Yamaguchi, A
Osaka J. Math.
33 (1996), 57-68

THE STRUCTURE OF THE HOPF ALGEBROID ASSOCIATED WITH THE ELLIPTIC HOMOLOGY THEORY

Dedicated to Professor Seiya Sasao on his 60th birthday

ATHUSHI YAMAGUCHI

(Received May 2, 1994)

Introduction

In representation theory of groups, there is an equivalence between the category of affine group schemes and the category of commutative Hopf algebras. This fact can be generalized to affine groupoid schemes, that is, a groupoid scheme is an internal groupoid in the category of schemes and there is an equivalence between the category of affine groupoid schemes and the category of Hopf algebroids.

On the other hand, by a work of J.F. Adams on generalized homology theory ([1], [2]) a commutative ring spectrum E such that E_*E is flat over $E_*=E_*(S^0)$ gives a Hopf algebroid (E_*, E_*E). One of the most important examples of such E is the complex cobordism spectrum MU. A theorem of Quillen enable us to identify the affine groupoid scheme represented by the Hopf algebroid defined from the complex cobordism theory with an affine groupoid scheme of formal groups and strict isomorphisms between them. Many homology theories related with complex cobordism (for example, BP-theory, Morava K-theory, elliptic homology) give closed (or locally closed) "subgroupoid schemes" of above groupoid scheme. This viewpoint is originally due to Jack Morava ([10]).

We give basic definitions and constructions on internal groupoids in sections 1 and 2. In section 3, we translate the language of previous sections in terms of Hopf algebroids. We give a general description of Hopf algebroids associated with complex oriented cohomology theories satisfying certain condition in Section 4. Finally, we determine the structure of the Hopf algebroid associated with the elliptic homology theory given by Landweber ([8]).

1. Internal categories and groupoids

DEFINITION 1.1. ([6]) Let $\mathscr{C}$ be a category with finite limits. An internal category $\boldsymbol{C}$ in $\mathscr{C}$ consists of the following objects and morphisms.

(1) A pair of objects C_0 (the *object-of-objects*) and C_1 (the *object-of-morphisms*).

(2) Four morphisms $\sigma: C_1 \to C_0$ (source), $\tau: C_1 \to C_0$ (target), $\varepsilon: C_0 \to C_1$ (iden-

tity), $\mu: C_1\times_{C_0}C_1\to C_1$ (composition), where C_1 on the right (resp. left) factor of the pull-back over C_0 is regarded as having structure map σ (resp. τ), such that $\sigma\varepsilon=\tau\varepsilon=id_{C_0}$ and the following diagrams commute.

$$\begin{array}{ccccc} C_1 & \xleftarrow{pr_1} & C_1\times_{C_0}C_1 & \xrightarrow{pr_2} & C_1 \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\tau} \\ C_0 & \xleftarrow{\sigma} & C_1 & \xrightarrow{\tau} & C_0 \end{array} \qquad \begin{array}{ccc} C_1\times_{C_0}C_1\times_{C_0}C_1 & \xrightarrow{\mu\times id_{C_1}} & C_1\times_{C_0}C_1 \\ \downarrow{\scriptstyle id_{C_1}\times\mu} & & \downarrow{\scriptstyle\mu} \\ C_1\times_{C_0}C_1 & \xrightarrow{\mu} & C_1 \end{array}$$

$$\begin{array}{ccccc} C_1\times_{C_0}C_0 & \xrightarrow{id_{C_1}\times\varepsilon} & C_1\times_{C_0}C_1 & \xleftarrow{\varepsilon\times id_{C_1}} & C_0\times_{C_0}C_1 \\ & \searrow{\scriptstyle pr_1} & \downarrow{\scriptstyle\mu} & \swarrow{\scriptstyle pr_2} & \\ & & C_1 & & \end{array}$$

We denote by $(C_1\rightrightarrows C_0)$ an internal category $\boldsymbol{C}$ whose object-of-objects and object-of-morphisms are C_0 and C_1, respectively.

A morphism $f: \boldsymbol{C}\to\boldsymbol{D}$ of internal categories (internal functor) consists of two morphisms $f_0: C_0\to D_0$ and $f_1: C_1\to D_1$ of $\mathscr{C}$ such that the following diagrams commute.

$$\begin{array}{ccccc} C_0 & \xleftarrow{\sigma} & C_1 & \xrightarrow{\tau} & C_1 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ D_0 & \xleftarrow{\sigma} & D_1 & \xrightarrow{\tau} & D_0 \end{array} \qquad \begin{array}{ccccc} C_1\times_{C_0}C_1 & \xrightarrow{\mu} & C_1 & \xleftarrow{\varepsilon} & C_0 \\ \downarrow{\scriptstyle f_1\times f_1} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ D_1\times_{D_0}D_1 & \xrightarrow{\mu} & D_1 & \xleftarrow{\varepsilon} & D_0 \end{array}$$

The above internal functor f is denoted by (f_1, f_0).

We denote by $\boldsymbol{cat}(\mathscr{C})$ the category of internal categories in $\mathscr{C}$.

A groupoid is a category whose morphisms are all isomorphisms. We also have an internal version of this notion.

DEFINITION 1.2. An internal groupoid $\boldsymbol{G}$ in $\mathscr{C}$ is an internal category in $\mathscr{C}$ with a morphism $\iota: G_1\to G_1$ (inverse) such that $\sigma\iota=\tau$, $\tau\iota=\sigma$ and the following diagram commutes.

$$\begin{array}{ccccc} G_1 & \xrightarrow{(id_{G_1},\,\iota)} & G_1\times_{G_0}G_1 & \xleftarrow{(\iota,\,id_{G_1})} & G_1 \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\tau} \\ G_0 & \xrightarrow{\varepsilon} & G_1 & \xleftarrow{\varepsilon} & G_0 \end{array}$$

An easy diagram chasing shows the following.

