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ON MULTIPLY TRANSITIVE GROUPS XII

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1. Introduction

The known 4-fold transitive groups are the symmetric groups $S_n$ ($n \geq 4$), the alternating groups $A_n$ ($n \geq 6$) and Mathieu groups $M_n$ ($n=11, 12, 23, 24$). The main purpose of this paper is to characterize these known 4-fold transitive groups. The result is as follows.

**Theorem.** Let $G$ be a 4-fold transitive group on $\Omega=\{1, 2, \ldots, n\}$. Assume that $t$ is the maximal number of fixed points of involutions of $G$.

Furthermore assume that $G$ contains a 2-subgroup $Q$ which satisfies the following conditions:

1. $|I(Q)| = t$ and $Q$ is a Sylow 2-subgroup of $G_{I(Q)}$,
2. $N(Q)^{I(\Omega)} = S_t$ or $A_t$.

Then $G$ is one of the following groups: $S_n$ ($n \geq 4$), $A_n$ ($n \geq 6$) or $M_n$ ($n=11, 12, 23, 24$).

This theorem is a generalization of theorems of M. Hall ([2], Theorem 5.8.1), H. Nagao [10] and the author [11]: the case $t<4$ has been proved by M. Hall, the case $t=4$ or 5 by H. Nagao and the case $t=6$ or 7 and $N(Q)^{I(\Omega)}=A_t$ by the author.

The followings are corollaries.

**Corollary 1.** Let $G$ be a 4-fold transitive group on $\Omega=\{1, 2, \ldots, n\}$, and $P$ a Sylow 2-subgroup of a stabilizer of four points in $G$. Assume that $n$ is even and $P \neq 1$.

1. If $I(P) = I(Z(P))$, where $Z(P)$ is the center of $P$, then $G$ is one of the following groups; $S_n$ ($n \geq 6$), $A_n$ ($n \geq 8$ and $n \equiv 0 \mod 4$) or $M_{12}$.
2. For any point $i$ of $\Omega-I(P)$ if $P_i$ is semiregular ($\pm 1$) on $\Omega-I(P_i)$ or 1, then $G$ is one of the following groups; $S_6$, $S_8$, $A_6$, $A_{10}$, $M_{12}$ or $M_{24}$.

**Corollary 2.** Let $G$ be a 4-fold transitive group on $\Omega=\{1, 2, \ldots, n\}$ and $P$ a Sylow 2-subgroup of a stabilizer of four points in $G$. If $P$ is a transitive group
Corollary 2 is a generalization of Theorem 1 and Theorem 2 in [7] and Theorem in [8]. In the proof of Corollary 1 we make use of the following

**Lemma.** Let $G$ be a 4-fold transitive group on $\Omega=\{1, 2, \cdots, n\}$. Assume that the maximal number of fixed points of involutions of $G$ is twelve. Then for any 2-subgroup $Q$ fixing exactly twelve points $N(Q)^{I(Q)} = M_{12}$.

We shall use the same notations in [12].

2. **Proof of the theorem**

We proceed by way of contradiction. From now on we assume that $G$ is a counter-example to our theorem of the least possible degree. Since there is no 4-fold transitive group of degree less than thirty-five except known ones ([2], P. 80), the degree $n$ of $G$ is not less than thirty-five. Let $I(Q) = \{1, 2, \cdots, t\}$ and $\Delta = \Omega - I(Q)$. For any point $t+i$ of $\Delta$ set $i' = t+i$, $1 \leq i \leq n-t$.

2.1. $t \geq 6$. In particular if $N(Q)^{I(Q)} = A_4$, then $t \geq 8$.

Proof. If $t<4$, then by a theorem of M. Hall ([2], Theorem 5.8.1) $G = S_5$, $S_6$, $A_5$, $A_6$, or $M_{11}$, which is a contradiction since $n \geq 35$. If $t=4$ or 5, then by a theorem of H. Nagao [10] $G = S_7$, $A_8$, $A_9$, or $M_{12}$, which is also a contradiction. Thus $t \geq 6$.

Suppose that $N(Q)^{I(Q)} = A_t$, $t=6$ or 7. Since $Q$ is a Sylow 2-subgroup of $G_{I(Q)}$, $Q$ is a Sylow 2-subgroup of a stabilizer of four points of $I(Q)$ in $G$. Hence by a theorem of [11] $G = M_{23}$, which is also a contradiction. Thus if $N(Q)^{I(Q)} = A_t$, then $t \geq 8$.

2.2. $|\Delta| \geq 17$.

Proof. $G$ is a 4-fold transitive group and $n \geq 35$. Hence by a theorem of W. A. Manning [5]

$$|\Delta| \geq \frac{n-1}{2} \geq \frac{35-1}{2} = 17.$$  

2.3. Let $R$ be a 2-subgroup of $N(Q)$ containing $Q$, and $X$ a 2-subgroup of $N(Q)$. If $\langle R, X \rangle^{I(Q)}$ is a 2-group, then there is a 2-subgroup $X'$ in $N(Q)$ such that $X^{I(Q)} = X'^{I(Q)}$, $\langle R, X \rangle$ is a 2-group and $\langle Q, X \rangle$ is conjugate to $\langle Q, X' \rangle$ in $N(Q)$.

Proof. Let $P$ be a Sylow 2-subgroup of $\langle R, X \rangle$ containing $R$. Since $\langle R, X \rangle^{I(Q)}$ is a 2-group, $P^{I(Q)} = \langle R, X \rangle^{I(Q)}$. Then $P$ contains a 2-group $X'$ such that $X^{I(Q)} = X'^{I(Q)}$. Then $\langle R, X' \rangle$ is a 2-subgroup of $P$. Since $Q$ is a Sylow 2-subgroup of $G_{I(Q)}$ and $\langle Q, X' \rangle^{I(Q)} = \langle Q, X \rangle^{I(Q)}$, both $\langle Q, X \rangle$ and $\langle Q, X' \rangle$ are
Sylow 2-subgroups of \( \langle Q, X, X' \rangle \). Hence \( \langle Q, X' \rangle \) is conjugate to \( \langle Q, X \rangle \) in \( \langle Q, X, X' \rangle \). Thus \( \langle Q, X' \rangle \) is conjugate to \( \langle Q, X \rangle \) in \( N(Q) \).

2.4. If \( N(Q)^{(Q)} = S_t \), then \( N(Q) \) has a 2-group \( \langle Q, x_1, \ldots, x_k \rangle \), where
\[
x_i = (1 \ 2) \cdots (2i-2) (2i-1 \ 2i) (2i+1) \cdots (t) \cdots ,
\]
\[
1 \leq i \leq k, \quad k = \frac{t}{2} \quad \text{if } t \text{ is even and } k = \frac{t-1}{2} \quad \text{if } t \text{ is odd}.
\]
Furthermore since \( N(Q)^{(Q)} = S_t \) or \( A_t \), \( N(Q) \) has a 2-group \( \langle Q, y_1, y_2, \ldots, y_n, y_1' \rangle \), where
\[
y_i = (1 \ 2) (3 \ 4) \cdots (2i) (2i+1 \ 2i+2) (2i+3) \cdots (t) \cdots ,
\]
\[
y_i' = (1 \ 3) (2 \ 4) (5 \ 6) \cdots (t) \cdots ,
\]
\[
1 \leq i \leq k, \quad k = \frac{t-2}{2} \quad \text{if } t \text{ is even and } k = \frac{t-3}{2} \quad \text{if } t \text{ is odd}.
\]

In either case \( k \geq 3 \).

Proof. Since \( N(Q)^{(Q)} = S_t \) or \( A_t \), this follows immediately from (2.1) and (2.3).

From now on we denote that \( \langle Q, x_1, x_2, \ldots, x_k \rangle \) and \( \langle Q, y_1, y_2, \ldots, y_n, y_1' \rangle \) are the groups in (2.4).

2.5. Suppose that \( N(Q) \) has the 2-group \( \langle Q, x_1, x_2, \ldots, x_k \rangle \) in (2.4), which is abelian and fixes a subset \( \Delta' \) of \( \Delta \). If \( \langle Q, x_1 \rangle \) is semiregular on \( \Delta' \), then \( \langle Q, x_1, x_2, \ldots, x_k \rangle \) is semiregular on \( \Delta' \).

Proof. Suppose that \( \langle Q, x_1, x_2, \ldots, x_i \rangle, \ i \geq 2 \), is semiregular on \( \Delta' \) and \( \langle Q, x_1, x_2, \ldots, x_i \rangle \) is not semiregular on \( \Delta' \). Then \( \langle Q, x_1, x_2, \ldots, x_i \rangle \) has an element \( x \) fixing a \( \langle Q, x_1, x_2, \ldots, x_i \rangle \)-orbit of length \( 2^i \cdot |Q| (\geq 2^{i+1}) \) in \( \Delta' \) pointwise since \( \langle Q, x_1, x_2, \ldots, x_i \rangle \) is abelian and \( \langle Q, x_1, x_2, \ldots, x_i \rangle \) is semiregular on \( \Delta' \). Then since \( x \) has at most \( i+1 \) 2-cycles in \( I(Q) \) and \( i \geq 2 \), \( |I(x)| \geq t-2(i+1)+2^{i+1} > t \), contrary to the assumption (*). Thus if \( \langle Q, x_1, x_2, \ldots, x_i \rangle, \ i \geq 2 \), is semiregular on \( \Delta' \), then \( \langle Q, x_1, x_2, \ldots, x_i \rangle \) is semiregular on \( \Delta' \). Then since \( \langle Q, x_1, x_i \rangle \) is semiregular on \( \Delta' \), this implies by induction that \( \langle Q, x_1, x_2, \ldots, x_k \rangle \) is semiregular on \( \Delta' \).

2.6. \( N(Q) \) has the 2-group \( \langle Q, y_1, y_2, \ldots, y_k \rangle \) in (2.4). Suppose that \( \langle Q, y_1, y_2, \ldots, y_i \rangle, \ i \geq 3 \), is semiregular on \( \Delta' \) and \( \langle Q, y_1, y_2, \ldots, y_i \rangle \) is not semiregular on \( \Delta' \). Then \( \langle Q, y_1, y_2, \ldots, y_i \rangle \) has an element \( y \) fixing a \( \langle Q, y_1, y_2, \ldots, y_i \rangle \)-orbit of length \( 2^i \cdot |Q| (\geq 2^{i+1}) \) in \( \Delta' \) pointwise.
since $\langle Q, y_1, y_2, \ldots, y_{i+1} \rangle$ is abelian and $\langle Q, y_1, y_2, \ldots, y_i \rangle$ is semiregular on $\Delta'$. Then since $y$ has at most $i+2$ 2-cycles in $I(Q)$ and $i \geq 3$, $|I(y)| \geq t-2(i+2)+2^{i+1} > t$, contrary to the assumption (*). Thus if $\langle Q, y_1, y_2, \ldots, y_i \rangle$, $i \geq 3$, is semiregular on $\Delta'$, then $\langle Q, y_1, y_2, \ldots, y_{i+1} \rangle$ is semiregular on $\Delta'$. Then since $\langle Q, y_1, y_2, y_i \rangle$ is semiregular on $\Delta'$, this implies by induction that $\langle Q, y_1, y_2, \ldots, y_k \rangle$ is semiregular on $\Delta'$.

2.7. $|\Delta| \equiv 0 \pmod{4}$.

Proof. Since $Q$ is semiregular ($\pm 1$) on $\Delta$, $|\Delta|$ is even, i.e., $|\Delta| \equiv 0$ or 2 (mod 4). Suppose by way of contradiction that $|\Delta| \equiv 2 \pmod{4}$. Then $|Q| = 2$. Hence we may assume that $Q = \langle a \rangle$ and

$$a = (1) (2) \cdots (t) (1' 2') (3' 4') \cdots (n-1 \ n).$$

Then $N(Q) = C(Q) = C(a)$ and $C(a)^{(a)} = S_t$ or $A_t$. We treat these cases separately.

(i) Suppose that $C(a)^{(a)} = S_t$. Then $C(a)$ has the 2-group $\langle a, x_1, x_2, \ldots, x_k \rangle$ in (2.4). First we show that $\langle a, x_1, x_2, \ldots, x_k \rangle$ has only one orbit $\Gamma$ of length $t$ in $\Delta$ and is semiregular on $\Delta - \Gamma$.

Since $|\Delta| \equiv 2 \pmod{4}$ and $\Delta$ is a union of $\langle a, x_1, x_2, \ldots, x_k \rangle$-orbits, $\langle a, x_1, x_2, \ldots, x_k \rangle$ has at least one orbit of length two in $\Delta$. Hence we may assume that $\langle a, x_1, x_2, \ldots, x_k \rangle$ is a group of length two in $\Delta$. Then we may assume that $\langle a, x_1, x_2, \ldots, x_k \rangle$ has exactly one orbit $\Gamma$ of length two in $\Delta$ and is semiregular on $\Delta - \Gamma$.

Suppose that $\langle a, x_1, x_2 \rangle$ is not semiregular on $\Delta - \{1', 2'\}$. Then $\langle a, x_1, x_2 \rangle$ has an orbit $\Delta'$ of length four in $\Delta - \{1', 2'\}$. Since $\langle a, x_1, x_2 \rangle$ is an elementary abelian group of order eight, there is exactly one element ($\pm 1$) in $\langle a, x_1, x_2 \rangle$ fixing $\Delta'$ pointwise. Since $\langle a, x_1 \rangle$ and $\langle a, x_2 \rangle$ are semiregular on $\Delta - \{1', 2'\}$, $\langle a, x_1 \rangle$, or $\langle a, x_2 \rangle$ fixes $\Delta'$ pointwise. Since $I(x_1, x_2)$ contains $I(a) - \{1, 2, 3, 4\} \cup \{1', 2'\}$ of length $t-2$, $x_1 x_2$ does not fix $\Delta'$ pointwise by the assumption (*). Hence $ax_1 x_2$ fixes $\Delta'$ pointwise. Then $|I(ax_1, x_2)| = t$ and so $ax_1 x_2$ has no fixed point in $\Delta - \{\{1', 2'\} \cup \Delta'\}$. This shows that $\langle a, x_1, x_2 \rangle$ is semiregular on $\Delta - \{\{1', 2'\} \cup \Delta'\}$. By (2.4) $k \geq 3$ and so $C(a)$ has $x_1$ in (2.4). Since $x_2$ normalizes $\langle a, x_1, x_2 \rangle$, $x_2$ fixes $\Delta'$. Then by the same argument as above $ax_1 x_2$ fixes $\Delta'$ pointwise. Thus $I(ax_1, x_2, ax_2) = I(x_1, x_2)$ contains $I(a) - \{3, 4, 5, 6\} \cup \{1', 2'\}$ of length $t+2$, contrary to the assumption (*). Thus $\langle a, x_1, x_2 \rangle$ is semiregular on $\Delta - \{\{1', 2'\} \cup \Delta'\}$. By (2.4) $k \geq 3$ and so $C(a)$ has $x_1$ in (2.4). Since $x_2$ normalizes $\langle a, x_1, x_2 \rangle$, $x_2$ fixes $\Delta'$. Then by the same argument as above $ax_1 x_2$ fixes $\Delta'$ pointwise. Thus $I(ax_1, x_2, ax_2) = I(x_1, x_2)$ contains $I(a) - \{3, 4, 5, 6\} \cup \{1', 2'\}$ of length $t+2$, contrary to the assumption (*). Thus $\langle a, x_1, x_2 \rangle$ is semiregular on $\Delta - \{\{1', 2'\} \cup \Delta'\}$.
gular on \( \Delta - \{1', 2'\} \). Hence by (2.5) \( \langle a, x_1, x_2, \ldots, x_k \rangle \) is semiregular on \( \Delta - \{1', 2'\} \).

On the other hand \( a \) normalizes \( G_1 \) and \( G_2 \), which is even order. Hence \( a \) commutes with an involution \( u \) of \( G_1 \) and \( G_2 \). Since \( C(a)^{(\alpha)} = S_\Gamma \), \( \langle a, x_1, x_2, \ldots, x_k \rangle \) has a subgroup which is conjugate to \( \langle a, u \rangle \) in \( C(a) \). Since \( u \) fixes at least four points of \( \Delta \), \( \langle a, x_1, x_2, \ldots, x_k \rangle \) has an element \((\pm 1)\) fixing at least four points in \( \Delta \), which is a contradiction. Thus \( C(a)^{(\alpha)} = S_\Gamma \).

(ii) Suppose that \( C(a)^{(\alpha)} = A_\Gamma \). Let \( y \) be a 2-element such that \( y^{(\alpha)} \) is an involution consisting two 2-cycles. Since \( |I(y)| \leq t \), \( |I(y) \cap \Delta| = 0, 2 \) or 4.

(ii.i) First assume that \( |I(y) \cap \Delta| = 4 \). By (2.4) \( C(a) \) has the 2-group \( \zeta = \zeta_1 \cdots \zeta_k \). Since \( \zeta_1 \cdots \zeta_k \) is conjugate to \( \zeta_1 \) in \( C(a) \), we may assume that \( y_1 = (1 2) (3 4) (5 6) \cdots \). Since \( \Delta - \{1', 2', 3', 4'\} \) is a union of \( \langle a, y_1 \rangle \)-orbits of length two in \( \Delta - \{1', 2', 3', 4'\} \) is odd. Hence we may assume that \( \{5, 6\} \) is the orbit of length two. Then \( y_1 = (5 6) \) on \( \{5, 6\} \), and \( \langle a, y_1 \rangle \) is semiregular on \( \Delta - \{1', 2', 3', 4'\} \) since \( I(ay_1) \subseteq \Delta \). Furthermore \( C(a) \) has a 2-element \( y_2 = (1 2) (3 4) (5 7) (6 8) (9) \cdots \).

By (2.3) we may assume that \( \langle a, y_1, y_2, y_3 \rangle \) is a 2-group. Then \( y_2, y_3 \) and \( y_2' \) normalize \( \langle a, y_1 \rangle \). So \( |I(y_1) \cup I(y_2) \cup I(y_3)| = |I(ay_1)| = |I(y_2)| = |I(y_3)| = 1 \). Thus \( y_2, y_3 \) and \( y_2' \) centralize \( \langle a, y_1 \rangle \), and so fix \( \{1', 2', 3', 4'\} \) and \( \{5', 6'\} \). Since \( y_2 \) or \( ay_2 \), \( i=2, 3 \), and \( y_2 \) or \( ay_2 \) fix \( \{5', 6'\} \) pointwise, we may assume that \( y_2, y_3 \) and \( y_2' \) fix \( \{5', 6'\} \) pointwise. Since \( I(y_2 \cdots y_3) \) contains \( I(a) \cup \{5', 6'\} \) of length \( t+2, 2 \leq t \leq 3 \), \( y_2 \cdots y_3 \) is of order \( t+2 \), contrary to the assumption \((*)\). Similarly \( y_2 \) is of order \( t+2 \), \( t \neq 1 \).

Thus \( y_2 \) and \( y_3 \) are elementary abelian. Since \( y_2, y_3 \) and \( y_2' \) fix \( \{1', 2', 3', 4'\} \), \( y_2, y_3 \) and \( y_2' \) are \( (1') (2') (3') (4'), (1') (2') (3'), (1') (2') (3' 4'), (1') (2') (3') (4') \), or \( (1') (2') (3' 4') \) on \( \{1', 2', 3', 4'\} \). Since \( I(y_2) \) contains \( I(a) - \{1, 2, 3, 4, 5, 6\} \cup \{5', 6'\} \) of length \( t-2 \), \( y_2 \) does not fix \( \{1', 2', 3', 4'\} \) pointwise. Similarly \( y_3 \) and \( y_3' \) do not fix \( \{1', 2', 3', 4'\} \) pointwise. If \( y_2 = (1') (2') (3' 4') \cdots \), then \( I(ay_2) \) contains \( I(a) - \{1, 2, 3, 4, 5, 6\} \cup \{1', 2', 3', 4'\} \) of length \( t+2 \), contrary to the assumption \((*)\). Thus \( y_2 \) is of order \( t' \) pointwise. Similarly \( y_3 \) and \( y_3' \) is of order \( t' \) pointwise. Next suppose that \( y_2 = (1') (2') (3') (4') \cdots \). The proof in the case \( y_2 = (1') (2') (3') (4') \cdots \) is similar. Since \( y_2 \) commutes with \( y_2, y_2' \) or \( (1') (2') (3') (4') \cdots \), \( I(y_2) \) contains \( I(a) - \{1, 2, 3, 4, 5, 6\} \cup \{1', 2', 3', 4'\} \) of length \( t+2 \), contrary to the assumption \((*)\). Thus \( y_2 = (1') (2') (3') (4') \cdots \). On the other hand as we have seen above \( y_2' = (1') (2') (3') (4') (5') (6'), (1') (2') (3') (4') (5'), (1') (2') (3' 4') (5') (6'), (1') (3') (2' 4') (5') (6') \) or \( (1') (2' 3') (5') (6') \) on \( \{1', 2', 3', 4'\} \). If \( y_2' \) is of the first form, then
\[(y_2, y'_2)^3\] is of even order and \(|I((y_2, y'_2)^3)| \geq t+2\), contrary to the assumption (\ast). If \(y'_2\) is of the second form, then \((y_2, y'_2)^3\) is of even order and \(|I((y_2, y'_2)^3)| \geq t+2\), contrary to the assumption (\ast). If \(y'_2\) is of the third or fourth form, then \((y_2, y'_2)^3\) is of even order and \(|I((y_2, y'_2)^3)| \geq t+2\), contrary to the assumption (\ast). Thus \(y'_2 = (1' 2') (3') (4') \cdots\) and so \(y'_2 = (1') (2') (3' 4') \cdots\). Finally suppose that \(y'_2 = (1' 3') (2' 4') \cdots\). The proof in the case \(y'_2 = (1' 4') (2' 3') \cdots\) is similar. Then by the same argument as is used for \(y'_2\), \(y'_2\) are \((1' 3') (2' 4')\) or \((1' 4') (2' 3')\) on \{1', 2', 3', 4'\}. If \(y_2\) or \(y'_2 = (1' 3') (2' 4') \cdots\), then \(|I(y_2 y_2)|\) or \(|I((y_2, y'_2)^3)|\) \(\geq t+2\) respectively, contrary to the assumption (\ast). Thus \(y_2\) and \(y'_2 = (1' 4') (2' 3') \cdots\). Then \((y_2, y'_2)^3\) is of even order and \(|I((y_2, y'_2)^3)| \geq t+2\), contrary to the assumption (\ast). Thus if \(y\) is a 2-element of \(C(a)\) such that \(y^{l(a)}\) is an involution consisting of two 2-cycles, then \(|I(y) \cap \Delta| = 0\) or 2. By (2.4) \(C(a)\) has the 2-group \(<a, y_1, y_2, \ldots, y_k>\). First we show that \(<a, y_1, y_2, \ldots, y_k>\) has exactly one orbit \(\Gamma\) of length two in \(\Delta\) and is semiregular on \(\Delta - \Gamma\).

Since \(|\Delta| \equiv 2 \pmod{4}\) and \(\Delta\) is a union of \(<a, y_1, y_2, \ldots, y_k>-orbits\), \(<a, y_1, y_2, \ldots, y_k>\) has at least one orbit of length two in \(\Delta\). We may assume that \{1', 2'\} is the \(<a, y_1, y_2, \ldots, y_k>-orbit\) of length two. Then \(y_i\) or \(a y_i, 1 \leq i \leq k\), fixes \{1', 2'\} pointwise. Hence we may assume that \(y_i\) fixes \{1', 2'\} pointwise. Since \(|I(y_i) \cap \Delta| = 0\) or 2, \(|I(y_i) \cap \Delta| = 1\'). Since \(|I(\Gamma y_i, y_i)|\) contains \(I(a) \cup \{1', 2'\}\) of length \(t+2\), \(1 \leq i, j \leq k\), \(y_i y_j y_i y_j = 1\) by the assumption (\ast). Hence \(y_i = 1\) and \(y_i y_j = y_j y_i\). Thus \(<a, y_1, y_2, \ldots, y_k>\) is an elementary abelian group.

Since \(a\) and \(y_i\) has no fixed point in \(\Delta - \{1', 2'\}\) and \(|\Delta - \{1', 2'\}| \equiv 0 \pmod{4}\), \(|I(a y_i) \cap (\Delta - \{1', 2'\})| \equiv 0 \pmod{4}\). Hence by (i.i) \(|I(a y_i) \cap (\Delta - \{1', 2'\})| = 0\). Thus \(<a, y_i>\) is semiregular on \(\Delta - \{1', 2'\}\).

Suppose that \(<a, y_1, y_2>\) is not semiregular on \(\Delta - \{1', 2'\}\). Then \(<a, y_1, y_2>\) has an orbit \(\Delta'\) of length four in \(\Delta - \{1', 2'\}\). Since \(<a, y_1, y_2>\) is an abelian group, there is an involution \(y'\) in \(<a, y_1, y_2>\) fixing \(\Delta'\) pointwise. Then \(y'^{l(a)}\) is an involution consisting of two 2-cycles and \(|I(y') \cap \Delta| = \Delta'\), contrary to (ii.i). Thus \(<a, y_1, y_2>\) is semiregular on \(\Delta - \{1', 2'\}\).

