

Theorems 1.1 and 1.2 are condensed versions of 3.8 and 3.10 , below.

## 2. The Main theory

We shall utilize the tecniques of Becker, McClendon, and the author with regards to constructions over, and over-and-under a fixed space. [2, 8, 12]

If $\pi: E \rightarrow B$ is a fibration and $f: X \rightarrow B$ is a map, let $[X ; E]_{f}$ be the set of fiber-homotopy classes of liftings of $f$ to $E$, and let $[X ; E ; f]=\left(\pi_{\#}\right)^{-1}[f] \subset[X \in:$ be the set of homotopy classes of liftings of $f$ to $E$. Recall that $[X ; E ; f]$ is the set of orbits of a left action [9, 15]:

$$
\mu: \pi_{1}\left(B^{X}, f\right) \times\left[\begin{array}{ll}
X & ; E]_{f} \rightarrow[X ; E]_{f}
\end{array}\right.
$$

Furthermore, if $\operatorname{dim} X \leqslant 2 n$, where each fiber of $\pi$ is $n$-connected (the metastable assumption), $[X ; E]_{f}$ is an Abelian affine group and $\mu$ is an affine action, i.e., $\mu(\alpha$,$) is an affine automorphism for each \alpha \in \pi_{1}\left(B^{X}, f\right)$. In that case, we
 Writing $a x$ for $\gamma(\alpha, x)$ for all $x \in[X ; E]_{f}{ }^{0}$, we have $\alpha(a+x)=a+\alpha x$ for all $a \in[X ; E]_{f}, x \in[X ; E]_{f}{ }^{0}$. More generally, if $h^{*}$ is any $B$-twisted cohomology theory, a left action

$$
\gamma: \pi_{1}\left(B^{X}, /\right) \times h^{*}(X, A, /) \rightarrow h^{*}(X, A, /)
$$

can always be defined, for a subcomplex $A$; and 7 is functorial in all the obvious ways.

Let $G=O, U$, or $S p$, and let $r=1,2$, or 4, respectively. Let $\pi: B G(n) \rightarrow B G$ be a fibration replacing the usual inclusion, and let $\mathcal{S}_{n}=\mathcal{S}_{n} G$ be the $\Omega_{B G^{-}}$ spectrum associated with $\pi$ (see [8]). Let $\Gamma^{n+1} \in H^{1}\left(B G S_{n}\right)$ be the single obstruction to section of $\pi$. If $\xi$ is a stable vector bundle over a complex $X$ of dimension $m$, classified by $/: X \rightarrow B G$, we can define a characteristic class $\Gamma^{n+1} \xi \in H^{1}\left(X ; f^{-1} \mathcal{S}_{n}\right)$ )he single (metastable) obstruction to $n$-dimensionality of $\xi$. We have [2, 8]:

REMARK 2.1. If $g . d . ~ \xi \leqslant n$ (g.d. $=$ geometric dimension), $\Gamma^{n+1} \xi=0$. This condition is sufficient if $m \leqslant 2 r n+2 r-3$.

Proof. The fiber of $B G(n) \rightarrow B G$ is $r(n+1)-2$ connected.
Now let $A_{n}(\xi ; X)=[X ; B G(n)]_{f}$, the set of $n$-plane bundles stabilized to $\xi$, an Abelian affine group in the metastable range, i.e., $m \leqslant 2 r n+2 r-4$.

REMARK 2.2. (I) If $a, b \in A_{n}(\xi ; X)$, a unique difference class $\Delta(a, b) \in$ $H^{0}\left(X ; f^{-1} \mathcal{S}_{n}\right)$ is defined, such that $\Delta(a, c)=\Delta(a, b)+\Delta(b, c)$ for all $a, b, c \in$ $A_{n}(\xi ; X)$ (II) If $m \leqslant 2 r n+2 r-4, \Delta(a, b)=0$ if and only if $a=b$; while for any $a \in A_{n}(\xi ; X)$ and $x \in H^{0}\left(X ; f^{-1} \mathcal{S}_{n}\right)$,there exists $b=a+x \in A_{n}(\xi ; X)$ such that $\Delta(a, b)=x$. It follows that (III) If $m \leqslant 2 r n+2 r-4, H^{0}\left(X ; f^{-1} \mathcal{S}_{n}\right)$ corresponds in a natural way to $A_{n}{ }^{\circ}(\xi ; X)$, the difference group of $A_{n}(\xi X)$; provided the latter is non-empty.

Recall that $\Lambda_{\xi}: K G^{-1}(X) \cong \pi_{1}\left(B G^{X} f\right)$ in a natural way [4]. We thus have a left action:

$$
\mu: K G^{-1}(X) \times A_{n}(\xi ; X) \rightarrow A_{n}(\xi \quad X)
$$

and $V_{n}(\xi X)$, the set of equivalence classes of $n$-plane bundles stably equivalent to $\xi$, corresponds in a natural way to the set of orbits of $\mu$. Write $\alpha a$ for $\mu(\alpha, a)$, for any $\alpha \in K G^{-1}(X), a \in A_{n}(\xi ; X)$. Thus $N_{n}(\xi ; X)$ is simply the cardinality of $V_{n}(\xi ; X)$.

We summarize our general results for classification of real bundles in low codimension cases. Throughout, let $\xi$ be a real stable bundle over a complex $X$ ofdimension $m$, classified by $f: X \rightarrow B O$; let $Y=K\left(Z_{2}, 1\right) \times K\left(Z_{2}, 2\right)$, and let $\beta: B O \rightarrow Y$ bea map such that $\beta^{*}\left(\iota_{1} \otimes 1\right)=w_{1}$ and $\beta^{*}\left(1 \otimes \iota_{2}\right)=w_{2}$. Without loss of generality, $X$ is connected, thus we may identify $H^{\circ}\left(X ; Z_{2}\right)$ with $Z_{2}$.

Theorem 2.3. Let $m \leqslant 2 n-2, m \leqslant n+2$. Then (I) There is an $\Omega_{Y}$-spectrum $\mathscr{I}_{n}$ such that $H^{i}\left(X ;(\beta f)^{-1} \mathscr{I}_{n}\right) \cong H^{i}\left(X ; f^{-1} \mathcal{S}_{n}\right) f d t i \geqslant 0$ (canonical isomorphism) (II) There is a universal characteristic class $\phi^{n+1} \in H^{1}\left(B O ; \beta^{-1} \mathscr{I}_{n}\right)$ such that $\phi \xi=$ $f^{*} \phi^{n+1}=\Gamma^{n+1} \xi$. (III) The constructions of $\mathscr{I}_{n}, \phi^{n+1}$ are independent of $X$ and $\xi$.

Proof. Let $\mathcal{S}_{n}{ }^{(2)}$ be the second stage of the Postnikov tower for $\mathcal{S}_{n} O \mathcal{S}_{n}{ }^{(2)}$ has homotopy width three or less. By McClendon [14], $\mathcal{S}_{n}{ }^{(2)}=\beta^{-1} \mathscr{I}_{n}$ for some
$\Omega_{Y}$-spectrum $\mathscr{I}_{n}$, since $\beta$ is a 3 -equivalence. Let $\phi^{n}=P_{\#} \Gamma^{n}$, where $P: \mathcal{S}_{n} \rightarrow S_{n}{ }^{(2)}$ is the projection. The remaining details are trivial, and we are done.

Let $h_{i}=\sigma w_{i+1} \in H^{i}\left(O Z_{2}\right)$ for all $i \geqslant 0$, where $\sigma$ is the looping suspension. We have a short exact sequence:

$$
\begin{equation*}
0 \rightarrow[X ; S p i n] \rightarrow K O^{-1}(X) \xrightarrow{h_{0}, h_{1}} H^{0}\left(X ; Z_{2}\right)+H^{1}\left(X ; Z_{2}\right) \cong \pi_{1}\left(Y^{X}, \beta f\right) \rightarrow 0 \tag{2-1}
\end{equation*}
$$

From 2.3 and [9, diagram (3-1)], we have
REMARK 2.4. Let $m \leqslant 2 n-2, m \leqslant n+2$. Then there is a homomorphism $\nu_{\xi}:[X ; \mathrm{Spin}] \rightarrow A_{n}{ }^{0}(\xi ; X)$ such that $\mu(\alpha, a)=a+\nu_{\xi} \alpha$ for all $\alpha \in[X ;$ Spin $]$, $a \in A_{\boldsymbol{n}}(\xi ; X)$.

We may express $\nu_{\boldsymbol{s}}$ in a very specific way. For any $\alpha \in K O^{-1}(X) \beta \quad F_{\overline{7}, \alpha}=$ $\beta \circ f \circ p_{X}$, where $F_{\xi, \alpha}: X \times S \rightarrow B O$ is a homotopy of $f$ representing $\Lambda_{\xi} \alpha$, and $p_{X}: X X S \rightarrow X$ is the projection. Thus

$$
H^{i}\left(X \times S ;\left(\beta \circ F_{\xi, \infty}\right)^{-1} \mathscr{I}_{n}\right)=H^{i}\left(X ;(\beta \circ f)^{-1} \mathscr{I}_{n}\right)+H^{i}\left(X \times S, X\left(\beta \circ f \circ p_{X}\right)^{-1} \mathscr{I}_{n}\right)
$$

and $\nu_{\xi} \alpha$ can be uniquely defined by the equation

$$
F_{\xi, \infty} * \phi^{n}=\phi^{n} \otimes 1+\nu_{\xi} \alpha \otimes \sigma
$$

where $\sigma \in \pi_{S}^{1}(S)$ is the fundamental class of $S$ in stable cohomotopy.
We now consider the action

$$
\gamma: K O^{-1}(X) \mathrm{X} A_{n}{ }^{0}(\xi X) \rightarrow A_{n}{ }^{0}(\xi X)
$$

where we write $a x$ for $\gamma(\alpha, x) . \quad 2.5$ can be restated as follows:
REMARK 2.5. For $m \leqslant 2 n-2, m \leqslant n+2$, and fixed $x \in A_{n}{ }^{\circ}(\xi X), \alpha x$ depends only on $h_{0} \alpha$ and $h_{1} \alpha$. Equivalently, If $h_{0} \alpha=h_{1} \alpha=0, \alpha x=x$ for all $x \in A_{n}{ }^{\circ}(\xi ; X)$.