Proposition 1.3. *Let C be an internal category in $\mathscr{C}$. Suppose that morphisms α, β, $\gamma : D \to C_1$ satisfy $\tau\alpha = \sigma\beta$, $\sigma\gamma = \tau\beta$ and make the following diagram commute, then $\alpha = \gamma$.*

$$\begin{array}{ccccc} D & \xrightarrow{(\beta,\ \gamma)} & C_1 \times_{C_0} C_1 & \xleftarrow{(\alpha,\ \beta)} & D \\ \downarrow{\scriptstyle \sigma\beta} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \tau\beta} \\ C_0 & \xrightarrow{\varepsilon} & C_1 & \xleftarrow{\varepsilon} & C_0 \end{array}$$

In particular, if ι_1, $\iota_2 : G_1 \to G_1$ are morphisms satisfying $\tau\iota_1 = \sigma$, $\sigma\iota_2 = \tau$ and making

$$\begin{array}{ccccc} G_1 & \xrightarrow{(id_{G_1},\ \iota_2)} & G_1 \times_{G_0} G_1 & \xleftarrow{(\iota_1,\ id_{G_1})} & G_1 \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \tau} \\ G_0 & \xrightarrow{\varepsilon} & G_1 & \xleftarrow{\varepsilon} & G_0 \end{array}$$

commute, we have $\iota_1 = \iota_2$. This implies the uniqueness of the morphism $\iota : G_1 \to G_1$ satisfying the above conditions. Moreover, if $f : \boldsymbol{G} \to \boldsymbol{H}$ is an internal functor and both $\boldsymbol{G}$ and $\boldsymbol{H}$ are internal groupoids, it can be shown that f_1 commutes with ι by applying the above proposition for $\alpha = \iota f_1$, $\beta = f_1$, $\gamma = f_1\iota$. It follows that we can regard the category of internal groupoids in $\mathscr{C}$ as a full subcategory of $\boldsymbol{cat}(\mathscr{C})$. We denote by $\boldsymbol{gr}(\mathscr{C})$ the category of internal groupoids in $\mathscr{C}$.

2. Pull-back of internal categories

Let $\mathscr{C}$ be a category with finite limits and $\boldsymbol{C} = (C_1 \rightrightarrows C_0)$ an internal category in $\mathscr{C}$.

For morphisms $f : D \to C_0$, $g : E \to C_0$ of $\mathscr{C}$, we denote by $C_{f,g}$ a limit of a diagram

$$D \xrightarrow{f} C_0 \xleftarrow{\sigma} C_1 \xrightarrow{\tau} C_0 \xleftarrow{g} E.$$

$C_{f,g}$ is also denoted by $D \times_{C_0} C_1 \times_{C_0} E$. Define $\sigma_{f,g} : C_{f,g} \to D$, $f * g : C_{f,g} \to C_1$ and $\tau_{f,g} : C_{f,g} \to E$ to be the projections onto each component. If $\boldsymbol{C}$ is an internal groupoid, let $\iota_{f,g} : C_{f,g} \to C_{g,f}$ be the morphism induced by $\tau_{f,g}$, $\iota(f * g) : C_{f,g} \to C_1$ and $\sigma_{f,g}$.

Let $h : F \to C_0$ a morphism of $\mathscr{C}$. We define $\mu_{f,g,h} : C_{f,g} \times_E C_{g,h} \to C_{f,h}$ to be the composition $C_{f,g} \times_E C_{g,h} \cong D \times_{C_0} C_1 \times_{C_0} C_1 \times_{C_0} F \xrightarrow{id_D \times \mu \times id_F} C_{f,h}$.

In the case $D = E = F$ and $f = g = h$, we set $C_{f,f} = D_{f1}$, $\sigma_{f,f} = \sigma_f$, $\tau_{f,f} = \tau_f$, $\mu_{f,f,f} = \mu_f$, $f * f = \tilde{f}$, ($\iota_{f,f} = \iota_f$ if $\boldsymbol{C}$ is an internal groupoid) and denote by $\varepsilon_f : D \to D_{f1}$ the morphism induced by $id_D : D \to D$ and $\varepsilon f : D \to C_1$. Then we have an internal category $\boldsymbol{C}_f=(D_{f1}\rightrightarrows D)$ with structure maps σ_f, τ_f, ε_f, μ_f and also have an internal functor $(\tilde{f}, f): \boldsymbol{C}_f \to \boldsymbol{C}$. We call $\boldsymbol{C}_f$ a pull-back of $\boldsymbol{C}$ along f. Note that $\boldsymbol{C}_f$ is an internal groupoid with inverse ι_f if $\boldsymbol{C}$ is so.

The following fact is easily verified.

Proposition 2.1. 1) *Let $\boldsymbol{D}=(D_1\rightrightarrows D_0)$ be an internal category in $\mathscr{C}$ and $(f_1, f_0): \boldsymbol{D}\to\boldsymbol{C}$ an internal functor, then there is a unique morphism $h: D_1\to D_{f_01}$ such that $(h, id_{D_0}): \boldsymbol{D}\to\boldsymbol{C}_f$ is an internal functor and $f_1=\tilde{f}_0 h$.*

2) *Let $f': D'\to D$ and $g': E'\to E$ be morphisms of $\mathscr{C}$. Then, there is a natural isomorphism $C_{ff',gg'}\cong(C_{f,g})_{f',g'}$ which commutes with various structure maps.*

Definition 2.2. An internal functor $f=(f_1, f_0): \boldsymbol{D}\to\boldsymbol{C}$ is said to be faithful (resp. fully faithful) if the induced morphism $h: D_1\to D_{f_01}$ is a monomorphism (resp. isomorphism). An internal subcategory $\boldsymbol{D}$ of $\boldsymbol{C}$ consists of subobjects $D_0\to C_0$, $D_1\to C_1$ such that these monomorphisms give an internal functor. If this internal functor is fully faithful, $\boldsymbol{D}$ is called a full subcategory.