Suppose that \(<a, y_1, y_2, y_3>\) is not semiregular on \(\Delta - \{1', 2'\}\). Then \(<a, y_1, y_2, y_3>\) has an orbit \(\Delta'\) of length eight in \(\Delta - \{1', 2'\}\). Since \(<a, y_1, y_2, y_3>\) is an abelian group of order sixteen, there is exactly one involution \(y'\) in \(<a, y_1, y_2, y_3>\) fixing \(\Delta'\) pointwise. Since \(|\Delta'| = 8\), \(y'\) has at least four 2-cycles on \(I(a)\). Thus \(y' = y_1 y_2 y_3\) or \(a y_1 y_2 y_3\). If \(y' = y_1 y_2 y_3\), then \(|I(y')|\) contains \((I(a) - \{1, 2, \ldots, 8\}) \cup \{1', 2'\} \cup \Delta'\) of length \(t+2\), contrary to the assumption (\ast). Thus \(y' = a y_1 y_2 y_3\). Then \(I(a y_1 y_2 y_3) = (I(a) - \{1, 2, \ldots, 8\}) \cup \Delta'\) since \(|I(a) - \{1, 2, \ldots, 8\}| = 8\). Furthermore this shows that \(<a, y_1, y_2, y_3>\) has no orbit of length eight in \(\Delta - \{1', 2'\} \cup \Delta'\). On the other hand \(C(a)\) has a 2-element

\[y'_1 = (1 3) (2 4) (5 6) \cdots (t) \cdots\]
By (2.3) we may assume that \( \langle a, y, y_2, y_3 \rangle \) is a 2-group. Then \( y_i \) normalizes \( \langle a, y, y_2, y_3 \rangle \) and so \( y_i \) fixes \( \{1', 2'\} \) and \( \Delta' \). Set \( R=\langle a, y, y_2, y_3, y_i \rangle \), where \( i \in \Delta' \). Then the order of \( R \) is four and so \( R \) is cyclic or elementary abelian. Since \( \langle a, y_i \rangle \) is contained in the center of \( \langle a, y, y_2, y_3, y_i \rangle \) and semiregular on \( \Delta' \), any element of \( R \) fixes at least four points of \( \Delta \). Suppose that \( R \) is a cyclic group generated by an element \( z \). Then since \( ay, y_2, y_3 \) is the involution of \( R \), \( z^2=ay, y_2, y_3 \). Thus \( z^{(a)} \) has two 4-cycles since \( (ay, y_2, y_3) \) has no such element.

Next suppose that \( R \) is elementary abelian. Since \( \langle a, y, y_2, y_3 \rangle \) is also an elementary abelian group of order four. Furthermore since any element of \( R \) fixes at least four points of \( \Delta \), every element \((+1)\) of \( R^{(a)} \) has at least three 2-cycles by the assumption (\( \ast \)) and (\( \nu, \tau \)). This is a contradiction since \( \langle a, y_2, y_3, y_i \rangle \) is semiregular on \( \Delta=\{1', 2'\} \). Hence by (2.6) \( \langle a, y, y_2, \ldots, y_n \rangle \) is semiregular on \( \Delta=\{1', 2'\} \).

On the other hand \( a \) commutes with an involution \( u \) of \( G, \langle y, y_2, \ldots, y_n \rangle \). Since \( C(a)^{(a)}=A, \langle a, y, y_2, \ldots, y_n \rangle \) has a subgroup which is conjugate to \( \langle a, u \rangle \) in \( C(a) \). Since \( u \) fixes at least four points of \( \Delta \), \( \langle a, y, y_2, \ldots, y_n \rangle \) has an element \((+1)\) fixing at least four points of \( \Delta \), which is a contradiction. Thus \( C(a)^{(a)}\neq A \). Hence \( |\Delta|=0 \) (mod 4).

2.8. Let \( x \) be a 2-element of \( N(Q) \) such that \( x^{(Q)} \) is an involution consisting of \( m \) 2-cycles. If \( x \) fixes \( r \) \( Q \)-orbits in \( \Delta \), then \( r \leq 2m \) and \( Qx \) has at least \( r \) involutions which have fixed points in \( \Delta \).

Proof. Assume that \( x \) fixes \( r \) \( Q \)-orbits \( \Delta_1, \Delta_2, \ldots, \Delta_r \) in \( \Delta \). Set \( \Gamma=\Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_r \). Then

\[
r \cdot |\langle Q, x \rangle| = \sum_{u \in Q_x} |I(u^r)|.
\]

Since \( \langle Q, x \rangle=Q+Qx \) and \( |Q|=|\Delta_1|=\cdots=|\Delta_r| \),

\[
r \cdot 2 \cdot |Q| = \sum_{u \in Q} |I(u^r)| = \sum_{u \in Q} |I((ux)^r)| = r \cdot |Q| + \sum_{u \in Q} |I((ux)^r)|.
\]

Hence

\[
\sum_{u \in Q} |I((ux)^r)| = r \cdot |Q|.
\]

On the other hand \( |I(x) \cap I(Q)|=t-2m \). Hence for any element \( u \) of \( Q \) \( |I(ux) \cap \Delta| \leq 2m \) by the assumption (\( \ast \)). Hence \( |I((ux)^r)| \leq 2m \). Suppose that \( Qx \) has \( s \) elements which have fixed points in \( \Gamma \). Then

\[
\sum_{u \in Q} |I((ux)^r)| \leq 2ms.
\]
Hence \( r \cdot |Q| \leq 2ms \). Thus \( \frac{r}{2m} \cdot |Q| \leq s \). Furthermore since \( s \leq |Q| \), \( \frac{r}{2m} \cdot |Q| \leq |Q| \). Hence \( r \leq 2m \).

Let \( x' \) be any element of \( Qx \) such that \( |I(x') \cap \Delta| \neq 0 \). Then \( |I(x'')| > t \). Hence \( x'' = 1 \) by the assumption (*).

We use the following notations: Assume that the \( Q \)-orbits on \( \Delta \) consist of \( \Delta_1, \Delta_2, \ldots, \Delta_r \). For any element \( x \in N(Q) \) let \( \pi \) be the permutation on \( \{ \Delta_1, \Delta_2, \ldots, \Delta_r \} \) induced by \( x \),

\[
\pi = \begin{pmatrix}
\Delta_1 & \Delta_2 & \cdots & \Delta_r \\
\Delta_1' & \Delta_2' & \cdots & \Delta_r'
\end{pmatrix}
\]

Then \( \pi \) form a permutation group \( N(Q) \) on \( \Delta = \{ \Delta_1, \Delta_2, \ldots, \Delta_r \} \).

2.9. Suppose that \( N(Q) \) has the 2-group \( \langle Q, x_1, x_2, \ldots, x_k \rangle \) as in (2.4), and \( \langle Q, x_1, x_2, \ldots, x_k \rangle \) fixes a subset \( \Delta' \) of \( \Delta \). If \( \langle Q, x_1, x_2, x_3, x_4 \rangle \) is semiregular on \( \Delta' \), then \( \langle Q, x_1, x_2, \ldots, x_k \rangle \) is semiregular on \( \Delta' \).

Proof. Suppose that \( \langle Q, x_1, x_2, \ldots, x_i \rangle, i \geq 4, \) is semiregular on \( \Delta' \) and \( \langle Q, x_1, x_2, \ldots, x_i, x_{i+1} \rangle \) is not semiregular on \( \Delta' \). Then \( \langle Q, x_1, x_2, \ldots, x_i, x_{i+1} \rangle \) has an element \( x \) having fixed points in \( \Delta' \). Since \( \langle x_1, x_2, \ldots, x_{i+1} \rangle \) is abelian and \( \langle x_1, x_2, \ldots, x_{i+1} \rangle \) is semiregular on the set of the \( Q \)-orbits contained in \( \Delta' \), \( x \) fixes at least \( 2i \) \( Q \)-orbits in \( \Delta' \). On the other hand since \( x \in \langle Q, x_1, x_2, \ldots, x_{i+1} \rangle \), \( x \) has at most \( i+1 \) 2-cycles on \( I(Q) \). Hence by (2.8) \( 2i \leq 2(i+1) \), so \( i \leq 3 \), which is a contradiction. Thus if \( \langle Q, x_1, x_2, \ldots, x_i, x_{i+1} \rangle \) is semiregular on \( \Delta' \), then \( \langle Q, x_1, x_2, \ldots, x_i, x_{i+1} \rangle \) is semiregular on \( \Delta' \). Since \( \langle Q, x_1, x_2, x_3, x_4 \rangle \) is semiregular on \( \Delta' \), this implies by induction that \( \langle Q, x_1, x_2, \ldots, x_k \rangle \) is semiregular on \( \Delta' \).

2.10. Suppose that \( \langle Q, y_1, y_2, \ldots, y_k \rangle \) as in (2.4) fixes a subset \( \Delta' \) of \( \Delta \). If \( \langle Q, y_1, y_2, \ldots, y_k, y_i \rangle \) is semiregular on \( \Delta' \), then \( \langle Q, y_1, y_2, \ldots, y_k, y_i \rangle \) is semiregular on \( \Delta' \).

Proof. Suppose that \( \langle Q, y_1, y_2, \ldots, y_i, y_i \rangle, i \geq 4, \) is semiregular on \( \Delta' \) and \( \langle Q, y_1, y_2, \ldots, y_i, y_i, y_i \rangle \) is not semiregular on \( \Delta' \). Then \( \langle Q, y_1, y_2, \ldots, y_i, y_i, y_i \rangle \) has an element \( y \) having fixed points in \( \Delta' \). Since \( \langle y_1, y_2, \ldots, y_i, y_i \rangle \) is abelian and \( \langle y_1, y_2, \ldots, y_i, y_i \rangle \) is semiregular on the set of the \( Q \)-orbits contained in \( \Delta' \), \( y \) fixes at least \( 2i \) \( Q \)-orbits in \( \Delta' \). On the other hand since \( y \in \langle Q, y_1, y_2, \ldots, y_i, y_i \rangle \), \( y \) has at most \( i+1 \) 2-cycles on \( I(Q) \). Hence by (2.8) \( 2i \leq 2(i+1) \), so \( i \leq 3 \), which is a contradiction. Thus if \( \langle Q, y_1, y_2, \ldots, y_i, y_i \rangle \) is semiregular on \( \Delta' \), then \( \langle Q, y_1, y_2, \ldots, y_i, y_i \rangle \) is semiregular on \( \Delta' \). Since \( \langle Q, y_1, y_2, y_3, y_4 \rangle \) is semiregular on \( \Delta' \), this implies by induction that \( \langle Q, y_1, y_2, \ldots, y_k \rangle \) is semiregular on \( \Delta' \).
MULTIPLY TRANSITIVE GROUPS XII

603

\[ \pi \in S_n \]

is semiregular on \( \Delta' \). Since \( \langle \pi, \varphi, \varphi^2, \varphi^3, \varphi^4 \rangle \) is semiregular on \( \Delta' \), this implies by induction that \( \langle \pi, \varphi, \varphi^2, \cdots, \varphi^k \rangle \) is semiregular on \( \Delta' \).

2.11. \textbf{G is not 5-fold transitive on } \( \Omega \).

Proof. If \( G \) is 5-fold transitive on \( \Omega \), then \( G_1 \) is 4-fold transitive on \( \Omega - \{1\} \) and satisfies the assumptions of the theorem. Hence by the minimal nature of the degree of \( G \), \( G \) contains \( A_{n-1} \), so \( G \) contains \( A_n \). This is a contradiction. Thus \( G \) is not 5-fold transitive.

2.12. \textbf{Let } \( x \) \textbf{be an involution of } \( N(Q) \). \textbf{If there is a } \( Q \)-orbit \( \Delta' \) \textbf{in } \( \Delta \) \textbf{such that } \( |I(x) \cap \Delta'| = 2 \), then } \( C(Q)^{(Q)} = A_t \) \textbf{or } \( S_t \).

Proof. Since \( x \) is an involution and \( |I(x) \cap \Delta'| = 2 \), \( x \) induces an involutory automorphism of \( Q \) which fixes exactly two elements. By a theorem of H. Zassenhaus \([16], \text{ Satz 5}\) \( Q \) contains a cyclic group of index two. Then the automorphism group of \( Q \) is \( S_t, S_t \) or a 2-group (cf. H. Zassenhaus \([17], \text{ IV, } \S 3, \text{ Exercise 4}\)). Since \( N(Q)^{(Q)} = A_t \) or \( S_t, t \geq 6 \) and \( N(Q)^{(Q)} / C(Q)^{(Q)} \) is involved in the automorphism group of \( Q \), \( C(Q)^{(Q)} \) contains \( A_t \).

2.13. \textbf{Let } \( x \) \textbf{be a } \( 2 \)-\textbf{element of } \( N(Q) \). \textbf{If } \( x^{(Q)} \) \textbf{is an involution consisting of exactly one } \( 2 \)-\textbf{cycle}, \textbf{then } \( |I(x) \cap \Delta| = 0 \).

Proof. Since \( |I(x)| \leq t, |I(x) \cap \Delta| = 0 \) or 2. Suppose by way of contradiction that \( x^{(Q)} \) is an involution consisting of exactly one \( 2 \)-cycle and \( |I(x) \cap \Delta| = 2 \). Then \( |I(x^2)| \geq t+2 \). Hence \( x^2 = 1 \). Since \( x^{(Q)} \) is an odd permutation, \( N(Q)^{(Q)} = S_t \). Furthermore by (2.12) \( C(Q)^{(Q)} = S_t \) or \( A_t \). We treat these cases separately.

(i) Suppose that \( C(Q)^{(Q)} = S_t \). Then \( C(Q) \) has a \( 2 \)-element \( x' \) such that \( x^{(Q)} = x^{(Q)} \). Since \( Q \) is a Sylow \( 2 \)-subgroup of \( G_{(Q)} \), \( \langle Q, x \rangle \) and \( \langle Q, x' \rangle \) are Sylow \( 2 \)-subgroups of \( \langle Q, x, x' \rangle \). Hence \( \langle Q, x \rangle \) is conjugate to \( \langle Q, x' \rangle \). Thus \( x \) is conjugate to \( x^c \), where \( c \in Q \), and so \( |I(x^c) \cap \Delta| = 2 \). Hence \( x^c \) commutes with exactly one element of \( Q \) other than 1, which is a central involution of \( Q \). On the other hand since \( x^c \in C(Q) \), \( x^c \) commutes with \( c \). Hence \( x^c \) commutes with \( c \). Thus \( c = 1 \) or a central involution of \( Q \). Hence \( x^c \in C(Q) \) and so \( Q \) is of order two. Set \( Q = \langle a \rangle \). Then we may assume that

\[ a = (1) (2) \cdots (t) (1' 2') (3' 4') \cdots (n-1 \ n). \]

Since \( |\Delta| \equiv 0 \) (mod 4) and \( |I(x) \cap \Delta| = 2 \), \( |I(ax) \cap \Delta| \equiv 2 \) (mod 4). Hence \( |I(ax) \cap \Delta| = 2 \) because \( |I(ax)| \leq t \). Since \( C(a)^{(a)} = S_t, C(a) \) has the 2-group \( \langle a, x_1, x_2, \cdots, x_k \rangle \) as in (2.4). Since \( \langle a, x_i \rangle, 1 \leq i \leq k, \) is conjugate to \( \langle a, x \rangle \) in \( C(a) \), \( \langle a, x_i \rangle \) is elementary abelian and \( |I(x_i) \cap \Delta| = |I(ax_i) \cap \Delta| = 2 \). Hence we may assume that
\[ x_i = (1 \ 2) (3 \ 4) \cdots (t) (1') (2') (3' \ 4') (5' \ 7') (6' \ 8') \cdots. \]

Then \( <a, x_i> \) is semiregular on \( \Delta - \{1', 2', 3', 4'\} \).

Now we show that \( <a, x_i, x_2, \ldots, x_k> \) is elementary abelian and semiregular on \( \Delta - \{1', 2', 3', 4'\} \), where \( \{1', 2'\} \) and \( \{3', 4'\} \) are \( <a, x_i, x_2, \ldots, x_k> \)-orbits of length two. Since \( x_2 \) normalizes \( <a, x_i, x_2> \), \( x_i x_2 = x_i \) or \( ax_i \). Suppose that \( x_i x_2 = ax_i \). Then \( (x_i x_2)^2 = a \). Hence \( <a, x_i> \) is a cyclic group of order four and contains \( a \).

On the other hand since \( C(a)^{(\sigma)} = S_r \), \( <a, x_i, x_2> \) is conjugate to \( <a, x_i, x_2> \) in \( C(a) \). Hence \( x_i x_2 = ax_i \). Thus \( x_i x_2 x_i = x_i \) and so \( x_i x_2 \) centralizes \( <a, x_i> \). Furthermore since \( I(x_i) \cap \Delta = \{1', 2'\} \) and \( I(ax_i) \cap \Delta = \{3', 4'\} \), \( x_i x_2 \) fixes \( \{1', 2'\} \) and \( \{3', 4'\} \). Thus \( I((x_i x_2)^t) \) contains \( I(a) \cap \{1', 2', 3', 4'\} \) of length \( t+4 \). Hence \( (x_i x_2)^t = 1 \). This is a contradiction since \( <a, x_i, x_2> \) is conjugate to the cyclic group \( <x_i, x_2> \). Thus \( x_2 \) commutes with \( x_i \) and so \( <a, x_i, x_2> \) is elementary abelian.

Furthermore \( <a, x_i, x_2> \) is conjugate to \( <a, x_i, x_j>, i \neq j \) and \( 1 \leq i, j \leq k \). Hence \( <a, x_i, x_j> \) is also elementary abelian. Thus \( <a, x_i, x_2, \ldots, x_k> \) is elementary abelian.

Since \( I(x_i) \cap \Delta = \{1', 2'\} \) and \( I(ax_i) \cap \Delta = \{3', 4'\} \), \( \{1', 2'\} \) and \( \{3', 4'\} \) are \( <a, x_i, x_2, \ldots, x_k> \)-orbits of length two. Since \( x_i \) or \( ax_i, 2 \leq i \leq k \), fixes \( \{1', 2'\} \) pointwise, we may assume that \( x_i \) fixes \( \{1', 2'\} \) pointwise.

Suppose that \( <a, x_i, x_2> \) is not semiregular on \( \Delta - \{1', 2', 3', 4'\} \). Then \( <a, x_i, x_2> \) has an orbit \( \Delta' \) of length four in \( \Delta - \{1', 2', 3', 4'\} \). Since \( <a, x_i, x_2> \) is an elementary abelian group of order eight, there is exactly one involution \( x' \) in \( <a, x_i, x_2> \) fixing \( \Delta' \) pointwise. Since \( \mid \Delta' \mid = 4 \), \( x' \) has at least two 2-cycles in \( I(a) \). Hence \( x' = x_i x_2 \) or \( ax_i x_2 \). If \( x' = x_i x_2 \), then \( I(x') \) contains \( I(a) - \{1, 2, 3, 4\} \cup \{1', 2'\} \cup \Delta' \) of length \( t+2 \), contrary to the assumption \( (\ast) \). Thus \( x' = ax_i x_2 \). Then \( I(ax_i x_2) = I(a) - \{1, 2, 3, 4\} \cup \Delta' \) since \( \mid I(a) - \{1, 2, 3, 4\} \cup \Delta' \mid = t \). This shows that \( <a, x_i, x_2> \) is semiregular on \( \Delta - \{1', 2', 3', 4'\} \cup \Delta' \). By (2.4) \( C(a) \) has \( x_i \). Then \( x_2 \) normalizes \( <a, x_i, x_2> \) and so fixes \( \Delta' \). Hence by the same argument as above \( ax_i x_2 \) fixes \( \Delta' \) pointwise.

Thus \( I(ax_i x_2, ax_i x_2) = I(x_i x_2) \) contains \( I(a) - \{3, 4, 5, 6\} \cup \{1', 2', 3', 4'\} \cup \Delta' \) of length \( t+4 \), contrary to the assumption \( (\ast) \). Thus \( <a, x_i, x_2> \) is semiregular on \( \Delta - \{1', 2', 3', 4'\} \).

Hence by (2.5) \( <a, x_i, x_2, \ldots, x_k> \) is semiregular on \( \Delta - \{1', 2', 3', 4'\} \).

On the other hand \( <a, x_i> \) normalizes \( G_{\sigma' \sigma' \ldots} \), which is even order. Hence \( a \) and \( x_i \) commute with an involution \( u \) of \( G_{\sigma' \sigma' \ldots} \). Since \( I(x_i) \cap \Delta = \{1', 2'\} \) and \( I(ax_i) \cap \Delta = \{3', 4'\} \), \( <a, u> \) has at least four orbits \( \{1', 2'\}, \{3', 4'\}, \{5', 6'\} \) and \( \{7', 8'\} \) of length two in \( \Delta \). Since \( C(a)^{(\sigma)} = S_r \), \( <a, x_i, x_2, \ldots, x_k> \) has a subgroup \( <a, u> \) which is conjugate to \( <a, u> \) in \( C(a) \). This is a contradiction since \( <a, u> \) has exactly two orbits \( \{1', 2'\} \) and \( \{3', 4'\} \) of length two in \( \Delta \). Thus \( C(Q)^{(\sigma)} = S_r \).

(ii) Suppose that \( C(Q)^{(\sigma)} = A_t \).

(ii.i) We show that \( x \) fixes exactly one \( Q \)-orbit in \( \Delta \). Since \( \mid I(x) \cap \Delta \mid = 2 \), \( x \) fixes at least one \( Q \)-orbit in \( \Delta \). On the other hand by (2.8) \( x \) fixes at most two \( Q \)-orbits. Suppose that \( x \) fixes exactly two \( Q \)-orbits \( \Delta_i \) and \( \Delta_j \) in \( \Delta \). Let \( u \) be
any element of $Q$. Then by (2.8) $ux$ is an involution having fixed points in $\Delta_1$ or $\Delta_2$. Since $ux$ consists of one 2-cycle on $I(Q)$, $ux$ fixes two points and these two points are contained in either $\Delta_1$ or $\Delta_2$. Hence $\langle Q, x \rangle$ is semiregular on $\Delta - (\Delta_1 \cup \Delta_2)$. Since $(ux)^2 = 1, u^x = u^{-1}$. In particular if $u$ is an involution, then $x$ commutes with $u$. On the other hand since $|I(x) \cap \Delta| = 2$, $x$ commutes with exactly one involution of $Q$. Hence $Q$ has exactly one involution and so $Q$ is a cyclic or generalized quaternion group. Let $u$ and $u'$ be any two elements of $Q$. Then $(uu')^2 = (uu')^{-1}$, and $(uu')^x = u^x u'^x = u^{-1} u'^{-1} = (u'u)^{-1}$. Hence $uu' = u'u$ and so $Q$ is a cyclic group. Furthermore since $C(Q)^{I(Q)} = A_4$, any 2-element of $N(Q)$ whose restriction on $I(Q)$ is an even permutation belongs to $C(Q)$.

$N(Q)$ has the 2-group $\langle Q, x, x_2, x_3 \rangle$ as in (2.4). Since $\langle Q, x_i \rangle$ is conjugate to $\langle Q, x \rangle$, we may assume that $x_i = x$.

$$x_i = (1\ 2)\ (3\ 4)\ \ldots (t\ 1')\ (2')\ (3'\ 4')\ \ldots$$

and $\{1', 2'\} \subset \Delta_i$. Since $x_i$ normalizes $\langle Q, x_i \rangle$ and $\langle Q, x_i \rangle$ has exactly two orbits $\Delta_1$ and $\Delta_2$ of length $|Q|$, $\Delta_1^x = \Delta_1$ or $\Delta_2$. First assume that $\Delta_1^x = \Delta_1$. Since $\langle Q, x_i, x_i \rangle$ is conjugate to $\langle Q, x_i, x_i \rangle$ in $N(Q)$, $\Delta_1^x = \Delta_1$. Hence $\Delta_1^x \Delta_2 = \Delta_1$. Next assume that $\Delta_1^x = \Delta_2$. Then similarly $\Delta_1^x = \Delta_2$. Hence $\Delta_1^x \Delta_2 = \Delta$. Thus in either case $\Delta_1^x \Delta_2 = \Delta$. Hence there is an element $y$ in $Qx_i x_i$ such that $|I(y) \cap \Delta_i| = 0$. Since $y^{I(Q)} = (3\ 4)\ (5\ 6), |I(y) \cap \Delta_1| = 2$ or 4. Furthermore as we have seen above $y \in C(Q)$. Hence $|Q| = 2$ or 4. However we assumed that $N(Q) \neq C(Q)$.

$N(Q)$ has the 2-group $\langle Q, x_i, x_i, x_2 \rangle$ as in (2.4). Since $\langle Q, x_i \rangle$ is conjugate to $\langle Q, x \rangle$, we may assume that $x_i = x$.

$$x_i = (1\ 2)\ (3\ 4)\ \ldots (t\ 1')\ (2')\ (3'\ 4')\ \ldots$$

and $\{1', 2'\} \subset \Delta_i$. Since $x_i$ normalizes $\langle Q, x_i \rangle$ and $\langle Q, x_i \rangle$ has exactly two orbits $\Delta_1$ and $\Delta_2$ of length $|Q|$, $\Delta_1^x = \Delta_1$ or $\Delta_2$. First assume that $\Delta_1^x = \Delta_1$. Since $\langle Q, x_i, x_i \rangle$ is conjugate to $\langle Q, x_i, x_i \rangle$ in $N(Q)$, $\Delta_1^x = \Delta_1$. Hence $\Delta_1^x \Delta_2 = \Delta_1$. Next assume that $\Delta_1^x = \Delta_2$. Then similarly $\Delta_1^x = \Delta_2$. Hence $\Delta_1^x \Delta_2 = \Delta$. Thus in either case $\Delta_1^x \Delta_2 = \Delta$. Hence there is an element $y$ in $Qx_i x_i$ such that $|I(y) \cap \Delta_i| = 0$. Since $y^{I(Q)} = (3\ 4)\ (5\ 6), |I(y) \cap \Delta_1| = 2$ or 4. Furthermore as we have seen above $y \in C(Q)$. Hence $|Q| = 2$ or 4. However we assumed that $N(Q) \neq C(Q)$. Hence $|Q| = 4$. Let $Q = \langle b \rangle$. Since $b^* = b^{-1}$, we may assume that $b = (1\ 2)\ \ldots (t\ 1')\ (2')\ (3'\ 4')\ (5'\ 6'\ 7'\ 8')\ \ldots$.