We now give very specific expressions for $\gamma$ in the case $m=n+1$, and the case $m=n+2$ for $m \equiv 3(4)$. We shall assume that the reader is familiar with the procedure for construction of the first two stages of the relative modified Postnikov tower for $B O(n) \rightarrow B O$ in all cases.

Generally, for any $u \in H^{1}\left(X ; Z_{2}\right)$, let $Z[u]$ be the sheaf of integers over $X$ twisted by $u$; thus $Z[0]=Z$, and $Z[u+v]=Z[u] \otimes Z[v]$. Let $\rho: H^{i}(X ; Z[u]) \rightarrow$ $H^{i}\left(X ; Z_{2}\right)$ be reduction modulo 2 , and let $\delta[u]: H^{i}\left(X ; Z_{2}\right) \rightarrow H^{i+1}(X ; Z[u])$ be the Bokstein homomorphism associated with the exact sequence of sheaves $Z[u] \rightarrow Z[u] \rightarrow Z_{2}$. Let $w_{i}=w_{i} \xi$, the $i^{t h}$ Stiefel Whitney class of $\xi$, for all $i \geqslant 0$.

Theorem 2.6. Let $n \equiv 1(4), n \geqslant 5, \operatorname{dim} X=m \leqslant n+1$, and assume $\dot{w}_{n+1}=0$. Let $\theta=S q^{2}+w_{2} \cup$. Then (I) We have an exact sequence:

$$
H^{n-1}\left(X ; Z_{2}\right) \xrightarrow{\theta} H^{n+1}\left(X ; Z_{2}\right) \xrightarrow{\lambda} A_{n}{ }^{0}(\xi \quad X) \xrightarrow{p} H^{n}\left(X Z_{2}\right) \rightarrow 0
$$

(II) If $x \in H^{n}\left(X ; Z_{2}\right)$ and $z \in p^{-1} x$, then $2 z=\lambda S q^{1} x$.
(III) If $x \in H^{n}\left(X Z_{2}\right)$ and $z \in p^{-1} x$, and if $\alpha \in K O^{-1}(X)$,

$$
\alpha z=z+\lambda\left(\left(h_{1} \alpha+w_{1} \cup h_{0} \alpha\right) \cup x\right)
$$

Theorem 2.7. Let $\mathrm{rc}^{\wedge} 3(4), n \geqslant 7, m \leqslant n+2$; and assume $w_{n+1}=0$. Let $\theta=\left(S q^{2}+w_{2} \mathrm{U}+w_{1} \mathrm{U}\right) S q^{1}$. (I) We have an exact sequence:

$$
H^{n-1}\left(X ; Z_{2}\right) \xrightarrow{\wedge} \mathbf{t f}^{v^{+2}}\left(X ; Z_{2}\right) \rightarrow A_{n}^{0}(\xi ; X) \xrightarrow{p} H^{n}\left(X ; Z_{2}\right) \rightarrow \mathbf{0}
$$

(II) If $x \in H^{n}\left(X Z_{2}\right)$ and $z \in p^{-1} x$, then $2 z=\lambda\left(S q^{2}+w_{2} \cup+w_{1}{ }^{2} \cup\right) x$
(III) If $x \in H^{n}\left(X Z_{2}\right)$ and $z \in p^{-1} x$, and $\alpha \in K O^{-1}(X)$, then

$$
\alpha z=z+\lambda\left(\left(h_{1} \alpha+w_{1} \cup h_{0} \alpha\right) \cup S q^{1} x\right)
$$

Theorem 2.8. Let $n=0(4), n \geqslant 4, m \leqslant n+1$; and assume $\delta\left[w_{1}\right] w_{:}=0$. Let $\theta(x, y)=\left(S q^{2}+w_{2} \mathrm{U}\right) \rho x+w_{1} \mathrm{U} y$. (I) We have an exact sequence:

$$
\begin{aligned}
& H^{n-1}\left(X ; Z\left[w_{1}\right]\right)+H^{n}\left(X ; Z_{2}\right)^{\theta} H^{n+1}\left(X ; Z_{2}\right){ }^{\lambda} A_{n}{ }^{0}(\xi ; X) \xrightarrow{p} \\
& H^{n}\left(X ; Z\left[w_{1}\right]\right)+H^{n+1}\left(X ; Z_{2}\right) \rightarrow 0
\end{aligned}
$$

(II) $/ / x \in H^{n}\left(X ; Z\left[w_{1}\right]\right), y \in H^{n+1}\left(X ; Z_{2}\right), 2 x=0$, and $z \in p^{-1}(x, y)$, pick $w \in H^{n-1}\left(X Z_{2}\right)$ such that $\delta_{1}\left[w_{1}\right] w=x$. Then $2 z=\lambda\left(S q^{1} \rho x+\left(S q^{2}+w_{2} \cup\right) w\right)$.
(III) If $x \in H^{n}\left(X ; Z\left[w_{1}\right]\right), y \in H^{n+1}\left(X ; Z_{2}\right)$ and $z \in p^{-1}(x, y)$, and if $\alpha \in K O^{-1}(X)$, then $\alpha z=(-1)^{h_{0}{ }^{\alpha}} z+\lambda\left(\left(h_{1} \alpha+w_{1} \cup h_{0} \alpha\right) \cup x+h_{0} \alpha \cup y\right)$ (Recall $H^{0}\left(X ; Z_{2}\right)=Z_{2}$; let $(-1)^{0}=1,(-1)^{1}=-1$.)

Theorem 2.9. Let $n=2(4), n \geqslant 6, m \leqslant n+1$; and assume $\delta\left[w_{1}\right] w_{n}=0$. Let $\theta(x, y)=\left(S q^{2}+w_{2} U\right) \rho x+S q^{1} y$. (I) We haz e an exact sequence:

$$
\begin{aligned}
& H^{n-1}\left(X ; Z\left[w_{1}\right]\right)+H^{n}\left(X ; Z_{2}\right) \xrightarrow{\theta} H^{n+1}\left(X ; Z_{2}\right) \xrightarrow{\lambda} A_{n}{ }^{0}(\xi \quad X) \xrightarrow{p} \\
& \quad H^{n}\left(X ; Z\left[w_{1}\right]\right)+H^{n+1}\left(X ; Z_{2}\right) \rightarrow 0
\end{aligned}
$$

(II) $/ / x \in H^{n}\left(X ; Z\left[w_{1}\right]\right), y \in H^{n+1}\left(X ; Z_{2}\right), 2 x=0$, and $z \in p^{-1}(x, y)$, pick $w \in H^{n-1}\left(X ; Z_{2}\right)$ rocA that $\delta\left[w_{1}\right] w=x$. Then

$$
2 z=\lambda\left(S q^{1} \rho x+\left(S q^{2}+w_{2} \mathrm{U}\right) w+y\right) .
$$

(III) // $\left.x \in H^{n}\left(X ; Z\left[w_{1}\right]\right), y \in H^{n+1} ; Z_{2}\right)$, and $z \in p^{-1}(x, y)$, then for any $\alpha \in K O^{-1}(X): \alpha z=(-1)^{h_{0} \alpha} z+\lambda\left(\left(h_{1} \alpha+w \cup h_{0} \alpha\right) \cup x+h_{0} \alpha \cup y\right)$

Proof of 2.6-2.9. (I) is obtained from the McClendon spectral sequence [12], while (II) is computed using the extension results of [11]. (III) follows from 4.2, 4.6, and 4.7. (Note that ${ }^{\infty} w_{1}=h_{0} \alpha,{ }^{\infty} w_{2}=h_{1} \alpha+w_{1} \mathrm{U} h_{0} \alpha$, in the notation of §4.)

In the range $m \leqslant 2 n-2, m \leqslant n+2$, knowledge of the three things suffices to enumerate $V_{n}(\xi ; X)$ : namely, $\nu_{\xi}, 7$, and a function $s_{a}: H^{0}\left(X ; Z_{2}\right)+H^{1}\left(X Z_{2}\right) \rightarrow$ $A_{n}{ }^{\circ}(\xi ; X)$, for any fixed $a \in A_{n}(\xi X)$. [9, Thm 3.1] We shall see that $s_{a}$ is determined by its values on generators, although it is not a homomorphism.