Remark 2.3. If f and g are monomorphisms, then $f*g: C_{f,g}\to C_1$ is also a monomorphism. If $\mathscr{C}$ is the category of sets, the image of $f*g$ consists of morphisms of C_0 whose sources and targets belong to the images of f and g, respectively.

3. Affine groupoid scheme

Let k be a commutative ring. We denote by $\boldsymbol{An}_k$ the category of commutative k-algebras and $\boldsymbol{An}_k^{\wedge}$ denotes the category of functors from $\boldsymbol{An}_k$ to the category of sets. Then, $\boldsymbol{An}_k^{\wedge}$ has finite limits. For an object A of $\boldsymbol{An}_k$, we denote by $h(A)$ the functor represented by A, that is, $h(A)(R)$ is the set of morphisms from A to R. We call a functor isomorphic to a such representable functor an affine k-scheme ([4], [5]).

Example 3.1. Let A, H be commutative k-algebras and put $G_0=h(A)$, $G_1=h(H)$. Then, giving (G_1, G_0) a structure of an internal groupoid in $\boldsymbol{An}_k^{\wedge}$ is equivalent to giving (A, H) a structure of a Hopf algebroid, that is, giving k-algebra homomorphisms σ_*, $\tau_*: A\to H$, $\varepsilon_*: H\to A$, $\iota_*: H\to H$ and $\mu_*: H\to H\otimes_A H$ such that $\varepsilon_*\sigma_*=\varepsilon_*\tau_*=id_A$, $\iota_*\sigma_*=\tau_*$, $\iota_*\tau_*=\sigma_*$ and the following diagrams commute. Here, we regard H as a left A-module by σ_* and a right A-module by τ_*.

$$
\begin{array}{ccc}
H\otimes_A H\otimes_A H & \xleftarrow{\mu_*\otimes 1} & H\otimes_A H \\
\uparrow{\scriptstyle 1\otimes\mu_*} & & \uparrow{\scriptstyle \mu_*} \\
H\otimes_A H & \xleftarrow{\mu_*} & H
\end{array}
\qquad
\begin{array}{ccccc}
H & \xleftarrow{(1,\ \iota_*)} & H\otimes_A H & \xrightarrow{(\iota_*,\ 1)} & H \\
\uparrow{\scriptstyle \sigma_*} & & \uparrow{\scriptstyle \mu_*} & & \uparrow{\scriptstyle \tau_*} \\
A & \xleftarrow{\varepsilon_*} & H & \xrightarrow{\varepsilon_*} & A
\end{array}
$$

$$
\begin{array}{ccccc}
H & \xrightarrow{i_1} & H\otimes_A H & \xleftarrow{i_2} & H \\
\uparrow{\scriptstyle \sigma_*} & & \uparrow{\scriptstyle \mu_*} & & \uparrow{\scriptstyle \tau_*} \\
A & \xrightarrow{\sigma_*} & H & \xleftarrow{\tau_*} & A
\end{array}
\qquad
\begin{array}{ccccc}
H\otimes_A A & \xleftarrow{1\otimes\varepsilon_*} & H\otimes_A H & \xrightarrow{\varepsilon_*\otimes 1} & A\otimes_A H \\
& {\scriptstyle j_1}\nwarrow & \uparrow{\scriptstyle \mu_*} & \nearrow{\scriptstyle j_2} & \\
& & H & &
\end{array}
$$

In the above diagrams, $i_1,\ i_2 : H\to H\otimes_A H$ and $(1,\ \iota_*),\ (\iota_*,\ 1) : H\otimes_A H\to H$ are maps defined by $i_1(x)=x\otimes 1$, $i_2(x)=1\otimes x$, $(1,\ \iota_*)(x\otimes y)=x\iota_*(y)$, $(\iota_*,\ 1)(x\otimes y)=\iota_*(x)y$. $j_1 : H\to H\otimes_A A$, $j_2 : H\to A\otimes_A H$ are isomorphisms defined by $j_1(x)=x\otimes 1$, $j_2(x)=1\otimes x$. In this case, we call $(G_1\rightrightarrows G_0)$ an affine k-groupoid scheme represented by a Hopf algebroid $(A,\ H)$. The full subcategory of $\boldsymbol{gr}(\boldsymbol{An}_k^\wedge)$ consisting of affine k-groupoid schemes is equivalent to the category of Hopf algebroids over k.

EXAMPLE 3.2. Let $\boldsymbol{G}=(G_1\rightrightarrows G_0)$ be an affine k-groupoid scheme represented by a Hopf algebroid $(A,\ H)$. We put $D=h(A_1)$, $E=h(A_2)$, $F=h(A_3)$ for k-algebras A_1, A_2, A_3 and Let $f : D\to G_0$, $g : E\to G_0$ and $h : F\to G_0$ be the morphisms induced by k-algebra homomorphisms $\varphi_i : A\to A_i$. Then, $G_{f,g}=h(A_1\otimes_A H\otimes_A A_2)$ and $\sigma_{f,g}$, $f*g$, $\tau_{f,g}$ are induced by the canonical maps $i_1 : A_1\to A_1\otimes_A H\otimes_A A_2$, $i_2 : H\to A_1\otimes_A H\otimes_A A_2$, $i_3 : A_2\to A_1\otimes_A H\otimes_A A_2$, respectively. $\iota_{f,g}$ is induced by $c : A_2\otimes_A H\otimes_A A_1\to A_1\otimes_A H\otimes_A A_2$ which maps $b\otimes x\otimes a$ to $a\otimes\iota_*(x)\otimes b$. $\mu_{f,g,h}$ is induced by composition $A_1\otimes_A H\otimes_A A_2 \xrightarrow{1\otimes\mu_*\otimes 1} A_1\otimes_A H\otimes_A H\otimes_A A_2 \xrightarrow{1\otimes j\otimes 1} (A_1\otimes_A H\otimes_A A_2)\otimes_{A_2}(A_2\otimes_A H\otimes_A A_3)$, where $j : H\otimes_A H\to H\otimes_A A_2\otimes_{A_2} A_2\otimes_A H$ maps $x\otimes y$ to $x\otimes 1\otimes 1\otimes y$.
In the case $A_1=A_2=A_3=B$ and $\varphi_1=\varphi_2=\varphi_3=\varphi$, ε_f is induced by composition $B\otimes_A H\otimes_A B \xrightarrow{1\otimes\varepsilon_*\otimes 1} B\otimes_A A\otimes_A B\cong B\otimes_A B \xrightarrow{prod} B$.