$\Delta_1 = \{1', 2', 3', 4'\}$ and $\Delta_2 = \{5', 6', 7', 8'\}$. Then

$$y = (1\ 2)\ (3\ 4)\ (5\ 6)\ (7\ 8)\ \ldots (t\ 1')\ (2')\ (3')\ (4')\ (5'\ 6')\ (7'\ 8')\ \ldots$$

On the other hand $C(Q)$ has a 2-element

$$y' = (1\ 2)\ (3\ 5)\ (4\ 6)\ (7\ 8)\ \ldots (t\ 1')\ (2')\ (3')\ (4')\ (5'\ 6')\ (7'\ 8')\ \ldots$$

By (2.3) we may assume that $\langle Q, x_i, y, y' \rangle$ is a 2-group. Since $\langle Q, x_i, y' \rangle$ is conjugate to $\langle Q, x_i, y \rangle$ in $N(Q)$, $\Delta_1' = \Delta_1$ and $\Delta_2' = \Delta_2$. Then $Qy'$ has an element

$$y'' = (1\ 2)\ (3\ 5)\ (4\ 6)\ (7\ 8)\ \ldots (t\ 1')\ (2')\ (3')\ (4')\ (5'\ 6')\ (7'\ 8')\ \ldots$$

Then $yy''$ is of even order and $I(yy'')$ contains $(I(Q) - \{3, 4, 5, 6\}) \cup \Delta_1 \cup \Delta_2$ of length $t + 4$, contrary to the assumption (*). Thus $x_i$ fixes exactly one $Q$-orbit in $\Delta$.

(ii.i) We show that $|Q| = 4$. Since $N(Q)^{I(Q)} \neq C(Q)^{I(Q)}$, $|Q| = 2$. Suppose by way of contradiction that $|Q| \geq 8$. By (2.4) $N(Q)$ has the 2-group $\langle Q, x_i, x_3, x_4 \rangle$.
\( x_i \). Since \( \langle Q, x_i \rangle \) is conjugate to \( \langle Q, x \rangle \), we may assume that \( x_i = x \) and

\[
x_i = (1 \ 2) \ (3 \ 4) \cdots (t-1) \ (2' \ 3' \ 4') \ (5' \ 7') \ (6' \ 8') \cdots .
\]

Then there is exactly one involution \( a \) in \( Q \) commuting with \( x_i \). Then we may assume that

\[
a = (1 \ 2) \cdots (t \ (1' \ 2') \ (3' \ 4') \ (5' \ 6') \ (7' \ 8') \cdots (n-1 \ n) .
\]

By (ii.i) there is exactly one \( Q \)-orbit \( \Delta \) in \( \Delta \) fixed by \( x_i \). Since \( |\Delta| = |Q| \geq 8 \), we may assume that \( \Delta \supseteq \{1', 2', \ldots, 8'\} \). Since \( x_i \) and \( x_j \) normalizes \( \langle Q, x_i \rangle \), \( x_i \) and \( x_j \) fix \( \Delta \). Thus \( Qx_i \) and \( Qx_j \) have elements fixing \( 1' \) of \( \Delta \). We may assume that \( x_i \) and \( x_j \) fix \( 1' \). Then \( I(x_i^x, x_j^x) \supseteq I(a) \cup \{1'\} \), \( 1 \leq i, j \leq 3 \). Hence \( x_i^x = x_j^x = 1 \) and \( x_i \) commutes with \( x_j \). Since \( I(x_i) \cap \Delta = \{1', 2'\} \) and \( |I(x_i)| \leq t, i=2, 3 \), \( I(x_i) \cap \Delta = \{1', 2'\} \). This implies that \( x_i \) and \( x_j \) commute with \( a \). Thus \( \langle a, x_i, x_j, x_i^x \rangle \) is elementary abelian. Furthermore \( I(ax_i) \cap \Delta = \{3', 4'\} \). Hence \( x_i \) and \( x_j = (1') (2') (3' 4') \) on \( \{1', 2', 3', 4'\} \). On the other hand \( |\Delta| - \{1', 2', 3', 4'\} | \equiv 4 \) (mod 8). Hence \( \langle a, x_i, x_j, x_i^x \rangle \) has an orbit of length four in \( \Delta = \{1', 2', 3', 4'\} \). Hence we may assume that \( \{5', 6', 7', 8'\} \) is the \( \langle a, x_i, x_j, x_i^x \rangle \)-orbit of length four. Since \( |\langle a, x_i, x_j, x_i^x \rangle| = 8, i=2, 3 \), there is an involution \( x_i' \in \langle a, x_i, x_i^x \rangle \) fixing \( \{5', 6', 7', 8'\} \) pointwise. Since \( |I(x_i')| \leq t, x_i' = x_i, a \) or \( x_i, x_j \). If \( x_i' = x_i, x_j \), then \( I(x_i) \cap \Delta \supseteq \{1', 2', \ldots, 8'\} \) and so \( |I(x_i)| \geq t+4 \), contrary to the assumption (\( \ast \)). Thus \( x_i' = ax_i, x_i \). Hence \( I(ax_i, x_j, ax_i, x_i) = I(x_i, x_j) \) contains \( (I(a) - \{3, 4, 5, 6\}) \cup \{1', 2', \ldots, 8'\} \) of length \( t+4 \), contrary to the assumption (\( \ast \)). Thus \( |Q| = 4 \).

(ii.iii) We show that \( |Q| = 4 \) implies a contradiction. \( N(Q) \) has the 2-group \( \langle Q, x_i, x_j, x_k \rangle \) as in (2.4). Since \( \langle Q, x_i \rangle \) is conjugate to \( \langle Q, x \rangle \), we may assume that \( x_i = x \) and

\[
x_i = (1 \ 2) \ (3 \ 4) \cdots (t-1) \ (2' \ 3' \ 4') \ (5' \ 7') \ (6' \ 8') \cdots .
\]

Let \( a \) be an involution of \( Q \) commuting with \( x_i \). Then we may assume that

\[
a = (1 \ 2) \cdots (t \ (1' \ 2') \ (3' \ 4') \ (5' \ 6') \ (7' \ 8') \cdots (n-1 \ n) .
\]

Then by (ii.i) and (ii.ii) \( \{1', 2', 3', 4'\} \) is a \( \langle Q, x_i \rangle \)-orbit and \( \langle Q, x_i \rangle \) is semiregular on \( \Delta = \{1', 2', 3', 4'\} \). Since \( x_i \) normalizes \( \langle Q, x_i \rangle, 2 \leq i \leq k, x_i \) fixes \( \{1', 2', 3', 4'\} \). Hence \( Qx_i \) has an element fixing \( 1' \). We may assume that \( x_i \) fixes \( 1' \). Then \( I(x_i, x_j) \), \( 1 \leq i, j \leq k \), contains \( I(Q) \cup \{1'\} \) of length \( t+1 \). Hence \( x_i^x = x_j \). Thus \( x_i^x = 1 \) and \( x_i = x_jx_i^x \). Furthermore \( I(x_i) \cap \Delta = \{1', 2'\} \). Hence \( I(x_i) \cap \Delta = \{1', 2'\} \), \( i \geq 2 \). This implies that \( x_i \) commutes with \( a \). Thus \( \langle a, x_i, x_j, x_k \rangle \) is elementary abelian and \( x_i = (1') (2') (3' 4') \) on \( \{1', 2', 3', 4'\} \). Furthermore since \( x_i = (1') (2') (3' 4') \) on \( \{1', 2', 3', 4'\} \) pointwise, \( \langle a, x_i, x_j \rangle < Z \langle Q, x_j, x_k \rangle \).

Now we show that \( \langle Q, x_i, x_j, \ldots, x_k \rangle \) is semiregular on \( \Delta = \{1', 2', 3', 4'\} \).
Suppose that \( \langle Q, x_1, x_2 \rangle \) is not semiregular on \( \Delta - \{ 1', 2', 3', 4' \} \). Then there is a \( \langle Q, x_1, x_2 \rangle \)-orbit \( \Delta' \) of length eight. Since \( \langle Q, x_1 \rangle \) and \( \langle Q, x_2 \rangle \) are semiregular on \( \Delta - \{ 1', 2', 3', 4' \} \), there is an element \( u \) in \( Q \) such that \( u x_1, x_2 \) has fixed points in \( \Delta' \). If \( u = 1 \) or \( a \), then \( u x_1, x_2 \in Z(\langle Q, x_1, x_2 \rangle) \). Thus \( u x_1, x_2 \) fixes \( \Delta' \) pointwise and so \( |I(u x_1, x_2)| \geq t+4 \), contrary to the assumption (\*). Thus \( u \neq 1, a \). Since \( 0 < |I(u x_1, x_2) \cap \Delta'| \leq 4 \) and \( u x_1, x_2 \in C(Q) \), \( u x_1, x_2 \) fixes exactly four points of \( \Delta' \). Since \( |\Delta'| = 8 \), there is an element \( u' \) in \( Q \) such that \( u' x_1, x_2 \) fixes exactly four points of \( \Delta' \) which are not fixed by \( u x_1, x_2 \). By the same reason as above \( u' \neq 1, a \). Hence \( u' = u a \). Furthermore this shows that \( \langle Q, x_1, x_2 \rangle \) is semiregular on \( \Delta - (\{ 1', 2', 3', 4' \} \cup \Delta') \). By (2.4) \( N(Q) \) has \( x_2 \). Then \( x_2 \) normalizes \( \langle Q, x_1, x_2 \rangle \) and so fixes \( \Delta' \). Hence by the same argument as above \( u'' x_1, x_3 \), where \( u'' = u \) or \( u a \), fixes the same points of \( \Delta' \) that \( u x_1, x_2 \) fixes. Then \( u x_1, x_2, u'' x_1, x_2 = uu'' x_1, x_2 \) has fixed points in \( \Delta' \). Since \( uu'' = u' \) or \( u' a \) and \( u'' = 1 \) or \( a, uu'' = 1 \) or \( a \). Hence \( uu'' x_1, x_2 \in C(\langle Q, x_1, x_2 \rangle) \) and so \( uu'' x_1, x_2 \) fixes \( \Delta' \) pointwise. Thus \( |I(uu'' x_1, x_2)| \geq t+4 \), contrary to the assumption (\*). Thus \( \langle Q, x_1, x_2 \rangle \) is semiregular on \( \Delta - \{ 1', 2', 3', 4' \} \).

Suppose that \( \langle Q, x_1, x_3, x_2 \rangle \) is not semiregular on \( \Delta - \{ 1', 2', 3', 4' \} \). Then there is an \( \langle Q, x_1, x_3, x_2 \rangle \)-orbit \( \Delta' \) of length sixteen. Since \( \langle Q, x_1, x_2 \rangle \) and \( \langle Q, x_2, x_3 \rangle \) are conjugate to \( \langle Q, x_1, x_2 \rangle \) in \( N(Q) \), \( \langle Q, x_1, x_3 \rangle \) and \( \langle Q, x_2, x_3 \rangle \) are semiregular on \( \Delta - \{ 1', 2', 3', 4' \} \). Hence there is an element \( x' \) in \( Q x_1, x_2, x_3 \) such that \( x' \) has fixed points in \( \Delta' \). Since \( \langle a, x_1, x_2, x_3 \rangle < Z(\langle Q, x_1, x_2, x_3 \rangle), x' \in C(\langle a, x_1, x_2, x_3 \rangle) \). On the other hand \( \langle Q, x_1, x_2 \rangle, \langle Q, x_2, x_3 \rangle \) and \( \langle Q, x_3, x_2 \rangle \) are semiregular on \( \Delta' \). Hence \( \langle a, x_1, x_2, x_3 \rangle \) is semiregular on \( \Delta' \). Since \( x' \) has fixed points in \( \Delta' \) and \( |\langle a, x_1, x_2, x_3 \rangle| = 8 \), \( x' \) fixes at least eight points of \( \Delta' \). Thus \( |I(x')| \geq t - 6 + 8 = t + 2 \), contrary to the assumption (\*). Thus \( \langle Q, x_1, x_2, x_3 \rangle \) is semiregular on \( \Delta - \{ 1', 2', 3', 4' \} \).

Suppose that \( \langle Q, x_1, x_2, x_3, x_4 \rangle \) is not semiregular on \( \Delta - \{ 1', 2', 3', 4' \} \). Then \( \langle Q, x_1, x_2, x_3, x_4 \rangle \) has an orbit \( \Delta' \) of length 21. Since \( \langle Q, x_1, x_2, x_3, x_4 \rangle \), \( \langle Q, x_1, x_2, x_4 \rangle \) and \( \langle Q, x_1, x_3, x_4 \rangle \) are conjugate to \( \langle Q, x_1, x_2, x_3, x_4 \rangle \) in \( N(Q) \), these groups are semiregular on \( \Delta - \{ 1', 2', 3', 4' \} \). Hence there is an element \( x' \) in \( Q x_1, x_2, x_3, x_4 \) such that \( x' \) has fixed points in \( \Delta' \). Since \( \langle Q, x_1, x_2, x_3, x_4 \rangle \subseteq C(Q), x' \subseteq C(Q) \). Furthermore since \( x_1, x_2, x_3, x_4 \in Z(\langle Q, x_1, x_2, x_3, x_4 \rangle), x_1, x_2, x_3, x_4 \) commute with \( x' \). Thus \( x' \subseteq C(\langle Q, x_1, x_2, x_3, x_4 \rangle) \). Since \( \langle Q, x_1, x_2, x_3, x_4 \rangle \) is semiregular on \( \Delta - \{ 1', 2', 3', 4' \} \) and of order \( 2^t \), \( x' \) fixes at least \( 2^t \) points in \( \Delta' \). Then \( |I(x')| \geq t - 2 \cdot 4 + 2^t = t + 8 \), contrary to the assumption (\*). Thus \( \langle Q, x_1, x_2, x_3, x_4 \rangle \) is semiregular on \( \Delta - \{ 1', 2', 3', 4' \} \). Hence by (2.9) \( \langle Q, x_1, x_2, x_3, \ldots, x_6 \rangle \) is semiregular on \( \Delta - \{ 1', 2', 3', 4' \} \).

On the other hand \( \langle a, x_1 \rangle \) normalizes \( G_{\bar{s'}, \bar{r'}, \bar{d}'} \), which is even order. Hence \( a \) and \( x_1 \) commute with an involution \( u \) of \( G_{\bar{s'}, \bar{r'}, \bar{d}'} \). Then \( \langle a, x_1, u \rangle \) normalizes \( G_{\bar{s}, \bar{r}, \bar{d}} \). Hence there is a Sylow 2-subgroup \( Q' \) of \( G_{\bar{s}, \bar{r}, \bar{d}} \) such that \( \langle a, x_1, u \rangle \) normalizes \( Q' \). Since \( Q' \) is conjugate to \( Q \) in \( G_{\bar{s}, \bar{r}, \bar{d}} \) and \( N(Q') = S_2, \langle Q', a, x_1, u \rangle \)
is conjugate to a subgroup of $\langle Q, x_1, x_2, \ldots, x_k \rangle$ in $N(G_{1/2})$. Then $\langle Q', a, x_1, u \rangle$ is semiregular on $\Delta = \{1', 2', 3', 4'\}$ since $I(x_1) \cap \Delta = \{1', 2'\}$ and $I(ax_1) \cap \Delta = \{3', 4'\}$. This is a contradiction since $I(u) \cap \Delta = \{5', 6', 7', 8'\}$. Thus $C(Q)^{\langle Q' \rangle} = A_i$ and so we complete the proof of (2.13).

2.14. Let $y$ be a 2-element of $N(Q)$. If $y^{\langle Q' \rangle}$ is an involution consisting of exactly two 2-cycles, then $|I(y) \cap \Delta| = 2$.

Proof. Suppose by way of contradiction that $y^{\langle Q' \rangle}$ is an involution consisting of exactly two 2-cycles and $|I(y) \cap \Delta| = 2$. Then $|I(y^2)| \geq t + 2$. Hence $y^2 = 1$. We may assume that

$$y = (1 \ 2 \ 3 \ 4 \ 5 \ 6) \cdots (t \ 1' \ 2' \ 3' \ 4') \cdots .$$

Then by (2.12) $C(Q)^{\langle Q' \rangle} = S_t$ or $A_t$. Then since $y^{\langle Q' \rangle}$ is an even permutation, $y^{\langle Q' \rangle} \in C(Q)^{\langle Q' \rangle}$. Thus there is an element $a$ of $Q$ such that $ay \in C(Q)$. Hence $ay$ commutes with $a$ and so $y$ commutes with $a$. On the other hand $y$ commutes with exactly one involution of $Q$, which is a central involution of $Q$. Hence $a \in Z(Q)$ and so $y \in C(Q)$. Thus $Q = 2$ and so $Q = \langle a \rangle$. Since $I(y) \cap \Delta = \{1', 2'\}$ and $|\Delta - \{1', 2'\}| \equiv 2 \pmod{4}$, $|I(ay) \cap \Delta| \equiv 2 \pmod{4}$. Hence $|I(ay) \cap \Delta| = 2$. Thus we may assume that

$$a = (1 \ 2) \cdots (t \ 1' \ 2') \cdots (n - 1 \ n).$$

Then $\langle a, y \rangle$ is semiregular on $\Delta = \{1', 2', 3', 4'\}$. Since $C(a)^{\langle a \rangle} \geq A_t$, there is an element $z$ in $C(Q)$ of the form

$$z = (1 \ 3 \ 2 \ 4 \ 5 \ 6) \cdots (t \cdots .$$

By (2.3) we may assume that $\langle a, y, z \rangle$ is a 2-group. Then $z^2 = y$ or $ay$, and so $I(z^2) \cap \Delta = \{1', 2'\}$ or $\{3', 4'\}$. Thus $z$ consists of 4-cycles on $\Delta = \{1', 2'\}$ or $\Delta = \{3', 4'\}$. Hence $|\Delta| \equiv 2 \pmod{4}$, contrary to (2.7). Thus we complete the proof.

2.15. Let $y$ be a 2-element of $N(Q)$. If $y^{\langle Q' \rangle}$ is an involution consisting of exactly two 2-cycles, then $|I(y) \cap \Delta| = 0$.

Proof. Since $|I(y) \cap I(Q)| = t - 4$, $|I(y) \cap \Delta| = 0$, 2 or 4. By (2.14) $|I(y) \cap \Delta| \geq 2$. Hence suppose by way of contradiction that $|I(y) \cap \Delta| = 4$. By (2.4) $N(Q)$ has the 2-group $\langle Q, y_1, y_2, \ldots, y_k \rangle$. Since $\langle Q, y_1 \rangle$ is conjugate to $\langle Q, y \rangle$, we may assume that $y_1 = y$.

First we show that $y_1$ fixes at least two $Q$-orbits in $\Delta$. Suppose by way of contradiction that $y_1$ fixes exactly one $Q$-orbit $\Delta_i$ in $\Delta$. Then $|I(y_1) \cap \Delta_i| = 4$, so $|Q| = |\Delta_i| \geq 4$.

Since $N(Q)^{\langle Q' \rangle} = S_t$ or $A_t$, first assume that $N(Q)^{\langle Q' \rangle} = S_t$. Then $N(Q)$ has
a 2-element

\[ x = (1\ 2\ (3\ (4)\ \cdots )) \cdots . \]

By (2.3) we may assume that \( \langle Q, y_1, x \rangle \) is a 2-group. Then \( x \) normalizes \( \langle Q, y_1 \rangle \). Hence \( x \) fixes \( \Delta_i \), contrary to (2.13). Thus \( N(Q)^{\langle Q \rangle} = \{e\} \).

Hence \( N(Q)^{\langle Q \rangle} = A_i \). First we show that \( \langle Q, y_1, y_2, \ldots, y_n, y'_1 \rangle \) fixes \( \Delta_i \) and is semiregular on \( \Delta - \Delta_i \). Since \( y_1' \) normalizes \( \langle Q, y_1, y'_1 \rangle \) fixes \( \Delta_i \). Since \( \langle Q, y_1, y'_1 \rangle \) and \( \langle Q, y_1, y'_{i_2} \rangle \) are conjugate to \( \langle Q, y'_1 \rangle \) in \( N(Q) \), \( \langle Q, y_1, y'_1 \rangle \) and \( \langle Q, y_1, y'_{i_2} \rangle \) are semiregular on \( \Delta - \Delta_i \). Thus \( \langle Q, y_1, y'_{i_2} \rangle \) are semiregular on \( \Delta - \Delta_i \).

Since \( (y_1, y_j)^{\langle Q \rangle} = (y_j, y_1)^{\langle Q \rangle} \), \( 1 \leq i, j \leq k \), \( y_1' \) is elementarily abelian. Similarly since \( (y_1, y'_i)^{\langle Q \rangle} = (y_i, y'_1)^{\langle Q \rangle} \) and \( (y_1, y'_i, y'_j)^{\langle Q \rangle} = (y'_i, y_1, y'_j)^{\langle Q \rangle} \), \( 2 \leq i, j \leq k \), \( y_1' \) is elementarily abelian. Since \( y_1' \) fixes exactly one \( Q \)-orbit \( \Delta \), \( (y_1', y_j, y_i, y_i') \) fixes \( \Delta \). Thus \( \Delta \) is the \( \langle Q, y_1, y_2, \ldots, y_k, y'_1 \rangle \)-orbit.

Suppose that \( \langle Q, y_1, y_2, y_3, y'_1 \rangle \) is not semiregular on \( \Delta - \Delta_i \). Then there is an element \( y' \) in \( \langle Q, y_1, y'_1 \rangle \) such that \( y' \) has fixed points in \( \Delta - \{\Delta_i\} \). Then \( y'^{\langle Q \rangle} \) is of order two or four. If \( y'^{\langle Q \rangle} \) is of order two, then \( y'^{\langle Q \rangle} \) consists of two 2-cycles. Thus \( \langle Q, y' \rangle \) is conjugate to \( \langle Q, y'_1 \rangle \) which fixes exactly one \( Q \)-orbit \( \Delta \). This is a contradiction. Thus \( y'^{\langle Q \rangle} \) is of order four and consists of one 4-cycle and one 2-cycle. Then \( y'^2 \) consists of two 2-cycles on \( I(Q) \) and fixes at least two \( Q \)-orbits in \( \Delta \), which is also a contradiction. Thus \( \langle Q, y_1, y_2, y_3, y'_1 \rangle \) is semiregular on \( \Delta - \Delta_i \).

Let \( a \) be an involution of \( Q \) commuting with \( y_1 \) and \( \{i_1, i_2, i_3, i_4\} \) be any
\[ \langle a, y_i \rangle \text{-orbit in } \Delta - \Delta. \] Then \( \langle a, y_i \rangle \) normalizes \( G_{i_1 i_2 i_3 i_4} \), which is of even order. Hence \( a \) and \( y_i \) commute with an involution \( \alpha \) of \( G_{i_1 i_2 i_3 i_4} \). Then the 2-group \( \langle y, \alpha \rangle \) normalizes \( G_{\frac{I}{Q}} \). Hence \( \langle y, \alpha \rangle \) normalizes a Sylow 2-subgroup \( Q' \) of \( G_{\frac{I}{Q}} \). Since \( Q' \) is conjugate to \( \langle q, y_1, y_2, \cdots, y_4 \rangle \) in \( N(G_{\frac{I}{Q}}) \). Hence \( I(y_i) \cap \Delta \) and \( \{ i_1, i_2, i_3, i_4 \} \) are contained in the same \( Q' \)-orbit. Since \( \{ i_1, i_2, i_3, i_4 \} \) is any \( \langle a, y_i \rangle \)-orbit in \( \Delta - \Delta, G_{\frac{I}{Q}} \) is transitive on \( \Delta. \) Hence \( G_{\frac{I}{Q}} \) is transitive or has two orbits \( \{ 5, 6, \cdots, t \} \) and \( \Delta \) on \( \Omega - \{ 1, 2, 3, 4 \} \). If \( G_{\frac{I}{Q}} \) is transitive on \( \Omega - \{ 1, 2, 3, 4 \} \), then \( G \) is 5-fold transitive on \( \Omega \), contrary to (2.11). Hence \( G_{\frac{I}{Q}} \) has two orbits \( \{ 5, 6, \cdots, t \} \) and \( \Delta \). Furthermore since \( G \) is 4-fold transitive, for any four points \( k_1, k_2, k_3, k_4 \) of \( \Omega \) \( G_{\frac{I}{Q}} \) has two orbits \( \Gamma_1 \) and \( \Gamma_2 \), where \( | \Gamma_1 | = \frac{t-4}{4} \) and \( | \Gamma_2 | = | \Delta |. \) By a theorem of W. A. Manning [5] \( | \Gamma_2 | > | \Gamma_1 |. \) Set \( \Gamma(k_1, k_2, k_3, k_4) = \Gamma_1 \cup \{ k_1, k_2, k_3, k_4 \}. \) Since \( I(y_i) \cap \Delta = 4 \) and \( y_i \) commutes with \( a, \) we may assume that

\[
\begin{align*}
 a &= (1)(2) \cdots (t)(1' 2')(3' 4') \cdots , \\
y_i &= (1 2)(3 4)(5 6) \cdots (t)(1') (2')(3')(4') \cdots .
\end{align*}
\]

Let \( i, j \) be any two points of \( I(Q) - \{ 1, 2, 3, 4 \} \). Then \( y_i \in G_{\frac{I}{Q}} i, j \) and \( a \) normalizes \( G_{\frac{I}{Q}} i, j \). Since \( I(1', 2', i, j) - \{ 1', 2', i, j \} \neq \Omega - \Gamma(1', 2', i, j), \) a fixes \( \Gamma(1', 2', i, j). \) Suppose that \( \Gamma(1', 2', i, j) \) contains \( \{ 1, 2 \} \). Then as we have seen above \( \Gamma(1, 2, i, j) \) contains \( \{ 1', 2' \}. \) This is a contradiction since \( \Gamma(1, 2, i, j) = I(Q) \). Similarly \( \Gamma(1', 2', i, j) \) does not contain \( \{ 3, 4 \}. \) On the other hand since \( N(G_{\frac{I}{Q}} i, j) = A_i, \) \( a \) and \( y_i \) are even permutations on \( \Gamma(1', 2', i, j). \) Hence \( \Gamma(1', 2', i, j) \) contains \( \{ 3', 4' \}. \) Hence \( \Gamma(1', 2', 3', 4') \) contains \( \{ i, j \}. \) Since \( i, j \) are any two points of \( I(Q) - \{ 1, 2, 3, 4 \} \), \( \Gamma(1', 2', 3', 4') \) contains \( I(Q) - \{ 1, 2, 3, 4 \}. \) By (2.1) \( | I(Q) | \geq 8. \) Hence \( I(Q) - \{ 1, 2, 3, 4 \} \) contains \( \{ 5, 6, 7, 8 \}, \) which is contained in \( \Gamma(1', 2', 3', 4'). \) Hence \( \Gamma(5, 7, 8, 6) \) contains \( \{ 1', 2', 3', 4' \}. \) This is a contradiction since \( \Gamma(5, 6, 7, 8) = I(Q). \) Thus \( y_i \) fixes at least two \( Q \)-orbits in \( \Delta. \)

Since \( C(Q)^{\frac{I}{Q}} = S_t, A_t \) or \( 1, \) we treat the following two cases separately:

Case 1. \( C(Q)^{\frac{I}{Q}} = S_t, \) or \( A_t. \)

Case 2. \( C(Q)^{\frac{I}{Q}} = 1. \)

Case 1. \( C(Q)^{\frac{I}{Q}} = S_t \) or \( A_t. \) Then we may assume that

\[
\begin{align*}
y_i &= (1 2)(3 4)(5 6) \cdots (t)(1')(2')(3')(4') \cdots , \\
a &= (1)(2) \cdots (t)(1' 2')(3' 4') \cdots (n - 1 n),
\end{align*}
\]

where \( a \) is a central involution of \( Q \) commuting with \( y_i. \)

(i) Assume that \( y_i \in C(Q). \) Since \( C(Q)^{\frac{I}{Q}} \geq A_t, \) there is an element \( b \) in \( Q \) such that \( b y_i \in C(Q). \) Then \( b y_i \) commutes with \( b, \) so \( y_i \) commutes with \( b. \)
Since \( y_1 \in C(Q) \) and \( b \in Z(Q) \). Thus \( Q \) is non-abelian and so \( |Q| > 4 \). Since \( b \) fixes \( \{1', 2', 3', 4'\} \) and commutes with \( a \), \( b \) is an involution or \( b^2 = a \). Furthermore \( Z(Q, y_1) \geq \langle a, by_1 \rangle \). Let \( y' \) be any element of \( Z(Q, y_1) \). Since \( I(y_1) \cap \Delta = \{1', 2', 3', 4'\} \), \( y' \) fixes \( \{1', 2', 3', 4'\} \). Furthermore since \( \langle a, b \rangle \) is regular on \( \{1', 2', 3', 4'\} \), \( y'^{1', 2', 3', 4'} \subseteq \langle a, b \rangle^{1', 2', 3', 4'} \). Hence there is an element \( u \) in \( \langle a, b \rangle \) such that \( uy' \) fixes \( \{1', 2', 3', 4'\} \) pointwise. Thus \( uy' \in \langle y_1 \rangle \) because \( Q, y_1, y = \langle y_1 \rangle \). Hence \( uy' = 1 \) or \( y_1 \). If \( uy' = 1 \), then \( y' \in \langle a, b \rangle \cap Z(Q, y_1) \) since \( y' \in Z(Q, y_1) \) and \( u \in \langle a, b \rangle \). Hence \( y' = a \) or \( 1 \). Next suppose that \( uy' = y_1 \). If \( u = a \) or \( 1 \), then \( y_1 = uy' \in C(Q) \) and \( y' \in C(Q) \). This is a contradiction since \( y_1 \in C(Q) \). Thus \( u = b \) or \( ab \). Hence \( y' = by_1 \) or \( aby_1 \). Thus in either case \( y' \in \langle a, by_1 \rangle \). Hence \( Z(Q, y_1) = \langle a, by_1 \rangle \).