Without loss of generality, $X$ is connected. Let $\rho \in K O^{-1}(X)$ be classified by a map which takes $X$ to a single point of $O$ which does not lie in the identity component. For any $u \in H^{1}\left(X ; Z_{2}\right)$, let $\psi[u] \in K O^{-1}(X$ be classified by the composition $X \xrightarrow{u} P_{\infty} \xrightarrow{W} S O$ where $W$ isthe Whitehead map: recall that $W^{*} h_{i}=u^{i}$ for all $\imath$. We define (for fixed $a \in A_{n}(\xi ; X)$ ), for $(x, y) \in H^{0}\left(X ; Z_{2}\right)+H^{1}\left(X ; Z_{2}\right)$;

$$
s_{a}(x, y)=\alpha_{x, y} a-a \quad \text { where } \alpha=\begin{array}{ll}
\{\downarrow\lceil v\rceil & \text { if } x=0 \\
\sim \mp \psi\lfloor y\rfloor & \text { if } x=\overline{1}
\end{array}
$$

For any $(x, y) \in H^{0}\left(X ; Z_{2}\right)+H^{1}\left(X Z_{2}\right)$, and any $z \in A_{n}{ }^{0}(\xi ; X)$, let $\gamma^{\prime}(x, y, z)=$ $(x, y) z=\rho_{x, y} z \in A_{n}{ }^{0}(\xi ; X)$. It is clear that $\gamma^{\prime}$ is an action if $m \leqslant 2 n-2$ and $m \leqslant n+2$.

From 2.4 and Theorem 3.1 of [9], we immediately have:
REMARK 2.10. Let $X$ be a connected complex of dimension $m$, where $m \leqslant 2 n-2, m \leqslant n+2$; and let $\xi$ be a real stable vector bundle over $X$ which has an $n$-dimensional stabilization $a$. Let $\Gamma \subset H^{1}\left(X ; Z_{2}\right)$ be any generating set. Then, in order to enumerate $V_{n}(\xi ; X)$, it is sufficient to compute
(i) the Abelian group $A_{n}{ }^{\circ}(\xi X)$
(ii) the homomorphism $\nu_{\xi}:[X ; S p i n] \rightarrow A_{n}{ }^{\circ}(\xi ; X)$
(iii) the action $\gamma^{\prime}: H^{0}\left(X ; Z_{2}\right)+H^{1}\left(X ; Z_{2}\right) \times A_{n}{ }^{\circ}(\xi ; X) \rightarrow A_{n}{ }^{0}(\xi ; X)$
(iv) $s_{a}(1,0)=s_{a} \rho \in A_{n}{ }^{0}(\xi ; X)$, and $s_{a}(0, u)=s_{a} \psi[u] \in A_{n}{ }^{0}(\xi ; X)$ for all $u \in \Gamma$.

The following lemma, which follows from James and Thomas [4, 1.4] will be a useful aid in computing $s_{a}$. Let $\rho^{\wedge}: A_{n}{ }^{0}(\xi ; X) \rightarrow H^{n}\left(X ; Z_{2}\right)$ be the reduction defined in the obvious way, to wit, if $x=a-b$, where $a$ and $b$ are stabilized vector bundles, classified by liftings $g_{a}, g_{b}: X \rightarrow B O(n)$ of $f: X \rightarrow B O, \rho^{\wedge} x$ is the difference class of $g_{a}$ and $g_{b}$ defined by $w_{n+1}$ in $\dot{B O}$.

Lemma 2.11. Let $\xi$ be a real stable bundle over a connected complex $X$ of dimension $m, m \leqslant 2 n-2$, and let $a \in A_{n}(\xi ; X)$. (I) If $\alpha \in K O^{-1}(X), \rho^{\wedge}(\alpha a-a)=$ $\sum_{i=0}^{n} h_{i} \alpha \cup w_{n-i} \xi$. (II) If $m \leqslant n+2 \rho \wedge s_{a} \rho=w_{n} \xi$, and $\rho^{\wedge} s_{a} \phi[u]=\sum_{i=1}^{n} u^{i} \cup w_{n-i} \xi$ for all $u \in H^{1}\left(X ; Z_{2}\right)$.

Proof. (I) follows immediately from the James and Thomas formula, and (II) is an immediate corollary of (I).

## 3. Applications to projective spaces

Tensor products. Let $\xi$ be a stable real, or complex vector bundle over a
complex $X$, and let $L$ be a line bundle over $X$. Let $\xi^{(n)}$ be the virtual ra-plane bundle representing $\xi$; we define $L \otimes_{n} \xi$ to be the stable bundle represented by the virtual $n$-plane bundle $L \otimes \xi^{(n)}$. Let $t$ and $t_{n}$, respectively, classify $\otimes$ and $\otimes_{n}$ such that we have a commutative diagram:


If $a \in A_{n}(\xi ; X)$, we can thus define $L \otimes a \in A_{n}\left(L \otimes_{n} \xi ; X\right)$ as follows: if $g: X \rightarrow$ $B G(n)$ classifies $a, t(l, g)$ classifies $L \otimes a$; where $/: X \rightarrow B G(1)$ classifies $L$.

REMARK 3.1. (I) $L \otimes: A_{n}(\xi ; X) \rightarrow A_{n}\left(L \otimes_{n} \xi ; X\right)$ is one-to-one and onto: in fact its inverse is $L^{\prime} \otimes$, where $L!$ is the line bundle conjugate to $L$ (note that $L^{\prime}=L$ in the real case). (II) In the metastable range, i.e., $\operatorname{dim} X \leqslant 2 r n+2 r-4$, $L ®$ is an affine isomorphism.

Proof. (I) is obvious; (II) requires some manipulation of the base spaces; we leave the details to the reader.

Let $l \otimes_{n}: B G^{X} \rightarrow B G^{X}$ be given by $l \otimes_{n} f^{\prime}=t_{n}\left(l / /^{\prime}\right)$ for any $f^{\prime}: X \rightarrow B G$. If $/: X \rightarrow B G$ classifies $\xi$, and $\Lambda_{\xi}: \pi_{1}\left(B G^{X} f\right) \cong K G^{-1}(X)$ is the James-Thomas isomorphism, let $\delta_{\xi, L, n}: K G^{-1}(X) \rightarrow K G^{-1}($ XD) e the composition

$$
K G^{-1}(X) \stackrel{\left(\Lambda_{\xi}\right)^{-1}}{\approx} \pi_{1}\left(B G^{X}, f\right) \stackrel{\left(l \otimes_{n}\right)_{\xi}}{\approx} \pi_{1}\left(B G^{X}, l \otimes_{n} f\right)^{\Lambda_{L \otimes_{n} \xi}} K G^{-1}(X)
$$

In the metastable range, define $L \otimes x \in A_{n}{ }^{\circ}\left(L \otimes_{n} \xi ; X\right)$ for all $x \in A_{n}{ }^{0}(\xi X)$ by $L \otimes(a-b)=L \otimes a-L \otimes b$, for all $a, b \in A_{n}(\xi ; X)$. By simply chasing the definitions, we can easily check that:

REMARK 3.2. (I) For all $a \in A_{n}(\xi ; X)$ and all $\alpha \in K G^{-1}(X), L \otimes \alpha a=$ $\delta_{\xi, L, n} \alpha(L \otimes a)$. (II) If $m \leqslant 2 n r+2 r-4, \quad L \otimes \alpha x=\delta_{\xi, L, n} \alpha(L \otimes x)$ for all $x \in$ $A_{n}{ }^{\circ}(\xi ; X)$ and $\alpha \in K G^{-1}(X)$.

And from the splitting theorem, we can compute
Lemma 3.3. In the real case, i.e., $G=O, t_{n}{ }^{*} w_{k}=\sum_{t=0}^{k}\binom{n-k+1}{i} w_{1}{ }^{i}$ $\otimes w_{k-i}$. (Where it is understood that if $a<0, b \geqslant 0,\binom{a}{( }$ is defined (modulo 2) to


The null-element. Let $\xi$ be real, henceforth. We define $\nu_{\xi, n} \in K O^{-1}(X$ ฤо be the element corresponding to $N_{\xi, n} \in \pi_{1}\left(B O^{\times}, f\right)$ where $N_{\xi, n}$ is the composition

$$
\mathbf{X x} \bar{S}-\quad \xrightarrow{{ }_{1} \lambda f} B O(1) \times B O \xrightarrow{t_{n}} B O
$$

where $\varepsilon_{1}$ represents the generator of $\pi_{1} B O(1)$, and $T$ reverses coordinates. We call the $\nu_{\xi, n}$ null-element of $\xi$ in dimension $n$.

Lemma 3.4. Let $\xi$ be real. Then (I) $\nu_{\xi, n}$ is functorial in $X$, i.e., if $g: X^{\prime} \rightarrow X$ is a map and $\xi^{\prime}=g^{-1} \xi$, then $\nu_{\xi^{\prime}, n}=g^{\prime} \nu_{\xi, n}$. (II) For all $a \in A_{n}(\xi ; X)$, $\nu_{\xi, n} a=a$. (III) For all $k \geqslant 0, h_{k} \nu_{\xi, n}=\binom{n-k}{1} w_{k} \xi$.

Proof. (I) is obvious. (II) holds since, if $g: X \rightarrow B O(n)$ is a lifting of $f$ which represents $a, N_{\xi, n}$ lifts to $t \circ\left(\varepsilon_{1} \times g\right) \circ T X \times S \rightarrow B O(n)$, a self-homotopy of $g$. (III) follows immediately from 3.3.