4. An application to complex cobordism

Let E be a commutative ring spectrum such that E_*E is a flat E_*-module and E_* is a k-algebra for a commutative ring k ($k=E_0$ for example). We assume that $E_n=E_nE=0$ if n is odd for simplicity. Then, $(E_*,\ E_*E)$ has a structure of a Hopf algebroid ([1]) and it represents an affine k-groupoid scheme. We denote this by $\boldsymbol{G}_E=(G_{E1}\rightrightarrows G_{E0})$. Let $\varphi : E_*\to A$ be a flat morphism of k-algebras, then a functor $X \mapsto A\otimes_{E_*}E_*(X)$ is a multiplicative homology theory. Hence there exists a ring spectrum E_φ and a ring map $E\to E_\varphi$ such that $E_{\varphi *}(X)=A\otimes_{E_*}E_*(X)$ for any spectrum X. Let $\psi: E_*\to B$ also be a flat morphism of k-algebras and put $g=h(\psi)$, then it is easy to verify the following.

Proposition 4.1. *$E_{\varphi*}E_\psi$ is isomorphic to $A\otimes_{E_*}E_*E\otimes_{E_*}B$. Hence $E_{\varphi*}E_\psi$ represents $(G_E)_{f,g}$. If $A=B$ and $\varphi=\psi$, $E_{\varphi*}E_\varphi$ is a flat $E_{\varphi*}$-module and $(E_{\varphi*}, E_{\varphi*}E_\varphi)$ represents the pull-back of $\boldsymbol{G}_E$ along f.*

We consider the case $E=MU$. By the theorem of Quillen, for each commutative ring R, $G_{MU0}(R)$ and $G_{MU1}(R)$ can be naturally identified with the set of formal group laws over R and the set of strict isomorphisms between them, respectively. We denote by $F_U(X, Y)\in MU_*[[X, Y]]$ the universal formal group law. Let $\varphi: MU_*\to A$ and $\psi: MU_*\to B$ be ring homomorphisms and put $F_\varphi(X, Y)=\varphi_*F_U(X, Y)\in A[[X, Y]]$, $F_\psi(X, Y)=\psi_*F_U(X, Y)\in B[[X, Y]]$. Suppose that both φ and ψ satisfy the following condition for $k=\boldsymbol{Z}$.

(*) *For a morphism $\theta: S\to T$ of $\boldsymbol{An}_k$, $h(\theta): h(T)(R)\to h(S)(R)$ is injective for any commutative k-algebra R.*

Then, we may regard $h(A)$ and $h(B)$ as subfunctors of G_{MU0} by $f=h(\varphi)$ and $g=h(\psi)$. Thus $h(A)(R)$ and $h(B)(R)$ are identified with $\{\theta_*F_\varphi(X, Y)|\theta\in h(A)(R)\}$ and $\{\zeta_*F_\psi(X, Y)|\zeta\in h(B)(R)\}$, respectively. It follows from (2.3) that $(G_{MU1})_{f,g}(R)$ is identified with the set of all strict isomorphisms from elements of $h(A)(R)$ to elements of $h(B)(R)$. Hence a formal power series $\gamma(X)=\sum_{j\geq 1}c_jX^j\in R[[X]]$ belongs to $(G_{MU1})_{f,g}(R)$ if and only if its coefficients c_i's satisfy $c_1=1$ and the following equality in $R[[X, Y]]$ for some $\theta\in h(A)(R)$, $\zeta\in h(B)(R)$.

$$\sum_{j\geq 1}c_j\theta_*F_\varphi(X, Y)^j=\zeta_*F_\psi(\textstyle\sum_{j\geq 1}c_jX^j, \sum_{j\geq 1}c_jY^j)$$