Since \( C(Q, y_1) \geq A_4 \), \( Qy_2 \) has an element which belongs to \( C(Q) \). Hence we may assume that \( y_2 \in C(Q) \). Since \( y_2 \) normalizes \( Q, y_1 \), \( y_2 \) normalizes the center \( \langle a, by_1 \rangle \) of \( Q, y_1 \). Hence \( (by_1)^{y_2} = by_1 \) or \( aby_1 \). First assume that \( (by_1)^{y_2} = by_1 \). Since \( y_2 \) commutes with \( b \), \( y_2 \) commutes with \( y_1 \). Hence \( y_2 \) fixes \( \{1', 2', 3', 4'\} \). Since \( \langle a, by_1, y_2 \rangle \) is an abelian group of order eight and \( \langle a, by_1 \rangle \) is regular on \( \{1', 2', 3', 4'\} \), there is an element \( u \) in \( \langle a, by_1 \rangle \) which fixes \( \{1', 2', 3', 4'\} \) pointwise. Thus \( u \) consists of exactly two 2-cycles on \( I(Q) \) and so \( I(u) \cap \Delta = \{1', 2', 3', 4'\} \) by the assumption (*). On the other hand \( \langle a, by_1, y_2 \rangle \leq C(Q) \). Hence \( u \in C(Q) \). Thus \( |Q| \leq 4 \), which is a contradiction. Next suppose that \( (by_1)^{y_2} = aby_1 \). Then by the same argument as is used for \( y_2 \) we may assume that \( y'_i \in C(Q) \) and \( (by_1)^{y'_i} = aby_1 \). Hence \( (by_1)^{y'_2} = by_1 \). Since \( y_2 y'_i \in C(Q) \), \( y_2 y'_i \) commutes with \( b \). Hence \( y_2 y'_i \) commutes with \( y_1 \). Thus \( y_2 y'_i \) fixes \( \{1', 2', 3', 4'\} \). Thus \( \langle a, by_1, y_2 y'_i \rangle \) is an abelian group fixing \( \{1', 2', 3', 4'\} \). Hence there is an element \( u \) (\( \pm 1 \)) in \( \langle a, by_1, y_2 y'_i \rangle \) which fixes \( \{1', 2', 3', 4'\} \) pointwise. Thus \( u \) consists of exactly two 2-cycles or one 4-cycle and one 2-cycle on \( I(Q) \). Hence \( |I(u) \cap \Delta| \leq 6 \) by the assumption (*). On the other hand \( u \in C(Q) \) and \( |Q| > 4 \). Hence \( |I(u) \cap \Delta| \geq 8 \), which is a contradiction. Thus \( y_1 \in C(Q) \). Hence \( |Q| = 4 \) or 2.

(ii) Assume that \( |Q| = 4 \). Then \( Q \) is elementary abelian or cyclic.

(ii.i) Assume that \( Q \) is elementary abelian. Then we may assume that \( Q = \langle a, b \rangle \) and

\[
\begin{align*}
a &= (1 \ 2 \cdots (l') \ 2' \ 3' \ 4' \ 5' \ 6' \ 7' \ 8') \cdots , \\
b &= (1 \ 2 \cdots (l') \ 2' \ 3' \ 4' \ 5' \ 7' \ 6' \ 8') \cdots .
\end{align*}
\]

As we have proved above, \( y_1 \) fixes at least two \( Q \)-orbits in \( \Delta \). Hence we may assume that

\[
y_1 = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \cdots (l') \ 2' \ 3' \ 4' \ 5' \ 6' \ 7' \ 8') \cdots .
\]

Since \( \langle Q, y_2 \rangle \) and \( \langle Q, y_1 y_2 \rangle \) are conjugate to \( \langle Q, y_1 \rangle \), both groups are elementary abelian. Hence \( \langle Q, y_1, y_2 \rangle \) is elementary abelian. Thus \( y_2 \) fixes \( \{1', 2', 3', 4'\} \) and \( \{5', 6', 7', 8'\} \). Hence \( Qy_2 \) has an element which fixes \( \{1', 2', 3', 4'\} \) point-
wise. We may assume that \( y_2 \) fixes \( \{1', 2', 3', 4'\} \) pointwise. Thus \( I(\gamma_2) = (I(Q) - \{1, 2, 5, 6\}) \cup \{1', 2', 3', 4'\} \) since \( |(I(Q) - \{1, 2, 5, 6\}) \cup \{1', 2', 3', 4'\}| = t \). Furthermore since \( |I(\gamma_1, \gamma_2)| \leq t \), \( \gamma_2 = (5' 7') (6' 8') \) or \( (5' 8') (6' 7') \) on \( \{5', 6', 7', 8'\} \). Since \( \langle Q, \gamma_2 \rangle \) and \( \langle Q, \gamma_1, \gamma_2 \rangle \) are conjugate to \( \langle Q, \gamma_2 \rangle, \langle Q, \gamma_1, \gamma_2 \rangle \) is elementary abelian and by the similar argument as above we may assume that \( \gamma_1' = (1') (2') (3') (4') (5' 7') (6' 8') \) or \( (1') (2') (3') (4') (5' 8') (6' 7') \) on \( \{1', 2', \ldots, 8'\} \). Then in either case the order of \( (y_1, y_2)^t \) is even and \( |I((y_1, y_2)^t)| \geq t + 4 \), contrary to the assumption (*). Thus \( Q \) is not an elementary abelian group.

(ii.ii) Assume that \( Q \) is cyclic. Then we may assume that \( \gamma_2 = \langle \gamma_1 \rangle = \langle \gamma_1 \rangle \). Since we have proved above, \( \gamma_1 \) fixes at least two \( \rho \)-orbits in \( \Delta \). Hence we may assume that \( \gamma_1 = (1 2) (3 4) (5 6) (7) (8) \). Then \( I(\gamma_1) \cap \Sigma = \{5', 6', 7', 8'\} \). Hence \( \langle Q, \gamma_1 \rangle \) is semiregular on \( \{9, 10, \ldots, n\} \). Since \( \gamma_1 \) normalizes \( \langle Q, \gamma_1 \rangle \), \( \gamma_1 \gamma_2 = \gamma_1 \) or \( \gamma_2 \). Suppose that \( \gamma_1 \gamma_2 = \gamma_1 \). Then \( \gamma_2 \) fixes \( \{1', 2', 3', 4'\} \) and \( \{5', 6', 7', 8'\} \). Furthermore since \( \langle Q, \gamma_2 \rangle \) is abelian, \( \langle Q, \gamma_2 \rangle \) has an element \( \gamma_2' = (1 2) (3) (4) (5 6) (7) (8) \). Then \( |I(\gamma_1, \gamma_2)^t| \geq t + 4 \), contrary to the assumption (*). Thus \( \gamma_1 \gamma_2 = \gamma_1 \gamma_2 \). Since \( \gamma_1, \gamma_2 \) is conjugate to \( \gamma_1, \gamma_2 \), \( \langle Q, \gamma_2 \rangle \) has an involution. Hence we may assume that \( \gamma_2 \) is an involution. Furthermore by the same argument as is used for \( \gamma_2 \), \( \gamma_2' = \gamma_1 \gamma_2 \) interchanges \( \{1', 2', 3', 4'\} \) and \( \{5', 6', 7', 8'\} \) as a set. Hence \( \gamma_1 \gamma_2 \gamma_2' = \gamma_1 \gamma_2 \). This means that \( \gamma_1 \gamma_2 \gamma_2' = \gamma_1 \gamma_2 \). Thus \( \gamma_1 \gamma_2 \gamma_2' = \gamma_1 \gamma_2 \). Hence \( |I(\gamma_1, \gamma_2)| \geq t + 4 \), contrary to the assumption (*). Thus \( Q \) is not cyclic. Hence \( |Q| \neq 4 \).

(iii) Assume that \( |Q| = 2 \). Then \( Q = \langle \alpha \rangle \). Since \( C(\alpha)^{\langle \alpha \rangle} = S_t \) or \( A_t \), we treat these cases separately.

(iii.i) Assume that \( C(\alpha)^{\langle \alpha \rangle} = S_t \). Then \( C(\alpha) \) has a 2-element

\[ x_t = (1 2) (3) (4) \ldots (t) \ldots. \]

By (2.3) we may assume that \( \langle a, x_t, y_1, y_2, \ldots, y_6, y_1' \rangle \) is a 2-group. Then \( x_t \)
normalizes \langle a, y \rangle. Hence \( y_i^{x_i} = ay_i \) or \( y_i \).

First suppose that \( y_i^{x_i} = ay_i \). Since \( x_i \in \langle a \rangle \), \( x_i^2 = 1 \) or \( a \). Suppose that \( x_i^2 = 1 \). Then \( \langle a, x_i \rangle \) is an elementary abelian group of order four. On the other hand since \( y_i^{x_i} = ay_i \), \( (x_i y_i)^2 = a \). Thus \( \langle x_i y_i \rangle \) is a cyclic group of order four. This is a contradiction since \( x_i y_i \) is conjugate to \( a, x_i \). Suppose that \( x_i^2 = a \). Then \( \langle x_i \rangle \) is a cyclic group of order four. On the other hand since \( y_i^{x_i} = ay_i \), \( (x_i y_i)^2 = 1 \). Thus \( \langle a, x, y_i \rangle \) is an elementary abelian group of order four. This is a contradiction since \( \langle a, x, y_i \rangle \) is conjugate to \( a, x_i \). Thus \( y_i^{x_i} \neq ay_i \).

Next suppose that \( y_i^{x_i} = y_i \). Then \( \langle a, x, y_i \rangle \) is an abelian group of order eight. By (2.14) |I(ay_i) ∩ Δ| = 0 or 4. Assume that |I(ay_i) ∩ Δ| = 4. Then we may assume that I(ay_i) ∩ Δ = \{5', 6', 7', 8'\} and

\[
y_i = (1\ 2\ 3\ 4\ 5\ 6...t') (1') (2') (3') (4') (5' 6') (7' 8')... .
\]

Then \( \langle a, y \rangle \) is semiregular on \{9', 10', ..., n\}. By (2.13) \( \langle a, x \rangle \) and \( \langle a, x_i, y \rangle \) are semiregular on \ Δ. Hence \( \langle a, x, y \rangle \) is semiregular on \{9', 10', ..., n\}. Since \( \langle a, y \rangle \) and \( \langle a, x_i, y \rangle \) are conjugate to \( \langle a, x_i \rangle \), \( \langle a, y \rangle \) and \( \langle a, y_i \rangle \) are elementary abelian. Hence \( \langle a, x, y \rangle \) is elementary abelian. Furthermore since \( \langle a, y_i \rangle \) is conjugate to \( \langle a, x_i, y \rangle \), \( \langle a, x_i, y_i \rangle \) is also abelian. Hence \( \langle a, x, y, y_i \rangle \) is abelian. Since \( \langle a, y \rangle \) is conjugate to \( \langle a, y_i \rangle \) in C(a), \( |I(y_i) ∩ Δ| = |I(ay_i) ∩ Δ| = 4 \). If \( y_2 \) has fixed points in \{9', 10', ..., n\}, then since \( y_2 \in C(\langle a, x, y_i \rangle) \), \( y_2 \) fixes at least eight points in \{9', 10', ..., n\}, contrary to the assumption (*). Similarly \( ay_i \) has no fixed point in \{9', 10', ..., n\}. Thus \( y_2 \) or \( ay_i \) fixes \( \{1', 2', 3', 4'\} \) pointwise. Hence \( y_2 \) or \( ay_i = (1') (2') (3') (4') (5' 6') (7' 8') \) on \{1', 2', ..., 8'\}. Thus \( |I(y_i, y_2)| \) or \( |I(ay_i, y_2)| \) ≥ t + 4, contrary to the assumption (*).

Hence \( |I(ay_i) ∩ Δ| = 0 \). Then \( \langle a, x, y \rangle \) is semiregular on \( \Delta - \{1', 2', 3', 4'\} \). Since \( \langle a, y \rangle \) and \( \langle a, y_i \rangle \), \( i \neq j \) and \( 1 \leq i, j \leq k \), are conjugate to \( \langle a, y \rangle \), \( \langle a, y_i \rangle \) and \( \langle a, y_i, y \rangle \) are elementary abelian. Hence \( \langle a, y, y_i, y \rangle \) is elementary abelian. Furthermore since \( \langle a, x, y \rangle \), \( 2 \leq i \leq k \), is conjugate to \( \langle a, x_i, y \rangle \), \( \langle a, x, y \rangle \) is abelian. Thus \( \langle a, x_i, y \rangle \), \( 2 \leq i \leq k \), is abelian. Hence \( y_i \) fixes \( \{1', 2', 3', 4'\} \), \( 1 \leq i \leq k \). Since \( \langle a, y \rangle \), \( 2 \leq i \leq k \), is conjugate to \( \langle a, y_i \rangle \), \( y_i \) or \( ay_i \) has fixed points \( \Delta \). Hence we may assume that \( y_i \) has fixed points in \( \Delta \). Since \( y_i \in C(\langle a, x, y_i \rangle) \) and \( \langle a, x, y_i \rangle \) is semiregular on \( \Delta - \{1', 2', 3', 4'\} \), if \( y_i \) has fixed points in \( \Delta - \{1', 2', 3', 4'\} \), then \( y_i \) fixes at least eight points of \( \Delta - \{1', 2', 3', 4'\} \), contrary to the assumption (*). Hence \( y_i \) fixes \( \{1', 2', 3', 4'\} \) pointwise.

Assume that \( \langle a, x, y_1, y_2, ..., y_i \rangle, i \geq 1 \), is semiregular on \( \Delta - \{1', 2', 3', 4'\} \). If \( \langle a, x, y_1, y_2, ..., y_i+1 \rangle \) is not semiregular on \( \Delta - \{1', 2', 3', 4'\} \), then \( \langle a, x, y_1, y_2, ..., y_i+1 \rangle \) has an element \( y' (\neq 1) \) fixing \langle a, x, y_1, y_2, ..., y_i \rangle-orbit of length \( 2i+2 \) pointwise. Then since \( y' \) consists of at most \( i+2 \) 2-cycles on \( I(a) \) and \( i \geq 1 \), \( |I(y')| = t - (2i + 2) > t \), contrary to the assumption (*). Thus \( \langle a, x, y_1, y_2, ..., y_i \rangle \) is semiregular on \( \Delta - \{1', 2', 3', 4'\} \) and this implies by induction that
$\langle a, x, y, z, \ldots, y \rangle$ is semiregular on $\Delta - \{1', 2', 3', 4'\}$.

Furthermore $y'$ fixes $\{1', 2', 3', 4'\}$. Suppose that $\langle a, x, y, z, \ldots, y, y' \rangle$ is not semiregular on $\Delta - \{1', 2', 3', 4'\}$. Then there is an element $y'$ in $\langle a, x, y, z, \ldots, y, y' \rangle$ which has fixed points in $\Delta - \{1', 2', 3', 4'\}$. Then $y'^{t(a)}$ is of order four or two. If $y'^{t(a)}$ is of order four, then $\langle a, y'^{a} \rangle = \langle a, y \rangle$ and $y'^{a}$ has fixed points in $\Delta - \{1', 2', 3', 4'\}$, which is a contradiction. Thus $y'^{t(a)}$ is of order two. Then $y'$ is $(13)(24)$ or $(14)(23)$ on $\{1, 2, 3, 4\}$. Hence $y' \in \langle a, y, x, y, z, \ldots, x, y, x \rangle$ or $\langle a, y, x, y, z, \ldots, x, y, x \rangle$ is semiregular on neither $\{1', 2', 3', 4'\}$ nor $\Delta - \{1', 2', 3', 4'\}$. This is a contradiction since $\langle a, y, x, y, z, \ldots, x, y, x \rangle$ and $\langle a, y, x, y, z, \ldots, x, y, x \rangle$ are conjugate to $\langle a, y, x, y, z, \ldots, x, y, x \rangle$ in $C(a)$ which is semiregular on $\Delta - \{1', 2', 3', 4'\}$. Thus $\langle a, x, y, z, \ldots, y, y \rangle$ is semiregular on $\Delta - \{1', 2', 3', 4'\}$.

On the other hand $\langle a, y \rangle$ normalizes $G_{d', e', r'}$, which is even order. Hence there is an involution $u$ in $G_{d', e', r'}$ commuting with $a$ and $y$. Since $C(a)^{t(a)} = S_4$, $\langle a, y, u \rangle$ is conjugate to a subgroup of $\langle a, x, y, z, \ldots, y, y \rangle$ in $C(a)$. This is a contradiction since for any point of $\{1', 2', 3', 4'\}$ of length eight $\langle a, y, u \rangle$ has an element $(\pm 1)$ fixing this point. Thus $C(a)^{t(a)} = S_4$.

(iii.ii) Assume that $C(a)^{t(a)} = A_4$. Since $\langle a, y, x, y \rangle$, $\langle a, y, x \rangle$, and $\langle a, y, y \rangle$ are conjugate to $\langle a, y_x \rangle$, these groups are elementary abelian. Hence $\langle a, y, x, y \rangle$ is elementary abelian. Since $I(y) \cap \Delta = \{1', 2', 3', 4'\}$, $y_x$ and $y_x$ fix $\{1', 2', 3', 4'\}$. Thus $y_x$ and $y_x$ are $(1')(2')(3')(4')$, $(1')(2')(3')(4')$, $(1')(3')(2')(4')$, $(1')(4')(2')(3')$, $(1')(2')(3')(4')$ or $(1')(2')(3')(4')$ on $\{1', 2', 3', 4'\}$. Furthermore by (2.14) $|I(y)| \cap \Delta | = 0$ or 4.

Assume that $|I(y)| \cap \Delta | = 4$. Then we may assume that

\[
\begin{align*}
a &= (1)(2)\cdots(t)(1')(2')(3')(4')\cdots(n-1)\, n, \\
y &= (1\, 2\, 3\, 4\, 5\, 6)\cdots(t)(1')(2')(3')(4')(5')(6')(7')(8')(9')(11') \\
& \quad (10')(12')(13')(15')(14')(16')\cdots. 
\end{align*}
\]

Suppose that $y_x=(1')(2')(3')(4')$ on $\{1', 2', 3', 4'\}$. The proof in the case $y_x=(1')(2')(3')(4')$ is similar since if $y_x=(1')(2')(3')(4')$ then $ay_x=(1')(2')(3')(4')\cdots$. Since $\langle a, y_x \rangle$ and $\langle a, y, y \rangle$ are conjugate to $\langle a, y_x \rangle$, any element of $\langle a, y, y \rangle - \langle a \rangle$ has four fixed points in $\Delta$. Hence we may assume that

\[
\begin{align*}
y_x &= (1\, 2\, 3\, 4\, 5\, 6\, 7\, 8)\cdots(t)(1')(2')(3')(4')(5')(6')(7')(8')(9')(10')(11')(12')(13')(15')(14')(16')\cdots.
\end{align*}
\]

Thus $\langle a, y_x \rangle$ has two orbits of length two and three orbits of length four in $\Delta$. The remaining $\langle a, y, y \rangle$-orbits are of length eight in $\Delta$. Since $\langle a, y_x \rangle$ is conjugate to $\langle a, y_x \rangle$, $y_x$ has four fixed points in $\Delta$. Since $\langle a, y_x \rangle$ is abelian, $y_x$ fixes $\{1', 2', 3', 4'\}$ or one of the $\langle a, y_x \rangle$-orbits of length four pointwise. Moreover $y_x$ fixes the $\langle a, y, y \rangle$-orbits of length four setwise. Thus $y_x$ fixes $\{1', 2', 3', 4'\}$ pointwise or has no fixed point in $\{1', 2', 3', 4'\}$. First suppose
that \(y_3\) fixes \(\{1', 2', 3', 4'\}\) pointwise. Then \(\langle y_1, y_2, y_3 \rangle\) fixes \(\{1', 2', 3', 4'\}\) pointwise, and \(\{5', 6', 7', 8'\}\) and \(\{9', 10', 11', 12'\}\) are \(\langle y_1, y_2, y_3 \rangle\)-orbits of length four. Hence \(\langle y_1, y_2, y_3 \rangle\) has exactly one element \(y'\) fixing \(\{5', 6', 7', 8'\}\) pointwise. Thus \(|I(y_1y_2y_3)| \geq t+4\), contrary to the assumption (*) .

Similarly \(\langle y_1, y_2, y_3 \rangle\) has exactly one element \((\pm 1)\) fixing \(\{9', 10', 11', 12'\}\) pointwise, which is also \(y_1y_2y_3\) .

Thus \(\langle y_1, y_2, y_3 \rangle\) has no orbit of length two in \(\Delta\) .

On the other hand \(C(a)\) has a 2-element
\[y' = (1) (2) (3) (4) (5 7) (6 8) (9) (10) \cdots (t) \cdots .\]

By (2.3) we may assume that \(\langle a, y_1, y_2y_3, y' \rangle\) is a 2-group .

Since \(\langle a, y_1, y' \rangle\) is conjugate to \(\langle a, y_1, y_2y_3 \rangle\) in \(C(a)\), \(\langle a, y_1, y' \rangle\) has no orbit of length two in \(\Delta\) .

Hence \(y_1 = (1' 2')(3' 4') \cdots \) since if \(y_1 = (1' 2')(3' 4') \cdots \) then \(ay_1 = (1')(2')(3')(4') \cdots \) .

Next suppose that \(y_1 = (1' 3')(2' 4') \cdots \) or \((1' 4')(2' 3') \cdots \) .

Since \(\langle a, y_1, y_3 \rangle\) is conjugate to \(\langle a, y_1, y_2y_3 \rangle\), \(\langle a, y_1, y_3 \rangle\) has exactly two orbits of length two in \(\Delta\) .

Hence \(y_1\) fixes \(\{5', 6'\}\) and \(\{7', 8'\}\) .

Then \(\langle a, y_1, y_2y_3 \rangle\) has no orbit of length two in \(\Delta\) .

By (2.3) we may assume that \(\langle a, y_1, y_2y_3, y' \rangle\) is a 2-group .

Since \(\langle a, y_1, y' \rangle\) is conjugate to \(\langle a, y_1, y_2y_3 \rangle\) in \(C(a)\), \(\langle a, y_1, y' \rangle\) has exactly one element \(y''(\pm 1)\) fixing four \(\langle a, y_1 \rangle\)-orbits of length two pointwise.
Then
\[ y'' = (12) (34) (57) (68) (9) (10) \cdots (1') (2') (3',4') \cdots . \]
Thus \( |I(y, y_2 y_3)^t| \geq t + 4 \), contrary to the assumption (*). Hence \( y_2 = (1')(2')(3',4') \cdots \) and so \( y_2 = (1'2')(3')(4') \cdots \).