Projective spaces. Henceforth, let $P_{r}$ be real projective $r$-space for any integer $r \geqslant 0$, and let $u \in H^{1}\left(P_{r} ; Z_{2}\right)$ be the generator, if $r \geqslant 1$. Let $\psi=\psi[u] \in$ $K O^{-1}\left(P_{r}\right)$. It is well-known and easily computable that

Lemma 3.5. Let $r \geqslant 1$. Then (I) if $r$ 末 $3(4), K O^{-1}\left(P_{r}\right) \cong Z_{2}+Z_{2}$,with generators $p$ and $\psi$. (II) If $r=3(4), K O^{-1}\left(P_{r}\right) \cong Z_{2}+Z_{2}+Z_{\psi \text { with }}$ generators $p$, $\psi$, and $\tau$, where $\tau \in\left[P_{r} ;\right.$ Spin $]$ is classified by $P_{r} \rightarrow P_{r} \mid P_{r-1}=\stackrel{\varepsilon}{S^{r-r}} \xrightarrow{\rightarrow}$ Spinwhere $\varepsilon_{r}$ represents the generator of $\pi_{r}(S p i n) \cong Z$. (III) $K O^{\circ}\left(P_{r}\right)=\left[P_{r} ; B O\right] \cong Z_{2^{s}}$, generated by $\eta$, the canonical line bundle over $P_{r}\left(\right.$ i.e., $\left.w_{1} \eta=u\right)$, where $s$ is the number of positive integers less than or equal to $r$ which are equivalent to $0,1,2$, or 4 modulo 8. (IV) For all $i \geqslant 1, h_{i} \rho=0$, and $h_{i} \psi=u^{i}$; while $h_{0} \rho=1$ and $h_{0} \psi=0$. If $\mathrm{r}=2$ (4), $h_{r} \tau=u^{r}$ for $r=2$ or 6 ; while $h_{i} \tau=0$ if $i \neq r$ or $r>6$. (V) For any $0 \leqslant k<2^{s}, w_{i}(k \eta)=\binom{k}{i} u^{i}$ for all $i \geqslant 0$.

Lemma 3.6. For $r \geqslant 1$ and $0 \leqslant k<2^{s}$ (where $s$ is computed as in 3.5 (III)), (I) $\delta_{k \eta, \eta, n} \psi=\psi$. (II) $\delta_{k n, n, n} \rho=\rho+\psi$. (III) If $r=2(4), \delta_{k \eta, \eta, n} \tau=\tau$.

Proof. Clearly $\delta_{k \eta, n, n} \psi$ and $\delta_{k n, \eta, n} \rho$ must be 2-torsion elements; the stated results for (I) and (II) are the only answers which agree with 3.3. To prove (III), consider the covering map $c: S^{r} \rightarrow P_{r}$. Now $c^{-1} \eta=1$, the trivial line bundle, and Ker $c^{\prime}$ is generated by $p$ and $\psi$. Thus $\delta_{k \eta, \eta, n} \boldsymbol{\tau}=\boldsymbol{\tau}+\lambda_{1} \rho+\lambda_{2} \psi$ for $\lambda_{1}, \lambda_{2} \in Z_{2}$. Again by 3.3, $\lambda_{1}=\lambda_{2}=0$.

Lemma 3.7. Let $r \geqslant 1,0 \leqslant k<2^{s}$. Then

$$
\nu_{k n, n}=\left\{\begin{array}{l}
0 \text { if } n \text { is even } \\
P \text { if } n \text { odd, } k \text { even } \\
\rho+\psi \text { if } n \text { odd, } k \text { odd }
\end{array}\right.
$$

Proof. For $r \neq 2(4)$, the stated result is the only possible which agrees with 3.3. If $r=2(4)$, the result still holds, by $3.4(\mathrm{I})$, since $k \eta$ lives in $P_{r+1}$.

Notation. We give standard names, $x_{01}, x_{00}$, and $x_{10}$, for generators of the Abelian group $A_{n}{ }^{\circ}\left(k \eta ; P_{r}\right)$ (which can be computed by one of the theorems 2.6-2.9), for $r=n+1, n \geqslant 3$. If $n \equiv 3(4)$, let $x_{01}=\lambda u^{n+1}$, while if $n=3(4)$, let $x_{01}=0$. Let $x_{00}$ be defined by the equation $p x_{00}=u^{n}$ if $n$ is odd, $p x_{00}=\left(\delta u^{n-1}, 0\right)$ if $n$ and $k$ are both even. If $n$ is even and $k$ is odd, $x_{00}$ shall not be defined. If $n$ is even, $x_{10}$ shall be defined by the equation $p x_{10}=\left(0, u^{n+1}\right)$. For odd $n, x_{10}$ is not defined.

Note that $x_{01}$ is always uniquely defined, but may be zero, while $x_{00}$ and $x_{10}$ are not always defined, have intederminacy $x_{01}$, and are never zero. Intuitively, if $e^{n}$ and $e^{n+1}$ are the bottom two cells of the Stiefel manifold $V_{n}$, and $\iota, \eta$ are the generators of the stable 0 - and 1 -stems in the homotopy of spheres, we may write $x_{00}=e^{n} \otimes \iota \otimes u^{n}, x_{01}=e^{n} \otimes \eta \otimes u^{n+1}$, and $x_{10}=e^{n+1} \otimes \eta \otimes u^{n+1}$.

Define endomorphisms $\chi_{0}$ and $\chi_{1}$ on $A_{n}{ }^{0}\left(k \eta ; P_{r}\right)$ for $r=n+l, n \geqslant 3$, as follows: $\chi_{0} x_{01}=\chi_{1} x_{01}=0$ in all cases; $\chi_{0} x_{00}=x_{01}$ and $\chi_{1} x_{00}=0$ if $x_{00}$ exists; and $\chi_{0} x_{10}=0$ and $\chi_{1} x_{10}=x_{01}$ if $x_{10}$ exists.

Theorem 3.8. Let $n \geqslant 3$, and let $k$ be any integer such that $\left(\begin{array}{ll}( & k\end{array}\right)$ is even. (I) If $n \neq 6$, the homomorphism $\nu_{k \eta}:\left[P_{n+1} ; S p i n\right] \rightarrow A_{n}{ }^{0}\left(k \eta ; P_{n+1}\right)$ is zero, while if $n=6, \nu_{k \eta} \tau= \pm x_{10}$. (II) The group $A_{n}{ }^{0}\left(k \eta ; P_{n+1}\right)$, the automorphisms $\gamma(\rho$, ) and $\gamma\left(\psi\right.$, ) on $A_{n}{ }^{0}\left(k \eta ; P_{n+1}\right)$, the elements $s_{a} \rho, s_{a} \psi \in A_{n}{ }^{0}\left(k \eta ; P_{n+1}\right)$ (for some choice of $\left.a \in A_{n}\left(k \eta ; P_{n+1}\right)\right)$, and the resulting value of $N_{n}\left(k \eta ; P_{n+1}\right)$ are as in Table $A$.

Proof. The groups $A_{n}{ }^{0}\left(k \eta ; P_{n+1}\right)$, with their generators and relations, come from theorems 2.5-2.8, (I) and (II). The actions $\gamma(\mathrm{p}$, ) and $\gamma(\psi$,$) are$ then from 2.5-2.8 (III). If $n \neq 2(4), \nu_{k n}=0$ since $\left[P_{n+1} ; S p i n\right]=0$. If $n=2(4)$, $\left[P_{n+1} ; S p i n\right] \cong$ Zgenerated by $r$ (cf. 3.5) and $\nu_{k n} \tau= \pm x_{10}$ by 2.11 if $n=6$. If $n \geqslant 14, \nu_{k n}=0$ since $P_{n+1} \rightarrow S^{n+1} \rightarrow \operatorname{Spin}$ can be lifted to $\operatorname{Spin}(n)$ (see Barratt and Mahowald [1]). If $n=10$ and $k=3(4), \nu_{k \eta}=0$ since $\rho^{\wedge}: A_{10}{ }^{0}\left(k \eta ; P_{11}\right) \rightarrow$ $H^{10}\left(P_{11} ; Z_{2}\right)$ is mono. Let $n=10, \mathrm{ft}=4 \hat{\mathfrak{i}}+\Lambda \quad 0 \leqslant j \leqslant 2$. Since $\pi_{12}(B S p) \rightarrow$ $\pi_{12}$ (BSpin) is onto, $\varepsilon_{12}: S^{12} \rightarrow B S$ pin classifies a quaternionic bundle. Let $\left(\Lambda_{k \eta}\right)^{-1}: K O^{-1}\left(P_{11}\right) \cong \pi_{1}\left(B O^{P_{11}}, f\right)$ e the James-Thomas isomorphism. Then $\left(\Lambda_{k \eta}\right)^{-1} \tau$ is represented by $F_{\tau, k \eta}: P_{11} \times S \rightarrow B O$ which classifies $Q \otimes j \eta$, where $Q$ is a bundle with a quaternionic structure. One may easily verify that $e_{3} Q=0$, hence g.d. $Q \leqslant 8$. Thus $F_{\tau, k \eta}$ lifts to $B O(10)$, whence $\nu_{k \eta}=0$.

The values of $s_{a} \rho$ and $s_{a} \psi$ can now be computed up to the natural indeterminacy caused by the choice of $a$, using $2.11,3.1,3.2,3.3,3.6$, and 3.7; and in the case $n=2(4)$, by lemma 3.9 below. This completes the proof of 3.8.

Lemma 3.9. (I) Let $n=2(4),\binom{k}{2}$ even, $\left(\begin{array}{cc}\prime \\ & 2 \\ i\end{array}\right)$ even. Then $\rho a=a$ for some $a \in A_{n}\left(\eta P_{n+1}\right)$. (II) Let $n=2(4), k \equiv 1(4),\binom{\dot{k}}{n+1}$ even. Then $(\rho+\psi) a=$ ofor some $a \in A_{n}\left(k \eta, P_{n+1}\right)$.