Let $i_1: A\to A\otimes B$ and $i_2: B\to A\otimes B$ be the canonical maps. We define a graded ring $H_{\varphi,\psi}$ by $H_{\varphi,\psi}=A\otimes B[u_1, u_2, \cdots, u_i, \cdots]/(u_1-1)+I_{\varphi,\psi}$, where $I_{\varphi,\psi}$ is the ideal generated by the coefficients of $\sum_{j\geq 1}u_ji_{1*}F_\varphi(X, Y)^j-i_{2*}F_\psi(\sum_{j\geq 1}u_jX^j, \sum_{j\geq 1}u_jY^j)\in A\otimes B[u_1, u_2, \cdots][[X, Y]]$ and $deg\ u_i=2i-2$. We denote by $\sigma_{f,g*}: A\to H_{\varphi,\psi}$ and $\tau_{f,g*}: B\to H_{\varphi,\psi}$ the compositions $A\xrightarrow{i_1}A\otimes B\hookrightarrow A\otimes B[u_1, u_2, \cdots]\xrightarrow{\text{proj}}H_{\varphi,\psi}$ and $B\xrightarrow{i_2}A\otimes B\hookrightarrow A\otimes B[u_1, u_2, \cdots]\xrightarrow{\text{proj}}H_{\varphi,\psi}$, respectively. Define a natural transformation $\Phi: h(H_{\varphi,\psi})\to(G_{MU1})_{f,g}$ as follows. For $\gamma\in h(H_{\varphi,\psi})(R)$, $\sum_{j\geq 1}\gamma(u_j)X^j\in R[[X]]$ is a strict isomorphism from $(\gamma\sigma_{f,g*})_*F_\varphi(X, Y)$ to $(\gamma\tau_{f,g*})_*F_\psi(X, Y)$ by the above construction. $\Phi_R: h(H_{\varphi,\psi})(R)\to(G_{MU1})_{f,g}(R)$ is defined to be the map which assigns $\sum_{j\geq 1}\gamma(u_j)X^j$ to γ. Conversely, for a strict isomorphism $\sum_{j\geq 1}c_jX^j\in R[[X]]$ from $\theta_*F_\varphi(X, Y)$ to $\zeta_*F_\psi(X, Y)$, define $\gamma: H_{\varphi,\psi}\to R$ by $\gamma(a\otimes b)=\theta(a)\zeta(b)$ $(a\in A,\ b\in B)$, $\gamma(u_j)=c_j$. Then, a correspondence $\sum_{j\geq 1}c_jX^j\mapsto\gamma$ gives the inverse of Φ. Thus $H_{\varphi,\psi}$ represents $(G_{MU1})_{f,g}$. It also follows that the source $\sigma_{f,g}: (G_{MU1})_{f,g}\to h(A)$ and the target $\tau_{f,g}: (G_{MU1})_{f,g}\to h(B)$

are induced by $\sigma_{f,g*}$ and $\tau_{f,g*}$.

Let $\xi: MU_*\to C$ be another ring homomorphism satisfying the condition (*). We set $h=h(\xi): h(C)\to G_{MU0}$. Recall that the composition $\mu_{f,g,h}: (G_{MU1})_{f,g}\times_{h(B)}(G_{MU1})_{g,h}\to(G_{MU1})_{f,h}$ is the composition of strict isomorphisms. On the other hand, the composition of $\gamma(X)=\sum_{j\geq 1}c_jX^j: \theta_*F_\varphi(X, Y)\to\zeta_*F_\psi(X, Y)$ and $\delta(X)=\sum_{l\geq 1}d_lX^l: \zeta_*F_\psi(X, Y)\to\eta_*F_\xi(X, Y)$ is given by $\delta\gamma(X)=\sum_{m\geq 1}e_mX^m$ for

$$e_m=\sum_{j_1+2j_2+\cdots=m}\frac{(j_1+j_2+\cdots)!}{j_1!j_2!\cdots}c_1^{j_1}c_2^{j_2}\cdots d_{j_1+j_2+\cdots}.$$

It is easily seen that the map $\mu_{f,g,h}$ is induced by a left A-linear right C-linear ring homomorphism $\mu_{f,g,h*}: H_{\varphi,\xi}\to H_{\varphi,\psi}\otimes_BH_{\psi,\xi}$ given by

$$\mu_{f,g,h*}(u_m)=\sum_{j_1+2j_2+\cdots=m}\frac{(j_1+j_2+\cdots)!}{j_1!j_2!\cdots}u_1^{j_1}u_2^{j_2}\cdots\otimes u_{j_1+j_2+\cdots}.$$

A formal power series $\sum_{j\geq 1}\bar{c}_jX^j\in R[[X]]$ is the inverse of a strict isomorphism $\gamma(X)=\sum_{j\geq 1}c_jX^j$ if and only if $\bar{c}_1=1$ and $\sum_{j_1+2j_2+\cdots=m}c_1^{j_1}c_2^{j_2}\cdots\bar{c}_{j_1+j_2+\cdots}=0$ for $m\geqq 2$. It follows that the inverse $\iota_{f,g}: (G_{MU1})_{f,g}\to(G_{MU1})_{g,f}$ is induced by $\iota_{f,g*}: H_{\psi,\varphi}\to H_{\varphi,\psi}$, $\iota_{f,g*}(b\otimes a)=a\otimes b$ $(a\in A,\ b\in B)$, $\iota_{f,g*}(u_j)=\bar{u}_j$ where $\bar{u}_j$'s are inductively given by $\bar{u}_1=1$ and $\sum_{j_1+2j_2+\cdots=m}u_1^{j_1}u_2^{j_2}\cdots\bar{u}_{j_1+j_2+\cdots}=0$ for $m\geqq 2$.

In the case $A=B$ and $\varphi=\psi$, it is obvious that the identity $\varepsilon_f: h(A)\to(G_{MU1})_f$ is induced by $\varepsilon_{f*}: H_{\varphi,\varphi}\to A$, $\varepsilon_{f*}(a\otimes 1)=\varepsilon_{f*}(1\otimes a)=a$, $\varepsilon_{f*}(u_j)=0$ for $j\geqq 2$.

Summarizing the argument so far, we have shown the following result.