Suppose that \( y_2 = (1'3')(2'4') \) on \{1', 2', 3', 4'\}. The proof in the case \( y_2 = (1'4')(2'3') \) on \{1', 2', 3', 4'\} is similar since if \( y_2 = (1'4')(2'3') \) then \( ay_2 = (1'3')(2'4') \cdots \). Since \( I(ay_2) \cap \Delta = \{5', 6', 7', 8'\} \), if \( y_2 \) or \( y_3 \) has fixed points in \{5', 6', 7', 8'\}, then by the same argument as above we have a contradiction. Hence we may assume that
\[ y_2 = (12) (34) (56) (78) (9) (10) \cdots (1'3')(2'4')(5'7')(6'8') \cdots . \]

Similarly \( y_3 \) or \( ay_3 \) is \( (1'3')(2'4') \) on \{1', 2', 3', 4'\}. Hence we may assume that \( y_3 = (1'3')(2'4') \) on \{1', 2', 3', 4'\}. Furthermore \( y_3 \) is \( (5'7')(6'8') \) or \( (5'8')(6'7') \) on \{5', 6', 7', 8'\}. Since \( |I(ay_3)| \leq t \),
\[ y_3 = (12) (34) (56) (78) (9) (10) \cdots (1'3')(2'4')(5'8')(6'7') \cdots , \]
and so
\[ y_1 y_2 y_3 = (12) (34) (56) (78) (9) (10) \cdots (1'3')(2'4')(5'8')(6'7') \cdots . \]

Hence by the same argument as in the case \( y_2 = (1')(2')(3',4') \), we have a contradiction. Thus \( y_2 = (1'3')(2'4') \cdots \) and so \( y_2 = (1'4')(2'3') \cdots \). Hence \( |I(ay_3) \cap \Delta| = 4 \).

Thus \( |I(ay_1) \cap \Delta| = 0 \). Then we may assume that
\[ y_1 = (12) (34) (56) (78) (9) (10) \cdots (1'3')(2'4')(5'7')(6'8') \cdots . \]
Since \( \langle a, y_2 \rangle \) is conjugate to \( \langle a, y_1 \rangle \) in \( C(a) \), either \( y_2 \) or \( ay_2 \) has four fixed points in \( \Delta \). Hence we may assume that \( y_2 \) has four fixed points in \( \Delta \). Then \( y_2 \) fixes \{1', 2', 3', 4'\} or one of the \( \langle a, y_2 \rangle \)-orbits of length four pointwise.

First suppose that \( y_2 \) fixes \{1', 2', 3', 4'\} pointwise. Since \( \langle a, y_2 \rangle \) and \( \langle a, y_1 y_2 \rangle \) are conjugate to \( \langle a, y_1 \rangle \) in \( C(a) \), \( \langle a, y_2 \rangle \) and \( \langle a, y_1 y_2 \rangle \) are semiregular on \( \Delta \) - \{1', 2', 3', 4'\}. Hence \( \langle a, y_1 \rangle \) is semiregular on \( \Delta \) - \{1', 2', 3', 4'\}. Since \( \langle a, y_1 \rangle \) and \( \langle a, y_1 y_2 \rangle \), \( i \neq i \) and \( 1 \leq i, j \leq k \), are conjugate to \( \langle a, y_1 \rangle \), \( \langle a, y_i \rangle \) and \( \langle a, y_1 y_2 \rangle \) are elementary abelian. Hence \( \langle a, y_1, y_2, \ldots, y_k \rangle \) is elementary abelian. Moreover \( y_3 \) or \( ay_3 \), \( 3 \leq i \leq k \), has four fixed points in \( \Delta \). Hence we may assume that \( y_3 \) has fixed points in \( \Delta \). Since \( y_1 \in C(\langle a, y_1, y_2 \rangle) \) and \( \langle a, y_1, y_2 \rangle \) is of order eight and semiregular on \( \Delta \) - \{1', 2', 3', 4'\}, \( y_1 \) fixes \{1', 2', 3', 4'\} pointwise.

Now we show that \( \langle a, y_1, y_2, \ldots, y_k \rangle \) is semiregular on \( \Delta \) - \{1', 2', 3', 4'\}. Suppose that \( \langle a, y_1, y_2, y_3 \rangle \) is not semiregular on \( \Delta \) - \{1', 2', 3', 4'\}. Then there is exactly one element \( y' \) \( (= 1) \) in \( \langle a, y_1, y_2, y_3 \rangle \) fixing a \( \langle a, y_1, y_2 \rangle \)-orbit \( \Delta' \) in \( \Delta \) - \{1', 2', 3', 4'\} pointwise. Since \( |\Delta'| = 8, |I(y') \cap I(a)| \leq t - 8 \). Hence
y' = y_1 y_2 y_3 or a y_1 y_2 y_3. If y' = y_1 y_2 y_3, then \( I(y') \) contains \( (I(a) - \{1, 2, \ldots, 8\}) \cup \{1', 2', 3', 4'\} \cup \Delta' \) of length \( t+4 \), contrary to the assumption (\ast). Thus y' = a y_1 y_2 y_3 and \( I(y') = (I(a) - \{1, 2, \ldots, 8\}) \cup \Delta' \) since \( |(I(a) - \{1, 2, \ldots, 8\}) \cup \Delta'| = t \). Furthermore this shows that \( \langle a, y_1, y_2, y_3 \rangle \) is semiregular on \( \Delta - \{1', 2', 3', 4'\} \cup \Delta' \). Hence \( \langle a, y_1, y_2, y_3 \rangle \) has two orbits \( \{1', 2'\} \) and \( \{3', 4'\} \) of length two and two orbits of length four whose union is \( \Delta' \) in \( \Delta \), and the remaining orbits in \( \Delta \) are of length eight. On the other hand \( C(a) \) has a 2-element

\[
y'' = (1) (2) (3) (4) (5 7) (6 8) (9) (10) \ldots (t) \ldots .
\]

By (2.3) we may assume that \( \langle a, y_1, y_2, y_3, y'' \rangle \) is a 2-group. Then \( y'' \) normalizes \( \langle a, y_1, y_2, y_3 \rangle \) and so \( y'' \) fixes \( \{1', 2', 3', 4'\} \) and \( \Delta' \). Since \( \langle a, y_1, y'' \rangle \) is conjugate to \( \langle a, y_1, y_2, y_3 \rangle \) in \( C(a) \), \( \langle a, y_1, y'' \rangle \) is elementary abelian and has two orbits \( \{1', 2'\} \) and \( \{3', 4'\} \) of length two and two orbits of length four in \( \Delta \). Hence we may assume that \( y'' \) fixes \( \{1', 2', 3', 4'\} \) pointwise and \( a y_1 y'' \) has eight fixed points in \( \Delta - \{1', 2', 3', 4'\} \). Furthermore since \( y'' \) fixes \( \Delta' \), \( a y_1 y'' \) fixes \( \Delta' \) pointwise or \( \langle a, y_1, y'' \rangle \) is regular on \( \Delta' \). If \( a y_1 y'' \) fixes \( \Delta' \) pointwise, then \( I(ay_1 y_2 y_3, ay_1 y'') = I(y_2 y_3 y'') \) contains \( (I(a) - \{5, 6, 7, 8\}) \cup \{1', 2', 3', 4'\} \cup \Delta' \) of length \( t+8 \), contrary to the assumption (\ast). Thus \( \langle a, y_1, y'' \rangle \) is regular on \( \Delta' \). On the other hand \( \langle a, y_1, y_3 \rangle \) is elementary abelian and regular on \( \Delta' \). Hence \( \langle a, y_1, y_3 \rangle \) has an element \( w \) such that \( w^a = y '' \). Thus \( \langle a, y_1, y_3, y'' \rangle \rangle \) and \( I(\langle a, y_1, y_3, y'' \rangle) \rangle \) contains \( \Delta' \) of length eight. Hence \( |I(\langle a, y_1, y_3, y'' \rangle) \rangle \leq t - 8 \). This is a contradiction since any element of \( \langle a, y_1, y_3, y'' \rangle \rangle \) fixes at least \( t - 6 \) points of \( I(a) \). Thus \( \langle a, y_1, y_2, y_3 \rangle \rangle \) is semiregular on \( \Delta - \{1', 2', 3', 4'\} \). Hence by (2.6) \( \langle a, y_1, y_2, y_3, \ldots, y_h \rangle \rangle \) is semiregular on \( \Delta - \{1', 2', 3', 4'\} \).

Since \( y_1 \) normalizes \( \langle a, y_1, y_2, \ldots, y_h, y_1' \rangle \rangle \), \( y_1' \) fixes \( \{1', 2', 3', 4'\} \). Suppose that \( \langle a, y_1, y_2, \ldots, y_h, y_1' \rangle \rangle \) is not semiregular on \( \Delta - \{1', 2', 3', 4'\} \). Then there is an element \( y' \) in \( \langle a, y_1, y_2, \ldots, y_h, y_1' \rangle \rangle \) which has fixed points in \( \Delta - \{1', 2', 3', 4'\} \). Then \( y''(a) \) is of order four or two. If \( y''(a) \rangle \) is of order four, then \( \langle a, y'' \rangle = \langle a, y_1 \rangle \rangle \) and \( y'' \) has fixed points in \( \Delta - \{1', 2', 3', 4'\} \), which is a contradiction. Hence \( y''(a) \rangle \) is of order two. Thus \( y' \) is (13) (24) or (14) (23) on \( \{1, 2, 3, 4\} \). Hence \( y' \in \langle a, y_1', y_2 y_3, y_2 y_4, \ldots, y_2 y_k \rangle \rangle \) or \( \langle a, y_1, y_1', y_3 y_5, y_2 y_4, \ldots, y_2 y_k \rangle \rangle \). Thus \( \langle a, y_1', y_2 y_3, y_2 y_4, \ldots, y_2 y_k \rangle \rangle \) or \( \langle a, y_1, y_1', y_3 y_5, y_2 y_4, \ldots, y_2 y_k \rangle \rangle \) is semiregular on neither the orbit \( \{1', 2', 3', 4'\} \rangle \) of length four nor \( \Delta - \{1', 2', 3', 4'\} \). This is a contradiction since these groups are conjugate to \( \langle a, y_1, y_3 y_5, y_2 y_4, \ldots, y_2 y_k \rangle \rangle \) in \( C(a) \rangle \) which is semiregular on \( \Delta - \{1', 2', 3', 4'\} \). Thus \( \langle a, y_1, y_2, y_3, \ldots, y_h, y_1' \rangle \rangle \) is semiregular on \( \Delta - \{1', 2', 3', 4'\} \).

On the other hand \( \langle a, y_1 \rangle \rangle \) normalizes \( G_{\gamma', \gamma''} \), which is of even order. Hence there is an involution \( u \) in \( G_{\gamma', \gamma''} \) commuting with \( a \) and \( y_1 \). Then \( \langle a, y_1, u \rangle \rangle \) is conjugate to a subgroup of \( \langle a, y_1, y_2, \ldots, y_h, y_1' \rangle \rangle \) in \( C(a) \rangle \). This is a contradiction since for any point of \( \{1', 2', \ldots, 8'\} \rangle \) of length eight \( \langle a, y_1, u \rangle \rangle \) has an element \( (\neq 1) \) fixing this point. Thus \( y_2 \neq (1) (2') (3') (4') \ldots \).
Next suppose that $y_{z}$ fixes a $\langle a, y_{z}\rangle$-orbit of length four pointwise. Then we may assume that $y_{z}$ fixes $\{5', 6', 7', 8'\}$ pointwise and

$$y_{z} = (12) (3) (4) (5 6) (7) (8) \cdots (t) (1') (2') (3') (5') (6') (7') (8') \cdots .$$

Since $\langle a, y_{z}, y_{z} \rangle$ is conjugate to $\langle a, y_{z}, y_{z} \rangle$, $y_{z}$ or $ay_{z}$ is $(1' 3') (2' 4')$ on $\{1', 2', 3', 4'\}$. Hence we may assume that $y_{z} = (1' 3') (2' 4') \cdots$. Since $\langle a, y_{z}, y_{z} \rangle$ is conjugate to $\langle a, y_{z}, y_{z} \rangle$, $y_{z}$ is $(5' 7') (6' 8')$ or $(5' 8') (6' 7')$ on $\{5', 6', 7', 8'\}$. On the other hand $C(a)$ has a 2-element

$$y_{z}' = (1) (2) (3) (4) (5 7) (6 8) (9) (10) \cdots (t) \cdots .$$

By (2.3) we may assume that $\langle a, y_{1}', y_{2}, y_{z}, y_{z}' \rangle$ is a 2-group. Since $\langle a, y_{1}' \rangle$ and $\langle a, y_{z}' \rangle$ are conjugate to $\langle a, y_{1}' \rangle$, $\langle a, y_{z}' \rangle$ and $\langle a, y_{z}' \rangle$ are elementary abelian. Since $\langle a, y_{1}' \rangle$ and $\langle a, y_{z}' \rangle$ are conjugate to $\langle a, y_{1}', y_{2}, y_{z} \rangle$ and $I(y_{1}) \cap \Delta = I(y_{2}, y_{z}) \cap \Delta = \{1', 2', 3', 4'\}$, $y_{1}'$ or $ay_{1}'$, $i = 1, 2$, fixes $\{1', 2', 3', 4'\}$ pointwise. Hence we may assume that $y_{1}'$ and $y_{z}'$ fix $\{1', 2', 3', 4'\}$ pointwise. Thus $y_{z}', y_{z}'$ and $y_{z}'$ fix $\{1', 2', 3', 4'\}$ pointwise. Hence $\langle a, y_{1}', y_{2}, y_{z}, y_{z}' \rangle$ is elementary abelian.

If $y_{1}'$ or $y_{z}'$ fixes $\{5', 6', 7', 8'\}$, then $(y_{z}y_{1}')^2$ or $(y_{z}y_{z})^2$ is of order two and fixes $(I(a) \cup \{1', 2', 3', 4'\}) \cap \{1', 2', \cdots , 8'\}$ of length $t + 4$ pointwise, contrary to the assumption $(*)$. Thus $\{5', 6', 7', 8'\} = \{5', 6', 7', 8'\}$, $i = 1, 2$.

Since $y_{z} = (5' 7') (6' 8') \cdots$ or $(5' 8') (6' 7') \cdots$, first suppose that $y_{z} = (5' 7') (6' 8') \cdots$. Then $I(y_{z}, y_{z}) \cap \Delta = \{1', 2', \cdots , 8'\}$. Since $I(y_{1}') \cap \Delta = \{1', 2', 3', 4'\}$ and $y_{1}'$ commutes with $y_{z}, y_{z}$, $y_{1}'$ fixes $\{5', 6', 7', 8'\}$, which is a contradiction. Next suppose that $y_{z} = (5' 8') (6' 7') \cdots$. Since $\{5', 6', 7', 8'\} = \{5', 6', 7', 8'\}$, we may assume that $\{5', 6', 7', 8'\} = \{9', 10', 11', 12'\}$, where $\{9', 10', 11', 12\}$ is a $\langle a, y_{1}' \rangle$-orbit. Since $ay_{z}, y_{z}$, fixes $\{5', 6', 7', 8'\}$ pointwise and commutes with $y_{1}'$, $ay_{z}, y_{z}$ fixes $\{9', 10', 11', 12\}$ pointwise. Then $I(ay_{z}, y_{z}) \cap \Delta = \{5', 6', \cdots , 12\}$ since $|I(ay_{z}, y_{z})| \leq t$. Furthermore $y_{1}'$ commutes with $ay_{z}, y_{z}$. Hence $\{5', 6', 7', 8'\} = \{9', 10', 11', 12\}$. Thus $\{5', 6', \cdots , 12\}$ is a $\langle y_{1}', y_{2}, y_{z}, y_{z}', y_{z}' \rangle$-orbit of length eight. Since the order of $\langle y_{1}', y_{2}, y_{z}, y_{z}, y_{z}' \rangle$ is sixteen, there is an element $y' (\neq 1)$ in $\langle y_{1}', y_{2}, y_{z}, y_{z}, y_{z}' \rangle$ fixing $\{5', 6', \cdots , 12\}$ pointwise. Moreover since $I(y_{1}', y_{2}, y_{z}, y_{z}, y_{z}') \supseteq \{1', 2', 3', 4'\}$, $I(y_{1}') \supseteq \{1', 2', 3', 4'\}$ and so $|I(y_{1}') \cap \Delta | \geq 12$. This contradicts the assumption $(*)$ since $y_{1}'$ is an involution consisting of at most four 2-cycles. Thus $C(Q)_{I(y_{1})} \supseteq A_{1}$.

Case 2. $C(Q)_{I(y_{1})} = 1.$

(i) Since $|I(y_{1}) \cap \Delta | = 4$, $I(y_{1}) \cap \Delta$ is contained in one or two $Q$-orbits in $\Delta$. If $I(y_{1}) \cap \Delta$ is contained in two $Q$-orbits, then $y_{1}$ fixes exactly two points of a $Q$-orbit. Then by (2.12) $C(Q)_{I(y_{1})} \supseteq A_{1}$, which is a contradiction. Thus $I(y_{1}) \cap \Delta$ is contained in one $Q$-orbit.

1) The proof in this case is due to the suggestion of Dr. E. Bannai. The proof was first more complicated.
(ii) Let $\Phi(Q)$ be the Frattini subgroup of $Q$. Then since $y_i$ is an automorphism of $Q$ and $\Phi(Q)$ by conjugation, $y_i$ induces an automorphism of $Q/\Phi(Q)$, which we denote by $y_i^*$. For an element $a$ of $Q$, $a^{-1}a^y$ is in $\Phi(Q)$ if and only if the image in $Q/\Phi(Q)$ of $a$ is in $C_{Q/\Phi(Q)}(y_i^*)$. Hence the number of elements $a$ in $Q$ such that $a^{-1}a^y$ is in $\Phi(Q)$ is $|C_{Q/\Phi(Q)}(y_i^*)| \cdot |\Phi(Q)|$. On the other hand, for elements $a$ and $b$ of $Q$, $ab^{-1}$ is in $C_Q(y_i^*)$ if and only if the image in $Q/\Phi(Q)$ of $a$ is in $C_{Q/\Phi(Q)}(y^*)$. Hence the number of elements $a$ in $Q$ such that $a^{-1}a^y$ is in $\Phi(Q)$ is at most $|C_{Q/\Phi(Q)}(y^*)| \cdot |\Phi(Q)|$. Thus $4 \cdot |\Phi(Q)| \geq |C_{Q/\Phi(Q)}(y_i^*)| \cdot |\Phi(Q)|$ and so $4 \geq |C_{Q/\Phi(Q)}(y_i^*)|$. Since $Q/\Phi(Q)$ is elementary abelian, $|Q/\Phi(Q)| \leq (2^2)^2 = 2^4$ by Lemma of [6]. Thus the automorphism group of $Q/\Phi(Q)$ is contained in $GL(4, 2)$.

Furthermore if an element of odd order in $Q(Q)$ acts trivially on $Q/\Phi(Q)$ by conjugation, then this element belongs to $C(Q)([1], \text{Theorem 5.1.4})$.

Since $C(Q)$ is a Sylow 2-subgroup of $H$, $|H| = 6$, which is a contradiction by [12]. Thus $x$ is an even permutation. Hence $x^*$ is an odd permutation. On the other hand since $x$ has no fixed point in $\Delta$ and $x^2 \in Q$, every cycle of $x$ in $\Delta$ has the same length and $x$ consists of 2-cycles. Thus $x$ consists of cycles of length $|Q|/2$ in $\Delta$ since $x^2$ is an odd permutation. Then $|x| = 2|Q|$. Hence $|x| = |Q|$. Since $x^2 \in Q$, $Q = \langle x^2 \rangle$. Hence the automorphism group of $Q$ is a 2-group. This is a contradiction since $N(Q)^{Q} = S_3$ and $N(Q)^{Q}$ is involved in the automorphism group of $Q$. Thus $N(Q)^{Q} + S_3$.

(iii) Suppose that $N(Q)^{Q} = S_3$. Let $H$ be the normal subgroup of $G$ consisting of all even permutations of $G$. Then for any point $i$ of $\Omega$, $H_i$ is normal in $G_i$. Since $G_i$ is 3-fold transitive on $\Omega - \{i\}$ and $|\Omega - \{i\}|$ is odd, $H_i$ is 3-fold transitive on $\Omega - \{i\}$ by a theorem of Wagner [15]. Hence $H$ is 4-fold transitive on $\Omega$. Let $x$ be a 2-element of $N_G(Q)$ such that

$$x = (1) (2) (3) (4) (56)\cdots.$$  

Then $x$ has no fixed point in $\Delta$ by (2.13). Hence the number of $Q$-orbits in $\Delta$ is even and so $Q \leq H$. If $x$ is an odd permutation, then $x \in N_H(Q)$. Hence $Q$ is a Sylow 2-subgroup of $H_{1234}$ and $|I(Q)| = 6$, which is a contradiction by [12]. Thus $x$ is an even permutation. Hence $x^2$ is an odd permutation. On the other hand since $x$ has no fixed point in $\Delta$ and $x^2 \in Q$, every cycle of $x$ in $\Delta$ has the same length and $x$ consists of 2-cycles. Thus $x$ consists of cycles of length $|Q|/2$ in $\Delta$ since $x^2$ is an odd permutation. Then $|x| = 2|Q|$. Hence $|x| = |Q|$. Since $x^2 \in Q$, $Q = \langle x^2 \rangle$. Hence the automorphism group of $Q$ is a 2-group. This is a contradiction since $N(Q)^{Q} = S_3$ and $N(Q)^{Q}$ is involved in the automorphism group of $Q$. Thus $N(Q)^{Q} + S_3$.

(v) Suppose that $N(Q)^{Q} = A_8$.

(v. i) $y_i^{(Q)}$ is an involution consisting of exactly two 2-cycles. Hence by (2.8) $y_i$ fixes at most four $Q$-orbits in $\Delta$. Furthermore we have proved that $y_i$ fixes at least two $Q$-orbits in $\Delta$. Thus $y_i$ fixes two, three or four $Q$-orbits in $\Delta$.

(v. ii) Suppose that $y_i$ fixes exactly four $Q$-orbits in $\Delta$. Then by (2.8) every element of $Qy_i$ is an involution. Since $\langle Q, y_2 \rangle$ and $\langle Q, y_1y_2 \rangle$ are conjugate to $\langle Q, y_i \rangle$, every element of $Qy_2$ and $Qy_1y_2$ is an involution. In particular $y_1y_2$ and $y_2$ are involutions. Hence $y_1$ and $y_2$ commute. Let $u$ be any element of $Q$. Then $uy_1$ and $uy_1y_2$ are also involutions. Hence $y_i$ commutes with $uy_1$ and
so commutes with $u$. Thus $y_i \in C(Q)$, which is a contradiction since $C(Q)^{y_i}=1$.

(v. iii) Suppose that $y_i$ fixes exactly three $Q$-orbits in $\Delta$. Then by (2.8) there are at least $\frac{3}{4} |Q|$ involutions in $Qy_i$. Since $y_2$ normalizes $\langle Q, y_i \rangle$, $y_2$ fixes at least one $\langle Q, y_i \rangle$-orbit of length $|Q|$. Then for a point $i$ of the $\langle Q, y_i \rangle$-orbit of length $|Q|$, $Qy_i$ and $Qy_2$ have elements fixing $i$. Hence we may assume that $y_1$ and $y_2$ fix $i$. Then $y_1^2=y_2^2=1$ and $y_1y_2=y_2y_1$. Let $T$ be a set of elements $u$ in $Q$ such that both $uy_1$ and $uy_2y_1$ are involutions. Since $\langle Q, y_1y_2 \rangle$ is conjugate to $\langle Q, y_i \rangle$, there are at least $\frac{3}{4} |Q|$ involutions in $Qy_1y_2$. Hence $|T| \geq \frac{1}{2} |Q|$. Since $y_2$ is an involution, $y_2$ commutes with $uy_1$, where $u \in T$. Furthermore $y_2$ commutes with $y_i$. Hence $y_2$ commutes with exactly four elements of $Q$. Thus $|T| \leq 4$. Hence $4 \geq |T| \geq \frac{1}{2} |Q|$ and so $8 \geq |Q|$. Then the automorphism group of $Q$ is a 2-group, $S_3$, $S_4$, or $SL(3,2)$ (see [3]). Since $N(Q)^{y_1}=A_8$ and $N(Q)^{y_2}$ is involved in the automorphism group of $Q$, we have a contradiction.

(v. iv) Thus $y_i$ fixes exactly two $Q$-orbits in $\Delta$. Then any 2-element of $N(Q)$ which is an involution consisting of exactly two 2-cycles on $I(Q)$ fixes two $Q$-orbits in $\Delta$. Set $\Delta=\{\Delta_1, \Delta_2, \ldots, \Delta_r\}$, where $\Delta=\Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_r$ and $\Delta_i, 1 \leq i \leq r$, is a $Q$-orbit. Then we may assume that

$$y_i = (\Delta_1) (\Delta_2) (\Delta_3 \Delta_4) (\Delta_5 \Delta_6) \cdots$$

and $y_i$ fixes four points $1'$, $2'$, $3'$, $4'$ of $\Delta_i$.