Table A: $n$-plane bundles over $P_{n+1}$ of stable type $k \eta$, for $n \geq 3,\binom{k}{n+1}$ even.

|  | $A_{n}{ }^{0}\left(k \eta P_{n+1}\right)$ | generators \& relations | $r(\rho$, | $r(\psi$, | $s_{a} \rho$ | $s_{a} \psi$ | $N_{n}\left(k \eta ; P_{n+1}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $n \equiv 3(4)$ | $Z_{2}$ | $\begin{aligned} & \# 00, \# 01 \\ & x_{01}=2 x_{00}=0 \end{aligned}$ | 1 | 1 | $\binom{k}{n} x_{00}$ | $\binom{k-1}{n-1} x_{00}$ | 2 if $\binom{k-1}{n-1}$ even <br> 1 if $\binom{k-1}{n-1}$ odd |
| $\begin{aligned} & n=1(4) \\ & k \equiv 0(4) \end{aligned}$ | $Z_{4}$ | $\begin{aligned} & x_{00,} \neq 01 \\ & 2 x_{00}=x_{01} \\ & 2 x_{01}=0 \end{aligned}$ | 1 | $1+\chi_{0}$ | 0 | $\binom{k-1}{n-1} x_{00}$ | 3 if $\binom{k-1}{n-1}$ even 2 if $\binom{k-1}{n-1}$ odd |
| $\begin{aligned} & n=1<4) \\ & k=1(4) \end{aligned}$ | $Z_{4}$ | $\begin{aligned} & \# 00, \# 01 \\ & 2 x=x_{01} \\ & 2 x=0 \end{aligned}$ | $1+!$ | $1+\chi_{0}$ | $\binom{k}{n} x_{00}$ | $\binom{k}{n} x_{00}$ | 3 if $\binom{k-1}{n-1}$ even <br> 2 if $\binom{k-1}{n-1}$ odd |
| $\begin{aligned} & n=1(4) \\ & k \equiv 2(4) \end{aligned}$ | $Z_{2}$ | $\begin{aligned} & \# 00, \# 01 \\ & 2 x_{00}=x_{01}=0 \end{aligned}$ | 1 | 1 | 0 | $\binom{k-1}{n-1} x_{00}$ | $2 \operatorname{if}\binom{k-1}{n-1}$ even <br> 1 if $\binom{k-1}{n-1}$ odd |
| $\begin{aligned} & n \equiv 1(4) \\ & A=3(4) \end{aligned}$ | $Z_{2}$ | $\begin{aligned} & \# 00, \# 01 \\ & 2 x_{00}=x_{01}=0 \end{aligned}$ | 1 | 1 | $\binom{k}{n} x_{00}$ | $\binom{k-1}{n-1} x_{00}$ | 2 if $\binom{k-1}{n-1}$ even <br> 1 if $\binom{k-1}{n-1}$ odd |
| $n \equiv 0(4)$ <br> $k$ odd | $Z_{2}$ | $\begin{aligned} & \# 10, \# 01 \\ & 2 x_{10}=x_{01}=0 \end{aligned}$ | 1 | 1 | 0 | 0 | 2 |
| $\begin{aligned} & n \equiv 0(4) \\ & k \equiv 0(4) \end{aligned}$ | $Z_{4}+Z_{2}$ | $\begin{aligned} & x_{00,}, \# 01, \# 10 \\ & 2 x_{00}=x_{01} \\ & 2 x_{10}=2 x_{01} \\ & =0 \end{aligned}$ | $1+\chi_{0}+\chi_{1}$ | $1+\chi_{0}$ | $\binom{k}{n} x_{00}$ | $\begin{aligned} & \binom{k-1}{n-1} x_{00} \\ & +\binom{k-1}{n} x_{10} \end{aligned}$ | 3 if $\binom{k-1}{n-1}$ odd <br> 2 if $\binom{k}{n}$ odd <br> $\&\binom{k-1}{n-1}$ even <br> 5 if both even |
| $\begin{aligned} & n \equiv 0(4) \\ & k=2(4) \end{aligned}$ | $Z_{2}+Z_{2}+Z_{2}$ | $\begin{gathered} \# 00, \# 10, \# 01 \\ 2 x_{00}=2 x_{10} \\ =2 x_{01}=0 \end{gathered}$ | $1+\chi_{1}$ | $1+\chi_{0}$ | $\binom{k}{n} x_{00}$ | $\left(n^{\prime}\right) x_{10}$ | 5 if 0 even 2 if $\binom{k}{n}$ odd |
| $\begin{aligned} & n=2(4) \\ & k=3(4) \end{aligned}$ | $Z_{2}$ | $\begin{aligned} & \# 10, \# 01 \\ & 2 x_{10}=x_{01}=0 \end{aligned}$ | 1 | 1 | 0 | 0 | $\begin{aligned} & 2 \text { if } n \geq 10 \\ & 1 \text { if } n=6 \end{aligned}$ |
| $\begin{aligned} & n \equiv 2(4) \\ & k \equiv 1(4) \end{aligned}$ | $Z_{4}$ | $\begin{aligned} & \# 10, \# 01 \\ & 2 x_{10}=x_{01} \\ & 2 x_{01}=0 \end{aligned}$ | 1 | 1 | 0 | 0 | $\begin{aligned} & 4 \text { if } n \geq 10 \\ & 1 \text { if } n=6 \end{aligned}$ |
| $\begin{aligned} & n \equiv 2(4) \\ & k \equiv 0(4) \end{aligned}$ | $Z_{4}+Z_{2}$ | $\begin{aligned} & \# 00, \# 01, \# 10 \\ & 2 x_{10}=x_{01} \\ & 2 x_{00}=2 x_{01} \\ & =0 \end{aligned}$ | 1 | $1+\chi_{0}$ | 0 | $\begin{aligned} & \binom{k-1}{n-1} x_{n \Omega} \\ & +\binom{k-1}{n} x_{10} \end{aligned}$ |  |
| $\begin{aligned} & n \equiv 2(4) \\ & k \equiv 2(4) \end{aligned}$ | $Z_{4}+Z_{2}$ | $\begin{gathered} \# 00, x_{01}, \# 10 \\ 2 x_{00}=2 x_{10} \\ =x_{01} \\ 2 x_{01}=0 \end{gathered}$ | $1+\chi_{0}$ | $1+\mathrm{Z}$. | $\binom{k}{n} x_{00}$ | $\binom{k-1}{n-1} x_{00}$ | $\begin{aligned} & 4 \text { if } \begin{array}{l} n \geq 10, \\ \binom{k-1}{n-1} \end{array} \text { odd } \\ & 1 \text { if } n=6, \\ & \binom{k-1}{n-1} \text { odd } \\ & \hline \end{aligned}$ |

Proof. (I) Let $\sum$ be the line bundle classified by $P_{n+1} \times S \xrightarrow{p_{s}} S \xrightarrow{\varkappa_{1}} B O(1)$. If $k=4 i+j j=0$ or $1,4 i \eta$ has a quaternionic structure, and $e_{n+2}(4 i \eta)=0$, thus g.d.kn$\leqslant n-1 . \quad F_{\rho, k \eta}: P_{n+1} \times S \rightarrow B O$ classifies $k \eta \oplus \sum$, hence can be lifted to $B O(n)$. Thus $\rho a=a$ for some $a$. (II) If $k \equiv 1(4),(k-1)\left(\eta \otimes \sum\right)$ has aquaternionic structure, hence $F_{\rho+\psi, k \eta}$, which classifies $k(\eta \otimes \Sigma)$, lifts to $B O(n)$ :thus $\rho a=\psi a$ for some $a$.

Theorem 3.10 ${ }^{1}$. Let $n=3(4), n \geqq 7$, and let $k$ be any integer such that $\binom{k}{\left(\begin{array}{l}+1\end{array}\right)}$ is even. Let $x_{00}$ be an element of $A_{0}{ }^{n}\left(k \eta ; P_{n+2}\right)$ with $p_{00} x=u^{n}$ and let $x_{11}=\lambda u^{n+2}$. Then
(i) $A_{n}{ }^{0}\left(k \eta ; P_{n+2}\right) \cong Z_{4}, 2 x_{00}=x_{11}$ fork $=0,3(4)$,

$$
A_{n}^{0}\left(k \eta ; P_{n+2}\right) \cong Z_{2}+Z_{2}, 2 x_{00}=2 x_{11}=0 \text { fok }=1,2(4) ;
$$

furthermore $A_{n}{ }^{0}\left(k \eta ; P_{n+2}\right) \rightarrow A_{n}{ }^{0}\left(k \eta ; P_{n+1}\right)$ is onto.
(ii) $\gamma\left(\psi, x_{00}\right)=x_{00}+x_{11}$, and $\gamma\left(\mathrm{p}, x_{00}\right)=x_{00}+\binom{k}{1} x_{11}$.
(iii) For any $a \in A_{n}\left(k \eta ; P_{n+2}\right), s_{a} \psi=\left(\begin{array}{l}k- \\ n-1\end{array}, x_{00}\right.$, with indeterminacy $x_{11}$ (based on the choice of $a$ ). Furthermore, $s_{a} \rho=0$ for $k$ even, $s_{a} \rho=s_{a} \psi$ for $k$ odd; both with no indeterminacy.