Theorem 4.2. *Let $\varphi: MU_*\to A$, $\psi: MU_*\to B$ and $\xi: MU_*\to C$ be ring homomorphisms satisfying condition (*). Set $f=h(\varphi): h(A)\to G_{MU0}$, $g=h(\psi): h(B)\to G_{MU0}$, $h=h(\xi): h(C)\to G_{MU0}$ and $F_\varphi(X, Y)=\varphi_*F_U(X, Y)$, $F_\psi(X, Y)=\psi_*F_U(X, Y)$.*

1) *$(G_{MU1})_{f,g}$ is represented by a graded ring $H_{\varphi,\psi}=A\otimes B[u_1, u_2, \cdots, u_i, \cdots]/(u_1-1)+I_{\varphi,\psi}$, where $I_{\varphi,\psi}$ is the ideal generated by the coefficients of $\sum_{j\geq 1}u_ji_{1*}F_\varphi(X, Y)^j-i_{2*}F_\psi(\sum_{j\geq 1}u_jX^j, \sum_{j\geq 1}u_jY^j)\in A\otimes B[u_1, u_2, \cdots][[X, Y]]$ and $\deg u_i=2i-2$.*

2) *The source $\sigma_{f,g}: (G_{MU1})_{f,g}\to h(A)$ and the target $\tau_{f,g}\to h(B)$ are induced by compositions $A\xrightarrow{i_1}A\otimes B\hookrightarrow A\otimes B[u_1, u_2, \cdots]\xrightarrow{proj}H_{\varphi,\psi}$ and $B\xrightarrow{i_2}A\otimes B\hookrightarrow A\otimes B[u_1, u_2, \cdots]\xrightarrow{proj}H_{\varphi,\psi}$, respectively.*

3) The composition $\mu_{f,g,h}: (G_{MU1})_{f,g}\times_{h(B)}(G_{MU1})_{g,h}\to(G_{MU1})_{f,h}$ is induced by a left A-linear right C-linear ring homomorphism $\mu_{f,g,h*}: H_{\varphi,\xi}\to H_{\varphi,\psi}\otimes_BH_{\psi,\xi}$ given by

$$\mu_{f,g,h*}(u_m)=\sum_{j_1+2j_2+\cdots=m}\frac{(j_1+j_2+\cdots)!}{j_1!j_2!\cdots}u_1^{j_1}u_2^{j_2}\cdots\otimes u_{j_1+j_2+\cdots}.$$

4) *The inverse $\iota_{f,g}: (G_{MU1})_{f,g}\to(G_{MU1})_{g,f}$ is induced by $\iota_{f,g*}: H_{\psi,\varphi}\to H_{\varphi,\psi}$,*

$\iota_{f,g*}(b\otimes a)=a\otimes b$ $(a\in A,\ b\in B)$, $\iota_{f,g*}(u_j)=\bar{u}_j$ *where* $\bar{u}$*'s are inductively given by* $\bar{u}_1=1$ *and* $\sum_{j_1+2j_2+\cdots=m}u_1^{j_1}u_2^{j_2}\cdots\bar{u}_{j_1+j_2+\cdots}=0$ *for* $m\geqq 2$.

5) *In the case* $A=B$ *and* $\varphi=\psi$, *the identity* $\varepsilon_f: h(A)\to(G_{MU1})_f$ *is induced by* $\varepsilon_{f*}: H_{\varphi,\varphi}\to A$, $\varepsilon_{f*}(a\otimes 1)=\varepsilon_{f*}(1\otimes a)=a$, $\varepsilon_{f*}(u_j)=0$ *for* $j\geqq 2$.

Applying (4.1) to the above, we have

Theorem 4.3. *If* $\varphi: MU_*\to A$ *and* $\psi: MU_*\to B$ *are flat and satisfy condition* (*), *then* $MU_{\varphi*}MU_\psi$ *is isomorphic to* $H_{\varphi,\psi}$. *Moreover, if* $A=B$ *and* $\varphi=\psi$, *Hopf algebroid* $(A,\ MU_{\varphi*}MU_\varphi)$ *is isomorphic to* $(A,\ H_{\varphi,\varphi})$.

5. Examples

EXAMPLE 5.1. ([3]) We set $A=\boldsymbol{Z}[t,t^{-1}]$ $(deg\ t=2)$. There exists a ring homomorphism $\varphi: MU_*\to A$ which maps $[CP^n]$ to $(-1)^n t^n$. φ is flat ([7]) and satisfy (*). In fact, $h(A)$ is a locally closed subscheme of G_{MU0}. $F_\varphi(X,\ Y)$ is the multiplicative formal group law $X+Y+tXY$. Put $i_1(t)=t\otimes 1=x$, $i_2(t)=1\otimes t=y$. K_*K is isomorphic to the quotient of $\boldsymbol{Z}[x,\ y,\ x^{-1},\ y^{-1},\ u_1,\ u_2,\ \cdots]$ by the ideal generated by u_1-1, $yu_iu_j-\sum_{n=\max\{i,j\}}^{i+j}\frac{n!}{(n-i)!(n-j)!(i+j-n)!}x^{i+j-n}u_n$ $(i,\ j\geqq 0)$.

REMARK 5.2. The relations of K_*K imply $n!u_n=(y-x)(y-2x)\cdots(y-(n-1)x)$. Since K_*K is flat over A and A is flat over $\boldsymbol{Z}$, K_*K is flat over $\boldsymbol{Z}$ hence torsion free. We can regard K_*K as a subalgebra of $K_*K\otimes\boldsymbol{Q}$. Then, in $K_*K\otimes\boldsymbol{Q}$, $u_n=\frac{1}{n!}(y-x)(y-2x)\cdots(y-(n-1)x)\in\boldsymbol{Q}[x,\ y]$. Moreover, put $p_n=x^{-n}yu_n\in K_*K$, then $p_n=\frac{1}{n!}\frac{y}{x}\left(\frac{y}{x}-1\right)\left(\frac{y}{x}-2\right)\cdots\left(\frac{y}{x}-(n-1)\right)\in K_*K\otimes\boldsymbol{Q}$ and K_*K is isomorphic to the quotient of $\boldsymbol{Z}[x,\ y,\ x^{-1},\ y^{-1},\ p_1,\ p_2,\ \cdots]$ by the ideal generated by xp_1-y, $p_ip_j-\sum_{n=\max\{i,j\}}^{i+j}\frac{n!}{(n-i)!(n-j)!(i+j-n)!}p_n$ $(i,\ j\geqq 0)$.