(v. v) Since $y_2$ normalizes $\langle Q, y_i \rangle$, $y_2$ fixes $\{\Delta_1, \Delta_2\}$, Assume that $y_2 = (\Delta_1 \Delta_2) \cdots$. Since $\langle Q, y_i \rangle$ and $\langle Q, y_2y_i \rangle$ are conjugate to $\langle Q, y_i \rangle$, $y_2$ and $y_2y_i$ fix exactly two $Q$-orbits in $\Delta$. Since $y_i = (\Delta_1) (\Delta_2) (\Delta_3 \Delta_4) (\Delta_5 \Delta_6) \cdots$ and $y_2$ commutes with $y_i$, we may assume that

$$y_2 = (\Delta_1 \Delta_2) (\Delta_3) (\Delta_4 \Delta_5) \cdots.$$ 

Then $\langle y_i, y_2 \rangle$ is semiregular on $\{\Delta_1, \Delta_4, \cdots\}$. Since $\langle y_i, y_2, y_3 \rangle$ is elementary abelian, $y_3$ fixes $\{\Delta_1, \Delta_3\}$, $\{\Delta_2, \Delta_5\}$ and $\{\Delta_6, \Delta_5\}$. Furthermore since $\langle Q, y_2y_3 \rangle$ and $\langle Q, y_2y_3 \rangle$ are conjugate to $\langle Q, y_i \rangle$, $y_iy_2y_3$ and $y_2y_3$ fix exactly two $Q$-orbits in $\Delta$. Hence

$$y_3 = (\Delta_1 \Delta_2) (\Delta_3 \Delta_4) (\Delta_5) \cdots.$$ 

Since $y_3$ fixes $\Delta_i$, there is an element in $Qy_3$ fixing $1'$ of $\Delta_i$. Hence we may assume that $y_3$ fixes $1'$. Then $I((y_3y_2)^t)$ and $I((y_2y_3)^t \cdot y_2y_3)$ contains $I(Q) \cup \{1'\}$ of length $t+1$. Hence by the assumption ($*$) $(y_2y_3)^t=1$ and $y_1y_2y_3 = y_2y_3y_1$. Let $T$ be a set of elements $u$ of $Q$ such that both $y_1y_2$ and $y_1y_2y_3$ are involutions. Since $y_1y_2$ fixes $\Delta_i$ and $\Delta_j$, by (2.8) there are at least $\frac{|Q|}{2}$ involu-
tions in $y_3y_4Q$ having fixed points in $\Delta$. Furthermore since $y_3, y_4, y_5$ fixes $\{\Delta_1, \Delta_2, \ldots, \Delta_8\}$ pointwise and $y_1y_2y_3$ consists of four 2-cycles on $I(Q)$, by (2.8) at least $\frac{3}{4}|Q|$ involutions of $y_1y_2y_3Q$ have fixed points in $\Delta$. Hence $|T| \geq \frac{1}{4}|Q|$. Since for any element $u$ of $T$ $y_1y_2u$ and $y_1y_2y_3u$ are involutions, $y_i$ commutes with $y_2y_3u$. Furthermore $y_i$ commutes with $y_2y_3$. Hence $y_i$ commutes with $u$. Since $|I(y_i) \cap \Delta|=4$, $y_i$ commutes with exactly four elements of $Q$. Hence $|T| \leq 4$. Thus $\frac{1}{4}|Q| \leq 4$ and so $|Q| \leq 16$. Since $C(Q)^{\langle Q \rangle} = 1$, $N(Q)^{\langle Q \rangle} = A$, $Q$ is involutive in the automorphism group of $Q$. Hence $Q$ is an elementary abelian group of order sixteen (see [3]). As we have seen above, at least $\frac{3}{4}|Q|$ elements of $y_1y_2y_3Q$ are involutions. Then since $y_1y_2y_3$ is an involution and $Q$ is elementary abelian, $y_1y_2y_3$ commutes with at least $\frac{3}{4}|Q|$ elements of $Q$. Hence $y_1y_2y_3$ centralizes $Q$. This is a contradiction since $C(Q)^{\langle Q \rangle} = 1$. Thus we may assume that $y_i = (\Delta_1, \Delta_2, \Delta_3, \Delta_4)$. Similarly $y_3$ fixes $\{\Delta_1, \Delta_2\}$ pointwise.

Suppose that $\langle y_1, y_2, y_3 \rangle$ is not semiregular on $\Delta - \{\Delta_1, \Delta_2\}$. Then we may assume that $y_3$ fixes $\{\Delta_3, \Delta_4, \Delta_5\}$. Then $y_iy_jy_3$ fixes $\{\Delta_3, \Delta_4, \ldots, \Delta_8\}$ pointwise. Hence by the same argument as above we have a contradiction. Thus $\langle y_1, y_2, y_3 \rangle$ is semiregular on $\Delta - \{\Delta_1, \Delta_2\}$.

Since $\langle Q, y_1 \rangle$ is conjugate to $\langle Q, y_i \rangle$, $y_i$ fixes exactly two $Q$-orbits in $\Delta$. Since $\langle y_1, y_2, y_3 \rangle$ is abelian and $\langle y_1, y_2, y_3 \rangle$ is semiregular on $\Delta - \{\Delta_1, \Delta_2\}$, $y_i$ fixes $\Delta_1$ and $\Delta_2$.

Suppose that $\langle y_1, y_2, y_3, y_4 \rangle$ is not semiregular on $\Delta - \{\Delta_1, \Delta_2\}$. Then there is an element $y'$ in $\langle Q, y_1, y_2, y_3 \rangle$ such that $y'$ has fixed points in $\Delta$ other than $\Delta_1$ and $\Delta_2$. Then $y'^{\langle Q \rangle}$ is of order four or two. If $y'^{\langle Q \rangle}$ is of order four, then $y'^2 = y_1$. This is a contradiction since $y_1$ has no fixed point in $\Delta - \{\Delta_1, \Delta_2\}$. If $y'^{\langle Q \rangle}$ is of order two, then $y'^{\langle Q \rangle}$ has exactly two or four 2-cycles. Hence $\langle Q, y \rangle$ is conjugate to $\langle Q, y_1 \rangle$ or $\langle Q, y_1y_2y_3 \rangle$. This is a contradiction since $y_1$ and $y_1y_2y_3$ have exactly two fixed points $\Delta_1$ and $\Delta_2$. Thus $\langle y_1, y_2, y_3, y_4 \rangle$ is semiregular on $\Delta - \{\Delta_1, \Delta_2\}$.

Since $y_2, y_3$ and $y_4$ fix $\Delta_1, Qy_2, Qy_3$ and $Qy_4$ have elements fixing $1'$ of $\Delta_1$. Hence we may assume that $y_2, y_3$ and $y_4$ fix $1'$. Then $\langle y_1, y_2, y_3 \rangle$ and $\langle y_1, y_2y_3, y_4 \rangle$ are elementary abelian. Since $I(y_1) \cap \Delta = \{1', 2', 3', 4'\}$, $\langle y_1, y_2, y_3, y_4 \rangle$ fixes $\{1', 2', 3', 4'\}$. Set $R = C_Q(y_1)$. Then $R$ is of order four and has an orbit $\{1', 2', 3', 4'\}$. Hence $\langle y_1, y_2, y_3, y_4 \rangle$ normalizes $R$. Since $y_1 \in C(Q)$, $|Q| > 4$. Hence the number of the $R$-orbit in $\Delta_1$ is even. Since $\langle y_1, y_3, y_3, y_4 \rangle$ fixes the $R$-orbit $\{1', 2', 3', 4'\}$ in $\Delta_1$, we may assume that $\langle y_1, y_2, y_3, y_4 \rangle$ fixes one more $R$-orbit $\{5', 6', 7', 8'\}$ in $\Delta_1$. 

MULTIPLY TRANSITIVE GROUPS XII 621
Let $a$ be an involution $R$ commuting with $y_1$, $y_2$ and $y_3$. Then
\[ \langle a, y_i \rangle \text{-orbits in } \Delta - (\Delta_i \cup \Delta_a) \text{ are of length four}. \]
Let $\{i_1, i_2, i_3, i_4\}$ be any $\langle a, y_i \rangle$-orbit in $\Delta - (\Delta_i \cup \Delta_a)$. Then $\langle a, y_i \rangle$ normalizes $G_{i_1 i_2 i_3 i_4}$. Hence there is an involution $u$ in $G_{i_1 i_2 i_3 i_4}$ commuting with $a$ and $y_i$. Then $\langle y_i, u \rangle$ normalizes $G_{i_1 i_2 i_3 i_4}$ and so a Sylow 2-subgroup $Q'$ of $G_{i_1 i_2 i_3 i_4}$. Since $N(Q') = A_{3}, \langle Q', y_i, u \rangle$ is conjugate to a subgroup of $\langle Q, y_i, y_2, y_3, y_4 \rangle$ in $N(G_{i_1 i_2 i_3 i_4})$. Hence $y_i$ fixes exactly two $Q'$-orbits $\Delta'$ and $\Delta''$ in $\Delta$ and $\{i_1, i_2, i_3, i_4\}$ is contained in $\Delta'$ or $\Delta''$. Furthermore since $\langle Q', y_i \rangle$ is conjugate to $\langle Q, y_i \rangle$ in $\langle Q', y_i \rangle$, there is an element $v$ in $\langle Q', y_i \rangle$ such that $\langle Q', y_i \rangle^v = \langle Q, y_i \rangle$. Then $\langle Q', y_i \rangle^v = \Delta' \cup \Delta''$. Since $\langle Q, y_i \rangle \cap \langle Q', y_i \rangle = \{1\}$ and $\langle Q, y_i \rangle^v = \langle Q, y_i \rangle$, we may assume that $\langle Q, y_i \rangle = \{1\}$. Then $v \in G_{i_1 i_2 i_3 i_4}$ and $\langle \Delta' \cup \Delta'' \rangle = \Delta_i \cup \Delta_a$. Thus $\{i_1, i_2, i_3, i_4\}$ is contained in a $G_{i_1 i_2 i_3 i_4}$-orbit which contains $\Delta_i$ or $\Delta_a$. Since $\{i_1, i_2, i_3, i_4\}$ is any $\langle a, y_i \rangle$-orbit in $\Delta - (\Delta_i \cup \Delta_a)$, any $\langle a, y_i \rangle$-orbit in $\Delta - (\Delta_i \cup \Delta_a)$ is contained in the $G_{i_1 i_2 i_3 i_4}$-orbit which contains $\Delta_i$ or $\Delta_a$. Hence $G_{i_1 i_2 i_3 i_4}$ is transitive or has two orbits $\Gamma_i$ and $\Gamma_a$ on $\Delta$, where $\Gamma_i \supseteq \Delta_i$ and $\Gamma_a \supseteq \Delta_a$.

Since $y_i$ fixes exactly two $Q$-orbits in $\Delta$, the number of $Q$-orbits in $\Delta$ is even. Hence $|\Delta| = 2|\Delta_i| = 2|Q|$. If $G_{i_1 i_2 i_3 i_4}$ is transitive on $\Delta$, then the order of $G_{i_1 i_2 i_3 i_4}$ is divisible by $2|\Delta|$. This is a contradiction since $Q$ is a Sylow 2-subgroup of $G_{i_1 i_2 i_3 i_4}$. Hence $G_{i_1 i_2 i_3 i_4}$ has two orbits $\Gamma_i$ and $\Gamma_a$ on $\Delta$.

Since $y_i \in C(Q)$, $|Q| > 4$. Hence $\langle Q, y_i, y_i' \rangle$ is a Sylow 2-subgroup of $G_{5678}$. Since $G$ is 4-fold transitive, any Sylow 2-subgroup $P$ of a stabilizer of four points in $G$ is conjugate to $\langle Q, y_i, y_i' \rangle$ and so has exactly one orbit of length four. Furthermore a stabilizer of a point of this orbit of length four in $P$ is conjugate to $Q$.

We may assume that
\[
y_i = (1 2) (3 4) (5 6) (7 8) (1') (2') (3') (4') (5' 6') (7' 8') \ldots,
\]
\[
a = (1) (2) \ldots (8) (1' 2') (3' 4') \ldots.
\]
Since $y_2$ and $y_i$ fix 1' and commute with $a$ and $y_i$, $y_2$ and $y_3$ are (1') (2') (3') (4') or (1') (2') (3') (4') on $\{1', 2', 3', 4'\}$.

Assume that $y_2 = (1') (2') (3') (4')$ on $\{1', 2', 3', 4'\}$. Since $|I(y_i y_3)| \leq 1$, we may assume that
\[
y_2 = (1 2) (3 4) (5 6) (7 8) (1') (2') (3') (4') (5' 7') (6' 8') \ldots.
\]
Thus $\langle y_i, y_2 \rangle$ is semiregular on $\{5', 6', \ldots, n\}$. Suppose that $y_3$ has fixed points in $\{5', 6', \ldots, n\}$. Since $\langle y_i, y_2, y_3 \rangle$ is abelian, $y_3$ has at least four fixed points in $\{5', 6', \ldots, n\}$. This is a contradiction since $I(y_3) \supseteq \{1'\}$ and $|I(y_3)| \leq 8$. Hence $y_3$ fixes $\{1', 2', 3', 4'\}$ pointwise. Since $\langle y_i, y_2, y_3 \rangle$ fixes the $R$-orbit $\{5', 6', 7', 8'\}$, there is an element $\{\pm 1\}$ in $\langle y_i, y_2, y_3 \rangle$ fixing $\{5', 6', 7', 8'\}$ pointwise. Since $I(\langle y_i, y_2, y_3 \rangle) \supseteq \{1', 2', 3', 4'\}$, this element is $y_i y_3 y_2$. Hence
\[
y_3 = (1 2) (3 4) (5 6) (7 8) (1') (2') (3') (4') (5' 8') (6' 7') \ldots.
\]
Then \( \langle y_1, y_2, y_3 \rangle \) normalizes \( G_{1234} \). Hence as we have seen above, \( \langle y_1, y_2, y_3 \rangle \) normalizes a 2-subgroup \( Q'' \) of \( G_{1234} \) which is conjugate to \( Q \). Then \( |I(Q'')| = 8 \) and \( N(Q''|Q'') = A_8 \). Hence \( y_i^{1(Q'')} \), \( y_2^{1(Q'')} \) and \( y_3^{1(Q'')} \) are even permutations.

Since \( y_1, y_2 \) and \( y_3 \) are \((1 2) (1') (2')\) on \( \{1, 2, 1', 2'\} \), \( y_1, y_2 \) and \( y_3 \) have exactly one more 2-cycle other than \((1 2)\) in \( I(Q'') \). This is impossible. Hence \( y_2 \) is \((1') (2') (3' 4')\)...

Similarly \( y_2 \) is \((1') (2') (3') (4')\)...

Thus \( y_2 \) and \( y_3 \) are \((1') (2') (3') (4')\) on \( \{1', 2', 3', 4'\} \). Since \( |R| = 4 \), \( R \) is cyclic or elementary abelian. First assume that \( R \) is cyclic. Then \( R = \langle b \rangle \) and

\[
b = (1) (2) \cdots (8) (1' 3' 2' 4') (5' 7' 6' 8') \cdots .
\]

Then \( \langle R, y_1, y_3 \rangle \) is semiregular on \( \{9', 10', \ldots, n\} \). Since \( \langle a, y_1, y_3 \rangle \) is abelian, if \( y_2 \) has fixed points in \( \{9', 10', \ldots, n\} \), then \( y_2 \) fixes at least four points of \( \{9', 10', \ldots, n\} \). This is a contradiction since \( I(y_2) \) contains \( \{3, 4, 7, 8\} \cup \{1'\} \) of length five. Thus \( y_2 \) has no fixed points in \( \{9', 10', \ldots, n\} \). Similarly \( y_2 \) has no fixed points in \( \{5', 6', 7', 8'\} \). Hence \( y_2 \) and \( y_3 \) have exactly two fixed points in \( \{5', 6', 7', 8'\} \). Next assume that \( R \) is elementary abelian. Then \( R = \langle a, b \rangle \) and

\[
b' = (1) (2) \cdots (8) (1' 3') (2' 4') \cdots .
\]

Then \( b'y_2 \) and \( b'y_3 \) are of order four and so 4-cycle on \( \{5', 6', 7', 8'\} \). Hence \( y_2 \) and \( y_3 \) have exactly two fixed points in \( \{5', 6', 7', 8'\} \). Thus in both cases we may assume that

\[
a = (1) (2) \cdots (8) (1' 2') (3' 4') (5' 6') (7' 8') \cdots ,
\]

\[
y_2 = (1 2) (3) (4) (5 6) (7 8) (1' 2') (3' 4') (5') (6') (7' 8') \cdots ,
\]

\[
y_3 = (1 2) (3) (4) (5) (6) (7 8) (1' 2') (3' 4') (5' 6') (7' 8') \cdots .
\]

Since \( \langle a, y_1, y_2, y_3 \rangle \) normalizes \( G_{1234} \), as we have seen above \( \langle a, y_1, y_2, y_3 \rangle \) normalizes a 2-subgroup \( Q'' \) of \( G_{1234} \) which is conjugate to \( Q \). Then \( |I(Q'')| = 8 \) and \( N(Q''|Q'') = A_8 \). Hence \( a^{1(Q'')} \), \( y_i^{1(Q'')} \), \( y_2^{1(Q'')} \) and \( y_3^{1(Q'')} \) are even permutations.

Since \( a = (1) (2) (1' 2') \) and \( y_i = (1 2) (1') (2') \), \( i = 1, 2, 3 \), on \( \{1, 2, 1', 2'\} \), \( a \) and \( y_i \) have exactly one more 2-cycle other than \((1' 2')\) and \((1 2)\) respectively in \( I(Q'') \). Since the lengths of \( \langle a, y_1, y_2, y_3 \rangle \)-orbits in \( \{9', 10', \ldots, n\} \) are at least eight, \( |I(Q'') \cap \{9', 10', \ldots, n\}| = 0 \). Hence \( I(Q'') = \{1, 2, 3, 4, 1', 2', 3', 4'\} \), \( \{1, 2, 5, 6, 1', 2', 5', 6'\} \), or \( \{1, 2, 7, 8, 1', 2', 7', 8'\} \).

First assume that \( I(Q'') = \{1, 2, 3, 4, 1', 2', 3', 4'\} \). Then a Sylow 2-subgroup of \( G_{1234} \) containing \( Q \) or \( Q'' \) has exactly one orbit \( \{5, 6, 7, 8\} \) or \( \{1', 2', 3', 4'\} \) of length four respectively. Since Sylow 2-subgroups of \( G_{1234} \) are conjugate, \( \{5, 6, 7, 8\} \) and \( \{1', 2', 3', 4'\} \) are contained in the same \( G_{1234} \)-orbit. Since \( G_{1234} \) is not 5-fold transitive. Hence \( G_{1234} \) has two orbits \( \{5, 6, 7, 8\} \cup \{1', 2', 3', 4'\} \) and \( \{1, 2, 3, 4\} \).

Next assume that \( I(Q'') = \{1, 2, 5, 6, 1', 2', 3', 6'\} \). Then by the same
argument as above $G_{1234}$ has two orbits $\{3, 4, 7, 8\} \cup \Gamma_1$ and $\Gamma_2$. Since $N(Q)^{\langle Q \rangle} = A$, there is an element $x=(1) (2) (3) (4) (5) (6) (7) (8) \cdots$. Then $G_{1234}^x = (G_{1234})^x$ has two orbits $\{5, 6, 7, 8\} \cup \Gamma_1^x$ and $\Gamma_2^x$. Since $\Gamma_1$ and $\Gamma_2$ are $\langle Q \rangle$-orbits, $\Gamma_1^x = \Gamma_1$ or $\Gamma_2$. On the other hand $G$ is 4-fold transitive on $\Omega$. Hence $G_{1234}$ has two orbits $\{3, 4, 5, 6\} \cup \Gamma_i$ and $\Gamma_j$, where $\{i,j\} = \{1, 2\}$. Since $z \in G_{1234}$, $z$ fixes $\Gamma_i$ and $\Gamma_j$. Hence $G_{1234}$ has two orbits $\{5, 6, 7, 8\} \cup \Gamma_1$ and $\Gamma_2$. Similarly if $I(Q') = \{1, 2, 7, 8, 1', 2', 7', 8'\}$, then $G_{1234}$ has two orbits $\{5, 6, 7, 8\} \cup \Gamma_1$ and $\Gamma_2$. Thus in any case $G_{1234}$ has the two orbits $\{5, 6, 7, 8\} \cup \Gamma_1$ and $\Gamma_2$.

On the other hand $\Delta_2$ is contained in $\Gamma_2$ and fixed by $y_i$. Hence there is an element in $Qy_i$ fixing four points of $\Delta_2$. Then by the same argument as above $\{5, 6, 7, 8\}$ and $\Gamma_2$ are contained in the same $G_{1234}$-orbit. Thus $G_{1234}$ is transitive on $\Omega - \{1, 2, 3, 4\}$, contrary to (2.11). Thus $N(Q)^{\langle Q \rangle} = A$. Hence we complete the proof of (2.15).

### 2.16. $N(Q)^{\langle Q \rangle} = S_t$.

Proof. Suppose by way of contradiction that $N(Q)^{\langle Q \rangle} = S_t$. Then by (2.4) $N(Q)$ has the 2-group $\langle Q, x_1, x_2, \cdots, x_k \rangle$. Now we show that $\langle Q, x_1, x_2, \cdots, x_k \rangle$ is semiregular on $\Delta$. By (2.13) and (2.15) $\langle Q, x_1, x_2, x_3 \rangle$ is semiregular on $\Delta$.

Suppose that $\langle Q, x_1, x_2, x_3 \rangle$ is not semiregular on $\Delta$. Then $x_3$ fixes a $\langle Q, x_1, x_2, x_3 \rangle$-orbit $\Delta'$ of length $|Q|/2$ in $\Delta$. Then by (2.13) and (2.15) $x, x_2, x_3$ fixes $Q$-orbits in $\Delta'$. Furthermore $\langle x, x_2, x_3 \rangle$ is abelian and $\langle x, x_2, x_3 \rangle$ is semiregular on $\Delta$. Hence $x, x_2, x_3$ fixes four $Q$-orbits in $\Delta'$. By (2.8) $x, x_2, x_3$ fixes at most six $Q$-orbits in $\Delta$. Hence $x, x_2, x_3$ does not fix any $Q$-orbit in $\Delta - \Delta'$. Hence $\langle Q, x_1, x_2, x_3 \rangle$ is semiregular on $\Delta - \Delta'$. Since $N(Q)^{\langle Q \rangle} = S_t$, $N(Q)$ has a 2-element $y_i' = (1) (3) (2) (4) (5) (6) \cdots (t) \cdots$.

By (2.3) we may assume that $\langle Q, x_1, x_2, x_3, x_4, y_i' \rangle$ is a 2-group. Then $y_i'$ normalizes $\langle Q, x_1, x_2, x_3, x_4, y_i' \rangle$. Hence $y_i'$ fixes the $\langle Q, x_1, x_2, x_3, y_i' \rangle$-orbit $\Delta'$. Thus $\Delta'$ is $\langle Q, x_1, x_2, x_3, y_i' \rangle$-orbit. Hence $\langle Q, x_1, x_2, x_3, y_i' \rangle$ has an element $x$ fixing a point of $\Delta'$. Then by (2.13) and (2.15) $x^{\langle Q \rangle}$ is of order four and has exactly one 4-cycle (1 3 2 4) or (1 4 2 3). Hence $(x^t)^{\langle Q \rangle} = (1) (2) (3) (4)$ and has fixed points in $\Delta$, contrary to (2.15). Thus $\langle Q, x_1, x_2, x_3, y_i' \rangle$ is semiregular on $\Delta$.

Suppose that $\langle Q, x_1, x_2, x_3, y_i' \rangle$ is not semiregular on $\Delta$. Then $x_1$ fixes a $\langle Q, x_1, x_2, x_3, y_i' \rangle$-orbit $\Delta'$ of length $|Q|/2$ in $\Delta$. Since $\langle x, x_2, x_3, x_4 \rangle$ is abelian and $\langle x, x_2, x_3 \rangle$ is semiregular on $\Delta$, by (2.8) $x, x_2, x_3, x_4$ fixes exactly eight $Q$-orbits in $\Delta$, whose union is $\Delta'$. Thus $\langle Q, x_1, x_2, x_3, x_4 \rangle$ is semiregular on $\Delta - \Delta'$. Since $N(Q)^{\langle Q \rangle} = S_t$, $N(Q)$ has a 2-element $y_i' = (1) (3) (2) (4) (5) (6) \cdots (t) \cdots$.

By (2.3) we may assume that $\langle Q, x_1, x_2, x_3, x_4, y_i' \rangle$ is a 2-group. Then $y_i'$ normalizes $\langle Q, x_1, x_2, x_3, x_4 \rangle$. Hence $y_i'$ fixes $\Delta'$. Then $\Delta'$ is a $\langle Q, x_1, x_2, x_3, y_i' \rangle$-
orbit. Hence there is an element $x$ in $\langle Q, x_1, x_2, x_3, y_1 \rangle$ fixing a point of $\Delta'$. Since $\langle Q, x \rangle$ is not conjugate to any subgroup of $\langle Q, x_1, x_2, x_3 \rangle$, $x^{I(Q)}$ is of order four and has exactly one 4-cycle $(1 \ 2 \ 3 \ 4)$ or $(1 \ 4 \ 2 \ 3)$. Hence $(x^4)^{I(Q)} = (1 \ 2 \ 3 \ 4)$ and $x^2$ has fixed points in $\Delta$, contrary to (2.15). Thus $\langle Q, x_1, x_2, x_3, y_1 \rangle$ is semiregular on $\Delta$. Hence by (2.9) $\langle Q, x_1, x_2, \ldots, x_9 \rangle$ is semiregular on $\Delta$.

On the other hand $Q$ has an involution $a = (1 \ 2 \ 3 \ 4)$. Let $a$ normalize $G_{1,2,3}$ and so commutes with an involution $u$ of $G_{1,2,3}$. Then $u$ normalizes a Sylow 2-subgroup $Q'$ of $G_{1,2,3}$.

Since $\langle Q, x \rangle$ is not conjugate to any subgroup of $\langle Q, x \rangle$, $x^2$ has fixed points in $\Delta$, contrary to (2.15). Thus $\langle Q, x_1, x_2, \ldots, x_9 \rangle$ is semi-

2.17. We show that $N(Q)^{I(Q)} = A_t$ and complete the proof of the theorem.

Proof. Suppose by way of contradiction that $N(Q)^{I(Q)} = A_t$. First suppose that $t = 8$ or $9$. Let $a = (1 \ 2 \ 3 \ 4)$ be an involution of $Q$. Then $a$ normalizes $G_{1,2,3}$ and so commutes with an involution $u$ of $G_{1,2,3}$. Since $N(Q)^{I(Q)} = N(G_{1,2,3})^t = A_t$ or $A_9$ and $\{u\} \leq A_t$, $u^{I(Q)}$ consists of exactly two 2-cycles. This contradicts (2.15) since $|\{u\} \cap \Delta| = 0$.