Proof. Similar to 3.8.
For the sake of uniformity, the above information is displayed in table B.
Table $\mathbf{B}^{2)}$ : $n$-plane bundles over $P_{n}+{ }_{2}$ of stable type $k$ for $n=3(4), n \geq 7,1$, ${ }^{\text {j }}$ even.

|  | $\left.{ }_{1}^{1} A_{n}{ }^{0}\left(k \eta P_{n+2}\right)\right)_{\text {\& }}^{\text {generators }}$ |  | $r(\rho$, | $r(\psi$, | $s_{a} \rho$ | $s_{a} \psi$ | $N_{n}\left(k \eta ; P_{n+2}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $k \equiv 0(4)$ | $Z_{4}$ | $\begin{aligned} & x_{00}, x_{11} \\ & 2 x_{00}=x_{11} \\ & 2 x_{11}=0 \end{aligned}$ | 1 | $1+x$ | 0 | $\binom{k-1}{n-1} x_{00}$ | 3 if $\binom{k-1}{n-1}$ even <br> 2 if $\binom{k-1}{n-1}$ odd |
| $k \equiv 1$ (4) | $Z_{2}+Z_{2}$ | $\begin{aligned} & \# 00, \# 11 \\ & 2 x_{00}=2 x_{11} \\ & =0 \end{aligned}$ | $1+$ Z | $1+z$ | $\binom{k}{n} x_{00}$ |  |  |
| $k \equiv 2(4)$ | $Z_{2}+Z_{2}$ | $\begin{aligned} & \# 00, \# 11 \\ & 2 x_{00}=2 x_{11} \\ & =0 \end{aligned}$ | 1 | $1+\chi$ | 0 |  |  |
| $k \equiv 3$ (4) | $Z_{4}$ | $\begin{aligned} & \# 00, \# 11 \\ & 2 x_{00}=x_{11} \\ & 2 x_{11}=0 \end{aligned}$ | $1+X$ | $1+\chi$ | $\binom{k}{n} x_{00}$ |  |  |

[^0]
## 4. Actions

Let A* be any $Y$-twisted cohomology theory satisfying the axioms given in [2]. Fix, for the moment, a C.W.-pair $(X, A)$ and a map $/: X \rightarrow Y$. We have a natural left action

$$
\gamma: \mathscr{H} \times h^{*}(X, A, f) \rightarrow h^{*}(X, A, f)
$$

where $\mathscr{H}=\pi_{1}\left(Y^{-\pi} f\right)$, defined as follows: if $\alpha \in \mathscr{H}$ is represented by $F_{\infty}: X \times S$ $\rightarrow Y$, where $F_{a}(x, *)=f(x)$ for all $x \in X$, let $F_{a}$ also denote the composition $X \times I \rightarrow X X \xrightarrow{p_{a}^{p}} Y$. We have isomorphisms, where $i_{0}$ and $i_{1}: X \rightarrow X \times I$ are the inclusions along 0 and 1 , respectively:

$$
h^{*}\left(X \mathrm{x} /, A \mathrm{x} /, F_{a}\right) \underset{i_{1}{ }^{*}}{\stackrel{i_{\mathrm{u}}{ }^{*}}{\longrightarrow}} h^{*}(X, A, f)
$$

Let $\gamma(\alpha, x)=\alpha x=i_{0}{ }^{*}\left(i_{1}^{*}\right)^{-1} x$ for all $x \in h^{*}(X, A, f)$.
Let $\varepsilon \in Z_{2}$ : we shall write $(-1)^{\varepsilon}= \pm 1 \in Z$. For $\alpha \in \mathscr{H}$, we shall define a long exact action sequence for the pair $(\varepsilon, a)$ and the theory $h^{*}$ :

$$
\begin{align*}
\cdots \xrightarrow{\nu} & h^{k-1}(X, A, f) \xrightarrow{\phi_{a}{ }^{\varepsilon}} h^{k-1}(X, A, f)  \tag{4-1}\\
& \xrightarrow{\chi} h^{k+1}\left(X \times T_{\varepsilon}, X \times j S \cup A \times T_{\varepsilon}, F_{a s}\right) \xrightarrow{\nu} h^{k}(X A, f) \\
& \xrightarrow{\phi_{a}{ }^{\varepsilon}} h^{k}(X, A, f) \xrightarrow{\chi} \cdots
\end{align*}
$$

where $T_{\varepsilon}$ is the torus if $\varepsilon=0$, the Klein bottle if $\varepsilon=1, S \rightarrow T_{\varepsilon} \xrightarrow{\pi} S$ is the fibration, and $j: S \rightarrow T$ is the section. By a slight abuse of notation, write $F_{a}: X \times T{ }^{1}-\times \tilde{X} \times S-\xrightarrow{F} Y$. For ${ }^{*} \in h^{*}(X, A, f), \phi_{a}{ }^{8} x$ is given to be $\alpha x-(-1)^{\varepsilon} x$. We let $\nu$ be the composition

$$
\begin{aligned}
\left.h^{k+1}\left(X \times T_{\mathrm{\varepsilon}}, X \times j S \cup A \times T_{\varepsilon}, F_{a}\right)-{\stackrel{(1 \times i)^{*}}{\longrightarrow}}_{\longrightarrow}^{h^{k+1}(X \mathrm{x}} i S, A \times i S \cup X, f \circ p_{X}\right) \\
\cong \mid s \\
h^{k}(X, A, f)
\end{aligned}
$$

where $p_{X}$ is projection to $X$ and $s$ is suspension. Let $X$ be defined by commutativity of the diagram:
where $q: X \times i S \times I \rightarrow X \times T$ is the obvious quotient map, $L=i S \times\{0,1\}$ U $\{*\} \times I \subset i S \times I$, and $F_{t}(x, y, u)=\left(F_{\infty} \circ q\right)(\wedge y, \quad 1-t(1-u))$ for all $t, u \in I, x \in X$, $y \in i S$.

It is fairly straightforward to prove that the action sequence is exact and natural with respect to stable $Y$-twisted cohomology operations, i.e., $\phi_{a}{ }^{8} \psi$ ty $\varphi и>\nu \psi=\psi \nu$, and $\chi \psi=\psi \chi$ for any such operation $\psi$. We leave the details to the reader.

Now suppose that ${ }^{\prime} h^{*}$, "A* are $Y$-twisted cohomology theories classified by $\Omega_{Y}$-spectra ${ }^{\prime} \mathcal{E},{ }^{\prime \prime} \mathcal{E}$, respectively. Let $\psi:{ }^{\prime} h^{*} \rightarrow{ }^{\prime \prime} h^{*}$ be a stable Y-twisted cohomology operation of degree, say, rf, classified by an $\Omega_{Y}$-spectrum map $\psi:{ }^{\prime} \mathcal{E} \rightarrow{ }^{\prime \prime} \mathcal{E}$. If we let $\mathrm{A}^{*}$ be the $Y$-twisted cohomology theory classified by $\mathcal{E}$, the fiber of $\psi$, we obtain the long exact sequence associated to $\psi$ :

$$
\begin{align*}
& \cdots^{\prime} h^{k-1}(X, A, /) \xrightarrow{\psi}{ }^{\prime \prime} h^{k+d-1}(X, A, /) \xrightarrow{\lambda} h^{k}(X, A, f)  \tag{4-2}\\
& \xrightarrow{p} h^{k}(X, A, f) \xrightarrow{\psi} h^{\prime \prime} h^{k+d}(X, A, f) \rightarrow \cdots
\end{align*}
$$

where $\lambda, p$ are now stable $Y$-twisted cohomology operations of degrees $d-1$ and 0 , respectively,

We now examine the following question. Suppose we know the action of $\mathscr{H}$ on $h^{*}(X, A, f)$ and ${ }^{\prime \prime} h^{*}(X, A, f)$. How can we determine the action of $\mathscr{H}$ on $h^{*}(X, A, f)$ ? We give a partial answer, which suffices for our applications to vector bundles.

Let tfeKer-v/r in the sequence above, and suppose $\alpha x=(-1)^{\varepsilon} x$ for some $\alpha \in \mathcal{H}, \varepsilon \in Z_{2}$. If $z \in h^{k}(X, A, /)$ and $p z=x$, it is clear that ( -1$)^{\varepsilon} z$ differsfrom $\alpha z$ by $\lambda w$ for some $w \in^{\prime \prime} h^{k+d-1}(X, A, f)$. This element $w=\Phi_{a}{ }^{8} x$ clearly has indeterminancy. In fact (as is trivial to check), we have a homomorphism

$$
\Phi_{a}{ }^{\varepsilon}=\lambda^{-1} \phi_{a^{2}} p^{-1}: \operatorname{Ker} \psi \cap \operatorname{Ker} \phi_{a}{ }^{8} \rightarrow{ }^{\prime \prime} h^{*}(X, A, f) / \operatorname{Im} \psi+\operatorname{Im} \phi_{a}{ }^{8}
$$

Analogous to diagram (3-2) of [11] and Fig. 2 of [10], we have a commutative diagram with exact rows and columns:

We define a homomorphism:

$$
* \Phi_{a}{ }^{\varepsilon}=\chi^{-1} \psi \nu \nu^{-1}: \operatorname{Ker} \psi \cap \operatorname{Ker} \phi_{a}{ }^{8} \rightarrow{ }^{\prime \prime} h^{*}(X, A, f) / \operatorname{Im} \psi+\operatorname{Im} \phi_{a}{ }^{8}
$$

and, analogous to Theorem 2.5 of [10] and Theorem 3.2 of [11], we have
Theorem 4.1. $\Phi_{\alpha}{ }^{\varepsilon}=-* \Phi_{\alpha}{ }^{8}$
Proof. Analogous to that of Theorem 2.5, [10].
The usefulness of $\Phi_{a}{ }^{8}$ as a computational tool is illustrated by the following remark:

REMARK 4.2. Let $\varepsilon: \mathcal{H} \rightarrow Z_{2}$ be a homomorphism such that (in sequence (4-2)) $\alpha x=(-1)^{e d} x$ and $\alpha y=(-1)^{2 a} y$ for all $\alpha \in \mathscr{H}$ and all $x \in^{\prime} h^{k}(X, A, f)$, $y \in{ }^{\prime \prime} h^{k+d-1}(X, A, /)$. If $\Gamma \subset \mathscr{H}$ is a generating set, knowledge of $\Phi_{a}{ }^{\text {eas }} x$ for all $x \in \operatorname{Ker} \psi \subset^{\prime} h^{k}(X, A, f)$ and all $\alpha \in \Gamma$ suffices to determine the action of $\mathscr{H}$ on $h^{k}(X, A, f)$.