Thus we have the following arithmetic result.

Proposition 5.3. *Let* $q_n(X)$ $(n\geqq 1)$ be a polynominal $\frac{1}{n!}X(X-1)(X-2)\cdots(X-(n-1))$. *Then, we have*

$$q_i(X)q_j(X)=\sum_{n=j}^{i+j}\frac{n!}{(n-i)!(n-j)!(i+j-n)!}q_n(X)\ \ if\ \ i\leqq j.$$

In order to determine the structure of the Hops algebroid associated with the elliptic homology theory, we make some preparations.

DEFINITION 5.4. A formal group law $F(X, Y)$ is said to be odd if $F(-X, -Y)=-F(X, Y)$.

Let $F(X, Y)$ be a formal group law over a ring R which is torsion free. Since the logarithm $\log_F X$ is given by the integral $\int_0^X \left(\frac{\partial F}{\partial Y}(T, 0)\right)^{-1} dT$, we observe the following fact.

Proposition 5.5. *$F(X, Y)$ is odd if and only if the logarithm $\log_F X$ is of the form $X+\sum_{j\geq 1} a_{2j+1}X^{2j+1}$.*

Let L_0 be the Lazard ring for odd formal group law, that is, the quotient of the Lazard ring $L=MU_*$ by an ideal generated by the coefficients of $F_U(-X, -Y)+F_U(X, Y)$. Since $L=MU_*$ is generated by the coefficients of $F_U(X, Y)$, it follows that $L_0\otimes \boldsymbol{Z}\left[\frac{1}{2}\right]$ is the quotient by the ideal of $MU\left[\frac{1}{2}\right]_*$ generated by elements of degree not a multiple of four. On the other hand, it is known that the map of spectra $\rho: MU\to MSO$, forgetting weakly almost complex structures, induces an epimorphism $\rho_*: MU\left[\frac{1}{2}\right]_*\to MSO\left[\frac{1}{2}\right]_*$ where kernel is generated by elements of degree not a multiple of four ([11, p. 179]). Hence $MSO\left[\frac{1}{2}\right]_*$ is isomorphic to $L_0\otimes \boldsymbol{Z}\left[\frac{1}{2}\right]$ and the map $MU\left[\frac{1}{2}\right]_*\to L_0\otimes \boldsymbol{Z}\left[\frac{1}{2}\right]$ obtained by localizing the projection $L=MU_*\to L_0$ away from 2 can be identified with ρ_*. Moreover, ρ_* maps $F_U(X, Y)$ to the universal odd formal group law over $\boldsymbol{Z}\left[\frac{1}{2}\right]$-algebras.

There is a natural isomorphism $MSO\left[\frac{1}{2}\right]_*(X)\cong MSO\left[\frac{1}{2}\right]_*\otimes_{MU\,MU\left[\frac{1}{2}\right]_*} MU\left[\frac{1}{2}\right]_*(X)$ for each spectrum X([7]) and this implies the following isomorphism.

(5.6)

$$MSO\left[\frac{1}{2}\right]_* MSO\left[\frac{1}{2}\right]\cong MSO\left[\frac{1}{2}\right]_*\otimes_{MU\left[\frac{1}{2}\right]_*} MU\left[\frac{1}{2}\right]_* MU\left[\frac{1}{2}\right]\otimes_{MU\left[\frac{1}{2}\right]_*} MSO\left[\frac{1}{2}\right]_*$$

Thus, by (3.2), we see

Proposition 5.7. $\left(MSO\left[\frac{1}{2}\right]_*, MSO\left[\frac{1}{2}\right]_* MSO\left[\frac{1}{2}\right]\right)$ *is a Hopf algebroid* *representing an affine groupoid scheme over* $\boldsymbol{Z}\left[\frac{1}{2}\right]$ *which associates each* $\boldsymbol{Z}\left[\frac{1}{2}\right]$*-algebra R a category of odd formal group laws and strict isomorphisms defined over R.*

Reacall that, if E is a complex oriented commutative ring spectrum such that E_* is a $\boldsymbol{Z}\left[\frac{1}{2}\right]$-algebra, there is an isomorphism $E_*(BSO_+)\cong E_*[p_1, p_2, \cdots]$ ($deg\ p_i=4i$), where p_i is the i-th Pontrjagin class. Applying the Thom isomorphism $E_*(MSO)\cong E_*(BSO_+)$, we obtain

Proposition 5.8 *$E_*(MSO)$ is isomorphic to $E_*[t_2, t_4, \cdots]$, where $deg\ t_{2i}=4i$. Hence if $E_n=0$ for $n\not\equiv 0$ modulo 4, $E_n(MSO)=0$ for $n\not\equiv 0$ modulo 4. In particular,* $MSO\left[\frac{1}{2}\right]_n MSO\left[\frac{1}{2}\right]=0$ *for $n\not\equiv 0$ modulo 4.*

Let $\varphi: MU_*\to A$ be a flat ring homomorphism which factors through $MU_*\to MSO\left[\frac{1}{2}\right]_*$, namely, φ is the classifying map of an odd formal group law. By (4.1), (5.6) and (5.8), we have the following fact.