Thus $t \geq 10$. Then by (2.4) $N(Q)$ has the 2-group $\langle Q, y_1, y_2, \ldots, y_k, y'_1 \rangle$, $k \geq 4$. Now we show that $\langle Q, y_1, y_2, \ldots, y_k, y'_1 \rangle$ is semiregular on $\Delta$. By (2.15) $\langle Q, y_1, y_2 \rangle$ is semiregular on $\Delta$.

Let $y$ be any element of $\langle Q, y_1, y_2, y'_1 \rangle$. Then $y^{I(Q)}$ is of order two or four. If $y^{I(Q)}$ is of order two, then $y^{I(Q)}$ consists of exactly two 2-cycles. Hence by (2.15) $y$ is semiregular on $\Delta$. If $y^{I(Q)}$ is of order four, then $(y^2)^{I(Q)} = y^{I(Q)}$. Hence $y$ is semiregular on $\Delta$. Thus $\langle Q, y_1, y_2, y'_1 \rangle$ is semiregular on $\Delta$.

Suppose that $\langle Q, y_1, y_2, y_3 \rangle$ is not semiregular on $\Delta$. Then by (2.15) $\bar{y}_1 \bar{y}_2 \bar{y}_3$ has fixed points in $\Delta$. Since $(y_1, y_2, y_3)^{I(Q)}$ is an involution consisting of exactly four 2-cycles $\bar{y}_1 \bar{y}_2 \bar{y}_3$ fixes at most eight $Q$-orbits by (2.8). On the other hand $\langle y_1, y_2, y_3 \rangle$ is abelian and $\langle y_1, y_2 \rangle$ is a semiregular group of order four. Hence $\bar{y}_1 \bar{y}_2 \bar{y}_3$ fixes four or eight $Q$-orbits. Thus $y_3$ fixes one or two $\langle Q, y_1, y_2 \rangle$-orbits in $\Delta$.

Assume that $y_3$ fixes exactly one $\langle Q, y_1, y_2 \rangle$-orbit $\Gamma$ in $\Delta$. Then since $y_1'$ normalizes $\langle Q, y_1, y_2, y_3, y'_1 \rangle$, $y_1'$ fixes $\Gamma$. Hence $\Gamma$ is also a $\langle Q, y_1, y_2, y_3, y'_1 \rangle$-orbit. This is a contradiction since $\langle Q, y_1, y_2, y'_1 \rangle$ is semiregular on $\Delta$. Thus $y_3$ fixes exactly two $\langle Q, y_1, y_2, y'_1 \rangle$-orbits in $\Delta$, say $\Gamma_1$ and $\Gamma_2$. Hence by (2.8) any element of $Qy_1y_2y_3$ is an involution and has exactly eight fixed points in $\Delta$.

Suppose that $\Gamma_i = \Delta_i \cup \Delta_5 \cup \Delta_6 \cup \Delta_7$ and $\Gamma_2 = \Delta_3 \cup \Delta_4 \cup \Delta_5 \cup \Delta_7$, where $\Delta_i, 1 \leq i \leq 8$, is a $Q$-orbit. Set $\Gamma_i = \{\Delta_i, \Delta_2, \Delta_3, \Delta_4\}$ and $\Gamma_2 = \{\Delta_3, \Delta_4, \Delta_5, \Delta_6\}$. Then we may assume that
\[ y_1 = (\Delta, \Delta_1) (\Delta_2, \Delta_3) (\Delta_4, \Delta_5) \cdots, \]
\[ y_2 = (\Delta_1, \Delta_2) (\Delta_3, \Delta_4) (\Delta_5, \Delta_6) \cdots, \]
\[ y_3 = (\Delta_1, \Delta_4) (\Delta_2, \Delta_3) (\Delta_5, \Delta_6) \cdots. \]

Since \( y_i, i \geq 4 \), normalizes \( \langle Q, y_1, y_2, y_3 \rangle \), \( \Gamma_i \gamma_i = \Gamma_1 \) or \( \Gamma_2 \). Suppose that \( \Gamma_i \gamma_i = \Gamma_1 \). Then \( \Gamma_i \) is a \( \langle Q, y_1, y_2, y_3 \rangle \)-orbit. Hence \( y_i y_2 y_i \) fixes a \( Q \)-orbit in \( \Gamma_i \) by (2.15). Since \( y_i y_2 y_i \) is the identity on \( \Gamma_i \), \( y_i y_2 y_i \) fixes a \( Q \)-orbit in \( \Gamma_i \), contrary to (2.15). Thus \( \Gamma_i \gamma_i = \Gamma_2 \).

Suppose that \( t \geq 12 \). Then \( N(Q) \) has \( y_2 \) and \( y_3 \). Since \( \langle y_1, y_2, y_3 \rangle \) is elementary abelian and \( \Gamma_i \gamma_i = \Gamma_2 \), we may assume that
\[ y_4 = (\Delta_1, \Delta_4) (\Delta_2, \Delta_5) (\Delta_6, \Delta_7) (\Delta_8, \Delta_9) \cdots. \]

Furthemore since \( \Gamma_i \gamma_i = \Gamma_2 \), \( \Gamma_1 \Gamma_2 \) is a \( \langle y_1, y_2, y_3 \rangle \)-orbit of length eight. Hence \( \langle y_1, y_2, y_3 \rangle \) has an element fixing \( \Gamma \), pointwise. Thus we may assume that
\[ y_5 = (\Delta_1, \Delta_4) (\Delta_2, \Delta_5) (\Delta_6, \Delta_7) (\Delta_8, \Delta_9) \cdots. \]

On the other hand \( N(Q) \) has two elements
\[ y_1' = (1) (2) (3 4) (5) (6) (7 8) (9 10 11) (12) (13) \cdots, \]
\[ y_2' = (1) (2) (3 4) (5) (6) (7 8) (9 10 11) (12) (13) (14) \cdots. \]

By (2.3) we may assume that \( \langle Q, y_1, y_2, y_3, y_1', y_2' \rangle \) is a 2-group. Then by the same argument as above \( \Gamma_i \gamma_i = \Gamma_2 \). If \( y_2' = (\Delta_1, \Delta_4) \cdots, i = 4, 5 \), then \( (y_2 y_1')^2 \) has the same form as \( y_1 \) on \( I(Q) \) and fixes \( \Delta_1 \), which is a contradiction. Similarly \( y_i' = (\Delta_1, \Delta_4) \cdots, i = 4, 5 \), since \( (y_2 y_1')^2 = \bar{y}_1 \). Hence we may assume that
\[ y_4' = (\Delta_1, \Delta_4) (\Delta_2, \Delta_5) (\Delta_6, \Delta_7) \cdots, \]
\[ y_5' = (\Delta_1, \Delta_4) (\Delta_2, \Delta_5) (\Delta_6, \Delta_7) \cdots. \]

Then \( y_i y_1 y_i y_i' \) consists of exactly two 2-cycles on \( I(Q) \) and fixes \( \Delta_1 \), contrary to (2.15).

Thus \( t = 10 \) or 11. Assume that \( t = 10 \). The proof in the case \( t = 11 \) is similar. Since \( \langle Q, y_1, y_2, y_3 \rangle \) is a 2-group. Then by the same argument as above \( \Gamma_i \gamma_i = \Gamma_2 \). If \( y_i' = (\Delta_1, \Delta_4) \cdots, i = 4, 5 \), then \( (y_2 y_1')^2 \) has the same form as \( y_1 \) on \( I(Q) \) and fixes \( \Delta_1 \), which is a contradiction. Similarly \( y_i' = (\Delta_1, \Delta_4) \cdots, i = 4, 5 \), since \( (y_2 y_1')^2 = \bar{y}_1 \). Hence we may assume that
\[ y_4' = (\Delta_1, \Delta_4) (\Delta_2, \Delta_5) (\Delta_6, \Delta_7) \cdots, \]
\[ y_5' = (\Delta_1, \Delta_4) (\Delta_2, \Delta_5) (\Delta_6, \Delta_7) \cdots. \]

Then \( y_4 y_1 y_4 y_1' \) consists of exactly two 2-cycles on \( I(Q) \) and fixes \( \Delta_1 \), contrary to (2.15).

Thus \( t = 10 \) or 11. Assume that \( t = 10 \). The proof in the case \( t = 11 \) is similar. Since \( \langle Q, y_1, y_2, y_3 \rangle \) is semiregular on \( \Delta \), the lengths of \( \langle Q, y_1, y_2, y_3 \rangle \)-orbits on \( \Delta \) are \( 8 | \Delta | \). On the other hand \( \langle Q, y_1, y_2, y_3 \rangle \) fixes 7, 8, 9, 10 and has two orbits \{1, 2, 3, 4\} and \{5, 6\} on \( I(Q) \). Hence \( \langle Q, y_1, y_2, y_3 \rangle \) is a Sylow 2-group of \( G_{78910} \). Furthemore in \( \langle Q, y_1, y_2, y_3 \rangle \) any element fixing ten points belongs to \( Q \). Since \( G \) is 4-fold transitive, this shows that any element fixing ten points is conjugate to an element of \( Q \). Set \( z_1 = y_2 y_1 y_3 \). By what we have proved above every element of \( Qz_1 \) is an involution. Hence for any element \( u \) of \( Q \) \( u^* = u^{-1} \). Furthermore \( N(Q) \) has a 2-element
\[ z_2 = (1 3) (2 4) (5 7) (6 8) (9 10) \cdots. \]

By (2.3) we may assume that \( \langle Q, z_2 \rangle \) is a 2-group. Since \( \langle Q, z_2 \rangle \) and \( \langle Q, z_1 z_2 \rangle \) are conjugate to \( \langle Q, z_1 \rangle \), every element of \( Qz_2 \) and \( Qz_1 z_2 \) is an
involution. Hence for any element \( u \) of \( Q \), \( u^2 = u^{-1} \) and \( u^3 \) is \( u^{-1} \). On the other hand \((u^3)^2 = (u^{-1})^2 = u \). Hence \( u = u^{-1} \). Thus \( Q \) is elementary abelian and \( z_i \), \( z_j \in C(Q) \). Then since \( N(Q)^{(Q) fix} = A_{10} \) and \( C(Q)^{(Q) fix} \) is a normal subgroup \((\pm 1)\), \( N(Q)^{(Q) fix} = C(Q)^{(Q) fix} \). In particular since \( Q \) is abelian, every 2-element of \( N(Q) \) belongs to \( C(Q) \).

Since \( y_i^2 \in Q \), the order of \( y_i \) is two or four. Suppose that \( y_i \) is of order two. Then for any 2-cycle \( (ij) \) of \( y_i \) in \( \Delta \), \( y_i \) normalizes \( G_{i,j} \). Hence \( y_i \) normalizes a 2-subgroup \( Q' \) of \( G_{i,j} \) which is conjugate to \( Q \). Since \( N(Q')^{(Q') fix} = A_{10} \), \( y_i \) consist of exactly two or four 2-cycles on \( I(Q') \). Suppose that \( y_i \) consists of exactly four 2-cycles on \( I(Q') \). Then \( \langle Q', y_i \rangle \) is conjugate to \( \langle Q, z_i \rangle \). Then \( |I(y_i)| = 10 \), which is a contradiction. Thus \( y_i \) consists of exactly two 2-cycles on \( I(Q') \). Then \( I(Q') = \{i, j, 1, 2, 5, 6, \ldots, 10\} \). Then \( Q \) and \( Q' \) are contained in \( G_{78910} \) and so conjugate in \( G_{78910} \). Thus \( G_{78910} \) has an element which takes \( \{1, 2, i, j\} \) into \( \{1, 2, \ldots, 6\} \). Since \( \{1, 2, \ldots, 6\} \) is contained in a \( G_{78910} \)-orbit and \( (ij) \) is any 2-cycle of \( y_i \) in \( \Delta \), \( G_{78910} \) is transitive on \( \Omega = \{7, 8, 9, 10\} \), contrary to (2.11). Thus \( y_i \) is of order four. Hence every involution of \( N(Q) - Q \) consists of exactly four 2-cycles on \( I(Q) \) and every involution of \( G \) fixes exactly ten points.

\( C(Q) \) has an involution

\[
z_3 = (1 3)(2 4)(5 6)(7 8)(9 10) \ldots
\]

By (2.3) we may assume that \( \langle Q, z_3, z_2 \rangle \) is a 2-group. Then since \( z_i z_3 \) consists of exactly four 2-cycles on \( I(Q) \), \( z_i z_3 \) is of order two. Hence \( z_i z_3 = z_3 z_i \). Since \( I(z_i) = I(z_3) \) and any Sylow 2-subgroup of \( G_{I(z_i)} \) is conjugate to \( Q \), \( z_i \) fixes exactly two points of \( I(z_i) \). Hence \( |I(z_i) \cap I(z_3)| \cap \Delta| = 2 \). Then since \( Q \) is semiregular on \( \Delta \), \( \langle z_i, z_3 \rangle < C(Q) \), \( Q \mid = 2 \). Set \( Q = \langle a \rangle \).

Since \( \langle a, y_3 y_i \rangle \) is conjugate to \( \langle a, y_i \rangle \), \( y_3 y_i \) is of order four and \( (y_3 y_i)^2 = a \). Let \( (ij k l) \) be any 4-cycle of \( y_3 y_i \) in \( \Delta \). Then \( y_3 y_i \) normalizes \( G_{i,j k,l} \). Hence \( y_3 y_i \), commutes with an involution \( z \) of \( G_{i,j k,l} \). Since \( z \) commutes with \( y_3 y_i \), \( z = a \), \( z \) fixes \( I(a) \). Thus \( y_3 y_i z \) is of order four and \( (y_3 y_i z)^{I(a)} \) is of order two. Hence \( y_3 y_i z \) consists of exactly two 2-cycles on \( I(a) \). Then since \( (y_3 y_i z)^{I(a)} = (7 8)(9 10) \) and \( z^{I(a)} \) consists of exactly four 2-cycles, \( z \) has 2-cycles \( (7 8) \) and \( (9 10) \). Hence \( y_3 y_i z \in G_{78910} \). Furthermore \( y_3 y_i z \) is \( (ij k l) \) on \( \{i, j, k, l\} \). Hence \( \{i, j, k, l\} \) is contained in a \( G_{78910} \)-orbit. Set \( z_4 = y_3 y_i z_4 \). Then \( z_4 \) has 2-cycles \( (7 8) \) and \( (9 10) \). Since \( C(a)^{I(a)} = A_6 \), \( C(a) \) has an involution \( \tau \) which is conjugate to \( z \) under \( C(a)^{I(a)} \) and has the same form as \( z_4 \) on \( I(a) \). Then \( \langle a, \tau \rangle \) and \( \langle a, z_4 \rangle \) are Sylow 2-subgroups of \( \langle a, z_0, \tau \rangle \) and \( \langle a, z_4 \rangle^{I(a)} = \langle a, \tau \rangle^{I(a)} \). Hence \( \langle a, \tau \rangle \) is conjugate to \( \langle a, z_4 \rangle \) under \( \langle a, z_4, \tau \rangle^{I(a)} \) and so \( \tau \) is conjugate to \( z_4 \) or \( a z_4 \) under \( \langle a, z_4, \tau \rangle^{I(a)} \). Thus \( \tau \) is conjugate to \( z_4 \) or \( a z_4 \) under \( C(a)^{I(a)} \). Since \( I(z) \cap \Delta \subset \{i, j, k, l\} \), there is an element in \( C(a)^{I(a)} \) which takes \( \{i, j, k, l\} \) into \( I(z_4) \cap \Delta \) or \( I(a z_4) \cap \Delta \). On the other hand \( z_4^2 = z_4 a \).
Hence \((I(z) \cap \Delta)\gamma' = I(az) \cap \Delta\). Thus \(C(a)_{78910}\) has an element taking \(\{i, j, k, l\}\) into \(I(z) \cap \Delta\). Furthermore \(\gamma'_i, \gamma'_2\) is of order eight and commutes with \(z\). Hence \(\gamma'_i, \gamma'_2\) consists of a 8-cycle on \(I(z) \cap \Delta\). Thus \(I(z) \cap \Delta\) is contained in a \(C(a)_{78910}\)-orbit. Since \((i, j, k, l)\) is any 4-cycle of \(\gamma'_i, \gamma'_2\) in \(\Delta, \Delta\) is contained in a \(C(a)_{78910}\)-orbit and so in a \(G_{78910}\)-orbit. By (2.11) \(G_{78910}\) is intransitive on \(\Omega - \{7, 8, 9, 10\}\). Hence \(G_{78910}\) has exactly two orbits \(\{1, 2, \cdots, 6\}\) and \(\Delta\) on \(\Omega - \{7, 8, 9, 10\}\). Since \(G\) is 4-fold transitive, any four points \(i_1, i_2, i_3, i_4\) of \(\Omega\) uniquely determine a subset \(\Delta(i_1, i_2, i_3, i_4)\) of \(\Omega\) which is the \(G_{12}\)-orbit of length six.

Let \(y\) be any element of \(\langle Q, y_1, y_2, y_3 \rangle \cap \langle Q, y'_1 \rangle\). Then \(y^{(Q)}\) is of order two or four. If \(y^{(Q)}\) is of order two, then \(y^{(Q)}\) consists of two or four 2-cycles. Hence \(\langle Q, y'\rangle\) is conjugate to a subgroup of \(\langle Q, y_1, y_2, y_3 \rangle\) in \(N(Q)\). Hence \(y'\) is semiregular on \(\Delta\). If \(y^{(Q)}\) is of order four, then \(y'^{(Q)} = y_1^{(Q)}\). Hence \(y'\) is semiregular on \(\Delta\). Thus \(\langle Q, y_1, y_2, y_3, y'_1 \rangle\) is semiregular on \(\Delta\). Hence by (2.10) \(\langle Q, y_1, y_2, \cdots, y_n \rangle\) is semiregular on \(\Delta\).

Let \(x\) be any 2-element of \(N(G_{12})\). Then \(x\) normalizes a Sylow 2-subgroup \(Q'\) of \(G_{12}\). Suppose by way of contradiction that there is a 2-group \(Q\) in \(G\) such that \(|I(Q)| = 12\) and \(N(Q)^{(Q)} = M_{12}\). Let \(Q\) be a Sylow 2-subgroup of \(G_{12}\). Since \(Q\) is a Sylow 2-subgroup of \(G_{12}\) and \(N(Q)^{(Q)} = A_{12}\), \(\langle Q', x \rangle\) is conjugate to a subgroup of \(\langle Q, y_1, y_2, \cdots, y_n \rangle\). Hence \(x\) is semiregular on \(\Delta\). On the other hand \(Q\) has an involution \(a = (1)(2) \cdots (i) \cdots\). Then \(a\) normalizes \(G_{12}\), and so commutes with an involution \(u\) of \(G_{12}\). Then \(u \in N(Q)^{(Q)}\) and \(|I(u) \cap \Delta| = 0\), which is a contradiction. Thus \(N(Q)^{(Q)} = A_{12}\).

Thus we complete the proof of the theorem.

### 3. Proof of the lemma

In this section we assume that \(G\) is a permutation group as in Lemma. Suppose by way of contradiction that there is a 2-group \(Q\) in \(G\) such that \(|I(Q)| = 12\) and \(N(Q)^{(Q)} = M_{12}\). Let \(Q\) be a Sylow 2-subgroup of \(G_{12}\). Since \(N(Q)^{(Q)} = N(G_{12})^{(Q)} \geq N(Q)^{(Q)} = M_{12}\), \(N(Q)^{(Q)} = S_{12}, A_{12}\) or \(M_{12}\). If \(N(Q)^{(Q)}\)
MULTIPLY TRANSITIVE GROUPS XII

629

= S_{12}, or A_{17}, then by Theorem \( G = S_{14} \) or \( A_{14} \). Hence \( N(Q)^{(\Omega)} = S_{12} \), which is a contradiction. Thus \( N(Q)^{(\Omega)} = M_{12} \). Hence we may assume that \( Q \) is a Sylow 2-subgroup of \( G_{12}^{(\Omega)} \).

Set \( I(Q) = \{1, 2, \ldots, 12\} \) and \( \Delta = \Omega - I(Q) \). Then \( n \geq 35 \) ([2], p. 80) and so \( |\Delta| \geq 23 \).

Since \( N(Q)^{(\Omega)} = M_{12} \), we may assume that \( N(Q) \) has 2-element

\[
\begin{align*}
\sigma &= (1) (2) (3) (4) (5 6) (7 8) (9 10) (11 12) \cdots, \\
\tau &= (1) (2) (3) (4) (5 7 6 8) (9 11 10 12) \cdots,
\end{align*}
\]

and \( \langle Q, \sigma, \tau \rangle \) is a 2-group (see (2.3)). Then \( \langle Q, \sigma^2 \rangle = \langle Q, \tau^2 \rangle = \langle Q, \rho \rangle \). Since \( Q \) is a normal subgroup of \( \langle Q, \sigma, \tau \rangle \), \( Q \) has a central involution \( a \) of \( \langle Q, \sigma, \tau \rangle \). Then we may assume that \( a = (1) (2) (12) (13 14) (15 16) \cdots (n-1 n) \).

3.1. First we show that \( \langle Q, \sigma, \tau \rangle \) has at least one orbit of length eight in \( \Delta \) on which \( \langle Q, \sigma, \tau \rangle \) is a quaternion group.

Proof. Suppose by way of contradiction that \( \langle Q, \sigma, \tau \rangle \) has no orbit of length eight in \( \Delta \) on which \( \langle Q, \sigma, \tau \rangle \) is a quaternion group. Then \( \{5, 6, \ldots, 12\} \) is the unique \( \langle Q, \sigma, \tau \rangle \)-orbit of length eight and on which \( \langle Q, \sigma, \tau \rangle \) is a quaternion group.

(i) We show that \( \langle Q, \sigma, \tau \rangle \) is a Sylow 2-subgroup of \( G_{1234} \) and \( Q \) is a characteristic subgroup of \( \langle Q, \sigma, \tau \rangle \). Let \( x \) be any 2-element of \( N(\langle Q, \sigma, \tau \rangle) \). Hence \( x \in N(Q) \). Since \( N(Q)^{(\Omega)} = \langle y, \sigma, \tau \rangle \), \( x \sigma \tau \in \langle y, \sigma, \tau \rangle \). Hence there is an element \( x' \) in \( \langle Q, \sigma, \tau \rangle \) such that \( x' \sigma \tau = x \sigma \tau \). Hence \( x' = x \). Thus \( x \in \langle Q, \sigma, \tau \rangle \). This shows that \( \langle Q, \sigma, \tau \rangle \) is a Sylow 2-subgroup of \( G_{1234} \). Furthermore since any automorphism of \( \langle Q, \sigma, \tau \rangle \) fixes \( I(Q) \) and \( \langle Q, \sigma, \tau \rangle_{I(Q)} = Q \), \( Q \) is a characteristic subgroup of \( \langle Q, \sigma, \tau \rangle \).

(ii) Let \( i, j, k, l \) be any four points of \( \Omega \) and \( X \) be a 2-group such that \( X \leq N(G_{i, j, k, l}) \). Then we show that \( G_{i, j, k, l} \) has an involution \( x \) such that \( X \leq C(x) \), \( |I(x)| = 12 \) and \( C(x)^{(\sigma)} \leq M_{12} \). Since \( X \leq N(G_{i, j, k, l}) \), \( X \) normalizes a Sylow 2-subgroup \( P \) of \( G_{i, j, k, l} \). Since \( G \) is 4-fold transitive, \( P \) is conjugate to \( \langle Q, \sigma, \tau \rangle \). Hence \( P \) has a characteristic subgroup \( Q' \) which is conjugate to \( Q \). Then \( X \leq N(Q') \). Hence there is an involution \( x \) in \( Q' \) such that \( X \leq C(x) \). Since \( |I(Q')| = 12 \) and \( N(Q')^{(\sigma)} = M_{12} \), \( |I(x')| = 12 \) and \( C(x)^{(\sigma)} \leq M_{12} \). We remark that if \( x \) is the unique involution of \( Q' \) then \( C(x)^{(\sigma)} = M_{12} \).

(iii) We show that \( Q \) is a cyclic or generalized quaternion group and \( C(Q)^{(\Omega)} = N(Q)^{(\Omega)} \). Suppose by way of contradiction that \( Q \) has an involution \( b \) other than \( a \). Then since \( a \) is a central involution of \( Q \), we may assume that \( b = (1) (2) \cdots (12) (13 15) (14 16) (17 19) (18 20) (21 23) (22 24) \cdots \).
Then \( \langle a, b \rangle \leq N(G_{13,14,15,16}) \). Hence by (ii) \( G_{13,14,15,16} \) has an involution \( u \) such that \( \langle a, b \rangle \leq C(u) \), \( |I(u)| = 12 \) and \( C(u)^{I(u)} \leq M_{12} \). Then \( |I(a) \cap I(u)| = 0 \) or 4. If \( |I(a) \cap I(u)| = 4 \), then \( b^{I(u)} \) fixes the same four points that \( a \) fixes and commutes with \( a^{I(u)} \). This is a contradiction since \( C(u)^{I(u)} \leq M_{12} \). Hence \( |I(a) \cap I(u)| = 0 \). Then we may assume that

\[
u = (1 \ 3 \ 2 \ 4 \ 5 \ 7 \ 6 \ 8 \ 9 \ 11 \ 10 \ 12 \ 13 \ 14 \cdots (24) \cdots .
\]

Since \( \langle a, u \rangle \leq N(G_{13,14,15,16}) \), by (ii) \( G_{13,14,15,16} \) has an involution \( \nu \) such that \( \langle a, u \rangle \leq C(\nu) \), \( |I(\nu)| = 12 \) and \( C(\nu)^{I(\nu)} \leq M_{12} \). Let \( R \) be a Sylow 2-subgroup of \( \langle a, b, u, \nu \rangle \) containing \( \langle a, b, u \rangle \). Then \( R^{I(\nu)} = \langle u, \nu^2 \rangle^{I(\nu)} \). Hence \( R \) has an element \( \nu' \) such that \( \nu'^{I(\nu)} = \nu^{I(\nu)} \) and \( \nu' \) is conjugate to \( \nu \). Since \( u \in Z(\langle a, b, u, \nu \rangle) \), \( \nu' \) fixes \( I(u) \).