Proof. We leave to the reader.
We proceed to compute $* \Phi_{\infty}$ in certain cases which are applicable to vector bundles. Henceforth, assume that $' h^{*}$ is an ordinary twisted cohomology theory of type $Z$ or $Z_{2}$, i.e., with coefficients in a sheaf $G$ over $Y$, where each stalk of $G$ is isomorphic to either $Z$ or $Z_{2}$. Thus $G=Z_{2}$ or $G=Z[u]$ for some $u \in H^{1}\left(Y ; Z_{2}\right)$. The same assumption shall be made concerning " $h^{*}$, namely that it is also an ordinary twisted theory of type $Z$ or $Z_{2}$, and is determined by a sheaf $H$ over $Y$.

We shall make specific computations of the homomorphisms

$$
\begin{aligned}
& \nu: h^{*}\left(X \times T_{\varepsilon}, X \times j S \cup A: T_{\varepsilon}, F_{a}\right) \rightarrow h^{\prime} h^{*}(X, A, f) \\
& \psi: h^{*}\left(X \times T_{\varepsilon}, X \times j S \cup A \times T_{\varepsilon}, F_{a}\right) \rightarrow{ }^{\prime \prime} h^{*}\left(X \times T_{\varepsilon}, X \times i S \cup A \times T_{\varepsilon}, F_{a}\right) \\
& \chi: h^{*}(X, A, f) \rightarrow \quad{ }^{\prime \prime} h^{*}\left(X \times T_{\varepsilon}, X \times j S \cup A \times T_{\varepsilon}, F_{a}\right) \\
& \phi_{a}{ }^{\varepsilon}: h^{*}(X, A, f) \rightarrow{ }^{\prime} h(X, A, f) \\
& \phi_{a}{ }^{\varepsilon}: h^{\prime \prime} h^{*}(X, A, f) \rightarrow{ }^{\prime \prime} h(X, A, f)
\end{aligned}
$$

in each possible case. If ${ }^{\prime} h^{*}$, " $h^{*}$ are in fact direct sums of ordinary theories, no additional complication ensues, since $\psi$ will be a matrix of ordinary operations, and $v, \chi$, and $\phi_{a}$ shall each be a vector consisting of the corresponding homomorphisms in the ordinary component cases.

Recall that $F_{a} ; X \times S \rightarrow Y$ classifies $\alpha \in \mathcal{H}$; as before, we let $F_{a}=F_{a j} \circ(1 X \pi)$ : $X X T_{\varepsilon} \rightarrow Y$. If $y \in{ }^{k} H\left(Y ; Z_{2}\right)$, let ${ }^{a} y \in H^{k-1}\left(X Z_{2}\right)$ be defined by the equation $F_{a}{ }^{*} y=f^{*} y \otimes 1+{ }^{a} y \otimes \sigma$ where $\sigma \in H^{1}(S ; Z)$ is the fundamental class of $S$. We shall assume $X$ is connected, thus ${ }^{\infty} u \in H^{0}\left(X ; Z_{2}\right)=Z_{2}$ if $u \in H^{1}\left(Y ; Z_{2}\right)$. If ${ }^{a} u=\varepsilon$, and $y \in H^{k}(Y ; Z[u])$, let ${ }^{a} y \in H^{k-1}\left(X ; Z\left[f^{*} u\right]\right)$ be defined by $F_{a}{ }^{*} y=$ $f^{*} y \otimes 1+{ }^{a} y \otimes \sigma$.

The cohomology of $T_{\varepsilon}$ is well-known and easily computed. Let $a=\rho \pi^{*} \sigma$ $\in H^{1}\left(T_{q} ; Z_{2}\right)$, and let $B \in H^{1}\left(T_{\varepsilon} ; Z[\varepsilon a]\right)$ be uniquely defined by the equation $j * B=0, i^{*} B=\sigma$; and let $b=\rho B$. We have:

REMARK 4.3. $H^{*}\left(T_{\varepsilon} ; \boldsymbol{Z}_{2}\right)$ is generated by $a$ and $b$, subject only to the relations $a^{2}=0$ and $b^{2}=\varepsilon a b$.

Lemma 4.4. We have isomorphisms (for any $\left.u \in H^{1}\left(Y ; Z_{2}\right)\right)$ :
(I) $\quad \iota: H^{k-1}\left(X, A ; Z_{2}\right)+H^{k-2}\left(X, A ; Z_{2}\right) \cong H^{k}\left(\lambda T_{\varepsilon}, X \times j S \cup A>T_{\varepsilon} ; Z_{2}\right)$, where $\iota\left(x, x^{\prime}\right)=x \otimes b+x^{\prime} \otimes a b$.
(II) $\quad L: H^{k-1}\left(X, A ; Z\left[f{ }^{*} u\right]\right)+H^{k^{-2}}(X A ; Z[f * u]) \cong H^{k}\left(\mathbb{X} T_{\varepsilon}, X \times j S \cup\right.$ $\left.A \times T_{\varepsilon} ; Z\left[f^{*} u\right]\right)$, where $\iota\left(x, x^{\prime}\right)=x \otimes B+x^{\prime} \otimes \pi^{*} \sigma \cup B ; i f{ }^{a} u=\varepsilon$.
(III) $\quad \iota: H^{k-2}\left(X, A ; Z_{2}\right) \cong H^{k}\left(X \times T X \times j S \cup A \times T_{\varepsilon} ; Z[f * u]\right)$, where $\iota x=\delta\left[F_{\omega}{ }^{*} u\right](x \otimes b) ;$ if ${ }^{a} u \neq \varepsilon$.

Lemma 4.5. Consider the action sequencefor $h^{*}(X, A, f)=H^{*}\left(X, A ; f^{-1} G\right)$, where $G=Z_{2}$ or $Z[u], u \in H^{1}\left(Y ; Z_{2}\right)$. Then
(I) $\quad / / G=Z_{2}, \phi_{a}{ }^{8} x=0 ; \chi_{x=\iota}(0, x)$ and $\nu \iota\left(x, x^{\prime}\right)=x$ forany $x, x^{\prime} \in H^{*}\left(X, A ; Z_{2}\right)$.
(II) If $G=Z[u],{ }^{\infty} u=\varepsilon$, then $\phi_{a}{ }^{\varepsilon} x=0, \chi x=\iota(0, x)$, and $\nu \iota\left(x, x^{\prime}\right)=x$ for any $x, x^{\prime} \in H^{*}\left(X, A ; Z\left[f^{*} u\right]\right)$.
(III) If $G=Z[u]^{\alpha}, u \neq \varepsilon$, then $\phi_{a}{ }^{8} x= \pm 2 x, \chi x=\iota \rho x$, and $\nu \iota z=\delta\left[f^{*} u\right]$ for any $x \in H^{*}\left(X, A ; Z\left[f^{*} u\right]\right), z \in H^{*}\left(X, A ; Z_{2}\right)$.

Proof of 4.4 and 4.5. Straightforward; by 4.3 and the definitions of $\boldsymbol{\phi}_{a^{\circ}}{ }^{\varepsilon}$, $\chi$, and $\nu$.

We give a useful way of expressing $\psi: H^{*}\left(X \times T_{\varepsilon}, X \times j S \cup A \backslash T_{\varepsilon} ; F_{a}{ }^{-1} G\right)$ $\rightarrow H^{*}\left(X \times T_{\varepsilon}, X \times j S \cup A T_{\varepsilon} ; F_{a}{ }^{-1} H\right)$, for $G$ and $H$ both $Z_{2}$ or $Z[u]$.

Let $\mathfrak{A}$ be the mod 2 Steenrod algebra. Let $\kappa: \mathfrak{N} \rightarrow \mathfrak{A}$ be the derivation of degree -1 defined by Kristensen [5]; we write $\theta^{\prime}=\kappa \theta$. The following property also defines $\theta^{\prime}$ : if $x \in H^{*}\left(W ; Z_{2}\right), y \in H^{1}\left(W ; Z_{2}\right)$, where $W$ is any space, $\theta(\chi \cup y)=\theta x \mathrm{U} y+\theta^{\prime} x \mathrm{U} S q^{1} y$.