Proposition 5.9. *$MU_{\varphi*}MU_\varphi$ is isomorphic to* $A\otimes_{MU[\frac{1}{2}]_*} MSO\left[\frac{1}{2}\right]_* MSO\left[\frac{1}{2}\right]\otimes_{MSO[\frac{1}{2}]_*}A$. *If $A_n=0$ for $n\not\equiv 0$ modulo 4, $MU_{\varphi n}MU_\varphi=0$ holds for $n\not\equiv 0$ modulo 4.*

EXAMPLE 5.10. Put $A=\boldsymbol{Z}\left[\frac{1}{2}\right][\delta, \varepsilon, \varepsilon^{-1}(\delta^2-\varepsilon)^{-2}]$ ($deg\ \delta=4$, $deg\ \varepsilon=8$). Consider a formal group law $F(X, Y)=(X(1-2\delta Y^2+\varepsilon Y^4)^{\frac{1}{2}}+Y(1-2\delta X^2+\varepsilon X^4)^{\frac{1}{2}})(1-\varepsilon X^2Y^2)^{-1}$ over A and let $\varphi: MU_*\to A$ be the classifying map of $F(X, Y)$, that is, $F(X, Y)=\varphi_*F_U(X, Y)$. It is known that φ is flat ([8]) and $h(\varphi): h(A)\to G_{MU0}$ is an embedding of schemes. Then, we have a ring spectrum E_φ which is denoted by Ell. Since $F(X, Y)$ is an odd formal group law and $A_n=0$ if $n\not\equiv 0$ modulo 4, (5.7) and (5.9) imply $Ell_nEll=0$ if $n\not\equiv 0$ modulo 4 and we can drop the generators u_{2i}'s of (4.2).

Set $c(k, l, X, Y)=\sum_{\frac{l-k}{2}\leq i\leq l-k}\begin{pmatrix}\frac{1}{2}\\ i\end{pmatrix}\begin{pmatrix}i\\ l-k-i\end{pmatrix}(-2X)^{2i+k-l}Y^{l-i}\in\boldsymbol{Z}\left[\frac{1}{2}\right][X, Y]$ for $0\leq k\leq l$, then we have $F(X, Y)=\sum_{0\leq k\leq l}c(k, l, \delta, \varepsilon)(X^{2k+l}Y^{2l}+X^{2l}Y^{2k+1})$. We put $i_1(\delta)=\delta\otimes 1=x$, $i_1(\varepsilon)=\varepsilon\otimes 1=y$, $i_2(\delta)=1\otimes\delta=z$, $i_2(\varepsilon)=1\otimes\varepsilon=w$ in

$A \otimes A$. If $(k, \mu, i_2, \cdots, i_s, \cdots)$ is a sequence of non-negative integers such that $i_s = 0$ for all but finitely many s's, put

$$A(k, \mu, i_1, i_2, \cdots, i_s, \cdots) = \sum_{\substack{\Sigma_s (2s-1) j_s = 2\mu - 1 \\ \Sigma_s j_s = 2k+1, 0 \leqq j_s \leqq i_s}} \frac{(2k+1)!(\sum_s i_s - 2k - 1)!}{\prod s j_s ! (i_s - j_s)!}.$$

Then $Ell_* Ell$ is isomorphic to
$\mathbf{Z}\left[\frac{1}{2}\right][x, y, z, w, (yw(x^2-y)^2(z^2-w)^2)^{-1}, u_1, u_3, u_5, \cdots]/(u_1 - 1) + (U(\mu, \nu) | \mu, \nu \geqq 1)$, where $U(\mu, \nu)$ is given by

$$\sum_{\substack{\Sigma((2k+1)\alpha_{k,l} + 2l\beta_{k,l}) = 2\mu - 1 \\ \Sigma(2l\alpha_{k,l} + (2k+1)\beta_{k,l}) = 2\nu}} \left(\frac{(\sum_{k \leqq l} (\alpha_{k,l} + \beta_{k,l}))!}{\prod_{k \leqq l} (\alpha_{k,l}! \beta_{k,l}!)} \prod_{k \leqq l} c(k, \quad l, \quad x, \quad y)^{\alpha_{k,l} + \beta_{k,l}} \right) u_{\Sigma_{k \leqq l}(\alpha_{k,l} + \beta_{k,l})}$$
$$- \sum_{\Sigma_s (2s-1) i_s = 2\mu + 2\nu - 1} \Big(\sum_{k + l = \frac{1}{2}(\Sigma_s i_s - 1), 0 \leqq k \leqq l} A(k, \mu, i_1, i_2, \cdots) c(k, l, z, w) \Big) u_1^{i_1} u_3^{i_2} u_5^{i_3} \cdots.$$

References

[1] Adams. J. F: *Lectures on generalised cohomology*, Lecture Notes in Math, vol. **99**, (1969), Springer-Verlag, Berlin-Heidelberg-New York, 1-138

[2] Adams. J. F: Stable Homotopy and Generalised Homology, University of Chicago Press, Chicago, 1974.

[3] Adams. J. F., Harris. A.S., Switzer. R.M.: *Hopf algebras of cooperations for real and complex K-theory*, Proc. London Math. Soc. (3) **23** (1971), 385-408.

[4] Demazure. M. and Gabriel. P., Groupes Algebriques, North-Holland, Amsterdam, 1970.

[5] Demazure. M. and Gabriel. P., Introduction to Algebraic Geometry and Algebraic Groups, North-Holland Mathematics studies 39, Amsterdam, 1980.

[6] Johnstone. P. T: Topos Theory, Academic Press, 1977.

[7] Landweber. P.S: *Homological properties of comodules over $MU_*(MU)$ and $BP_*(BP)$*, Amer. J. Math., **98** (1976), 591-610.

[8] Landweber. P.S: *Elliptic cohomology and modular forms*, Lecture Notes in Math., **1326** (1988) Springer-Verlag, 55-68.

[9] Mac Lane. S: Categories for the Working Mathematician, Granduate Texts in Math., vol. **5**, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[10] Morava. J: *Structure theorem for cobordism comodules*, Preprint.

[11] Stong. R.E: Notes on cobordism, Math. Notes, Princeton Univ. Press, Princeton, N.J. 1968.

[12] Yamaguchi. A: *A note on algebraic geometry over graded rings and representation theory of groupoid schemes*, Preprint.

Depertment of Mathematics
University of Osaka Prefecture
Sakai, Osaka 593
Japan