Since \( \nu' \) fixes 1,3 which are not contained in \( I(u) \) and \( |I(\nu')| = 12 \), \( \nu' \) does not fix \( I(u) \) pointwise. Furthermore \( I(u) \) is a union of of \( \langle a, b, u, \nu \rangle \)-orbits and \( \nu' \) is conjugate to \( \nu \) which has fixed points in \( I(u) \). Hence \( \nu' \) has fixed points in \( I(u) \) and so \( \nu' \) fixes exactly four points of \( I(u) \). Since \( (b\nu')^{I(u)} \) is a 2-element of \( C(u)^{I(u)} \leq M_{12} \), \( (b\nu')^{I(u)} \) is of order two, four or eight. If \( (b\nu')^{I(u)} \) is of order two, then \( b \) commutes with \( \nu' \). Hence \( \langle a, b, \nu \rangle^{I(u)} \) is a four group and \( |I(\langle a, b, \nu \rangle^{I(u)})| = 4 \). This is a contradiction since \( M_{12} \) has no such subgroup. If \( (b\nu')^{I(u)} \) is of order four or eight, then \( (b\nu')^{I(u)} \) is an involution fixing four points and \( I((b\nu')^2) \) or \( I((b\nu')^4) \) contains \( \{1, 2, \ldots, 12\} \) and four points of \( I(u) \), contrary to the assumption. Thus \( Q \) has exactly one involution and so \( Q \) is a cyclic or generalized quaternion group. Hence the automorphism group of \( Q \) is a 2-group or \( S_4 \). Since \( N(Q)^{I(Q)} = M_{12} \) and \( N(Q)^{I(Q)} / C(Q)^{I(Q)} \) is involved in the automorphism group of \( Q \), \( C(Q)^{I(Q)} = N(Q)^{I(Q)} \).

(iv) Thus \( a \) is the unique involution of \( Q \). Since \( a \in N(G_{13,14,15,16}) \), \( G_{13,14,15,16} \) has an involution \( x \) such that \( ax = xa \), \( |I(x)| = 12 \) and \( C(x)^{I(x)} = M_{12} \) by (ii). Then we may assume that \( x = x_1 \) and

\[
x_1 = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14 \cdots (20) \cdots .
\]

Since \( \langle a, x_1 \rangle \leq N(G_{5,6,13,14}) \), \( G_{5,6,13,14} \) has an involution \( x_2 \) such that \( \langle a, x_1 \rangle \leq C(x_2) \), \( |I(x_2)| = 12 \) and \( C(x_2)^{I(x_2)} = M_{12} \) by (ii). Then \( x_1, x_2 \) normalizes a Sylow 2-subgroup of \( G_{5,6,13,14} \) containing \( a \). Hence we may assume that \( x_1, x_2 \) normalizes \( Q \). Furthermore since \( N(Q)^{I(Q)} = M_{12} \) and \( C(x_1)^{I(x_1)} = M_{12} \), we may assume that

\[
x_2 = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15 \ 16 \ 17 \ 18 \ 19 \ 20) \cdots .
\]

or

\[
x_2 = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 11 \ 10 \ 12 \ 13 \ 14 \ 15 \ 16 \ 17 \ 19 \ 18 \ 20) \cdots .
\]

(v) We show that \( x_1, x_2 \in C(Q) \). Suppose by way of contradiction that \( x_1 \notin C(Q) \). Since \( \langle Q, x_1 \rangle \) is conjugate to \( \langle Q, x_2 \rangle \) in \( N(Q) \), there is an element
u in Q such that $x_u$ is conjugate to $x_i$ in $N(Q)$. Then $x_u \in C(Q)$ and $|I(x_u)| = 12$. Hence $x_u$ commutes with $u$ and so $x_u$ commutes with $u$. Since $x_u$ and $x_u$ are of order two, $u^2 = 1$. Hence $u = a$ or $1$. Thus $x_a \in C(Q)$. Since $\langle x_i, x_a \rangle \subset C(Q)$ and $|I(x_i) \cap I(x_a) \cap \Delta| = 2$ or $4$, $Q$ is of order two or four. Thus $Q$ is abelian. Then since $N(Q)^{(o)} = C(Q)^{(o)}$ by (iii), $y_i \in C(Q)$, $i = 1, 2$. Since $y_i^2 \in \langle Q, x_i \rangle$, there is an element $y_i$ in $Q$ such that $y_i^2 = u_i x_i$. Then $y_i$ commutes with $u_i x_i$. Since $y_i$ commutes with $u_i$, $x_i$ commutes with $t_i$. Hence $t_i$ fixes $I(x_i) \cap \Delta$. Furthermore since $x_i \in C(Q)$, $Q$ fixes $I(x_i) \cap \Delta$. Thus $I(x_i) \cap \Delta$ is a union of $\langle Q, y_i, y_a \rangle$-orbits.

Suppose that $Q$ is of order four. Since $\langle Q, y_i, y_a \rangle^{(x_i)}$ is not a quaternion group and $C(x_i)^{(x_i)} = M_{12}$, $\langle Q, y_i, y_a \rangle^{(x_i)} = M_{12}$. Hence $|\langle Q, y_i, y_a \rangle^{(x_i)}| = 8$ and so $Q_{y_i}$, $i = 1, 2$, has an element $y_i^2$ fixing $I(x_i) \cap \Delta$ pointwise. Then $I(y_i^2) = I(x_i)$. Since $N(G_{y_i})^{(x_i)} = C(x_i)^{(x_i)} = M_{12}$, for the four points 1, 2, 3, 4 of $I(x_a)$ a Sylow 2-subgroup of $G_{1234}$ containing $\langle y_i^2 \rangle$ is of order at least 8. This is a contradiction since $\langle Q, y_i, y_a \rangle$ is a Sylow 2-subgroup of $G_{1234}$ and of order 8-4.

Next suppose that $Q$ is of order two. Then by the same reason as above $\langle Q, y_i, y_a \rangle^{(x_i)}$ is a cyclic group of order two or four. Hence $\langle Q, y_i, y_a \rangle$ has an element $y$ which is of order four and fixes $I(x_i) \cap \Delta$ pointwise. Then by the same argument as above $G_{1234}$ has a Sylow 2-subgroup containing $y$ and of order at least 8-4. This is a contradiction since $\langle Q, y_i, y_a \rangle$ is a Sylow 2-subgroup of $G_{1234}$ and of order 8-2. Thus $x_i \notin C(Q)$. Similarly $x_a \notin C(Q)$.

(vi) Since $C(Q)^{(o)} = N(Q)^{(o)}$ and $x_i \notin C(Q)$, $Q$ is nonabelian. Hence by (iii) $Q$ is a generalized quaternion group. Moreover there are elements $b_i$ and $c_i$ in $Q$ such that $b_i x_i$ and $b_i x_i$ belong to $C(Q)$. Then $b_i x_i$ commutes with $b_i$, $i = 1, 2$. Hence $x_i$ commutes with $b_i$. Thus $b_i$ fixes $I(x_i)$. Since $|I(x_i) \cap I(Q)| = 4$ and $C(x_i)^{(x_i)} = M_{12}$, $b_i$ fixes exactly four points of $I(x_i)$ and so $b_i$ is of order two or four. If $b_i$ is of order two, then $b_i = a$ since $a$ is the unique involution of $Q$. This is a contradiction since $x_i \notin C(Q)$. Thus $b_i$ is of order four. Furthermore this shows that $\langle Q, y_i, y_a \rangle$ has exactly one central involution $a$.

Suppose that $Q$ is of order at least sixteen. Then we may assume that $Q = \langle c, d \rangle$, where $c^2 = d^{2r} = 1$ and $r \geq 3$. Suppose that $b_i \in \langle d \rangle$. Then since $d$ commutes with $b_i x_i$, $d$ commutes with $x_i$. Then $d$ fixes $I(x_i) \cap \Delta$ of length eight. Since $d$ is of order at least eight, $d$ is of order eight. Thus $d^{(x_i)}$ has four fixed points and one 8-cycle, which is a contradiction since $C(x_i)^{(x_i)} = M_{12}$. Thus $b_i \notin \langle d \rangle$ and so $Q = \langle b_i, d \rangle$. Similarly $Q = \langle b_i, d \rangle$. Hence $d^{b_i} = d^{-1}$, $i = 1, 2$, and so $d^{b_i x_i} = (d^{-1})^{x_i}$. On the other hand since $b_i x_i \in C(Q)$. Hence $d^{b_i x_i} = d$. Thus $d = d^2$ and so $d^{x_i} = d$. Since $|I(x_i, x_a)| = 12$, $|I(x_i, x_a) \cap I(Q)| = 4$ and $I(x_i, x_a) \cap \Delta \supseteq \{13, 14\}$, $2 \leq |I(x_i, x_a) \cap \Delta| \leq 8$. Then since $d$ is of order at least eight, $|I(x_i, x_a) \cap \Delta| = 8$ and $d$ is of order eight. Thus $|I(x_i, x_a)| = 12$ and $d^{(x_i, x_a)}$ has four fixed points and one 8-cycle. This implies that $C(x_i, x_a)^{(x_i, x_a)} = M_{12}$. 

MULTIPLY TRANSITIVE GROUPS XII

631
On the other hand for any four points $i, j, k, l$ of $I(x, x_2)$ let $P'$ be a Sylow 2-subgroup of $G_{i, j, k, l}$ containing $x, x_2$. Then since $G$ is 4-fold transitive, $P'$ is conjugate to $\langle Q, y_1, y_2 \rangle$. Hence $P'$ has the unique central involution $a'$ which is conjugate to $a$. Then $P'_{I(a')}^a$ is conjugate to $Q$ and $C(a')^Q = M_{12}$. If $x, x_2 = a'$, then $C(x, x_2)^{\langle x, x_2 \rangle} = M_{12}$, which is a contradiction. Hence $x, x_2 \neq a'$. Then since $P'_{I(a')}^a$ has exactly one involution $a'$, $x, x_2 \in P'_{I(a')}^a$. Hence $I(x, x_2) \cap I(a') = \{i, j, k, l\}$ because $C(a')^Q = M_{12}$. Thus $a^{I(a')}{x_2}$ fixes exactly four points $i, j, k, l$. Then by a lemma of Livingstone and Wanger [4] $C(x, x_2)^{\langle x, x_2 \rangle}$ is 4-fold transitive on $I(x, x_2)$. Since $C(x, x_2)^{\langle x, x_2 \rangle} = M_{12}$, $C(x, x_2)^{\langle x, x_2 \rangle} \geq A_{12}$. Then by Theorem $G = S_{14} \text{ or } A_{16}$, which is a contradiction.

Thus $Q$ is a quaternion group. Since $C(Q)^{\langle Q \rangle} = N(Q)^{\langle Q \rangle}$, $y_1$ has an element which belongs to $C(Q)$. Hence we may assume that $y_1, y_2 \in C(Q)$. Hence $y_1, y_2 \in C(Q) \cap Q \neq \langle a \rangle$. Thus $y_1 = b_1 x_1$ or $ab_1 x_1$ and so $y_1$ is of order eight. Furthermore $y_1$ commutes with $a$ and $b$. Hence $y_1$ commutes with $x_2$. Thus $y_1$ fixes $I(x_2)$ and so $y_1^{I(x_2)}$ has four fixed points and one 8-cycle. This is a contradiction since $C(x, x_2)^{\langle x, x_2 \rangle} = M_{12}$. Thus we complete the proof of (3.1).

3.2. Next we show that $Q$ is of order two and $Qx_1$ has an involution $x_1'$ such that $|I(x_1')| = 12$ and $C(x_1')^{I(x_2)} = M_{12}$.

Proof. By (3.1) $\langle Q, y_1, y_2 \rangle$ has an orbit $\Gamma$ in $\Delta$ such that $|\Gamma| = 8$ and $\langle Q, y_1, y_2 \rangle^\Gamma$ is a quaternion group. Then $Q$ is a quaternion group or a cyclic group of order four or two. Hence the automorphism group of $Q$ is $S_4$ or a 2-group. Furthermore $N(Q)^{\langle Q \rangle} = M_{12}$ and $N(Q)^{\langle Q \rangle}/C(Q)^{\langle Q \rangle}$ is involved in the automorphism group of $Q$. Hence $N(Q)^{\langle Q \rangle} = C(Q)^{\langle Q \rangle}$.

Suppose that $Q$ is a cyclic group of order four. Then since $N(Q)^{\langle Q \rangle} = C(Q)^{\langle Q \rangle}$ and $Q$ is abelian, any 2-element of $N(Q)$ is contained in $C(Q)$. Thus $Z(\langle Q, y_1, y_2 \rangle) \geq Q$. On the other hand $\langle Q, y_1, y_2 \rangle^\Gamma$ is a quaternion group. Hence $Q$ has an element $b$ of order four and $b^2 \in Z(\langle Q, y_1, y_2 \rangle^\Gamma)$, which is a contradiction. Thus the order of $Q$ is not four.

Since $\langle Q, y_1, y_2 \rangle^\Gamma$ is a quaternion group and $\langle Q, y_1, y_2 \rangle$ is of order at least 8·2, $\langle Q, y_1, y_2 \rangle^\Gamma$ has an involution, which is contained in $Qx_1$. Hence we may assume that $x_1 \in \langle Q, y_1, y_2 \rangle$. Then $x_1 \in Z(\langle Q, y_1, y_2 \rangle)$ and $|I(x_1)| = 12$. Let $x$ be any involution of $\langle Q, y_1, y_2 \rangle$ other than $a$ and $x_1$. Since $Q$ has exactly one involution $a$, $x \in Q$. Hence $x \in Qx_1$. Thus $x^{\langle Q \rangle} = x_1^{\langle Q \rangle}$ and so $xx_1$ is an involution of $Q$. Hence $xx_1 = a$ and so $x = ax_1$. Thus $\langle Q, y_1, y_2 \rangle$ has exactly three involution $a, x_1$, and $ax_1$, which are contained in $Z(\langle Q, y_1, y_2 \rangle)$.

Assume that $\langle Q, y_1, y_2 \rangle$ is a Sylow 2-subgroup of $G_{i, j, k, l}$. For any four points, $i, j, k, l$ of $I(x_1)$ let $P'$ be a Sylow 2-subgroup of $G_{i, j, k, l}$ containing $x_1$. Since $G$ is 4-fold transitive, $P'$ is conjugate to $\langle Q, y_1, y_2 \rangle$. Since any involution of $\langle Q, y_1, y_2 \rangle$ is contained in the center of $\langle Q, y_1, y_2 \rangle$, $x_1$ is contained in the center of $P'$. Thus $P'^{\langle x_1 \rangle} \leq C(x_1)^{\langle x_1 \rangle}$ and $P'^{\langle x_1 \rangle}$ fixes exactly four points $i, j,$
Then by a lemma of Livingstone and Wagner [4] \(C(x)^{\langle \sigma \rangle} = M_{12}\) by Theorem.

Assume that \(\langle Q, y, y \rangle\) is not a Sylow 2-subgroup of \(G_{1234}\). Then \(N(\langle Q, y, y \rangle)_{1234}\) has a 2-element \(x\) such that \(x^2 \in \langle Q, y, y \rangle\). If \(x\) fixes \(I(Q)\), then \(x^{1/2} \in \langle Q, y, y \rangle_{1234}\) since \(N(G_{1234}) = M_{12}\). Hence there is an element \(x''\) in \(\langle Q, y, y \rangle\) such that \(x'' = x^{1/2}\). Thus \(x''^{-1} \in \langle Q, y, y \rangle\) and so \(x'' \in \langle Q, y, y \rangle\), which is a contradiction. Thus \(x''\) does not fix \(I(Q)\).

Hence \(\langle Q, y \rangle \) has an element \(x_1\), where \(x_1 = x \) or \(ax_1\), such that \(|I(x_1)| = 12\) and \(C(x_1)^{\langle \sigma \rangle} = M_{12}\).

Since \(N(Q)\) has a 2-element \(*_2 = (1) (2) (3 4) (5) (6) (7 8) (9 10) (11 12)\),

and \(\langle Q, y, y \rangle\) is a 2-group. Then \(\langle Q, x_2 \rangle\) is conjugate to \(\langle Q, x_1 \rangle\). Hence we may assume that \(|I(x_2)| = 12\), \(x_2 \in C(Q)\), \(|I(x_2')| = 12\) and \(C(x_2')^{\langle \sigma \rangle} = M_{12}\), where \(x_2 = x_2\) or \(ax_2\).

Since \(x_2 \in N(\langle Q, y, y \rangle)\), \(x_2^2 = x_1\) or \(ax_1\). Suppose that \(x_2^2 = ax_1\). If \(Q\) is of order two, then \(\langle Q, x_2 \rangle\) is an elementary abelian group of order four. On the other hand \(\langle Q, x_2 \rangle\) is conjugate to \(\langle Q, x_1 \rangle\) and \(x_2\) is of order four, which is a contradiction. Thus \(Q\) is a quaternion group. Set \(\Gamma = I(ax_1) \cap \Delta\).

Then \(I(x_2) \cap \Delta) = \Gamma\). Hence \(|\Gamma'| = 8\) and \(\langle Q, y, y \rangle\) is a quaternion group. Since \(\langle Q, y, y \rangle\) has an element \(y\) fixing \(\Gamma\) pointwise. Then \(y' \in C(Q)\). Since \(Q\) is a quaternion group, \(y\) is the identity or an involution. Hence \(y\) is not the identity and fixes \(\{1, 2, 3, 4\} \cup \Gamma \cup \Gamma'\) pointwise. This is a contradiction since \(|\{1, 2, 3, 4\} \cup \Gamma \cup \Gamma'\}| = 20\). Thus \(x_2^2 = x_1\).

Then \(x_2\) and \(x_2\) commute. Since \(C(x_2')^{\langle \sigma \rangle} = M_{12}\), \(I(x_2') \cap I(x_2') = \{1, 2, i, j\}\), where \(\{i, j\} \subset \Delta\). Thus \(x_2\) fixes exactly two points \(i, j\) of \(\Delta\). Then since \(x_2\) fixes exactly two points \(i, j\) of \(\Delta\), \(Q\) is of order two.

3.3. Finally we show that \(|Q| = 2\) and complete the proof.

Proof. By (3.2) \(|Q| = 2\), and so \(Q = \langle a \rangle\) and \(\langle a, x_1 \rangle\) is an elementary abelian group of order four. Furthermore we may assume that \(C(x)^{\langle \sigma \rangle} = M_{12}\) and \(I(x_1) = \{1, 2, 3, 4, 13, 14, \ldots, 20\}\). Since \(N(Q)^{\langle \sigma \rangle} = C(a)^{\langle \sigma \rangle} = M_{12}\) and \(C(a)^{\langle \sigma \rangle} > \langle y_1, y_2 \rangle\), \(C(a)\) has 2-elements

\[
\begin{align*}
x_2 &= (1) (2) (3 4) (5) (6) (7 8) (9 11) (10 12) \ldots, \\
x_3 &= (1 2) (3 4) (5) (6) (7) (8) (9 10) (11 12) \ldots.
\end{align*}
\]

Then we may assume that \(\langle a, y, y, x_2, x_3 \rangle\) is a 2-group (see (2.3)). Since \(\langle a, x_1 \rangle\) is conjugate to \(\langle a, x_1 \rangle\) in \(C(a)\), \(i = 2, 3\), we may assume that \(|I(x_1)| = 12\) and \(C(x_1)^{\langle \sigma \rangle} = M_{12}\). Furthermore since \(\langle a, x_1 x_2 \rangle, i \neq j \) and \(1 \leq i, j \leq 3\), is conjugate to \(\langle a, x_1 \rangle x_1 x_j\) is of order two. Thus \(x_1\) and \(x_j\) commute and so \(\langle a, x_1, x_2, x_3 \rangle\) is elementary abelian.
Since \( a^{(\sigma)} = (1) (2) (3) (4) (13) (14) (15) (16) (17) (18) (19) (20) \) and \( C(x_i)^{(\sigma)} = M_{12} \), we may assume that \( x_i^{(\sigma)} = (1) (2) (3) (4) (13) (14) (15) (16) (17) (18) (19) (20) \) and \( x_i^{(\sigma)} = (1) (2) (3) (4) (13) (14) (15) (16) (17) (18) (19) (20) \). Since \( |I(x_i)| = 12 \), we may assume that \( I(x_i) = \{1, 2, 5, 6, 13, 14, 15, 16, 17, 18, 19, 20\} \). Then since \( a^{(\sigma)} = (1) (2) (3) (4) (13) (14) (15) (16) (17) (18) (19) (20) \) and \( C(x_i)^{(\sigma)} = M_{12} \), we may assume that \( \sigma = (1) (2) (3) (4) (13) (14) (15) (16) (17) (18) (19) (20) \), and \( \sigma = (1) (2) (3) (4) (13) (14) (15) (16) (17) (18) (19) (20) \). Since \( |I(x_i)| = 12 \), we may assume that \( I(x_i) = \{5, 6, 7, 8, 13, 14, 15, 16, 27, 28, 29, 30\} \). Then since \( a^{(\sigma)} = (1) (2) (3) (4) (13) (14) (15) (16) (17) (18) (19) (20) \), we may assume that \( \sigma = (1) (2) (3) (4) (13) (14) (15) (16) (17) (18) (19) (20) \). Then \( a \) is of order two and \( I(ax) \) contains \( \{9, 10, 11, 12, 17, 18, 19, 20, 23, 24, \ldots, 30\} \) of length sixteen, which is a contradiction. Thus we complete the proof of the lemma.

4. Proof of Corollary 1

In this section we assume that \( G \) is a 4-fold transitive group on \( \Omega = \{1, 2, \ldots, n\} \) and \( n \) is even. Let \( P \) be a Sylow 2-subgroup of a stabilizer of four points in \( G \). Then \( |I(P)| = 4 \) by Corollary of [13].

Proof of (1) of Corollary 1. We proceed by way of contradiction. We assume that \( G \) is a counter-example to (1) of Corollary 1 of the least possible degree. Then \( n \geq 35 \) ([2], p.80). Set \( I(P) = \{1, 2, 3, 4\} \). Let \( t \) be the maximal number of fixed points of involutions of \( G \) and \( Q \) be a Sylow 2-subgroup of \( G/\omega \) such that \( |I(Q)| = t \). For any four points \( i, j, k, l \) of \( I(Q) \) let \( P' \) be a Sylow 2-subgroup of \( G_{i,j,k,l} \) containing \( P \). Since \( G \) is 4-fold transitive, \( P' \) is conjugate to \( P \). Hence by the assumption \( I(P') = I(Z(P')) = \{i, j, k, l\} \). Thus \( C(Q)^{(\sigma)} \geq Z(P')^{(\sigma)} \) and \( I(Z(P'))^{(\sigma)} = \{i, j, k, l\} \). Hence by a lemma of Livingstone and Wagner [4], \( C(Q)^{(\sigma)} \) is 4-fold transitive on \( I(Q) \). If \( (C(Q)^{(\sigma)})_{i,j,k,l} \) is of odd order, then \( |I(Q)| = 4 \). Hence by a theorem of H. Nagao [10] \( G = S_6, A_8 \) or \( M_{12} \), which is a contradiction since \( n \geq 35 \). Hence \( (C(Q)^{(\sigma)})_{i,j,k,l} \) is of even order. Then \( C(Q)^{(\sigma)} \) satisfies the assumption of (1) of Corollary 1. Hence by the minimal nature of the degree of \( G \), \( C(Q)^{(\sigma)} = S_6, A_8 \) or \( M_{12} \). By Lemma \( C(Q)^{(\sigma)} \neq M_{12} \). If \( C(Q)^{(\sigma)} = S_6, A_8 \), then by Theorem \( G \geq A_6 \), which is a contradiction. Thus we complete the proof.

Proof of (2) of Corollary 1. If \( P_i = 1 \), then by a theorem of H. Nagao [10] \( G = S_6, A_8 \) or \( M_{12} \). Suppose that there is a point \( i \) of \( \Omega - I(P) \) such that \( P_i \neq 1 \). Let \( t \) be the maximal number of fixed points of involutions of \( G \). Since \( P_i \) is semiregular \( (\pm 1) \), we may assume that \( |I(P_i)| = t \). For any four points \( i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8 \) of \( I(P_i) \) let \( P' \) be a Sylow 2-subgroup of \( G_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8} \) containing \( P_i \). Then \( N_{P_i}(P_i)^{(\sigma)} \) is semiregular \( (\pm 1) \) and fixes exactly four points \( i_1, i_2, i_3, i_4 \). Hence by a lemma of Livingstone and Wagner [4] \( N_{P_i}(P_i)^{(\sigma)} \) is 4-fold transitive on \( I(P_i) \).
and by a theorem of H. Nagao [10] \( N(P_i)_{P} = S_6, A_8 \) or \( M_{12} \). Hence by Theorem and Lemma, \( G = S_6 \) or \( A_8 \). Thus we complete the proof.

5. Proof of Corollary 2

In this section we assume that \( G \) is a permutation group as in Corollary 2. We may assume that \( P \) is a Sylow 2-subgroup of \( G_{1234} \). Then by a corollary of [13] \( |I(P)| = 4, 5 \) or \( 7 \).

Suppose that \( |I(P)| = 4 \). Then \( n \) is even. Furthermore since \( P \) is transitive on \( \Omega - I(P) \), \( I(P) = I(Z(P)) \). Hence by Corollary 1, \( G = S_2^{n+1} (k \geq 1), A_2^{n+1} (k \geq 2) \) or \( M_{12} \).

Next suppose that \( |I(P)| = 5 \). Since \( P \) is transitive on \( \Omega - I(P) \), by a theorem of H. Nagao [9] \( G_{1234} \) is doubly transitive on \( \Omega - \{1, 2, 3, 4\} \). Then \( G \) satisfies the assumption of Corollary 2 and \( |I(P) - \{1\}| = 4 \). Hence by what we have proved above, \( G \) is one of the groups listed above. Hence \( G = S_2^{n+5} (k \geq 1) \) or \( A_2^{n+5} (k \geq 2) \).

Finally suppose that \( |I(P)| = 7 \). Then by a theorem of [12] \( G = M_{23} \). Thus we complete the proof.

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References