Lemma 4.6. (I) Let $G=H=Z_{2}, \psi=y \cup \theta$ foisome $y \in H^{*}\left(Y ; Z_{2}\right), \overrightarrow{\mathcal{Y}}$. Thenfor $x \in H^{k^{-1}}\left(X, A \quad Z_{2}\right), x^{\prime} \in H^{k^{-2}}\left(X, A \quad Z_{2}\right), \psi \iota(x, t f)=\iota\left(f_{y}^{*} \quad \mathrm{U} \theta x^{\prime}, f^{*} \mathrm{U} \theta x^{\prime}\right.$ $\left.+{ }^{a} y \mathrm{U} \theta x+\varepsilon f^{*} \mathrm{U} \theta^{\prime} x\right)$. (II) Let $G=Z[u], H=Z_{2}, \psi=\rho$, reduction modulo 2. // $u=\varepsilon, \psi \iota(x, t f)=\iota\left(p x, \quad \rho x^{\prime}\right)$ for $x \in H^{k^{-1}}(X, A ; Z[f * u]), x^{\prime} \in H^{k^{-2}}(X, A ; Z[f * u])$. If ${ }^{a} u \neq \varepsilon, \psi \iota\left(x, x^{\prime}\right)=\iota\left(x+\left(S q^{1}+f^{*} u \cup\right) x^{\prime}\right)$. (III) Let $G=Z_{2}, H=Z[v], \psi=\delta[v]$. If ${ }^{a} v \neq \varepsilon, \psi \iota\left(x, x^{\prime}\right)=\iota\left(x+S q^{1} x^{\prime}+f * v \cup x^{\prime}\right)$ far $x \in H^{k-1}\left(X, A ; Z_{2}\right), x^{\prime} \in H^{k-2}(X$, $\left.A ; Z_{2}\right)$. If $v^{\infty}=\varepsilon, \psi \iota(x, t f)=\imath\left(\delta\left[f^{*} v\right] x, \quad \delta\left[f^{*} v\right] x^{\prime}\right)$ or $x \in H^{k^{-1}}\left(X, A ; Z_{2}\right), x^{\prime} \in$ $H^{k^{-2}}\left(X, A ; Z_{2}\right)$. (IV) Let $G=Z[u], H=Z[v], \psi=D \cup$, where $D \in H^{*}(Y ; Z[u$ $+v])$. If ${ }^{a} u=\varepsilon, \quad x \in H^{k-1}(X, A ; Z[f * u])$, and $x^{\prime} \in H^{k-2}(X, A ; Z[f * u])$ then $\psi \iota\left(x, x^{\prime}\right)=\iota\left(f * D \cup x, f^{*} D \cup x^{\prime}+{ }^{a} D \cup x \downarrow f{ }^{\omega} v=\varepsilon\right.$, while ty $\iota\left(x, x^{\prime}\right)=\iota\left({ }^{a}(\rho D) x \cup x\right.$
$\left.+f^{*}(\rho D) \cup x^{\prime}\right) i f v^{a} \neq \varepsilon$. On the other hand, if ${ }^{a} u \neq \varepsilon$, and $x \in H^{k-2}\left(X, A ; Z_{2}\right)$, $\psi c x=\iota\left(\delta\left[f^{*} v\right]\left(\rho f^{*} D \cup x \delta\left[f^{*} v\right]\left({ }^{( }(\rho D) \cup x\right)\right)\right.$ far ${ }^{\omega} v=\varepsilon$, while if ${ }^{\omega} v \neq \varepsilon$, $\psi \iota x=$ $\iota\left(\rho f * D \mathrm{U} x+{ }^{\infty} D \mathrm{U} f^{*} u \mathrm{U} x+{ }^{*} D \mathrm{U} S q^{1} x\right)$.

Proof. Case (I) is elementary. For cases (II) and (III), the formulas given are the only ones consistent with case (I) in the respective universal examples. Case (IV) is proved somewhat similarly (it is necessary to observe that if ${ }^{\infty} u \neq{ }^{\alpha} v,\left(S q^{1}+f^{*} u \mathrm{U}+f^{*} v \mathrm{U}\right)^{\infty}(\rho D)=f^{*}(\rho D)$ we leave the details to the reader. If ${ }^{\prime} h^{*}$, " $h^{*}$ can both be expressed as direct sums of ordinary twisted cohomologies of types $Z_{2}$ and $\mathrm{Z}, \Phi_{a}{ }^{\varepsilon}$ can be easily computed using 4.1, 4.4, 4.5, and 4.6. We write the specific results in all cases where each has only one summand.

Corollary 4.7. Let $f: X \rightarrow Y,(X, A)$ a C.W.-pair, $X$ connected, $\alpha \in \mathcal{H}=$ $\pi_{1}\left(Y^{*}, f\right)$, and $\varepsilon \in Z_{2}$. Let ' $h^{*}$, " $h^{*}$ be twisted cohomology theories determined by sheaves of local coefficients $G$ and $H$, respectively, over $Y$, and let $\psi: h^{*} \rightarrow{ }^{\prime \prime} h^{*}$ be a stable $Y$-twisted cohomology operation. Let $x \in H^{*}\left(X, A f^{-1} G\right)$, such that $\psi x=0, \phi_{\varepsilon}{ }^{w} \alpha=0$. Then (where in each case, $y_{\lambda} \in H^{*}\left(Y ; Z_{2}\right), \theta_{\lambda} \in \mathfrak{N}$ for each $\lambda \in \Lambda$, some finite indexing set; and $\left.u, v \in H^{1}\left(Y ; Z_{2}\right)\right)$ : (I) If $G=H=Z_{2}$ and $\psi=\sum_{\lambda \in \Lambda} y_{\lambda} \mathrm{U} \theta_{\lambda}$. Then $\Phi_{a}{ }^{2} x=\sum_{\lambda \in \Lambda}{ }^{a} y_{\lambda} \mathrm{U} \theta_{\lambda} x+\varepsilon f^{*}{ }^{*} \mathrm{U} \theta_{\lambda}{ }^{\prime} x$. (II) $/ / G=Z[u]$, $H=Z_{2}$, and $\psi=\sum_{\lambda \in \Lambda} y_{\lambda} \cup \theta_{\lambda} \rho$, then

$$
\Phi_{a}{ }^{\varepsilon} x=\left\{\begin{array}{l}
\sum_{\lambda \in \Lambda}{ }^{a} y_{\lambda} \cup \theta_{\lambda} \rho x \quad \text { if } \varepsilon={ }^{a} u \\
\sum_{\lambda \in \Lambda}{ }^{a} y_{\lambda} \cup \theta_{\lambda} \rho x+f^{*} y \cup \theta_{\lambda}^{\prime} \rho x+f^{*} y_{\lambda} \cup \theta_{\lambda} z \quad \text { if } \varepsilon \neq{ }^{\alpha} \varepsilon
\end{array}\right.
$$

uwhere, if $\varepsilon \neq{ }^{\infty} u, z \in H^{*}(X, A ; Z)$ is chosen such that $\delta\left[f^{*} u\right] z=x$. (III) $G=Z_{2}$, $H=Z[v], \psi=\delta[v] \sum_{\lambda \in \Delta} y_{\lambda} \mathrm{U} \theta_{\lambda}$. Then
where, if $\varepsilon \neq{ }^{\infty} v, w \in H^{*}\left(X, A ; Z\left[f^{*} v\right] ;>\right.$ chosen such that $\rho w=\sum_{\lambda \in \Lambda} f^{*} y_{\lambda} \mid \theta_{\lambda} x$. (IV) If $G=Z[u], \quad H=Z[v], \quad$ and $\psi=D \cup+\delta[v] \sum_{\lambda \in \Lambda} y_{\lambda} \cup \theta_{\lambda}$ for some $D \in$ $H^{*}(Y ; Z[u+v])$, then

$$
\Phi_{\infty}{ }^{\varepsilon} x=\left\{\begin{array}{c} 
\pm{ }^{\alpha} D \cup x+\delta\left[f^{*} v\right] \sum_{\lambda \in \Lambda}{ }^{a} y_{\lambda} \cup \theta_{\lambda} \rho x+\varepsilon f^{*} y_{\lambda} \cup \theta_{\lambda}{ }^{\prime} \rho x \quad \text { if } \varepsilon={ }^{a} u={ }^{a} v \\
w+\delta\left[f^{*} v\right] \sum_{\lambda \in \Lambda}{ }^{\alpha} y_{\lambda} \cup \theta_{\lambda} \varepsilon \rho x+\varepsilon f^{*} y_{\lambda} \theta_{\lambda}{ }^{\prime} \rho x \quad \text { if } £={ }^{\infty} u \Phi^{\alpha} v \\
\delta\left[f^{*} v\right]\left({ }^{\alpha}(\rho D) \mathrm{U} z+\sum_{\lambda \in \Lambda} f^{*} y_{\lambda} \cup \theta_{\lambda} z+{ }^{\alpha} y_{\lambda} \cup \theta_{\lambda} z+\varepsilon f^{*} y_{\lambda} \cup \theta_{\lambda}{ }^{\prime} \rho x\right. \\
\text { if } \varepsilon \neq{ }^{\infty} u
\end{array}\right.
$$

where (when necessary), $w \in H^{*}(X, A ; Z[f * v])$ and $z \in H^{*}\left(X, A ; Z_{2}\right)$ are chosen by the equations $\rho w={ }^{\infty}(\rho D) \mathrm{U} \rho x+\sum_{\lambda \in \Lambda} f^{*} y_{\lambda} \mathrm{U} \theta_{\lambda} \rho x$, and $\delta[f * u] z=x$.

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[^0]:    1), 2) The corrected statement of Theorem 3.10, and a corresponding correction of Table B, were supplied by the referee.

